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Essays on Prior-Free Mechanism Design

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Economics

by

Pavel Andreyanov

2019

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ABSTRACT OF THE DISSERTATION

Essays on Prior-Free Mechanism Design

by

Pavel Andreyanov
Doctor of Philosophy in Economics
University of California, Los Angeles, 2019
Professor John William Asker, Co-chair
Professor Tomasz Marek Sadzik, Co-chair

My dissertation contributes to the literature on prior-free (robust) mechanism design. Prior-freeness can be interpreted differently, but a common feature is that certain mechanisms can be ranked above the others without the exact knowledge of distributions and/or utilities. According to the Wilson critique, the knowledge of fine details of the setting such as distributions and utilities is an unrealistic assumption and, moreover, optimal mechanisms in the classic (Bayesian) sense are often too complex to be implemented in reality.

In the first chapter I study a scoring auction and the welfare implications of switching between the two leading designs of the scoring rule: linear ("weighted bid") and log-linear ("adjusted bid"), when the designer's preferences for quality and money are unknown. Motivated by the empirical application, I formulate a new model of scoring auctions, with two key elements: exogenous quality and a reserve price, and characterize the equilibrium for a rich set of scoring rules. The data is drawn from the Russian public procurement sector in which the linear scoring rule was applied from 2011 to 2013. I estimate the underlying distribution of firms' types nonparametrically and simulate the equilibria for both scoring rules with different weights. The empirical results show that for any log-linear scoring rule, there exists a linear one, yielding a higher expected quality and rebate. Hence, at least with risk-neutral preferences, the linear design is superior to the log-linear.

In the second chapter[1] I study robust allocation of a divisible public good among n agents with quasi-linear utilities, when the budget is exactly balanced. Under several additional assumptions, we prove that such mechanism is equivalent to a distribution over *simple posted prices*. A *robustly optimal* mechanism minimizes expected welfare loss among robust divisible ones. For any prior belief, we show that a simple posted prices is robustly optimal. This justifies a restriction to binary allocations commonly found in the mechanism design literature. Robustness comes at a high cost. For certain beliefs, I show that the expected welfare loss of an optimal posted price is as big as $1/2$ of the expected welfare in the corresponding optimal Bayesian mechanism, independently of the size of the economy. This bound is tight for the special case of two agents.

In the third chapter[2] I provide mechanisms for exchange economies with private information and interdependent values, which are ex-post individually rational, incentive compatible, generate budget surplus and are ex-post nearly efficient, when there are many agents. Our framework is entirely prior-free, and I make no symmetry restrictions. The mechanisms can be implemented using a novel *discriminatory conditional double auction*, without knowledge of information structure or utility functions. I also show that no other mechanism satisfying the constraints can generate inefficiency of smaller order.

[1] Co-authored with Jernej Copic and Byeong-hyeon Jeong, UCLA

[2] Co-authored with Tomasz Sadzik, UCLA

The dissertation of Pavel Andreyanov is approved.

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2019

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ACKNOWLEDGMENTS

I am grateful to John Asker, Denis Chetverikov, Martin Hackmann, Elisabeth Honka, Hugo Hopenhayn, Moritz Meyer-Ter-Vehn, Tomasz Sadzik, Robert Zeithammer, the participants of the Theory and IO proseminars at UCLA, and also my brother, Boris Andreianov, for useful comments and suggestions.

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CHAPTER 1

Mechanism Choice in Scoring Auctions

1.1 Introduction

A first-score auction is often used in procurement to determine the best supplier among a variety of candidates. This mechanism weighs the price bid together with the non-price attributes of the firm using a publicly known scoring rule, and the participant with the highest score wins the contract at the price equal to its bid.

There are multiple shapes of the scoring rule, see [DPV06] and [MY07] for an overview, but two of them are especially widespread. The linear ("weighted bid") rule is used, for example, in Delaware, Idaho, and Oregon, while the log-linear ("adjusted bid") rule is used in Alaska, Colorado, and Florida, for various public works contracts. The latter, also known as "price-quality ratio", is used extensively in Japan. It is unclear how a scoring rule is chosen in any given case, and the existing literature offers limited guidance on how to rank them. The theoretical literature was focused primarily on the optimal mechanism design and the differences between various auction formats, such as first-score and second-score, see [Che93], [AC08] and [AC10], but not on the comparisons of the existing shapes of the scoring rule.

