UNIVERSITY OF CALIFORNIA SAN DIEGO

On two variant models of branching Brownian motion

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The dissertation of Jiaqi Liu is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

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DEDICATION

Dedicated to my family.

EPIGRAPH

人生天地间，忽如远行客。

——【汉】佚名《古诗十九首》

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ABSTRACT OF THE DISSERTATION

On two variant models of branching Brownian motion

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Branching Brownian motion is a random particle system which incorporates both the tree-like structure and the diffusion process. We consider two variant models of branching Brownian motion, branching Brownian motion with absorption and branching Brownian motion with an inhomogeneous branching rate.

In the first variant model, branching Brownian motion with absorption, particles move as Brownian motions with drift $-\rho$, undergo dyadic branching at rate 1, and are killed when they reach the origin. Kesten (1978) first introduced this model and showed that $\rho = \sqrt{2}$ is the critical value separating the supercritical case $\rho < \sqrt{2}$ and the subcritical case $\rho > \sqrt{2}$. We study the transition of the model from the slightly subcritical regime to the critical regime. Write $\rho = \sqrt{2 + 2\varepsilon}$. We obtain a Yaglom type asymptotic result, showing that the long-run expected number of particles conditioned on survival grows exponentially

as $1/\sqrt{\varepsilon}$ as the process gets closer to being critical.

In the second variant model, branching Brownian motion with an inhomogeneous branching rate, each particle independently moves as Brownian motion with negative drift, each particle can die or undergo dyadic fission, and the difference between the birth rate and the death rate is proportional to the particle's location. This model was first considered by Roberts and Schweinsberg (2021) and models the evolution of populations undergoing selection. Aiming to understand the distribution of fitness levels of individuals in a large population undergoing selection, we study the particle configurations of this model from the left edge to the right edge. We show that, under certain assumptions, after a sufficiently long time, the distribution of individual fitnesses from the least fit individuals to the most fit individuals is approximately a traveling wave with a profile related to the Airy function. Our work, complements the results in Roberts and Schweinsberg (2021), giving a fuller picture of the fitness distribution.

Chapter 1

Introduction

Branching Brownian motion (BBM) is a random particle system which incorporates both the tree-like structure and the diffusion process. BBM has a natural interpretation as a population model. Variations of BBM can be used to model the evolution of populations under different constraints, and therefore provide mathematical justifications for biological observations. Beyond the biological aspect, BBM also has intrinsic relations with partial differential equations and statistical physics. In this chapter, we first introduce the definitions of BBM and its variant models. In Section 1.2, we explain how BBM can be viewed as a population model. In Section 1.3, we give a brief review of the results on the frontier of BBM, showing how BBM is related to one type of partial differential equation, the F-KPP equation. The chapter ends with two techniques that are widely used in the study of BBM.

1.1 Definition

The study of branching processes originated from the work of Bienaymé [18], and Galton and Watson [86]. They considered stochastic models which record the number of alive descendants in each generation, with the key feature that in every generation, each individual gives birth to a random number of descendants independently of the others. In 1962, in order to describe both random growth and random dispersal of a population in continuous spaces at the same time, Adke and Moyal [5] introduced BBM. The following definition is similar to the one given in [5], and we refer to it as the ordinary BBM. Let $r > 0$ be a fixed constant and $(p_k)_{k=0}^{\infty}$ be a probability law with support on non-negative integers.

Definition 1. *The ordinary BBM is a continuous-time stochastic defined as follows.*

- *At time* 0*, there is a single particle at* $x \in \mathbb{R}$*.*
- *During its lifetime, each particle independently moves according to one-dimensional Brownian motion.*
- *Each particle has an exponentially distributed lifetime with rate* r*. The lifetime of each particle is independent of its position and of all other particles.*
- *At the end of the lifetime, each particle independently splits into a random number of offspring according to the probability law* $(p_k)_{k\in\mathbb{N}}$*.*

Here, r is the branching rate and $(p_k)_{k\in\mathbb{N}}$ is the branching distribution (or reproduction law). In general, there are three places where we can modify the above definition and generate variant models of BBM.

First, the motion of particles can be generalized to other stochastic processes other than the standard Brownian motion. For example, particles can move as Brownian motion with an inhomogeneous variance. To be more precise, let $A(x)$ be the function which characterizes the inhomogeneous variance of Brownian motion. The function $A(x)$ is increasing and right-continuous with $A(0) = 0$ and $A(1) = 1$. Let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion. Fix a time horizon T and define $(B_t^A)_{0\leq t\leq T}$ to be a time change of the standard Brownian motion on $[0, T]$

$$B_t^A = B_{TA(t/T)}.$$

We see that $(B_t^A)_{0\leq t\leq T}$ is the Brownian motion with inhomogeneous variance $A(x)$. BBM with inhomogeneous variance is defined as the ordinary BBM except that particles move as Brownian motion with inhomogeneous variance $A(x)$. The case $A(x) = x$ corresponds to the ordinary BBM with homogeneous variance. Fang and Zeitouni [37], and Bovier and Hartung [20] studied the case where $A(x)$ is a piecewise linear function. The case studied by Fang and Zeitouni [38], and Maillard and Zeitouni [64] falls in the regime where $A(x) > x$ for some $x \in (0, 1)$. Bovier and Hartung [21] considered the weak correlation regime, where $A(x) < x$ for all $x \in (0, 1)$. In [22], Bovier and Hartung studied the transitional behavior of the process of by considering functions $A(x)$ that lie slightly above or below $A(x) = x$. Another well-studied variant model is the branching Ornstein-Uhlenbeck process. As the name suggests, particles move according to the Ornstein-Uhlenbeck process satisfying the stochastic differential equation

$$dY(t) = -\mu Y(t)dt + dB_t,$$

where $\mu > 0$. This model is interesting for two reasons. First, it was conjectured by Cortines and Mallein [30] that branching Ornstein-Uhlenbeck process has unusual genealogical behavior when undergoing selection (particles killed at a barrier). Second, the Ornstein-Uhlenbeck process has a stationary distribution. After a period of time, positions of particles are approximately independent. Therefore, this branching process is know to satisfy a Law of Large Numbers (see Enderle and Hering [35]). As for the Central Limit Theorems, the normalization and limit depends on the relation between the branching parameter r and the parameter of the Ornstein-Uhlenbeck process μ. See e.g. Adamczak and Miłoś [2, 3].

Second, selection criteria can be added to the branching particle system. For example, particles hitting certain boundaries can be removed from the system. The most well-studied model falls into this category is BBM with absorption, in which particles move as Brownian motion with drift $-\rho$ ($\rho \in \mathbb{R}$), undergo dyadic branching at rate 1, and are killed when they reach the origin. Kesten [56] first studied this model in 1978 and delineated the regions where the process is subcritical, critical or supercritical. He showed that $\rho = \sqrt{2}$ is the critical value separating the supercritical case $\rho < \sqrt{2}$ and the subcritical case $\rho > \sqrt{2}$.

The critical behavior of the process was studied by [12, 13, 56, 61, 63]. Berestycki et al. [10, 11] studied the slightly supercritical regime. The long run survival probability in the subcritical case was studied by [48]. Another model with killing at barriers is BBM in a strip, in which particles are killed not only at 0, but also at some $K > 0$. This model was first studied by [51]. Another selection criterion is keeping the total number of particles in the system fixed. Because of the strong interactions between particles, BBM processes under such selection criteria are usually difficult to analyze rigorously. The model N-BBM is one of them, in which the total number of particles is kept constant equal to N by killing the left-most particle at each branching event. A discrete version of this model was first considered by Brunet and Derrida [24] while studying the velocity of the traveling wave of an F-KPP type equation. As a natural continuous time version of the previous process, N-BBM was proposed by Maillard [62], and later on studied by De Masi et al. [31], and Berestycki et al. [17]. A multi-dimensional version of this model is called Brownian bees, in which particles independently move as Brownian motions in $\mathbb{R}^2$ and branch at rate 1, and the total number of particles is kept constant equal to N by killing the particle that is furthest away from the origin at each branching event. The stationary distribution and hydrodynamic limit of the Brownian bees was fully understood by Berestycki et al. [15, 16]. A similar model, which is called Barycentric Brownian bees was studied by Addario-Berry et al. [4].

Third, the branching rate can be inhomogeneous. For example, particles can breed at a rate dependent on its position. BBM with a space-dependent branching rate was first introduced by Harris and Harris [49]. In their model, each particle branches at a rate $\beta|x|^p$, which is proportional to its distance from the origin raised to the power $p \in [0, 2]$. Later, Roberts and Schweinsberg [75] used BBM with an inhomogeneous branching rate to model the evolution of populations undergoing selection.

Variant models of BBM can be considered with different physical and biological motivations. In this thesis, we are particularly interested in two variant models, BBM with absorption and BBM with an inhomogeneous branching rate.

1.2 Branching Brownian motion as a population model

BBM has a natural interpretation as a population model, where branching events represent births, particles represent individuals in the population, positions of particles represent fitnesses of individuals and movements of particles represent changes in fitnesses over generations. See e.g. [11, 12, 25, 26, 62]. In Chapter 3, we use a variant model of BBM, BBM with an inhomogeneous branching rate to model the evolution of populations undergoing selection. In this section, we introduce a discrete population model which motivates the construction of BBM with an inhomogeneous branching rate model in Chapter 3.

The simplest discrete population model is the Moran model, introduced by Moran [67] in 1958. In the Moran model, there are N individuals in the population at all times. Every individual independently lives for an exponentially distributed time. When an individual dies, it is replaced by a new individual, whose parent is chosen uniformly at random from the population. Note that in the Moran model, no selection acts on the population. To incorporate selection, we construct a variation of the Moran model, which was considered in [75]. We refer to it as the discrete population model with selection.

Definition 2. *We construct the discrete population model with selection as follows.*

- *There are N individuals in the population at all time.*
- *Each individual independently acquires mutations at rate μ_N.*
- *All mutations are beneficial and the fitness of an individual depends on the number of mutations this individual has acquired. Let s_N be the selection rate. The fitness level of an individual with j mutations at time t is*

$$\max\left\{1 + s_N(j - M(t)), 0\right\}$$

where $M(t)$ is the average number of mutations of the population at time t.

- *Each individual independently lives for an exponentially distributed time, then dies and gets replaced by a new individual. The parent of the new individual is chosen at random with probability proportional to the fitness.*

In this model, when the rate of beneficial mutations μ_N is large but the selective advantage s_N resulting from each mutation is small, an individual's fitness level will evolve like a random walk. With proper scaling, the fitness of each individual will move as a Brownian motion with drift. Motivated by this discrete model, Roberts and Schweinsberg [75] constructed BBM with an inhomogeneous branching rate model in which particles independently move as Brownian motion with drift, particles can die or undergo dyadic fission, and the difference between the birth rate and the death rate is proportional to the particle's location. Particles with higher locations are more likely to branch, which implies that individuals with higher fitnesses are more likely to reproduce offspring. Therefore, inhomogeneous branching depicts the most important feature of populations undergoing selection. If we relate the parameters μ_N, s_N and N in the discrete model with the branching rate and the drift of Brownian motion, we believe BBM with an inhomogeneous branching rate can serve as a good approximation of the discrete population model with selection.

The discrete population model with selection is just one example of discrete population genetics models in which the effect of the natural selection is considered. Others include the Wright-Fisher model with selection and the Moran model with selection defined in Etheridge [36], and some other variations of the discrete population model with selection studied in [34, 54, 80, 88].

1.3 F-KPP equation and the frontier of BBM

There has been a long-standing interest in the extremal position of BBM. In this section, we will review some of the most exciting results regarding the maximal displacement of the ordinary BBM. For simplicity, throughout this section, we assume that the branching rate $r = 1$ and the branching distribution is $p_2 = 1$.

1.3.1 F-KPP equation

The Fisher-Kolmogorov-Petrovskii-Piskounov equation, known as F-KPP equation is a semilinear partial differential equation of the form

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = F(u),$$

where $F(u)$ is a sufficiently smooth function satisfying

$$F(0) = F(1) = 0, \qquad F(u) > 0 \quad \text{for all } u \in (0,1),$$

and

$$F'(0) = \beta > 0, \qquad F'(u) \leq \beta \quad \text{for all } u \in (0,1].$$

The F-KPP equation is one of the simplest reaction-diffusion system that can be used to model population growth and wave propagation.

The F-KPP equation is closely related to BBM. In 1975, McKean [65] first gave a solution of the F-KPP equation expressed in terms of BBM. Consider the ordinary dyadic BBM started from a single particle at x. We denote by E^x the expectation under the probability measure of this process. Let $\mathcal{N}_t$ be the set of particles at time t and $\{X_u(t) : u \in \mathcal{N}_t\}$ be the set of positions of particles at time t.

Theorem 3. *Let $g \in C^2(\mathbb{R})$ with $0 \leq g(x) \leq 1$ for all x. If $u : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfies $u \in [0,1]$ and solves the F-KPP equation with initial condition g,*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u, \qquad u(0,x) = g(x), \tag{1.1}$$

then we have the representation

$$u(t, x) = E^x \left[\prod_{u \in \mathcal{N}_t} g(X_u(t)) \right].$$

As a special example, if we let the initial condition to be the indicator $g(x) = 1_{\{x \geq 0\}}$, then we have that the distribution function of the maximal displacement of BBM satisfies the F-KPP equation. To be more precise, define M_t to be the maximal displacement of the ordinary BBM. By Theorem 3,

$$u(t, x) = E^x \left[\prod_{u \in \mathcal{N}_t} 1_{\{X_u(t) \geq 0\}} \right]$$

satisfies (1.1) with initial condition $g(x)$. Note that after translation,

$$u(t, x) = P^0(M_t \leq x),$$

which is the cumulative density function of the maximal displacement at time t. In fact, this result can be seen directly from the branching property and the fact that the infinitesimal generator of Brownian motion is $\Delta/2$, where Δ is the Laplacian operator.

1.3.2 Frontier of BBM

Let $u(t, x)$ be the distribution function of the maximal displacement. It is known that the solution of the F-KPP equation converges to a traveling wave. That is to say, there exist functions $m(t)$ and $w(x)$ such that w is a probability distribution function and

$$u(t, m(t) + x) \to w(x) \text{ uniformly in } x \text{ as } t \to \infty,$$

and the traveling wave solution w solves the ordinary differential equation

$$\frac{1}{2}w'' + \sqrt{2}w' + w(w-1) = 0.$$

Therefore, $M_t - m(t)$ converges weakly to a limit whose distribution is given by $w(x)$. The exact asymptotic expression of the median position $m(t)$ of the maximal displacement was first given by Bramson [23] in 1978. Using the connection between the maximal displacement M_t and the F-KPP equation, he showed that there exists a constant C such that

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + C + o(1) \quad as\ t \to \infty.$$

If we compare $m(t)$ with the median of the maximum of e^t independently and identically distributed random variables with normal distribution $N(0,t)$, then they have the same first leading orders, but different second leading orders. Therefore, Bramson's result reflects a deep understanding of the branching structure. Bramson's proof was further simplified by Roberts [74] in 2013. Roberts also proved that

$$-\frac{3}{2\sqrt{2}} = \liminf_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} \le \limsup_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{1}{2\sqrt{2}}.$$

This result was first established by Hu and Shi [52] in the discrete setting branching random walk.

In 1987, Lalley and Sellke [58] noticed that despite the weak convergence of $M_t - m(t)$, the empirical distribution of the centered maximal displacement $M_t - m(t)$ does not converge to $w(x)$ in the limit of large times. Instead, they proved a weak limit theorem which relates the maximal displacement with a certain martingale in the limit of large times. It turns out that this martingale encodes the fluctuation of the beginning of the BBM. Afterwards, the behavior of the maximal displacement is approximately dominated by the extreme-value mechanism. Let

$$Z(t) = \sum_{u \in \mathcal{N}_t} \big(\sqrt{2}t - X_u(t)\big) e^{\sqrt{2}X_u(t) - 2t}.$$

Lalley and Sellke showed that $(Z(t), t \geq 0)$ is a martingale and

$$\lim_{t \to \infty} Z(t) = Z \qquad a.s.$$

where Z is strictly positive and almost surely finite. The process $(Z(t), t \geq 0)$ is often called the derivative martingale. Lalley and Sellke proved that there exists some constant c such that

$$w(x) = \lim_{t \to \infty} P\left(M_t \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + x \right) = E\left[e^{-cZe^{-\sqrt{2}x}} \right]. \tag{1.2}$$

That is, conditioned on Z, the asymptotic distribution of $M_t - m(t)$ is the Gumbel distribution with random shift $\log Z$.

Lalley and Sellke's result suggests a random shift picture for the frontier of BBM. The following question would be if we look at the BBM from its right-most particle, what is the picture of the whole process. This question was answered in two papers with different proofs, [6, 7]. Define the point measure

$$\mathcal{E}_t = \sum_{u \in \mathcal{N}_t} \delta_{X_u(t) - m(t)}.$$

As $t \to \infty$, the random measure $\mathcal{E}_t$ converges in law to a random intensity decorated Poisson point process $\mathcal{E}$. The distribution of $\mathcal{E}$ is determined by an intensity measure ν, which is a random σ-finite measure on $\mathbb{R}$ and a random point process $\mathcal{D}$ on $\mathbb{R}$. Let

$$\nu = c\sqrt{2} Z e^{-\sqrt{2}x},$$

and

$$\mathcal{D}(\cdot) = \lim_{t \to \infty} P\left(\sum_{u \in \mathcal{N}_t} \delta_{\{X_u(t) - M_t\}} \in \cdot \,\middle|\, M_t \geq \sqrt{2}t \right).$$

where c is the constant in (1.2). Conditionally on ν, we first construct a Poisson point process with intensity ν, whose atoms are denoted by $(x_i)_{i=1}^{\infty}$. For each x_i, we attach a point measure $\mathcal{D}_i$ where $\mathcal{D}_i$ is an independent copy of $\mathcal{D}$. We denote the points of $\mathcal{D}_i$ as $(d_{i,j})_{j=1}^{\infty}$. Then

$$\mathcal{E} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{x_i + d_{i,j}}.$$

1.4 Techniques

This section will introduce two extremely useful techniques in the study of branching processes, the spine decomposition and the many-to-one type lemmas. The spine decomposition is the cornerstone of the proofs in Chapter 2 and the many-to-one type lemmas are extensively used in Chapter 3.

1.4.1 Spine decomposition

As the name suggests, the idea of the spine decomposition is to identify one particular line of descent from the root as the spine and view all the other particles as descendants branching off the spine. This idea first appeared in Chauvin and Rouault's work [28] in 1988, and was later developed and generalized by Kurtz et al. [57], Lyons [59], Lyons et al. [60], and Hardy and Harris [47]. The spine decomposition is usually done via the change of measure. With different changes of measure, different spine decompositions can be obtained and will be helpful in calculating the probability of certain events. Here we will present one spine decomposition using the additive martingale, which follows from Chauvin and Rouault [28].

Denote by P^x the probability measure of the ordinary BBM started from a single particle at x. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of this process up to time t and $m = \sum_{k=1}^{\infty} kp_k - 1$. For the branching distribution, assume $p_0 = 0$ and $0 < m < \infty$. For every

$\lambda \in \mathbb{R}$, define

$$W_\lambda(t) = \sum_{u \in \mathcal{N}_t} e^{\lambda X_u(t)} e^{-rmt - \lambda^2 t/2}.$$

It is well known that $(W_\lambda(t), t \geq 0)$ is a positive martingale called the additive martingale, and its limit $W_\lambda = \lim_{t\to\infty} W_\lambda(t)$ exists almost surely. When $|\lambda| \geq \sqrt{2rm}$, $W_\lambda = 0$ almost surely and when $|\lambda| < \sqrt{2rm}$, W_λ is almost surely positive and is also the L^1 limit. We define the new measure Q^x_λ via $(W_\lambda, s \geq 0)$,

$$\left.\frac{dQ^x_\lambda}{dP^x}\right|_{\mathcal{F}_t} = \frac{W_\lambda(t)}{W_\lambda(0)}.$$

Under the new measure Q^x_λ, there is one chosen particle which is the spine whose law is altered and all subtrees branching off the spine behave like the ordinary BBM. The spine moves as a Brownian motion with drift λ starting from x. With accelerated rate $(1+m)r$, the spine splits into a random number of offspring according to the probability law $(\widehat{p}_k)_{k=1}^\infty$ where

$$\widehat{p}_k = \frac{kp_k}{m+1}, \qquad k = 1, 2, \ldots.$$

The spine is chosen uniformly from the offspring, and the remaining offspring initiate independent copies of the ordinary BBM starting from its birth position.

1.4.2 Many-to-one and many-to-two lemmas

Most of the proofs in BBM involve delicate moment estimates. Usually, the first moment estimate gives the average value and the second moment estimate controls the fluctuation. By many-to-one type lemmas, moment estimates for BBM can be transformed into the moment calculations of a single particle, which will greatly simplify the calculation. We present a many-to-one lemma and a many-to-two lemma for the general branching Markov process. The many-to-one lemma follows from Theorem 8.5 in [47], and the many-to-two lemma is adapted from Theorem 2.2 in [78].

Consider a branching Markov process in which each particle independently moves according to a one-dimensional Markov process $(\Xi_t, t \geq 0)$. For a particle at location x, it dies at rate $R(x)$. To be more precise, if the particle $u \in \mathcal{N}_t$ is alive at time t, then the probability that it will die in the interval $[t, t+dt)$ is $R(X_u(t))dt + o(dt)$. At the end of the lifetime, each particle independently splits into a random number of offspring according to the probability law $(p_k)_{k=1}^{\infty}$ on the positive integers with finite mean. Let $m = \sum_{k=1}^{\infty} kp_k - 1$. As before, we denote by $\mathcal{N}_t$ the set of particles at time t, $\{X_u(t) : u \in \mathcal{N}_t\}$ the set of positions of particles at time t and E^x the expectation under this branching Markov process. We assume that the function $R(x)$ and the process $(\Xi_t, t \geq 0)$ are smooth enough that $R(\Xi_s)$ is integrable as a function of s over the interval $[0, t]$ for all t.

Lemma 4 (Many-to-one lemma). *If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, then*

$$E^x\left[\sum_{u \in \mathcal{N}_t} f(X_u(t))\right] = E^x\left[e^{\int_0^t mR(\Xi_s)ds} f(\Xi_t)\right].$$

We further assume that the Markov process $(\Xi_t, t \geq 0)$ is time homogeneous. Let $p_t(x, y)$ be the density for the branching Markov process. That is to say, if there is a single particle at x at time 0, then the expected number of particles in the Borel set $B \subset \mathbb{R}$ is given by $\int_B p_t(x, y)dy$. The density $p_t(x, y)$ can be calculated from Lemma 4

$$p_t(x, y)dy = E^x\left[e^{\int_0^t mR(\Xi_s)ds}; \Xi_t \in dy\right].$$

Lemma 5 (Many-to-two lemma). *If $f : \mathbb{R} \to [0, \infty)$ is a measurable function, then*

$$E^x\left[\left(\sum_{u \in \mathcal{N}_t} f(X_u(t))\right)^2\right] = \int_0^\infty f(y)^2 p_t(x, y)dy + 2\int_0^t \int_0^\infty p_s(x, z)\left(\int_0^\infty f(y)p_{t-s}(z, y)dy\right)^2 dzds.$$

1.5 Outline

This thesis focuses on two variant models of BBM, BBM with absorption and BBM with an inhomogeneous branching rate. In Chapter 2, aiming to understand the transition from the near critical regime to the critical regime for BBM with absorption, we study a Yaglom type asymptotic result in the slightly subcritical regime. In Chapter 3, we use BBM with an inhomogeneous branching rate to model the evolution of populations undergoing selection and provides mathematical descriptions of the population evolution.

Chapter 2

Branching Brownian motion with absorption

We consider a slightly subcritical branching Brownian motion (BBM) with absorption, where particles move as Brownian motions with drift $-\sqrt{2+2\varepsilon}$, undergo dyadic fission at rate 1, and are killed when they reach the origin. We obtain a Yaglom type asymptotic result, showing that the long run expected number of particles conditioned on survival grows exponentially as $1/\sqrt{\varepsilon}$ as the process approaches criticality.

2.1 Introduction

BBM with absorption was first introduced by Kesten [56] in 1978. In this model, particles move as Brownian motions with drift $-\rho$ ($\rho \in \mathbb{R}$), undergo dyadic branching at rate 1, and are killed when they reach the origin. Kesten [56] showed that when $\rho \geq \sqrt{2}$, BBM with absorption dies out almost surely while when $\rho < \sqrt{2}$, there is a positive probability of survival. Therefore, $\rho = \sqrt{2}$ is the critical value separating the supercritical case $\rho < \sqrt{2}$ and the subcritical case $\rho > \sqrt{2}$. We denote by $P^x_{-\rho}$ the probability measure for BBM started from a single particle at $x > 0$ with drift $-\rho$ and absorbed at 0.

There has been a long-standing interest in problems related to the asymptotic behavior of the survival probability. In the critical case, after introducing the model, Kesten [56] obtained upper and lower bounds on the probability that the process survives until some large time. Kesten's result was further improved by Berestycki et al. [12]. Finally, Maillard and Schweinsberg [63] established the precise asymptotic for the long run survival probability. Let ζ be the extinction time. They showed that there exists a positive constant α such that

$$\lim_{t\to\infty} \frac{P^x_{-\sqrt{2}}(\zeta > t)}{\alpha x} e^{-\sqrt{2}x+(3\pi^2 t)^{1/3}} = 1.$$

They also provided an expression for the constant α, which is related to the tail of the derivative martingale of BBM. The asymptotic result for the survival probability in the supercritical case was obtained by Harris et al. [50] through studying the FKPP equation associated with this process. Derrida and Simon [32] gave a quite precise prediction for the survival probability in the slightly supercritical case through nonrigorous PDE methods, where the drift ρ is slightly below the critical value. Rigorous probabilistic proofs were provided by [10]. In this chapter, we are interested in a nearly critical case, where ρ approaches the critical value $\sqrt{2}$ from above. For notational simplicity, we write $\rho^2/2 - 1 = \varepsilon$ where $0 < \varepsilon < 1$ and ε approaches to 0. We denote by $\mathcal{N}_t^{-\rho}$ the set of surviving particles under $P^x_{-\rho}$ at time t. The set of positions of particles at time t under $P^x_{-\rho}$ is $\{Y_u(t) : u \in \mathcal{N}_t^{-\rho}\}$ and the number of particles at time t under $P^x_{-\rho}$ is $N_t^{-\rho}$.

In the subcritical case, almost surely, the process becomes extinct. However, it is interesting to consider the behavior of the process conditioned on survival up to a large time. This type of result is called a Yaglom theorem and has been considered by Yaglom [87] in ordinary branching processes. A similar question was studied by Chauvin and Rouault [28] in the setting of BBM without absorption. Let P be the law of an ordinary BBM started from 0 without drift and absorption and $\mathcal{N}_t$ be the set of particles at time t. Chauvin and Rouault first gave an asymptotic expression for the probability of existence of particles to the right of $\rho t + x$ at some large time, $P(\exists u \in \mathcal{N}_t : Y_u(t) > \rho t + x)$. Then they obtained a limit distribution for the number of particles that drift above $\rho t + x$ at time t conditioned on the presence of such particles for $\rho > \sqrt{2}$. Harris and Harris [48] obtained related results for BBM with absorption. They derived a large-time asymptotic formula for the survival probability in the subcritical case. They proved that for $\rho > \sqrt{2}$ and $x > 0$, there exists a constant K_ε that is independent of x but dependent on the drift ρ, and therefore on ε, such that,

$$\lim_{t\to\infty} P^x_{-\rho}(N_t^{-\rho} > 0)\frac{\sqrt{2\pi t^3}}{x}e^{-\rho x+\varepsilon t} = K_\varepsilon, \tag{2.1}$$

and furthermore,

$$\lim_{t\to\infty} \frac{P^x_{-\rho}(N_t^{-\rho} > 0)}{E^x_{-\rho}[N_t^{-\rho}]} = \frac{1}{2}\rho^2 K_\varepsilon. \tag{2.2}$$

Comparing this with Chauvin and Rouault's result, as t goes to infinity, $P^x_{-\rho}(N_t^{-\rho} > 0)$ and $P(\exists u \in \mathcal{N}_t : Y_u(t) > \rho t + x)$ are the same on the exponential scale but different in terms of the polynomial corrections. The constant K_ε plays an important role in calculating the limiting expected number of particles alive conditioned on at least one surviving. In fact, it is pointed out by Harris and Harris in [48] that as a direct consequence of (2.2), we have

$$\lim_{t\to\infty} E^x_{-\rho}[N_t^{-\rho} | N_t^{-\rho} > 0] = \frac{2}{\rho^2 K_\varepsilon}. \tag{2.3}$$

Furthermore, by using the method of Chauvin and Rouault [28], it follows from (2.2) that there is a probability distribution $(\pi_j)_{j\geq 1}$ such that

$$\lim_{t\to\infty} P^x_{-\rho}(N_t^{-\rho} = j | N_t^{-\rho} > 0) = \pi_j.$$

Our main result, which is Theorem 8 below, analyzes the asymptotic behavior of (2.3) as ε goes to 0. We show that the long-run expected number of particles conditioned on survival grows exponentially as the process gets closer to being critical.

Theorem 6. *There exist positive constants C_1 and C_2 such that for ε small enough,*

$$e^{C_1/\sqrt{\varepsilon}} \leq \lim_{t\to\infty} E^x_{-\rho}[N_t^{-\rho} | N_t^{-\rho} > 0] \leq e^{C_2/\sqrt{\varepsilon}}. \tag{2.4}$$

Kesten [56] had a result of this type in the critical case. Recently, Maillard and Schweinsberg [63] proved Yaglom-type limit theorems for more specific behaviors of the process in the critical case. They derived the asymptotic distributions of the extinction time ζ, the number of particles $N_t^{-\sqrt{2}}$ and the position of the right-most particle at time t, $M_t^{-\sqrt{2}}$ for the process conditioned on survival for a long time. To be more precise, let V be an exponentially distributed random variable with parameter 1. They proved that conditioned on $\zeta > t$, as $t \to \infty$ and $\omega = (3\pi^2)^{1/3}/\sqrt{2}$,

$$\left(t^{-2/3}(\zeta - t), t^{-2/9} \log N_t^{-\sqrt{2}}, t^{-2/9} M_t^{-\sqrt{2}}\right) \Rightarrow \left(\frac{3V}{\sqrt{2}\omega}, \left(\frac{3\omega^2 V}{\sqrt{2}}\right)^{1/3}, \left(\frac{3\omega^2 V}{\sqrt{2}}\right)^{1/3}\right),$$

where $\Rightarrow$ represents the weak convergence. In the setting of supercritical branching random walk (BRW), Gantert et al. [43] and Pain [69] considered problems with similar flavor. Let γ be the asymptotic speed of the right-most position in the BRW. Gantert et al. [43] studied the probability that there exists an infinite ray which stays above the line of slope $\gamma - \varepsilon$ as ε goes to 0. Having an infinite ray staying above the line with slope $\gamma - \varepsilon$ can be viewed as survival with slightly supercritical drift. They proved that when $\varepsilon \to 0$, this probability decays as $\exp(-(C + o(1))/\sqrt{\varepsilon})$ where C is a positive constant depending on the distribution of the branching random walk. In [69], Pain studied the near-critical Gibbs measure and the partition function of parameter β on the $n-$th generation of the BRW. In his setting, the inverse temperature β is a function of n and approaches to the critical value 1 both from above and below. Our setting can be viewed as a iterated limit where we first let time t go to infinity and then let the process approach to criticality, while Pain's setting can be viewed as a double limit where the process approaches to criticality at the same time when the generation goes to infinity.

It is important to point out that Theorem 6 does not imply that as the process approaches criticality, we have $\log N_t^{-\rho} = O(\varepsilon^{-1/2})$ conditioned on survival up to time t in a typical realization of the process. We conjecture that there is a big difference between the expected number and the typical number of surviving particles because the expectation is dominated by rare events where an unusually large number of particles survive. We further conjecture that for ε sufficiently small, the logarithm of the number of particles at time t conditioned on survival up to time t is typically around $\varepsilon^{-1/3}$.

The proof of Theorem 6 relies on a better understanding of K_ε as the drift approaches the critical value. According to (2.1), studying K_ε boils down to finding an asymptotic expression for the survival probability in the slightly subcritical regime. Here we apply a spinal decomposition to transform survival probability to expectation of the reciprocal of a martingale.

As in Harris and Harris [48], define

$$V(t) := \sum_{u \in \mathcal{N}_t^{-\rho}} Y_u(t) e^{\rho Y_u(t) + \varepsilon t}.$$

Lemma 2 in [48] shows that $\{V(t)\}_{t \geq 0}$ a martingale under $P_{-\rho}^x$. We can define a new measure Q^x on the same probability space as $P_{-\rho}^x$ via $\{V(s)\}_{s \geq 0}$,

$$\left.\frac{dQ^x}{dP_{-\rho}^x}\right|_{\mathcal{F}_s} = \frac{V(s)}{V(0)}. \tag{2.5}$$

Under the measure Q^x, there is one chosen particle which is called the spine whose law is altered and all subtrees branching off the spine behave like the original BBM with absorption. The spine moves as a Bessel-3 process starting from x. With accelerated rate 2, the initial ancestor undergoes binary fission. The spine is chosen uniformly from the two offspring, and the remaining offspring initiates an independent copy of the original BBM with absorption. In this chapter Q^x is used both for probability and expectation. Representing K_ε under Q^x in (2.1),

$$K_\varepsilon = \lim_{t \to \infty} Q^x \left[\frac{V(0)}{V(t)}; N_t^{-\rho} > 0 \right] \frac{\sqrt{2\pi t^3}}{x} e^{-\rho x + \varepsilon t} = \lim_{t \to \infty} \sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_{u \in \mathcal{N}_t^{-\rho}} Y_u(t) e^{\rho Y_u(t)}} \right].$$

As a result, Theorem 6 follows from the following proposition.

Proposition 7. *There exist positive constants C_1 and C_2 such that for ε small enough,*

$$\limsup_{t\to\infty} \sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_{u\in\mathcal{N}_t^{-\rho}} Y_u(t) e^{\rho Y_u(t)}} \right] \le e^{-C_1/\sqrt{\varepsilon}}, \tag{2.6}$$

$$\liminf_{t\to\infty} \sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_{u\in\mathcal{N}_t^{-\rho}} Y_u(t) e^{\rho Y_u(t)}} \right] \ge e^{-C_2/\sqrt{\varepsilon}}. \tag{2.7}$$

We point out here that we cannot specify choices of C_1 and C_2. We are only able to determine upper and lower bounds for $\lim_{t\to\infty} E^x_{-\rho}[N_t^{-\rho} | N_t^{-\rho} > 0]$.

The rest of this chapter is organized as follows. In Section 2.2, results related to Brownian motion and the Bessel-3 process will be summarized. Sections 2.3 and 2.4 will be devoted to the proofs of the upper bound and lower bound in Proposition 7 respectively. Throughout this chapter, for two nonzero functions $f(t)$ and $g(t)$, we use the notation $f(t) \sim g(t)$ as $t \to a$ to mean that $\lim_{t\to a} f(t)/g(t) = 1$. We summarize some of the notation that are used throughout the chapter in Table 2.1.

Table 2.1: *Index of notation in Chapter 2*

$P^x_{-\rho}$	The probability measure of the BBM started from a single particle at $x > 0$ with drift $-\rho$ and absorbed at 0.
$\mathcal{N}_t^{-\rho}$	The set of surviving paticles under $P^x_{-\rho}$ at time t.
Q^x	The probability measure on the same probability space as $P^x_{-\rho}$ defined via a spine change of measure.
$\{\xi_t\}_{t\ge 0}$	The trajectory of the spine.
$\{\zeta_s\}_{0\le s\le t}$	The reversed trajectory of the spine up to time t, i.e. $\{\xi_{t-s}\}_{0\le s\le t}$.
$Q^{x,t,z}$	The probability measure of the branching process under Q^x whose spine starts from x and is conditioned to end up at z at time t, i.e. $Q^x(\cdot\|\xi_t = z)$ or $Q^x(\cdot\|\zeta_0 = z)$.

Table 2.1: *Index of notation in Chapter 2, Continued*

$\{B_t^{x,u,y}\}_{0\leq t\leq u}$	Brownian bridge from x to y over time u.
$\{X_t^{x,u,y}\}_{0\leq t\leq u}$	Bessel bridge from x to y over time u. If clear from the context, we will write $\{X_r\}_{0\leq r\leq u}$ for simplicity.
$p_t^{x,u,y}(\cdot)$	The transition density of a Bessel process from x to y over time u at time t.
$\{R_r^z\}_{r\geq 0}$	Bessel-3 process started from z.
$p_t(x,\cdot)$	The transition density of a Bessel process started from x at time t.

2.2 Preliminary results

In this section, we will summarize results pertaining to Brownian motion and the Bessel-3 process which will be used later in the proof. For further properties of the Brownian motion and the Bessel process, we refer the reader to Borodin and Salminen [19].

Let$\{B_t\}_{t\geq 0}$ be standard Brownian motion and $\{B_t^{x,u,y}\}_{0\leq t\leq u}$ be a Brownian bridge from x to y over time u. Standard Brownian bridge refers to the Brownian bridge from 0 to 0 in time 1, $\{B_t^{0,1,0}\}_{0\leq t\leq 1}$. Reflected Brownian bridge is the absolute value of the Brownian bridge, $\{|B_t^{x,u,y}|\}_{0\leq t\leq u}$. Now we will be able to state the following lemma. Lemma 8 derives the limit of the probability that a reflected standard Brownian bridge always stays below a line $at+b$ as $b(a+b)$ approaches 0. We will prove by first obtaining the explicit probability formula written as an infinite sum and then analyzing its limiting behavior through Jacobi theta functions.

Lemma 8. *For $a\geq 0$ and $b>0$, we have*

$$P\Big(\sup_{0\leq t\leq 1}\Big(|B_t^{0,1,0}|-at\Big)<b\Big)\sim\sqrt{\frac{2\pi}{b(a+b)}}e^{-\frac{\pi^2}{8b(a+b)}}\quad \textit{as}\ \ b(a+b)\downarrow 0.$$

Proof. According to Theorem 7 in [77], we have

$$P\Big(\sup_{0\le t\le 1}\Big(|B_t^{0,1,0}|-at\Big)<b\Big)=\sum_{k=-\infty}^{\infty}(-1)^k e^{-2k^2b(a+b)}. \tag{2.8}$$

To have a better understanding of this expression for small values of $b(a+b)$, we need to introduce the Jacobi theta functions of type 2, $\vartheta_2(z|\tau)$ and type 4, $\vartheta_4(z|\tau)$ and their relationship. A good reference would be Section 16 of [1]. We have

$$\vartheta_2(z|\tau):=2e^{i\pi\tau/4}\sum_{k=0}^{\infty}e^{i\pi\tau k(k+1)}\cos((2k+1)z),$$

$$\vartheta_4(z|\tau):=\sum_{k=-\infty}^{\infty}(-1)^k e^{i\pi\tau k^2}e^{2kiz}.$$

As a special case of Jacobi's imaginary transformation,

$$\vartheta_4(0|\tau)=(-i\tau)^{-1/2}\vartheta_2\Big(0|-\frac{1}{\tau}\Big).$$

Then (2.8) can be written in terms of Jacobi theta functions,

$$P\Big(\sup_{0\le t\le 1}\Big(|B_t^{0,1,0}|-at\Big)<b\Big)=\vartheta_4\Big(0|\frac{2b(a+b)i}{\pi}\Big)=\sqrt{\frac{\pi}{2b(a+b)}}\vartheta_2\Big(0|\frac{\pi i}{2b(a+b)}\Big).$$

We want to explore the limiting behavior of $P(\sup_{0\le t\le 1}|B_t^{0,1,0}|-at<b)$ as $b(a+b)$ approaches 0. By the series representation for the theta function ϑ_2, if $e^{i\pi\tau}\in\mathbb{R}$ and $e^{i\pi\tau}\to 0$, then

$$\vartheta_2(0|\tau)\sim 2e^{i\pi\tau/4}.$$

Therefore, as $b(a+b)$ approaches 0 from above,

$$P\Big(\sup_{0\le t\le 1}\Big(|B_t^{0,1,0}|-at\Big)<b\Big)=\sqrt{\frac{\pi}{2b(a+b)}}\vartheta_2\Big(0|\frac{\pi i}{2b(a+b)}\Big)\sim\sqrt{\frac{2\pi}{b(a+b)}}e^{-\frac{\pi^2}{8b(a+b)}}.$$

□

Below we will present a stochastic dominance relation between Brownian bridges with the same length but different endpoints.

Lemma 9. *For every $t > 0$, if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $\{B_r^{x_1,t,y_1}\}_{0\leq r\leq t}$ stochastically dominates $\{B_r^{x_2,t,y_2}\}_{0\leq r\leq t}$. In other words, these two processes can be constructed on some probability space such that almost surely for all $r \in [0,t]$,*

$$B_r^{x_1,t,y_1} \geq B_r^{x_2,t,y_2}.$$

Proof. According to IV.21 of [19], after some computations, $\{B_r^{x_1,t,y_1}\}_{0\leq r\leq t}$ and $\{B_r^{x_2,t,y_2}\}_{0\leq r\leq t}$ can be expressed in terms of $\{B_r^{0,t,0}\}_{0\leq r\leq t}$,

$$B_r^{x_i,t,y_i} = \frac{t-r}{t}x_i + \frac{r}{t}y_i + B_r^{0,t,0}, \quad \text{for } 0 \leq r \leq t \text{ and } i = 1,2.$$

Since $x_1 \geq x_2$ and $y_1 \geq y_2$, $\{B_r^{x_1,t,y_1}\}_{0\leq r\leq t}$ stochastically dominates $\{B_r^{x_2,t,y_2}\}_{0\leq r\leq t}$ from the above coupling. □

Next we are going to introduce results pertaining to the Bessel-3 process. The Bessel-3 process is defined to be the radial part of a three-dimensional Brownian motion. Since only the Bessel-3 process will be considered in this chapter, below we will write the Bessel process for convenience. Also, the Bessel process is identical in law to a one dimensional Brownian motion conditioned to avoid the origin. Let $p_t(x,y)$ be the transition density of a Bessel process started from x at time t. We have

$$p_t(x,y) = \frac{y}{x}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}\left(1 - e^{-2xy/t}\right).$$

Similarly to the Brownian motion setting, define $\{X_t^{x,u,y}\}_{0\leq t\leq u}$ as a Bessel bridge from x to y over time u and $p_t^{x,u,y}(z)$ as its transition density at time t. Specifically, $\{X_t^{0,1,0}\}_{0\leq t\leq 1}$ is a Brownian excursion. It is shown in the proof of Lemma 7 in [48] that a Bessel bridge is identical in law to a Brownian bridge that is conditioned to avoid the origin. Since a time-reversed Brownian bridge is also a Brownian bridge, we see that a time-reversed Bessel bridge is also a Bessel bridge. To be more precise,

$$\{X_{t-s}^{x,t,z}\}_{0\le s\le t} \stackrel{d}{=} \{X_s^{z,t,x}\}_{0\le s\le t}.$$

As in the Brownian motion case, there is also a stochastic dominance relation between Bessel bridges. The technical tool we use here is the comparison theorem for solutions of stochastic differential equations and a good reference for it is [53].

Lemma 10. *If $x_1 \ge x_2 \ge 0$ and $y_1 \ge y_2 \ge 0$, then $\{X_r^{x_1,1,y_1}\}_{0\le r\le 1}$ stochastically dominates $\{X_r^{x_2,1,y_2}\}_{0\le r\le 1}$. In other words, these two processes can be constructed on some probability space such that almost surely for all $r \in [0,1]$,*

$$X_r^{x_1,1,y_1} \ge X_r^{x_2,1,y_2}.$$

Proof. It is sufficient to show that for every $0<\delta<1$, the process $\{X_r^{x_1,1,y_1}\}_{0\le r\le 1-\delta}$ stochastically dominates $\{X_r^{x_2,1,y_2}\}_{0\le r\le 1-\delta}$. Note that the Bessel bridge is nonnegative. Instead of working with Bessel bridges directly, we will prove the lemma for squared Bessel bridges, for which the comparison theorem can be applied readily. Define squared Bessel bridges for $0\le r\le 1-\delta$,

$$Y_r^{x_i^2,1,y_i^2} := (X_r^{x_i,1,y_i})^2, \quad \text{for } i=1,2.$$

By (0.27) of [72] and Ito's formula, letting $\{B_r\}_{r\ge 0}$ be a standard Brownian motion, squared Bessel bridges $\{Y_r^{x_1^2,1,y_1^2}\}_{0\le r\le 1-\delta}$ and $\{Y_r^{x_2^2,1,y_2^2}\}_{0\le r\le 1-\delta}$ can be respectively represented as pathwise unique solutions over $[0,1-\delta]$ of the stochastic differential equations

$$Y_0^{x_i^2,1,y_i^2} = x_i^2, \quad dY_r^{x_i^2,1,y_i^2} = \left(3 + \frac{2y_i\sqrt{Y_r^{x_i^2,1,y_i^2}} - 2Y_r^{x_i^2,1,y_i^2}}{1-r}\right)dr + 2\sqrt{Y_r^{x_i^2,1,y_i^2}}\,dB_r$$

where $i = 1, 2$. Set

$$b_i(t,x) = 3 + \frac{2y_i\sqrt{x} - 2x}{1-t} \text{ for } i = 1,2, \quad \sigma(t,x) = 2\sqrt{x}.$$

We see that for $x, y \in \mathbb{R}$ and $t \geq 0$,

$$|\sigma(t,x) - \sigma(t,y)| = 2|\sqrt{x} - \sqrt{y}| \leq 2\sqrt{|x-y|} =: \phi(|x-y|)$$

where ϕ is an increasing function such that $\phi(0) = 0$ and

$$\int_{0^+} \phi(x)^{-2} dx = \infty.$$

Furthermore, because $b_i(t,x)$ for $i = 1, 2$ and $\sigma(t,x)$ are continuous on $[0, 1-\delta) \times \mathbb{R}$, we have $\{Y_r^{x_1^2, 1, y_1^2}\}_{0 \leq r \leq 1-\delta}$ stochastically dominates $\{Y_r^{x_2^2, 1, y_2^2}\}_{0 \leq r \leq 1-\delta}$ by Theorem 1.1 of [53]. Finally after taking the square root, the lemma holds for Bessel bridges. $\square$

There is also one more fact on the relationship between the Bessel bridge and Bessel process, which is borrowed from Lemma 7 of [48].

Lemma 11. *As $t \to \infty$, the Bessel bridge converges to the Bessel process in the Skorokhod topology on $D[0, \infty)$, i.e.*

$$P_{BES}^{z,t,x} \Rightarrow P_{BES}^{z}.$$

2.3 Upper bound

2.3.1 Proof outline

In this section, we show the upper bound (2.6). Throughout this section, $P_{-\rho}^x$ is the probability measure for the BBM started from a single particle at $x > 0$ with drift $-\rho$ and absorbed at 0. Let $\mathcal{N}_t^{-\rho}$ be the set of surviving particles at time t. The configuration of particles at time t under $P^x_{-\rho}$ is written as $\{Y_u(t) : u \in \mathcal{N}_t^{-\rho}\}$. For a particle $u \in \mathcal{N}_t^{-\rho}$, denote by O_u the time that the ancestor of u branches off the spine. By convention, if u is the spinal particle, $O_u = t$. Defined in (2.5), Q^x is the law of a branching diffusion with the spine which initiates from a single particle at $x > 0$. Under the measure Q^x, let $\{\xi_t\}_{t\geq 0}$ be the trajectory of the spinal particle which diffuses as a Bessel-3 process. Define $\{\zeta_s\}_{0\leq s\leq t} = \{\xi_{t-s}\}_{0\leq s\leq t}$ to be the reversed trajectory of the spinal particle. We denote by $Q^{x,t,z}$ the law of the branching process whose spine starts from x and is conditioned to end up at z at time t, i.e.

$$Q^{x,t,z}(\cdot) = Q^x(\cdot|\xi_t = z) = Q^x(\cdot|\zeta_0 = z).$$

First we will control the case where the position of the spinal particle at time t is greater than $\varepsilon^{-1/2}$, which is Lemma 12 below.

Lemma 12. *For all t and all ε sufficiently small, there exists a positive constant C_3 such that*

$$\sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}; \xi_t \geq \varepsilon^{-1/2} \right] \leq e^{-C_3/\sqrt{\varepsilon}}.$$

As a result, we only need to deal with the case where the spine ends up near the origin. To prove (2.6), it is sufficient to show that there exists a constant C_4 such that for ε small enough,

$$\limsup_{t\to\infty} \sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}; \xi_t \leq \varepsilon^{-1/2} \right] \leq e^{-C_4/\sqrt{\varepsilon}}. \tag{2.9}$$

It remains to prove equation (2.9). To obtain an upper bound, we only take particles that branch off the spine within the last $\varepsilon^{-3/2}$ time into account. Conditioned on the spine being at y at time $t - \varepsilon^{-3/2}$, we restart the process from y and let the process run for time $\varepsilon^{-3/2}$. Essentially, we will work on bounding

$$Q^y \left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} \middle| \xi_{\varepsilon^{-3/2}} = z \right]. \tag{2.10}$$

Considering the reversed trajectory of the spine, let

$$M = \sup_{0 \le s \le \varepsilon^{-3/2}} \Big(\varepsilon \zeta_s - \frac{1}{\rho} \varepsilon^2 s \Big).$$

For any positive constant $C > 2\pi$, we will divide the proof for (2.9) into the small M and large M cases,

$$\begin{aligned}(2.10) = Q^y & \left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} 1_{\{M \ge 2C\sqrt{\varepsilon}\}} \middle| \xi_{\varepsilon^{-3/2}} = z \right] \\ & + Q^y \left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} 1_{\{M < 2C\sqrt{\varepsilon}\}} \middle| \xi_{\varepsilon^{-3/2}} = z \right]. \end{aligned} \tag{2.11}$$

For the large M case, the main strategy is as follows:

- If $M \ge 2C\sqrt{\varepsilon}$, then $\varepsilon\zeta_s - \frac{1}{\rho}\varepsilon^2 s$ stays above $C\sqrt{\varepsilon}$ for a while. In other words, the position of the spine at time $t - s$ satisfies $\xi_{t-s} \ge C/\sqrt{\varepsilon} + \varepsilon s/\rho$ for some time. During that time, many particles branch off the spine.

- For $s \in [0, \varepsilon^{-3/2}]$, each particle that branches off the spine at time $t - s$ and is located to the right of $C/\sqrt{\varepsilon} + \varepsilon s/\rho$ will have a descendant at time t above $C/(4\sqrt{\varepsilon})$ with some nonzero probability which is independent of ε and s. When this occurs, it will follow that, for sufficiently small ε,

$$\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}} \le \frac{4\sqrt{\varepsilon}}{C} e^{-C\rho/(4\sqrt{\varepsilon})} \le e^{-C\rho/(5\sqrt{\varepsilon})}.$$

- Taking the number of branching events of the spine into consideration, the probability that there exists at least one particle which stays to the right of $C/(4\sqrt{\varepsilon})$ at time t converges to 1 as ε goes to 0.

From the above strategy, we can see why the $\varepsilon^{-3/2}$ time period is considered here. The length of this time period has to be large enough such that a considerable number of particles branch off the spine and also small enough such that if a particle branches off the spine during that time, the position of its descendant at time t won't be too far from its branching position. Essentially, for $C > 2\pi$, we need to prove the following two lemmas.

Lemma 13. *Let $\{t - t_i\}_{i=1}^{N_\varepsilon}$ be the set of times that particles branch off the spine between time $t - \varepsilon^{-3/2}$ and t. Then there exists a positive constant C_5 such that for ε sufficiently small, for every $y \in (0,\infty)$ and $z \in (0, \varepsilon^{-1/2}]$, we have*

$$Q^y\left(\left\{\sum_{i=1}^{N_\varepsilon} 1_{\{\varepsilon\zeta_{t_i} - \frac{1}{\rho}\varepsilon^2 t_i \geq C\sqrt{\varepsilon}\}} \leq \frac{1}{\sqrt{\varepsilon}}\right\} \cap \{M \geq 2C\sqrt{\varepsilon}\} \middle| \xi_{\varepsilon^{-3/2}} = z\right) \leq \left(6 + \frac{2}{yz}\right) e^{-C_5/\sqrt{\varepsilon}}.$$

Lemma 14. *There exists a positive constant C_6 such that for all sufficiently small ε and $s \in [0, \varepsilon^{-3/2}]$,*

$$P_{-\rho}^{C/\sqrt{\varepsilon}+\varepsilon s/\rho}\left(\exists u \in \mathcal{N}_s^{-\rho} : Y_u(s) > \frac{C}{4\sqrt{\varepsilon}}\right) > C_6. \tag{2.12}$$

With the help of Lemmas 13 and 14, we will be able to state the result regarding the large M case.