This paper is the first to compare the two leading designs of the scoring rule — linear and log-linear — in a first-score environment, and to rank them in terms of welfare using a structural approach. To achieve this goal, I formulate a novel model of scoring auctions, which allows for both linear and log-linear cases, and derive a tractable equilibrium characterization. I employ the equilibrium structure to analyze a novel dataset on bidding and quality assessments in Russian procurement auctions, and propose a computationally feasi-

ble procedure for nonparametric estimation and simulation of counterfactuals. Finally using the simulated equilibria, I compare the two designs in terms of welfare.

I leverage a novel dataset on bidding behavior and relevant quality measurements in scoring auctions using a public archive of Russian procurement auctions which took place from 2011 to 2013. Through this period, 46,387 contracts were assigned using the scoring auction format, across multiple industries and regions of the country. I select 3,228 auctions, based on criteria, such as uniformity of the scoring procedure and low market concentration. The data are split into ten parts based on the type of economic activity and the weight in the scoring formula.

The data exhibit two distinct features. First, all auctions in the sample use quality measurements such as a firm's experience, qualification of firm's personnel, or previous successful contracts. These characteristics cannot be exchanged for cost reduction in the short run, which I will exploit in the model by treating them as exogenous (nonstrategic). Second, the reserve price is chosen by a significant portion of bidders, a phenomenon that is often referred to as *bunching*.

Clearly, the model should reflect the patterns observed in the data. However, none of the existing models feature a reserve price, and the quality is always assumed to be a strategic choice variable. While the endogenous quality could be, to some degree, considered as a generalization of exogenous quality, the addition of the reserve price creates a distinct theoretical challenge due to its interaction with the scoring rule. Indeed, from the viewpoint of the score, the constraint is variable, so it might be binding for some firms, thus producing the bunching.

I build a novel model of a scoring auction in which quality is exogenous and the reserve price is explicitly set. The key to solving the model is to think of the firm's strategy as an unconstrained choice of the score, which is later censored at the reserve price constraint. Due to the feedback loop between the strategic shading and the expectations about the bunch, an additional level of endogeneity appears that was not featured in classic auction models. Despite its complexity, the model remains tractable with a wide set of scoring rules, which

I call affine, nesting both linear and log-linear rules.

A construction of the equilibrium and a proof of its existence and uniqueness are the main theoretical results in this paper. At the core of the equilibrium is a one-dimensional strategy that captures the unconstrained choice of the score, pinned down by a simple (ordinary) differential equation. Moreover, this strategy is similar to the bidding strategy in a classic first-price auction, allowing the traditional nonparametric estimators to be applied.

To estimate the joint distribution of firm's cost and quality pairs from the observed bid and quality pairs, I use an algorithm similar to the two-step smoothing procedure in [GPV00]. In the first step, I use kernel smoothing and the optimality conditions derived from the model to estimate the amount of strategic shading from the observed scores.

In the second step, I use kernel smoothing to recover the joint distribution of bids and quality, while ignoring the bids equal to the reserve price. I use the boundary correction advocated in [HH15] to estimate the density at the boundary, that is, the bids just below the reserve price. Finally, by reversing the shading, I recover the joint distribution of costs and quality wherever it is fully identified.

The last piece of the puzzle is the part of the distribution of costs and quality that maps into the reserve price, which I could not recover from the second step. Since there are multiple distributions that are consistent with the data, I pick a simple one. Namely, I extrapolate the missing part of the distribution from the boundary, that is, from the bids that were placed just below the reserve price. The final estimator therefore has a semiparametric flavor.

To relax some of the restrictions that the model puts on the data, I investigate the implications of auction-level heterogeneity. By assuming a multiplicative error term, I show that it has no impact on the estimation as long as it affects the reserve price in the same way as it affects the cost. This means that instead of the actual bids, I can simply use the normalized bids on a scale between 0 and 100, where 100 stands for the reserve price. For this to be empirically justified, I need evidence that the reserve price serves as a signal of the scale of the contract. I find such evidence in the regulation behind the data.