Lemma 15. *There exists a positive constant C_7 such that for all sufficiently small ε, for all $y \in (0,\infty)$ and $z \in (0, \varepsilon^{-1/2}]$,*

$$Q^y\left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} 1_{\{M \geq 2C\sqrt{\varepsilon}\}} \middle| \xi_{\varepsilon^{-3/2}} = z\right] \leq \left(\frac{2}{yz^2} + \frac{7}{z} + 1\right) e^{-C_7/\sqrt{\varepsilon}}.$$

As for the small M case, Lemma 16 provides an upper bound. The key step is to bound the probability that M is less than $2C\sqrt{\varepsilon}$.

Lemma 16. *There exists a positive constant C_8 such that for sufficiently small ε, for all $y \in (0,\infty)$ and $z \in (0, \varepsilon^{-1/2}]$,*

$$Q^y\left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} 1_{\{M < 2C\sqrt{\varepsilon}\}} \middle| \xi_{\varepsilon^{-3/2}} = z\right] \leq \frac{1}{z} e^{-C_8/\sqrt{\varepsilon}}. \tag{2.13}$$

In the end, the upper bound (2.6) is proved in Section 2.3.2 by combining Lemmas 12, 15 and 16.

In Section 2.3.2, we will gather all the lemmas to obtain the upper bound (2.6) and in Section 2.3.3, we will provide proofs for the lemmas above.

2.3.2 Proof of upper bound

To begin with, conditioning on the end point of the spinal trajectory, we have

$$\begin{aligned}
&\sqrt{2\pi t^3}Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}};\xi_t \le \varepsilon^{-1/2}\right] \\
&= \sqrt{2\pi t^3}\int_0^{\varepsilon^{-1/2}} Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\middle|\xi_t = z\right]\frac{z}{x}\frac{1}{\sqrt{2\pi t}}e^{-(x-z)^2/2t}(1-e^{-2xz/t})dz \\
&\le \sqrt{2\pi t^3}\int_0^{\varepsilon^{-1/2}} Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\middle|\xi_t = z\right]\frac{z}{x}\frac{1}{\sqrt{2\pi t}}e^{-(x-z)^2/2t}\frac{2xz}{t}dz \\
&= \int_0^{\varepsilon^{-1/2}} 2z^2e^{-(x-z)^2/2t}Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\middle|\xi_t = z\right]dz. \qquad (2.14)
\end{aligned}$$

Next, knowing that a reversed Bessel bridge is still a Bessel bridge, if $\{\xi_s\}_{0\le s\le t}$ is a Bessel bridge from x to z within time t, then $\{\zeta_s\}_{0\le s\le t}$ is also a Bessel bridge from z to x within time t. Since we are going to obtain an upper bound, it is enough to only look at the set of living particles at time t that branch off the spine in the last $\varepsilon^{-3/2}$ time. For clarification, under Q^x, the set $\{u \in \mathcal{N}_t : O_u \ge t-\varepsilon^{-3/2}\}$ includes the spinal particle. We have

$$\begin{aligned}
&Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\middle|\xi_t = z\right] \\
&\quad\le Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u\ge t-\varepsilon^{-3/2}\}}}\right] \\
&\quad= \int_0^\infty Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u\ge t-\varepsilon^{-3/2}\}}}\middle|\zeta_{\varepsilon^{-3/2}} = y\right]p^{z,t,x}_{\varepsilon^{-3/2}}(y)dy. \qquad (2.15)
\end{aligned}$$

According to the Markov property of BBM and the Bessel process,

$$Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u\ge t-\varepsilon^{-3/2}\}}}\middle|\zeta_{\varepsilon^{-3/2}} = y\right] = Q^{y,\varepsilon^{-3/2},z}\left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2})e^{\rho Y_u(\varepsilon^{-3/2})}}\right]. \qquad (2.16)$$

Note that $1-e^{-x} \le x$ for all $x \ge 0$ and $1-e^{-x} \ge x/2$ for $0 \le x \le 1$. For any fixed ε, if t is large enough such that $t/(t-\varepsilon^{-3/2}) \le 2$, then for all $z \in (0, \varepsilon^{-1/2}]$ and $y \in (0, \infty)$,

$$
\begin{aligned}
p^{z,t,x}_{\varepsilon^{-3/2}}(y) &= \frac{p_{\varepsilon^{-3/2}}(z,y)p_{t-\varepsilon^{-3/2}}(y,x)}{p_t(z,x)} \\
&= \frac{1}{\sqrt{2\pi\varepsilon^{-3/2}}} \cdot \frac{y}{z} e^{-(y-z)^2/(2\varepsilon^{-3/2})}(1-e^{-2yz\varepsilon^{3/2}}) \\
&\quad \times \frac{\frac{1}{\sqrt{2\pi(t-\varepsilon^{-3/2})}} \cdot \frac{x}{y} e^{-(y-x)^2/2(t-\varepsilon^{-3/2})}(1-e^{-2yx/(t-\varepsilon^{-3/2})})}{\frac{1}{\sqrt{2\pi t}} \cdot \frac{x}{z} e^{-(z-x)^2/2t}(1-e^{-2xz/t})} \\
&\le \frac{1}{\sqrt{2\pi\varepsilon^{-3/2}}} \frac{y}{z} e^{-(y-z)^2/(2\varepsilon^{-3/2})} 2yz\varepsilon^{3/2} \sqrt{\frac{t}{t-\varepsilon^{-3/2}}} \frac{z}{y} e^{(z-x)^2/2t} \frac{2xy}{t-\varepsilon^{-3/2}} \frac{t}{xz} \\
&= \sqrt{\frac{8}{\pi}} \varepsilon^{9/4} \left(\frac{t}{t-\varepsilon^{-3/2}} \right)^{3/2} y^2 e^{-(y-z)^2/(2\varepsilon^{-3/2})} e^{(z-x)^2/2t} \\
&\le \frac{8}{\sqrt{\pi}} \varepsilon^{9/4} y^2 e^{-(y-z)^2/(2\varepsilon^{-3/2})} e^{(z-x)^2/2t}.
\end{aligned} \tag{2.17}
$$

By (2.14), (2.15), (2.16) and (2.17), for every fixed ε, if t is large enough, we have

$$
\begin{aligned}
&\sqrt{2\pi t^3} Q^x \left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}; \xi_t \le \varepsilon^{-1/2} \right] \\
&\quad \le \int_0^{\varepsilon^{-1/2}} \int_0^\infty \frac{16}{\sqrt{\pi}} \varepsilon^{9/4} Q^{y,\varepsilon^{-3/2},z} \left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} \right] z^2 y^2 e^{-(y-z)^2/(2\varepsilon^{-3/2})} dy dz.
\end{aligned}
$$

Furthermore, by Lemmas 15 and 16, it follows that for sufficiently small ε, if t is large enough,

$$
\begin{aligned}
&\sqrt{2\pi t^3}Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}};\xi_t \le \varepsilon^{-1/2}\right] \\
&\le \int_0^{\varepsilon^{-1/2}}\int_0^\infty \frac{16}{\sqrt{\pi}}\varepsilon^{9/4}\left[\left(\frac{2}{yz^2}+\frac{7}{z}+1\right)e^{-C_7/\sqrt{\varepsilon}}+\frac{1}{z}e^{-C_8/\sqrt{\varepsilon}}\right]z^2y^2e^{-(y-z)^2/(2\varepsilon^{-3/2})}dydz \\
&\le \frac{16}{\sqrt{\pi}}\varepsilon^{9/4}\Big\{\int_0^{\varepsilon^{-1/2}}\int_0^\infty 2e^{-C_7/\sqrt{\varepsilon}}ye^{-(y-z)^2/(2\varepsilon^{-3/2})}dydz \\
&\quad +\int_0^{\varepsilon^{-1/2}}\int_0^\infty\left(7e^{-C_7/\sqrt{\varepsilon}}+e^{-C_8/\sqrt{\varepsilon}}\right)zy^2e^{-(y-z)^2/(2\varepsilon^{-3/2})}dydz \\
&\quad +\int_0^{\varepsilon^{-1/2}}\int_0^\infty e^{-C_7/\sqrt{\varepsilon}}z^2y^2e^{-(y-z)^2/(2\varepsilon^{-3/2})}dydz\Big\} \\
&=: \frac{16}{\sqrt{\pi}}\varepsilon^{9/4}(I_1+I_2+I_3). \qquad (2.18)
\end{aligned}
$$

For the first term, by substitution, we obtain that

$$
\begin{aligned}
I_1 &= 2e^{-C_7/\sqrt{\varepsilon}}\int_0^{\varepsilon^{-1/2}}\int_{-z}^\infty (u+z)e^{-u^2/(2\varepsilon^{-3/2})}dudz \\
&\le 2e^{-C_7/\sqrt{\varepsilon}}\left(\int_0^{\varepsilon^{-1/2}}\int_0^\infty ue^{-u^2/(2\varepsilon^{-3/2})}dudz+\int_0^{\varepsilon^{-1/2}}z\int_{-\infty}^\infty e^{-u^2/(2\varepsilon^{-3/2})}dudz\right) \\
&= \left(2\varepsilon^{-2}+\sqrt{2\pi}\varepsilon^{-7/4}\right)e^{-C_7/\sqrt{\varepsilon}}. \qquad (2.19)
\end{aligned}
$$

As for I_2 and I_3, because

$$
\begin{aligned}
\int_0^\infty y^2e^{-(y-z)^2/(2\varepsilon^{-3/2})}dy &\le \sqrt{2\pi\varepsilon^{-3/2}}\int_{-\infty}^\infty y^2\frac{1}{\sqrt{2\pi\varepsilon^{-3/2}}}e^{-(y-z)^2/(2\varepsilon^{-3/2})}dy \\
&= \sqrt{2\pi\varepsilon^{-3/2}}\left(\varepsilon^{-3/2}+z^2\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
I_2 &\le \sqrt{2\pi}\varepsilon^{-3/4}\left(7e^{-C_7/\sqrt{\varepsilon}}+e^{-C_8/\sqrt{\varepsilon}}\right)\int_0^{\varepsilon^{-1/2}}z(\varepsilon^{-3/2}+z^2)dz \\
&\le \sqrt{2\pi}\varepsilon^{-13/4}\left(7e^{-C_7/\sqrt{\varepsilon}}+e^{-C_8/\sqrt{\varepsilon}}\right), \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\le \sqrt{2\pi}\varepsilon^{-3/4}e^{-C_7/\sqrt{\varepsilon}}\int_0^{\varepsilon^{-1/2}} z^2(\varepsilon^{-3/2}+z^2)dz \\
&\le \sqrt{2\pi}\varepsilon^{-15/4}e^{-C_7/\sqrt{\varepsilon}}. \qquad (2.21)
\end{aligned}
$$

Setting $0 < C_4 < \min\{C_7, C_8\}$, equation (2.9) follows from (2.18), (2.19), (2.20) and (2.21). Finally, according to Lemma 12 and equation (2.9), the upper bound (2.6) is proved by letting $0 < C_1 < \min\{C_3, C_4\}$.

2.3.3 Proofs of Lemmas

Proof of Lemma 12. For all ε and all t, we have a trivial bound for the expectation

$$
\begin{aligned}
&\sqrt{2\pi t^3}Q^x\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}};\xi_t \ge \varepsilon^{-1/2}\right] \\
&\le \sqrt{2\pi t^3}Q^x\left[\frac{1}{\xi_t e^{\rho\xi_t}};\xi_t \ge \varepsilon^{-1/2}\right] \\
&= \sqrt{2\pi t^3}\int_{\varepsilon^{-1/2}}^{\infty}\frac{1}{ze^{\rho z}}p_t(x,z)dz \\
&= \sqrt{2\pi t^3}\int_{\varepsilon^{-1/2}}^{\infty}\frac{1}{ze^{\rho z}}\frac{z}{x}\frac{1}{\sqrt{2\pi t}}e^{-(x-z)^2/2t}(1-e^{-2xz/t})dz \\
&\le \sqrt{2\pi t^3}\int_{\varepsilon^{-1/2}}^{\infty}\frac{1}{ze^{\rho z}}\frac{z}{x}\frac{1}{\sqrt{2\pi t}}\frac{2xz}{t}dz \\
&= 2\int_{\varepsilon^{-1/2}}^{\infty} ze^{-\rho z}dz \\
&= \frac{2}{\rho}e^{-\rho/\sqrt{\varepsilon}}\left(\frac{1}{\sqrt{\varepsilon}}+\frac{1}{\rho}\right).
\end{aligned}
$$

Letting $0 < C_3 < \sqrt{2} \le \rho$, Lemma 12 is established. □

For the rest of this section we will denote by $X^{a,t,b}$ a Bessel bridge from a to b in time t and $B^{a,t,b}$ a Brownian bridge from a to b in time t.

Proof of Lemma 13. Observe that since $C + 1/\rho \le 3C/2$, for $0 \le t_i \le \varepsilon^{-3/2}$,

$$Q^y\left(\left\{\sum_{i=1}^{N_\varepsilon} 1_{\{\varepsilon\zeta_{t_i} - \frac{1}{\rho}\varepsilon^2 t_i \ge C\sqrt{\varepsilon}\}} \le \frac{1}{\sqrt{\varepsilon}}\right\} \cap \left\{M \ge 2C\sqrt{\varepsilon}\right\} \middle| \xi_{\varepsilon^{-3/2}} = z\right)$$
$$\le Q^{y,\varepsilon^{-3/2},z}\left(\left\{\sum_{i=1}^{N_\varepsilon} 1_{\{\zeta_{t_i} \ge 3C/(2\sqrt{\varepsilon})\}} \le \frac{1}{\sqrt{\varepsilon}}\right\} \cap \left\{M \ge 2C\sqrt{\varepsilon}\right\}\right).$$

So it is sufficient to show that there exists a constant C_5 such that for every $y \in (0, \infty)$ and $z \in (0, \varepsilon^{-1/2}]$,

$$Q^{y,\varepsilon^{-3/2},z}\left(\left\{\sum_{i=1}^{N_\varepsilon} 1_{\{\zeta_{t_i} \ge 3C/(2\sqrt{\varepsilon})\}} \le \frac{1}{\sqrt{\varepsilon}}\right\} \cap \left\{M \ge 2C\sqrt{\varepsilon}\right\}\right) \le \left(4 + \frac{1}{yz}\right) e^{-C_5/\sqrt{\varepsilon}}. \tag{2.22}$$

According to the spinal decomposition and the formula for expectations of additive functionals of Poisson point processes,

$$Q^{y,\varepsilon^{-3/2},z}\left(\sum_{i=1}^{N_\varepsilon} 1_{\{\zeta_{t_i} \ge 3C/(2\sqrt{\varepsilon})\}} \middle| \{\zeta_s\}_{0 \le s \le \varepsilon^{-3/2}}\right) = 2\int_0^{\varepsilon^{-3/2}} 1_{\{\zeta_s \ge 3C/(2\sqrt{\varepsilon})\}} ds.$$

Below for simplicity, we denote

$$\sum_{i=1}^{N_\varepsilon} 1_{\{\zeta_{t_i} \ge 3C/(2\sqrt{\varepsilon})\}} =: X,$$

$$2\int_0^{\varepsilon^{-3/2}} 1_{\{\zeta_s \ge 3C/(2\sqrt{\varepsilon})\}} ds =: Y.$$

The proof of (2.22) can be separated into two parts with the help of the above conditional expectation,

$$
\begin{aligned}
&Q^{y,\varepsilon^{-3/2},z}\Big(\Big\{X \le \frac{1}{\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\Big)\\
&\quad= Q^{y,\varepsilon^{-3/2},z}\Big(\Big\{X \le \frac{1}{\sqrt{\varepsilon}}\Big\} \cap \Big\{Y \ge \frac{2}{\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\Big)\\
&\qquad+ Q^{y,\varepsilon^{-3/2},z}\Big(\Big\{X \le \frac{1}{\sqrt{\varepsilon}}\Big\} \cap \Big\{Y \le \frac{2}{\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\Big)\\
&\quad\le Q^{y,\varepsilon^{-3/2},z}\Big(\Big\{X \le \frac{1}{\sqrt{\varepsilon}}\Big\} \cap \Big\{Y \ge \frac{2}{\sqrt{\varepsilon}}\Big\}\Big) + Q^{y,\varepsilon^{-3/2},z}\Big(\Big\{Y \le \frac{2}{\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\Big)\\
&\quad=: J_1 + J_2.
\end{aligned}
\tag{2.23}
$$

First, we will show that conditioned on the trajectory of the spine, the number of particles that branch off the spine at a position and time (ζ_{t_i}, t_i) satisfying $\zeta_{t_i} \ge 3C/(2\sqrt{\varepsilon})$ isn't far from its conditional expectation, which gives an upper bound for J_1. Next we will find an upper bound for J_2 through analysis of the behavior of the spine.

For the first part, we will apply the following bound for the Poisson distribution (see, e.g. [27]). For a Poisson distributed random variable Z with expectation λ, for any $v > 0$, we have

$$
P(|Z - \lambda| \ge v) \le 2e^{-v^2/2(\lambda+v)}. \tag{2.24}
$$

We know that under $Q^{y,\varepsilon^{-3/2},z}$, the conditional distribution of X given $\{\zeta_s\}_{0\le s\le \varepsilon^{-3/2}}$ is a Poisson distribution with parameter Y. Applying (2.24) under the conditional expectation with $\lambda = Y$ and $v = Y/2$, we have

$$
\begin{aligned}
J_1 &\le Q^{y,\varepsilon^{-3/2},z}\Big(1_{\{Y\ge 2/\sqrt{\varepsilon}\}} Q^{y,\varepsilon^{-3/2},z}\big(\big|X - Y\big| \ge Y/2 \big| \{\zeta_s\}_{0\le s\le\varepsilon^{-3/2}}\big)\Big)\\
&\le Q^{y,\varepsilon^{-3/2},z}\Big(1_{\{Y\ge 2/\sqrt{\varepsilon}\}} 2\exp\Big\{-\frac{(Y/2)^2}{2(Y+Y/2)}\Big\}\Big)\\
&\le 2Q^{y,\varepsilon^{-3/2},z}\Big(1_{\{Y\ge 2/\sqrt{\varepsilon}\}} e^{-Y/12}\Big)\\
&\le 2e^{-1/(6\sqrt{\varepsilon})}.
\end{aligned}
\tag{2.25}
$$

As for the second part, we have

$$J_2 \le Q^{y,\varepsilon^{-3/2},z}\left(\left\{\int_0^{\varepsilon^{-3/2}} 1_{\{\zeta_s \ge 3C/(2\sqrt{\varepsilon})\}} ds \le \frac{1}{\sqrt{\varepsilon}}\right\} \cap \left\{\sup_{0\le s\le \varepsilon^{-3/2}} \varepsilon\zeta_s \ge 2C\sqrt{\varepsilon}\right\}\right)$$
$$= Q^{y,\varepsilon^{-3/2},z}\left(\left\{\int_0^1 1_{\{\varepsilon^{3/4}\zeta_{\varepsilon^{-3/2}r} \ge 3C\varepsilon^{1/4}/2\}} dr \le \varepsilon\right\} \cap \left\{\sup_{0\le r\le 1} \varepsilon^{3/4}\zeta_{\varepsilon^{-3/2}r} \ge 2C\varepsilon^{1/4}\right\}\right).$$

Notice that under $Q^{y,\varepsilon^{-3/2},z}$, the process $\{\zeta_s\}_{0\le s\le \varepsilon^{-3/2}}$ is a Bessel bridge from z to y in time $\varepsilon^{-3/2}$. After scaling, $\{\varepsilon^{3/4}\zeta_{\varepsilon^{-3/2}r}\}_{0\le r\le 1}$ is a Bessel bridge from $\varepsilon^{3/4}z$ to $\varepsilon^{3/4}y$ within time 1. Recall that $\{X_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\}_{0\le r\le 1}$ is a Bessel bridge from $\varepsilon^{3/4}z$ to $\varepsilon^{3/4}y$ within time 1. For simplicity, we will write $\{X_r\}_{0\le r\le 1}$ in place of $\{X_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\}_{0\le r\le 1}$. Therefore, we have

$$J_2 \le P\left(\left\{\int_0^1 1_{\{X_r \ge 3C\varepsilon^{1/4}/2\}} dr \le \varepsilon\right\} \cap \left\{\sup_{0\le r\le 1} X_r \ge 2C\varepsilon^{1/4}\right\}\right). \tag{2.26}$$

Define $\{\bar{X}_r\}_{0\le r\le 1} = \{X_{1-r}\}_{0\le r\le 1}$ to be the time reversed process of $\{X_r\}_{0\le r\le 1}$. Then $\{\bar{X}_r\}_{0\le r\le 1}$ is a Bessel bridge from $\varepsilon^{3/4}y$ to $\varepsilon^{3/4}z$ in time 1. Thus the intersection of the events in (2.26) is contained in the union of two events. One of the events is that $\{X_r\}_{0\le r\le 1}$ first reaches $2C\varepsilon^{1/4}$ before time $1/2$ and then comes down below $3C\varepsilon^{1/4}/2$ in time less than ε. The other event is that $\{\bar{X}_r\}_{0\le r\le 1}$ first reaches $2C\varepsilon^{1/4}$ before time $1/2$ and then comes down below $3C\varepsilon^{1/4}/2$ in time less than ε. Define

$$\tau = \inf\left\{r \ge 0 : X_r \ge 2C\varepsilon^{1/4}\right\}, \quad \bar{\tau} = \inf\left\{r \ge 0 : \bar{X}_r \ge 2C\varepsilon^{1/4}\right\}.$$

We see that

$$P\left(\left\{\int_0^1 1_{\{X_r \ge 3C\varepsilon^{1/4}/2\}} dr \le \varepsilon\right\} \cap \left\{\sup_{0\le r\le 1} X_r \ge 2C\varepsilon^{1/4}\right\}\right)$$
$$\le P\left(\left\{\tau \le \frac{1}{2}\right\} \cap \left\{\min_{0\le r\le \varepsilon} X_{\tau+r} \le \frac{3C\varepsilon^{1/4}}{2}\right\}\right)$$
$$+ P\left(\left\{\bar{\tau} \le \frac{1}{2}\right\} \cap \left\{\min_{0\le r\le \varepsilon} \bar{X}_{\bar{\tau}+r} \le \frac{3C\varepsilon^{1/4}}{2}\right\}\right). \tag{2.27}$$

Therefore, the proof for the second part of Lemma 13 boils down to Lemma 17, whose statement and proof is deferred until later.

Letting $0 < C_5 < \min\{1/6, C_9\}$, with equations (2.23), (2.25), (2.26), (2.27) and Lemma 17, formula (2.22) is proved and thus the proof of Lemma 13 is finished.

□

Below, we will state and prove Lemma 17.

Lemma 17. *There exists a positive constant C_9 such that for ε sufficiently small, for all $z \in (0,\infty)$ and $y \in (0,\infty)$,*

$$P\left(\left\{\tau \leq \frac{1}{2}\right\} \cap \left\{\min_{0\leq r\leq\varepsilon} X_{\tau+r} \leq \frac{3C\varepsilon^{1/4}}{2}\right\}\right) \leq \left(\frac{1}{yz}+2\right)e^{-C_9/\sqrt{\varepsilon}}. \tag{2.28}$$

Proof of Lemma 17. Let's first consider the case when $z \in (2C\varepsilon^{-1/2}, \infty)$. Under this scenario, $\tau = 0$ and thus

$$P\left(\left\{\tau < \frac{1}{2}\right\} \cap \left\{\min_{0\leq r\leq\varepsilon} X_{\tau+r} \leq \frac{3C\varepsilon^{1/4}}{2}\right\}\right) = P\left(\min_{0\leq r\leq\varepsilon} X_r \leq \frac{3C\varepsilon^{1/4}}{2}\right). \tag{2.29}$$

According to Lemma 10, the process $\{X_r\}_{0\leq r\leq 1}$ stochastically dominates $\{X_r^{2C\varepsilon^{1/4},1,\varepsilon^{3/4}y}\}_{0\leq r\leq 1}$, which is a Bessel bridge from $2C\varepsilon^{1/4}$ to $\varepsilon^{3/4}y$ in time 1. Therefore

$$\begin{aligned}P\left(\min_{0\leq r\leq\varepsilon} X_r \leq \frac{3C\varepsilon^{1/4}}{2}\right) &\leq P\left(\min_{0\leq r\leq\varepsilon} X_r^{2C\varepsilon^{1/4},1,\varepsilon^{3/4}y} \leq \frac{3C\varepsilon^{1/4}}{2}\right)\\ &= P\left(\left\{\tau \leq \frac{1}{2}\right\} \cap \left\{\min_{0\leq r\leq\varepsilon} X_{\tau+r}^{2C\varepsilon^{1/4},1,\varepsilon 3/4y} \leq \frac{3C\varepsilon^{1/4}}{2}\right\}\right).\end{aligned}$$

Therefore, it is sufficient to only consider the case when $z \in (0, 2C\varepsilon^{-1/2}]$.

For $y,z > 0$, denote by $\{B_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\}_{0\leq r\leq 1}$ a Brownian bridge from $\varepsilon^{3/4}z$ to $\varepsilon^{3/4}y$ within time 1. Define

$$\tau_0 = \tau_0(y,z) := \inf\left\{r \in [0,1] : B_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y} = 0\right\}$$

and τ' to be τ under the setting of Brownian bridge

$$\tau' := \inf\Big\{r \geq 0 : B_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y} \geq 2C\varepsilon^{1/4}\Big\}.$$

By convention, $\inf\emptyset = \infty$. We know that (see, e.g., page 86 of [48]) the probability that a Brownian bridge avoids the origin is

$$P(\tau_0 = \infty) = 1 - e^{-2\varepsilon^{3/2}yz}.$$

Furthermore, according to the first part of the proof of Lemma 7 in [48], a Brownian bridge that is conditioned to avoid the origin has the same law as a Bessel bridge. Together with the inequality

$$\frac{1}{1-e^{-x}} \leq \frac{2}{x}1_{\{0<x<1\}} + 2\cdot 1_{\{x\geq 1\}} \leq \frac{2}{x} + 2,$$

we have for ε sufficiently small,

$$\begin{aligned}
&P\bigg(\Big\{\tau \leq \frac{1}{2}\Big\} \cap \Big\{\min_{0\leq r\leq\varepsilon} X_{\tau+r} \leq \frac{3C\varepsilon^{1/4}}{2}\Big\}\bigg)\\
&\quad= P\bigg(\Big\{\tau \leq \frac{1}{2}\Big\} \cap \Big\{\min_{0\leq r\leq\varepsilon}\Big(X_{\tau+r} - X_\tau\Big) \leq -\frac{C\varepsilon^{1/4}}{2}\Big\}\bigg)\\
&\quad= P\bigg(\Big\{\tau' \leq \frac{1}{2}\Big\} \cap \Big\{\min_{0\leq r\leq\varepsilon}\Big(B_{\tau'+r}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y} - B_{\tau'}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\Big) \leq -\frac{C\varepsilon^{1/4}}{2}\Big\}\bigg|\tau_0 = \infty\bigg)\\
&\quad= \frac{P\Big(\big\{\tau' \leq \frac{1}{2}\big\} \cap \Big\{\min_{0\leq r\leq\varepsilon}\Big(B_{\tau'+r}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y} - B_{\tau'}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\Big) \leq -C\varepsilon^{1/4}/2\Big\} \cap \{\tau_0 = \infty\}\Big)}{P(\tau_0 = \infty)}\\
&\quad\leq \bigg(\frac{1}{\varepsilon^{3/2}yz} + 2\bigg)P\bigg(\Big\{\tau' \leq \frac{1}{2}\Big\} \cap \Big\{\min_{0\leq r\leq\varepsilon}(B_{\tau'+r}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y} - B_{\tau'}^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}) \leq -\frac{C\varepsilon^{1/4}}{2}\Big\}\bigg).
\end{aligned} \tag{2.30}$$

Now we are going to bound the probability of the above event under the setting of the Brownian bridge. Let $\mathcal{F}_{\tau'}$ be the $\sigma-$field generated by the stopping time τ'. Conditioning on $\mathcal{F}_{\tau'}$,

$$P\biggl(\Bigl\{\tau' \le \frac{1}{2}\Bigr\} \cap \Bigl\{\min_{0\le r\le\varepsilon}(B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'+r} - B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'}) \le -\frac{C\varepsilon^{1/4}}{2}\Bigr\}\biggr)$$
$$= E\biggl[1_{\{\tau'\le\frac{1}{2}\}}P\biggl(\min_{0\le r\le\varepsilon}\Bigl(B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'+r} - B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'}\Bigr) \le -\frac{C\varepsilon^{1/4}}{2}\bigg|\mathcal{F}_{\tau'}\biggr)\biggr]. \tag{2.31}$$

Since the Brownian bridge is a strong Markov process (see, e.g., Proposition 1 of [41]), the conditional distribution of $\min_{0\le r\le\varepsilon}\bigl(B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'+r} - B^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}_{\tau'}\bigr)$ given $\tau' = 1-u$ is the same as the distribution of $\min_{0\le r\le\varepsilon} B^{0,u,\varepsilon^{3/4}y-2C\varepsilon^{1/4}}_r$ and is independent of $\mathcal{F}_{\tau'}$. Therefore, given $\tau' = 1-u$, the probability inside equation (2.31) can be written as

$$P\biggl(\min_{0\le r\le\varepsilon} B^{0,u,\varepsilon^{3/4}y-2C\varepsilon^{1/4}}_r \le -\frac{C\varepsilon^{1/4}}{2}\biggr) = P\biggl(\max_{0\le r\le\varepsilon} B^{0,u,-\varepsilon^{3/4}y+2C\varepsilon^{1/4}}_r \ge \frac{C\varepsilon^{1/4}}{2}\biggr). \tag{2.32}$$

To bound the probability inside the expectation, we will consider the cases where $y \in (0,\varepsilon^{-1/2}]$ and $y \in (\varepsilon^{-1/2},\infty)$ separately. For $y \in (0,\varepsilon^{-1/2}]$, we will apply Theorem 2.1 of [9], which gives the distribution of the maximum of the beginning period of a Brownian bridge. Let $\beta = C\varepsilon^{1/4}/2$, $\eta = 2C\varepsilon^{1/4} - \varepsilon^{3/4}y$ and $s = \varepsilon$. We have

$$P\biggl(\max_{0\le r\le\varepsilon} B^{0,u,-\varepsilon^{3/4}y+2C\varepsilon^{1/4}}_r \ge \frac{C\varepsilon^{1/4}}{2}\biggr)$$
$$= \exp\Bigl\{-\frac{2\beta(\beta-\eta)}{u}\Bigr\}\int_{-\infty}^{(2\beta s-\eta s-\beta u)/\sqrt{us(u-s)}} \frac{e^{-v^2/2}}{\sqrt{2\pi}}dv + \int_{(\beta u-\eta s)/\sqrt{us(u-s)}}^{\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}}dv. \tag{2.33}$$

On the event $\{\tau' \le 1/2\}$, we have $1/2 \le u < 1$. Combined with the fact that $C > 2\pi$, we can derive the following limits as ε approaches 0 for $y \in (0,\varepsilon^{-1/2}]$,

$$\frac{2\beta(\beta-\eta)}{u} < 0, \quad \frac{2\beta(\beta-\eta)}{u} = O(\varepsilon^{1/2}), \tag{2.34}$$

$$\frac{2\beta s-\eta s-\beta u}{\sqrt{us(u-s)}} < 0, \quad \frac{2\beta s-\eta s-\beta u}{\sqrt{us(u-s)}} = O(\varepsilon^{-1/4}), \tag{2.35}$$

$$\frac{\beta u - \eta s}{\sqrt{us(u-s)}} > 0, \quad \frac{\beta u - \eta s}{\sqrt{us(u-s)}} = O(\varepsilon^{-1/4}). \tag{2.36}$$

Note that none of the asymptotic rates above depend on y. Moreover, it can be easily shown that

$$\int_x^\infty \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv \leq \frac{e^{-x^2/2}}{x\sqrt{2\pi}}. \tag{2.37}$$

By (2.33), (2.34), (2.35), (2.36) and (2.37), we see that there exists a positive constant C_{10} such that for ε sufficiently small, for all $y \in (0, \varepsilon^{-1/2}]$, given $\tau' = 1-u \leq 1/2$,

$$P\left(\max_{0\leq r\leq \varepsilon} B_r^{0,u,-\varepsilon^{3/4}y+2C\varepsilon^{1/4}} \geq \frac{C\varepsilon^{1/4}}{2}\right) \leq e^{-C_{10}/\sqrt{\varepsilon}}. \tag{2.38}$$

On the other hand, if $y \in (\varepsilon^{-1/2}, \infty)$, given $\tau' = 1-u \leq 1/2$, by Lemma 9,

$$\begin{aligned} P\left(\max_{0\leq r\leq \varepsilon} B_r^{0,u,-\varepsilon^{3/4}y+2C\varepsilon^{1/4}} \geq \frac{C\varepsilon^{1/4}}{2}\right) &\leq P\left(\max_{0\leq r\leq \varepsilon} B_r^{0,u,(2C-1)\varepsilon^{1/4}} \geq \frac{C\varepsilon^{1/4}}{2}\right) \\ &\leq e^{-C_{10}/\sqrt{\varepsilon}}. \end{aligned} \tag{2.39}$$

As a result, when $z \in (0, 2C\varepsilon^{-1/2}]$, taking $C_9 < C_{10}$, equation (2.28) follows from (2.30), (2.31), (2.32), (2.38) and (2.39).

□

Proof of Lemma 14. We first transform (2.12) from the setting of BBM with absorption and drift into standard BBM. Let P be the law of a standard BBM started from 0 without drift and absorption. We have for $s \in [0, \varepsilon^{-3/2}]$,

$$\begin{aligned} &P_{-\rho}^{C/\sqrt{\varepsilon}+\varepsilon s/\rho}\left(\exists u \in \mathcal{N}_s^{-\rho} : Y_u(r) > 0 \; \forall\, r \leq s,\; Y_u(s) > \frac{C}{4\sqrt{\varepsilon}}\right) \\ &\quad = P\left(\exists u \in \mathcal{N}_s : Y_u(r) + \frac{C}{\sqrt{\varepsilon}} + \frac{\varepsilon s}{\rho} - \rho r > 0 \; \forall r \leq s,\; Y_u(s) + \frac{C}{\sqrt{\varepsilon}} + \frac{\varepsilon s}{\rho} - \rho s > \frac{C}{4\sqrt{\varepsilon}}\right) \\ &\quad \geq P\left(\exists u \in \mathcal{N}_s : Y_u(r) > \rho r - \frac{\varepsilon s}{\rho} - \frac{3C}{4\sqrt{\varepsilon}} \; \forall r \leq s\right). \end{aligned} \tag{2.40}$$

Then we will apply Theorem 1 in Roberts [73], which gives the explicit formula of a curve such that at least one particle stays above this curve all the time with nonzero probability. Borrowing notations from [73], we let $A_c = 3^{4/3}\pi^{2/3}2^{-7/6}$ and

$$g(s) = \sqrt{2}s - A_c s^{1/3} + \frac{A_c s^{1/3}}{\log^2(s+e)} - 1.$$

Theorem 1 in Roberts [73] states that there exists some nonzero absolute constant C_6, such that

$$P(\forall s \geq 0, \exists u \in \mathcal{N}_s : Y_u(r) > g(r)\ \forall r \leq s) > C_6.$$

Together with our choice of $C > 2\pi$ and the Taylor expansion for ρ, we have for ε sufficiently small, for all $r \leq s$,

$$\rho r - \frac{\varepsilon s}{\rho} - \frac{3C}{4\sqrt{\varepsilon}} = \sqrt{2}r + \frac{\varepsilon r}{\rho} + O(\varepsilon^2)r - \frac{\varepsilon s}{\rho} - \frac{3C}{4\sqrt{\varepsilon}} \leq \sqrt{2}r - \frac{3C}{4\sqrt{\varepsilon}} + O(\varepsilon^{1/2}),$$

$$g(r) \geq \sqrt{2}r - A_c(\varepsilon^{-3/2})^{1/3} - 1 \geq \sqrt{2}r - \frac{3C}{4\sqrt{\varepsilon}} + O(\varepsilon^{1/2}).$$

As a result, for all $s \in [0, \varepsilon^{-3/2})$,

$$\begin{aligned} P\bigg(\exists u \in & \mathcal{N}_s : Y_u(r) > \rho r - \frac{\varepsilon s}{\rho} - \frac{3C}{4\sqrt{\varepsilon}}\ \forall r \leq s\bigg) \\ &\geq P\big(\forall s \leq 0, \exists u \in \mathcal{N}_s : Y_u(r) > g(r)\ \forall r \leq s\big) \\ &> C_6. \end{aligned} \tag{2.41}$$

The lemma follows from (2.40) and (2.41). □

Proof of Lemma 15. From Lemma 14, we know that if a particle starts from $C/\sqrt{\varepsilon} + \varepsilon s/\rho$, it will have a descendant at time s which stays to the right of $C/(4\sqrt{\varepsilon})$ with probability at least C_6. So if we have a particle branching off the spine at a position and time $(t - t_i, \zeta_{t_i})$

satisfying $0 \le t_i \le \varepsilon^{-3/2}$ and $C/\sqrt{\varepsilon} + \varepsilon t_i/\rho \le \zeta_{t_i}$, then

$$P_{-\rho}^{\zeta_{t_i}}\Big(\exists u \in \mathcal{N}_{t_i}^{\rho} : Y_u(t_i) > \frac{C}{4\sqrt{\varepsilon}}\Big) \ge P_{-\rho}^{C/\sqrt{\varepsilon}+\varepsilon t_i/\rho}\Big(\exists u \in \mathcal{N}_{t_i}^{\rho} : Y_u(t_i) > \frac{C}{4\sqrt{\varepsilon}}\Big) \ge C_6.$$

Combined with Lemmas 13 and 14 and the branching property, we have

$$\begin{aligned}
&Q^{y,\varepsilon^{-3/2},z}\bigg(\Big\{\forall u \in \mathcal{N}_{\varepsilon^{-3/2}}^{-\rho}, Y_u(\varepsilon^{-3/2}) \le \frac{C}{4\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\bigg) \\
&\le Q^{y,\varepsilon^{-3/2},z}\bigg(\{M \ge 2C\sqrt{\varepsilon}\} \cap \Big\{\sum_{i=1}^{N_\varepsilon} 1_{\{\varepsilon\zeta_{t_i} - \frac{1}{\rho}\varepsilon^2 t_i \ge C\sqrt{\varepsilon}\}} \le \frac{1}{\sqrt{\varepsilon}}\Big\}\bigg) \\
&\quad + Q^{y,\varepsilon^{-3/2},z}\bigg(\forall u \in \mathcal{N}_{\varepsilon^{-3/2}}^{-\rho}, Y_u(\varepsilon^{-3/2}) \le \frac{C}{4\sqrt{\varepsilon}}\bigg| \{M \ge 2C\sqrt{\varepsilon}\} \cap \Big\{\sum_{i=1}^{N_\varepsilon} 1_{\{\varepsilon\zeta_{t_i} - \frac{1}{\rho}\varepsilon^2 t_i \ge C\sqrt{\varepsilon}\}} \\
&\quad \ge \frac{1}{\sqrt{\varepsilon}}\Big\}\bigg) \times Q^{y,\varepsilon^{-3/2},z}\bigg(\Big\{\sum_{i=1}^{N_\varepsilon} 1_{\{\varepsilon\zeta_{t_i} - \frac{1}{\rho}\varepsilon^2 t_i \ge C\sqrt{\varepsilon}\}} \ge \frac{1}{\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\bigg) \\
&\le \left(\frac{2}{yz} + 6\right) e^{-C_5/\sqrt{\varepsilon}} + (1 - C_6)^{1/\sqrt{\varepsilon}}.
\end{aligned}$$

Note that if there exists a $u \in \mathcal{N}_{\varepsilon^{-3/2}}^{-\rho}$ such that $Y_u(\varepsilon^{-3/2}) \ge C/(4\sqrt{\varepsilon})$, then for ε small enough, there exists a $0 < C_{11} < C\rho/4$ satisfying

$$\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} \le \frac{4\sqrt{\varepsilon}}{C} e^{-C\rho/(4\sqrt{\varepsilon})} \le e^{-C_{11}/\sqrt{\varepsilon}}.$$

As a result,

$$\begin{aligned}
&Q^y\Big[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2}) e^{\rho Y_u(\varepsilon^{-3/2})}} 1_{\{M \ge 2C\sqrt{\varepsilon}\}}\Big| \xi_{\varepsilon^{-3/2}} = z\Big] \\
&\le \frac{1}{ze^{\rho z}} Q^{y,\varepsilon^{-3/2},z}\bigg(\Big\{\forall u \in \mathcal{N}_{\varepsilon^{-3/2}}^{-\rho}, Y_u(\varepsilon^{-3/2}) \le \frac{C}{4\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\bigg) \\
&\quad + e^{-C_{11}/\sqrt{\varepsilon}} Q^{y,\varepsilon^{-3/2},z}\bigg(\Big\{\exists u \in \mathcal{N}_{\varepsilon^{-3/2}}^{-\rho}, Y_u(\varepsilon^{-3/2}) > \frac{C}{4\sqrt{\varepsilon}}\Big\} \cap \{M \ge 2C\sqrt{\varepsilon}\}\bigg) \\
&\le \frac{2}{yz^2} e^{-C_5\sqrt{\varepsilon}} + \frac{6}{z} e^{-C_5/\sqrt{\varepsilon}} + \frac{1}{z}(1 - C_6)^{1/\sqrt{\varepsilon}} + e^{-C_{11}/\sqrt{\varepsilon}}.
\end{aligned} \tag{2.42}$$

Letting $0 < C_7 < \min\{C_5, -\log(1 - C_6), C_{11}\}$, the lemma is proved. □

Proof of Lemma 16. First note that if $y \in [(2C + 1/\rho)\varepsilon^{-1/2}, \infty)$, then $M \geq 2C\sqrt{\varepsilon}$ and therefore the inequality (2.13) holds trivially. It only remains to consider the case where $y \in (0, (2C + 1/\rho)\varepsilon^{-1/2})$.

Observe that there is a simple upper bound for (2.13)

$$Q^y\left[\frac{1}{\sum_u Y_u(\varepsilon^{-3/2})e^{\rho Y_u(\varepsilon^{-3/2})}}1_{\{M<2C\sqrt{\varepsilon}\}}\middle|\xi_{\varepsilon^{-3/2}} = z\right] \leq \frac{1}{z}Q^{y,\varepsilon^{-3/2},z}(M < 2C\sqrt{\varepsilon}). \tag{2.43}$$

Furthermore, because $1/\rho < C$,

$$\begin{aligned}
Q^{y,\varepsilon^{-3/2},z}(M < 2C\sqrt{\varepsilon}) &= Q^{y,\varepsilon^{-3/2},z}\Big(\sup_{0\leq s\leq\varepsilon^{-3/2}}(\varepsilon\zeta_s - \frac{1}{\rho}\varepsilon^2 s) < 2C\sqrt{\varepsilon}\Big)\\
&\leq Q^{y,\varepsilon^{-3/2},z}\left(\sup_{0\leq s\leq\varepsilon^{-3/2}}\zeta_s < \frac{3C}{\sqrt{\varepsilon}}\right)\\
&= Q^{y,\varepsilon^{-3/2},z}\Big(\sup_{0\leq r\leq 1}\varepsilon^{3/4}\zeta_{\varepsilon^{-3/2}r} < 3C\varepsilon^{1/4}\Big).
\end{aligned}$$

Notice that under $Q^{y,\varepsilon^{-3/2},z}$, the process $\{\varepsilon^{3/4}\zeta_{\varepsilon^{-3/2}r}\}_{0\leq r\leq 1}$ is a Bessel bridge from $\varepsilon^{3/4}z$ to $\varepsilon^{3/4}y$ in time 1. Recall that $\{X_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\}_{0\leq r\leq 1}$ denotes a Bessel bridge from $\varepsilon^{3/4}z$ to $\varepsilon^{3/4}y$ in time 1. For simplicity, below we will omit the superscript of $\{X_r^{\varepsilon^{3/4}z,1,\varepsilon^{3/4}y}\}_{0\leq r\leq 1}$. Therefore, we have

$$Q^{y,\varepsilon^{-3/2},z}(M < 2C\sqrt{\varepsilon}) \leq P\Big(\sup_{0\leq r\leq 1}X_r < 3C\varepsilon^{1/4}\Big). \tag{2.44}$$

According to (0.22) of [72], let $B_{(1)}^{0,1,0}, B_{(2)}^{0,1,0}, B_{(3)}^{0,1,0}$ be three independent standard Brownian bridges, we have

$$X^{0,1,0} \overset{d}{=\!=} \sqrt{\left(B_{(1)}^{0,1,0}\right)^2 + \left(B_{(2)}^{0,1,0}\right)^2 + \left(B_{(3)}^{0,1,0}\right)^2}.$$

According to Lemma 10 and the above formula, letting $\{X_r^{0,1,0}\}_{0\leq r\leq 1}$ be a Bessel bridge from 0 to 0 in time 1 and $\{B_r^{0,1,0}\}_{0\leq r\leq 1}$ be a Brownian bridge from 0 to 0 in time 1, for

$z \in (0, \varepsilon^{-1/2}]$ and $y \in (0, (2C + 1/\rho)\varepsilon^{-1/2})$, we get

$$P\Big(\sup_{0\le r\le 1} X_r < 3C\varepsilon^{1/4}\Big) \le P\Big(\sup_{0\le r\le 1} X_r^{0,1,0} < 3C\varepsilon^{1/4}\Big) \le \Big[P\Big(\sup_{0\le r\le 1} |B_r^{0,1,0}| < 3C\varepsilon^{1/4}\Big)\Big]^3. \tag{2.45}$$

From Lemma 8, for ε sufficiently small,

$$P\Big(\sup_{0\le r\le 1} |B_r^{0,1,0}| < 3C\varepsilon^{1/4}\Big) \le C^{-1}\varepsilon^{-1/4}\exp\Big\{-\frac{\pi^2}{72C^2\sqrt{\varepsilon}}\Big\}. \tag{2.46}$$

In the end, setting $0 < C_8 < \pi^2/(24C^2)$, by (2.43)–(2.46), Lemma 16 is proved. □

2.4 Lower bound

2.4.1 Proof of the Lower bound

In this section, we will prove the lower bound (2.7). We first state two lemmas, which are the key ingredients in the proof of the lower bound.

We observe that for ε sufficiently small, the probability that particles which branch off the spine before a large time have descendants at time t is small. As a result, in order to deal with the lower bound, we only need to consider particles that branch off the spine after a large time. We will start by finding this cutoff time t^*.

Let $0 < \delta_1 < \delta_2 < 1/4$. We denote

$$t^* := t - \left(\frac{4}{\varepsilon}\right)^{2/(1-2\delta_1)}, \quad t' := t - t^{1/2+\delta_2}.$$

Define V_1 to be the event that particles that branch off the spine before time t' have descendants alive at time t and the spine stays below $(t')^{1/2+\delta_1}$ for all $s \le t'$. Define V_2 to be the event that particles that branch off the spine before time t' have descendants alive at time t and the spine crosses the curve $(t')^{1/2+\delta_1}$ for some $s \le t'$. Define V_3 to be the event that particles that branch off the spine between time t' and t^* have descendants alive at time t

and the spine stays below the curve $s^{1/2+\delta_1}$ for all $s \in (t', t^*]$. Define V_4 to be the event that particles that branch off the spine between time t' and t^* have descendants alive at time t and the spine crosses the curve $s^{1/2+\delta_1}$ for some $s \in (t', t^*]$. More precisely,

$$V_1 = \{\exists u \in \mathcal{N}_t : O_u \le t'\} \cap \{\xi_s \le (t')^{1/2+\delta_1},\ \forall s \le t'\}, \tag{2.47}$$

$$V_2 = \{\exists u \in \mathcal{N}_t : O_u \le t'\} \cap \{\exists\ s \le t' : \xi_s > (t')^{1/2+\delta_1}\}, \tag{2.48}$$

$$V_3 = \{\exists u \in \mathcal{N}_t : t' < O_u \le t^*\} \cap \{\xi_s \le s^{1/2+\delta_1},\ \forall s \in (t', t^*]\}, \tag{2.49}$$

$$V_4 = \{\exists u \in \mathcal{N}_t : t' < O_u \le t^*\} \cap \{\exists\ s \in (t', t^*] : \xi_s > s^{1/2+\delta_1}\}. \tag{2.50}$$

Then we have,

$$\{\exists u \in \mathcal{N}_t : O_u \le t^*\} = V_1 \cup V_2 \cup V_3 \cup V_4.$$

Lemma 18. *For any $0 < \delta < 1/2$, if ε is sufficiently small, then for all $z \in (0, \varepsilon^{-1/2}]$,*

$$\limsup_{t\to\infty} Q^{x,t,z}\left(\bigcup_{i=1}^{4} V_i\right) < \delta. \tag{2.51}$$

Note that as δ_1 goes to 0, $2/(1-2\delta_1)$ goes to 2. Roughly speaking, this lemma shows that only particles which branch off the spine within the last ε^{-2} time will contribute significantly to our expectation in Proposition 7. For simplicity, letting $\kappa = 4\delta_1/(1-2\delta_1) > 0$, we will also write the cutoff time t^* as

$$t^* = t - \left(\frac{4}{\varepsilon}\right)^{2+\kappa}.$$

We need one more lemma to finish the proof of (2.7). Define

$$M' = \sup_{0 \le s \le (4/\varepsilon)^{2+\kappa}} \left(\varepsilon\zeta_s - \frac{1}{\rho}\varepsilon^2 s\right).$$

Similarly to the proof of upper bound, we will divide the space into two parts, $\{M' \le C\sqrt{\varepsilon}\}$

and $\{M' > C\sqrt{\varepsilon}\}$ for some constant $C > 2\sqrt{3}$. Since this time we focus on the lower bound, it is enough to consider only one of them.

Lemma 19. *Let $C > 2\sqrt{3}$. There exists a positive constant C_{12} such that for ε sufficiently small, for all $z \in (0, \varepsilon^{-1/2}]$ and $y \in (0, \varepsilon^{-1-\kappa}]$, we have*

$$Q^{y,(4/\varepsilon)^{2+\kappa},z}\Big(M' \le C\sqrt{\varepsilon}\Big) \ge \varepsilon^{-3/4} e^{-C_{12}/\sqrt{\varepsilon}}.$$

Below, we will apply above lemmas, together with Jensen's inequality and the martingale property to prove the lower bound.

Proof of (2.7). Conditioned on the endpoint of the spinal particle, we have

$$\begin{aligned}
&\sqrt{2\pi t^3} Q^x\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}\right] \\
&\quad \ge \sqrt{2\pi t^3} Q^x\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}; \xi_t \le \varepsilon^{-1/2}\right] \\
&\quad = \sqrt{2\pi t^3} \int_0^{\varepsilon^{-1/2}} Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}\right] \frac{1}{\sqrt{2\pi t}} \frac{z}{x} e^{-(x-z)^2/(2t)} (1 - e^{-2xz/t}) dz.
\end{aligned}$$

For every ε and x, there exists a $T(\varepsilon, x)$ such that for all $t \ge T(\varepsilon, x)$ and $z \in (0, \varepsilon^{-1/2}]$,

$$e^{-(x-z)^2/(2t)} \ge \frac{1}{2}, \quad 1 - e^{-2xz/t} \ge \frac{1}{2} \cdot \frac{2xz}{t} = \frac{xz}{t}.$$

Therefore,

$$\sqrt{2\pi t^3} Q^x\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}\right] \ge \frac{1}{2} \int_0^{\varepsilon^{-1/2}} Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}\right] z^2 dz. \tag{2.52}$$

Next, we restrict the integrand to the case where all particles branch off the spine after t^*. Letting

$$V = \big\{\forall u \in \mathcal{N}_t : O_u > t^*\big\} = \bigcap_{i=1}^{4} V_i^c,$$

we have

$$Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\right] \geq Q^{x,t,z}\left[\frac{1_V}{\sum_u Y_u(t)e^{\rho Y_u(t)}}\right] = Q^{x,t,z}\left[\frac{1_V}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}}\right]. \quad (2.53)$$

Define $\mathcal{G}_t$ to be the $\sigma-$field generated by V and the whole trajectory of the spine, $\{\xi_s\}_{0\leq s\leq t}$. In other words, $\mathcal{G}_t$ contains all the information regarding the movement of the spine and the event that all descendants alive at time t branch off the spine after t^*. Conditioning on $\mathcal{G}_t$, Jensen's inequality for conditional expectation gives

$$\begin{aligned}
&Q^{x,t,z}\left[\frac{1_V}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}}\right]\\
&\quad\geq Q^{x,t,z}\left[\frac{1_{V\cap\{M'\leq C\sqrt{\varepsilon}\}}}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}}\right]\\
&\quad= Q^{x,t,z}\left[1_{V\cap\{M'\leq C\sqrt{\varepsilon}\}}Q^{x,t,z}\left[\frac{1}{\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}}\middle|\mathcal{G}_t\right]\right]\\
&\quad\geq Q^{x,t,z}\left[\frac{1_{V\cap\{M'\leq C\sqrt{\varepsilon}\}}}{Q^{x,t,z}\left[\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}\middle|\mathcal{G}_t\right]}\right]. \qquad (2.54)
\end{aligned}$$

To deal with the denominator, we need to use the fact that for every ε, $\left\{\sum_u Y_u(t)e^{\rho Y_u(t)+\varepsilon t}\right\}_{t\geq 0}$ is a martingale for the original BBM with absorption. Under the measure Q, particles branch off the spine with rate 2 and initiate independent copies of the original BBM with absorption. Note that $\xi_s = \zeta_{t-s}$ for $0\leq s\leq t$. Then by the spinal decomposition and the formula for expectations of additive functionals of Poisson point processes, we have

$$\begin{aligned}
Q^{x,t,z}\left[\sum_u Y_u(t)e^{\rho Y_u(t)}1_{\{O_u>t^*\}}\middle|\mathcal{G}_t\right] &= 2\int_{t^*}^{t}\xi_r e^{\rho\xi_r-\varepsilon(t-r)}dr + ze^{\rho z}\\
&= 2\int_0^{(4/\varepsilon)^{2+\kappa}}\zeta_s e^{\rho\zeta_s-\varepsilon s}ds + ze^{\rho z}. \qquad (2.55)
\end{aligned}$$

Moreover, on the event $\{M'\leq C\sqrt{\varepsilon}\}$, if ε is sufficiently small, for all $0\leq s\leq(4/\varepsilon)^{2+\kappa}$,

$$\zeta_s \leq \frac{1}{\rho}\left(\frac{4}{\varepsilon}\right)^{2+\kappa}\varepsilon + \frac{C}{\sqrt{\varepsilon}} \leq 4^{2+\kappa}\varepsilon^{-1-\kappa},$$

and

$$\rho\zeta_s - \varepsilon s = \frac{\rho}{\varepsilon}(\varepsilon\zeta_s - \frac{1}{\rho}\varepsilon^2 s) \leq \frac{\rho}{\varepsilon}C\sqrt{\varepsilon} = \frac{C\rho}{\sqrt{\varepsilon}}.$$

Thus, when $M' \leq C\sqrt{\varepsilon}$, for all $z \in (0, \varepsilon^{-1/2}]$,

$$\begin{aligned} 2\int_0^{(4/\varepsilon)^{2+\kappa}} \zeta_s e^{\rho\zeta_s - \varepsilon s} ds + z e^{\rho z} &\leq 2\left(\frac{4}{\varepsilon}\right)^{2+\kappa} \cdot 4^{2+\kappa}\varepsilon^{-1-\kappa} \cdot e^{C\rho/\sqrt{\varepsilon}} + \varepsilon^{-1/2} e^{\rho/\sqrt{\varepsilon}} \\ &\leq 2^{10+4\kappa}\varepsilon^{-3-2\kappa} e^{C\rho/\sqrt{\varepsilon}}. \end{aligned} \tag{2.56}$$

Combining (2.54), (2.55) and (2.56), we have

$$Q^{x,t,z}\left[\frac{1_V}{\sum_u Y_u(t) e^{\rho Y_u(t)} 1_{\{O_u > t^*\}}}\right] \geq 2^{-10-4\kappa}\varepsilon^{3+2\kappa} e^{-C\rho/\sqrt{\varepsilon}} Q^{x,t,z}\Big(V \cap \{M' \leq C\sqrt{\varepsilon}\}\Big). \tag{2.57}$$

It remains to find a lower bound for the probability of the above event. Because $\{\xi_s\}_{0\leq s\leq t}$ is a Markov process under $Q^{x,t,z}$, we have $\{\xi_s\}_{0\leq s\leq t^*}$ is conditionally independent of $\{\xi_s\}_{t^*\leq s\leq t}$ given ξ_{t^*}. Furthermore, note that V is the event that particles which branch off the spine before time t^* all become extinct before time t and once a particle branches off the spine, it initiates a BBM independent of the future trajectory of the spine. As a result, conditioned on ξ_{t^*}, the events V and $\{M' \leq C\sqrt{\varepsilon}\}$ are independent. By Lemma 19, we obtain

$$\begin{aligned} &Q^{x,t,z}\Big(V \cap \{M' \leq C\sqrt{\varepsilon}\}\Big) \\ &\quad = \int_0^\infty Q^{x,t,z}\Big(V \cap \{M' \leq C\sqrt{\varepsilon}\}\Big|\xi_{t^*} = y\Big) p_{t^*}^{x,t,z}(y) dy \\ &\quad = \int_0^\infty Q^{x,t,z}\big(V\big|\xi_{t^*} = y\big) Q^{x,t,z}\big(M' \leq C\sqrt{\varepsilon}\big|\xi_{t^*} = y\big) p_{t^*}^{x,t,z}(y) dy \\ &\quad \geq \int_0^{\varepsilon^{-1-\kappa}} Q^{x,t,z}\big(V\big|\xi_{t^*} = y\big) Q^{y,(4/\varepsilon)^{2+\kappa},z}\big(M' \leq C\sqrt{\varepsilon}\big) p_{t^*}^{x,t,z}(y) dy \\ &\quad \geq \varepsilon^{-3/4} e^{-C_{12}/\sqrt{\varepsilon}} Q^{x,t,z}\Big(V \cap \big\{\xi_{t^*} \leq \varepsilon^{-1-\kappa}\big\}\Big) \\ &\quad = \varepsilon^{-3/4} e^{-C_{12}/\sqrt{\varepsilon}}\Big[Q^{x,t,z}\Big(\xi_{t^*} \leq \varepsilon^{-1-\kappa}\Big) - Q^{x,t,z}\Big(V^c \cap \big\{\xi_{t^*} \leq \varepsilon^{-1-\kappa}\big\}\Big)\Big]. \end{aligned} \tag{2.58}$$

As for the first term, note that $\{\xi_{t^*} \le \varepsilon^{-1-\kappa}\} = \{\zeta_{(4/\varepsilon)^{2+\kappa}} \le \varepsilon^{-1-\kappa}\}$, where $\{\zeta_s\}_{0\le s\le t}$ is a Bessel bridge from z to x in time t under $Q^{x,t,z}$. Define $\{R_r^z\}_{r\ge 0}$ to be a Bessel process starting from z. We apply Lemma 11 to obtain,

$$\lim_{t\to\infty} Q^{x,t,z}\Big(\xi_{t^*} \le \varepsilon^{-1-\kappa}\Big) = P\Big(R^z_{(4/\varepsilon)^{2+\kappa}} \le \varepsilon^{-1-\kappa}\Big).$$

According to the scaling property of the Bessel process, we have for ε sufficiently small, for all $z \in (0, \varepsilon^{-1/2}]$,

$$\begin{aligned} P\Big(R^z_{(4/\varepsilon)^{2+\kappa}} \le \varepsilon^{-1-\kappa}\Big) &= P\bigg(\Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2} R^z_{(4/\varepsilon)^{2+\kappa}} \le \Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2} \varepsilon^{-1-\kappa}\bigg) \\ &= P\bigg(R_1^{z(\varepsilon/4)^{1+\kappa/2}} \le \frac{\varepsilon^{-\kappa/2}}{4^{1+\kappa/2}}\bigg) \\ &> \frac{1}{2}. \end{aligned}$$

Therefore, for ε small enough, for all $z \in (0, \varepsilon^{-1/2}]$, if t is large enough, we have

$$Q^{x,t,z}\Big(\xi_{t^*} \le \varepsilon^{-1-\kappa}\Big) \ge \frac{1}{2}. \tag{2.59}$$

As for the second term, according to Lemma 18, for ε sufficiently small, for all $z \in (0, \varepsilon^{-1/2}]$, if t is large enough, then

$$Q^{x,t,z}\Big(V^c \cap \big\{\xi_{t^*} \le \varepsilon^{-1-\kappa}\big\}\Big) \le Q^{x,t,z}\Big(V^c\Big) < \frac{1}{4}. \tag{2.60}$$

In the end, by (2.52), (2.53), (2.57)–(2.60) and Fatou's Lemma, we proved that for ε

small enough,

$$\begin{aligned}
&\liminf_{t\to\infty} \sqrt{2\pi t^3} Q^x\left[\frac{1}{\sum_u Y_u(t) e^{\rho Y_u(t)}}\right] \\
&\geq \frac{1}{2}\int_0^{\varepsilon^{-1/2}} 2^{-10-4\kappa}\varepsilon^{2+2\kappa} e^{-C\rho/\sqrt{\varepsilon}} \liminf_{t\to\infty} Q^{x,t,z}\Big(V\cap\{M'\leq C\sqrt{\varepsilon}\}\Big) z^2 dz \\
&\geq 2^{-13-4\kappa}3^{-1}\varepsilon^{3/4+2\kappa} e^{-(C\rho+C_{12})/\sqrt{\varepsilon}}.
\end{aligned}$$

Consequently, the lower bound in Theorem 6 is proved as long as

$$C_2 > 2C + C_{12} \geq C\rho + C_{12}.$$

□

2.4.2 Proof of Lemmas

Before proving Lemma 18, we need one more ingredient. Recall that $\{R_r^z\}_{z\geq 0}$ is a Bessel process starting from z.

Lemma 20. *For every fixed ε, we have*

$$\lim_{t\to\infty} Q^{x,t,z}\left(\exists r\geq\left(\frac{4}{\varepsilon}\right)^{2/(1-2\delta_1)} : \zeta_r \geq r^{1/2+\delta_1}\right) = P\left(\exists r\geq\left(\frac{4}{\varepsilon}\right)^{2/(1-2\delta_1)} : R_r^z \geq r^{1/2+\delta_1}\right).$$

Proof. According to Lemma 11, the Bessel bridge converges to the Bessel process in the Skorokhod topology. Recall that under $Q^{x,t,z}$, the process $\{\zeta_r\}_{0\leq r\leq t}$ is a Bessel bridge from z to x in time t. Since both Bessel bridges and the Bessel process are continuous, the Skorokhod topology in this case coincides with the uniform topology. Thus, it is sufficient to prove that for a Bessel process $\{R_r^z\}_{r\geq 0}$ starting from z under P, for every constant $c\geq 1$, the event

$$A := \left\{\exists r\geq c : R_r^z \geq r^{1/2+\delta_1}\right\}$$

is a continuity set under the uniform topology. That is to say, letting ∂A denote the boundary set of A under the uniform topology, essentially, we want to prove that

$$P(\partial A) = P\big(\{\omega : \{R_r^z(\omega)\}_{r\geq 0} \in \partial A\}\big) = 0. \tag{2.61}$$

We first consider elements in ∂A, which can be approached both from A and A^c under the uniform topology. Note that $A^c = \{\forall r \geq c, R_r^z < r^{1/2+\delta_1}\}$. For $\{R_r^z(\omega)\}_{r\geq 0} \in \partial A$, if there exists an $r \geq c$ such that $R_r^z(\omega) > r^{1/2+\delta_1}$, then $\{R_r^z(\omega)\}_{r\geq 0}$ cannot be approached from A^c. As a result, $R_r^z(\omega) \leq r^{1/2+\delta_1}$ for all $r \geq c$. Furthermore, if $\inf_{r\geq c}(r^{1/2+\delta_1} - R_r^z(\omega)) > 0$, then it cannot be approached from A. Thus, $\inf_{r\geq c}(r^{1/2+\delta_1} - R_r^z(\omega)) = 0$. Indeed, this infimum must be attained at some finite value of r because of the law of iterated logarithm of the Bessel process at infinity (see, e.g., IV.40 of [19]). More precisely, letting $\sigma = \inf\{r \geq c : R_r^z = r^{1/2+\delta_1}\}$, we see that

$$\begin{aligned} P(\{\sigma = \infty\} \cap \partial A) &\leq P\Big(\{\sigma = \infty\} \cap \big\{\inf_{r\geq c}(r^{1/2+\delta_1} - R_r^z) = 0\big\}\Big) \\ &\leq P\Big(\lim_{r\to\infty} \big(r^{1/2+\delta_1} - R_r^z\big) = 0\Big) \\ &= 0. \end{aligned}$$

Note that σ is a stopping time and let $\mathcal{F}_\sigma$ be the $\sigma-$ field generated by σ. By the strong Markov property of the Bessel process, we have

$$\begin{aligned} P(\partial A) &= P(\partial A \cap \{\sigma < \infty\}) \\ &\leq P\Big(\Big\{\forall r \geq c : R_r^z \leq r^{1/2+\delta_1}\Big\} \cap \{\sigma < \infty\}\Big) \\ &= E\Big[1_{\{\sigma<\infty\}} P\Big(\forall r \geq 0, R_r^{\sigma^{1/2+\delta_1}} \leq (r+\sigma)^{1/2+\delta_1}\Big)\Big]. \end{aligned} \tag{2.62}$$

Using the same method as the proof of Lemma 10, it can be shown that the Bessel process

$\{R^z_r\}_{r\geq 0}$ stochastically dominates Brownian motion $\{B^z_r\}_{r\geq 0}$. Thus, conditioned on $\mathcal{F}_\sigma$,

$$P\Big(\forall r \geq 0, R_r^{\sigma^{1/2+\delta_1}} \leq (r+\sigma)^{1/2+\delta_1}\Big) \leq P\Big(\forall r \geq 0, B_r^{\sigma^{1/2+\delta_1}} \leq (r+\sigma)^{1/2+\delta_1}\Big)$$
$$= P\Big(\forall r \geq 0, B_r \leq (r+\sigma)^{1/2+\delta_1} - \sigma^{1/2+\delta_1}\Big). \tag{2.63}$$

By the law of the iterated logarithm at 0 for Brownian motion (see, e.g., IV.5 of [19]), we have almost surely

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t \ln\ln(1/t)}} = 1. \tag{2.64}$$

Conditioned on $\mathcal{F}_\sigma$, since $0 < \delta_1 < 1/4$ and $\sigma \geq c \geq 1$, a Taylor expansion gives

$$(r+\sigma)^{1/2+\delta_1} - \sigma^{1/2+\delta_1} = \left(\frac{1}{2} + \delta_1\right)\sigma^{-1/2+\delta_1} r + o(r) \leq \left(\frac{1}{2} + \delta_1\right) r + o(r).$$

Since $0 < \delta_1 < 1/4$, conditioned on $\mathcal{F}_\sigma$, there exists an α such that for all $r \leq \alpha$,

$$(r+\sigma)^{1/2+\delta_1} - \sigma^{1/2+\delta_1} < \frac{3}{4}\sqrt{2r \ln\ln\left(\frac{1}{r}\right)}. \tag{2.65}$$

From (2.64) and (2.65), conditioned on $\mathcal{F}_\sigma$, we have

$$P(\forall r \geq 0, B_r \leq (r+\sigma)^{1/2+\delta_1} - \sigma^{1/2+\delta_1}) = 0. \tag{2.66}$$

By (2.62), (2.63) and (2.66), equation (2.61) is proved and the lemma follows. □

Proof of Lemma 18. Let $u(t,x)$ be the probability of survival at time t for a BBM starting from x under $P^x_{-\rho}$. It is pointed out in equation (5) of [48] that

$$u(t,x) \leq e^{\rho x - \varepsilon t}. \tag{2.67}$$

Moreover, we write $0 \leq \tau_1 < \tau_2 < ... \leq t$ for the successive branching times along the spine. Note that under Q^x, particles branch off the spine at rate 2. We define $\mathcal{N}^i_t$ to be the the set of surviving particles at time t which have branched off the spine at time τ_i. Inheriting notations from Section 2.3, we denote by $p_s(x,y)$ the transition probability of a Bessel process and $p_s^{x,t,z}(y)$ the transition probability of a Bessel bridge from x to z within time t.

Start with V_1 which is defined in (2.47). Applying (2.67), we have

$$\begin{aligned}
Q^{x,t,z}(V_1) &\leq Q^{x,t,z}\left[\sum_{i:\tau_i\leq t'} 1_{\{\mathcal{N}_t^i\neq\emptyset\}} 1_{\{\xi_s\leq (t')^{1/2+\delta_1},\ \forall s\leq t'\}}\right] \\
&\leq 2\int_0^{t'}\int_0^{(t')^{1/2+\delta_1}} u(t-s,y)p_s^{x,t,z}(y)dyds \\
&\leq 2\int_0^{t'}\int_0^{(t')^{1/2+\delta_1}} e^{\rho y-\varepsilon(t-s)}p_s^{x,t,z}(y)dyds \\
&\leq 2e^{\rho(t')^{1/2+\delta_1}}\int_0^{t'} e^{-\varepsilon(t-s)}ds \\
&\leq \frac{2}{\varepsilon}e^{-\varepsilon(t-t')+\rho(t')^{1/2+\delta_1}}.
\end{aligned} \tag{2.68}$$

For every fixed ε, since $0<\delta_1<\delta_2<1/4$, we have

$$\lim_{t\to\infty}\frac{\varepsilon(t-t')}{\rho(t')^{1/2+\delta_1}} = \infty.$$

Therefore, for every fixed ε,

$$\lim_{t\to\infty} Q^{x,t,z}(V_1) = 0. \tag{2.69}$$

As for V_2 and V_4, which are defined in equations (2.48) and (2.50) respectively, notice that the process $\{\xi_s\}_{0\leq s\leq t}$ is a Bessel bridge from x to z in time t under $Q^{x,t,z}$. According to the scaling property of the Bessel bridge, $\{\xi_{rt}/\sqrt{t}\}_{0\leq r\leq 1}$ is a Bessel bridge from $x/\sqrt{t}$ to $z/\sqrt{t}$ within time 1. Define $\{X_r^{x/\sqrt{t},1,z/\sqrt{t}}\}_{0\leq r\leq 1}$ to be a Bessel bridge from $x/\sqrt{t}$ to $z/\sqrt{t}$

within time 1. Then we have $0 \le Q^{x,t,z}(V_2)$ and

$$\begin{aligned} Q^{x,t,z}(V_2) &\le Q^{x,t,z}\Big(\exists s \le t', \xi_s > (t')^{1/2+\delta_1}\Big) \\ &= P\bigg(\exists r \le \frac{t'}{t}, X_r^{x/\sqrt{t},1,z/\sqrt{t}} > \frac{(t')^{1/2+\delta_1}}{t^{1/2}}\bigg). \end{aligned}$$

Note that

$$\frac{(t')^{1/2+\delta_1}}{t^{1/2}} = \frac{(t-t^{1/2+\delta_2})^{1/2+\delta_1}}{t^{1/2}} \to \infty, \quad \text{as } t \to \infty.$$

Accordingly, for every fixed ε,

$$0 \le \lim_{t\to\infty} Q^{x,t,z}(V_2) \le \lim_{t\to\infty} P\bigg(\exists r < 1, X_r^{x/\sqrt{t},1,z/\sqrt{t}} > \frac{(t')^{1/2+\delta_1}}{t^{1/2}}\bigg) = 0. \tag{2.70}$$

Similarly for V_4,

$$\begin{aligned} Q^{x,t,z}(V_4) &\le Q^{x,t,z}\Big(\exists s \in (t', t^*], \xi_s > s^{1/2+\delta_1}\Big) \\ &= P\bigg(\exists r \in \bigg(\frac{t'}{t}, \frac{t^*}{t}\bigg], X_r^{x/\sqrt{t},1,z/\sqrt{t}} > \frac{(rt)^{1/2+\delta_1}}{t^{1/2}}\bigg). \end{aligned}$$

Because as $t \to \infty$, for all $r \in (t'/t, t^*/t]$, $(rt)^{1/2+\delta_1}/t^{1/2} \to \infty$, we have for fixed x and ε, for all $z \in (0, \varepsilon^{-1/2}]$,

$$0 \le \lim_{t\to\infty} Q^{x,t,z}(V_4) \le \lim_{t\to\infty} P\bigg(\exists r \in \bigg(\frac{t'}{t}, \frac{t^*}{t}\bigg], X_r^{x/\sqrt{t},1,z/\sqrt{t}} > \frac{(rt)^{1/2+\delta_1}}{t^{1/2}}\bigg) = 0. \tag{2.71}$$

It remains to work on $Q^{x,t,z}(V_3)$. Recall that V_3 is defined in (2.49). We will separate $Q^{x,t,z}(V_3)$ into two parts and show both of them have small probability as $t \to \infty$. For

$z \in (0, \varepsilon^{-1/2}]$,

$$\begin{aligned} Q^{x,t,z}(V_3) &= Q^{x,t,z}\Big(\{\exists u \in \mathcal{N}_t : t' < O_u \le t^*\} \cap \{\xi_s \le s^{1/2+\delta_1},\ \forall s \in (t', t^*]\}\Big) \\ &\le Q^{x,t,z}\Big(\{\exists u \in \mathcal{N}_t : t' < O_u \le t^*\} \cap \{\xi_s \le (t-s)^{1/2+\delta_1},\ \forall s \in (t', t^*]\}\Big) \\ &\quad + Q^{x,t,z}\Big(\exists s \in (t', t^*] : \xi_s \ge (t-s)^{1/2+\delta_1}\Big) \\ &=: H_1 + H_2. \end{aligned} \tag{2.72}$$

For H_1, we have

$$\begin{aligned} H_1 &\le Q^{x,t,z}\left[\sum_{i: t' < \tau_t \le t^*} 1_{\{\mathcal{N}_t^i \neq \emptyset\}} 1_{\{\xi_s \le (t-s)^{1/2+\delta_1},\ \forall s \in (t', t^*]\}}\right] \\ &\le 2\int_{t'}^{t^*}\int_0^{(t-s)^{1/2+\delta_1}} u(t-s, y) p_s^{x,t,z}(y) dy ds. \end{aligned}$$

Letting $r = t - s$ and noting that a time-reversed Bessel bridge is also a Bessel bridge, we get

$$\begin{aligned} H_1 &\le 2\int_{t-t^*}^{t^{1/2+\delta_2}}\int_0^{r^{1/2+\delta_1}} u(r, y) p_r^{z,t,x}(y) dy dr \\ &\le 2\int_{t-t^*}^{t^{1/2+\delta_2}}\int_0^{r^{1/2+\delta_1}} e^{\rho y - \varepsilon r} p_r(z, y) \frac{p_{t-r}(y, x)}{p_t(z, x)} dy dr. \end{aligned} \tag{2.73}$$

We further observe that for large time t, the difference between the probability density functions of the Bessel bridge and the Bessel process are negligible. Note that $1 - e^{-x} \le x$ for $x \ge 0$ and $1 - e^{-x} \ge x/2$ for $0 \le x \le 1$. Then for every fixed ε, when t is large enough, we have for all $z \in (0, \varepsilon^{-1/2}]$, $y \in (0, r^{1/2+\delta_1}]$, and $r \in [t - t^*, t^{1/2+\delta_2}]$ uniformly,

$$\begin{aligned}\frac{p_{t-r}(y,x)}{p_t(z,x)} &= \frac{\frac{1}{\sqrt{2\pi(t-r)}} \cdot \frac{x}{y} e^{-(y-x)^2/2(t-r)}(1 - e^{-2yx/(t-r)})}{\frac{1}{\sqrt{2\pi t}} \cdot \frac{x}{z} e^{-(z-x)^2/2t}(1 - e^{-2xz/t})} \\ &\leq \sqrt{\frac{t}{t-r}} \frac{\frac{x}{y} \cdot \frac{2yx}{t-r}}{\frac{x}{z} \cdot \frac{2}{3} \cdot \frac{xz}{t}} \\ &= 3\left(\frac{t}{t-r}\right)^{3/2} \\ &\leq 4.\end{aligned}$$

Also see that for $r \geq t - t^*$, $\rho r^{1/2+\delta_1} \leq \rho\varepsilon r/4$. Based on the above two observations, we have for sufficiently small ε, if t is large enough, then

$$\begin{aligned}2\int_{t-t^*}^{t^{1/2+\delta_2}} \int_0^{r^{1/2+\delta_1}} e^{\rho y - \varepsilon r} p_r(z,y) \frac{p_{t-r}(y,x)}{p_t(z,x)} dydr &\leq 8 \int_{t-t^*}^{t^{1/2+\delta_2}} \int_0^{r^{1/2+\delta_1}} e^{\rho y - \varepsilon r} p_r(z,y) dydr \\ &\leq 8 \int_{t-t^*}^{t^{1/2+\delta_2}} e^{\rho r^{1/2+\delta_1} - \varepsilon r} dr \\ &\leq 8 \int_{t-t^*}^{t^{1/2+\delta_2}} e^{(\rho\varepsilon r/4) - \varepsilon r} dr \\ &\leq \frac{8}{\varepsilon(1-\rho/4)} e^{-\varepsilon(1-\rho/4)(4/\varepsilon)^{2/(1-2\delta_1)}}. \qquad (2.74)\end{aligned}$$

Since $0 < \delta_1 < 1/4$, together with (2.73) and (2.74), we have for any $0 < \delta < 1$, if ε is sufficiently small,

$$\limsup_{t\to\infty} H_1 < \frac{16}{\varepsilon} e^{-1/\varepsilon} < \frac{\delta}{2}. \qquad (2.75)$$

As for H_2, we will apply the law of the iterated logarithm for the Bessel process (see, e.g., IV.40 of [19]) for all $z \in (0, \varepsilon^{-1/2}]$,

$$P\left(\limsup_{t\to\infty} \frac{R_t^z}{\sqrt{2t \ln \ln t}} = 1\right) = 1.$$

Then it follows that for all $z \in (0, \varepsilon^{-1/2}]$,

$$\lim_{t\to\infty} P\Big(R_s^z < s^{1/2+\delta_1},\ \forall s \ge t\Big) = 1. \tag{2.76}$$

Recall that $\{\zeta_s\}_{0\le s\le t}$ denotes the time-reversed Bessel bridge, which is a Bessel bridge from z to x in time t under $Q^{x,t,z}$. From Lemma 20 and (2.76), if ε is sufficiently small, we have

$$\begin{aligned}
\limsup_{t\to\infty} H_2 &= \limsup_{t\to\infty} Q^{x,t,z}\Big(\exists\, t \in [t', t^*] : \xi_s \ge (t-s)^{1/2+\delta_1}\Big) \\
&\le \limsup_{t\to\infty} Q^{x,t,z}\left(\exists\, r > \left(\frac{4}{\varepsilon}\right)^{2/(1-2\delta_1)} : \zeta_r \ge r^{1/2+\delta_1}\right) \\
&= P\left(\exists\, r > \left(\frac{4}{\varepsilon}\right)^{2/(1-2\delta_1)} : R_r^z \ge r^{1/2+\delta_1}\right) \\
&< \frac{\delta}{2}.
\end{aligned} \tag{2.77}$$

Consequently, by (2.72), (2.75) and (3.273), for sufficiently small ε,

$$\limsup_{t\to\infty} Q^{x,t,z}(V_3) < \delta. \tag{2.78}$$

Together with (2.69), (2.70) and (2.71), the lemma follows. □

Proof of Lemma 19. Under $Q^{y,(4/\varepsilon)^{2+\kappa},z}$, the reversed trajectory of the spine $\{\zeta_s\}_{0\le s\le (4/\varepsilon)^{2+\kappa}}$ is a Bessel bridge from z to y within time $(4/\varepsilon)^{2+\kappa}$. After scaling, $\left\{(\varepsilon/4)^{1+\kappa/2}\zeta_{(4/\varepsilon)^{2+\kappa}r}\right\}_{0\le r\le 1}$ is a Bessel bridge from $(\varepsilon/4)^{1+\kappa/2}z$ to $(\varepsilon/4)^{1+\kappa/2}y$ within time 1. Recall that the process $\{X_r^{(\varepsilon/4)^{1+\kappa/2}z,1,(\varepsilon/4)^{1+\kappa/2}y}\}_{0\le r\le 1}$ is a Bessel bridge from $(\varepsilon/4)^{1+\kappa/2}z$ to $(\varepsilon/4)^{1+\kappa/2}y$ in time 1. For simplicity, we will write $\{X_r\}_{0\le r\le 1}$ in place of $\{X_r^{(\varepsilon/4)^{1+\kappa/2}z,1,(\varepsilon/4)^{1+\kappa/2}y}\}_{0\le r\le 1}$. Accordingly, we have

$$
\begin{aligned}
&Q^{y,(4/\varepsilon)^{2+\kappa},z}\Big(M' \le C\sqrt{\varepsilon}\Big)\\
&\quad= Q^{y,(4/\varepsilon)^{2+\kappa},z}\bigg(\sup_{0\le s\le(4/\varepsilon)^{2+\kappa}}\Big(\varepsilon\zeta_s - \frac{1}{\rho}\varepsilon^2 s\Big) \le C\sqrt{\varepsilon}\bigg)\\
&\quad= Q^{y,(4/\varepsilon)^{2+\kappa},z}\bigg(\sup_{0\le r\le 1}\bigg(\Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2}\zeta_{(4/\varepsilon)^{2+\kappa}r} - \frac{4^{1+\kappa/2}}{\rho}\varepsilon^{-\kappa/2}r\bigg) \le \frac{C}{4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg)\\
&\quad= P\bigg(\sup_{0\le r\le 1}\Big(X_r - \frac{4^{1+\kappa/2}}{\rho}\varepsilon^{-\kappa/2}r\Big) \le \frac{C}{4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg).
\end{aligned}
$$

By (0.22) of [72], we can represent $\{X_r\}_{0\le r\le 1}$ in terms of three independent standard Brownian bridges, $B^{0,1,0}_{(1)}, B^{0,1,0}_{(2)}, B^{0,1,0}_{(3)}$,

$$
X_r \overset{d}{=} \sqrt{\bigg(\Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2}z(1-r) + \Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2}yr + B^{0,1,0}_{(1)}(r)\bigg)^2 + \Big(B^{0,1,0}_{(2)}(r)\Big)^2 + \Big(B^{0,1,0}_{(3)}(r)\Big)^2}.
$$

According to the two formulas above, because $C > 2\sqrt{3}$, we have for all $z \in (0, \varepsilon^{-1/2}]$ and $y \in (0, \varepsilon^{-1-\kappa}]$,

$$
\begin{aligned}
&Q^{y,(4/\varepsilon)^{2+\kappa},z}\Big(M' \le C\sqrt{\varepsilon}\Big)\\
&\ge P\bigg(\sup_{0\le r\le 1}\bigg(\Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2}z + \Big(\frac{\varepsilon}{4}\Big)^{1+\kappa/2}yr + \Big|B^{0,1,0}_{(1)}(r)\Big| - \frac{4^{1+\kappa/2}}{\sqrt{3}\rho}\varepsilon^{-\kappa/2}r\bigg) \le \frac{C\varepsilon^{(\kappa+1)/2}}{\sqrt{3}\cdot 4^{1+\kappa/2}}\bigg)\\
&\quad\times\bigg[P\bigg(\sup_{0\le r\le 1}\Big(\Big|B^{0,1,0}_{(2)}(r)\Big| - \frac{4^{1+\kappa/2}}{\sqrt{3}\rho}\varepsilon^{-\kappa/2}r\Big) \le \frac{C}{\sqrt{3}\cdot 4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg)\bigg]^2\\
&\ge P\bigg(\sup_{0\le r\le 1}\Big(\Big|B^{0,1,0}_{(1)}(r)\Big| - \frac{1}{2}\varepsilon^{-\kappa/2}r\Big) \le \frac{1}{4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg)\\
&\quad\times\bigg[P\bigg(\sup_{0\le r\le 1}\Big(\Big|B^{0,1,0}_{(2)}(r)\Big| - \frac{4^{1+\kappa/2}}{\sqrt{3}\rho}\varepsilon^{-\kappa/2}r\Big) \le \frac{C}{\sqrt{3}\cdot 4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg)\bigg]^2.
\end{aligned}
\tag{2.79}
$$

According to Lemma 8, for ε sufficiently small, we have

$$
P\bigg(\sup_{0\le r\le 1}\Big(\Big|B^{0,1,0}_{(1)}(r)\Big| - \frac{1}{2}\varepsilon^{-\kappa/2}r\Big) \le \frac{1}{4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\bigg) \ge \varepsilon^{-1/4}\exp\bigg\{-\frac{4\pi^2}{\sqrt{\varepsilon}}\bigg\}, \tag{2.80}
$$

$$P\left(\sup_{0\le r\le 1}\left(\left|B_{(2)}^{0,1,0}(r)\right| - \frac{4^{1+\kappa/2}}{\sqrt{3}\rho}\varepsilon^{-\kappa/2}r\right) \le \frac{C}{\sqrt{3}\cdot 4^{1+\kappa/2}}\varepsilon^{(\kappa+1)/2}\right)$$
$$\ge \sqrt{\frac{3\rho\pi}{C}}\varepsilon^{-1/4}\exp\left\{-\frac{3\rho\pi^2}{8C\sqrt{\varepsilon}}\right\}. \qquad (2.81)$$

Letting $C_{12} > 4\pi^2 + 12\pi^2/(8C) \ge 4\pi^2 + 6\rho\pi^2/(8C)$, the Lemma follows from equations (2.79), (2.80) and (2.81). □

2.5 Acknowledgement

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Chapter 3

Branching Brownian motion with an inhomogeneous branching rate

Aiming to understand the distribution of fitness levels of individuals in a large population undergoing selection, we study the particle configurations of branching Brownian motion (BBM) where each particle independently moves as Brownian motion with negative drift, particles can die or undergo dyadic fission, and the difference between the birth rate and the death rate is proportional to the particle's location. Under some assumptions, we obtain the limit in probability of the number of particles in any given interval and an explicit formula for the asymptotic empirical density of the fitness distribution. We show that after a sufficiently long time, the fitness distribution from the lowest to the highest fitness levels approximately evolves as a traveling wave with a profile which is asymptotically related to the Airy function. Our work complements the results in Roberts and Schweinsberg (2021), giving a fuller picture of the fitness distribution.

3.1 Introduction

To understand the evolution of populations undergoing selection, we study the distribution of fitness levels of individuals. There is a well-known observation in the biology literature that in a large population where various beneficial mutations compete for fixation simultaneously, the distribution of the fitness is well approximated by a traveling wave. This observation goes back at least to the work of Tsimring, Levin and Kessler [84]. In a recent paper by Melissa et al. [66], they showed using non-rigorous methods that the fitness distribution within a population can be described as a traveling wave with a profile defined by the Airy function. Roberts and Schweinsberg [75] used BBM with an inhomogeneous branching rate to model a population undergoing selection. They showed that the empirical distribution of fitness levels of individuals is approximately Gaussian. Our work complements the results in [75], giving a fuller picture of the fitness distribution and providing a mathematically rigorous justification of the biology conjecture in [66]. Based on the model in [75], we show that after a sufficiently long time, the fitness distribution from the lowest to the highest fitness levels approximately evolves as a traveling wave with a profile asymptotically equivalent to the profile that is expressed in terms of the Airy function. This Airy traveling wave profile has been obtained using nonrigorous methods in the biology literature. See e.g. [29, 66, 68, 84].

The most intuitive model of fitness is the fitness landscape, which is a mapping from the multidimensional genotype space to a real valued fitness space. This model is constructed in a high-dimensional space where the number of dimensions is equal to the number of nucleotides in the genome. Each point represents a particular genome and each genome is assigned with a fitness level. Although the fitness landscape visualizes the relationship between genotypes and fitness, limited quantitative analyses can be done in this model due to its high dimensional construction, see e.g. [82]. In order to study the evolution on a smooth fitness landscape, Tsimring, Levine and Kessler [84] introduced the traveling wave model in a one-dimensional fitness space where they characterized the population density as a function of time and the fitness level. Since then, the traveling wave model of fitness has become an important model in adaptation and has been studied in the evolutionary context for more than two decades with different model assumptions, see e.g. [8, 33, 34, 39, 44, 45, 55, 66, 68, 71, 70, 76]. Some of these works demonstrated that in a large asexual population where the influx of beneficial mutations is large enough, the distribution of the fitness will settle into a bell-shaped density moving to higher fitness as a traveling wave and the population adapts. For a complete summary of the dynamical behavior of the traveling wave fitness models under different mutation settings (which is also known as mutation kernels or distribution of fitness effects), see [46].

In the mathematics literature, most of the work related to the dynamical behavior of fitness has been done under the framework of Moran model where the number of individuals in the population is N at all times and the individual fitness level can only change discretely. In the strong selection and weak mutation regime, one selective sweep occurs at a time. This regime was mentioned in Desai and Fisher [33]. For rigorous analyses of this process, see [81]. Yu, Etheridge and Cuthbertson [88], followed by Kelly [54] considered the very fast mutation case and established the upper and lower bounds for the rate of increase of mean fitness in the population. Durrett and Mayberry [34] first rigorously established the non-Gaussian traveling wave behavior of fitness when the selection rate is constant and the mutation rate is $N^{-\alpha}$ for $0 < \alpha < 1$. Schweinsberg [80] considered slightly faster mutation rates and showed that the distribution of fitness has a Gaussian-like tail behavior, though it does not actually converge to a Gaussian distribution. Schweinsberg [80] rigorously proved the results in earlier work of Desai and Fisher [33]. For Moran model, the explicit expression for the distribution of the fitness has not been established precisely.

The idea of modeling the population evolution using branching processes has a long history and rich literature. In the last decade or so, there has been some work that used BBM to model the evolution of populations, see e.g. [11, 13, 25, 26, 62]. In these works, the branching rate is homogeneous in space and selection was not fully considered. Recently, Roberts and Schweinsberg [75] studied the evolution of a large population undergoing selection using an inhomogeneous BBM model. They considered the case where the rate of beneficial mutations is large but the selective advantage of each mutation is relatively small.

In this scenario, since each individual acquires many mutations with a small selective advantage, the individual fitness level will behave like a continuous-time random walk. After proper scaling, the fitness of each individual will move according to Brownian motion. To incorporate the stochastic dynamics of discrete replicating individuals, they constructed an inhomogeneous BBM model where each particle independently moves according to Brownian motion with negative drift, particles can die or undergo dyadic fission, and the difference between the birth rate and the death rate is proportional to the particle's location. We will work under the same setup and assumptions as [75]. We are interested in the bulk distribution of individual fitness levels from the least fit individuals to the most fit individuals, or in other words, the particle configurations from the left edge to the right edge.

BBM with a space-dependent branching rate model was first introduced by Harris and Harris in [49]. In their model, a particle at location $y \in \mathbb{R}$ will split into two particles at rate $\beta|y|^p$, where $\beta > 0$ and $p \in [0, 2]$. They didn't include the case $p > 2$ because the process will explode in finite time if $p > 2$. Our model is related to their model with $p = 1$. They studied the right-most position for different values of p using martingales and the related spine changes of measure. They proved that for $p \in [0, 2)$, the maximal displacement grows polynomially while for $p = 2$, the maximal displacement grows exponentially. Later on, Berestycki et al. in [14] studied the particle configurations of such models for all $p \in [0, 2)$. By studying the large deviations probabilities for particles following certain rescaled paths, they obtained the logarithmic order of the expected and almost sure number of particles whose rescaled trajectories follow paths in some set. Tourniaire [83] studied a space-homogeneous BBM with absorption model where particles branch at rate $\rho/2$ in the interval $[0, 1]$ for some $\rho > 1$, and at rate $1/2$ in $(1, \infty)$. This branching particle system is an analytically tractable system which can be used to study the so-called semi-pushed traveling waves. This branching particle system can be viewed as a biological model which describes the invasion of an uncolonized habitat by a species. Particle configurations for some of the BBM models have been understood completely. Berestycki et al. in [13] studied configurations of particles in BBM with absorption at the origin. They proved that if the process settles into an equilibrium configuration and there are N particles in total, then the density of particles near y is roughly proportional to $e^{-\sqrt{2}y}\sin(\sqrt{2}\pi y/\log N)$. Some of the ideas in this chapter are inspired by [13].

3.1.1 The model

Consider a sequence of models indexed by n. In the n-th model, each particle moves independently as Brownian motion with drift $-\rho_n$. For a particle at position x, it will either die at rate $d_n(x)$ or branch into two particles at rate $b_n(x)$, where

$$b_n(x) - d_n(x) = \beta_n x.$$

Here, each particle corresponds to an individual in the population. The positions of particles represent fitness levels of individuals and the movements of particles illustrate changes in fitness levels over generations. Branching events represent births. If the birth rate is less than the death rate, unfit individuals will die and the corresponding lineage will become extinct.

We assume that

$$\lim_{n\to\infty} \frac{\rho_n^3}{\beta_n} = \infty, \tag{3.1}$$

$$\lim_{n\to\infty} \rho_n = 0, \tag{3.2}$$

and there exists $\alpha \in (0,1)$ such that

$$d_n(x) \geq \alpha \text{ for all } x \in \mathbb{R},\ n \in \mathbb{N} \quad \text{and} \quad b_n(x) \leq 1/\alpha \text{ for all } x \leq 1/\beta_n, n \in \mathbb{N}. \tag{3.3}$$

We assume that (3.1) and (3.2) hold true throughout the rest of this chapter, even when they are not specifically stated. In the n-th model, we denote by $N_{t,n}$ the total number of particles at time t, $\mathcal{N}_{t,n}$ the set of particles alive at time t and $N_{t,n}(\mathcal{I})$ the number of particles in the interval $\mathcal{I}$ at time t. The set of positions of particles at time t is written as $\{X_{i,n}(t), i \in \mathcal{N}_{t,n}\}$. For $i \in \mathcal{N}_{t,n}$, we denote $\{X_{i,n}(r), 0 \leq r \leq t\}$ the past trajectory of the particle i alive at time t. To state further assumptions, we need to introduce the Airy function

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{y^3}{3} + xy\right) dy.$$

The Airy function has an infinite number of zeros, all of which are negative. We denote the zeros of the Airy function by $(\gamma_k)_{k=1}^\infty$ such that $\cdots < \gamma_2 < \gamma_1 < 0$. Specifically,

$$\gamma_1 \approx -2.338. \tag{3.4}$$

We define

$$L_n^* = \frac{\rho_n^2}{2\beta_n}, \qquad L_n^\dagger = -\frac{5\rho_n^2}{8\beta_n}. \tag{3.5}$$

Roughly speaking, most particles will stay within $[L_n^\dagger, L_n^*]$. We call L_n^* the right edge and $L_n^\dagger$ the left edge. We define a boundary L_n that is slightly larger than L_n^* so that almost no particles will exceed L_n,

$$L_n = \frac{\rho_n^2}{2\beta_n} - (2\beta_n)^{-1/3}\gamma_1. \tag{3.6}$$

Let

$$Y_n(t) = \sum_{i \in \mathcal{N}_{t,n}} e^{\rho_n X_{i,n}(t)}, \tag{3.7}$$

and

$$Z_n(t) = \sum_{i \in \mathcal{N}_{t,n}} e^{\rho_n X_{i,n}(t)} Ai\big((2\beta_n)^{1/3}(L_n - X_{i,n}(t)) + \gamma_1)\big) 1_{\{X_{i,n}(t) < L_n\}}. \tag{3.8}$$

We make the following assumptions regarding the initial configuration expressed in terms of $Y_n(0)$ and $Z_n(0)$. We assume that

$$\rho_n^2 e^{-\rho_n L_n} Y_n(0) \to_p 0, \tag{3.9}$$

and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for n sufficiently large,

$$P\left(\delta \frac{\beta_n^{1/3}}{\rho_n^3} e^{\rho_n L_n} \leq Z_n(0) \leq \frac{1}{\delta} \frac{\beta_n^{1/3}}{\rho_n^3} e^{\rho_n L_n}\right) > 1 - \varepsilon. \tag{3.10}$$

It is mentioned in [75] that $Z_n(t)$ provides a natural measure of "size" of the process at time t. Roughly speaking, assumption (3.9) requires that the "size" of the process at early time will not be dominated by the descendants of a single particle in the initial configuration or particles that are far from L_n at time 0. This assumption is biologically natural because otherwise, a single lineage will quickly take over the whole population and we cannot expect the population adapts and reach an evolutionary stable equilibrium. Assumption (3.10) is roughly saying that the "size" of the initial configuration will be around $\beta_n^{1/3} e^{\rho_n L_n}/\rho_n^3$.

3.1.2 Main results

We will first introduce some notation that will be used throughout the chapter. For two sequences of positive numbers $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, if x_n/y_n is bounded above by a positive constant, we write $x_n \lesssim y_n$ and if $\lim_{n\to\infty} x_n/y_n = 0$, we write $x_n \ll y_n$. We define $x_n \gtrsim y_n$ and $x_n \gg y_n$ correspondingly. Moreover, the notation $x_n \asymp y_n$ means that x_n/y_n is bounded above and below by positive constants, and the notation $x_n \sim y_n$ means that $\lim_{n\to\infty} x_n/y_n = 1$. We write $x_n = O(y_n)$ if the sequence $(x_n/y_n)_{n=1}^{\infty}$ is bounded and $x_n = o(y_n)$ if $\lim_{n\to\infty} x_n/y_n = 0$.

Before stating our new results, we briefly recall the main results that Roberts and Schweinsberg [75] established for this model. Under assumptions (3.1), (3.2), (3.3), (3.9) and (3.10), they showed that if $\rho_n^{2/3}/\beta_n^{8/9} \ll t_n - \rho_n/\beta_n \lesssim \rho_n/\beta_n$, then most particles are near the origin at time t_n and the scaled empirical distribution of particles at time t_n is Gaussian. More precisely, define the random probability measure which represents the empirical distribution of the particle locations at time t_n, scaled in space, to be

$$\zeta_n(t_n) = \frac{1}{N_{t_n,n}} \sum_{i \in \mathcal{N}_{t_n,n}} \delta_{X_{i,n}(t_n)\sqrt{\beta_n/\rho_n}}. \tag{3.11}$$

They showed as $n \to \infty$, that the random measures $\zeta_n(t_n)$ converge weakly to the standard normal distribution in the Polish space of probability measures on $\mathbb{R}$ equipped with the weak topology. From the scaling in (3.11), this result implies that the empirical distribution of particle locations at time t is approximately normal with mean 0 and variance ρ_n/β_n. In particular, this result describes the configuration of particles whose distance to the origin is $O(\sqrt{\rho_n/\beta_n})$.

Roberts and Schweinsberg [75] also provided an explicit characterization of the empirical distribution of particles close to the right edge. They considered the empirical measure where a particle at x is weighted by $e^{\rho_n x}$. Define the random probability measure

$$\xi_n(t_n) = \frac{1}{Y_n(t_n)} \sum_{i \in \mathcal{N}_{t_n,n}} e^{\rho_n X_{i,n}(t_n)} \delta_{(2\beta_n)^{-1/3}(L_n - X_{i,n}(t_n))} \tag{3.12}$$

Thus, particles with a higher fitness level will contribute more to $\xi_n(t_n)$. Let μ be the probability measure on $(0, \infty)$ with probability density function

$$h(y) = \frac{Ai(y+\gamma_1)}{\int_0^\infty Ai(z+\gamma_1)dz}.$$

Roberts and Schweinsberg proved that under assumptions (3.1), (3.2), (3.3), (3.9) and (3.10), if

$$\beta_n^{-2/3} \log^{1/3}\left(\frac{\rho_n}{\beta_n^{1/3}}\right) \ll t_n \lesssim \frac{\rho_n}{\beta_n},$$

then as $n \to \infty$, we have

$$\xi_n(t_n) \Rightarrow \mu, \tag{3.13}$$

where $\Rightarrow$ refers to weak convergence in the Polish space of probability measures on $\mathbb{R}$

equipped with the weak topology. From the scaling in (3.12), we see that this convergence result describes the configuration of particles whose distance from the right edge L_n^* is $O(\beta_n^{-1/3})$.

Our goal in this chapter is to obtain a fuller understanding of the particle configurations from the left edge $L_n^\dagger$ to the right edge L_n^*. In other words, for this model, we aim to characterize the long-run empirical distribution of the fitness levels of individuals in a large population.

Consider a sequence of intervals $\{[a_n, b_n]\}_{n=1}^\infty$, where $-\infty \le a_n < b_n \le \infty$, satisfying the following three conditions:

$$b_n - a_n \gg 1 \tag{3.14}$$

$$L_n^* - a_n \gg \beta_n^{-1/3} \tag{3.15}$$

$$b_n - L_n^\dagger \gg \beta_n^{-1/3}. \tag{3.16}$$

We are interested in the number of particles in the intervals $[a_n, b_n]$. We include the conditions (3.15) and (3.16) because we do not expect our results to describe the configuration of particles which are within distance $O(\beta_n^{-1/3})$ of the right edge L_n^* or the left edge $L_n^\dagger$. Also, the particles within $O(\beta_n^{-1/3})$ distance of the right edge were studied in Theorem 1.2 in [75]. Define

$$z_n = \begin{cases} a_n & \text{if } a_n \ge 0, \\ 0 & \text{if } a_n < 0,\ b_n > 0, \\ b_n & \text{if } b_n \le 0. \end{cases} \tag{3.17}$$

Note that $z_n \in (L_n^\dagger, L_n^*)$, and the restrictions (3.15) and (3.16) are equivalent to

$$L_n^* - z_n \gg \beta_n^{-1/3}, \qquad z_n - L_n^\dagger \gg \beta_n^{-1/3}. \tag{3.18}$$

Later, we will see that the asymptotic density of the number of particles in $[a_n, b_n]$ reaches its maximum at z_n. As a result, the number of particles near z_n dominates the total number of particles in $[a_n, b_n]$.

For every n, we will define two important functions in the domain $(-\infty, L_n^*]$. First, we let

$$t_n(y) = \sqrt{\frac{2}{\beta_n}(L_n^* - y)}. \tag{3.19}$$

We will later see that particles near y are most likely descended from ancestors that were near the right edge approximately $t_n(y)$ time units in the past. For every n, we observe that $t_n(y)$ is a decreasing function of y. We have $t_n(0) = \rho_n/\beta_n$. If $L_n^* - z_n \gg \beta_n^{-1/3}$, then

$$t_n(z_n) \gg \beta_n^{-2/3}. \tag{3.20}$$

Also, for $y \in (-\infty, L_n^*]$, we define

$$g_n(y) = \rho_n(L_n^* - y) - \frac{2\sqrt{2\beta_n}}{3}(L_n^* - y)^{3/2}. \tag{3.21}$$

We will see shortly that in the long-run, the number of particles located near y is roughly proportional to $e^{g_n(y)}$. Note that $g_n(y)$ is decreasing in $[0, L_n^*]$ and increasing in $(-\infty, 0]$. The functions $g_n(y)$ and $t_n(y)$ were previously obtained in [75].

We now state our main result, which describes the configuration of particles from the left edge to the right edge.

Theorem 21. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold. For every sequence of intervals* $\{[a_n, b_n]\}_{n=1}^\infty$ *satisfying (3.14)-(3.16), define* z_n *according to (3.17). If*

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(z_n) \ll \frac{\rho_n}{\beta_n}, \tag{3.22}$$

then as $n \to \infty$,

$$N_{t_n,n}\big([a_n,b_n]\big)\Big/\left(\frac{1}{Ai'(\gamma_1)^2}Z_n(0)e^{-\rho_n L_n^*}\int_{[a_n,b_n]\cap(-\infty,L_n^*]}\frac{1}{\sqrt{2\pi t_n(y)}}e^{g_n(y)}dy\right)\to_p 1. \tag{3.23}$$

If

$$t_n - t_n(z_n) \asymp \frac{\rho_n}{\beta_n}, \tag{3.24}$$

then as $n \to \infty$,

$$N_{t_n,n}\big([a_n,b_n]\big)\Big/\left(\frac{1}{Ai'(\gamma_1)^2}Z_n\big(t_n - t_n(z_n)\big)e^{-\rho_n L_n^*}\int_{[a_n,b_n]\cap(-\infty,L_n^*]}\frac{1}{\sqrt{2\pi t_n(y)}}e^{g_n(y)}dy\right)\to_p 1. \tag{3.25}$$

Theorem 21 describes the number of particles in any given interval in the long run. The randomness is characterized by the stochastic process $\{Z_n(t), t \geq 0\}$, which measures how the overall "size" of the process changes over time. The deterministic part has a density formula proportional to $e^{g_n(y)}/\sqrt{2\pi t_n(y)}$. To be more precise, shortly after time $t_n(z_n)$, the number of particles in the interval $[a_n, b_n]$ depends on the initial configuration of particles through the value of $Z_n(0)$. For much later times t_n, when $t_n - t_n(z_n)$ is of the order ρ_n/β_n, the number of particles in the interval $[a_n, b_n]$ depends on $Z(t_n - t_n(z_n))$, which is the "size" of the process $t_n(z_n)$ time units in the past. Here z_n is the point where the density of the number of particles in $[a_n, b_n]$ is maximized. Later it will be shown in the proof that the number of particles in any interval $[a_n, b_n]$ is dominated by the number of particles that are close to z_n. The proof of Theorem 21 indeed shows that most of the particles in the interval $[a_n, b_n]$ at time t_n are descendants of particles that are close to the right edge $t_n(z_n)$ time units in the past. This also explains why the number of particles in $[a_n, b_n]$ depends on $Z(t_n - t_n(z_n))$.

Corollary 22. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold. For every sequence of intervals* $\{[a_n, b_n]\}_{n=1}^{\infty}$ *satisfying (3.14)-(3.16), define* z_n *according to (3.17).*

Suppose

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - \max\big\{t_n(z_n), t_n(0)\big\} \lesssim \frac{\rho_n}{\beta_n}. \tag{3.26}$$

For $y \in (-\infty, L_n^*]$, *define*

$$f_n(y) = \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y) - \rho_n^3/6\beta_n}.$$

The sequence

$$(D_n)_{n=1}^{\infty} := \left\{ \frac{N_{t_n,n}\big([a_n, b_n]\big)}{N_{t_n}} \Big/ \left(\int_{[a_n,b_n]\cap(-\infty,L^*]} f_n(y)dy \right) \right\}_{n=1}^{\infty}$$

is tight. If $0 \in [a_n, b_n]$ *for all* n, *then* D_n *converges to* 1 *in probability as* $n \to \infty$.