Finally, I am able to simulate the equilibria in both the default and the counterfactual

scoring designs and, additionally, pick an arbitrary weight in the scoring formula. However, two problems prevent us from making decision on the mechanism choice right away. First, it is not absolutely clear which log-linear scoring rule to pick, as they vary by the weight in the scoring formula. Second, the welfare of the scoring auction is measured, at the very least, in two dimensions: expected quality and expected rebate, and so the two scoring rules might not necessarily be successfully ranked.

To overcome these problems, I propose the following approach. For each of the two scoring designs, I construct a frontier in the space of expected quality and rebate, spanned by the weight in the scoring formula, see Section 1.1. The weight is a natural parameter to vary as it controls the trade-off between quality and rebate. By fixing a target level of expected quality, it is then possible to rank the two mechanisms in the remaining dimension.

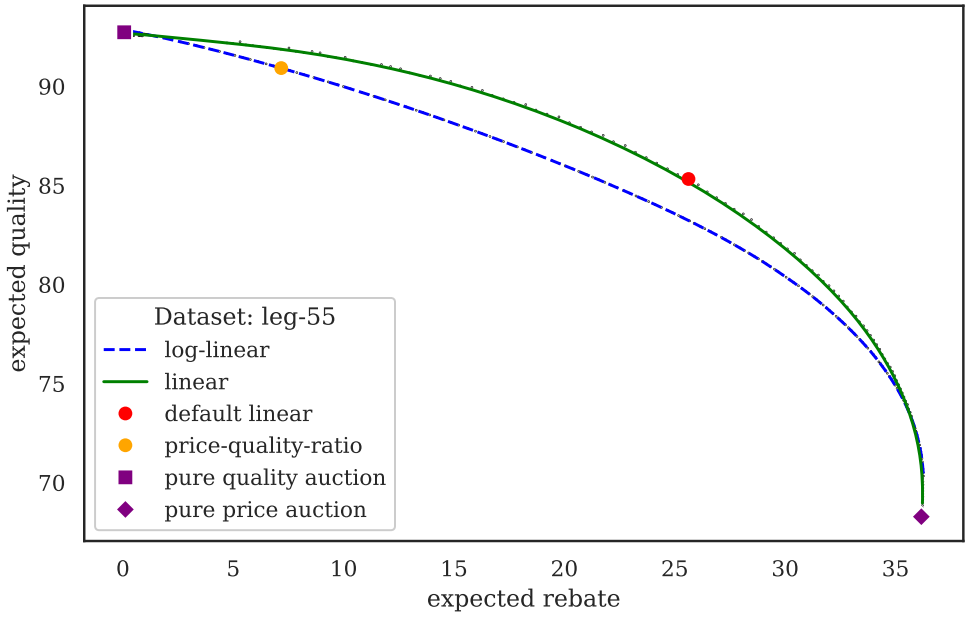

Figure 1.1: Welfare frontiers for the linear and log-linear families

My main empirical finding is that, based on the available data, for any log-linear scoring rule, there exists a linear one that is better in terms of rebate while having the same quality. In other words, the linear frontier is above the log-linear frontier, as seen on Section 1.1. This it is true for every part of my data, see Figure 1.21 and Figure 1.22. The monetary loss associated with a switch to a log-linear rule with the same quality varies between 0.4% and 4%.

This ranking may seem surprising, as I am comparing two equally plausible shapes of the scoring rule. However, there is a strong reason for such results. If the firms were to bid truthfully in a first-score auction, the mechanism would always pick the score-efficient firm. As a result, the (truthful) linear frontier would be the frontier of first-best mechanisms in the space of expected quality and rebate, and the (truthful) log-linear frontier would be below. The actual frontiers, of course, look slightly different due to the distortions associated with strategic shading and bunching. First, shading is pushing the truthful frontiers to the left, due to the informational rents paid to the firms. Second, bunching is pushing them towards the origin due to the inefficient screening. My empirical findings demonstrate that these distortions are not sufficient to change the ranking, at least with the available distributions.

Of course, these results implicitly rely on the risk-neutrality of the designer as I am using expected quality and expected rebate as measures of auction success. Mechanism design literature has long used expected revenue for ranking various classic auction mechanisms and I am simply following this tradition. It is possible, however, that when higher moments are considered, the ranking will change.

1.1.1 Related literature

The theoretical literature on scoring auctions is sparse compared to that of the classic auctions. Apart from [AC08] and [HHN16], only a few attempts have been made in — [Bra97], [Das14] and [Nis15] — to generalize the original [Che93] framework. None of them, however, deals with the reserve price.

The closest model to ours can be found in [HHN16]. The authors show how to solve for the symmetric equilibrium with the most general scoring rule. While losing much of the tractability of the earlier models, the only new scoring rule (used in practice) that this generality buys is, again, the log-linear. As in [Che93] and [AC08], the quality in this paper is endogenous, and there is no reserve price.

We stress that our model is not a special case of the model of [HHN16]; rather, we study different aspects of the scoring auction. While their focus is on nonlinearities in the scoring

rule, our focus is on the interaction between the scoring rule and the reserve price. Also, our equilibrium characterization is more tractable due to the exogenous quality and the affine structure of the scoring rule.

Similar in spirit to our empirical finding of the superiority of the linear design are the theoretical results on the optimality of quasi-linear scoring rules in [Che93] and [Nis15]. These papers, however, deal with a special environment, in which the distribution of firms' costs and quality is singular and they are already sorted in terms of productivity. The role of the auction mechanism then narrows to picking the efficient one. In our environment, the situation is more complex, as sorting is endogenous due to the full support of firm's cost and quality and, moreover, imperfect due to the bunching at the reserve price. Consequently, neither the linear nor the log-linear scoring rules can be generically considered as optimal, even from the perspective of the score.

The empirical literature comprises of studies of delegation and favoritism in scoring auctions: [DM14], [AV16], [Hua16] and several reduced form studies of the choice between scoring and price only designs: [LB11], [KM14] and [ADZ08]. Only a few papers contribute to the structural modeling of a scoring auction: [HHN16], [Nak13], however, the questions investigated in these papers do not directly overlap with ours.

Finally, our model of an affine scoring rule could be used to model bid preferences, such as the ones studied in [Mar07] and [KS11]. Indeed, the intercept and the slope of bid in the scoring formula could be thought of as additive and multiplicative preferences. A firm enjoying the preferential treatment can be interpreted as a high-quality firm in a scoring auction. However, since our model is symmetric, we require that the firms treat each other's bid preferences as random, which might be rejected by the data if the pool of participants and the preferences are made public prior to the bidding stage.

The rest of the paper is organized as follows. In Section 1.2, I discuss the institutional background and the features of our data. In Section 1.3 and Section 1.4, I explain the theoretical and empirical framework for my analysis. In Section 1.5, we discuss potential heterogeneity problems and how we address them. In Section 1.6, I present the empirical

findings, and I conclude in Section 1.7.

1.2 Data

We access an electronic archive of all public procurement contracts issued by the Russian federal government and municipalities between January 2011 and December 2013. Out of the 46,387 contracts that were assigned within the scoring auctions format, we select 3,228 contracts based on such criteria as similarity of contracts, uniformity of the scoring procedure and low market concentration. These auctions comprise a total award value of 219.6 million dollars[1].

The scoring auction is the most complex and regulated format among all used in Russian procurement. The contractee, which is typically not a private firm but a public body, that wishes to start a scoring auction has to follow the detailed instructions written by the Ministry of Economic Development. The auction process is supervised by a special commission and each step is made public. The whole process can take more than a month, depending on the size and complexity of the contract, see Section 1.2.

Step	Timing
1. Formation of the supervising commission and development of the notification and relevant tender documentation.	Prior to the publication.
2. Publication of the upcoming auction notification and relevant tender documentation. Starting bid collecting period.	-
3. Ending bid collecting period, opening quotes and publication of opening protocols.	1-4 weeks since the start of bid collection period.
4. Evaluation of bids and determining the winner. Publication of evaluation protocols and winner announcement.	Up to 10 days since the quotes we opened.
5. Signing the contract	Up to 10 days since the publication of auction protocols.