Corollary 22 shows that the ratio of the number of particles in any given interval to the total number of particles is comparable to the integral of $f_n(y)$ over the given interval. We can therefore regard $f_n(y)$ as the density of the limiting empirical distribution of the process, or in other words, the asymptotic empirical density of the fitness levels of individuals.

To understand the connection with results on traveling waves in the biology and physics literature, consider a translation of the model where each particle independently moves as standard Brownian motion without drift. A particle at location y can either die or split into two particles, and the difference between the birth rate and the death rate is $\beta_n(y - \rho_n t)$. Corollary 22 shows that after a sufficiently long time, the empirical density of individual fitness levels is

$$f_n^*(t,y) = f_n(y - \rho_n t) = \frac{1}{\sqrt{2\pi t_n(y - \rho_n t)}} e^{g_n(y-\rho_n t) - \rho_n^3/6\beta_n}, \quad \text{for } y \in (L_n^\dagger + \rho_n t, L_n^* + \rho_n t),$$

which is a traveling wave with profile $f_n(y)$.

The asymptotic empirical density $f_n(y)$ is closely related to the Airy function. For $y < L_n^*$, define

$$f_n^A(y) = (2\beta_n)^{1/3} e^{-\rho_n y + \rho_n^3/3\beta_n} Ai\big((2\beta_n)^{1/3}(L_n^* - y)\big). \tag{3.27}$$

According to (2.45) in [85],

$$\lim_{x\to\infty} 2\sqrt{\pi}x^{1/4}e^{(2/3)x^{3/2}}Ai(x) = 1. \tag{3.28}$$

Therefore, if $L_n^* - y_n \gg \beta_n^{-1/3}$, then as $n \to \infty$,

$$f_n(y_n) \sim f_n^A(y_n).$$

Note that the restriction $L_n^* - y_n \gg \beta_n^{-1/3}$ is consistent with our requirement (3.15) on the interval. The appearance of this Airy density function is not a coincidence. Using nonrigorous methods, Melissa et al. in [66] showed that the fitness distribution within population can be described as a traveling wave which has a steady-state shape $\tilde{f}_n^A(y)$

$$\tilde{f}_n^A(y) = e^{-\rho_n y}Ai\Big((2\beta_n)^{1/3}\Big(\frac{\rho_n^2}{2\beta_n} - y\Big)\Big), \quad \text{for } y < L_n \tag{3.29}$$

after matching parameters. We will explain the derivation of equation (3.29) in more details in Section 3.1.3. Note that $\tilde{f}_n^A(y)$ is proportional to $f_n^A(y)$. Therefore, Corollary 22 shows that, under certain assumptions, after a sufficiently long time, the bulk distribution of fitness levels of individuals from the least fit individuals to the most fit individuals is approximately a traveling wave with a profile asymptotically equivalent to the profile defined in citeMelissa, providing mathematically rigorous justification for the result in [66]. The idea that the traveling wave profile should have a shape given by the Airy function goes back to the early work of Tsimring, Levine, and Kessler [84], and this Airy shape also appears, for example, in [29, 68, 66]. Theorem 21 and Corollary 22 therefore provide rigorous justification for this result in the biology and physics literature.

We also observe that the shape of $f_n(y)$ near 0 is very much like the Gaussian density

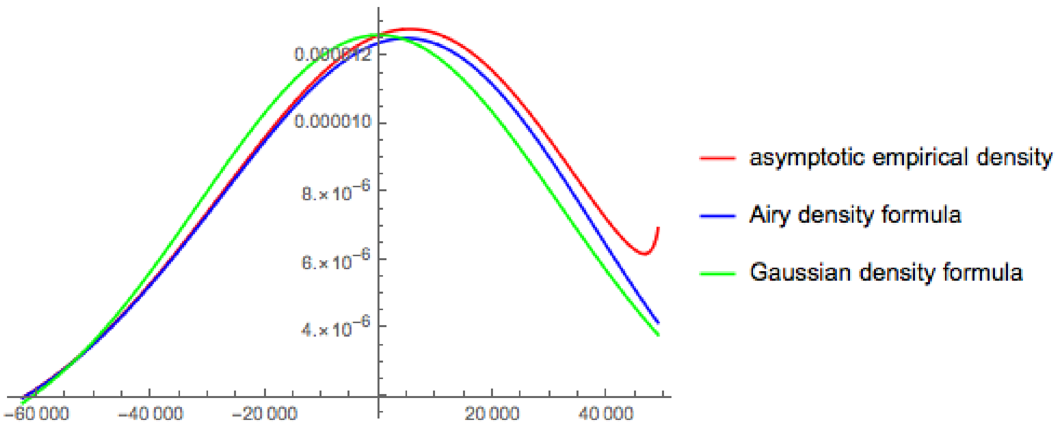

Figure 3.1: Graph of the symptotic empirical density, Airy density formula and Gaussian density formula when $\rho_n = 10^{-4}$ and $\beta_n = 10^{-13}$

function with standard deviation $\sqrt{\rho_n/\beta_n}$. Let

$$f_n^G(y) = \frac{1}{\sqrt{2\pi\rho_n/\beta_n}} \exp\left(- \frac{\beta_n y^2}{2\rho_n}\right). \tag{3.30}$$

As noted in [75], the Taylor expansions of $g_n(y)$ and $t_n(y)$ around 0 give

$$g_n(y) \approx \frac{\rho_n^3}{6\beta_n} - \frac{\beta_n y^2}{2\rho_n}, \qquad t_n(y) \approx \sqrt{\frac{\rho_n}{\beta_n}}.$$

Therefore, the asymptotic empirical density $f_n(y)$ can be approximated by the Gaussian density formula $f_n^G(y)$. This is consistent with Theorem 1.1 in [75].

Figure 3.1 illustrates the graphs of the asymptotic empirical density $f_n(y)$, the Airy density formula $f_n^A(y)$ and the Gaussian density formula $f_n^G(y)$ from $L_n^\dagger$ to L_n^* when $\rho_n = 10^{-4}$ and $\beta_n = 10^{-13}$. We see that all three functions have similar shapes. The asymptotic empirical density is very close to the Airy density formula in the bulk, especially in the negative real line where y is far away from the right boundary L^*.

Let $M_{t,n} = \max\{X_{i,n}(t), i \in \mathcal{N}_{t,n}\}$ be the position of the right-most particle at time t and $m_{t,n} = \min\{X_{i,n}(t), i \in \mathcal{N}_{t,n}\}$ be the position of the left-most particle at time t. Propositions 23 and 24 show that, under certain assumptions, with high probability the right-most particle is close to L_n^* and the left-most particle is close to $L_n^\dagger$. This explains why we are able to refer to L_n^* as the right edge and $L_n^\dagger$ as the left edge of the process.

Proposition 23. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold and* $(t_n)_{n=1}^\infty$ *satisfies*

$$\beta_n^{-2/3}\log^{1/3}\left(\frac{\rho_n}{\beta_n^{1/3}}\right) \ll t_n \lesssim \frac{\rho_n}{\beta_n}. \tag{3.31}$$

For any positive constant C_1*, we have*

$$\lim_{n\to\infty} P\left(M_{t_n,n} \geq L_n - \frac{C_1}{\beta_n^{1/3}}\right) = 1. \tag{3.32}$$

If in addition, the birth rate $b_n(x)$ *is non-decreasing and the death rate* $d_n(x)$ *is non-increasing, then for any constant* $C_2 \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(M_{t_n,n} \leq L_n + \frac{C_2}{\rho_n}\right) = 1. \tag{3.33}$$

Therefore, we have as $n \to \infty$,

$$\frac{M_{t_n,n}}{L_n^*} \to_p 1. \tag{3.34}$$

Define

$$\bar{L}_n = -\frac{5}{8}\frac{\rho_n^2}{\beta_n} + 2(2\beta_n)^{-1/3}\gamma_1, \tag{3.35}$$

which is slightly smaller than $L_n^\dagger$. The following proposition shows that $\bar{L}_n$ is the approximate position of the left-most particle.

Proposition 24. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold and* $(t_n)_{n=1}^\infty$ *satisfies*

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(\bar{L}_n) \lesssim \frac{\rho_n}{\beta_n}. \tag{3.36}$$

For any $\kappa > 0$*, there exists a positive constant* C_3 *such that for* n *large enough,*

$$P\left(m_{t_n,n} \leq \bar{L}_n + \frac{C_3}{\beta_n^{1/3}}\right) > 1 - \kappa. \tag{3.37}$$

If in addition, the birth rate $b_n(x)$ is non-decreasing and the death rate $d_n(x)$ is non-increasing, then for any $\kappa > 0$, there exists a positive constant C_4 such that for n large enough,

$$P\left(m_{t_n,n} \geq \bar{L}_n - \frac{C_4}{\rho_n}\right) > 1 - \kappa. \tag{3.38}$$

Therefore, we have as $n \to \infty$,

$$\frac{m_{t_n,n}}{L_n^\dagger} \to_p 1. \tag{3.39}$$

3.1.3 Ideas behind the proof

We have two perspectives to understand the expression of the asymptotic empirical density $f_n(y)$ heuristically. The first one is through the large deviations techniques in [14] and the second one is through the traveling wave approach in [29, 66, 68, 84].

The heuristics proposed in [14] inspired the derivation of the functions $g_n(z_n)$ and $t_n(z_n)$, although the techniques they used are not sufficient to derive the exact asymptotic rate of the number of particles like what we did in Theorem 21. Roberts and Schweinsberg in [75] first derived the explicit formulas of $g_n(z_n)$ and $t_n(z_n)$ using the method of [14]. They conjectured that the number of particles near z_n in the long run is proportional to $e^{g_n(z_n)}$ and proved this conjecture when $|z_n|$ is around the origin ($z_n \lesssim \sqrt{\rho_n/\beta_n}$). Since these heuristics are essential in understanding the behavior of the process and the main ideas of the proof, we will briefly recall their calculations.

For every n, consider a large time t_n and a path $f_n : [0, t_n] \to \mathbb{R}$. By Schilder's theorem and the many-to-one lemma, if the process starts with one particle at $f_n(0)$, the expected number of particles that stay "close" to f_n during $[0, t_n]$ is approximately

$$\exp\left(\int_0^{t_n} \left(\beta_n f_n(u) - \frac{1}{2}(f_n'(u) + \rho_n)^2\right) du\right). \tag{3.40}$$

Note that if $f_n(u) \equiv \rho_n^2/2\beta_n$, then the integrand is 0. Thus the number of particles around $\rho_n^2/2\beta_n$ is $O(1)$ and the right-most particle should stay close to $\rho_n^2/2\beta_n$, which is the right edge L_n^*. We next consider the optimal trajectory $f_n^{z_n}$ followed by particles that are near z_n at time t_n. This path is optimal in the sense that particles which end up near z_n must follow this trajectory to achieve the maximum almost sure growth rate. According to Theorem 7 in [14], there exists a cutoff time $t_n(z_n)$ such that the optimal path will follow the trajectory of the right-most particle up to some time $t_n - t_n(z_n)$ and then moves towards z_n by following a path that satisfies a certain differential equation. Therefore, $f_n^{z_n}$ satisfies

$$f_n^{z_n}(u) = \rho_n^2/2\beta_n \text{ for } u \in [0, t_n - t_n(z_n)],$$

$$(f_n^{z_n})''(u) = -\beta_n \text{ for } u \in [t_n - t_n(z_n), t_n] \ ,$$

$$f_n^{z_n}(t_n) = z_n.$$

Solving the above equations, we get the expression (3.19) for $t_n(z_n)$. Together with (3.40), the number of particles near z_n at time t_n is approximately

$$\exp(g_n(z_n)) = \exp\left(\int_0^{t_n} \left(\beta f_n^{z_n}(u) - \frac{1}{2}\big((f_n^{z_n})'(u) + \rho_n\big)^2\right) du \right),$$

which gives (3.21) for all z_n. Figure 3.2 is an illustration of the trajectory of $f_n^{z_n}$. It is also worth mentioning that the expressions for the left edge $L_n^\dagger$ and the right edge L_n^* emerge from $g_n(z_n)$. Solving $g_n(z_n) = 0$, we get two solutions

$$z_n = -\frac{5\rho_n^2}{8\beta_n}, \qquad z_n = \frac{\rho_n^2}{2\beta_n},$$

which correspond to $L_n^\dagger$ and L_n^* respectively. These heuristics also explain why we need assumptions (3.1) and (3.2). Note that the Taylor expansion of $e^{g(z)}$ is proportional to the Gaussian density with mean 0 and variance ρ_n/β_n. The standard deviation of the Gaussian distribution should be much smaller than the right-most position, which leads to (3.1). Moreover, the branching rate should be small around the right edge $\rho_n^2/2\beta$, which leads to

(3.2).

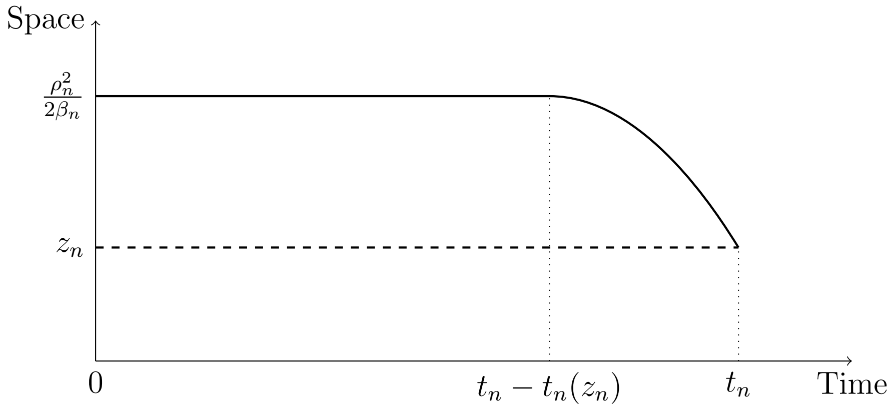

Figure 3.2: Trajectory of $f_n^{z_n}$

According to the discussion after Corollary 22, we know that the asymptotic empirical density $f_n(y)$ is closely related to the Airy density formula $f_n^A(y)$, which is $\tilde{f}^A(y)$ after matching parameters. Below we will follow the derivation of $\tilde{f}^A(y)$ in [66]. We assume that there are N individuals in the population. Each individual subject to new mutations at rate μ and the selective advantage s of each mutation is random and has a distribution with probability density function $\rho(s)$. Let $q(x,t)$ be the density of particles with fitness x at time t, which gives the distribution of fitnesses within the population. Define $m(t)$ to be the average fitness at time t with $m(0) = 0$. Let $\nu(s) = \mu\rho(s)$. Then equation (4) in [66] shows that $q(x,t)$ will satisfy the following nonlinear stochastic differential equation,

$$\frac{\partial}{\partial t} q(x,t) = (x - m(t))q(x,t) + \int \big(p(x-s,t) - p(x,t)\big)\nu(s)ds + \sqrt{\frac{q(x,t)}{N}}\eta(x,t), \quad (3.41)$$

where η is a Brownian noise term. By stochastic simulations, it was conjectured by multiple biology literatures (e.g. [84]) that equation (3.41) has a traveling wave solution of the form

$$q(x,t) = \omega(x - vt),$$

which moves at an unknown average rate of mean fitness change $v = m(t)/t$. Writing

$y = x - vt$ for the relative fitness and neglecting the noise term, equation (3.41) becomes

$$-v\omega'(y) = y\omega(y) + \int \big(\omega(y-s) - \omega(y)\big)\nu(s)ds.$$

Since the selective advantage s is sufficiently small, we can use Taylor expansion to approximate

$$\omega(y-s) - \omega(y) \approx -s\omega'(y) + \frac{1}{2}s^2\omega''(y).$$

which leads to

$$-v\omega'(y) = y\omega(y) - \mu E[s]\omega'(y) + \frac{1}{2}\mu E[s^2]\omega''(y).$$

According to the Fisher's Fundamental Theorem of Natural Selection [40], the speed of the traveling wave v is the summation of the variance in the fitness distribution, written as σ^2 and the direct contribution from mutations, written as D. Letting $D = \mu E[s^2]/2$, we get

$$D\omega''(y) + \sigma^2\omega'(y) + y\omega(y) = 0. \tag{3.42}$$

We note that the above equation will lead to a solution which take negative values at sufficiently large y, which is impossible for a fitness distribution. To avoid this, we assume that there is a cutoff value y_{cut}, which can be understood as the maximum fitness of the individuals in the population, such that

$$\omega(y) = 0, \qquad \text{for } y > y_{cut}. \tag{3.43}$$

As a result, the solution of equations (3.42) and (3.43) is

$$\omega(y) = e^{-\sigma^2 y/2D} Ai\Big(\frac{\sigma^4}{4D^{4/3}} - \frac{y}{D^{1/3}}\Big), \quad \text{for } y < \frac{\rho^2}{2} - 2^{-1/3}\beta^{2/3}\gamma_1. \tag{3.44}$$

It remains to relate parameters σ^2 and D with ρ_n and β_n in our model. Recall that the limiting distribution of (3.11) is the standard Gaussian distribution. Therefore, in our model, the empirical distribution of particles has variance ρ_n/β_n. Because the branching rate, which measures the fitness of the population, increases in the unit of β_n, our model can be viewed as the model in [66] scaled by β_n. Thus the variance in the fitness distribution

$$\sigma^2 = \beta_n^2 \frac{\rho_n}{\beta_n} = \rho_n \beta_n.$$

As for D, equation (1.20) in [75] states that $\beta_n = s\sqrt{\mu}$, which leads to

$$D = \frac{\mu E[s^2]}{2} \approx \frac{\mu s^2}{2} = \frac{\beta_n^2}{2}.$$

Plugging the above two formulas into (3.44) and taking the scaling into consideration, we get

$$\omega(\beta_n y) = e^{-\rho_n y} Ai\Big(\frac{\rho^2}{2^{2/3}\beta^{2/3}} - 2^{1/3}\beta^{1/3}y\Big), \quad \text{for } y < L_n,$$

which is exactly the expression of $\tilde{f}_n^A(y)$ in (3.29). According to (3.28), $\tilde{f}_n^A(y)$ is essentially the same as the asymptotic empirical density $f_n(y)$ expressed in terms of $t_n(y)$ and $g_n(y)$ in the limit when $L_n^* - y \gg \beta_n^{-1/3}$.

From the large deviations analysis, for n sufficiently large, particles near the right edge should be in a stable configuration at time $t_n - t_n(z_n)$ according to (3.13), and a law of large numbers behavior is expected. As a result, we are going to use a first and second moment argument to prove Theorem 21. The argument consists of two parts. First, we compute the expectation. Next, we control the fluctuations and show that $N_{t_n,n}([a_n, b_n])$ is concentrated around its mean. Moreover, $N_{t_n,n}([a_n, b_n])$ is concentrated around the number of particles that are close to z_n. As for the first moment estimate, we denote by $p_t^n(x, y)$ the density at location y and time t for the process which starts from a single particle at x at time 0. This means that if there is a single particle located at x at time 0, then the expected number of particles in the measurable set U at time t is

$$\int_U p_t(x,y)dy.$$

According to formula (2.11) in [75], by the many-to-one lemma

$$p_t^n(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(\rho_n x - \rho_n y - \frac{(x-y)^2}{2t} - \frac{\rho_n^2 t}{2} + \frac{\beta_n(x+y)t}{2} + \frac{\beta_n^2 t^3}{24}\right). \tag{3.45}$$

The first moment estimates involve expressing $p_{t_n}^n(x, z_n)$ in terms of $g_n(z_n)$ and some other controllable terms. As for the second moment, we will only control the fluctuations of the number of particles that are close to z_n. The remaining particles in the interval $[a_n, b_n]$ can be dealt with using the first moment estimate. A direct second moment computation will fail because without truncation, the expected number of particles around z_n is dominated by rare events in which one particle drifts very far to the right and generates a large number of descendants around z_n. As a result, we need to do a truncated second moment estimate where particles are killed upon hitting the boundary L_n. This boundary L_n needs to be large enough that particles which contribute most won't hit this boundary and small enough that we can control the second moment of the number of particles. It is pointed out in [75] that (3.6) is an appropriate choice. We show that the major contribution of the first moment comes from particles which stay close to L_n at time $t_n - t_n(z_n)$ and don't hit the right boundary L_n during time $[t_n - t_n(z_n), t_n]$. Since the proofs of the second moment estimates are quite tedious, we defer them until Section 3.5.

3.1.4 Table of notation

We summarize some of the notation that is used throughout the rest of this chapter in the following table.

Table 3.1: *Index of notation in Chapter 3*

n	Index of a sequence of processes.
ρ_n	Particles move according to Brownian motion with drift $-\rho_n$.
β_n	Selection parameter. The difference between the birth rate and the death rate for a particle at x is $\beta_n x$.
$N_{t,n}$	Total number of particles at time t.
$\mathcal{N}_{t,n}$	The set of particles alive at time t.
$N_{t,n}(\mathcal{I})$	Number of particles in the interval $\mathcal{I}$ at time t.
$X_{i,n}(t)$	Positions of the particle i at time t for $i \in \mathcal{N}_{t,n}$.
γ_1	The largest zero of the Airy function.
L_n	The approximate position of right-most particle, $L_n = \rho_n^2/\beta_n - (2\beta_n)^{-1/3}\gamma_1$.
L_n^A	Defined to equal $L_n - A/\rho_n$ for $A \in \mathbb{R}$.
$\bar{L}_n$	The approximate position of left-most particle, $\bar{L}_n = -5\rho_n^2/8\beta_n - 2(2\beta_n)^{-1/3}\gamma_1$.
L_n^*	The position that is near the position of the right-most particle. We call it the right edge. Explicitly, $L_n^* = \rho_n^2/\beta_n$.
$L_n^\dagger$	The position that is near the position of the left-most particle. We call it the left edge. Explicitly, $L_n^\dagger = -5\rho_n^2/8\beta_n$.
$Y_n(t)$	Sum of $e^{\rho_n X_{i,n}(t)}$ for all $i = 1, ..., N_{t,n}$. Defined in (3.7).
$Z_n(t)$	Weighted sum used to characterize the size of the configuration at time t. Defined in (3.8).
$\lesssim$	Write $x_n \lesssim y_n$ if x_n/y_n is bounded above by a positive constant. Define $\gtrsim$ similarly.
$\ll$	Write $x_n \ll y_n$ if $\lim_{n\to\infty} x_n/y_n = 0$. Define $\gg$ similarly.
$\asymp$	Write $x_n \asymp y_n$ if x_n/y_n is bounded above and below by positive constants.
O	Write $x_n = O(y_n)$ if the sequence $(x_n/y_n)_{n=1}^\infty$ is bounded.

Table 3.1: *Index of notation in Chapter 3, Continued*

o	Write $x_n = o(y_n)$ if $\lim_{n\to\infty} x_n/y_n = 0$.
$[a_n, b_n]$	Interval satisfying (3.14)-(3.16).
z_n	Roughly speaking, the asymptotic density of the number of particles in $[a_n, b_n]$ is maximized at z_n. Defined in (3.17).
l_n	Measures the length of the interval in which we are counting the number of particles.
$t_n(y)$	For particles near y at time t_n, $t_n - t_n(y)$ is the time when their ancestors start to leave the right boundary and drift toward y. Defined in (3.19).
$g_n(y)$	Function used to approximate the density of particles. Defined in (3.21).
$M_{t,n}$	Position of the left-most particle at time t.
$p_t^n(x, y)$	Density at location y and time t for the process which starts from a single particle at x at time 0. Defined in (3.45).
$p_t^{L_n}(x, y)$	Density at location y and time t for the process where there is only one particle at x at time 0 and particles are killed upon hitting L_n.
$c_{0,n}$	The ratio between z_n and L_n^*. Defined in (3.48).
c_n	Measures the distance between z_n and L_n^*. Defined in (3.87).
$r_x^{L_n}(v)$	Rate at which particles hit L_n at time v. Defined in 3.1.4.
$N_t^{L_n}(\mathcal{I})$	Number of particles in the interval $\mathcal{I}$ at time t for the process in which particles are killed at L_n.
$(\mathcal{F}_t, t \geq 0)$	Natural filtration associated with the BBM process.
d	Used to divide the length of the interval in the proof of Proposition 26. Defined in (3.135).
s	Constants used to adjust time. In the proof of Propositions 25 and 26, we define $s = C_1\beta_n^{-2/3}$.

Table 3.1: *Index of notation in Chapter 3, Continued*

s_y	Constants used to adjust time for each $y \in [z-l, z+l]$ based on the choice of s. Defined to be $t(y) - t(z) + s$.
u_1	The first cutoff time in the second moment calculation. See Lemma 42.
u_2	The second cutoff time in the second moment calculation. Defined in (3.284).

3.1.5 Organization of the chapter

The rest of this chapter is organized as follows. In Section 3.2, we show how to obtain Theorem 21 and Corollary 22 from two other propositions, one of which controls the number of particles in narrow intervals and one of which controls the number of particles in longer intervals. In Section 3.3, we prove Propositions 23 and 24, and give the most important arguments for the proofs of the two propositions that lead to Theorem 21. Proofs of some technical lemmas are postponed until Section 3.4, and the second moment calculations are presented in Section 3.5.

3.2 Structure of the proofs

In this section, we show how Theorem 21 and Corollary 22 follow from Propositions 25 and 26 below. We also introduce some notation that will be used throughout the chapter.

3.2.1 Division into larger and smaller intervals

The proof of Theorem 21 will be divided into two cases. First, we will deal with intervals with smaller length. In such intervals, we will control the number of particles using a second moment argument. Indeed, we will show that most particles that end up near z_n

at time t_n stay close to L_n up to time $t_n - t_n(z_n)$ and then drift towards z_n. Trajectories of such particles are illustrated in Figure 3.2. Second, we will consider longer intervals. We will show that the number of particles in the interval $[a_n, b_n]$ that are far away from z_n is negligible using a first moment argument, allowing us to estimate the number of particles in the entire interval by the number of particles in a smaller interval around z_n. The first step will lead to Proposition 25 while the second step will lead to Proposition 26.

Consider a sequence $(z_n)_{n=1}^{\infty}$ satisfying (3.18) such that

$$|z_n| \gtrsim \sqrt{\frac{\rho_n}{\beta_n}} \qquad \text{or} \qquad |z_n| \ll \sqrt{\frac{\rho_n}{\beta_n}}. \tag{3.46}$$

We further assume that

$$z_n \geq 0 \quad \text{for all } n \qquad \text{or} \qquad z_n \leq 0 \quad \text{for all } n. \tag{3.47}$$

Denote

$$c_{0,n} = \frac{z_n}{L_n^*}. \tag{3.48}$$

We consider intervals of the forms $[z_n, z_n + l_n]$ and $[z_n - l_n, z_n]$ where l_n is the length of the interval. By convention, if $l_n = \infty$, then $[z_n, z_n + l_n] = [z_n, \infty)$ and $[z_n - l_n, z_n] = (-\infty, z_n]$.

Proposition 25. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold. For every sequence* $(z_n)_{n=1}^{\infty}$ *satisfying (3.18), (3.46) and (3.47), choose* $(l_n)_{n=1}^{\infty}$ *such that*

$$\begin{cases} 1 \ll l_n \lesssim \frac{1}{|c_{0,n}|\rho_n} & \text{if } |z_n| \gtrsim \sqrt{\frac{\rho_n}{\beta_n}}, \\ 1 \ll l_n \lesssim \sqrt{\frac{\rho_n}{\beta_n}} & \text{if } |z_n| \ll \sqrt{\frac{\rho_n}{\beta_n}}. \end{cases} \tag{3.49}$$

Consider intervals of the form

$$\mathcal{I}_n = \begin{cases} [z_n, z_n + l_n] & \text{if } z_n \geq 0, \\ [z_n - l_n, z_n] & \text{if } z_n \leq 0. \end{cases} \tag{3.50}$$

If t_n satisfies

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(z_n) \ll \frac{\rho_n}{\beta_n}, \tag{3.51}$$

then for any $\kappa > 0$, we have

$$\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(0) \int_{\mathcal{I}_n} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy \leq N_{t_n,n}(\mathcal{I}_n) \\ \leq \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(0) \int_{\mathcal{I}_n} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy \bigg) = 1. \tag{3.52}$$

Proposition 26. *Suppose assumptions (3.1), (3.2), (3.3), (3.9) and (3.10) hold. For every sequence of $(z_n)_{n=1}^{\infty}$ satisfying (3.18), (3.46) and (3.47), choose $(l_n)_{n=1}^{\infty}$ such that*

$$\begin{cases} l_n \gg \frac{1}{|c_{0,n}|\rho_n} & \text{if } |z_n| \gtrsim \sqrt{\frac{\rho_n}{\beta_n}}, \\ l_n \gg \sqrt{\frac{\rho_n}{\beta_n}} & \text{if } |z_n| \ll \sqrt{\frac{\rho_n}{\beta_n}}. \end{cases} \tag{3.53}$$

Consider intervals of the form

$$\mathcal{J}_n = \begin{cases} [z_n, z_n + l_n] & \text{if } z_n \geq 0, \\ [z_n - l_n, z_n] & \text{if } z_n \leq 0. \end{cases} \tag{3.54}$$

If t_n satisfies (3.51), then for any $\kappa > 0$, we have

$$\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(0) \int_{\mathcal{J}_n\cap(-\infty,L_n^*]} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy \leq N_{t_n,n}(\mathcal{J}_n) \\ \leq \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(0) \int_{\mathcal{J}_n\cap(-\infty,L_n^*]} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy \bigg) = 1. \tag{3.55}$$

Next, we will explain heuristically why the interval length l_n is divided into the above two cases (3.49) and (3.53). Let us take the case $z_n \geq 0$ as an example. The case when $z_n \leq 0$ is essentially the same. Our hope is to find a cutoff length l_n depending on z_n such that the number of particles in $[z_n, \infty)$ is dominated by the number of particles in $[z_n, z_n + l_n]$. Since the number of particles near z_n is approximately proportional to $e^{g_n(z_n)}/\sqrt{2\pi t_n(z_n)}$, this boils down to finding a cutoff length l_n such that for any $\eta > 0$, if n is sufficiently large, then

$$\int_{z_n}^{\infty} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy < (1+\eta) \int_{z_n}^{z_n+l_n} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy.$$

It turns out that if $z_n \gtrsim \sqrt{\rho_n/\beta_n}$, then we can take $l_n \asymp 1/c_{0,n}\rho_n$, as shown in Lemma 36, while if $z_n \ll \sqrt{\rho_n/\beta_n}$, then we take $l_n \asymp \sqrt{\rho_n/\beta_n}$.

3.2.2 Proof of Theorem 21

In this subsection, we deduce Theorem 21 from Propositions 25 and 26. We first review an important result from [75] which will be needed in the proof.

Remark 27. *Proposition 2.3 in [75] states that if $\beta_n t_n/\rho_n$ converges to a positive real number as n goes to infinity, then with probability tending to 1 as $n \to \infty$, conditions (3.9) and (3.10) hold with $Y_n(t_n)$ and $Z_n(t_n)$ in place of $Z_n(0)$ and $Y_n(0)$ respectively. Furthermore, if $t_n \asymp \rho_n/\beta_n$, then for every subsequence $(n_j)_{j=1}^{\infty}$, there exists a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{\beta_{n_{j_k}} t_{n_{j_k}}}{\rho_{n_{j_k}}} = \tau \in (0, \infty).$$

Consequently, by Proposition 2.3 in [75], with probability tending to 1 as $k \to \infty$, conditions (3.9) and (3.10) hold with $Y_{n_{j_k}}(t_{n_{j_k}})$ and $Z_{n_{j_k}}(t_{n_{j_k}})$ in place of $Z_n(0)$ and $Y_n(0)$ respectively.

Proof of Theorem 21. First, we consider the case when t_n satisfies (3.22). To prove (3.23), it suffices to show that for every subsequence $(n_j)_{j=1}^{\infty}$, there exists a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$, such that for any $0 < \kappa < 1$,

$$\lim_{k\to\infty} P\Bigg(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_{n_{j_k}} L^*_{n_{j_k}}} Z_{n_{j_k}}(0) \int_{[a_{n_{j_k}}, b_{n_{j_k}}]\cap(-\infty, L^*_{n_{j_k}}]} \frac{1}{\sqrt{2\pi t_{n_{j_k}}(y)}} e^{g_{n_{j_k}}(y)} dy$$
$$\le N_{t_{n_{j_k}}, n_{j_k}}\big([a_{n_{j_k}}, b_{n_{j_k}}]\big)$$
$$\le \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_{n_{j_k}} L^*_{n_{j_k}}} Z_{n_{j_k}}(0) \int_{[a_{n_{j_k}}, b_{n_{j_k}}]\cap(-\infty, L^*_{n_{j_k}}]} \frac{1}{\sqrt{2\pi t_{n_{j_k}}(y)}} e^{g_{n_{j_k}}(y)} dy \Bigg) = 1. \tag{3.56}$$

Given a subsequence $(n_j)_{j=1}^\infty$, there exists a further subsequence $(n_{j_k})_{k=1}^\infty$ such that one of the following holds:

1. We have $a_{n_{j_k}} \ge 0$ for all k. Let $z_{n_{j_k}} = a_{n_{j_k}}$ and $l_{n_{j_k}} = b_{n_{j_k}} - a_{n_{j_k}}$. The subsequence $(z_{n_{j_k}})_{k=1}^\infty$ satisfies (3.18), (3.46) and (3.47), and the subsequence $(l_{n_{j_k}})_{k=1}^\infty$ satisfies (3.49) or (3.53).

2. We have $b_{n_{j_k}} \le 0$ for all k. Let $z_{n_{j_k}} = b_{n_{j_k}}$ and $l_{n_{j_k}} = b_{n_{j_k}} - a_{n_{j_k}}$. The subsequence $(z_{n_{j_k}})_{k=1}^\infty$ satisfies (3.18), (3.46) and (3.47), and the subsequence $(l_{n_{j_k}})_{k=1}^\infty$ satisfies (3.49) or (3.53).

3. We have $a_{n_{j_k}} < 0$ and $b_{n_{j_k}} > 0$ for all k. Let $z_{n_{j_k}} = 0$, $l_{1,n_{j_k}} = -a_{n_{j_k}}$ and $l_{2,n_{j_k}} = b_{n_{j_k}}$. Both the subsequences $(l_{1,n_{j_k}})_{k=1}^\infty$ and $(l_{2,n_{j_k}})_{k=1}^\infty$ satisfy (3.49) or (3.53).

In cases 1 and 2, since $[a_{n_{j_k}}, b_{n_{j_k}}]$ satisfies the hypotheses of either Proposition 25 or Proposition 26, equation (3.56) follows from (3.52) or (3.55). As for case 3, we see that both $[a_{n_{j_k}}, 0]$ and $[0, b_{n_{j_k}}]$ satisfy the hypotheses of Proposition 25 or Proposition 26. Thus, both $[a_{n_{j_k}}, 0]$ and $[0, b_{n_{j_k}}]$ satisfy (3.56) with $[a_{n_{j_k}}, 0]$ and $[0, b_{n_{j_k}}]$ in place of $[a_{n_{j_k}}, b_{n_{j_k}}]$ respectively. Consequently, equation (3.56) also holds in this case. Therefore, equation (3.23) follows.

Next, consider the case when t_n satisfies (3.24). Choose a sequence $(h_n)_{n=1}^\infty$ for which

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll h_n \ll \frac{\rho_n}{\beta_n}. \tag{3.57}$$

Let

$$r_n = t_n - t_n(z_n) - h_n. \tag{3.58}$$

Note that $r_n \asymp \rho_n/\beta_n$. By Remark 27, for every subsequence $(n_j)_{j=1}^\infty$, we can choose a sub-subsequence $(n_{j_k})_{k=1}^\infty$ such that assumptions (3.9) and (3.10) hold when $Y_n(0)$ and $Z_n(0)$ are replaced by $Y_{n_{j_k}}(r_{n_{j_k}})$ and $Z_{n_{j_k}}(r_{n_{j_k}})$. By using the Markov property at time $r_{n_{j_k}}$ and applying the previous argument, there exists a further sub-subsequence $(n_{j_{k_m}})_{m=1}^\infty$ such that equation (3.56) holds with $Z_{n_{j_{k_m}}}(r_{n_{j_{k_m}}})$ in place of $Z(0)$. As a result, we have for any $0 < \kappa < 1$,

$$\begin{aligned}\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(r_n) \int_{[a_n,b_n]\cap(-\infty,L_n^*]} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy \leq N_{t_n,n}\big([a_n,b_n]\big) \\ \leq \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n L_n^*} Z_n(r_n) \int_{[a_n,b_n]\cap(-\infty,L_n^*]} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy\bigg) = 1. \end{aligned} \tag{3.59}$$

Note that equation (3.59) holds for all choices of $(t_n)_{n=1}^\infty$ satisfying (3.24) and $(h_n)_{n=1}^\infty$ satisfying (3.57). Thus for every $(z_n)_{n=1}^\infty$ satisfying (3.18), $(t_n)_{n=1}^\infty$ satisfying (3.24) and any two sequences $(h_{1,n})_{n=1}^\infty$, $(h_{2,n})_{n=1}^\infty$ satisfying (3.57), we have

$$\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{1+\kappa} Z_n(r_{1,n}) \leq Z_n(r_{2,n}) \leq \frac{1+\kappa}{1-\kappa} Z_n(r_{1,n})\bigg) = 1, \tag{3.60}$$

where $r_{i,n} = t_n - t_n(z_n) - h_{i,n}$ for $i = 1, 2$. Choose $(z_n^*)_{n=1}^\infty$ satisfying (3.18), $(t_n^*)_{n=1}^\infty$ satisfying (3.24) and $(h_{1,n})_{n=1}^\infty$, $(h_{2,n})_{n=1}^\infty$ satisfying (3.57) such that

$$t_n - t_n(z_n) = t_n^* - t_n(z_n^*) - h_{1,n}, \qquad t_n - t_n(z_n) - h_n = t_n^* - t_n(z_n^*) - h_{2,n}.$$

For example, for any sequence of $(h_{1,n})_{n=1}^\infty$ satisfying (3.57), we can take $z_n^* = z_n$, $t_n^* = t_n + h_{1,n}$ and $h_{2,n} = h_{1,n} + h_n$. By (3.58) and (3.60), we have

$$\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{1+\kappa} Z_n\big(t_n - t_n(z_n)\big) \leq Z_n(r_n) \leq \frac{1+\kappa}{1-\kappa} Z_n\big(t_n - t_n(z_n)\big)\bigg) = 1. \tag{3.61}$$

Finally, equation (3.25) follows from (3.59) and (3.61). □

Remark 28. *The argument leading to (3.61) can be modified to show that for* $t_n \asymp \rho_n/\beta_n$ *and* $h_n \ll \rho_n/\beta_n$*, as* $n \to \infty$*,*

$$\frac{Z_n(t_n)}{Z_n(t_n + h_n)} \to_p 1. \tag{3.62}$$

To see this, note that we can choose $(z_n^*)_{n=1}^\infty$ *satisfying (3.18),* $(t_n^*)_{n=1}^\infty$ *satisfying (3.24), and* $(h_{1,n})_{n=1}^\infty$ *and* $(h_{2,n})_{n=1}^\infty$ *satisfying (3.57) such that*

$$t_n = t_n^* - t_n(z_n^*) - h_{1,n}, \qquad t_n + h_n = t_n^* - t_n(z_n^*) - h_{2,n}.$$

For example, we can take $(h_{2,n})_{n=1}^\infty$ *to be any sequence satisfying (3.57) and* $(z_n^*)_{n=1}^\infty$ *to be any sequence satisfying (3.18) such that* $t_n(z_n^*) \ll \rho_n/\beta_n$*. Then we let* $h_{1,n} = h_{2,n} + h_n$ *and* $t_n^* = t_n + t_n(z_n^*) + h_{1,n}$*. Letting* $r_{i,n} = t_n^* - t_n(z_n^*) - h_{i,n}$ *for* $i = 1, 2$*, equation (3.62) follows from (3.60).*

As a byproduct of the proof of Theorem 21, the following lemma shows that the number of particles in any given interval will not change much on a time scale shorter than ρ_n/β_n.

Lemma 29. *Suppose (3.1), (3.2), (3.3), (3.9) and (3.10) hold. For every sequence* $\{[a_n, b_n]\}_{n=1}^\infty$ *satisfying (3.14)-(3.16), define* z_n *according to (3.17). Suppose*

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(z_n) \lesssim \frac{\rho_n}{\beta_n}, \qquad \frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t'_n - t_n(z_n) \lesssim \frac{\rho_n}{\beta_n}. \tag{3.63}$$

If

$$|t_n - t'_n| \ll \frac{\rho_n}{\beta_n}, \tag{3.64}$$

then as $n \to \infty$*,*

$$\frac{N_{t_n,n}([a_n, b_n])}{N_{t'_n,n}([a_n, b_n])} \to_p 1. \tag{3.65}$$

Proof of Lemma 29. First, we consider the case $\rho_n^{2/3}/\beta_n^{8/9} \ll t_n - t_n(z_n) \ll \rho_n/\beta_n$. From (3.64), we also have $\rho_n^{2/3}/\beta_n^{8/9} \ll t_n' - t_n(z_n) \ll \rho_n/\beta_n$. Therefore, both $N_{t_n,n}([a_n, b_n])$ and $N_{t_n',n}([a_n, b_n])$ satisfy (3.23) and equation (3.65) follows.

It remains to consider the case $t_n - t_n(z_n) \asymp \rho_n/\beta_n$. By (3.64), we also have $t_n' - t_n(z_n) \asymp \rho_n/\beta_n$. Then both $N_{t_n,n}([a_n, b_n])$ and $N_{t_n',n}([a_n, b_n])$ satisfy (3.25). As a result, equation (3.65) follows from (3.25) and (3.62). □

3.2.3 Proof of Corollary 22

In this subsection, we show how to obtain Corollary 22 from Theorem 21. We first recall that Proposition 2.2 in [75] proves that if

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(0) \ll \frac{\rho_n}{\beta_n}, \tag{3.66}$$

then for all $\kappa > 0$,

$$\lim_{n\to\infty} P\left(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n^3/3\beta_n} Z_n(0) \le N_{t_n,n} \le \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n^3/3\beta_n} Z_n(0)\right) = 1. \tag{3.67}$$

Equation (3.67) also follows from (3.23) with $a_n = -\infty$ and $b_n = \infty$, once it is established that for n sufficiently large, we have

$$1 - \eta < e^{-\rho_n^3/6\beta_n} \int_{-\infty}^{L_n^*} \frac{1}{\sqrt{2\pi t_n(y)}} e^{g_n(y)} dy < 1 + \eta. \tag{3.68}$$

One can obtain (3.158) by comparing the density f_n defined in the statement of Corollary 22 to the Gaussian density f_n^G defined in (3.30). We omit the details.

Proof of Corollary 22. By applying the same argument leading to (3.25) in the proof of

Theorem 21, equation (3.67) implies that if $t_n - t_n(0) \asymp \rho_n/\beta_n$, then

$$\lim_{n\to\infty} P\left(\frac{1-\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n^3/3\beta_n} Z_n\big(t_n - t_n(0)\big) \le N_{t_n,n} \le \frac{1+\kappa}{Ai'(\gamma_1)^2} e^{-\rho_n^3/3\beta_n} Z_n\big(t_n - t_n(0)\big) \right) = 1. \tag{3.69}$$

If

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(z_n) \ll \frac{\rho_n}{\beta_n}, \qquad \frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(0) \ll \frac{\rho_n}{\beta_n},$$

then by (3.23) and (3.67), we have for any $\kappa > 0$

$$\lim_{n\to\infty} P\left(\frac{1-\kappa}{1+\kappa} < D_n < \frac{1+\kappa}{1-\kappa} \right) = 1. \tag{3.70}$$

Thus, the sequence $(D_n)_{n=1}^{\infty}$ is tight. If

$$t_n - t_n(z_n) \asymp \frac{\rho_n}{\beta_n}, \qquad \frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(0) \ll \frac{\rho_n}{\beta_n},$$

then by (3.25) and (3.67), we have for any $\kappa > 0$,

$$\lim_{n\to\infty} P\left(\frac{1-\kappa}{1+\kappa} \frac{Z_n(t_n - t_n(z_n))}{Z_n(0)} < D_n < \frac{1+\kappa}{1-\kappa} \frac{Z_n(t_n - t_n(z_n))}{Z_n(0)} \right) = 1. \tag{3.71}$$

Note that $Z_n(0)$ satisfies (3.10) and $Z_n(t_n - t_n(z_n))$ also satisfies (3.10) by Remark 27. Equation (3.71) thus implies that $(D_n)_{n=1}^{\infty}$ is tight. The remaining two cases

$$\frac{\rho_n^{2/3}}{\beta_n^{8/9}} \ll t_n - t_n(z_n) \ll \frac{\rho_n}{\beta_n}, \qquad t_n - t_n(0) \asymp \frac{\rho_n}{\beta_n},$$

and

$$t_n - t_n(z_n) \asymp \frac{\rho_n}{\beta_n}, \qquad t_n - t_n(0) \asymp \frac{\rho_n}{\beta_n},$$

follow by essentially the same argument as the second case, using (3.69) in place of (3.67). If $0 \in [a_n, b_n]$, then (3.70) holds true and thus $D_n \to 1$ in probability as $n \to \infty$. □

3.3 Proof of Propositions 23, 24, 25, and 26

In this section, we give the main arguments in the proofs of Propositions 23, 24, 25, and 26. We defer the proofs of several technical lemmas until Section 3.4 and the proof of the second moment estimates until Section 3.5.

3.3.1 A review of results from [75]

In this subsection, we will collect some of the results in [75] that will be used in the proofs. Suppose (3.9) and (3.10) hold.

3.1.1 Lemma 6.1 in [75] shows that

$$\lim_{r\to\infty} e^{-r^3/3}\int_0^\infty e^{r(\gamma_1+z)}Ai(\gamma_1+z)dz = 1. \tag{3.72}$$

Based on equations (6.5), (6.6) and Lemma 6.1 in [75], for any $\eta > 0$, there exists a constant $C_5 \geq 2$ sufficiently large such that

$$\left(1-\frac{\eta}{2}\right)e^{C_5^3/6} \leq \int_0^\infty e^{2^{-1/3}C_5(\gamma_1+y)}Ai(\gamma_1+y)dy \leq (1+\eta)e^{C_5^3/6}, \tag{3.73}$$

and

$$e^{-C_5^3/6}\frac{(1+\eta)\sqrt{2\pi}}{Ai'(\gamma_1)^2}\int_0^\infty Ai(\gamma_1+y)dy < \eta \tag{3.74}$$

hold. Furthermore, there exists a constant $C_6 \geq -2^{-1/3}\gamma_1$ sufficiently large such that

$$\int_{2^{1/3}C_6}^\infty e^{2^{-1/3}C_5(\gamma_1+y)}Ai(\gamma_1+y)dy < \frac{\eta}{2}e^{C_5^3/48}. \tag{3.75}$$

3.1.2 Fix $A \in \mathbb{R}$. Define

$$L_n^A = L_n - \frac{A}{\rho_n}. \tag{3.76}$$

Lemma 5.1 in [75] proves that there exists a constant C_7 such that the probability that some particle that is to the right of L_n^A at time 0 has a descendant alive in the population at time $C_7\rho_n^{-2}$ tends to 0 as $n \to \infty$. Moreover, according to the argument leading to (5.9) in [75], for $t_n \ll \rho_n/\beta_n$, it follows that with probability tending to 1 as $n \to \infty$, no particle that hits L_n^A before time $t_n - C_7\rho_n^{-2}$ has descendants alive at time t_n.

3.1.3 Consider the process in which particles are killed upon hitting L_n. If this process starts from a single particle at x, we denote the density of this process at time t by $p_t^{L_n}(x,y)$. Lemma 2.5 in [75] implies that if $x, y < L_n$ and

$$(2\beta_n)^{1/6}\big((L_n - x)^{1/2} + (L_n - y)^{1/2}\big) - 2^{-1/3}\beta_n^{2/3}t_n \to -\infty, \tag{3.77}$$

then there exits a constant C_8 such that

$$p_{t_n}^{L_n}(x,y) \leq C_4\beta_n^{1/3}e^{\rho_n x}Ai((2\beta_n)^{1/3}(L_n - x) + \gamma_1)e^{-\rho_n y}Ai((2\beta_n)^{1/3}(L_n - y) + \gamma_1). \tag{3.78}$$

Define

$$H_n(t) = L_n - \frac{\beta_n t^2}{9}. \tag{3.79}$$

Equation (5.5) in [75] states that if $x \leq H_n(t_n)$, $0 \leq \zeta_n \leq \beta_n t_n/2$ and $\beta_n^{-2/3} \ll t_n \ll \rho_n/\beta_n$, then

$$\int_{-\infty}^{L_n} p_{t_n}^{L_n}(x,y)e^{(\rho_n - \zeta_n)y}dy \ll e^{\rho_n x}e^{-\beta_n^2 t_n^3/73}. \tag{3.80}$$

Equation (5.6) in [75] states that if $x < L_n$, $0 \leq \zeta_n \leq \beta_n t_n/2$ and $\beta_n^{-2/3} \ll t_n \ll \rho_n/\beta_n$, then

$$\int_{-\infty}^{H_n(t_n)} p_{t_n}^{L_n}(x,y)e^{(\rho_n - \zeta_n)y}dy \ll e^{\rho_n x}e^{-\beta_n^2 t_n^3/73}. \tag{3.81}$$

3.1.4 Consider the process in which particles are killed upon hitting K. If the process starts with a single particle at $x < K$, we denote by $r_{x,n}^K(v)$ the rate at which particles hit K at time v and $r_{x,n}^K(u,t)$ the expected number of particles that are killed at K between times u and t. Then

$$r^K_{x,n}(u,t) = \int_u^t r^K_{x,n}(v)dv.$$

If $K = L_n$, then by (6.29) in [75], there exists a constant C_9 such that

$$r^{L_n}_{x,n}(v) \leq \frac{C_9(L_n - x)}{v^{3/2}} \exp\left(\rho_n x - \rho_n L_n - \frac{(L_n - x)^2}{2v} - 2^{-1/3}\beta_n^{2/3}\gamma_1 v\right). \tag{3.82}$$

Furthermore, for $K = L^A_n$, define $A_- = \max\{-A, 0\}$. By Lemma 2.13 in [75], we have for all $x < L_n$ and $0 \leq u < t$,

$$r^{L_n}_{x,n}(u,t) \lesssim e^{\rho_n x} e^{-\rho_n L^A_n} e^{-\beta_n^2 u^3/9} + \beta_n^{2/3}(t-u)e^{-\rho_n L^A_n} e^{\beta_n A_- t/\rho_n} e^{\rho_n x} Ai\big((2\beta_n)^{1/3}(L_n - x) + \gamma_1\big). \tag{3.83}$$

3.1.5 Suppose

$$\beta^{-2/3}\log^{1/3}\left(\frac{\rho}{\beta^{1/3}}\right) \ll t_n \ll \frac{\rho_n}{\beta_n}. \tag{3.84}$$

Let $f : \mathbb{R} \to [0, \infty)$ be a bounded measurable function. Define

$$\Phi_n(f) = \sum_{i \in \mathcal{N}_{t_n,n}} e^{\rho X_{i,n}(t_n)} f\big((2\beta_n)^{1/3}(L_n - X_{i,n}(t_n))\big). \tag{3.85}$$

According to (5.8) in [75], we have for any $\kappa > 0$,

$$\lim_{n\to\infty} P\bigg(\frac{1-\kappa}{Ai'(\gamma_1)^2}\left(\int_0^\infty f(z)Ai(\gamma_1 + z)\right)Z_n(0) < \Phi_n(f) < \frac{1+\kappa}{Ai'(\gamma_1)^2}\left(\int_0^\infty f(z)Ai(\gamma_1 + z)\right)Z_n(0)\bigg) = 1. \tag{3.86}$$

3.1.6 For $\beta_n^{-1/3} \ll x_n \ll \beta_n^{-1}$, consider the process started from a single particle at x_n. According to Lemma 2.14 in [75], there is a positive constant C_{10} such that for large enough n, the probability that the process survives until time $C_{10}/(\beta_n x_n)$ is bounded above by $2\beta_n x_n/\alpha$. Here, α is the constant appearing in assumption (3.3).