Table 1.1: Scoring auction timing and regulation

The archive consists of three types of files: notifications, protocols and tender documentation. The first two are standardized tables stored in the .xml format. The notification

[1] using the approximate exchange rate of 30 roubles per dollar in 2012

file contains an announcement of the upcoming auction together with a brief description of the contract, the industry code and the reserve price. The official site allows anybody to search the upcoming auctions by the information contained in the notification, thus reducing participation costs. The protocol file is created after the auction and contains all submitted bids together with the announcement of the winner. Finally, the documentation is a detailed description of the contract and the quality assessment, compiled by the contractee, typically in a form of one or several .doc files.

To link each contract with a certain type of economic activity, we use a 7-digit industry code called OKPD, extracted from the notifications. This code is an archaic classification of industries, currently replaced by a newer version OKPD2. We focus on five groups of economic activity: education, scientific research, legal services, technical services, and security.

dataset name	industry name	OKPD code	price weight	total auctions	total bids	total awards in mln $	total reserve in mln $
Edu-80	Education	80*****	0.80	89	225	4.6	5.8
Edu-55			0.55	272	712	13.3	16.1
Sci-80	Scientific Research	73*****	0.80	162	487	11.7	17.8
Sci-55			0.55	298	854	30.9	38.0
Leg-80	Legal Services	75*****	0.80	929	3048	5.2	10.9
Leg-55			0.55	55	159	5.6	7.4
Tec-80	Technical Design	71*****	0.80	936	2917	99.5	158.8
Tec-55			0.55	47	129	7.3	9.3
Sec-80	Security	84*****	0.80	409	1181	33.4	42.0
Sec-55			0.55	31	94	8.1	9.6
Total				3228	9806	219.6	315.6

Table 1.2: Auction aggregate statistics

The first group (Edu) consists of contracts for citizen education programs. These types of services are often purchased by the government when a factory is shut down or experiences a sharp decrease in labor demand. The type of quality that is required from the firm is measured in years of operation on the market and the number of people that went through the education program. The second group (Sci) consists of contracts for sociological, statistical, economic and other scientific research, with quality being typically some sort of experience, government accreditation, or the qualification (such as a doctoral degree) of personnel. The

dataset name	average bidder quality score	average bidder price score	average bidder score	average winning rebate %	average winning quality %	average bidders
Edu-80	66.2	23.6	32.1	30.7	79.7	2.5
Edu-55	72.9	17.0	42.2	19.6	87.9	2.6
Sci-80	61.5	25.0	32.3	32.8	73.8	3.0
Sci-55	64.6	18.8	39.4	16.7	85.4	2.9
Leg-80	74.7	41.1	47.8	52.3	82.8	3.3
Leg-55	59.0	23.0	39.2	17.3	94.5	2.9
Tec-80	60.2	28.2	34.6	36.3	74.0	3.1
Tec-55	66.2	21.0	41.4	20.8	87.7	2.7
Sec-80	74.0	15.7	27.4	19.9	84.7	2.9
Sec-55	56.2	15.0	33.6	13.4	76.6	3.0

Table 1.3: Auction average statistics

third group (Leg) consists of various legal services, for the most part related to the mandatory yearly inspection (audit) of firms. Here, quality is measured by the number of audits in the past, positive reviews from the clients, and also the number of personnel with a proper government accreditation. The fourth group (Tec) mostly consists of contracts for development of project documentation and estimates, with quality measurements very similar to (Leg). The fifth group (Sec) has contracts for maintaining security in various civil buildings and facilities. In this group, the quality is measured by the experience of the personnel and the availability of special equipment.

In each of our auctions, a linear scoring formula was used, which can be described as:

$$score = price\ weight \cdot price\ score + (1 - price\ weight\ ) \cdot quality\ score \tag{1.1}$$

$$price\ score = 100 \cdot (reserve\ price - price\ bid\ )/reserve\ price \tag{1.2}$$

where the price weight parameter is either 0.8 or 0.55. The quality score is assigned to each firm independently and strictly according to the evaluation criteria described in the contract documentation. These criteria vary by contract, but the final quality score is always between 0 and 100. Overall, we have 10 different datasets, which we summarize in Section 1.2.