3.3.2 Notation

Here we introduce one more piece of notation which will be used throughout the rest of this chapter. Recall that $c_{0,n} = z_n/L_n^*$. We denote

$$c_n = \sqrt{1 - c_{0,n}}. \tag{3.87}$$

We can now write

$$t_n(z_n) = \frac{c_n \rho_n}{\beta_n}, \qquad L_n^* - z_n = \frac{c_n^2 \rho_n^2}{2\beta_n}. \tag{3.88}$$

The notation $c_{0,n}$ and c_n will be useful in simplifying expressions involving z_n, $L_n^* - z_n$ and $t_n(z_n)$. Therefore, we list some of the most useful formulas involving $c_{0,n}$ and c_n below. We see that for $z_n \in (L_n^\dagger, L_n^*)$,

$$-\frac{5}{4} < c_{0,n} < 1, \qquad 0 < c_n < \frac{3}{2}. \tag{3.89}$$

We also have the following equivalent asymptotic expressions:

$$|1 - c_n| = \frac{|c_{0,n}|}{1 + c_n} \asymp |c_{0,n}|, \tag{3.90}$$

$$|z_n| \gtrsim \sqrt{\frac{\rho_n}{\beta_n}} \Longleftrightarrow |c_{0,n}| \gtrsim \frac{\beta_n^{1/2}}{\rho_n^{3/2}}, \tag{3.91}$$

$$L_n^* - z_n \gg \frac{1}{\beta_n^{1/3}} \Longleftrightarrow c_n \gg \frac{\beta_n^{1/3}}{\rho_n}, \tag{3.92}$$

$$z_n - L_n^\dagger \gg \frac{1}{\beta_n^{1/3}} \Longleftrightarrow c_{0,n} + \frac{5}{4} \gg \frac{\beta_n^{2/3}}{\rho_n^2} \Longleftrightarrow \frac{9}{4} - c_n^2 \gg \frac{\beta_n^{2/3}}{\rho_n^2} \Longleftrightarrow \frac{3}{2} - c_n \gg \frac{\beta_n^{2/3}}{\rho_n^2}.$$

Moreover, if $|z_n| \gtrsim \sqrt{\rho_n/\beta_n}$ and z_n satisfies assumption (3.18), then

$$|c_n c_{0,n}| \gtrsim \frac{\beta_n^{1/2}}{\rho_n^{3/2}}. \tag{3.93}$$

Also if $|z_n| \ll \sqrt{\rho_n/\beta_n}$, then

$$c_n \asymp 1, \qquad |c_{0,n}| \ll \frac{\beta_n^{1/2}}{\rho_n^{3/2}}. \tag{3.94}$$

Table 3.2 might be helpful in keeping track of the asymptotic behavior of $c_{0,n}$ and c_n.

Table 3.2: *Asymptotic behavior of* $c_{0,n}$ *and* c_n

z_n	$c_{0,n}$	c_n	Other
$\|z_n\| \ll \sqrt{\rho_n/\beta_n}$	$\|c_{0,n}\| \ll \beta_n^{1/2}/\rho_n^{3/2}$	$c_n \asymp 1$	
$z_n \gtrsim \sqrt{\rho_n/\beta_n}$, $L_n^* - z_n \gg \beta_n^{-1/3}$	$c_{0,n} \gtrsim \beta_n^{1/2}/\rho_n^{3/2}$	$\beta_n^{1/3}/\rho_n \ll c_n \leq 1$	$c_n c_{0,n} \gtrsim \beta_n^{1/2}/\rho_n^{3/2}$
$-z_n \gtrsim \sqrt{\rho_n/\beta_n}$, $z_n - L_n^\dagger \gg \beta_n^{-1/3}$	$\|c_{0,n}\| \gtrsim \beta_n^{1/2}/\rho_n^{3/2}$	$3/2 - c_n \gg \beta_n^{2/3}/\rho_n^2$	$\|c_n c_{0,n}\| \gtrsim \beta_n^{1/2}/\rho_n^{3/2}$

In the rest of this chapter, to lighten the burden of notation, we will usually omit the subscript n in the notation. For example, we will write ρ in place of ρ_n, z in place of z_n and $g(z)$ in place of $g_n(z_n)$. However, it is important to keep in mind that these quantities do depend on n.

3.3.3 Proof of Proposition 25

In this subsection, we will prove Proposition 25 using first and second moment estimates. First, we have the following lemma which shows that $y \in [z-l, z+l]$ satisfies the restriction (3.18) with y in place of z.

Lemma 30. *For every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). For all $y \in [z-l, z+l]$, we have*

$$L^* - y \gg \beta^{-1/3} \tag{3.95}$$

and

$$y - L^{\dagger} \gg \beta^{-1/3}. \tag{3.96}$$

Next, we have the following lemmas which control the difference between $t(z)$ and $t(y)$, and $g(z)$ and $g(y)$ for $y \in [z-l, z+l]$.

Lemma 31. *For every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). For all $y \in [z-l, z+l]$, we have*

$$|t(y) - t(z)| = o(\beta^{-2/3}). \tag{3.97}$$

Moreover, uniformly for all $y \in [z-l, z+l]$,

$$\lim_{n\to\infty} \frac{t(y)}{t(z)} = 1. \tag{3.98}$$

Lemma 32. *For every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). Then for all $y \in [z-l, z+l]$, we have*

$$|g(y) - g(z)| \lesssim 1. \tag{3.99}$$

The following lemma controls the first moment.

Lemma 33. *For every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). Let $s \geq 0$ and*

$$t = t(z) - s, \quad x = L^* - w, \quad s_y = t(y) - t(z) + s.$$

For all $w \in \mathbb{R}$, $s < t(z)$ and $y \in \mathbb{R}$,

$$p_t(x, y) \leq \frac{1}{\sqrt{2\pi t}} \exp\left(g(y) - \rho w + \beta w s_y - \frac{\beta^2}{6} s_y^3\right). \tag{3.100}$$

Furthermore, if $s \asymp \beta^{-2/3}$, then for all $|w| \lesssim \beta^{-1/3}$ and $y \in [z-l, z+l]$,

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(g(y) - \rho w + \beta w s_y - \frac{\beta^2}{6} s_y^3 + o(1) \right). \tag{3.101}$$

A key step in the proof of Proposition 25 is the following second moment estimate. Note that it is rare for a particle to drift to the right of L but once it does so, it will generate a large number of descendants in the interval $\mathcal{I}$ at time t, which ruins the second moment argument. Therefore, we need to consider a truncated second moment estimate where particles are killed at L. For this process, we denote by $N_t^L(\mathcal{I})$ the number of particles in the interval $\mathcal{I}$ at time t.

Lemma 34. *Consider the process which starts from a single particle at x such that $0 \le L - x \lesssim \beta^{-1/3}$. For every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). Consider intervals $\mathcal{I}$ defined in (3.50). Suppose*

$$s \asymp \beta^{-2/3}, \qquad t = t(z) - s.$$

Then for the process in which particles are killed upon hitting L, we have

$$E[N_t^L(\mathcal{I})^2] \lesssim \frac{\beta^{2/3}}{\rho^4} e^{\rho x + \rho L - 2\rho L^*} \left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \right)^2. \tag{3.102}$$

To prove Proposition 25, we need one more technical lemma. Let $\eta > 0$. Choose constants $C_5 \ge 2$ and $C_6 \ge -2^{-1/3}\gamma_1$ such that (3.73)-(3.75) hold. Then, C_5 and C_6 satisfy

$$(1-\eta) e^{C_5^3/6} \le \int_0^{2^{1/3} C_6} e^{2^{-1/3} C_5 (\gamma_1 + y)} Ai(\gamma_1 + y) dy \le (1+\eta) e^{C_5^3/6}. \tag{3.103}$$

The next lemma is a slight generalization of Lemma 6.2 in [75].

Lemma 35. *Suppose (3.9) and (3.10) hold. Let $\eta > 0$, and choose positive constants C_5 and C_6 such that (3.73)-(3.75) and (3.103) hold. Let $s = C_5 \beta^{-2/3}$ and $u = t - t(z) + s$ for* *all z. If*

$$\beta^{-2/3}\log^{1/3}\left(\frac{\rho}{\beta^{1/3}}\right) \ll u \ll \frac{\rho}{\beta}, \tag{3.104}$$

then

$$\lim_{n\to\infty} P\left(\frac{1-2\eta}{Ai'(\gamma_1)^2}Z(0) \le \exp\left(\frac{\rho^2 s}{2} - \frac{\beta^2 s^3}{6}\right)\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s)X_j(u)}1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}} \le \frac{1+2\eta}{Ai'(\gamma_1)^2}Z(0)\right) = 1. \tag{3.105}$$

Moreover, for every z satisfying (3.18), (3.46) and (3.47), choose l according to (3.49). For every $y \in [z-l, z+l]$, let $s_y = t(y) - t(z) + s$ and

$$\Gamma_y = \exp\left(\frac{\rho^2 s_y}{2} - \frac{\beta^2 s_y^3}{6}\right)\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)}1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}}. \tag{3.106}$$

If (3.104) holds, then uniformly for all $y \in [z-l, z+l]$,

$$\lim_{n\to\infty} P\left(\frac{1-3\eta}{Ai'(\gamma_1)^2}Z(0) \le \Gamma_y \le \frac{1+3\eta}{Ai'(\gamma_1)^2}Z(0)\right) = 1. \tag{3.107}$$

Lemmas 31, 33 and 35 will be proved in Section 3.4. Since the proof of Lemma 34 is rather technical and tedious, we defer it until Section 3.5. With the help of the above lemmas, we will follow the same strategy as the proof of Proposition 2.2 of [75] to prove Proposition 25.

Proof of Proposition 25. Let

$$s = C_5\beta^{-2/3}, \qquad u = t - t(z) + s. \tag{3.108}$$

By (3.20), we have $t - t(z) < u < t$ for n sufficiently large. For all $y \in \mathcal{I}$, denote

$$s_y = t(y) - t(z) + s.$$

By Lemma 31, for all $y \in \mathcal{I}$, we see that $s_y = (C_5 \pm o(1))\beta^{-2/3}$ or equivalently, uniformly for all $y \in \mathcal{I}$,

$$\lim_{n\to\infty} \beta^{2/3} s_y = C_5. \tag{3.109}$$

Figure 3.3 might be helpful for keeping track of notation.

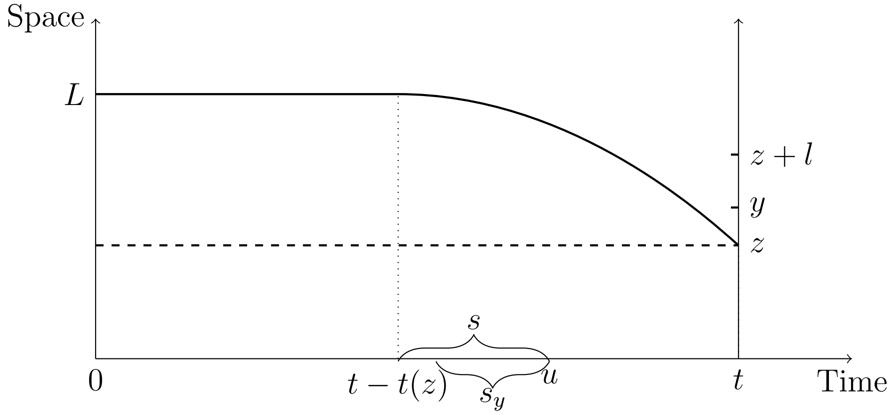

Figure 3.3: Notation when $z > 0$

Recall that $H(u) = L - \beta u^2/9$ by (3.79). For a particle $i \in \mathcal{N}_t$, recall that $\{X_i(v), 0 \le v \le t\}$ denotes its past trajectory. Define

$$S_1 = \big\{i \in \mathcal{N}_t : X_i(u) \le H(u), X_i(v) < L \text{ for all } v \in [0, u]\big\},$$
$$S_2 = \Big\{i \in \mathcal{N}_t : H(u) < X_i(u) \le L - C_6\beta^{-1/3}, X_i(v) < L \text{ for all } v \in [0, u]\Big\},$$
$$S_3 = \Big\{i \in \mathcal{N}_t : L - C_6\beta^{-1/3} < X_i(u) < L, X_i(v) > L \text{ for some } v \in (u, t)\Big\},$$
$$S_4 = \Big\{i \in \mathcal{N}_t : L - C_6\beta^{-1/3} < X_i(u) < L, X_i(v) \le L \text{ for all } v \in (u, t)\Big\},$$
$$S_5 = \mathcal{N}_t \setminus \big(S_1 \cup S_2 \cup S_3 \cup S_4\big).$$

For $j = 1, ..., 5$, write

$$\Theta_j = \sum_{i \in S_j} 1_{\{X_i(t) \in \mathcal{I}\}}.$$

Then

$$N_t(\mathcal{I}) = \sum_{j=1}^{5} \Theta_j.$$

We are going to show that the major contribution comes from Θ_4, and Θ_4 is concentrated around its mean. Define $(\mathcal{F}_t, t \geq 0)$ to be the natural filtration associated with the BBM process.

Let us first consider Θ_1. By inequality (3.100) and Tonelli's theorem, we have

$$\begin{aligned} E[\Theta_1 | \mathcal{F}_u] &= \sum_{j \in \mathcal{N}_u} \int_{\mathcal{I}} p_{t-u}(X_j(u), y) 1_{\{\forall v \in [0,u], X_j(v) < L\}} 1_{\{X_j(u) \leq H(u)\}} dy \\ &\leq \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi(t-u)}} \exp\left(g(y) - \rho L^* + \frac{\rho^2 s_y}{2} - \frac{\beta^2 s_y^3}{6} \right) \\ &\quad \times \sum_{j \in \mathcal{N}_u} e^{(\rho - \beta s_y) X_j(u)} 1_{\{\forall v \in [0,u], X_j(v) < L\}} 1_{\{X_j(u) \leq H(u)\}} dy. \end{aligned} \tag{3.110}$$

We denote by H_1 the summation on the last line of (3.110). By (3.51), we have $\beta^{-2/3} \ll u \ll \rho/\beta$. Furthermore, by equation (3.97), we see that $0 \leq \beta s_y \ll \beta u$ for all $y \in \mathcal{I}$. Therefore, by (3.81), we have for all $y \in \mathcal{I}$,

$$E\left[H_1 | \mathcal{F}_0\right] \ll e^{-\beta^2 u^3 / 73} Y(0). \tag{3.111}$$

Since $t - u = t(z) - s \gg \beta^{-2/3}$, equation (3.98) implies that uniformly for all $y \in \mathcal{I}$,

$$\lim_{n \to \infty} \frac{t(y)}{t-u} = 1. \tag{3.112}$$

Also, for all $y \in \mathcal{I}$, we see from (3.51) that

$$\rho^2 s_y / 2 \ll \beta^2 u^3. \tag{3.113}$$

Thus, from (3.110)–(3.113), we have

$$E[\Theta_1|\mathcal{F}_0] \ll e^{-\rho L^* - \beta^2 u^3/74} Y(0) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy.$$

Then by the conditional Markov's inequality, we can deduce that for any $\eta > 0$, if n is sufficiently large,

$$P\Bigg(\Theta_1 > \eta e^{-\rho L^*} Z(0) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \Bigg| \mathcal{F}_0\Bigg) \leq \frac{Y(0)}{\eta Z(0)} e^{-\beta^2 u^3/74}.$$

Based on assumptions (3.9) and (3.10), we have that for any $\eta > 0$,

$$\lim_{n\to\infty} P\Bigg(\Theta_1 > \eta e^{-\rho L^*} Z(0) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\Bigg) = 0. \tag{3.114}$$

We next consider Θ_2. By (3.100) and Tonelli's theorem again, we get

$$\begin{aligned} E[\Theta_2|\mathcal{F}_0] \leq \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi(t-u)}} \exp\left(g(y) - \rho L^* + \frac{\rho^2 s_y}{2} - \frac{\beta^2 s_y^3}{6}\right) \\ \times E\left[\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)} 1_{\{\forall v\in[0,u], X_j(v)<L\}} 1_{\{H(u)<X_j(u)\leq L-C_6\beta^{-1/3}\}} \middle| \mathcal{F}_0\right] dy. \end{aligned} \tag{3.115}$$

We can separate the expectation in the integrand into two parts by writing

$$\begin{aligned} &E\left[\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)} 1_{\{\forall v\in[0,u], X_j(v)<L\}} 1_{\{H(u)<X_j(u)\leq L-C_6\beta^{-1/3}\}} \middle| \mathcal{F}_0\right] \\ &= E\left[\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)} 1_{\{\forall v\in[0,u], X_j(v)<L\}} 1_{\{X_j(0)\leq H(u)\}} 1_{\{H(u)<X_j(u)\leq L-C_6\beta^{-1/3}\}} \middle| \mathcal{F}_0\right] \\ &\quad + E\left[\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)} 1_{\{\forall v\in[0,u], X_j(v)<L\}} 1_{\{H(u)<X_j(0)<L\}} 1_{\{H(u)<X_j(u)\leq L-C_6\beta^{-1/3}\}} \middle| \mathcal{F}_0\right] \\ &=: E[H_2|\mathcal{F}_0] + E[H_3|\mathcal{F}_0]. \end{aligned} \tag{3.116}$$

Note that when $H(u) < x < L$ and $H(u) < y < L - C_6\beta^{-1/3}$,

$$(2\beta)^{1/6}\big((L-x)^{1/2} + (L-y)^{1/2}\big) \le (2\beta)^{1/6} \cdot 2 \cdot (L - H(u))^{1/2} = 2^{7/6}3^{-1}\beta^{2/3}u.$$

Since $2^{7/6}3^{-1} < 2^{-1/3}$ and $\beta^{2/3}u \gg 1$, equation (3.77) is satisfied. Thus, in (3.116), we can upper bound the first expectation by (3.80) and upper bound the second expectation by (3.78). We have that for all $y \in \mathcal{I}$,

$$E[H_2 + H_3|\mathcal{F}_0] \le e^{-\beta^2u^3/73}Y(0) + C_8Z(0)\int_{H(u)}^{L-C_6\beta^{-1/3}} e^{-\beta s_y v}\beta^{1/3}Ai\Big((2\beta)^{1/3}(L-v)+\gamma_1\Big)dv. \tag{3.117}$$

Substituting v with $L-(2\beta)^{-1/3}r$, by (3.75) and (3.109), we have for n sufficiently large, for all $y \in \mathcal{I}$,

$$\begin{aligned}
&\int_{H(u)}^{L-C_6\beta^{-1/3}} e^{-\beta s_y v}\beta^{1/3}Ai\Big((2\beta)^{1/3}(L-v)+\gamma_1\Big)dv \\
&\quad\le 2^{-1/3}e^{-\rho^2 s_y/2}\int_{2^{1/3}C_6}^{\infty} e^{2^{-1/3}\beta^{2/3}s_y(\gamma_1+r)}Ai(\gamma_1+r)dr \\
&\quad\le (1+\eta)2^{-1/3}e^{-\rho^2 s_y/2}\int_{2^{1/3}C_6}^{\infty} e^{2^{-1/3}C_5(\gamma_1+r)}Ai(\gamma_1+r)dr \\
&\quad\le (1+\eta)2^{-1/3}e^{-\rho^2 s_y/2}\frac{\eta}{2}e^{C_5^3/48}.
\end{aligned} \tag{3.118}$$

Combining the above formula with (3.115) and (3.117), we have

$$\begin{aligned}
E[\Theta_2|\mathcal{F}_0] \le\ & e^{-\beta^2u^3/73}Y(0)e^{-\rho L^*}\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi(t-u)}}\exp\left(g(y) + \frac{\rho^2 s_y}{2} - \frac{\beta^2 s_y^3}{6}\right)dy \\
&+ \frac{C_8\eta(1+\eta)}{2^{4/3}}Z(0)e^{-\rho L^*}\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi(t-u)}}\exp\left(g(y) - \frac{\beta^2 s_y^3}{6} + \frac{C_5^3}{48}\right)dy.
\end{aligned}$$

By (3.113), if n is sufficiently large, then for all $y \in \mathcal{I}$,

$$\frac{\rho^2 s_y}{2} - \frac{\beta^2u^3}{73} \le -\frac{\beta^2u^3}{74}. \tag{3.119}$$

Also note that for n large enough, $\beta^2 s_y^3/6 \geq C_5^3/48$ for all $y \in \mathcal{I}$. Then by (3.112), we obtain that for n sufficiently large

$$E[\Theta_2|\mathcal{F}_0] \leq \left(e^{-\beta^2 u^3/74}Y(0) + \frac{C_8\eta(1+\eta)}{2^{4/3}}Z(0)\right)e^{-\rho L^*}\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy.$$

By (3.51), we have $u \gg \rho^{2/3}/\beta^{8/9}$, and therefore (3.9) and (3.10) imply that $e^{-\beta^2 u^3/74}Y(0)/Z(0) \to_p 0$. Thus, by the conditional Markov's inequality,

$$\limsup_{n\to\infty} P\left(\Theta_2 > \eta^{1/2}e^{-\rho L^*}Z(0)\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\right) \leq \frac{C_8\eta^{1/2}(1+\eta)}{2^{4/3}}. \tag{3.120}$$

We now consider Θ_3 and Θ_4. According to (3.101), (3.112) and Tonelli's theorem, there is a sequence of $\mathcal{F}_u$-measurable random variables $\{\theta_n\}_{n=1}^\infty$ which converges uniformly to 0 as n goes to infinity such that

$$\begin{aligned}E[\Theta_3+\Theta_4|\mathcal{F}_u] = (1+\theta_n)\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}}\exp\left(g(y) - \rho L^* + \frac{\rho^2 s_y}{2} - \frac{\beta^2 s_y^3}{6}\right)\\ \times \sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_y)X_j(u)}1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}}dy.\end{aligned}$$

We can deduce from (3.107) that

$$\begin{aligned}\lim_{n\to\infty} P\bigg(\frac{1-4\eta}{Ai'(\gamma_1)^2}e^{-\rho L^*}Z(0)\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy \leq E[\Theta_3+\Theta_4|\mathcal{F}_u]\\ \leq \frac{1+4\eta}{Ai'(\gamma_1)^2}e^{-\rho L^*}Z(0)\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\bigg) = 1.\end{aligned} \tag{3.121}$$

Next, we will estimate Θ_3 and Θ_4 individually. Note that Θ_3 accounts for particles that reach L between times u and t and then drift back to $\mathcal{I}$. Consider a process which starts from a single particle at $L - C_6\beta^{-1/3} < x < L$. Suppose we kill particles upon hitting L. For $v \in [0, t-u]$, recall $r_x^L(v)$ is the rate at which particles hit L at time v. We further denote by $m^z(v)$ the expected number of descendants in $\mathcal{I}$ at time t of a particle that reaches L at time $u+v$. Consider the process in which there is one particle at x at time u without killing. Then the expected number of particles in $\mathcal{I}$ at time t whose trajectories cross L between times u and t is

$$\int_0^{t-u} r_x^L(v) m^z(v) dv.$$

From the definition of $m^z(v)$, we have

$$m^z(v) = \int_{\mathcal{I}} p_{t-u-v}(L, y) dy.$$

Setting $w = (2\beta)^{-1/3}\gamma_1$ and substituting $s+v$ in place of s in equation (3.100), we have

$$\begin{aligned} m^z(v) \leq \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi(t-u-v)}} \exp\Big(& g(y) - \rho(2\beta)^{-1/3}\gamma_1 + \beta(2\beta)^{-1/3}\gamma_1\big(t(y) - (t-u-v)\big) \\ & - \frac{\beta^2}{6}\big(t(y) - (t-u-v)\big)^3\Big) dy. \end{aligned} \tag{3.122}$$

Note that $t(y) - (t-u-v) = s_y + v$. Combining (3.122) with (3.82), we have

$$\begin{aligned} r_x^L(v) m^z(v) \leq \frac{C_9(L-x)}{v^{3/2}} \frac{1}{\sqrt{2\pi(t-u-v)}} \int_{\mathcal{I}} \exp\Big(& \rho x - \rho L^* - \frac{(L-x)^2}{2v} + g(y) \\ & + 2^{-1/3}\beta^{2/3}\gamma_1 s_y - \frac{\beta^2}{6}(s_y+v)^3 \Big) dy. \end{aligned}$$

We are going to deal with the terms involving s_y by using an argument similar to the one leading to (3.118). Notice that $(s_y+v)^3 \geq s_y^3$. By (3.109), the dominated convergence theorem and Tonelli's theorem, for n large enough, we obtain

$$\begin{aligned} & \int_0^{t-u} r_x^L(v) m^z(v) dv \\ & \leq \frac{C_9(1+\eta)}{\sqrt{2\pi}} \exp\left(\rho x - \rho L^* + 2^{-1/3}\beta^{2/3}\gamma_1 s - \frac{\beta^2 s^3}{6}\right) \int_{\mathcal{I}} e^{g(y)} \\ & \quad \times \left(\int_0^{(t-u)/2} \frac{L-x}{v^{3/2}} \frac{1}{\sqrt{(t-u-v)}} \exp\left(- \frac{(L-x)^2}{2v}\right) dv + \int_{(t-u)/2}^{t-u} \frac{L-x}{v^{3/2}} \frac{1}{\sqrt{t-u-v}} dv \right). \end{aligned} \tag{3.123}$$

Lemma 4.1 in [75] states that for $a > 0$ and $b > 0$,

$$\int_0^\infty \frac{1}{v^{3/2}} e^{-b^2/av} dv = \frac{\sqrt{\pi a}}{b}.$$

Thus we have

$$\begin{aligned}
&\int_0^{(t-u)/2} \frac{L-x}{v^{3/2}} \frac{1}{\sqrt{(t-u-v)}} \exp\Big(-\frac{(L-x)^2}{2v}\Big) dv \\
&\quad \le \sqrt{\frac{2}{t-u}} \int_0^\infty \frac{L-x}{v^{3/2}} \exp\Big(-\frac{(L-x)^2}{2v}\Big) dv \\
&\quad = \frac{2\sqrt{\pi}}{\sqrt{t-u}}.
\end{aligned} \tag{3.124}$$

Noticing that $L - x \le C_6 \beta^{-1/3} \ll \sqrt{t-u}$, we see that for n sufficiently large

$$\int_{(t-u)/2}^{t-u} \frac{L-x}{v^{3/2}\sqrt{t-u-v}} dv \le \frac{2^{3/2}(L-x)}{(t-u)^{3/2}} \int_{(t-u)/2}^{t-u} \frac{1}{\sqrt{t-u-v}} dv = \frac{4(L-x)}{t-u} \le \frac{1}{\sqrt{t-u}}. \tag{3.125}$$

By equations (3.112), (3.123), (3.124) and (3.125), since $\gamma_1 < 0$, we have for n large enough,

$$\begin{aligned}
&\int_0^{t-u} r_x^L(v) m^z(v) dv \\
&\quad \le \big(2\sqrt{\pi}+1\big) C_9 (1+\eta) \exp\bigg(\rho x - \rho L^* + 2^{-1/3}\beta^{2/3}\gamma_1 s - \frac{\beta^2 s^3}{6}\bigg) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \\
&\quad \le \big(2\sqrt{\pi}+1\big) C_9 (1+\eta) \exp\bigg(\rho x - \rho L^* - \frac{\beta^2 s^3}{6}\bigg) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy.
\end{aligned}$$

Summing over all particles at time u, we have for n large enough

$$E[\Theta_3 | \mathcal{F}_u] \le \big(2\sqrt{\pi}+1\big) C_9 (1+\eta) Y(u) \exp\bigg(-\rho L^* - \frac{\beta^2 s^3}{6}\bigg) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \tag{3.126}$$

Furthermore, equation (6.31) in [75] states that

$$\lim_{n\to\infty} P\bigg(Y_n(u) < \frac{1+\eta}{Ai'(\gamma_1)^2} Z_n(0) \int_0^\infty Ai(\gamma_1 + z) dz \bigg) = 1. \tag{3.127}$$

Recall that $s = C_5\beta^{-2/3}$. Combining (3.126) with (3.74) and (3.127), we have

$$\lim_{n\to\infty} P\Big(E[\Theta_3|\mathcal{F}_u] \geq \Big(\sqrt{2}+\frac{1}{\sqrt{2\pi}}\Big)C_9\eta(1+\eta)e^{-\rho L^*}Z(0)\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\Big) = 0. \quad (3.128)$$

By the conditional Markov's inequality,

$$\limsup_{n\to\infty} P\Big(\Theta_3 > \eta^{1/2}e^{-\rho L^*}Z(0)\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t_y}}e^{g(y)}dy\Big) \leq \Big(\sqrt{2}+\frac{1}{\sqrt{2\pi}}\Big)C_9\eta^{1/2}(1+\eta). \quad (3.129)$$

Next, we are going to show that Θ_4 is concentrated around its mean. By (3.121) and (3.128), letting

$$C(\eta) = 4\eta + \Big(\sqrt{2}+\frac{1}{\sqrt{2\pi}}\Big)C_9\eta(1+\eta)Ai'(\gamma_1)^2,$$

we see that

$$\lim_{n\to\infty} P\Big(E[\Theta_4|\mathcal{F}_u] \geq \frac{1-C(\eta)}{Ai'(\gamma_1)^2}e^{-\rho L^*}Z(0)\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\Big) = 1. \quad (3.130)$$

Considering the process in which particles are killed upon hitting L, we can bound the conditional variance of Θ_4 by Lemma 34. We get

$$\begin{aligned}
\mathrm{Var}(\Theta_4|\mathcal{F}_u) &\leq \sum_{j\in\mathcal{N}_u} E_{X_j(u)}\big[N_t^L(\mathcal{I})^2\big]1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}} \\
&\lesssim \sum_{j\in\mathcal{N}_u}\frac{\beta^{2/3}}{\rho^4}e^{\rho X_i(u)+\rho L-2\rho L^*}\Big(\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\Big)^2 1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}} \\
&\leq \frac{\beta^{2/3}}{\rho^4}e^{-2\rho L^*+\rho L}Y(u)\Big(\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\Big)^2.
\end{aligned}$$

Then by Chebyshev's inequality,

$$P\Big(\Big|\Theta_4 - E[\Theta_4|\mathcal{F}_u]\Big| > \eta e^{-\rho L^*}Z(0)\int_{\mathcal{I}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\Big|\mathcal{F}_u\Big) \leq \frac{\beta^{2/3}e^{\rho L}}{\eta^2\rho^4}\frac{Y(u)}{Z(0)^2}. \quad (3.131)$$

On account of (3.10) and (3.127), as $n \to \infty$,

$$\frac{\beta^{2/3} e^{\rho L}}{\rho^4} \frac{Y(u)}{Z(0)^2} \to_p 0. \tag{3.132}$$

As a result, by equations (3.121), (3.130), (3.131) and (3.132), we obtain

$$\lim_{n\to\infty} P\Bigg(\bigg|\Theta_4 - \frac{1}{Ai'(\gamma_1)^2} e^{-\rho L^*} Z(0) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\bigg| \\ \leq \left(\eta + \frac{C(\eta)}{Ai'(\gamma_1)^2}\right) e^{-\rho L^*} Z(0) \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\Bigg) = 1. \tag{3.133}$$

It remains to consider Θ_5. Define S_5^* to be the set consists of particles whose trajectories cross L before time u, so

$$S_5^* = \big\{ i \in \mathcal{N}_t : X_i(v) \geq L \text{ for some } v \in [0, u] \big\}.$$

We observe that $S_5 \subseteq S_5^*$. Note that $u + 2C_7\rho^{-2} \leq t$ since $t - u = t(z) - s \gg \beta^{-2/3}$ by (3.20). According to 3.1.2, the probability that particles that either are to the right of L at time 0 or hit L before time u have descendants alive at time t goes to 0 as n goes to infinity. Therefore,

$$\lim_{n\to\infty} P(\Theta_5 = 0) = 1. \tag{3.134}$$

Consequently, for any $\kappa > 0$, by choosing η appropriately, equation (3.52) follows from (3.114), (3.120), (3.129), (3.133) and (3.134). □

3.3.4 Proof of Proposition 26

In this subsection, we will prove Proposition 26 with the help of Proposition 25. Before starting the proof, we need two more lemmas to control the number of particles that are far away from z.

Lemma 36. *Consider z such that (3.18) holds and $|z| \gtrsim \sqrt{\rho/\beta}$. For any $\eta > 0$, there exists a constant C_{11} large enough such that for*

$$d = \frac{C_{11}}{|c_0|\rho}, \tag{3.135}$$

the following hold:

1. *The constant satisfies*

$$C_{11} > 4, \qquad e^{-C_{11}/2} < \eta. \tag{3.136}$$

2. *If $z > 0$ for all n, then for n sufficiently large*

$$\int_{z+d}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < \eta \int_{z}^{z+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \tag{3.137}$$

3. *If $z < 0$ for all n, then for n sufficiently large*

$$\int_{-\infty}^{z-d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < \eta \int_{z-d}^{z} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \tag{3.138}$$

Since we expect that the density of the number of particles near y is roughly proportional to $e^{g(y)}/\sqrt{2\pi t(y)}$, Lemma 36 indicates that most particles in $[z, L^*]$ are in $[z, z+d]$, while most particles in $(-\infty, z]$ are in $[z-d, z]$.

Suppose (3.18) holds and $|z| \gtrsim \sqrt{\rho/\beta}$. For any $\eta > 0$, choose d according to Lemma 36. Note that (3.49) holds with d in place of l, and for l satisfying (3.53), we have $2d < l$ for n sufficiently large. Denote

$$\zeta = \begin{cases} z + 2d & \text{if } z > 0, \\ z - 2d & \text{if } z < 0, \end{cases}$$

Lemma 37. *Consider z such that (3.47) holds and $|z| \gtrsim \sqrt{\rho/\beta}$. Let*

$$s \asymp \beta^{-2/3}, \quad t = t(z) - s, \quad x \leq L.$$

1. *Suppose $z > 0$ for all n, and z satisfies (3.18). Choose d according to (3.135). For all $y \in [\zeta, \infty)$, we have for n sufficiently large,*

$$p_t(x,y) \leq p_t(x,\zeta)\exp\bigg(-\frac{\rho}{2}(1-c)(y-\zeta)\bigg). \tag{3.139}$$

2. *Suppose $z < 0$ for all n. Write $x = L^* - w$. For all y, we have for n sufficiently large,*

$$p_t(x,y) \leq \frac{1}{\sqrt{2\pi t}}\exp\bigg(g(z) - (c-1)\rho(z-y) - \rho w + \beta s w - \frac{\beta^2 s^3}{6}\bigg). \tag{3.140}$$

Furthermore, let $s_\zeta = t(\zeta) - t(z) + s$. If $y \leq \zeta$, then for n sufficiently large,

$$p_t(x,y) \leq \frac{1}{\sqrt{2\pi t}}\exp\bigg(g(\zeta) - (c-1)\rho(\zeta-y) - \rho w + \beta s_\zeta w - \frac{\beta^2 s_\zeta^3}{6}\bigg). \tag{3.141}$$

Proof of Proposition 26. Let us first consider the case $|z| \gtrsim \sqrt{\rho/\beta}$. Define s, u and $H(u)$ as in (3.108) and (3.79). Denote

$$\mathcal{K} = \begin{cases} [z,\zeta] & \text{if } z > 0, \\ [\zeta, z] & \text{if } z < 0. \end{cases}$$

Define

$$\begin{aligned} S_1 &= \big\{i \in \mathcal{N}_t : X_i(t) \in \mathcal{K}\big\}, \\ S_2 &= \big\{i \in \mathcal{N}_t \setminus S_1 : X_i(v) \geq L \text{ for some } v \in [0,u]\big\}, \\ S_3 &= \big\{i \in \mathcal{N}_t \setminus (S_1 \cup S_2) : X_i(u) \leq L - C_6\beta^{-1/3}\big\}, \\ S_4 &= \big\{i \in \mathcal{N}_t \setminus (S_1 \cup S_2) : L - C_6\beta^{-1/3} < X_i(u) < L\big\}. \end{aligned}$$

For $j = 1, ..., 4$, write

$$\Xi_j = \sum_{i \in S_j} 1_{\{X_i(t) \in \mathcal{J}\}}.$$

Then

$$N_t(\mathcal{J}) = \sum_{j=1}^{4} \Xi_j.$$

We will show that compared with Ξ_1, the terms Ξ_2, Ξ_3 and Ξ_4 are negligible.

We first consider Ξ_1. Since $2d$ satisfies the restriction (3.49) in Proposition 25, according to (3.52), (3.137) and (3.138)

$$\lim_{n\to\infty} P\bigg(\frac{1-\eta}{(1+\eta)Ai'(\gamma_1)^2}e^{-\rho L^*}Z(0)\int_{\mathcal{J}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy \le \Xi_1 \\ \le \frac{1+\eta}{Ai'(\gamma_1)^2}e^{-\rho L^*}Z(0)\int_{\mathcal{J}}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\bigg) = 1. \quad (3.142)$$

For Ξ_2, since $u + C_7\rho^{-2} \le t$, according to 3.1.2, the probability that particles that either are to the right of L at time 0 or hit L before time u have descendants alive at time t goes to 0 as n goes to infinity. Therefore,

$$\lim_{n\to\infty} P(\Xi_2 = 0) = 1. \quad (3.143)$$

It remains to consider Ξ_3 and Ξ_4. Let us first consider the case when $z > 0$. Recall the definition of H_1 in (3.110) and H_2, H_3 in (3.116). Since (3.49) holds with $2d$ in place of l, by inequalities (3.100), (3.139) and Tonelli's theorem, for n large enough, we have

$$\begin{aligned} E[\Xi_3|\mathcal{F}_u] &= \sum_{j\in\mathcal{N}_u}\int_{\zeta}^{z+l} p_{t-u}(X_j(u),y)1_{\{\forall v\in[0,u],X_j(v)<L\}}1_{\{X_j(u)\le L-C_6\beta^{-1/3}\}}dy \\ &\le \sum_{j\in\mathcal{N}_u} p_{t-u}(X_j(u),\zeta)1_{\{\forall v\in[0,u],X_j(v)<L\}}1_{\{X_j(u)\le L-C_6\beta^{-1/3}\}}\int_{\zeta}^{z+l} e^{-\rho(1-c)(y-\zeta)/2}dy \\ &\le \frac{1}{\sqrt{2\pi(t-u)}}\exp\left(g(\zeta)-\rho L^* + \frac{\rho^2 s_\zeta}{2} - \frac{\beta^2 s_\zeta^3}{6}\right)\int_{\zeta}^{z+l} e^{-\rho(1-c)(y-\zeta)/2}dy \\ &\quad \times (H_1+H_2+H_3) \end{aligned} \quad (3.144)$$

According to the choice of d, since $c \in (0,1)$ when $z > 0$, we have

$$\frac{2}{\rho(1-c)} = \frac{2(1+c)}{\rho c_0} \leq \frac{4}{\rho c_0} < d.$$

Also, by (3.98), uniformly for all $y \in [z, \zeta]$, we have $t - u > t(y)/(1+\eta)$ for sufficiently large n. Combining this observation with the fact that $g(y)$ is decreasing on $(0, L^*)$ and (3.137), we have

$$\begin{aligned}\frac{e^{g(\zeta)}}{\sqrt{2\pi(t-u)}} \int_{\zeta}^{z+l} e^{-\rho(1-c)(y-\zeta)/2} dy &\leq \frac{1}{\sqrt{2\pi(t-u)}} \frac{2}{\rho(1-c)} e^{g(\zeta)} \\ &\leq \int_{\zeta-d}^{\zeta} \frac{\sqrt{1+\eta}}{\sqrt{2\pi t(y)}} e^{g(y)} dy \\ &\leq \eta\sqrt{1+\eta} \int_{\mathcal{J}\cap(-\infty, L^*]} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \qquad (3.145)\end{aligned}$$

Since (3.49) in Proposition 25 holds with $2d$ in place of l, equations (3.111), (3.117) and (3.118) hold with s_ζ in place of s_y. By (3.111), (3.117), (3.118), (3.144) and (3.145), along with (3.119) with ζ in place of y, we get

$$E[\Xi_3|\mathcal{F}_0] < \eta\sqrt{1+\eta} e^{-\rho L^*} \left(2e^{-\beta^2 u^3/74} Y(0) + \frac{C_8 \eta(1+\eta)}{2^{4/3}} Z(0)\right) \int_{\mathcal{J}\cap(-\infty, L^*]} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \qquad (3.146)$$

Therefore, equations (3.9), (3.10), (3.146) and the conditional Markov's inequality imply

$$\limsup_{n\to\infty} P\left(\Xi_3 > \sqrt{\eta(1+\eta)} e^{-\rho L^*} Z(0) \int_{\mathcal{J}\cap(-\infty, L^*]} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\right) \leq \frac{C_8 \eta^{3/2}(1+\eta)}{2^{4/3}}. \qquad (3.147)$$

As for Ξ_4, according to the argument leading to (3.144), we have

$$E[\Xi_4|\mathcal{F}_u] \leq \frac{1}{\sqrt{2\pi(t-u)}} e^{g(\zeta)-\rho L^*} \int_{\zeta}^{z+l} e^{-\rho(1-c)(y-\zeta)/2} dy \times \Gamma_\zeta,$$

where Γ_ζ was defined in (3.106). By equations (3.107) and (3.145), and the conditional

Markov's inequality, we get

$$\limsup_{n\to\infty} P\left(\Xi_4 > \sqrt{\eta(1+\eta)}e^{-\rho L^*} Z(0) \int_{\mathcal{J}\cap(-\infty,L^*]} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\right) \le \frac{(1+3\eta)\sqrt{\eta}}{Ai'(\gamma_1)^2}. \quad (3.148)$$

As a result, when $z > 0$, for any $\kappa > 0$, by choosing η appropriately, equation (3.55) follows from (3.142), (3.143), (3.147) and (3.148).

When $z < 0$, by (3.141) and Tonelli's theorem, for n large enough, we have

$$\begin{aligned} E[\Xi_3|\mathcal{F}_u] &= \sum_{j\in\mathcal{N}_u} \int_{z-l}^{\zeta} p_{t-u}(X_j(u), y) 1_{\{\forall v\in[0,u], X_j(v)<L\}} 1_{\{X_j(u)\le L - C_6\beta^{-1/3}\}} dy \\ &\le \frac{1}{\sqrt{2\pi(t-u)}} \exp\left(g(\zeta) - \rho L^* + \frac{\rho^2 s_\zeta}{2} - \frac{\beta^2 s_\zeta^3}{6}\right) \int_{z-l}^{\zeta} e^{-\rho(c-1)(\zeta-y)} dy \\ &\quad \times (H_1 + H_2 + H_3), \end{aligned} \quad (3.149)$$

where H_1, H_2, and H_3 are defined as in (3.110) and (3.116) but with ζ in place of y. By (3.98), uniformly for all $y \in [\zeta, z]$, we have $t - u > t(y)/(1+\eta)$ for sufficiently large n. Combining this observation with the fact that $g(y)$ is increasing on $(-\infty, 0)$, we have

$$\frac{e^{g(\zeta)}}{\sqrt{2\pi(t-u)}} \int_{z-l}^{\zeta} e^{-\rho(c-1)(\zeta-y)} dy \le \frac{1}{\sqrt{2\pi(t-u)}} \frac{e^{g(\zeta)}}{\rho(c-1)} \le \frac{1}{d\rho(c-1)} \int_{\zeta}^{\zeta+d} \frac{\sqrt{1+\eta}}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \quad (3.150)$$

By (3.89), (3.90) and (3.136), we see that

$$(c-1)\rho d = \frac{(c-1)C_{11}}{|c_0|} = \frac{C_{11}}{1+c} > \frac{4}{1+3/2} = \frac{8}{5}. \quad (3.151)$$

Also, by (3.138), we have

$$\int_{\zeta}^{\zeta+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}\, dy \le \int_{-\infty}^{z-d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}\, dy \le \eta \int_{z-d}^{z} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}\, dy. \quad (3.152)$$

Therefore, by (3.150), (3.151) and (3.152), we obtain that for n sufficiently large,

$$\frac{e^{g(\zeta)}}{\sqrt{2\pi(t-u)}}\int_{z-l}^{\zeta} e^{-\rho(c-1)(\zeta-y)dy} \leq \frac{5\eta\sqrt{1+\eta}}{8}\int_{\mathcal{J}\cap(-\infty,L^*]}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy. \tag{3.153}$$

Since $2d$ satisfies the restriction (3.49) in Proposition 25, equations (3.111), (3.117) and (3.118) hold with s_ζ in place of s_y. By (3.111), (3.117), (3.118), (3.149) and (3.153), we get

$$E[\Xi_3|\mathcal{F}_0] < \frac{5\eta\sqrt{1+\eta}}{8}e^{-\rho L^*}\left(2e^{-\beta^2u^3/74}Y(0)+\frac{C_8\eta(1+\eta)}{2^{4/3}}Z(0)\right)\int_{\mathcal{J}\cap(-\infty,L^*]}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy. \tag{3.154}$$

Therefore, equations (3.9), (3.10), (3.154) and the conditional Markov's inequality imply

$$\limsup_{n\to\infty} P\left(\Xi_3 > \frac{5\sqrt{\eta(1+\eta)}}{8}e^{-\rho L^*}Z(0)\int_{\mathcal{J}\cap(-\infty,L^*]}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\right) \leq \frac{C_8\eta^{3/2}(1+\eta)}{2^{4/3}}. \tag{3.155}$$

As for Ξ_4, according to the argument leading to (3.149), we have

$$E[\Xi_4|\mathcal{F}_u] \leq \frac{1}{\sqrt{2\pi(t-u)}}e^{g(\zeta)-\rho L^*}\int_{z-l}^{\zeta} e^{-\rho(c-1)(z-y)}dy \times \Gamma_\zeta.$$

By equations (3.107) and (3.153), and the conditional Markov's inequality, we get

$$\limsup_{n\to\infty} P\left(\Xi_4 > \frac{5\sqrt{\eta(1+\eta)}}{8}e^{-\rho L^*}Z(0)\int_{\mathcal{J}\cap(-\infty,L^*]}\frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy\right) \leq \frac{(1+3\eta)\sqrt{\eta}}{Ai'(\gamma_1)^2}. \tag{3.156}$$

As a result, when $z<0$, for any $\kappa>0$, by choosing η appropriately, equation (3.55) follows from (3.142), (3.143), (3.155) and (3.156).

It remains to consider the case $|z| \ll \sqrt{\rho/\beta}$. Below we will only prove the result under the scenario $z>0$. The scenario when $z<0$ can be proved using the same argument.

The interval $[z, z+l]$ can be divided into two intervals

$$[z, z+l] = \Big[z, z + \sqrt{\frac{\rho}{\beta}}\Big] \cup \Big[z + \sqrt{\frac{\rho}{\beta}}, z+l\Big].$$

It is obvious that the first interval fits in the setting of Proposition 25. We further claim that the second interval fits in the setting of the previous case. Indeed, according to Lemma 31, we know that

$$t(z) - t\Big(z + \sqrt{\frac{\rho}{\beta}}\Big) = o(\beta^{-2/3}).$$

Thus (3.51) holds with $z + \sqrt{\rho/\beta}$ in place of z. Also, letting $c_0^* = (z + \sqrt{\rho/\beta})/L^*$, which is the same as c_0 but with $z + \sqrt{\rho/\beta}$ in place of z, the length of the second interval satisfies

$$l - \sqrt{\frac{\rho}{\beta}} \gg \sqrt{\frac{\rho}{\beta}} \asymp \frac{1}{c_0^* \rho}.$$

According to Proposition 25 and the previous case, equation (3.52) holds with $[z, z + \sqrt{\rho/\beta}]$ in place of $\mathcal{I}$ and equation (3.55) holds with $[z + \sqrt{\rho/\beta}, z+l]$ in place of $\mathcal{J}$. Combining these two equations, (3.55) follows. □

Remark 38. *Proposition 26 gives another proof for Equation (3.67) coming from [75]. Let $z = 0$. Note that z satisfies (3.18), (3.46) and (3.47), and also the assumption (3.66) is exactly the same as the assumption (3.66) in Proposition 26. Applying (3.55) to intervals $[0, \infty)$ and $(-\infty, 0]$, we get for any $\eta > 0$,*

$$\lim_{n\to\infty} P\bigg(\frac{1-\eta}{Ai'(\gamma_1)^2} e^{-\rho L^*} Z(0) \int_{-\infty}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \le N_t \\ \le \frac{1+\eta}{Ai'(\gamma_1)^2} e^{-\rho L^*} Z(0) \int_{-\infty}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy\bigg) = 1. \qquad (3.157)$$

We claim that for n sufficiently large,

$$(1-\eta)^2 < e^{-\rho^3/6\beta} \int_{-\infty}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < (1+\eta)^2. \qquad (3.158)$$

As a result, for any $\kappa > 0$, by choosing η appropriately, equations (3.157) and (3.158) implies (3.67).

It remains to prove the claim. First, we choose a constant C_7 large enough such that (3.136), (3.137), (3.138) and

$$1 - \eta < \int_{-C_7}^{C_7} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy < 1 \tag{3.159}$$

hold. By (3.137) and (3.138), we have for n sufficiently large,

$$\int_{\sqrt{\rho/\beta}}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < (1+\eta) \int_{\sqrt{\rho/\beta}}^{(1+C_7/2)\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy,$$

$$\int_{-\infty}^{-\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < (1+\eta) \int_{-(1+C_7/2)\sqrt{\rho/\beta}}^{-\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy.$$

Thus, by (3.136), we get for n sufficiently large,

$$\int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < \int_{-\infty}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy < (1+\eta) \int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \tag{3.160}$$

Next, we are going to show that $e^{g(y)-\rho^3/6\beta}/\sqrt{2\pi t(y)}$ can be well approximated by $f_n^G(y)$ defined in (3.30) which comes from the Taylor expansions of $g(y)$ and $t(y)$ around 0. Note that for $0 < x < 1$, we have $(1-x)^{3/2} = 1 - 3x/2 + 3x^2/8 + O(x^3)$. For all $y \in [-C_7\sqrt{\rho/\beta}, C_7\sqrt{\rho/\beta}]$,

$$\begin{aligned} g(y) - \frac{\rho^3}{6\beta} + \frac{\beta y^2}{2\rho} &= \rho(L^* - y) - \frac{\rho^3}{3\beta}\left(1 - \frac{y}{L^*}\right)^{3/2} - \frac{\rho^3}{6\beta} + \frac{\beta y^2}{2\rho} \\ &= \frac{\rho^3}{3\beta} - \rho y + \frac{\beta y^2}{2\rho} - \frac{\rho^3}{3\beta}\left(1 - \frac{3}{2}\frac{y}{L^*} + \frac{3}{8}\left(\frac{y}{L^*}\right)^2 + O\left(\left(\frac{y^3}{L^*}\right)^3\right)\right) \\ &= O\left(\frac{\beta^2}{\rho^3} y^3\right) \\ &= o(1). \end{aligned}$$

Combining with (3.98), we get for n sufficiently large

$$(1-\eta)\int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} f_n^G(y)dy < e^{-\rho^3/6\beta}\int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} \frac{1}{\sqrt{2\pi t(y)}}e^{g(y)}dy < (1+\eta)\int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} f_n^G(y)dy. \tag{3.161}$$

Observe that

$$\int_{-C_7\sqrt{\rho/\beta}}^{C_7\sqrt{\rho/\beta}} f_n^G(y)dy = \int_{-C_7}^{C_7} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy. \tag{3.162}$$

The claim (3.158) follows from (3.159)–(3.162).

3.3.5 Proof of Proposition 23

In this subection, we will prove Proposition 23, which gives the maximal displacement of the process. For any constant $C_2 \in \mathbb{R}$, define

$$A = \begin{cases} 0 & C_2 > 0 \\ 1 & C_2 = 0 \\ -2C_2 & C_2 < 0 \end{cases} \quad \text{and} \quad C_2' = \begin{cases} C_2 & C_2 > 0 \\ 1 & C_2 = 0 \\ -C_2 & C_2 < 0. \end{cases} \tag{3.163}$$

The proof of Proposition 23 requires the following lemma which concerns the maximal displacement of a slightly supercritical BBM with constant branching rate.