The participation pattern is similar in all 10 datasets. Approximately 1/2 of the auctions

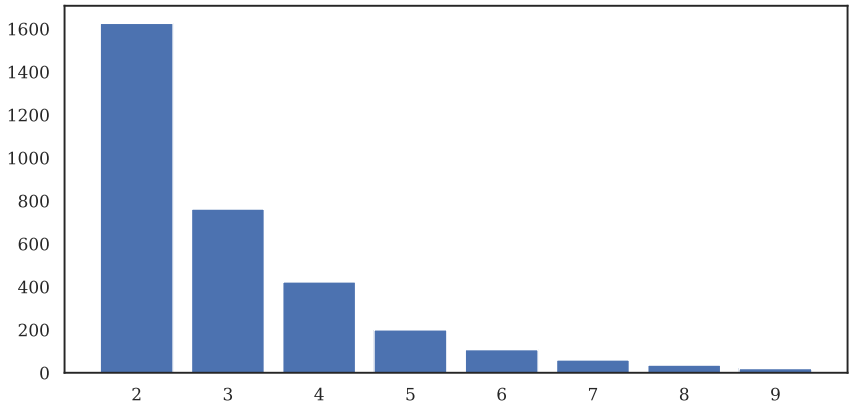

Figure 1.2: Aggregate participation histogram

have 2 bidders, 1/4 with 3 bidders, 1/8 with 4 bidders, etc., see Section 1.2. The are a few dozens of auctions that have a single participant, which we ignore. Since it is a sealed-bid format, there is no public signal about the number of firms in the upcoming auction, so we will treat it is unknown.

The rest of the section is organized as follows. First, in Section 1.2.1, we discuss the institutional background of our data. Then, in Section 1.2.2, we discuss an example of how quality is assessed. Finally, in Section 1.2.3, we discuss certain distributional properties of our data.

1.2.1 Institutional background

In contemporary Russia, the procurement regulation is very similar to the one in the European Union and the US, with the only exception that it is more centralized. However, it did not always look the same, rather it was constantly evolving in accordance with a sequence of federal laws and presidential decrees, gradually shaping the national procurement system over the last 25 years, see Section 1.2.1.

Our data spans the middle of that time period, beginning with the earliest historical records in year 2011, available at the central e-procurement website, and ending with the major overhaul of procurement after the adoption of a new federal law in year 2014.

federal law	year	general regulation	scoring auction regulation
FZ-60, UP-305, FZ-97	1994	First attempts are made to replace central planning with competitive procurement	Scoring auctions are mostly unregulated, leaving room for corruption and preferential treatment
FZ-94	2006 2011	Upcoming auction notifications become available online, on local websites. Explicit procedures for determining the reserve price are introduced.	The scoring auction format becomes limited to a small range of economic activities -
FZ-44	2014	A central auctioning platform is created, with access to all past (starting from 2011) and upcoming auction notifications and protocols.	The linear formula is replaced with the minimal bid formula. All time-related quality criteria are banned.

Table 1.4: Procurement regulation

In our data, one of the most important details is the way the reserve price is determined. More precisely, each time a government official sets a reserve price in a procurement auction, he has to follow the precise instructions recommended by the Russian Ministry of Economic Development. Typically, the reserve price is an average of three of more anonymous market offers from the firms in the industry, or the costs of similar contracts completed in the past.

Crucially, all the calculations and references have to be made public, together with the rest of the tender documentation, prior to the auction. The results of the auction could

be even contested in court if the bidder believed the reserve price to be misleading. This indicates that the reserve price is also a signaling device that helps firms calibrate their bids.

1.2.2 Quality of the firm

The quality score is assigned to the firm based on several quality criteria, such as experience, number of successful contracts, qualification, etc. Even within an industry, there is a significant variation in the sets of criteria selected for each particular auction and in the evaluation scheme. This gives rise to an almost infinite variety of schemes for quality evaluation, leaving us no option but to take the final quality score at face value.