Lemma 39. *Consider a BBM started from a single particle at L^A. Each particle moves as standard Brownian motion. Each particle independently dies at rate $d(2L)$, and splits into two particles at rate $b(2L)$. Let M_t^* be the maximal position that is ever reached by a particle before time t. For any constant $C_2 \in \mathbb{R}$, define $C_2' > 0$ as in (3.163). There exists a constant C_{12} such that if n is sufficiently large, then for all t,*

$$P\left(M_t^* > L^A + \frac{C_2'}{\rho}\right) \le C_{12}\rho^2. \tag{3.164}$$

Proof of Lemma 39. In this process, each individual lives for an exponentially distributed time with parameter $b(2L) + d(2L)$, and then gives birth to 0 offspring with probability $d(2L)/(b(2L) + d(2L))$ and 2 offspring with probability $b(2L)/(b(2L) + d(2L))$. Therefore, the generating function for the offspring distribution is

$$f(s) = \frac{d(2L)}{b(2L) + d(2L)} + \frac{b(2L)}{b(2L) + d(2L)} s^2.$$

Let B be the event of survival. By (3.3) and the formula for the survival probability of the Galton-Watson process, there exists a constant C_{13} such that for all n,

$$P(B) = \frac{b(2L) - d(2L)}{b(2L)} \leq C_{13}\rho^2.$$

For any time t, we get

$$\begin{aligned} P\left(M_t^* > L^A + \frac{C_2'}{\rho}\right) &\leq P\left(M_t^* > L^A + \frac{C_2'}{\rho}\middle| B^c\right) P(B^c) + P(B) \\ &\leq P\left(M_t^* > L^A + \frac{C_2'}{\rho}\middle| B^c\right) + C_{15}\rho^2. \end{aligned} \tag{3.165}$$

We are interested in the behavior of the process conditioned on the event B^c of extinction. According to equation (4) of Gadag and Rajarshi [42], the conditioned process is equivalent to a subcritical branching process with generating function

$$\hat{f}(s) = \frac{b(2L) f(sd(2L)/b(2L))}{d(2L)} = \frac{b(2L)}{d(2L) + b(2L)} + \frac{d(2L)}{d(2L) + b(2L)} s^2.$$

Thus, in the conditioned process, there is a single particle at L^A at the beginning. Each individual moves as standard Brownian motion. It lives for an exponentially distributed time with parameter $b(2L) + d(2L)$, and then gives birth to 0 offspring with probability $b(2L)/(b(2L) + d(2L))$ and 2 offspring with probability $d(2L)/(b(2L) + d(2L))$. Consider a critical branching process started from a single particle at L^A. Each individual moves as standard Brownian motion. Each particle lives for an exponentially distributed time with parameter $b(2L) + d(2L)$, and then gives birth to 0 offspring with probability 1/2 and 2 offspring with probability 1/2. We observe that the right-most position that is ever reached by particles up to time t in the conditioned process is stochastically dominated by the right-most position that is ever reached by particles up to time t in the critical process. Letting M be the all-time maximal displacement of the critical process, we have for all $C_2 > 0$ and all time t,

$$P\left(M_t^* > L^A + \frac{C_2'}{\rho}\middle| B^c\right) \leq P\left(M > L^A + \frac{C_2'}{\rho}\right) \tag{3.166}$$

According to equation (1.7) of Sawyer and Fleischman [79], we have for n large enough,

$$P\left(M > L^A + \frac{C_2'}{\rho}\right) \leq \frac{6}{(C_2')^2}\rho^2. \tag{3.167}$$

Letting $C_{12} = C_{13} + 6/(C_2')^2$, equations (3.165)-(3.167) imply (3.164). □

Proof of Proposition 23. Let us first consider the case when

$$\beta^{-2/3}\log^{1/3}\left(\frac{\rho}{\beta^{1/3}}\right) \ll t \ll \frac{\rho}{\beta}.$$

We start with the proof of equation (3.32), which follows directly from results in [75]. For any constant $C_1 > 0$, define

$$f(x) = \begin{cases} 1 & x < 2^{1/3}C_1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Phi(f)$ as in (3.85). By 3.1.5, we see that $\Phi(f)$ satisfies (3.86). For all $C_1 > 0$ and

$0 < \kappa < 1$, if n is sufficiently large, then

$$P\left(M_t \geq L - \frac{C_1}{\beta^{1/3}}\right) = P\left(\Phi(f) \geq e^{\rho L - C_1\rho/\beta^{1/3}}\right)$$
$$\geq P\left(\frac{1-\kappa}{Ai'(\gamma_1)^2}\left(\int_0^\infty f(z)Ai(\gamma_1+z)dz\right)Z(0) \geq e^{\rho L - C_1\rho/\beta^{1/3}}\right) - \frac{\kappa}{2}. \tag{3.168}$$

By (3.10), we have for n sufficiently large

$$P\left(\frac{1-\kappa}{Ai'(\gamma_1)^2}\left(\int_0^\infty f(z)Ai(\gamma_1+z)dz\right)Z(0) \geq e^{\rho L - C_1\rho/\beta^{1/3}}\right) \geq 1 - \frac{\kappa}{2}. \tag{3.169}$$

Equation (3.32) follows from (3.168) and (3.169).

We next prove equation (3.33) under the additional assumption that the birth rate function $b(x)$ is non-decreasing and the death rate function $d(x)$ is non-increasing. For any constant C_2, define A and C_2' as in (3.163). We divide particles at time t into the following categories:

$$S_1 = \left\{i \in \mathcal{N}_t : X_i(v) < L^A \text{ for all } v \in [0,t]\right\},$$
$$S_2 = \left\{i \in \mathcal{N}_t \setminus S_1 : X_i(v) \geq L^A \text{ for some } v \in [0, t - C_7\rho^{-2}]\right\},$$
$$S_3 = \mathcal{N}_t \setminus (S_1 \cup S_2).$$

For $j = 1, 2, 3$, write

$$M_t^{S_j} = \max\left\{X_i(t), i \in S_j\right\}.$$

Note that

$$M_t = \max\left\{M_t^{S_1}, M_t^{S_2}, M_t^{S_3}\right\}. \tag{3.170}$$

For S_1, it is obvious that for all constants C_2,

$$P\left(M_t^{S_1} \leq L + \frac{C_2}{\rho}\right) = P\left(M_t^{S_1} \leq L^A + \frac{C_2'}{\rho}\right) = 1. \tag{3.171}$$

For S_2, according to 3.1.2, with probability tending to 1, particles that either are to the right of L^A at time 0 or hit L^A before time $t - C_7\rho^{-2}$ will not have descendants alive at time t. Thus for all constant C_2,

$$\lim_{n\to\infty} P\left(M_t^{S_2} \le L + \frac{C_2}{\rho}\right) \ge \lim_{n\to\infty} P(S_2 = 0) = 1. \tag{3.172}$$

It remains to deal with S_3, which consists of particles whose trajectories cross L^A in the last $C_7\rho^{-2}$ units of time. Consider the process in which particles are killed upon hitting L^A. Let R be the number of particles that first hit L^A between $t - C_7\rho^{-2}$ and t. We denote by $\{r_i\}_{i=1}^R$ the sequence of hitting times. For the process started from a single particle at x, recall that $r_x^{L^A}(u,t)$ is the expected number of particles hitting L^A between time u and t. By (3.83), taking all the particles at time 0 into consideration, we have

$$E[R|\mathcal{F}_0] = \sum_{i\in\mathcal{N}_0} r_{X_i(0)}^{L^A}\left(t - \frac{C_7}{\rho^2}, t\right) \lesssim \exp\left(-\rho L^A - \frac{\beta^2(t - C_7\rho^{-2})^3}{9}\right) Y(0) + \frac{C_7}{\rho^2}\beta^{2/3} e^{-\rho L^A} Z(0).$$

From (3.9) and (3.10), it now follows that for any $\kappa > 0$, there exists a constant C_{14} such that for n sufficiently large,

$$P\left(E[R|\mathcal{F}_0] < C_{14}\frac{\beta}{\rho^3}\frac{1}{\rho^2}\right) > 1 - \frac{\kappa}{4}.$$

Thus by the conditional Markov's inequality, we have for n sufficiently large,

$$P\left(R > C_{14}\frac{\beta^{2/3}}{\rho^2}\frac{1}{\rho^2}\right) \le E\left[\frac{E[R|\mathcal{F}_0]}{C_{14}\beta^{2/3}\rho^{-4}} 1_{\{E[R|\mathcal{F}_0] < C_{14}\beta/\rho^5\}}\right] + P\left(E[R|\mathcal{F}_0] \ge C_{14}\frac{\beta}{\rho^3}\frac{1}{\rho^2}\right) < \frac{\kappa}{2}. \tag{3.173}$$

Therefore, with probability at least $1 - \kappa/2$, the number of particles that hit L^A in the last $C_7\rho^{-2}$ unit of time is at most $C_{14}\beta^{2/3}/\rho^4$, which is $o(\rho^{-2})$.

For every $i = 1, ..., R$, we consider three BBMs. All three processes start from a single particle at L^A at time r_i. The first process has inhomogeneous birth rate $b(x)$ and death rate $d(x)$. Each particle moves as Brownian motion with drift $-\rho$. The second process is constructed based on the first process with the extra restriction that particles are killed upon hitting $2L$. To be more precise, in the second process, particles give birth at rate $b(x)$ and die at rate $d(x)$. Particles move as Brownian motion with drift $-\rho$ and are absorbed at $2L$. In the third process, the birth rate is the constant $b(2L)$ and the death rate is the constant $d(2L)$. Each particle moves as standard Brownian motion. We denote by $\bar{M}_{t-r_i}$, $\bar{M}^{2L}_{t-r_i}$ and $M^*_{t-r_i}$ the maximal positions that are ever reached by particles before time t in the three processes respectively. Because of the monotonicity of $b(x)$ and $d(x)$, we observe that $M^*_{t-r_i}$ stochastically dominates $\bar{M}^{2L}_{t-r_i}$. By Lemma 39, we have for sufficiently large n,

$$P\bigg(\bar{M}^{2L}_{t-r_i} > L^A + \frac{C_2'}{\rho}\bigg) \leq P\bigg(M^*_{t-r_i} > L^A + \frac{C_2'}{\rho}\bigg) \leq C_{12}\rho^2.$$

Note that $L^A + C_2'/\rho < 2L$ for n sufficiently large. Thus, if $\bar{M}^{2L}_{t-r_i} \leq L^A + C_2'\rho^{-1}$, then the first process is identical to the second process up to time $t - r_i$. Therefore, for sufficiently large n,

$$P\bigg(\bar{M}_{t-r_i} > L + \frac{C_2}{\rho}\bigg) = P\bigg(\bar{M}_{t-r_i} > L_A + \frac{C_2'}{\rho}\bigg) = P\bigg(\bar{M}^{2L}_{t-r_i} > L^A + \frac{C_2'}{\rho}\bigg) \leq C_{12}\rho^2. \quad (3.174)$$

Combining (3.173) with (3.174), for any $\kappa > 0$, we have for n sufficiently large,

$$P\bigg(M^{S_3}_t > L + \frac{C_2}{\rho}\bigg) \leq \frac{\kappa}{2} + C_{14}\frac{\beta^{2/3}}{\rho^2}\frac{1}{\rho^2} \cdot C_{12}\rho^2 < \frac{3\kappa}{4}. \quad (3.175)$$

As a result, equation (3.33) follows from (3.170), (3.171), (3.172) and (3.175).

Now let us consider the case when t satisfies (3.31). It suffices to show that for every subsequence $(n_j)_{j=1}^{\infty}$, there exists a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$, such that

$$\lim_{k\to\infty} P\bigg(M_{t_{n_{j_k}}, n_{j_k}} \geq L_{n_{j_k}} - \frac{C_1}{\beta^{1/3}_{n_{j_k}}}\bigg) = 1 \quad (3.176)$$

and under the additional assumption on the birth rate and the death rate,

$$\lim_{k\to\infty} P\left(M_{t_{n_{j_k}},n_{j_k}} \le L_{n_{j_k}} + \frac{C_2}{\rho_{n_{j_k}}}\right) = 1. \tag{3.177}$$

By (3.31), given a subsequence $(n_j)_{j=1}^{\infty}$, we can choose a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$ for which

$$\lim_{k\to\infty} \frac{\beta_{n_{j_k}} t_{n_{j_k}}}{\rho_{n_{j_k}}} = \tau \in [0,\infty).$$

If $\tau = 0$, then according to the previous argument, (3.176) and (3.177) hold. If $\tau > 0$, choose times $(v_{n_{j_k}})_{k=1}^{\infty}$ for which

$$\beta_{n_{j_k}}^{-2/3} \log^{1/3}\left(\frac{\rho_{n_{j_k}}}{\beta_{n_{j_k}}^{1/3}}\right) \ll v_{n_{j_k}} \ll \frac{\rho_{n_{j_k}}}{\beta_{n_{j_k}}}.$$

Let $r_{n_{j_k}} = t_{n_{j_k}} - v_{n_{j_k}}$. By Remark 27, assumptions (3.9) and (3.10) hold with $Y_{n_{j_k}}(r_{n_{j_k}})$ and $Z_{n_{j_k}}(r_{n_{j_k}})$ in place of $Z(0)$ and $Y(0)$ respectively. Replacing $Y(0)$ and $Z(0)$ by $Y_{n_{j_k}}(r_{n_{j_k}})$ and $Z_{n_{j_k}}(r_{n_{j_k}})$, the previous argument also works. Therefore, equations (3.176) and (3.177) also hold in this case. Equation (3.34) follows from (3.32) and (3.33). □

3.3.6 Proof of Proposition 24

In this subsection, we will prove Proposition 24, which gives the position of the left-most particle of the process. Denote

$$t_1 = t - \frac{2C_{10}}{\rho^2}, \qquad t_2 = t + \frac{2C_{10}}{\rho^2},$$

where C_{10} is defined in 3.1.6. We have the following lemma which controls the number of particles below $\bar{L}$ at any time between t_1 and t_2.

Lemma 40. *Suppose*

$$\frac{\rho^{2/3}}{\beta^{8/9}} \ll t - t(\bar{L}) \ll \frac{\rho}{\beta}. \tag{3.178}$$

For any $\varepsilon > 0$, if n is sufficiently large, then

$$P\bigg(N_{t_1}\big((-\infty, \bar{L})\big) \leq \frac{1}{\rho^2}\frac{\beta^{3/4}}{\rho^{9/4}}\bigg) > 1 - \varepsilon. \tag{3.179}$$

Moreover, for any $\varepsilon > 0$, if n is sufficiently large, then there exists an event $B \in \mathcal{F}_{t_1}$ satisfying

$$P(B) > 1 - \varepsilon \tag{3.180}$$

such that

$$E\bigg[\int_{t_1}^{t_2} N_r\big((-\infty, \bar{L})\big)dr \cdot 1_B\bigg] \leq \frac{1}{\rho^4}\frac{\beta^{3/4}}{\rho^{9/4}}. \tag{3.181}$$

Proof of Lemma 40. Define s as in (3.108) and

$$u = t_2 - t(\bar{L}) + s = t - t(\bar{L}) + \frac{2C_{10}}{\rho^2} + \frac{C_5}{\beta^{2/3}}.$$

Since $2C_{10}\rho^{-2} + C_5\beta^{-2/3} \ll \rho/\beta$, by (3.178), we have $\rho^{2/3}/\beta^{8/9} \ll u \ll \rho/\beta$. For any $r \in [t_1, t_2]$, define

$$s_r = s - r + t_2.$$

Note that

$$s_r \asymp s, \qquad r - u = t(\bar{L}) - s_r. \tag{3.182}$$

For every $r \in [t_1, t_2]$, denote

$$\begin{aligned}
S_1(r) &= \{i \in \mathcal{N}_r : \exists v \in [0, u], X_i(v) \geq L\}, \\
S_2(r) &= \{i \in \mathcal{N}_r : X_i(u) \leq L - C_6\beta^{-1/3}, X_i(v) < L \text{ for all } v \in [0, u]\}, \\
S_3(r) &= \{i \in \mathcal{N}_r : L - C_6\beta^{-1/3} < X_i(u) < L, X_i(v) < L \text{ for all } v \in [0, u]\}.
\end{aligned}$$

For $j = 1, 2, 3$, write

$$\Sigma_j(r) = \sum_{i \in S_j(r)} 1_{\{X_i(r) \leq \bar{L}\}}.$$

Then

$$N_r\big((-\infty,\bar{L})\big) = \Sigma_1(r) + \Sigma_2(r) + \Sigma_3(r). \tag{3.183}$$

We first consider $\Sigma_1(r)$. Let B_1 be the event that particles that are to the right of L before time u have descendants alive at time t_1. Then $B_1 \in \mathcal{F}_{t_1}$. Since $u + C_7\rho^{-2} \ll t_1$, by 3.1.2, we have that for any $\eta > 0$, if n is sufficiently large, then

$$P(B_1) < \eta.$$

Note that $\{\Sigma_1(r) \neq 0\}$ is a subset of B_1 for all $r \in [t_1, t_2]$. Therefore, we have for sufficiently large n, for all $r \in [t_1, t_2]$,

$$P(\Sigma_1(r) \neq 0) \leq P(B_1) < \eta. \tag{3.184}$$

We now consider $\Sigma_2(r)$. Denote

$$c_{\bar{L}} = \sqrt{1 - \frac{\bar{L}}{L^*}}.$$

By (3.182), we see that for all $r \in [t_1, t_2]$, equation (3.140) holds with $r - u$ in place of t, $\bar{L}$ in place of z, $c_{\bar{L}}$ in place of c and s_r in place of s. By (3.140) and Tonelli's theorem, for n large enough, we have for all $r \in [t_1, t_2]$,

$$\begin{aligned} E[\Sigma_2(r)|\mathcal{F}_u] \leq \frac{1}{\sqrt{2\pi(r-u)}} &\exp\left(g(\bar{L}) - \rho L^* + \frac{\rho^2 s_r}{2} - \frac{\beta^2 s_r^3}{6}\right) \int_{-\infty}^{\bar{L}} e^{-\rho(c_{\bar{L}}-1)(\bar{L}-y)} dy \\ &\times (H_1 + H_2 + H_3), \end{aligned} \tag{3.185}$$

where H_1, H_2 and H_3 are defined as in (3.110) and (3.116) but with s_r in place of s_y. Since $s_r \asymp s$, the upper bounds on H_1, H_2 and H_3 in (3.111), (3.117), (3.118) also hold here. For any $\eta > 0$, since $r - u \sim t(\bar{L}) \sim 3\rho/2\beta$ and $c_{\bar{L}} \sim 3/2$, we have for n sufficiently large,

$$\frac{1}{\sqrt{2\pi(r-u)}} \int_{-\infty}^{\bar{L}} e^{-\rho(c_{\bar{L}}-1)(\bar{L}-y)} dy = \frac{1}{\sqrt{2\pi(r-u)}} \frac{1}{\rho(c_{\bar{L}}-1)} \leq \frac{2(1+\eta)}{\sqrt{3\pi}} \frac{\beta^{1/2}}{\rho^{3/2}}. \tag{3.186}$$

Also, because $(1-x)^{3/2} \geq 1 - 3x/2$, we have

$$\begin{aligned} g(\bar{L}) &= \rho\left(L^* - \bar{L}\right) - \frac{2\sqrt{2\beta}}{3}\left(L^* - \bar{L}\right)^{3/2} \\ &= \frac{9\rho^3}{8\beta} - \frac{2^{2/3}\gamma_1\rho}{\beta^{1/3}} - \frac{9\rho^3}{8\beta}\left(1 - \frac{8\beta}{9\rho^2}\frac{2^{2/3}\gamma_1}{\beta^{1/3}}\right)^{3/2} \\ &\leq \rho(2\beta)^{-1/3}\gamma_1. \end{aligned} \tag{3.187}$$

We get from equations (3.111), (3.117), (3.118), (3.185)–(3.187) that for all $r \in [t_1, t_2]$, if n is sufficiently large, then

$$E[\Sigma_2(r)|\mathcal{F}_0] < \frac{2(1+\eta)}{\sqrt{3\pi}}\frac{\beta^{1/2}}{\rho^{3/2}}e^{-\rho L}\left(2e^{-\beta^2u^3/74}Y(0) + \frac{C_8\eta(1+\eta)}{2^{4/3}}Z(0)\right). \tag{3.188}$$

By (3.9) and (3.10), for any $\eta > 0$, there exists a $\delta > 0$ such that for n sufficiently large

$$P\left(\left\{Y(0) > \frac{1}{\rho^2}e^{\rho L}\right\} \cup \left\{Z(0) > \frac{1}{\delta}\frac{\beta^{1/3}}{\rho^3}e^{\rho L}\right\}\right) < \eta. \tag{3.189}$$

Define B_2 to be the union of the previous two events. We see that $B_2 \in \mathcal{F}_0 \subset \mathcal{F}_{t_1}$. Note that $e^{-\beta^2u^3/74} \ll \beta^{1/3}/\rho$. From equation (3.188), we have for all $r \in [t_1, t_2]$,

$$E[\Sigma_2(r)1_{B_2^c}] \lesssim \frac{1}{\rho^2}\frac{\beta^{5/6}}{\rho^{5/2}} \ll \frac{1}{\rho^2}\frac{\beta^{3/4}}{\rho^{9/4}}. \tag{3.190}$$

Specifically, for $r = t_1$, by (3.189) and (3.190), we have for n large enough

$$P\left(\Sigma_2(t_1) > \frac{1}{3\rho^2}\frac{\beta^{3/4}}{\rho^{9/4}}\right) < 2\eta. \tag{3.191}$$

Similarly for $\Sigma_3(r)$, by (3.140), (3.186), (3.187) and Tonelli's theorem, we have for n sufficiently large

$$E[\Sigma_3(r)|\mathcal{F}_u] < \frac{2(1+\eta)}{\sqrt{3\pi}}\frac{\beta^{1/2}}{\rho^{3/2}}e^{-\rho L}\exp\left(\frac{\rho^2 s_r}{2} - \frac{\beta^2 s_r^3}{6}\right)\sum_{j\in\mathcal{N}_u} e^{(\rho-\beta s_r)X_j(u)}1_{\{L-C_6\beta^{-1/3}<X_j(u)<L\}}.$$

Because $s \le s_r \le s + 4C_{10}/\rho^2$ for all $r \in [t_1, t_2]$, we have

$$E[\Sigma_3(r)|\mathcal{F}_u] \lesssim \frac{\beta^{1/2}}{\rho^{3/2}} e^{-\rho L} \exp\left(\frac{\rho^2 s}{2} - \frac{\beta^2 s^3}{6}\right) \sum_{j \in \mathcal{N}_u} e^{(\rho - \beta s) X_j(u)} 1_{\{L - C_6 \beta^{-1/3} < X_j(u) < L\}}. \quad (3.192)$$

By (3.10) and (3.105), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for n sufficiently large,

$$P\left(\exp\left(\frac{\rho^2 s}{2} - \frac{\beta^2 s^3}{6}\right) \sum_{j \in \mathcal{N}_u} e^{(\rho - \beta s) X_j(u)} 1_{\{L - C_6 \beta^{-1/3} < X_j(u) < L\}} > \frac{1}{\delta} \frac{1 + 2\eta}{Ai'(\gamma_1)^2} \frac{\beta^{1/3}}{\rho^3} e^{\rho L} \right) < \eta. \quad (3.193)$$

Define B_3 to be the event in the previous equation. Then $B_3 \in \mathcal{F}_u \subset \mathcal{F}_{t_1}$. From (3.192), we have for all $r \in [t_1, t_2]$,

$$E[\Sigma_3(r) 1_{B_3^c}] \lesssim \frac{1}{\rho^2} \frac{\beta^{5/6}}{\rho^{5/2}} \ll \frac{1}{\rho^2} \frac{\beta^{3/4}}{\rho^{9/4}}. \quad (3.194)$$

Specifically, for $r = t_1$, by (3.193) and (3.194), we have for n sufficiently large,

$$P\left(\Sigma_3(t_1) > \frac{1}{3\rho^2} \frac{\beta^{3/4}}{\rho^{9/4}} \right) < 2\eta. \quad (3.195)$$

As a result, for any $\varepsilon > 0$, by choosing η appropriately, equation (3.179) follows from (3.183), (3.184), (3.191) and (3.195). Let

$$B = B_1^c \cap B_2^c \cap B_3^c.$$

Then $B \in \mathcal{F}_{t_1}$, and by (3.184), (3.189) and (3.193), for n large enough,

$$P(B) = 1 - P(B_1 \cup B_2 \cup B_3) > 1 - 3\eta.$$

From (3.183), (3.184), (3.190) and (3.194), we have for n sufficiently large

$$\begin{aligned}
&E\left[\int_{t_1}^{t_2} N_r\big((-\infty,\bar{L})\big)dr\cdot 1_B\right]\\
&\quad\le E\left[\int_{t_1}^{t_2}\Sigma_1(r)dr\cdot 1_{B_1^c}\right]+E\left[\int_{t_1}^{t_2}\Sigma_2(r)dr\cdot 1_{B_2^c}\right]+E\left[\int_{t_1}^{t_2}\Sigma_3(r)dr\cdot 1_{B_3^c}\right]\\
&\quad\le \frac{1}{\rho^4}\frac{\beta^{3/4}}{\rho^{9/4}}
\end{aligned}$$

Letting $\eta=\varepsilon/3$, equations (3.180) and (3.181) follow. □

Proof of Proposition 24. Let us first consider the case when $\rho^{2/3}/\beta^{8/9}\ll t-t(\bar{L})\ll\rho/\beta$. We start with the proof of equation (3.37). For any sequence $(d_n)_{n=1}^{\infty}$ satisfying $\beta^{-1/3}\ll d\ll\rho^2/\beta$, we claim that

$$\lim_{n\to\infty}P\big(m_t\le\bar{L}+d\big)=1. \tag{3.196}$$

To prove the claim, we first note that $\bar{L}+d$ satisfies assumptions (3.18), (3.46) and (3.47) in Proposition 26. Furthermore, according to the Taylor expansion $\sqrt{1-x}=1-x/2+O(x^2)$, we get

$$\begin{aligned}
t(\bar{L})-t(\bar{L}+d)&=\sqrt{\frac{2}{\beta}(L^*-\bar{L})}\left(1-\sqrt{1-\frac{d}{L^*-\bar{L}}}\right)\\
&=\sqrt{\frac{2}{\beta}(L^*-\bar{L})}\left(1-1+\frac{d}{2(L^*-\bar{L})}+O\left(\frac{d^2}{(L^*-\bar{L})^2}\right)\right)\\
&=\frac{d}{\sqrt{2\beta(L^*-\bar{L})}}+O\left(\frac{d^2}{\sqrt{\beta}(L^*-\bar{L})^{3/2}}\right).
\end{aligned}\tag{3.197}$$

Since $d\ll\rho^2/\beta$, we have

$$\frac{d^2}{\sqrt{\beta}(L^*-\bar{L})^{3/2}}\ll\frac{d}{\sqrt{2\beta(L^*-\bar{L})}}\ll\frac{\rho}{\beta}.$$

Therefore,

$$\frac{\rho^{2/3}}{\beta^{8/9}}\ll t-t(\bar{L}+d)=t-t(\bar{L})+\Big(t(\bar{L})-t(\bar{L}+d)\Big)\ll\frac{\rho}{\beta},$$

which is assumption (3.51). Since for $0 < x < 1$, $(1-x)^{3/2} = 1 - 3x/2 + O(x^2)$, we have for n large enough,

$$\begin{aligned} g\left(L^\dagger + \frac{d}{2}\right) &= \rho\left(L^* - L^\dagger - \frac{d}{2}\right) - \frac{2\sqrt{2\beta}}{3}(L^* - L^\dagger)^{3/2}\left(1 - \frac{d}{2(L^* - L^\dagger)}\right)^{3/2} \\ &= \rho(L^* - L^\dagger) - \frac{\rho d}{2} - \frac{2\sqrt{2\beta}}{3}(L^* - L^\dagger)^{3/2}\left(1 - \frac{3}{2}\frac{d}{2(L^* - L^\dagger)} + O\left(\frac{d^2}{(L^* - L^\dagger)^2}\right)\right) \\ &= -\frac{\rho d}{2} + \frac{3\rho d}{4} + O\left(\frac{\beta d^2}{\rho}\right) \\ &\geq \frac{\rho d}{8}. \end{aligned}$$

Thus for n sufficiently large,

$$\int_{(-\infty, \bar{L}+d)} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \geq \int_{[L^\dagger + d/2, \bar{L}+d)} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \geq e^{\rho d/16}. \tag{3.198}$$

By Proposition 26 and equations (3.10) and (3.198), for any $\varepsilon > 0$, if n is sufficiently large, then

$$P\left(m_t > \bar{L} + d\right) = P\left(N_t\left((-\infty, \bar{L} + d]\right) = 0\right) < \varepsilon, \tag{3.199}$$

which implies (3.196). Now we use (3.196) to prove (3.37). Suppose (3.37) is not true. Then there exists $\kappa > 0$, such that for any constant $C_{15} > 0$, we have for infinitely many n,

$$P\left(m_t \leq \bar{L} + \frac{C_{15}}{\beta^{1/3}}\right) \leq 1 - \kappa.$$

We can therefore choose a sequence of positive integers $(n_j)_{j=1}^\infty$ and another sequence of positive constants $(C_{15,j})_{j=1}^\infty$ satisfying

$$n_j \ll 1 \qquad \text{and} \qquad 1 \ll C_{15,j} \ll \rho_{n_j}^2 / \beta_{n_j}^{2/3}$$

such that for all j,

$$P\left(m_{t_{n_j},n_j} \le \bar{L}_{n_j} + \frac{C_{15,j}}{\beta_{n_j}^{1/3}}\right) \le 1-\kappa. \tag{3.200}$$

Let $d_{n_j} = C_{15,j}\beta_{n_j}^{-1/3}$. Note that $\beta_{n_j}^{-1/3} \ll d_{n_j} \ll \rho_{n_j}^2/\beta_{n_j}$. Then (3.200) contradicts (3.196), and (3.37) follows.

We next prove equation (3.38) under the additional assumption that for all n, the birth rate function $b(x)$ is non-decreasing and the death rate function $d(x)$ is non-increasing. Define

$$\begin{aligned} S_1 &= \left\{i \in \mathcal{N}_t : X_i(t_1) < \bar{L}\right\}, \\ S_2 &= \left\{i \in \mathcal{N}_t \setminus S_1 : X_i(v) \ge \bar{L} \text{ for all } v \in [t_1, t_2]\right\}, \\ S_3 &= \mathcal{N}_t \setminus (S_1 \cup S_2). \end{aligned}$$

For $j = 1, 2, 3$, we further denote

$$m_t^{S_i} = \min\{X_i(t), i \in S_i\}.$$

We have

$$m_t = \min\left\{m_t^{S_1}, m_t^{S_2}, m_t^{S_3}\right\}. \tag{3.201}$$

For S_1, we will show that particles below $\bar{L}$ at time t_1 will not have descendants survive until time t. For $x < \bar{L}$, consider one process starting from a single particle at x at time t_1, and another process starting from a single particle at L^* at time t_1. Because of the monotonicity of the birth rate and the death rate, the probability that the first process will survive until time t is dominated by the probability that the second process will survive until time t, which is at most ρ^2/α by 3.1.6. Thus by (3.179), for any $\varepsilon > 0$ and all positive constant C_4, if n is sufficiently large,

$$P\left(m_t^{S_1} \leq \bar{L} - \frac{C_4}{\rho}\right) \leq P(S_1 \neq \emptyset) \leq \frac{\rho^2}{\alpha}\frac{1}{\rho^2}\frac{\beta^{3/4}}{\rho^{9/4}} + \varepsilon < 2\varepsilon. \tag{3.202}$$

For S_2, it is obvious that for all C_4,

$$P\left(m_t^{S_2} \leq \bar{L} - \frac{C_4}{\rho}\right) = 0. \tag{3.203}$$

We next deal with S_3. Consider the process in which particles are killed when they hit $\bar{L}$ between times t_1 and t. Let R be the number of particles that are killed at $\bar{L}$ between times t_1 and t. For $i \in 1, 2, ..., R$, let $r_i \in [t_1, t]$ be the time that this particle is killed at $\bar{L}$. For the ith particle that hits $\bar{L}$ between t_1 and t, consider a process without killing starts from this particle and let $K_i(v)$ be the number of descendants of this particle to the left of $\bar{L}$ at time $r_i + v$. Define

$$K_i = \int_0^{2C_{10}/\rho^2} K_i(v)dv.$$

Then

$$\sum_{i=1}^{R} K_i \leq \int_{t_1}^{t_2} N_r\big((-\infty, \bar{L})\big)dr. \tag{3.204}$$

For all $i = 1, 2, ..., R$, by Tonelli's theorem and (3.45), and interchanging the roles of y and $\bar{L} - y$ in the last step, we have for n large enough,

$$\begin{aligned}
E[K_i] &= \int_0^{2C_{10}/\rho^2} E[K_i(v)]dv \\
&= \int_0^{2C_{10}/\rho^2} \int_{-\infty}^{\bar{L}} p_v(\bar{L}, y)dydv \\
&= \int_0^{2C_{10}/\rho^2} \int_{-\infty}^{\bar{L}} \frac{1}{\sqrt{2\pi v}} \exp\left(\rho\bar{L} - \rho y - \frac{(\bar{L} - y)^2}{2v} - \frac{\rho^2 v}{2} + \frac{\beta(\bar{L} + y)v}{2} + \frac{\beta^2 v^3}{24}\right)dydv \\
&\geq \int_0^{2C_{10}/\rho^2} \int_0^{\infty} \frac{1}{\sqrt{2\pi v}} \exp\left(\rho y - \frac{y^2}{2v} - C_{10} + \frac{\beta(2\bar{L} - y)v}{2}\right)dydv. \qquad (3.205)
\end{aligned}$$

Since $\rho \gg C_{10}\beta/\rho^2$, for n sufficiently large, we have for all $y > 0$ and $v \leq 2C_{10}/\rho^2$,

$$\rho y + \frac{\beta(2\bar{L} - y)v}{2} \geq -\frac{5}{4}C_{10} + o(1). \tag{3.206}$$

Thus by (3.205) and (3.206), we get for n sufficiently large,

$$E[K_i] \geq \int_0^{2C_{10}/\rho^2} \frac{1}{2} e^{-9C_{10}/4+o(1)} dv \geq \frac{C_{10}}{\rho^2} e^{-5C_{10}/2}. \tag{3.207}$$

Note that random variables $\{K_i\}_{i=1}^R$ are independently and identically distributed. Moreover, conditioned on $\mathcal{F}_{t_1}$, the random variables R and K_i are independent. Let $B \in \mathcal{F}_{t_1}$ be the event defined in Lemma 40. By (3.204), we have

$$E[R|\mathcal{F}_{t_1}]E[K_i|\mathcal{F}_{t_1}]1_B = E\left[\sum_{i=1}^R K_i \middle| \mathcal{F}_{t_1}\right] 1_B \leq E\left[\int_{t_1}^{t_2} N_r\big((-\infty, \bar{L})\big) dr \cdot 1_B \middle| \mathcal{F}_{t_1}\right]. \tag{3.208}$$

Note that $E[K_i|\mathcal{F}_{t_1}] = E[K_i]$. By (3.181), (3.207) and (3.208), we have for n sufficiently large,

$$E[R1_B] \leq \frac{1}{C_{10}} e^{5C_{10}/2} \frac{1}{\rho^2} \frac{\beta^{3/4}}{\rho^{9/4}}. \tag{3.209}$$

For every $i = 1, ..., R$, we consider three BBMs. All three processes start from a single particle at $\bar{L}$ at time r_i. The first process has inhomogeneous birth rate $b(x)$ and death rate $d(x)$. Each particle moves as Brownian motion with drift $-\rho$. The second process is constructed based on the first process with the extra restriction that particles are killed upon hitting 0. In the third process, the birth rate is the constant $b(0)$ and the death rate is the constant $d(0)$. Each particle moves as standard Brownian motion without drift. We denote by m_{t-r_i} and $m^0_{t-r_i}$ the minimal displacement at time t in the first and second processes respectively. We let $\bar{M}^0_{t-r_i}$ be the maximal position that is ever reached by a particle before time t in the second process. Furthermore, for the third process, we denote by m the all-time minimum and M the all-time maximum. Because of the monotonicity of $b(x)$ and $d(x)$, we can couple the second and third processes such that M stochastically dominates $\bar{M}^0_{t-r_i}$. Taking the drift into consideration, we also have that $m^0_{t-r_i} + \rho(t - r_i)$ stochastically dominates m. Note that in the third process, since $b(0) = d(0)$, the branching is critical and the process dies out eventually. According to equation (1.7) of Sawyer and Fleischman [79], we have for x large enough

$$P(m < \bar{L} - x) \leq \frac{6}{x^2}, \qquad P(M > \bar{L} + x) \leq \frac{6}{x^2}. \tag{3.210}$$

Thus, by the construction of the first and the second processes, we have

$$\begin{aligned} P\Big(m_{t-r_i} < \bar{L} - \frac{C_4}{\rho}\Big) \\ &\leq P\Big(\Big\{m_{t-r_i} < \bar{L} - \frac{C_4}{\rho}\Big\} \cap \Big\{\bar{M}^0_{t-r_i} < \bar{L} + \frac{C_4}{\rho}\Big\}\Big) + P\Big(\bar{M}^0_{t-r_i} \geq \bar{L} + \frac{C_4}{\rho}\Big) \\ &= P\Big(m^0_{t-r_i} < \bar{L} - \frac{C_4}{\rho}\Big) + P\Big(\bar{M}^0_{t-r_i} \geq \bar{L} + \frac{C_4}{\rho}\Big). \end{aligned}$$

Note that $t - r_i \leq 2C_{10}/\rho^2$ for all r_i. By the stochastic dominance relations and equation (3.210), for $C_4 > 2C_{10}$, we have for n large enough,

$$P\Big(m_{t-r_i} < \bar{L} - \frac{C_4}{\rho}\Big) \leq P\Big(m < \bar{L} - \frac{C_4 - 2C_{10}}{\rho}\Big) + P\Big(M \geq \bar{L} + \frac{C_4}{\rho}\Big) \leq \frac{12}{(C_4 - 2C_{10})^2}\rho^2. \tag{3.211}$$

From (3.180), (3.209) and (3.211), we can choose constant C_4 large enough such that for n large enough,

$$\begin{aligned} P\Big(m^{S_3}_t \leq \bar{L} - \frac{C_4}{\rho}\Big) &\leq E\Big[\sum_{i=1}^{R} P\Big(m_{t-r_i} < \bar{L} - \frac{C_4}{\rho}\Big)1_B\Big] + \varepsilon \\ &\leq \frac{48}{C_{10}(C_4 - 2C_{10})^2} e^{5C_{10}/2} \frac{\beta^{3/4}}{\rho^{9/4}} + \varepsilon \\ &< 2\varepsilon. \end{aligned} \tag{3.212}$$

For any $\kappa > 0$, by choosing ε appropriately, equation (3.38) follows from (3.201), (3.202), (3.203) and (3.212).

Next we consider the case when $\beta(t-t(\bar{L}))/\rho \to \tau \in (0,\infty)$ as $n \to \infty$. Choose time $v < t$ for which

$$\frac{\rho^{2/3}}{\beta^{8/9}} \ll v - t(\bar{L}) \ll \frac{\rho}{\beta}.$$

Let $r = t - v$. By Remark 27, the previous argument still holds with $Z(r)$ in place of $Z(0)$ and $Y(r)$ in place of $Y(0)$. As a result, equations (3.37) and (3.38) follow in this case.

Finally, when t satisfies (3.36), we prove equations (3.37) and (3.38) by contradiction. Since the proofs of equations (3.37) and (3.38) are essentially the same, we only prove equation (3.37). Suppose equation (3.37) does not hold true. Then there exists $\kappa > 0$ such that for all positive constants C_3, we have for infinitely many n,

$$P\left(m_{t_n,n} \leq \bar{L}_n + \frac{C_3}{\beta_n^{1/3}}\right) \leq 1-\kappa.$$

We can therefore choose a sequence of positive integers $(n_j)_{j=1}^{\infty}$ and another sequence of positive constants $(C_{3,j})_{j=1}^{\infty}$, both of which tend to infinity as $j \to \infty$, such that

$$P\left(m_{t_{n_j},n_j} \leq \bar{L}_{n_j} + \frac{C_{3,j}}{\beta_{n_j}^{1/3}}\right) \leq 1-\kappa. \tag{3.213}$$

For every subsequence $(n_j)_{j=1}^{\infty}$, there exists a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{\beta_{n_{j_k}}\left(t_{n_{j_k}} - t_{n_{j_k}}\left(\bar{L}_{n_{j_k}}\right)\right)}{\rho_{n_{j_k}}} = \tau \in [0,\infty). \tag{3.214}$$

Fix any positive constant C_3. Then $C_3 < C_{3,j_k}$ for k sufficiently large. From (3.213), we have for k large enough,

$$P\left(m_{t_{n_{j_k}},n_{j_k}} \leq \bar{L}_{n_{j_k}} + \frac{C_3}{\beta_{n_{j_k}}^{1/3}}\right) \leq 1-\kappa. \tag{3.215}$$

On the other hand, note that $(t_{n_{j_k}})_{k=1}^{\infty}$ satisfies the assumptions of one of the previous two cases by (3.214). Therefore, equation (3.37) holds with n_{j_k} in place of n, which contradicts

(3.215). Thus, equation (3.37) holds true. Equation (3.39) follows from (3.37) and (3.38). □

3.4 Proofs of lemmas

In this section, we will prove all the lemmas except Lemma 34, whose proof is deferred until Section 3.5.

Proof of Lemma 30. First, let us prove equation (3.95). When $z \le 0$, equation (3.95) holds automatically. It remains to consider the case $z > 0$. If $z \ll \sqrt{\rho/\beta}$, then

$$l \lesssim \sqrt{\frac{\rho}{\beta}} \ll \frac{\rho^2}{\beta} \asymp L^* - z.$$

If $z \gtrsim \sqrt{\rho/\beta}$, then according to (3.88), (3.92) and (3.93)

$$l \lesssim \frac{1}{c_0 \rho} \ll \frac{c^2 \rho^2}{\beta} \asymp L^* - z.$$

For z satisfying (3.18) and $y \in [z-l, z+l]$, we have $L^* - y \ge L^* - z - l \gg \beta^{-1/3}$, which proves equation (3.95).

We next prove equation (3.96). When $z \ge 0$, equation (3.96) is obvious. It remains to consider the case $z < 0$. If $-z \ll \sqrt{\rho/\beta}$, then

$$l \lesssim \sqrt{\frac{\rho}{\beta}} \ll \frac{\rho^2}{\beta} \asymp z - L^\dagger.$$

If $\sqrt{\rho/\beta} \lesssim -z \ll \rho^2/\beta$, then according to (3.91),

$$l \lesssim \frac{1}{|c_0|\rho} \lesssim \sqrt{\frac{\rho}{\beta}} \ll z - L^\dagger.$$

If $-z \asymp \rho^2/\beta$, then

$$l \lesssim \frac{1}{|c_0|\rho} \asymp \frac{1}{\rho} \ll \frac{1}{\beta^{1/3}} \ll z - L^\dagger.$$

For z satisfying (3.18) and $y \in [z-l, z+l]$, we have $y - L^\dagger \geq z - L^\dagger - l \gg \beta^{-1/3}$, which proves equation (3.96). □

Proof of Lemma 31. Following similar calculations to those in (3.197), with the help of the Taylor expansion $\sqrt{1-x} = 1 - x/2 + O(x^2)$, we have for all $y \in [z-l, z+l]$,

$$\begin{aligned}|t(y) - t(z)| &= \left|\sqrt{\frac{2}{\beta}(L^* - z)}\left(1 - \frac{y-z}{2(L^*-z)} + O\left(\left(\frac{y-z}{L^*-z}\right)^2\right) - 1\right)\right| \\ &\leq \frac{l}{\sqrt{2\beta(L^*-z)}} + O\left(\frac{l^2}{\sqrt{\beta}(L^*-z)^{3/2}}\right).\end{aligned}$$

Expressing the above formula in terms of c according to (3.87), we obtain for all $y \in [z, z+l]$,

$$|t(y) - t(z)| \leq \frac{l}{c\rho} + O\left(\frac{\beta l^2}{c^3\rho^3}\right).$$

If $|z| \gtrsim \sqrt{\rho/\beta}$, then $l \lesssim 1/(|c_0|\rho)$. By formulas (3.92) and (3.93), we get

$$\frac{l}{c\rho} \lesssim \frac{1}{c|c_0|\rho^2} \ll \beta^{-2/3}, \qquad \frac{\beta l^2}{c^3\rho^3} \lesssim \frac{\beta}{c_0^2 c^3 \rho^5} \ll \beta^{-2/3}.$$

If $|z| \ll \sqrt{\rho/\beta}$, then $l \lesssim \sqrt{\rho/\beta}$ and $c \asymp 1$. We get

$$\frac{l}{c\rho} \lesssim \frac{1}{c\sqrt{\rho\beta}} \ll \beta^{-2/3}, \qquad \frac{\beta l^2}{c^3\rho^3} \lesssim \frac{1}{c^3\rho^2} \ll \beta^{-2/3}.$$

Combining the above three formulas, equation (3.97) follows. Moreover, by (3.95), we have $t(y) \gg \beta^{-2/3}$ for all $y \in [z-l, z+l]$ and equation (3.97) implies (3.98). □

Proof of Lemma 32. Note that for $0 < x < 1$, we have $(1-x)^{3/2} = 1 - 3x/2 + 3x^2/8 + O(x^3)$.

For all y,

$$
\begin{aligned}
g(y)-g(z)&\\
&= \rho(z-y)+\frac{2\sqrt{2\beta}}{3}(L^*-z)^{3/2}\left(1-\left(1-\frac{y-z}{L^*-z}\right)^{3/2}\right)\\
&= \rho(z-y)+\frac{2\sqrt{2\beta}}{3}(L^*-z)^{3/2}\left(1-1+\frac{3(y-z)}{2(L^*-z)}-\frac{3(y-z)^2}{8(L^*-z)^2}+O\left(\frac{|y-z|^3}{(L^*-z)^3}\right)\right)\\
&= \rho(z-y)-\sqrt{2\beta(L^*-z)}(z-y)-\sqrt{\frac{\beta}{8(L^*-z)}}(z-y)^2+O\left(\frac{\beta^{1/2}|z-y|^3}{(L^*-z)^{3/2}}\right).
\end{aligned}
$$

Because $L^*-z=c^2\rho^2/(2\beta)$, the above equation can be expressed in terms of c as

$$
g(y)-g(z)=\rho(1-c)(z-y)-\frac{\beta}{2c\rho}(z-y)^2+O\left(\frac{\beta^2|z-y|^3}{c^3\rho^3}\right). \tag{3.216}
$$

For all $y\in[z-l,z+l]$, we have

$$
|g(y)-g(z)|\le \rho l|1-c|+\frac{\beta l^2}{2c\rho}+O\left(\frac{\beta^2 l^3}{c^3\rho^3}\right). \tag{3.217}
$$

If $|z|\gtrsim\sqrt{\rho/\beta}$, then according to (3.90), (3.91) and (3.93), we see that

$$
\frac{\beta l^2}{2c\rho}\lesssim \rho l|1-c|\lesssim\frac{|1-c|}{|c_0|}=\frac{1}{1+c}\asymp 1,\qquad \frac{\beta^2 l^3}{c^3\rho^3}\lesssim\frac{\beta^2}{|cc_0|^3\rho^6}\lesssim\frac{\beta^{1/2}}{\rho^{3/2}}\ll 1. \tag{3.218}
$$

If $|z|\ll\sqrt{\rho/\beta}$, then according to (3.90) and (3.94), we see that

$$
\rho l|1-c|\lesssim\frac{\rho^{3/2}}{\beta^{1/2}}|c_0|\ll 1,\qquad \frac{\beta^2 l^3}{c^3\rho^3}\ll\frac{\beta l^2}{2c\rho}\lesssim\frac{1}{c}\asymp 1. \tag{3.219}
$$

The lemma follows from (3.217), (3.218), and (3.219). □

Proof of Lemma 33. We are going to express $p_t(x,y)$ in terms of s_y and w. Writing $t=$

$t(y) - s_y$ and using (3.45), we have

$$\begin{aligned}
p_t(x,y) &= \frac{1}{\sqrt{2\pi t}} \exp\Bigg(\rho(L^* - w) - \rho y - \frac{(L^* - y - w)^2}{2t(y)} \sum_{k=0}^{\infty} \left(\frac{s_y}{t(y)}\right)^k - \frac{\rho^2}{2}(t(y) - s_y) \\
&\quad + \frac{\beta(y + L^* - w)(t(y) - s_y)}{2} + \frac{\beta^2(t(y) - s_y)^3}{24}\Bigg) \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(\rho L^* - \rho y - \frac{(L^* - y)^2}{2t(y)} - \frac{\rho^2 t(y)}{2} + \frac{\beta(y + L^*)t(y)}{2} + \frac{\beta^2 t(y)^3}{24}\right) \\
&\quad \times \exp\Bigg(- \rho w - \frac{\beta w t(y)}{2} + \frac{\beta(L^* - y)s_y}{2} + \frac{\beta w s_y}{2} - \frac{\beta^2 s_y^3}{24} + \frac{\beta^2 t(y) s_y^2}{8} - \frac{\beta^2 t(y)^2 s_y}{8} \\
&\quad - \frac{(L^* - y)^2}{2t(y)} \left(\frac{s_y}{t(y)} + \frac{s_y^2}{t(y)^2} + \frac{s_y^3}{t(y)^3}\right) + \frac{w(L^* - y)}{t(y)}\left(1 + \frac{s_y}{t(y)}\right) \\
&\quad - \frac{(L^* - y)^2}{2t(y)} \sum_{k=4}^{\infty} \left(\frac{s_y}{t(y)}\right)^k - \frac{w^2}{2t(y)} \sum_{k=0}^{\infty} \left(\frac{s_y}{t(y)}\right)^k + \frac{w(L^* - y)}{t(y)} \sum_{k=2}^{\infty} \left(\frac{s_y}{t(y)}\right)^k \Bigg).
\end{aligned} \tag{3.220}$$

Using that $t(y) = \sqrt{2(L^* - y)/\beta}$ and $y + L^* = \rho^2/\beta - (L^* - y)$, we get

$$\begin{aligned}
&\rho L^* - \rho y - \frac{(L^* - y)^2}{2t(y)} - \frac{\rho^2 t(y)}{2} + \frac{\beta(y + L^*)t(y)}{2} + \frac{\beta^2 t(y)^3}{24} \\
&\quad = \rho(L^* - y) - \frac{\beta^{1/2}(L^* - y)^{3/2}}{2\sqrt{2}} - \frac{\rho^2 t(y)}{2} + \left(\frac{\rho^2 t(y)}{2} - \frac{\beta^{1/2}(L^* - y)^{3/2}}{\sqrt{2}}\right) + \frac{\beta^{1/2}(L^* - y)^{3/2}}{6\sqrt{2}} \\
&\quad = \rho(L^* - y) - \frac{2\sqrt{2\beta}}{3}(L^* - y)^{3/2} \\
&\quad = g(y).
\end{aligned} \tag{3.221}$$

Also notice that

$$\begin{aligned}
&- \frac{(L^* - y)^2}{2t(y)} \sum_{k=4}^{\infty} \left(\frac{s_y}{t(y)}\right)^k - \frac{w^2}{2t(y)} \sum_{k=0}^{\infty} \left(\frac{s_y}{t(y)}\right)^k + \frac{w(L^* - y)}{t(y)} \sum_{k=2}^{\infty} \left(\frac{s_y}{t(y)}\right)^k \\
&\quad = -\frac{1}{2(t(y) - s_y)} \left((L^* - y)\left(\frac{s_y}{t(y)}\right)^2 - w\right)^2 \\
&\quad = -\frac{(\beta s_y^2 - 2w)^2}{8t}.
\end{aligned} \tag{3.222}$$

For all w, according to (3.220), (3.221) and (3.222), replacing $t(y)$ with $\sqrt{2(L^* - y)/\beta}$, we have

$$\begin{aligned}
p_t(x,y) &= \frac{1}{\sqrt{2\pi t}}\exp\bigg(g(y) - \rho w - \frac{\beta w t(y)}{2} + \frac{\beta(L^*-y)s_y}{2} + \frac{\beta w s_y}{2} - \frac{\beta^2 s_y^3}{24} \\
&\quad + \frac{\beta^2 t(y) s_y^2}{8} - \frac{\beta^2 t(y)^2 s_y}{8} - \frac{(L^*-y)^2}{2t(y)}\bigg(\frac{s_y}{t(y)} + \frac{s_y^2}{t(y)^2} + \frac{s_y^3}{t(y)^3}\bigg) \\
&\quad + \frac{w(L^*-y)}{t(y)}\bigg(1 + \frac{s_y}{t(y)}\bigg) - \frac{(\beta s_y^2 - 2w)^2}{8t}\bigg) \\
&= \frac{1}{\sqrt{2\pi t}}\exp\bigg(g(y) - \rho w + \beta w s_y - \frac{\beta^2}{6}s_y^3 - \frac{(\beta s_y^2 - 2w)^2}{8t}\bigg).
\end{aligned} \tag{3.223}$$

For all $w \in \mathbb{R}$, $s < t(z)$ and $y \in \mathbb{R}$, since $-(\beta s_y^2 - 2w)^2/8t \leq 0$, formula (3.100) follows. Furthermore, if $s \asymp \beta^{-2/3}$, then by Lemma 31, we have $s_y \asymp \beta^{-2/3}$ for all $y \in [z-l, z+l]$. Then for $|w| \lesssim \beta^{-1/3}$, we get

$$\frac{(\beta s_y^2 - 2w)^2}{8t} = o(1).$$

and therefore (3.223) implies (3.101). □

Proof of Lemma 35. Equation (3.105) follows from Lemma 6.2 in [75] directly. We use a similar strategy as in the proof of Lemma 6.2 in [75] to prove (3.107). For every $y \in [z-l, z+l]$, define

$$f_y(x) = \begin{cases} e^{2^{-1/3}\beta^{2/3}s_y x} & 0 < x < 2^{1/3}C_6 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} e^{2^{-1/3}C_5 x} & 0 < x < 2^{1/3}C_6 \\ 0 & \text{otherwise.} \end{cases}$$

We can express Γ_y in terms of the function f_y by writing

$$\begin{aligned}
\Gamma_y &= \exp\bigg(2^{-1/3}\beta^{2/3}s_y\gamma_1 - \frac{\beta^2 s_y^3}{6}\bigg)\sum_{j\in\mathcal{N}_u} e^{\rho X_j(u)} e^{\beta s_y(L - X_j(u))} 1_{\{L - C_6\beta^{-1/3} < X_j(u) < L\}} \\
&= \exp\bigg(2^{-1/3}\beta^{2/3}s_y\gamma_1 - \frac{\beta^2 s_y^3}{6}\bigg)\sum_{j\in\mathcal{N}_u} e^{\rho X_j(u)} f_y\Big((2\beta)^{1/3}\big(L - X_j(u)\big)\Big).
\end{aligned} \tag{3.224}$$

According to Lemma 31, we see that $\beta^{2/3}s_y \to C_5$ uniformly for $y \in [z-l, z+l]$. Thus

$$\exp\left(2^{-1/3}\beta^{2/3}s_y\gamma_1 - \frac{\beta^2 s_y^3}{6}\right) \to \exp\left(2^{-1/3}\gamma_1 C_5 - \frac{C_5^3}{6}\right), \quad \text{as } n \to \infty. \tag{3.225}$$

Also, for every $\eta > 0$, if n is sufficiently large, then for all x,

$$\sup_{y \in [z, z+l]} |f_y(x) - f(x)| < \eta f(x).$$

Therefore, for every $\eta > 0$, if n is large enough, then for all $y \in [z-l, z+l]$,

$$\begin{aligned} &\left| \sum_{j \in \mathcal{N}_u} e^{\rho X_j(u)} f\Big((2\beta)^{1/3}\big(L - X_j(u)\big)\Big) - \sum_{j \in \mathcal{N}_u} e^{\rho X_j(u)} f_y\Big((2\beta)^{1/3}\big(L - X_j(u)\big)\Big) \right| \\ &\qquad \le \eta \sum_{j \in \mathcal{N}_u} e^{\rho X_j(u)} f\Big((2\beta)^{1/3}\big(L - X_j(u)\big)\Big). \end{aligned} \tag{3.226}$$

Furthermore, since u satisfies (3.84), equation (3.86) implies that

$$\frac{1}{Z(0)} \sum_{j \in \mathcal{N}_u} e^{\rho X_j(u)} f\Big((2\beta)^{1/3}\big(L - X_j(u)\big)\Big) \to_p \frac{1}{Ai'(\gamma_1)^2} \int_0^{2^{1/3}C_6} e^{2^{-1/3}C_5 y} Ai(\gamma_1 + y)dy. \tag{3.227}$$

As a result, equation (3.107) follows from (3.103), (3.224)–(3.227). □

Proof of Lemma 36. First, let us summarize properties of the function $g(y)$ that will be useful throughout the proof. For $y \in (-\infty, L^*)$, we have

$$g'(y) = -\rho + \sqrt{2\beta(L^* - y)}, \quad g''(y) = -\sqrt{\frac{\beta}{2(L^* - y)}} < 0.$$

The function $g(y)$ is increasing in the interval $(-\infty, 0)$ and decreasing in the interval $[0, L^*)$. The derivative of $g(y)$ is decreasing and $g(y)$ is a concave function. Thus $g(y)$ is bounded above by its first order Taylor approximation. We have for all $x_1, x_2 \in (-\infty, L^*]$,

$$g(x_2) \le g(x_1) + g'(x_1)(x_2 - x_1). \tag{3.228}$$

First consider the case $z \ge 0$. By (3.228) and the fact that the derivative of g is decreasing, we have for all $y \in [z, z+d]$,

$$g(y) \ge g(z) + g'(y)(y-z) \ge g(z) + g'(z+d)(y-z).$$

Also noticing that $t(y)$ is a decreasing function of y, we have

$$\begin{aligned}\int_z^{z+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy &\ge \frac{1}{\sqrt{2\pi t(z)}} e^{g(z)} \int_z^{z+d} e^{g'(z+d)(y-z)} dy \\ &= \frac{1}{\sqrt{2\pi t(z)}} e^{g(z)} \frac{1}{|g'(z+d)|} (1 - e^{dg'(z+d)}).\end{aligned}$$

According to the definitions of c_0 in (3.48) and c in (3.87), we get

$$dg'(z+d) \le dg'(z) = -\frac{C_{11}}{1+c} \le -\frac{C_{11}}{2}.$$

Therefore,

$$\int_z^{z+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \ge \frac{1}{\sqrt{2\pi t(z)}} e^{g(z)} \frac{1}{|g'(z+d)|} (1 - e^{-C_{11}/2}). \tag{3.229}$$

Moreover, since $t(z) = c\rho/\beta$ and $|g'(z+d)| \le \rho$, we see that

$$\int_z^{z+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \gtrsim \frac{\beta^{1/2}}{c^{1/2}\rho^{3/2}} e^{g(z)}. \tag{3.230}$$

On the other hand, we will separate the integral on the left hand side of (3.137) into two parts and upper bound each of them. Define

$$h = \beta^{-1/6}\sqrt{L^* - z} = \frac{c\rho}{\sqrt{2}\beta^{2/3}}. \tag{3.231}$$

We claim that $d \ll h$ and $h \ll L^* - z$. Indeed, since $z \gtrsim \sqrt{\rho/\beta}$, equation (3.93) gives

$$c_0 c \gtrsim \frac{\beta^{1/2}}{\rho^{3/2}} \gg \frac{\beta^{2/3}}{\rho^2},$$

which implies

$$d \asymp \frac{1}{c_0\rho} \ll \frac{c\rho}{\beta^{2/3}} \asymp h.$$

Furthermore, because $c \gg \beta^{1/3}/\rho$, we have

$$h \asymp \frac{c\rho}{\beta^{2/3}} \ll \frac{c^2\rho^2}{\beta} \asymp L^* - z. \tag{3.232}$$

We denote

$$\int_{z+d}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy = \int_{z+d}^{z+h} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy + \int_{z+h}^{L^*} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy =: K_1 + K_2. \tag{3.233}$$

We first consider K_1. By (3.228), we have $g(y) \le g(z+d) + g'(z+d)(y-z-d)$ for $y \in [z+d, z+h]$. Hence,

$$K_1 \le \frac{1}{\sqrt{2\pi t(z+h)}} e^{g(z+d)} \int_{z+d}^{z+h} e^{g'(z+d)(y-z-d)} dy \le \frac{1}{\sqrt{2\pi t(z+h)}} e^{g(z+d)} \frac{1}{|g'(z+d)|}. \tag{3.234}$$

Since $z \gtrsim \sqrt{\rho/\beta}$, by (3.90) and (3.93), we have

$$\rho d(1-c) \gg \frac{\beta^2 d^3}{c^3\rho^3}. \tag{3.235}$$

According to equations (3.90) and (3.216), we have for n sufficiently large,

$$\begin{aligned}g(z+d) &= g(z) - \rho d(1-c) - \frac{\beta d^2}{2c\rho} + O\left(\frac{\beta^2 d^3}{c^3\rho^3}\right)\\ &\le g(z) - \frac{\rho d(1-c)}{2}\\ &= g(z) - \frac{C_{11}}{2(1+c)}\\ &\le g(z) - \frac{C_{11}}{4}.\end{aligned} \tag{3.236}$$

Also by (3.232), we have for n large

$$t(z+h) = \sqrt{1 - \frac{h}{L^* - z}}t(z) \ge \frac{1}{2}t(z).$$

Combining the above two observations with (3.234), we have for n large enough,

$$K_1 \le e^{-C_{11}/4}\frac{1}{\sqrt{\pi t(z)}}e^{g(z)}\frac{1}{|g'(z+d)|}. \tag{3.237}$$

We next consider K_2. Recalling that the function $g(y)$ is decreasing when $y \in [0, L^*]$ and $t(y) = \sqrt{2/\beta}\sqrt{L^* - y}$, we get

$$K_2 \le e^{g(z+h)}\int_z^{L^*}\frac{\beta^{1/4}}{2^{3/4}\sqrt{\pi}}(L^* - y)^{-1/4}dy = \frac{2^{5/4}\beta^{1/4}}{3\sqrt{\pi}}(L^* - z)^{3/4}e^{g(z+h)}. \tag{3.238}$$

We are going to apply the same argument that led to (3.236). Because $z \gtrsim \sqrt{\rho/\beta}$, we have $\rho c_0 \gg \beta^{2/3}/(c\rho)$ by (3.93). Thus by (3.90) and (3.231), we get

$$\rho h(1-c) \asymp \rho h c_0 \gg \frac{\beta^{2/3}}{c\rho}h \asymp \frac{\beta^2 h^3}{c^3\rho^3}.$$

According to (3.216) and (3.231), since $cc_0 \gg \beta^{7/12}/\rho^{7/4}$ by (3.93), we have for n sufficiently large,

$$g(z+h) \le g(z) - \frac{\rho h(1-c)}{2} = g(z) - \frac{cc_0\rho^2}{2\sqrt{2}\beta^{2/3}(1+c)} \le g(z) - \frac{\rho^{1/4}}{\beta^{1/12}}.$$

Combining this result with (3.238), we get for n large,

$$K_2 \leq \frac{2^{5/4}\beta^{1/4}}{3\sqrt{\pi}}(L^* - z)^{3/4} \exp\left(g(z) - \frac{\rho^{1/4}}{\beta^{1/12}}\right) \asymp \frac{c^{3/2}\rho^{3/2}}{\beta^{1/2}} \exp\left(g(z) - \frac{\rho^{1/4}}{\beta^{1/12}}\right). \tag{3.239}$$

Furthermore, since $c \leq 1$ and $\rho^3 \gg \beta$, we notice that

$$\frac{c^{3/2}\rho^{3/2}}{\beta^{1/2}} \exp\left(g(z) - \frac{\rho^{1/4}}{\beta^{1/12}}\right) \ll \frac{\beta^{1/2}}{c^{1/2}\rho^{3/2}} e^{g(z)}, \tag{3.240}$$

As a result, equations (3.242), (3.239) and (3.240) imply

$$K_2 \ll \int_z^{z+d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy. \tag{3.241}$$

For any $\eta > 0$, choosing C_{11} large enough such that $\eta(1 - e^{-C_{11}/2})/\sqrt{2} > e^{-C_{11}/4}$, equation (3.137) follows from (3.229), (3.233), (3.237) and (3.241).

Next consider the case $z \leq 0$. The proof is similar to the previous case. By (3.228) and the fact that the derivative of g is decreasing, we have for all $y \in [z - d, z]$,

$$g(y) \geq g(z) - g'(y)(z - y) \geq g(z) - g'(z - d)(z - y).$$

Also, note that $t(y)$ is a decreasing function of y. Thus,

$$\begin{aligned} \int_{z-d}^{z} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy &\geq \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z)} \int_{z-d}^{z} e^{-g'(z+d)(z-y)} dy \\ &= \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z)} \frac{1}{g'(z-d)} (1 - e^{-dg'(z-d)}). \end{aligned}$$

According to the definitions of c_0 in (3.48) and c in (3.87), since $c \in [1, 3/2)$, we get

$$-dg'(z - d) \leq -dg'(z) = -\frac{C_{11}}{1 + c} \leq -\frac{2C_{11}}{5}.$$

Therefore,

$$\int_{z-d}^{z} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \geq \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z)} \frac{1}{g'(z-d)} (1 - e^{-2C_{11}/5}). \tag{3.242}$$

On the other hand, since $g(y) \leq g(z-d) + g'(z-d)(y-z+d)$ by (3.228) and $t(y)$ is decreasing, we get

$$\begin{aligned}\int_{-\infty}^{z-d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy &\leq \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z-d)} \int_{-\infty}^{z-d} e^{g'(z-d)(y-z+d)} dy \\ &\leq \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z-d)} \frac{1}{g'(z-d)}. \end{aligned} \tag{3.243}$$

We will apply the argument that led to (3.236) again. Note that under the current scenario $c_0 < 0$ and $1 \leq c < 3/2$. From (3.90), (3.216) and (3.235), we get for n sufficiently large,

$$g(z-d) \leq g(z) + \frac{\rho d(1-c)}{2} = g(z) - \frac{C_{11}}{2(1+c)} \leq g(z) - \frac{C_{11}}{5}.$$

Combining the above formula with (3.243), we get for n large

$$\int_{-\infty}^{z-d} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \leq \frac{1}{\sqrt{2\pi t(z-d)}} e^{g(z)} \frac{1}{g'(z-d)} e^{-C_{11}/5}. \tag{3.244}$$

For any $\eta > 0$, choosing C_{11} large enough such that $\eta(1 - e^{-2C_{11}/5}) > e^{-C_{11}/5}$, equation (3.138) follows from (3.242) and (3.244). Therefore, for any $\eta > 0$, we can choose C_{11} large enough such that (3.136)–(3.138) all hold and the lemma follows. □

Proof of Lemma 37. First consider the case $z > 0$. For $x \leq L$ and $y \in [\zeta, \infty)$, from (3.45),

$$\frac{p_t(x,y)}{p_t(x,\zeta)} = \exp\Bigg(-(y-\zeta)\Big(\rho - \frac{\beta t}{2} - \frac{2x-\zeta-y}{2t}\Big)\Bigg). \tag{3.245}$$

Using that $x \le L$ and $y \ge \zeta$, we have

$$\rho - \frac{\beta t}{2} - \frac{2x - \zeta - y}{2t} \ge \rho - \frac{\beta t(z)}{2} + \frac{\beta s}{2} - \frac{2(L - z - 2d)}{2t}$$
$$\ge \rho - \frac{\beta t(z)}{2} + \frac{\beta s}{2} - \frac{L^* - z}{t} + \frac{2^{-1/3}\beta^{-1/3}\gamma_1}{t}.$$

Note that we can expand $1/t$ as a geometric sum and thus

$$\frac{L^* - z}{t} = \frac{L^* - z}{t(z)} + \frac{(L^* - z)s}{t(z)^2} + \frac{(L^* - z)s^2}{t(z)^3} \sum_{k=0}^{\infty} \left(\frac{s}{t(z)} \right)^k = \frac{L^* - z}{t(z)} + \frac{(L^* - z)s}{t(z)^2} + \frac{(L^* - z)s^2}{t(z)^2 t}.$$

Recall from (3.88) that $t(z) = c\rho/\beta$ and $L^* - z = c^2\rho^2/(2\beta)$. Therefore, from the above two formulas, we get

$$\rho - \frac{\beta t}{2} - \frac{2x - \zeta - y}{2t} \ge \rho(1 - c) - \frac{\beta s^2}{2t} + \frac{2^{-1/3}\beta^{-1/3}\gamma_1}{t}.$$

Since $z \gtrsim \sqrt{\rho/\beta}$, we have $\rho(1 - c) \gg \beta^{2/3}/(c\rho)$ by (3.90) and (3.93). Thus,

$$\left| - \frac{\beta s^2}{2t} + \frac{2^{-1/3}\beta^{-1/3}\gamma_1}{t} \right| \asymp \frac{\beta^{2/3}}{c\rho} \ll \rho(1 - c).$$

Therefore, for n sufficiently large, we have for all $x \le L$ and $y \in [\zeta, \infty)$,

$$\rho - \frac{\beta t}{2} - \frac{2x - \zeta - y}{2t} \ge \frac{\rho}{2}(1 - c). \tag{3.246}$$

Equation (3.139) follows from (3.245) and (3.246).

Next consider the case $z < 0$. We are going to apply an argument that is similar to the proof of (3.100). Writing $r = z - y$ and expressing $p_t(x, y)$ in terms of s, w, r and z, we have

$$
\begin{aligned}
&p_t(x,y) \\
&= \frac{1}{\sqrt{2\pi t}} \exp\Bigg(\rho(L^*-w)-\rho(z-r)-\frac{(L^*-z+r-w)^2}{2t(z)}\sum_{k=0}^{\infty}\left(\frac{s}{t(z)}\right)^k-\frac{\rho^2}{2}(t(z)-s) \\
&\quad + \frac{\beta(z-r+L^*-w)(t(z)-s)}{2}+\frac{\beta^2(t(z)-s)^3}{24}\Bigg) \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(\rho L^*-\rho z-\frac{(L^*-z)^2}{2t(z)}-\frac{\rho^2 t(z)}{2}+\frac{\beta(z+L^*)t(z)}{2}+\frac{\beta^2 t(z)^3}{24}\right) \\
&\quad \times \exp\Bigg(-\rho w+\rho r-\frac{\beta(r+w)t(z)}{2}+\frac{\beta(L^*-z)s}{2}+\frac{\beta(r+w)s}{2}-\frac{\beta^2 s^3}{24}+\frac{\beta^2 t(z)s^2}{8} \\
&\quad - \frac{\beta^2 t(z)^2 s}{8}-\frac{(L^*-z)^2}{2t(z)}\left(\frac{s}{t(z)}+\frac{s^2}{t(z)^2}+\frac{s^3}{t(z)^3}\right)-\frac{(r-w)(L^*-z)}{t(z)}\left(1+\frac{s}{t(z)}\right) \\
&\quad - \frac{(L^*-z)^2}{2t(z)}\sum_{k=4}^{\infty}\left(\frac{s}{t(z)}\right)^k-\frac{(r-w)^2}{2t(z)}\sum_{k=0}^{\infty}\left(\frac{s}{t(z)}\right)^k-\frac{(r-w)(L^*-z)}{t(z)}\sum_{k=2}^{\infty}\left(\frac{s}{t(z)}\right)^k\Bigg).
\end{aligned}
\tag{3.247}
$$

By a computation similar to the one leading to (3.222), we get

$$
\begin{aligned}
&-\frac{(L^*-z)^2}{2t(z)}\sum_{k=4}^{\infty}\left(\frac{s}{t(z)}\right)^k-\frac{(r-w)^2}{2t(z)}\sum_{k=0}^{\infty}\left(\frac{s}{t(z)}\right)^k-\frac{(r-w)(L^*-z)}{t(z)}\sum_{k=2}^{\infty}\left(\frac{s}{t(z)}\right)^k \\
&\quad = -\frac{(\beta s^2+2r-2w)^2}{8t},
\end{aligned}
$$

which is negative. Combining the previous two formulas with (3.88) and (3.221), we have

$$
\begin{aligned}
p_t(x,y) &\le \frac{1}{\sqrt{2\pi t}} \exp\Bigg(g(z)-\rho w+\rho r-\frac{\beta(r+w)t(z)}{2}+\frac{\beta(L^*-z)s}{2}+\frac{\beta(r+w)s}{2}-\frac{\beta^2 s^3}{24} \\
&\quad + \frac{\beta^2 t(z)s^2}{8}-\frac{\beta^2 t(z)^2 s}{8}-\frac{(L^*-z)^2}{2t(z)}\left(\frac{s}{t(z)}+\frac{s^2}{t(z)^2}+\frac{s^3}{t(z)^3}\right) \\
&\quad - \frac{(r-w)(L^*-z)}{t(z)}\left(1+\frac{s}{t(z)}\right)\Bigg) \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(g(z)-(c-1)\rho r-\rho w+\beta s w-\frac{\beta^2 s^3}{6}\right),
\end{aligned}
$$

which is (3.140). Furthermore, let

$$c_\zeta = \sqrt{1 - \frac{\zeta}{L^*}}.$$

Note that that $t = t(\zeta) - s_\zeta$ and $s_\zeta \asymp \beta^{-2/3}$ by (3.97). Thus for $y \leq \zeta$, equation (3.140) holds with ζ in place of z, c_ζ in place of c and s_ζ in place of s, so we have

$$p_t(x,y) \leq \frac{1}{\sqrt{2\pi t}} \exp\left(g(\zeta) - (c_\zeta - 1)\rho(\zeta - y) - \rho w + \beta s_\zeta w - \frac{\beta^2 s_\zeta^3}{6} \right). \tag{3.248}$$

Since $\zeta \leq z < 0$, we have $c_\zeta > c > 1$. Therefore, equation (3.141) follows from (3.248). □

3.5 Second moment estimate

3.5.1 Proof of Lemma 34

A key step in the proof of Lemma 34 is the following second moment estimate, which will be proved in Section 3.5.2. Recall that $p_t^L(x,y)$ is the density of the process in which particles are killed at L.

Lemma 41. *For every z satisfying (3.18), (3.46) and (3.47), let*

$$0 \leq s \lesssim \beta^{-2/3}, \qquad t = t(z) - s.$$

Suppose $0 \leq L - x \lesssim \beta^{-1/3}$. Then

$$\int_0^t \int_{-\infty}^L p_u^L(x,r)\left(p_{t-u}^L(r,z)\right)^2 dr du \lesssim \frac{\beta^{2/3}}{t(z)\rho^4} \exp\left(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^* - z)^{3/2} \right). \tag{3.249}$$

Note that equation (3.249) means that the ratio between the left hand side and the right hand side is bounded above by a positive constant uniformly for all n and all z

satisfying (3.18), (3.46) and (3.47).

Proof of Lemma 34. According to the standard second moment formula (see e.g. Theorem 2.2 in [78]), we have

$$E[N_t^L(\mathcal{I})^2] \lesssim \int_{\mathcal{I}} p_t^L(x,y)dy + 2\int_0^t \int_{-\infty}^L p_u^L(x,r)\left(\int_{\mathcal{I}} p_{t-u}^L(r,y)dy\right)^2 drdu. \tag{3.250}$$

Regarding the first term in (3.250), we upper bound $p_t^L(x,y)$ by $p_t(x,y)$ and then apply (3.100) to get

$$\int_{\mathcal{I}} p_t^L(x,y)dy \leq \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t}} \exp\left(g(y) - \rho(L^* - x) + \beta(L^* - x)(t(y) - t) - \frac{\beta^2}{6}(t(y) - t)^3\right)dy.$$

On account of (3.97), we observe that $0 \leq t(y) - t \asymp \beta^{-2/3}$ for $y \in \mathcal{I}$. Also notice that $|L^* - x| \lesssim \beta^{-1/3}$. Therefore, we get

$$\begin{aligned}
&\int_{\mathcal{I}} p_t^L(x,y)dy \\
&\quad \lesssim \int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} \exp\left(g(y) - \rho L^* + \rho x\right)dy \\
&\quad = \frac{\beta^{2/3}}{\rho^4} e^{\rho x + \rho L - 2\rho L^*}\left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}dy\right)^2 \frac{\rho^4}{\beta^{2/3}} e^{-\rho L + \rho L^*}\left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}dy\right)^{-1}.
\end{aligned} \tag{3.251}$$

For z satisfying (3.18), we see that $g(z) \geq 0$ and $t(y) \leq 2t(z)$ for all $y \in \mathcal{I}$ when n is large enough. Also note that $c < 3/2$, $\rho/\beta^{1/3} \gg 1$ and $\gamma_1 < 0$. By (3.88), we get for n large,

$$\begin{aligned}
\frac{\rho^4}{\beta^{2/3}} e^{-\rho L + \rho L^*}\left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}dy\right)^{-1} &\leq \frac{\rho^4}{\beta^{2/3}} e^{-\rho L + \rho L^*}\left(\frac{l}{2\sqrt{\pi t(z)}}\right)^{-1} \\
&= \frac{2\sqrt{\pi}c^{1/2}\rho}{l}\left(\frac{\rho}{\beta^{1/3}}\right)^{7/2} \exp\left(\frac{\gamma_1 \rho}{2^{1/3}\beta^{1/3}}\right) \\
&\ll 1.
\end{aligned} \tag{3.252}$$

By (3.251) and (3.252), we have

$$\int_{\mathcal{I}} p_t^L(x,y)dy \ll \frac{\beta^{2/3}}{\rho^4} e^{\rho x + \rho L - 2\rho L^*} \left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t_y}} e^{g(y)} dy \right)^2. \tag{3.253}$$

Regarding the second part of (3.250), by the Cauchy-Schwarz inequality and Tonelli's theorem, we have

$$\int_0^t \int_{-\infty}^L p_u^L(x,r) \left(\int_{\mathcal{I}} p_{t-u}^L(r,y)dy \right)^2 dr du \leq l \int_{\mathcal{I}} \int_0^t \int_{-\infty}^L p_u^L(x,r) \big(p_{t-u}^L(r,y)\big)^2 dr du dy. \tag{3.254}$$

We want to apply Lemma 41 to upper bound the above expression. First, by Lemma 30, we know that (3.18) holds with y in place of z. Also, by Lemma 31, for all $y \in \mathcal{I}$,

$$t(y) = t(z) \pm o(\beta^{-2/3}).$$

As a result, the assumptions in Lemma 41 are satisfied, and we can apply Lemma 41 to get

$$\begin{aligned} & l \int_{\mathcal{I}} \int_0^t \int_{-\infty}^L p_u^L(x,r) \big(p_{t-u}^L(r,y)\big)^2 dr du dy \\ & \quad \lesssim \frac{l\beta^{2/3}}{\rho^4} \int_{\mathcal{I}} \frac{1}{2\pi t(y)} \exp\left(\rho x - 2\rho y + \rho L - \frac{4\sqrt{2\beta}}{3} (L^* - y)^{3/2} \right) dy \\ & \quad = \frac{\beta^{2/3}}{\rho^4} e^{\rho x + \rho L - 2\rho L^*} l \int_{\mathcal{I}} \frac{1}{2\pi t(y)} e^{2g(y)} dy. \end{aligned} \tag{3.255}$$

According to Lemma 32, we have for all $y \in \mathcal{I}$,

$$e^{g(z)} \asymp e^{g(y)}.$$

From the previous equation and (3.98), we get

$$l \int_{\mathcal{I}} \frac{1}{2\pi t(y)} e^{2g(y)} dy \asymp \left(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)} dy \right)^2. \tag{3.256}$$

By equations (3.254), (3.255) and (3.256), we obtain

$$\int_0^t \int_{-\infty}^L p_u^L(x,r)\Big(\int_{\mathcal{I}} p_{t-u}^L(r,y)dy\Big)^2 drdu \lesssim \frac{\beta^{2/3}}{\rho^4} e^{\rho x+\rho L-2\rho L^*}\Big(\int_{\mathcal{I}} \frac{1}{\sqrt{2\pi t(y)}} e^{g(y)}dy\Big)^2. \tag{3.257}$$

The lemma follows from (3.250), (3.253) and (3.257). □

3.5.2 Proof of Lemma 41

The proof of Lemma 41 will be divided into the following three lemmas.

Lemma 42. *For every z satisfying (3.18), (3.46) and (3.47), let*

$$0 \le s \lesssim \beta^{-2/3}, \qquad t = t(z) - s, \qquad u_1 = \beta^{-7/12}(L^* - z)^{1/4}.$$

Suppose $0 \le L - x \lesssim \beta^{-1/3}$. *Then*

$$\begin{aligned} I_1 := &\int_0^{u_1}\int_{-\infty}^L p_u^L(x,r)\Big(p_{t-u}^L(r,z)\Big)^2 drdu \\ &\lesssim \frac{\beta^{2/3}}{t(z)\rho^4}\exp\Big(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^*-z)^{3/2}\Big). \end{aligned} \tag{3.258}$$

Lemma 43. *For every z satisfying*

$$L^* - z \gg \beta^{-1/3}\log^{4/3}(\rho/\beta^{1/3}), \qquad z - L^\dagger \gg \beta^{-1/3}, \tag{3.259}$$

let

$$0 \le s \lesssim \beta^{-2/3}, \qquad t = t(z) - s, \qquad u_1 = \beta^{-7/12}(L^* - z)^{1/4}.$$

Suppose $0 \le L - x \lesssim \beta^{-1/3}$. *Then*

$$I_2 := \int_{u_1}^t\int_{-\infty}^L p_u^L(x,r)\Big(p_{t-u}^L(r,z)\Big)^2 drdu \lesssim \frac{\beta^{2/3}}{t(z)\rho^4}\exp\Big(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^*-z)^{3/2}\Big). \tag{3.260}$$

Lemma 44. *For every positive z satisfying*

$$\beta^{-1/3} \ll L^* - z \lesssim \beta^{-1/3}\log^{4/3}(\rho/\beta^{1/3}), \tag{3.261}$$

let

$$0 \le s \lesssim \beta^{-2/3}, \qquad t = t(z) - s, \qquad u_1 = \beta^{-7/12}(L^* - z)^{1/4}.$$

Suppose $0 \le L - x \lesssim \beta^{-1/3}$. *Then (3.260) holds.*

Proof of Lemma 41. It suffices to show that for every subsequence $(n_j)_{j=1}^{\infty}$, there exists a sub-subsequence $(n_{j_k})_{k=1}^{\infty}$, such that

$$\begin{aligned}\int_0^{t_{n_{j_k}}}\int_{-\infty}^{L_{n_{j_k}}} & p_u^{L_{n_{j_k}}}(x_{n_{j_k}}, r)\Big(p_{t_{n_{j_k}}-u_{n_{j_k}}}^{L_{n_{j_k}}}(r, z_{n_{j_k}})\Big)^2 dr du \\ &\lesssim \frac{\beta_{n_{j_k}}^{2/3}}{t_{n_{j_k}}(z_{n_{j_k}})\rho_{n_{j_k}}^4}\exp\left(\rho_{n_{j_k}}x_{n_{j_k}} - 2\rho_{n_{j_k}}z_{n_{j_k}} + \rho_{n_{j_k}}L_{n_{j_k}} - \frac{4\sqrt{2\beta_{n_{j_k}}}}{3}(L_{n_{j_k}}^* - z_{n_{j_k}})^{3/2}\right).\end{aligned} \tag{3.262}$$

Given a subsequence $(n_j)_{j=1}^{\infty}$, there exists a further subsequence $(n_{j_k})_{k=1}^{\infty}$ such that one of the following holds:

1. $L_{n_{j_k}}^* - z_{n_{j_k}} \gg \beta_{n_{j_k}}^{-1/3}\log^{4/3}(\rho_{n_{j_k}}/\beta_{n_{j_k}}^{1/3})$ and $z_{n_{j_k}} - L_{n_{j_k}}^{\dagger} \gg \beta_{n_{j_k}}^{-1/3}$.
2. $\beta_{n_{j_k}}^{-1/3} \ll L_{n_{j_k}}^* - z_{n_{j_k}} \lesssim \beta_{n_{j_k}}^{-1/3}\log^{4/3}(\rho_{n_{j_k}}/\beta_{n_{j_k}}^{1/3})$.

In case 1, equation (3.262) follows from Lemmas 42 and 43. In case 2, equation (3.262) follows from Lemmas 42 and 44. □

The second moment estimate relies on delicate estimates of the density. Different approximations to the density $p_t^L(x, y)$ were obtained in [75]. The following results come from Lemmas 2.6, 2.7 and 2.8 in [75].

Lemma 45. *For all $t \geq 0$ and $x, y < L$, we have*

$$p_t^L(x,y) \lesssim \min\left\{\frac{1}{t^{1/2}}, \frac{(L-x)(L-y)}{t^{3/2}}\right\} \exp\left(\rho x - \rho y - \frac{(y-x)^2}{2t} - \frac{\rho^2 t}{2} + \beta L t\right). \quad (3.263)$$

Moreover, when $t \geq 2\beta^{-2/3}$, $0 \leq L - x \lesssim \beta^{-1/3}$ and $y < L$, we have

$$\begin{aligned} p_t^L(x,y) \lesssim & \frac{\beta^{1/3}(L-x)}{\sqrt{t}} \max\left\{1, \frac{1}{\beta^{1/3}t}\Big(L - y - \frac{\beta t^2}{2}\Big)\right\} \\ & \times \exp\left(\rho x - \rho y - \frac{(y-x)^2}{2t} - \frac{\rho^2 t}{2} + \frac{\beta(x+y)t}{2} + \frac{\beta^2 t^3}{24} + \frac{1}{2\beta^{1/3}t}\Big(L - y - \frac{\beta t^2}{2}\Big)\right). \end{aligned} \quad (3.264)$$

It remains now to prove Lemmas 42, 43 and 44.

Proof of Lemma 42. For all $u \in [0, u_1]$, we see that $t - u \geq t/2 \gg \beta^{-2/3}$ for n sufficiently large. We will bound $p_u^L(x,r)$ by equation (3.263) and $p_{t-u}^L(r,z)$ by equations (3.45) and (3.264). We have

$$\begin{aligned} I_1 \lesssim & \int_0^{u_1} \int_{-\infty}^{L} \min\left\{\frac{1}{u^{1/2}}, \frac{(L-x)(L-r)}{u^{3/2}}\right\} \exp\left(\rho x - \rho r - \frac{(x-r)^2}{2u} - \frac{\rho^2 u}{2} + \beta L u\right) \\ & \times \frac{1}{t-u}\Bigg(1_{\{L-r>\beta^{-1/3}\}} + \beta^{2/3}(L-r)^2\bigg(\max\left\{1, \frac{1}{\beta^{1/3}(t-u)}\Big(L - z - \frac{\beta(t-u)^2}{2}\Big)\right\}\bigg)^2 \\ & \quad \times \exp\left(\frac{1}{\beta^{1/3}(t-u)}\Big(L - z - \frac{\beta(t-u)^2}{2}\Big)\right) 1_{\{0 \leq L-r \leq \beta^{-1/3}\}}\Bigg) \\ & \times \exp\left(2\rho r - 2\rho z - \frac{(r-z)^2}{t-u} - \rho^2(t-u) + \beta(r+z)(t-u) + \frac{\beta^2(t-u)^3}{12}\right) dr du. \end{aligned}$$

Denote

$$M(u, r, x) = \min\left\{\frac{1}{u^{1/2}}, \frac{(L-x)r}{u^{3/2}}\right\},$$

and

$$N(u,r,z) = 1_{\{r>\beta^{-1/3}\}} + \beta^{2/3} r^2 \left(\max\left\{1, \frac{1}{\beta^{1/3}(t-u)}\left(L - z - \frac{\beta(t-u)^2}{2}\right)\right\}\right)^2$$
$$\times \exp\left(\frac{1}{\beta^{1/3}(t-u)}\left(L - z - \frac{\beta(t-u)^2}{2}\right)\right) 1_{\{0\le r\le \beta^{-1/3}\}}.$$

Interchanging the roles of r and $L-r$, we have

$$I_1 \lesssim \int_0^{u_1}\int_0^\infty \frac{1}{t-u} M(u,r,x)N(u,r,z)\exp\Bigg(\rho x - 2\rho z + \rho L - \rho r - \frac{(x-L+r)^2}{2u}$$
$$- \frac{(L-r-z)^2}{t-u} + \frac{\rho^2 u}{2} - \rho^2 t + \beta L u + \beta(L-r)(t-u) + \beta z(t-u) + \frac{\beta^2(t-u)^3}{12}\Bigg) dr du.$$

Since $t = t(z) - s$, $L = L^* - (2\beta)^{-1/3}\gamma_1$ and

$$\frac{1}{t-u} = \frac{1}{t(z)} \sum_{k=0}^\infty \left(\frac{u+s}{t(z)}\right)^k, \tag{3.265}$$

we can express $-(L-r-z)^2/(t-u)$ as

$$- \frac{(L^*-z)^2}{t(z)} \sum_{k=0}^\infty \left(\frac{u+s}{t(z)}\right)^k - \frac{((2\beta)^{-1/3}\gamma_1 + r)^2}{t-u} + \frac{2(L^*-z)r}{t-u} + \frac{2(L^*-z)(2\beta)^{-1/3}\gamma_1}{t-u}$$
$$\le -\frac{(L^*-z)^2}{t(z)} - \frac{(L^*-z)^2}{t(z)^2}(u+s) - \frac{(L^*-z)^2}{t(z)^3}(u+s)^2 - \frac{(L^*-z)^2}{t(z)^4}(u+s)^3$$
$$+ \frac{2(L^*-z)r}{t-u} + \frac{2(L^*-z)(2\beta)^{-1/3}\gamma_1}{t-u}. \tag{3.266}$$

Rearranging all the terms, I_1 can be further bounded as follows:

$$I_1 \lesssim \exp\left(\rho x - 2\rho z + \rho L - \rho^2 t(z) + \beta L^* t(z) + \beta z t(z) + \frac{\beta^2 t(z)^3}{12} - \frac{(L^*-z)^2}{t(z)}\right)$$
$$\int_0^{u_1}\int_0^{\infty} \frac{1}{t-u} \times M(u,r,x)N(u,r,z)\exp\Bigg(-\rho r - \frac{((L-x)-r)^2}{2u} - \frac{(L^*-z)^2}{t(z)^2}(u+s)$$
$$-\frac{(L^*-z)^2}{t(z)^3}(u+s)^2 - \frac{(L^*-z)^2}{t(z)^4}(u+s)^3 + \frac{2(L^*-z)r}{t-u} + \frac{2(L^*-z)(2\beta)^{-1/3}\gamma_1}{t-u} + \frac{\rho^2 u}{2}$$
$$+\rho^2 s - \beta(2\beta)^{-1/3}\gamma_1 t - \beta(t-u)r - \beta z(u+s) - \beta L^* s - \frac{\beta^2(u+s)^3}{12}$$
$$-\frac{\beta^2 t(z)^2(u+s)}{4} + \frac{\beta^2 t(z)(u+s)^2}{4}\Bigg)drdu. \tag{3.267}$$

Note that

$$-\rho^2 t(z) + \beta L^* t(z) + \beta z t(z) + \frac{\beta^2 t(z)^3}{12} - \frac{(L^*-z)^2}{t(z)} = -\frac{4\sqrt{2\beta}}{3}(L^*-z)^{3/2}. \tag{3.268}$$

Also,

$$\begin{aligned}
&\frac{2(L^*-z)(2\beta)^{-1/3}\gamma_1}{t-u} - \beta(2\beta)^{-1/3}\gamma_1 t \\
&\quad < \frac{2(L^*-z)(2\beta)^{-1/3}\gamma_1}{t(z)} - \beta(2\beta)^{-1/3}\gamma_1 t(z) + \beta(2\beta)^{-1/3}\gamma_1 s \\
&\quad = \beta(2\beta)^{-1/3}\gamma_1 s \\
&\quad < 0,
\end{aligned}$$

and

$$\frac{\rho^2 u}{2} + \rho^2 s - \beta z(u+s) - \beta L^* s = \beta(L^*-z)(u+s).$$

Combining the above four formulas, we get

$$I_1 \lesssim \frac{1}{t}\exp\left(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^* - z)^{3/2}\right)$$
$$\int_0^{u_1}\int_0^{\infty} M(u,r,x)N(u,r,z)\exp\Bigg(- r\left(\rho + \beta(t-u) - \frac{2(L^*-z)}{t-u}\right) - \frac{((L-x)-r)^2}{2u}$$
$$- \frac{(L^*-z)^2}{t(z)^2}(u+s) - \frac{(L^*-z)^2}{t(z)^3}(u+s)^2 - \frac{(L^*-z)^2}{t(z)^4}(u+s)^3 + \beta(L^*-z)(u+s)$$
$$- \frac{\beta^2(u+s)^3}{12} - \frac{\beta^2 t(z)^2(u+s)}{4} + \frac{\beta^2 t(z)(u+s)^2}{4}\Bigg)drdu.$$

Observe that

$$-\frac{(L^*-z)^2}{t(z)^2}(u+s) + \beta(L^*-z)(u+s) - \frac{\beta^2 t(z)^2(u+s)}{4} = 0,$$

$$-\frac{(L^*-z)^2}{t(z)^3}(u+s)^2 + \frac{\beta^2 t(z)(u+s)^2}{4} = 0,$$
$$-\frac{(L^*-z)^2}{t(z)^4}(u+s)^3 - \frac{\beta^2(u+s)^3}{12} \leq -\frac{\beta^2 u^3}{3}.$$

Therefore,

$$I_1 \lesssim \frac{1}{t}\exp\left(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^* - z)^{3/2}\right)\int_0^{u_1}\int_0^{\infty} M(u,r,x)N(u,r,z)$$
$$\exp\left(- r\left(\rho + \beta(t-u) - \frac{2(L^*-z)}{t-u}\right) - \frac{((L-x)-r)^2}{2u} - \frac{\beta^2 u^3}{3}\right)drdu. \qquad (3.269)$$

Denote the double integral as J. Write $J = J_1 + J_2$ where J_1 is the portion of the double integral for which $r > \beta^{-1/3}$ and J_2 is the portion of the double integral for which $0 \leq r \leq \beta^{-1/3}$. We first estimate J_1. Since $u \leq u_1 = \beta^{-7/12}(L^* - z)^{1/4}$ and $s \ll u_1$, we see that

$\beta(u+s) \ll \rho$. Thus, according to (3.265), for n sufficiently large we get

$$\begin{aligned}
\rho + \beta(t-u) - \frac{2(L^*-z)}{t-u} &= \rho + t(z)\beta - \beta(u+s) - \frac{2(L^*-z)}{t(z)} \sum_{k=0}^{\infty} \left(\frac{u+s}{t(z)}\right)^k \\
&= \rho - \beta(u+s) - \frac{2(L^*-z)}{t(z)} \sum_{k=1}^{\infty} \left(\frac{u+s}{t(z)}\right)^k \\
&= \rho - \beta(u+s) - \frac{\beta t(z)(u+s)}{t-u} \\
&\geq \frac{\rho}{2}. \qquad (3.270)
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_1 &\lesssim \int_0^{u_1} \int_{\beta^{-1/3}}^{\infty} M(u,r,x)N(u,r,z)e^{-\rho r/2} drdu \\
&\leq \int_0^{\rho^{-2}} \frac{1}{u^{1/2}} \int_{\beta^{-1/3}}^{\infty} e^{-\rho r/2} drdu + \int_{\rho^{-2}}^{u_1} \int_{\beta^{-1/3}}^{\infty} \frac{(L-x)r}{u^{3/2}} e^{-\rho r/2} drdu \\
&\leq e^{-\rho/2\beta^{1/3}} \left(\frac{4}{\rho^2} + \frac{4(L-x)}{\beta^{1/3}} + \frac{8(L-x)}{\rho} \right).
\end{aligned}$$

Since $L - x \lesssim \beta^{-1/3}$, for n sufficiently large,

$$e^{-\rho/2\beta^{1/3}} \left(\frac{4}{\rho^2} + \frac{4(L-x)}{\beta^{1/3}} + \frac{8(L-x)}{\rho} \right) \lesssim e^{-\rho/2\beta^{1/3}} \beta^{-2/3} = \frac{\beta^{2/3}}{\rho^4} \left(\frac{\rho}{\beta^{1/3}}\right)^4 e^{-\rho/2\beta^{1/3}} \ll \frac{\beta^{2/3}}{\rho^4}.$$

Combining the above two equations, we have

$$J_1 \ll \frac{\beta^{2/3}}{\rho^4}. \qquad (3.271)$$

Next, we estimate J_2. Note that

$$\frac{1}{\beta^{1/3}(t-u)}\Big(L-z-\frac{\beta(t-u)^2}{2}\Big)=\frac{1}{\beta^{1/3}(t-u)}\Big(-(2\beta)^{-1/3}\gamma_1-\frac{\beta(u+s)^2}{2}+\beta t(z)(u+s)\Big)$$
$$=\frac{1}{\beta^{1/3}(t-u)}\Big(-\frac{\beta(u+s)^2}{2}+\beta t(z)(u+s)\Big)+o(1). \tag{3.272}$$

We will expand $1/(t-u)$ as a geometric sum. Using that $t=t(z)-s$, $u\le u_1=\beta^{-7/12}(L^*-z)^{1/4}$ and $s\lesssim\beta^{-2/3}$, we see that $u+s\ll t(z)$. Using also (3.88) and (3.265), we get

$$\begin{aligned}\frac{1}{\beta^{1/3}(t-u)}\Big(L-z-\frac{\beta(t-u)^2}{2}\Big)&\le\beta^{2/3}(u+s)\sum_{k=0}^{\infty}\Big(\frac{u+s}{t(z)}\Big)^k+o(1)\\&=\beta^{2/3}(u+s)\Big(1+O\Big(\frac{u+s}{t(z)}\Big)\Big)+o(1)\\&=\beta^{2/3}u+O(1).\end{aligned}$$

Equation (3.270) and the previous formula imply that

$$\begin{aligned}J_2&\lesssim\int_0^{u_1}\int_0^{\beta^{-1/3}}M(u,r,x)\beta^{2/3}r^2\Big(\max\big\{1,\beta^{2/3}u\big\}\Big)^2\\&\quad\times\exp\Big(-\frac{\rho r}{2}-\frac{(L-x-r)^2}{2u}-\frac{\beta^2u^3}{3}+\beta^{2/3}u\Big)drdu\\&\lesssim\int_0^{\rho^{-2}}\frac{1}{u^{1/2}}\int_0^{\beta^{-1/3}}\beta^{2/3}r^2e^{-\rho r/2}drdu\\&\quad+\int_{\rho^{-2}}^{3\beta^{-2/3}}\frac{1}{u^{3/2}}\int_0^{\beta^{-1/3}}(L-x)\beta^{2/3}r^3\exp\Big(-\frac{\rho r}{2}-\frac{(L-x-r)^2}{2u}\Big)drdu\\&\quad+\int_{3\beta^{-2/3}}^{u_1}\frac{1}{u^{3/2}}\int_0^{\beta^{-1/3}}(L-x)\beta^{2/3}r^3\exp\Big(-\frac{\rho r}{2}-\frac{(L-x-r)^2}{2u}\Big)\\&\qquad\times\exp\Big(-\frac{(\beta^{2/3}u)^3}{3}+\beta^{2/3}u+2\log(\beta^{2/3}u)\Big)drdu.\end{aligned}$$

When $x \geq 3$, we have $-x^3/3 + x + 2\log x \leq 0$. Therefore,

$$\begin{aligned}
J_2 &\lesssim \int_0^{\rho^{-2}} \frac{1}{u^{1/2}} \int_0^{\beta^{-1/3}} \beta^{2/3} r^2 e^{-\rho r/2} dr du \\
&\quad + \int_{\rho^{-2}}^{u_1} \frac{1}{u^{3/2}} \int_0^{\beta^{-1/3}} (L-x)\beta^{2/3} r^3 \exp\left(-\frac{\rho r}{2} - \frac{(L-x-r)^2}{2u} \right) dr du \\
&\lesssim \frac{\beta^{2/3}}{\rho^4} + (L-x)\beta^{2/3} \int_{\rho^{-2}}^{u_1} \frac{1}{u^{3/2}} \int_0^{\beta^{-1/3}} r^3 \exp\left(-\frac{\rho r}{2} - \frac{(L-x-r)^2}{2u} \right) dr du. \qquad (3.273)
\end{aligned}$$

Note that $u_1 \lesssim \rho^{1/2}\beta^{-5/6}$. By applying exactly the same calculation as in equations (8.54), (8.55) and (8.56) of [75], we see that

$$(L-x)\beta^{2/3} \int_{\rho^{-2}}^{u_1} \frac{1}{u^{3/2}} \int_0^{\beta^{-1/3}} r^3 \exp\left(-\frac{\rho r}{2} - \frac{(L-x-r)^2}{2u} \right) dr du \lesssim \frac{\beta^{2/3}}{\rho^4}. \qquad (3.274)$$

Combining equations (3.269), (3.271), (3.273) and (3.274), we have

$$I_1 \lesssim \frac{\beta^{2/3}}{t\rho^4} \exp\left(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^* - z)^{3/2} \right),$$

which implies (3.258). □

Proof of Lemma 43. The restriction (3.259) is equivalent to

$$c \gg \frac{\beta^{1/3}}{\rho} \log^{2/3}\left(\frac{\rho}{\beta^{1/3}} \right), \qquad \frac{3}{2} - c \gg \frac{\beta^{2/3}}{\rho^2}. \qquad (3.275)$$

In particular, $c \gg \beta^{1/3}\rho^{-1}$ and $0 < c < 3/2$. For $u_1 \leq u \leq t$, we will bound both $p_u^L(x,r)$ and $p_{t-u}^L(r,z)$ by equation (3.45). Similar to the calculation for I_1, interchanging the roles of r and $L-r$, we have

$$\begin{aligned}I_2 \lesssim \exp\Big(\rho x - 2\rho z + \rho L - \rho^2 t + \beta L t + \beta z t\Big)\\ \times \int_{u_1}^t u^{-1/2}(t-u)^{-1} \exp\left(\frac{\rho^2 u}{2} + \frac{\beta x u}{2} - \beta z u - \frac{\beta L u}{2} + \frac{\beta^2 u^3}{24} + \frac{\beta^2 (t-u)^3}{12}\right)\\ \int_0^\infty \exp\left(-r\left(\rho + \beta t - \frac{\beta u}{2}\right) - \frac{(L-x-r)^2}{2u} - \frac{(L-z-r)^2}{t-u}\right) dr du. \qquad (3.276)\end{aligned}$$

Using (3.88), to prove (3.260), it is equivalent to show that

$$I_2 \lesssim \frac{\beta^{5/3}}{c\rho^5} \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right). \qquad (3.277)$$

To simplify I_2, we are going to estimate K in two ways depending on the value of u. This cutoff value u_2 will be defined based on y_c introduced below. Denote

$$\Delta = \sqrt{\frac{9c^2}{4} + c + 1}, \qquad y_c = \frac{3}{2} + \frac{1}{c} - \frac{1}{c}\Delta. \qquad (3.278)$$

Note that $y_c < 1$ since for $0 < c \le 3/2$,

$$\begin{aligned}1 - y_c &= -\frac{1}{2} + \frac{(\Delta-1)(\Delta+1)}{c(\Delta+1)}\\ &= \frac{9c}{4(\Delta+1)} - \left(\frac{1}{2} - \frac{1}{\Delta+1}\right)\\ &= \frac{9c}{4(\Delta+1)} - \frac{(\Delta-1)(\Delta+1)}{2(\Delta+1)^2}\\ &= \frac{9c}{4(\Delta+1)} - \frac{c}{2(\Delta+1)^2} - \frac{9c^2}{8(\Delta+1)^2}\\ &\ge \frac{1}{(\Delta+1)^2}\left(\frac{9c}{2} - \frac{c}{2} - \frac{9c^2}{8}\right)\\ &> 0. \qquad (3.279)\end{aligned}$$

Choose a constant $C_{16} > 0$ small enough such that the following hold:

$$\frac{1}{3} - \frac{3C_{16}}{2} > C_{16}, \tag{3.280}$$

$$-\frac{4}{3(3+\sqrt{17})(1+\sqrt{17}/2)} + 2C_{16} < -C_{16}, \tag{3.281}$$

$$-\frac{7}{2(3+\sqrt{17})(1+\sqrt{17}/2)} + \frac{3}{2}C_{16} + 6C_{16} < 0, \tag{3.282}$$

$$\frac{-8}{9(3\cdot\frac{3}{2}+2\cdot\frac{11}{4})(\frac{11}{4}+1)} + 9C_{16} < -C_{16}. \tag{3.283}$$

Then define

$$u_2 = \left(\frac{3}{2} + \frac{1}{c} - \frac{1}{c}\sqrt{\frac{9c^2}{4} + c + 1} - C_{16}c\right)t. \tag{3.284}$$

Since $y_c < 1$ by (3.279), we see that $u_2 < t$. Also, when $0 < c \le 3/2$, one can show that $3/2 + 1/c - \Delta/c - C_{16}c$ is a decreasing function of c. Thus by (3.280),

$$u_2 \ge \left(\frac{3}{2} + \frac{2}{3} - \frac{2}{3}\sqrt{\frac{9}{4}\cdot\frac{9}{4} + \frac{3}{2} + 1} - \frac{3C_{16}}{2}\right)t = \left(\frac{1}{3} - \frac{3C_{16}}{2}\right)t > C_{16}t \tag{3.285}$$

and $u_2 > u_1$ for sufficiently large n.

Denote the inner integral in (3.276) as K. When $u_1 < u \le u_2$, letting $a = (\rho + \beta t - \beta u/2)(t-u)$ and $b = L - z$, K can be written as

$$K = \int_0^\infty \exp\left(-\frac{(r-(b-a/2))^2 + ab - a^2/4}{t-u} - \frac{(L-x-r)^2}{2u}\right)dr. \tag{3.286}$$

For $u_1 \le u \le u_2$, we claim that $b - a/2 \le 0$. Because $b - a/2$ is an increasing function of u, it is sufficient to show that for $u = u_2$, we have $b - a/2 \le 0$. Recall from (3.88) that $t = c\rho/\beta - s$ and $L^* - z = c^2\rho^2/2\beta$. Writing

$$y_2 = \frac{u_2}{t} = y_c - C_{16}c, \tag{3.287}$$

we have that

$$\begin{aligned} b-\frac{a}{2} &= (1-c_0)\frac{\rho^2}{2\beta}-(2\beta)^{-1/3}\gamma_1-\frac{1}{2}\bigg((1+c)\rho-\beta s-\frac{cy_2\rho}{2}+\frac{\beta y_2 s}{2}\bigg)(1-y_2)\bigg(c\frac{\rho}{\beta}-s\bigg) \\ &= \frac{\rho^2}{\beta}\bigg(-\frac{c^2y_2^2}{4}+y_2\Big(\frac{3c^2}{4}+\frac{c}{2}\Big)-\frac{c}{2}\bigg)+\rho s(1-y_2)\bigg(\frac{1}{2}+c-\frac{cy_2}{2}\bigg)+O(\beta^{-1/3}). \end{aligned} \tag{3.288}$$

We observe that y_c is one root of

$$-\frac{c^2y^2}{4}+y\Big(\frac{3c^2}{4}+\frac{c}{2}\Big)-\frac{c}{2}=0. \tag{3.289}$$

Therefore, according to (3.287) and (3.289), equation (3.288) implies that

$$b-\frac{a}{2}=-\frac{c^2\rho^2}{\beta}\bigg(\frac{C_{16}^2c^2}{4}+\frac{C_{16}\Delta}{2}\bigg)+\rho s\bigg(1-y_c+C_{16}c\bigg)\bigg(\frac{1}{2}+c-\frac{cy_2}{2}\bigg)+O(\beta^{-1/3}).$$

Because $1-y_c\le 9c/4(\Delta+1)$ by the second line of (3.279) and $1/2+c-cy_2/2\le 2$ for $0<c\le 3/2$, it follows that

$$b-\frac{a}{2}\le -\frac{C_{16}\Delta c^2\rho^2}{2\beta}+2\rho sc\bigg(\frac{9}{4(\Delta+1)}+C_{16}\bigg)+O(\beta^{-1/3}).$$

Since $c\gg\beta^{1/3}\rho^{-1}$ by (3.259) and $s\lesssim\beta^{-2/3}$, we see that $c^2\rho^2/\beta\gg\rho sc$ and $c^2\rho^2/\beta\gg\beta^{-1/3}$. As a result, for $u=u_2$, and thus for all $u_1<u\le u_2$, for n sufficiently large,

$$b-\frac{a}{2}\le 0. \tag{3.290}$$

From (3.286) and (3.290), we obtain that for $u_1<u\le u_2$,

$$K\le\int_0^\infty \exp\bigg(-\frac{(L-z)^2}{t-u}-\frac{(L-x-r)^2}{2u}\bigg)dr\le\sqrt{2\pi u}\exp\bigg(-\frac{(L-z)^2}{t-u}\bigg). \tag{3.291}$$

When $u_2\le u\le t$, we upper bound K by the formula for the moment generating function of the normal distribution to get

$$
\begin{aligned}
K \leq & \int_{-\infty}^{\infty} \exp\Big(- r\Big(\rho + \beta t - \frac{\beta u}{2} - \frac{L-x}{u} - \frac{2(L-z)}{t-u}\Big) \\
& - \frac{r^2}{2}\frac{t+u}{u(t-u)} - \frac{(L-x)^2}{2u} - \frac{(L-z)^2}{t-u}\Big) dr \\
= & \sqrt{\frac{2\pi u(t-u)}{t+u}} \exp\left(\frac{u(t-u)}{2(t+u)}\Big(\rho + \beta t - \frac{\beta u}{2} - \frac{L-x}{u} - \frac{2(L-z)}{t-u}\Big)^2 \right. \\
& \left. - \frac{(L-x)^2}{2u} - \frac{(L-z)^2}{t-u}\right). \qquad (3.292)
\end{aligned}
$$

According to (3.276), (3.291) and (3.292), we have

$$
\begin{aligned}
I_2 \lesssim & \exp\left(\rho x - 2\rho z + \rho L - \rho^2 t + \beta L t + \beta z t\right) \\
& \times \int_{u_1}^{u_2} (t-u)^{-1} \exp\left(\frac{\rho^2 u}{2} + \frac{\beta x u}{2} - \beta z u - \frac{\beta L u}{2} + \frac{\beta^2 u^3}{24} + \frac{\beta^2 (t-u)^3}{12} - \frac{(L-z)^2}{t-u}\right) du \\
& + \exp\left(\rho x - 2\rho z + \rho L - \rho^2 t + \beta L t + \beta z t\right) \\
& \times \int_{u_2}^{t} \sqrt{\frac{1}{(t-u)(t+u)}} \exp\left(\frac{\rho^2 u}{2} + \frac{\beta x u}{2} - \beta z u - \frac{\beta L u}{2} + \frac{\beta^2 u^3}{24} + \frac{\beta^2 (t-u)^3}{12}\right) \\
& \times \exp\left(\frac{u(t-u)}{2(t+u)}\Big(\rho + \beta t - \frac{\beta u}{2} - \frac{L-x}{u} - \frac{2(L-z)}{t-u}\Big)^2 - \frac{(L-x)^2}{2u} - \frac{(L-z)^2}{t-u}\right) du \\
=: & R_1 + R_2. \qquad (3.293)
\end{aligned}
$$

We first estimate R_1. Let $y_1 = u_1/t$ and $y_2 = u_2/t$. After making the change of variables $u = yt$ and writing $z = c_0 L^*$ and $t = c\rho/\beta - s$, we obtain

$$
\begin{aligned}
R_1 = \exp & \left(\rho x - 2\rho z + \rho L - \rho^2\Big(c\frac{\rho}{\beta} - s\Big) + \beta\Big(\frac{\rho^2}{2\beta} - (2\beta)^{-1/3}\gamma_1\Big)\Big(c\frac{\rho}{\beta} - s\Big) \right. \\
& \left. + \beta c_0 \frac{\rho^2}{2\beta}\Big(c\frac{\rho}{\beta} - s\Big)\right) \int_{y_1}^{y_2} \frac{1}{1-y} \exp\left(\frac{\rho^2 y}{2}\Big(c\frac{\rho}{\beta} - s\Big) - \frac{\beta(L-x)y}{2}\Big(c\frac{\rho}{\beta} - s\Big) \right. \\
& - \beta c_0 \frac{\rho^2}{2\beta}\Big(c\frac{\rho}{\beta} - s\Big) y + \frac{\beta^2 y^3}{24}\Big(c\frac{\rho}{\beta} - s\Big)^3 + \frac{\beta^2 (1-y)^3}{12}\Big(c\frac{\rho}{\beta} - s\Big)^3 \\
& \left. - \frac{((1-c_0)\rho^2/2\beta - (2\beta)^{-1/3}\gamma_1)^2}{(c\rho/\beta - s)(1-y)}\right) dy.
\end{aligned}
$$

Observing that

$$-\frac{((1-c_0)\rho^2/2\beta-(2\beta)^{-1/3}\gamma_1)^2}{(c\rho/\beta-s)(1-y)} \leq -\frac{(1-c_0)^2\rho^3}{4c\beta(1-y)}\sum_{k=0}^{\infty}\Big(\frac{s\beta}{c\rho}\Big)^k$$
$$\leq -\frac{(1-c_0)^2\rho^3}{4c\beta(1-y)}-\frac{(1-c_0)^2\rho^2 s}{4c^2(1-y)}-\frac{(1-c_0)^2\rho\beta s^2}{4c^3(1-y)},$$

we get

$$\begin{aligned}R_1 \lesssim \exp\bigg(&\rho x-2\rho z+\rho L-\frac{c^3}{2}\frac{\rho^3}{\beta}+\frac{c^2\rho^2 s}{2}-2^{-1/3}\gamma_1 c\frac{\rho}{\beta^{1/3}}\bigg)\\ &\int_{y_1}^{y_2}\frac{1}{1-y}\exp\bigg(\frac{\rho^3}{\beta}\Big(\frac{cy}{2}-\frac{cc_0y}{2}+\frac{c^3y^3}{24}+\frac{c^3(1-y)^3}{12}-\frac{(1-c_0)^2}{4c(1-y)}\Big)\\ &-\frac{\beta(L-x)y}{2}\Big(c\frac{\rho}{\beta}-s\Big)+\rho^2 s\Big(-\frac{y}{2}+\frac{c_0y}{2}-\frac{c^2y^3}{8}-\frac{c^2(1-y)^3}{4}-\frac{(1-c_0)^2}{4c^2(1-y)}\Big)\\ &+c\rho\beta s^2\Big(\frac{y^3}{8}+\frac{(1-y)^3}{4}\Big)-\beta^2s^3\Big(\frac{y^3}{24}+\frac{(1-y)^3}{12}\Big)-\frac{(1-c_0)^2\rho\beta s^2}{4c^3(1-y)}\bigg)dy. \qquad (3.294)\end{aligned}$$

Note that for $s \lesssim \beta^{-2/3}$ and all $y \in [y_1, y_2]$,

$$0 < -2^{-1/3}\gamma_1 c\frac{\rho}{\beta^{1/3}}+c\rho\beta s^2\Big(\frac{y^3}{8}+\frac{(1-y)^3}{4}\Big)=O\Big(\frac{c\rho}{\beta^{1/3}}\Big).$$

Since $\sqrt{1-c_0}=c$, the upper bound of R_1 in (3.294) can further be expressed as

$$\begin{aligned}R_1 \lesssim \exp\bigg(&\rho x-2\rho z+\rho L-\frac{2c^3}{3}\frac{\rho^3}{\beta}+O\Big(\frac{c\rho}{\beta^{1/3}}\Big)\bigg)\int_{y_1}^{y_2}\frac{1}{1-y}\exp\bigg(-\frac{c\rho\beta s^2}{4(1-y)}\bigg)\\ &\exp\bigg(\frac{c^3\rho^3}{4\beta}\Big(-\frac{y^3}{6}+y^2+y+1-\frac{1}{1-y}\Big)\\ &+c^2\rho^2 s\Big(\frac{1}{2}-\frac{y}{2}-\frac{y^3}{8}-\frac{(1-y)^3}{4}-\frac{1}{4(1-y)}\Big)\bigg)dy. \qquad (3.295)\end{aligned}$$

Let

$$h(y)=\frac{1}{2}-\frac{y}{2}-\frac{y^3}{8}-\frac{(1-y)^3}{4}-\frac{1}{4(1-y)}$$

Since for $y \in [0,1]$,

$$h'(y) = -\frac{1}{2} - \frac{3y^2}{8} + \frac{3(1-y)^2}{4} - \frac{1}{4(1-y)^2} = \frac{1}{4}\left(1 - \frac{1}{(1-y)^2}\right) + \frac{3y}{2}\left(\frac{y}{4} - 1\right) \leq 0,$$

we get $h(y) \leq h(0) = 0$. Also for all $y \in [y_1, y_2]$,

$$\frac{c^3}{4}\Big(-\frac{y^3}{6} + y^2 + y + 1 - \frac{1}{1-y}\Big) = -\frac{c^3 y^3}{4}\Big(\frac{1}{6} + \frac{1}{1-y}\Big) \leq -\frac{c^3 y_1^3}{24}.$$

Thus,

$$R_1 \lesssim \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} + O\Big(\frac{c\rho}{\beta^{1/3}}\Big) - \frac{c^3 y_1^3 \rho^3}{24\beta}\right)\int_{y_1}^{y_2} \frac{1}{1-y}\exp\left(- \frac{c\rho\beta s^2}{4(1-y)}\right) dy. \tag{3.296}$$

By (3.275), we have that

$$\frac{c^3 y_1^3 \rho^3}{24\beta} = \frac{c^3\rho^3}{24\beta}\Big(\frac{\beta^{-7/12}(L^* - z)^{1/4}}{t}\Big)^3 \sim \frac{c^3\rho^3}{24\beta}\Big(\frac{\beta^{-7/12}(c^2\rho^2/2\beta)^{1/4}}{c\rho/\beta}\Big)^3 = \frac{c^{3/2}\rho^{3/2}}{24\cdot 2^{3/4}\cdot \beta^{1/2}} \gg \frac{c\rho}{\beta^{1/3}}.$$

Also, because $c\rho\beta s^2 \gg 1$, after changing variables twice, we get

$$\begin{aligned}\int_{y_1}^{y_2} \frac{1}{1-y}\exp\left(- \frac{c\rho\beta s^2}{4(1-y)}\right) dy &\leq \int_0^1 \frac{1}{y}\exp\left(- \frac{c\rho\beta s^2}{4y}\right) dy \\ &= \int_1^\infty \frac{1}{y}\exp\left(- \frac{c\rho\beta s^2 y}{4}\right) dy \\ &\ll 1.\end{aligned} \tag{3.297}$$

Therefore, in equation (3.296), the term $O(c\rho/\beta^{1/3})$ can be absorbed into $-c^3 y_1^3\rho^3/24\beta$ in the exponent and the integral can be neglected. By (3.275), we conclude that

$$\begin{aligned}R_1 &\lesssim \frac{\beta^{5/3}}{c\rho^5}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right)\frac{c\rho^5}{\beta^{5/3}}\exp\left(- \frac{c^{3/2}\rho^{3/2}}{48\beta^{1/2}}\right) \\ &\ll \frac{\beta^{5/3}}{c\rho^5}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right).\end{aligned} \tag{3.298}$$

Here we want to point out that this is the only place in the proof of Lemma 43 where we need to use the assumption $c \gg \beta^{1/3}\rho^{-1}\log^{2/3}(\rho\beta^{-1/3})$ instead of the weaker one $c \gg \beta^{1/3}\rho^{-1}$.

Next, we estimate R_2. Letting $u = yt$, by similar computations as for R_1, we have

$$
\begin{aligned}
R_2 \lesssim \exp\bigg(&\rho x - 2\rho z + \rho L - \frac{c^3}{2}\frac{\rho^3}{\beta} + \frac{c^2\rho^2 s}{2} - 2^{-1/3}\gamma_1 c\frac{\rho}{\beta^{1/3}}\bigg) \\
&\int_{y_2}^1 \frac{1}{\sqrt{(1+y)(1-y)}} \exp\bigg(\frac{c^3\rho^3}{\beta}\Big(\frac{y}{2} + \frac{y^3}{24} + \frac{(1-y)^3}{12}\Big) - c^2\rho^2 s\Big(\frac{y}{2} + \frac{y^3}{8} + \frac{(1-y)^3}{4}\Big) \\
&+ c\rho\beta s^2\Big(\frac{y^3}{8} + \frac{(1-y)^3}{4}\Big) - \beta^2 s^3\Big(\frac{y^3}{24} + \frac{(1-y)^3}{12}\Big)\bigg) \\
&\times \exp\bigg(\frac{ty(1-y)}{2(1+y)}\Big(\rho + \beta t - \frac{\beta y t}{2} - \frac{L-x}{yt} - \frac{2(L-z)}{t(1-y)}\Big)^2 - \frac{(L-x)^2}{2yt} - \frac{(L-z)^2}{t(1-y)}\bigg) dy.
\end{aligned}
\tag{3.299}
$$

Denote the exponent on the last line of (3.299) as A. For $x < L$ and $y \in [y_2, 1]$, we get

$$
\begin{aligned}
A &= \frac{ty(1-y)}{2(1+y)}\bigg[\Big(\rho + \beta t - \frac{\beta y t}{2}\Big)^2 + \frac{(L-x)^2}{y^2t^2} + \frac{4(L-z)^2}{t^2(1-y)^2} - \frac{2(L-x)}{yt}\Big(\rho + \beta t - \frac{\beta y t}{2}\Big) \\
&\quad - \frac{4(L-z)}{t(1-y)}\Big(\rho + \beta t - \frac{\beta y t}{2}\Big) + \frac{4(L-x)(L-z)}{t^2 y(1-y)}\bigg] - \frac{(L-x)^2}{2yt} - \frac{(L-z)^2}{t(1-y)} \\
&= \frac{t}{2}\frac{y(1-y)}{1+y}\Big(\rho + \beta t - \frac{\beta y t}{2}\Big)^2 - \frac{(L-x)^2}{t(1+y)} - \frac{(L-z)^2}{t(1+y)} - \frac{1-y}{1+y}(L-x)\Big(\rho + \beta t - \frac{\beta y t}{2}\Big) \\
&\quad - \frac{2y}{1+y}(L-z)\Big(\rho + \beta t - \frac{\beta y t}{2}\Big) + \frac{2(L-x)(L-z)}{t(1+y)} \\
&\le \frac{t}{2}\frac{y(1-y)}{1+y}\Big(\rho + \beta t - \frac{\beta y t}{2}\Big)^2 - \frac{(L-z)^2}{t(1+y)} - \frac{2y}{1+y}(L-z)\Big(\rho + \beta t - \frac{\beta y t}{2}\Big) \\
&\quad + \frac{2(L-x)(L-z)}{t(1+y)}.
\end{aligned}
$$

Recalling that $t = c\rho/\beta - s$ and $L^* - z = c^2\rho^2/2\beta$, we have

$$
\begin{aligned}
A \le \frac{y(1-y)}{2(1+y)}\Big(\frac{c\rho}{\beta} - s\Big)\Big(\rho + \beta\Big(\frac{c\rho}{\beta} - s\Big) - \frac{\beta y}{2}\Big(\frac{c\rho}{\beta} - s\Big)\Big)^2 - \frac{(c^2\rho^2/2\beta - (2\beta)^{-1/3}\gamma_1)^2}{(1+y)(c\rho/\beta - s)} \\
- \frac{2y}{1+y}\frac{c^2\rho^2}{2\beta}\Big(\rho + \beta\Big(\frac{c\rho}{\beta} - s\Big) - \frac{\beta y}{2}\Big(\frac{c\rho}{\beta} - s\Big)\Big) + \frac{2(L-x)(c^2\rho^2/2\beta - (2\beta)^{-1/3}\gamma_1)}{(c\rho/\beta - s)(1+y)}.
\end{aligned}
\tag{3.300}
$$

Because $L - x \lesssim \beta^{-1/3}$ and $s \lesssim \beta^{-2/3}$, we see that for $y \in [y_2, 1]$,

$$\frac{2(L-x)(c^2\rho^2/2\beta - (2\beta)^{-1/3}\gamma_1)}{(c\rho/\beta - s)(1+y)} \lesssim \frac{c\rho}{\beta^{1/3}}.$$

We also observe that $\rho\beta s^2 \lesssim \rho/\beta^{1/3}$, $\beta^2 s^3 \ll \rho/\beta^{1/3}$ and

$$-\frac{(c^2\rho^2/2\beta - (2\beta)^{-1/3}\gamma_1)^2}{(1+y)(c\rho/\beta - s)} \le -\frac{c^4\rho^4/4\beta^2}{(1+y)c\rho/\beta}\sum_{k=0}^{\infty}\left(\frac{s}{c\rho/\beta}\right)^k \le -\frac{c^3\rho^3}{4\beta(1+y)} - \frac{c^2\rho^2 s}{4(1+y)}.$$

Therefore, equation (3.300) implies that

$$\begin{aligned} A \le &\frac{\rho^3}{\beta}\left[\frac{y(1-y)}{2(1+y)}c\left(1+c-\frac{cy}{2}\right)^2 - \frac{c^3}{4(1+y)} - \frac{y}{1+y}c^2\left(1+c-\frac{cy}{2}\right)\right] + \rho^2 s\left[-\frac{c^2}{4(1+y)}\right.\\ &\left. - \left(1-\frac{y}{2}\right)\left(1+c-\frac{cy}{2}\right)\frac{cy(1-y)}{1+y} - \frac{y(1-y)}{2(1+y)}\left(1+c-\frac{cy}{2}\right) + \frac{y}{1+y}\left(1-\frac{y}{2}\right)c^2\right]\\ &+ O\left(\frac{\rho}{\beta^{1/3}}\right). \end{aligned} \tag{3.301}$$

By (3.299) and (3.301), we have

$$\begin{aligned} R_2 \lesssim &\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} + O\left(\frac{\rho}{\beta^{1/3}}\right)\right)\int_{y_2}^{1}\frac{1}{\sqrt{(1+y)(1-y)}}\\ &\times\exp\left(\frac{\rho^3}{\beta}\left(\frac{c^3 y}{4} + \frac{c^3 y^2}{4} - \frac{c^3 y^3}{24} + \frac{c^3}{4} + \frac{y(1-y)}{2(1+y)}c\left(1+c-\frac{cy}{2}\right)^2 - \frac{c^3}{4(1+y)}\right.\right.\\ &\left. - \frac{y}{1+y}c^2\left(1+c-\frac{cy}{2}\right)\right) + \rho^2 s\left(\frac{c^2}{2} - \frac{c^2 y}{2} - \frac{c^2 y^3}{8} - \frac{c^2(1-y)^3}{4} - \frac{c^2}{4(1+y)}\right.\\ &\left.\left. - \left(1-\frac{y}{2}\right)\left(1+c-\frac{cy}{2}\right)\frac{cy(1-y)}{1+y} - \frac{y(1-y)}{2(1+y)}\left(1+c-\frac{cy}{2}\right)^2 + \frac{y}{1+y}\left(1-\frac{y}{2}\right)c^2\right)\right)dy. \end{aligned} \tag{3.302}$$

Define

$$\phi(y) = -\frac{2y^3}{3} + \left(\frac{10}{3} + \frac{2}{c}\right)y^2 - \left(\frac{2}{c^2} + \frac{6}{c}\right)y + \frac{2}{c^2}, \tag{3.303}$$

and

$$\psi(y) = \frac{c^2y^3}{2} - \Big(\frac{5c^2}{2} + c\Big)y^2 + \Big(2c^2 + 3c + \frac{1}{2}\Big)y - 2c - \frac{1}{2}. \tag{3.304}$$

After algebraic calculation, equation (3.302) is equivalent to

$$\begin{aligned} R_2 \lesssim \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} + O\Big(\frac{\rho}{\beta^{1/3}}\Big)\right) \int_{y_2}^1 \frac{1}{\sqrt{(1+y)(1-y)}} \\ \times \exp\left(\frac{c^3\rho^3}{\beta}\frac{y}{4(1+y)}\phi(y) + \rho^2 s \frac{y}{1+y}\psi(y)\right) dy. \end{aligned} \tag{3.305}$$

Below we will obtain the upper bounds for $\phi(y)$ and $\psi(y)$ in the cases $z \geq 0$ and $z < 0$.

Let us first study $\phi(y)$. Note that for every c, we have $\phi''(y) > 0$ for all $y \in [y_2, 1]$. Therefore, for $y \in [y_2, 1]$, the function $\phi(y)$ reaches its maximum either at 1 or at y_2. When $y = 1$, we have for all $c \in (0, 3/2)$,

$$\phi(1) = \frac{8}{3} - \frac{4}{c} = -\frac{8(3/2 - c)}{3c} < 0. \tag{3.306}$$

For $c \in (0, 3/2)$, since $0 < y_2 = y_c - C_{16}c \leq 1$, after rearranging terms, we have

$$\begin{aligned} \phi(y_2) &= \frac{2}{3}y_2^2(1 - y_c + C_{16}c) + \Big(\frac{8}{3} + \frac{2}{c}\Big)y_2^2 - \Big(\frac{2}{c^2} + \frac{6}{c}\Big)(y_c - C_{16}c) + \frac{2}{c^2} \\ &\leq \frac{2}{3}y_c^2(1 - y_c) + \Big(\frac{8}{3} + \frac{2}{c}\Big)y_c^2 - \Big(\frac{2}{c^2} + \frac{6}{c}\Big)y_c + \frac{2}{c^2} + \frac{2C_{16}c}{3} + \frac{2C_{16}}{c} + 6C_{16} \\ &= \phi(y_c) + \frac{2C_{16}c}{3} + \frac{2C_{16}}{c} + 6C_{16}. \end{aligned}$$

Since y_c satisfies (3.289), we have $y_c^2 = y_c(3 + 2/c) - 2/c$. Replacing y_c^2 with $y_c(3 + 2/c) - 2/c$

in the first step and replacing y_c with $3/2 + 1/c - \Delta/c$ in the second step, we get

$$\begin{aligned}
\phi(y_c) &= 4y_c - \frac{2}{3c^2}y_c - \frac{8}{3c} + \frac{2}{3c^2} \\
&= 6 - \frac{4\Delta}{c} + \frac{4}{3c} - \frac{1}{3c^2} - \frac{2}{3c^3} + \frac{2\Delta}{3c^3} \\
&= \frac{2(9c^2 - 4\Delta^2)}{c(3c + 2\Delta)} + \frac{4}{3c} - \frac{1}{3c^2} + \frac{2(\Delta^2 - 1)}{3c^3(\Delta + 1)} \\
&= \frac{-8c - 8}{(3c + 2\Delta)c} + \frac{4}{3c} + \frac{3}{2c(\Delta + 1)} - \frac{1}{3c^2} + \frac{2}{3c^2(\Delta + 1)}.
\end{aligned}$$

For $c \in (0, 3/2)$, we have $\Delta > 1$. According to the above two formulas, we have

$$\begin{aligned}
\phi(y_2) &\leq \frac{-8c - 8}{(3c + 2\Delta)c} + \frac{4}{3c} + \frac{3}{2c(\Delta + 1)} + \frac{2C_{16}c}{3} + \frac{2C_{16}}{c} + 6C_{16} \\
&= \frac{-24\Delta c + 3c - 14\Delta - 48 + 16\Delta^2}{6c(3c + 2\Delta)(\Delta + 1)} + \frac{2C_{16}c}{3} + \frac{2C_{16}}{c} + 6C_{16}.
\end{aligned} \tag{3.307}$$

If $z \geq 0$, then $c \in (0, 1]$ and $1 < \Delta \leq \sqrt{17}/2$. We have

$$-24\Delta c + 3c < -21c,$$

and

$$\begin{aligned}
-14\Delta - 48 + 16\Delta^2 &= 14\Delta(\Delta - 1) - 48 + 2\Delta^2 \\
&\leq 14 \cdot \frac{\sqrt{17}}{2}\left(\frac{\sqrt{17}}{2} - 1\right) - 48 + 2\left(\frac{\sqrt{17}}{2}\right)^2 \\
&< -8.
\end{aligned}$$

Therefore, combining the above two observations with equations (3.281), (3.282) and (3.307), we obtain

$$\begin{aligned}\phi(y_2) &\le \Big(-\frac{4}{3(3+\sqrt{17})(1+\sqrt{17}/2)}+2C_{16}\Big)\frac{1}{c}-\frac{7}{2(3+\sqrt{17})(1+\sqrt{17}/2)}+\frac{2}{3}C_{16}+6C_{16}\\ &< -\frac{C_{16}}{c}. \end{aligned} \tag{3.308}$$

According to equations (3.306) and (3.308), we get when $z \ge 0$, or equivalently $c \in (0,1]$,

$$\max_{y\in[y_2,1]}\phi(y) \le \max\Big\{-\frac{8(3/2-c)}{3c}, -\frac{C_{16}}{c}\Big\} = -\frac{C_{16}}{c}. \tag{3.309}$$

If $z<0$, then $c \in (1,3/2)$ and $\sqrt{17}/2 < \Delta < 11/4$. We have

$$\begin{aligned}-24\Delta c+3c-14\Delta-48+16\Delta^2 &< -21\Delta c+14\Delta(\Delta-1)+2\Delta^2-48\\ &< -21\cdot\frac{\sqrt{17}}{2}\cdot 1+14\cdot\frac{11}{4}\Big(\frac{11}{4}-1\Big)+2\cdot\frac{11}{4}\cdot\frac{11}{4}-48\\ &< -8.\end{aligned}$$

Therefore, combining the above two observations with equations (3.283) and (3.307), since $c \in (0,3/2)$, we obtain

$$\phi(y_2) \le \frac{-8}{9(3\cdot\frac{3}{2}+2\cdot\frac{11}{4})(\frac{11}{4}+1)}+9C_{16} < -C_{16}.$$

According to equations (3.306) and (3.308), we get when $z \le 0$, or equivalently $c \in (1,3/2)$,

$$\max_{y\in[y_2,1]}\phi(y) \le \max\Big\{-\frac{8(3/2-c)}{3c}, -C_{16}\Big\}. \tag{3.310}$$

We next study $\psi(y)$. Let us first consider the case $z \ge 0$, or equivalently, $c \in (0,1]$. For all $y \in [y_2,1]$, we have

$$\psi(y) \le \frac{c^2y^2}{2}-\Big(\frac{5c^2}{2}+c\Big)y^2+(2c^2+c)y = c(2c+1)y(1-y) \le 3c. \tag{3.311}$$

If $z<0$, we claim that for all $y \in [y_2, 1]$ and $c \in (0, 3/2)$, we have

$$\psi(y) \le -3c^2\phi(y)/4. \tag{3.312}$$

Indeed, for every y, we can view $-3c^2\phi(y)/4 - \psi(y)$ as a quadratic function of c:

$$-\frac{3c^2\phi(y)}{4} - \psi(y) = -2yc^2 + c\bigg(-\frac{1}{2}y^2 + \frac{3}{2}y + 2\bigg) + y - 1 =: \varphi(c).$$

Note that for every $y \in (0, 1]$, the quadratic function $\varphi(c)$ for $c \in [1, 3/2]$ reaches its minimum at either $c = 1$ or $c = 3/2$. Since for all $y \in (0, 1]$, we have

$$\varphi(1) = -\frac{1}{2}(y^2 - y - 2) > 0, \qquad \varphi\Big(\frac{3}{2}\Big) = -\frac{1}{4}(3y^2 + 5y - 8) \ge 0,$$

the claim follows.

Now it remains to apply the upper bounds of $\phi(y)$ and $\psi(y)$ in (3.305). By (3.285), for all $y \in [y_2, 1]$, we have

$$C_{16}/2 \le y/(1+y) \le 1. \tag{3.313}$$

When $z \ge 0$, combining (3.305) with (3.309), (3.311) and (3.313), we obtain

$$R_2 \lesssim \exp\bigg(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{C_{17}^2}{8}\frac{c^2\rho^3}{\beta} + 3c\rho^2 s + O\Big(\frac{\rho}{\beta^{1/3}}\Big)\bigg). \tag{3.314}$$

Recalling that $c \gg \beta^{1/3}/\rho$ and $s \lesssim \beta^{-2/3}$, we have

$$\frac{c^2\rho^3}{\beta} \gg c\rho^2 s, \quad \frac{c^2\rho^3}{\beta} \gg \frac{\rho}{\beta^{1/3}}, \quad \exp\bigg(-\frac{C_{17}^2}{16}\frac{c^2\rho^3}{\beta}\bigg) \ll \frac{\beta^{5/3}}{c\rho^5}.$$

Consequently,

$$R_2 \lesssim \frac{\beta^{5/3}}{c\rho^5}\exp\bigg(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\bigg). \tag{3.315}$$

If $z < 0$, since $c^3\rho^3/\beta \gg c^2\rho^2 s$, according to (3.305), (3.310), (3.312) and (3.313), we have for n sufficiently large

$$\begin{aligned} R_2 &\lesssim \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} + O\Big(\frac{\rho}{\beta^{1/3}}\Big)\right) \\ &\quad \times \int_{y_2}^{1} \frac{1}{\sqrt{(1+y)(1-y)}} \exp\left(\frac{c^3\rho^3}{\beta}\frac{y\phi(y)}{8(1+y)}\right) dy \\ &\lesssim \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{C_{16}}{16}\frac{c^3\rho^3}{\beta}\min\left\{\frac{8(3/2-c)}{3c}, C_{16}\right\} + O\Big(\frac{\rho}{\beta^{1/3}}\Big)\right). \end{aligned}$$

By (3.275), since $c > 1$, we have

$$\frac{c^3\rho^3}{\beta}\min\left\{\frac{8(3/2-c)}{3c}, C_{16}\right\} \gg \frac{\rho}{\beta^{1/3}}, \quad \exp\left(-\frac{c^3\rho^3}{\beta}\min\left\{\frac{8(3/2-c)}{3c}, C_{16}\right\}\right) \ll \frac{\beta^{5/3}}{c\rho^5}.$$

Therefore, equation (3.315) also holds when $z < 0$.

Finally, combining (3.298) and (3.315), equation (3.277) is proved and the lemma follows. □

Proof of Lemma 44. Recall that in the proof of Lemma 43, the only place where we used the assumption $L^* - z \gg \beta^{-1/3}\log^{4/3}(\rho/\beta^{1/3})$ is equation (3.298). Thus to prove Lemma 44, it is sufficient to prove that for z satisfying $\beta^{-1/3} \ll L^* - z \lesssim \beta^{-1/3}\log^{4/3}(\rho/\beta^{1/3})$, or equivalently, $\beta^{1/3}\rho^{-1} \ll c \lesssim \beta^{1/3}\rho^{-1}\log^{2/3}(\rho/\beta^{1/3})$, we have

$$I_2' := \int_{u_1}^{u_2}\int_{-\infty}^{L} p_u^L(x,r)\Big(p_{t-u}^L(r,z)\Big)^2 dr du \lesssim \frac{\beta^{5/3}}{c\rho^5}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right). \quad (3.316)$$

The portion of the double integral in I_2 for which $u_2 \le u \le t$ has been dealt with in Lemma 43. By (3.4), we can choose constant $C_{17} > 0$ such that

$$2^{-1/3}\gamma_1 + C_{17} + 1 < -\frac{1}{2} \quad (3.317)$$

According to equations (3.88) and (3.287), and the fourth equality in (3.279), we get

$$t - u_2 = t(1 - y_2) = t(1 - y_c + C_{16}c) \leq t(z)\left(\frac{9c}{4(\Delta+1)} + C_{16}c\right) \asymp \frac{c^2\rho}{\beta} \ll \beta^{-2/3}. \quad (3.318)$$

Thus $t - u_2 < 2\beta^{-2/3}$ for n large enough. Therefore, for n large enough, we can write

$$I_2' = P_1 + P_2 + P_3,$$

where P_1 is the part of the double integral for which $L - C_{17}\beta^{-1/3} \leq r \leq L$ and $u_1 \leq u \leq t - 2\beta^{-2/3}$, P_2 is the part of the double integral for which $L - C_{17}\beta^{-1/3} \leq r \leq L$ and $t - 2\beta^{-2/3} < u \leq u_2$, and P_3 is the part of the double integral for which $r < L - C_{17}\beta^{-1/3}$ and $u_1 \leq u \leq u_2$.

To bound P_1, we are going to bound $p_u^L(x,r)$ by (3.263) and $p_{t-u}^L(r,z)$ by (3.264). We get

$$\begin{aligned}
P_1 \lesssim & \int_{u_1}^{t-2\beta^{-2/3}} \int_{L-C_{17}\beta^{-1/3}}^{L} \frac{(L-x)(L-r)}{u^{3/2}} \exp\left(\rho x - \rho r - \frac{(x-r)^2}{2u} - \frac{\rho^2 u}{2} + \beta L u\right) \\
& \times \frac{\beta^{2/3}(L-r)^2}{t-u}\left(\max\left\{1, \frac{1}{\beta^{1/3}(t-u)}\left(L - z - \frac{\beta(t-u)^2}{2}\right)\right\}\right)^2 \\
& \times \exp\left(2\rho r - 2\rho z - \frac{(r-z)^2}{t-u} - \rho^2(t-u) + \beta(r+z)(t-u) + \frac{\beta^2(t-u)^3}{12}\right. \\
& \left. + \frac{1}{\beta^{1/3}(t-u)}\left(L - z - \frac{\beta(t-u)^2}{2}\right)\right) dr du.
\end{aligned}$$

Note that $L - x \lesssim \beta^{-1/3}$ and $t = t(z) - s$. Interchanging the roles of r and $L - r$, we have

$$P_1 \lesssim \int_{u_1}^{t-2\beta^{-2/3}} \frac{\beta^{1/3}}{u^{3/2}(t-u)} \left(\max\left\{1, \frac{1}{\beta^{1/3}(t-u)} \left(L - z - \frac{\beta(t-u)^2}{2} \right) \right\} \right)^2$$
$$\exp\left(\rho x - 2\rho z + \rho L - \frac{\rho^2 u}{2} + \beta L u - \rho^2 \big(t(z) - s - u\big) + \frac{\beta^2 (t(z) - s - u)^3}{12} \right.$$
$$+ \frac{1}{\beta^{1/3}(t-u)} \left(L - z - \frac{\beta(t-u)^2}{2} \right) \Bigg) \int_0^{C_{17}\beta^{-1/3}} r^3 \exp\bigg(-\rho r - \frac{(L-x-r)^2}{2u}$$
$$\left. - \frac{(L-r-z)^2}{t-u} + \beta(L - r + z)(t(z) - s - u) \right) dr du. \tag{3.319}$$

We first estimate the term $(L - z - \beta(t-u)^2/2)/(\beta^{1/3}(t-u))$. For $u \geq u_1$, we see that

$$\frac{\beta(u+s)^2}{2} \geq \frac{\beta u_1^2}{2} = \frac{\beta}{2} \big(\beta^{-7/12} (L^* - z)^{1/4} \big)^2 \asymp \frac{c\rho}{\beta^{2/3}} \gg \beta^{-1/3} \asymp (2\beta)^{-1/3} |\gamma_1|.$$

Thus by (3.272), we have for n large enough

$$\frac{1}{\beta^{1/3}(t-u)} \left(L - z - \frac{\beta(t-u)^2}{2} \right) \leq \frac{\beta t(z)(u+s)}{\beta^{1/3}(t-u)} = \frac{\beta^{2/3} t(z)(u+s)}{t-u}. \tag{3.320}$$

Furthermore, we note that

$$\frac{\beta^{2/3} t(z)(u+s)}{t-u} \geq \frac{\beta^{2/3} t(z) u_1}{t(z)} = \beta^{2/3} u_1 \asymp \frac{c^{1/2} \rho^{1/2}}{\beta^{1/6}} \gg 1.$$

Moreover, we will upper bound the term $-(L-r-z)^2/(t-u)$ by (3.266). By (3.266), (3.319) and (3.320), after rearranging terms, we get for n large

$$
\begin{aligned}
P_1 \lesssim \exp\Bigg(&\rho x - 2\rho z + \rho L - \rho^2 t(z) + \beta L^* t(z) + \beta z t(z) + \frac{\beta^2 t(z)^3}{12} - \frac{(L^*-z)^2}{t(z)}\Bigg) \\
&\int_{u_1}^{t-2\beta^{-2/3}} \frac{\beta^{5/3} t(z)^2 (u+s)^2}{u^{3/2}(t-u)^3} \exp\Bigg(- \frac{\rho^2 u}{2} + \beta L u + \rho^2(u+s) - \frac{\beta^2 (u+s)^3}{12} \\
&+ \frac{\beta^2 t(z)(u+s)^2}{4} - \frac{\beta^2 t(z)^2 (u+s)}{4} + \frac{\beta^{2/3} t(z)(u+s)}{t-u} - \beta L(u+s) - 2^{-1/3}\beta^{2/3}\gamma_1 t(z) \\
&- \beta z(u+s) - \frac{(L^*-z)^2(u+s)}{t(z)^2} - \frac{(L^*-z)^2(u+s)^2}{t(z)^3} - \frac{(L^*-z)^2(u+s)^3}{t(z)^4}\Bigg) \\
&\int_0^{C_{17}\beta^{-1/3}} r^3 \exp\Bigg(- \rho r - \beta r(t(z)-s-u) - \frac{(L-x-r)^2}{2u} \\
&+ \frac{2(L^*-z)((2\beta)^{-1/3}\gamma_1 + r)}{t-u}\Bigg) dr du.
\end{aligned}
$$

Notice that since $t(z) = \sqrt{2/\beta}\sqrt{L^*-z}$, $L = L^* - (2\beta)^{-1/3}\gamma_1$ and $s \lesssim \beta^{-2/3}$, we observe that

$$
\begin{aligned}
&- \frac{\rho^2 u}{2} + \beta L u + \rho^2(u+s) - \frac{\beta^2 t(z)^2(u+s)}{4} - \beta L(u+s) - \beta z(u+s) - \frac{(L^*-z)^2(u+s)}{t(z)^2} \\
&= \frac{\rho^2 u}{2} + \rho^2 s - \beta L s - \beta z(u+s) - \beta(L^*-z)(u+s) \\
&= \Big(\frac{\rho^2}{2} - \beta L\Big)s \\
&= O(1).
\end{aligned}
$$

Also

$$
\frac{\beta^2 t(z)(u+s)^2}{4} - \frac{(L^*-z)^2(u+s)^2}{t(z)^3} = 0
$$

and

$$
-\frac{\beta^2(u+s)^3}{12} - \frac{(L^*-z)^2(u+s)^3}{t(z)^4} = -\frac{\beta^2(u+s)^3}{3}.
$$

By the above four equations and (3.268), we can further bound P_1 as follows:

$$P_1 \lesssim \exp\left(\rho x - 2\rho z + \rho L - \frac{4\sqrt{2\beta}}{3}(L^* - z)^{3/2}\right) \int_{u_1}^{t-2\beta^{-2/3}} \frac{\beta^{5/3} t(z)^2 (u+s)^2}{u^{3/2}(t-u)^3}$$
$$\times \exp\bigg(- \frac{\beta^2(u+s)^3}{3} + \frac{\beta^{2/3} t(z)(u+s)}{t-u} - 2^{-1/3}\beta^{2/3}\gamma_1 t(z)$$
$$+ \frac{2(L^*-z)(2^{-1/3}\gamma_1 + C_{17})\beta^{-1/3}}{t-u}\bigg) \times \int_0^{C_{17}\beta^{-1/3}} r^3 e^{-\rho r} dr du.$$

Recall that $c \gg \beta^{1/3}/\rho$. By equation (3.88), for n sufficiently large, for all $u_1 \le u \le t - 2\beta^{-2/3}$, we have

$$-\frac{\beta^2(u+s)^3}{3} - 2^{-1/3}\beta^{2/3}\gamma_1 t(z) \le -\frac{\beta^2 u_1^3}{3} - \frac{\gamma_1 c\rho}{2^{1/3}\beta^{1/3}} = -\frac{c^{3/2}\rho^{3/2}}{3\cdot 2^{3/4}\beta^{1/2}} - \frac{\gamma_1 c\rho}{2^{1/3}\beta^{1/3}} \le -\frac{c^{3/2}\rho^{3/2}}{6\beta^{1/2}}.$$

Also, by equations (3.88) and (3.317), since $u + s \le t(z) = c\rho/\beta$, we have for all $u_1 \le u \le t - 2\beta^{-2/3}$,

$$\begin{aligned}
&\frac{\beta^{2/3} t(z)(u+s)}{t-u} + \frac{2(L^*-z)(2^{-1/3}\gamma_1 + C_{17})\beta^{-1/3}}{t-u} \\
&= \frac{c\rho}{\beta^{1/3}(t-u)}\left((u+s) + \frac{c\rho}{\beta}\left(2^{-1/3}\gamma_1 + C_{17}\right)\right) \\
&\le \frac{c^2\rho^2}{\beta^{4/3}(t-u)}\left(1 + 2^{-1/3}\gamma_1 + C_{17}\right) \\
&\le -\frac{c^2\rho^2}{2\beta^{4/3}(t-u)}.
\end{aligned}$$

Combining the above three equations with (3.88), after some standard calculations, we get

$$P_1 \lesssim \frac{c^2}{\rho^2\beta^{1/3}} \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{c^{3/2}\rho^{3/2}}{6\beta^{1/2}}\right)$$
$$\int_{u_1}^{t-2\beta^{-2/3}} \frac{(u+s)^2}{u^{3/2}(t-u)^3} \exp\left(- \frac{c^2\rho^2}{2\beta^{4/3}(t-u)}\right) du.$$

Note that for n large, we have $u + s \le 2u$ for all $u_1 \le u \le t - 2\beta^{-2/3}$. Let $v = t - u$. The integral in the previous equation can be upper bounded by

$$
\begin{aligned}
4\int_{u_1}^{t-2\beta^{-2/3}} \frac{u^{1/2}}{(t-u)^3}\exp\Big(-\frac{c^2\rho^2}{2\beta^{4/3}(t-u)}\Big)du &\leq \frac{4c^{1/2}\rho^{1/2}}{\beta^{1/2}}\int_{2\beta^{-2/3}}^{t-u_1} v^{-3}\exp\Big(-\frac{c^2\rho^2}{2\beta^{4/3}v}\Big)dv \\
&\leq \frac{4c^{1/2}\rho^{1/2}}{\beta^{1/2}}\int_{0}^{\infty} v^{-3}\exp\Big(-\frac{c^2\rho^2}{2\beta^{4/3}v}\Big)dv \\
&= \frac{4c^{1/2}\rho^{1/2}}{\beta^{1/2}}\cdot\frac{4\beta^{8/3}}{c^4\rho^4}. \qquad (3.321)
\end{aligned}
$$

Combining the above two formulas, because $c\rho/\beta^{1/3} \gg 1$, we get

$$
\begin{aligned}
P_1 &\lesssim \frac{\beta^{11/6}}{c^{3/2}\rho^{11/2}}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{c^{3/2}\rho^{3/2}}{6\beta^{1/2}}\right) \\
&\ll \frac{\beta^{5/3}}{c\rho^5}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right). \qquad (3.322)
\end{aligned}
$$

We next estimate P_2. We are going to bound both $p_u^L(x,r)$ and $p_{t-u}^L(r,z)$ by (3.263). Interchanging the roles of r and $L-r$, we get

$$
\begin{aligned}
P_2 \lesssim\ & (L-x)(L-z)^2\exp\left(\rho x - 2\rho z + \rho L - \rho^2 t + 2\beta L t\right) \\
&\times \int_{t-2\beta^{-2/3}}^{u_2} \frac{1}{u^{3/2}(t-u)^3}\exp\left(\frac{\rho^2 u}{2} - \beta L u\right) \\
&\int_0^{C_{17}\beta^{-1/3}} r^3\exp\left(-\rho r - \frac{(L-x-r)^2}{2u} - \frac{(L-r-z)^2}{t-u}\right)dr du. \qquad (3.323)
\end{aligned}
$$

By (3.317), we have for all $0 \leq r \leq C_{17}\beta^{-1/3}$,

$$
-\frac{(L-r-z)^2}{t-u} = -\frac{(L^*-z)^2}{t-u} - \frac{((2\beta)^{-1/3}\gamma_1 + r)^2}{t-u} + \frac{2(L^*-z)((2\beta)^{-1/3}\gamma_1 + r)}{t-u} \leq -\frac{(L^*-z)^2}{t-u}.
$$

Thus for $t - 2\beta^{-2/3} \leq u \leq u_2$, the inner integral in (3.323) can be upper bounded by

$$
\begin{aligned}
\int_0^{\infty} r^3\exp\Big(-\rho r - \frac{(L^*-z)^2}{t-u}\Big)dr &\lesssim \frac{1}{\rho^4}\exp\Big(-\frac{(L^*-z)^2}{t-u}\Big) \\
&\leq \frac{1}{\rho^4}\exp\left(-\frac{(L^*-z)^2}{2(t-u)} - \frac{(L^*-z)^2}{2\cdot 2\beta^{-2/3}}\right).
\end{aligned}
$$

Then equation (3.323) becomes

$$P_2 \lesssim \frac{(L-x)(L-z)^2}{\rho^4} \exp\left(\rho x - 2\rho z + \rho L - \rho^2 t + 2\beta L t - \frac{\beta^{2/3}(L^* - z)^2}{4}\right)$$
$$\int_{t-2\beta^{-2/3}}^{u_2} \frac{1}{u^{3/2}(t-u)^3} \exp\left(\frac{\rho^2 u}{2} - \beta L u - \frac{(L^*-z)^2}{2(t-u)}\right) du.$$

Expressing $L - z$ and t in terms of c, since $L - x \lesssim \beta^{-1/3}$, we get

$$P_2 \lesssim \frac{c^4}{\beta^{7/3}} \exp\left(\rho x - 2\rho z + \rho L - 2^{2/3}\gamma_1 \frac{c\rho}{\beta^{1/3}} - \frac{c^4\rho^4}{16\beta^{4/3}}\right)$$
$$\times \int_{t-2\beta^{-2/3}}^{u_2} \frac{1}{u^{3/2}(t-u)^3} \exp\left(-\frac{c^4\rho^4}{8\beta^2(t-u)}\right) du.$$

By applying the same argument as in (3.321), the integral in the previous equation can be upper bounded by

$$\begin{aligned}\frac{1}{(t-2\beta^{-2/3})^{3/2}} \int_{t-2\beta^{-2/3}}^{u_2} \frac{1}{(t-u)^3} \exp\left(-\frac{c^4\rho^4}{8\beta^2(t-u)}\right) &\lesssim \frac{\beta^{3/2}}{c^{3/2}\rho^{3/2}} \int_0^\infty \frac{1}{v^3} \exp\left(-\frac{c^4\rho^4}{8\beta^2 v}\right) dv \\ &\asymp \frac{\beta^{11/2}}{c^{19/2}\rho^{19/2}}.\end{aligned}$$

Combining the above two equations, since $c\rho/\beta^{1/3} \gg 1$, we have

$$\begin{aligned}P_2 &\lesssim \frac{\beta^{19/6}}{c^{11/2}\rho^{19/2}} \exp\left(\rho x - 2\rho z + \rho L - 2^{2/3}\gamma_1 \frac{c\rho}{\beta^{1/3}} - \frac{c^4\rho^4}{16\beta^{4/3}}\right) \\ &\ll \frac{\beta^{5/3}}{c\rho^5} \exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right). \end{aligned} \tag{3.324}$$

It remains to estimate P_3. We are going to bound both $p_u^L(x,r)$ and $p_{t-u}^L(r,z)$ by

(3.45). By a similar calculation as in (3.276), we get

$$\begin{aligned}
P_3 &\lesssim \exp\Big(\rho x - 2\rho z + \rho L - \rho^2 t + \beta L t + \beta z t\Big)\\
&\quad \times \int_{u_1}^{u_2} u^{-1/2}(t-u)^{-1}\exp\Big(\frac{\rho^2 u}{2} + \frac{\beta x u}{2} - \beta z u - \frac{\beta L u}{2} + \frac{\beta^2 u^3}{24} + \frac{\beta^2 (t-u)^3}{12}\Big)\\
&\qquad \int_{C_{17}\beta^{-1/3}}^{\infty} \exp\Big(-r\Big(\rho + \beta t - \frac{\beta u}{2}\Big) - \frac{(L-x-r)^2}{2u} - \frac{(L-z-r)^2}{t-u}\Big) dr du\\
&\le \exp\Big(\rho x - 2\rho z + \rho L - \rho^2 t + \beta L t + \beta z t - \frac{C_{17}\rho}{\beta^{1/3}}\Big)\\
&\quad \times \int_{u_1}^{u_2} (t-u)^{-1}\exp\Big(\frac{\rho^2 u}{2} + \frac{\beta x u}{2} - \beta z u - \frac{\beta L u}{2} + \frac{\beta^2 u^3}{24} + \frac{\beta^2 (t-u)^3}{12}\Big) du.
\end{aligned}$$

Note that the above upper bound is very similar to R_1 defined in (3.293). Therefore, by carrying out the same calculation as for R_1 in (3.295), we obtain

$$\begin{aligned}
P_3 &\lesssim \exp\Big(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{C_{17}\rho}{\beta^{1/3}} + O\Big(\frac{c\rho}{\beta^{1/3}}\Big)\Big)\\
&\quad \times \int_{y_1}^{y_2} \frac{1}{1-y}\exp\Big(\frac{c^3\rho^3}{4\beta}\Big(-\frac{y^3}{6} + y^2 + y + 1\Big) + c^2\rho^2 s\Big(\frac{1}{2} - \frac{y}{2} - \frac{y^3}{8} - \frac{(1-y)^3}{4}\Big)\Big) dy,
\end{aligned} \tag{3.325}$$

where $y_1 = u_1/t$ and $y_2 = u_2/t$. We see that for $c \lesssim \beta^{1/3}\rho^{-1}\log^{2/3}(\rho/\beta^{1/3})$ and $y \in [y_1, y_2]$,

$$\frac{c^3\rho^3}{4\beta}\Big(-\frac{y^3}{6} + y^2 + y + 1\Big) \le \frac{3c^3\rho^3}{4\beta} \ll \frac{\rho}{\beta^{1/3}},$$

$$c^2\rho^2 s\Big(\frac{1}{2} - \frac{y}{2} - \frac{y^3}{8} - \frac{(1-y)^3}{4}\Big) \lesssim \frac{c^2\rho^2}{\beta^{2/3}} \ll \frac{\rho}{\beta^{1/3}}.$$

Thus the integral in (3.325) can be bounded by

$$\exp\Big(o\Big(\frac{\rho}{\beta^{1/3}}\Big)\Big)\int_{y_1}^{y_2}\frac{1}{1-y}dy \le \exp\Big(o\Big(\frac{\rho}{\beta^{1/3}}\Big)\Big)\int_{1-y_2}^{1}\frac{1}{v}dv = \exp\Big(o\Big(\frac{\rho}{\beta^{1/3}}\Big)\Big)\log\Big(\frac{1}{1-y_2}\Big). \tag{3.326}$$

According to (3.278) and (3.284), we see that $1 - y_2 = 1 - y_c + C_{16}c \ge C_{16}c$. Thus, since

$c \gg \beta^{1/3}\rho^{-1}$, we have

$$\log\left(\frac{1}{1-y_2}\right) \leq \log\left(\frac{1}{C_{16}c}\right) \lesssim \log\left(\frac{\rho}{\beta^{1/3}}\right). \tag{3.327}$$

Combining (3.326) and (3.327) with (3.325), since $c \lesssim \beta^{1/3}\rho^{-1}\log^{2/3}(\rho/\beta^{1/3})$, we get

$$\begin{aligned} P_3 &\lesssim \log\left(\frac{\rho}{\beta^{1/3}}\right)\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta} - \frac{C_{17}\rho}{\beta^{1/3}} + O\Big(\frac{c\rho}{\beta^{1/3}}\Big) + o\Big(\frac{\rho}{\beta^{1/3}}\Big)\right) \\ &\ll \frac{\beta^{5/3}}{c\rho^5}\exp\left(\rho x - 2\rho z + \rho L - \frac{2c^3}{3}\frac{\rho^3}{\beta}\right). \end{aligned} \tag{3.328}$$

Finally, equation (3.316) follows from (3.322), (3.324) and (3.328) and the lemma follows.

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