A typical formula for the quality score consists of two criteria:

$$\textit{quality score} = \textit{first criterion score} + \textit{second criterion score}.$$

For example, in the Education industry, these are often the number of customers (graduates) and the number of years that the firm has been operating on the market:

first criterion		*second criterion*	
less than 200 customers	20 points	less than 3 years	10 points
between 201 and 500 customers	30 points	between 4 and 7 years	20 points
more than 501 customers	60 points	between 8 and 10 years	30 points
		more than 11 years	40 points

Table 1.5: Quality criteria example 1

The quality evaluation scheme may consist of more than two criteria, or sometimes criteria have sub-divisions. In the following example from Scientific Research industry, quality is measured using two criteria: number of past contracts and qualification. However, the first criterion is split into three sub-criteria that correspond to different types of contracts:

Note that in these examples, the evaluation instructions are fixed and do not depend on the characteristics of actual bidders. This is not always the case in our data. For certain auctions, the maximal points are assigned to the firm with the maximal among the current bidders number of customers (years), and all other bidders get proportionally less. We do not include these auctions in our datasets since they would require a more powerful structural

first criterion						*second criterion*	
type 1 contracts		*type 2 contracts*		*type 3 contracts*		*qualified personnell*	
$\geqslant 5$	30 points	$\geqslant 2$	25 points	$\geqslant 1$	20 points	$\geqslant 4$	25 points
4	24 points	1	10 points	0	0 points	3	18 points
3	18 points	0	0 points			2	12 points
2	12 points					1	6 points
1	6 points					0	0 points
0	0 points						

Table 1.6: Quality criteria example 2

model, able to deal with scoring formulas that depend on the characteristics of many bidders.

In our datasets, we have pooled together a large number of auctions with qualities of different nature. Since we take the final score at face value, we essentially assume that the utility of the designer (as a function of various quality criteria) is aligned with the quality evaluation scheme. For example, in Section 1.2.2, that would mean that 10 quality points are approximately equivalent to 3 years of experience, or 200 additional customers.

Due to the variety of the schemes and their non-linearity, we will abstain from exact interpretations of quality and treat it as an ordinal measure. In other words, we care about the fact that more quality is better than less quality, but not about what that quality means.

1.2.3 Spikes and bunches

In the examples of the previous section, the functions that map contracts or years into quality points are bounded from both sides. This means that two firms can be formally different in terms of their age and experience, but still have the same maximum (or minimum) quality points. This gives rise to two pronounced spikes in the quality score histogram, see Section 1.2.3, with the right spike containing approximately 25% of the data (quality between 99 and 100).

Since our model is continuous, we will have to make sure that these distributional anomalies are smoothed out in the estimation phase. But this will happen automatically since we do not perform boundary correction in the q dimension. While this might seem like a bad idea because of over-smoothing at the boundary, it will be, in fact, beneficial for the model because sharp increases in the estimated density of quality might lead to non-existence of continuous equilibria.

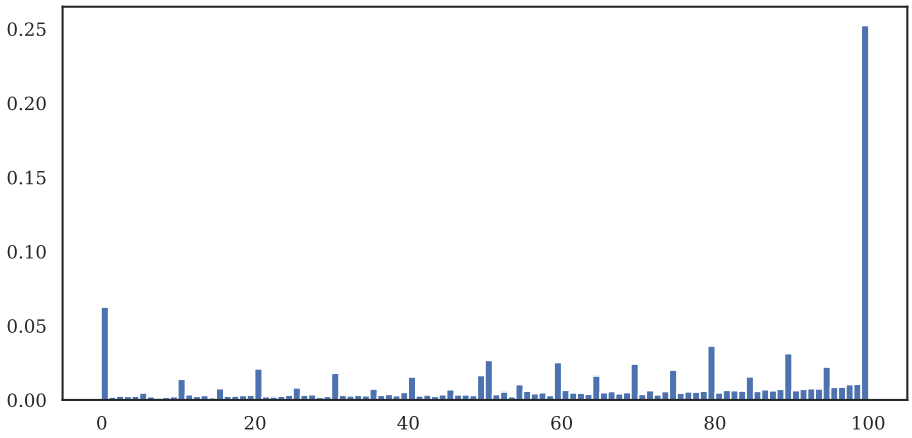

Figure 1.3: Aggregate histogram of quality scores.

Another distributional anomaly happens when bidders set their bid exactly at the reserve price, which we refer to as a bunch at the reserve price, see Section 1.2.3. Approximately 7% of the data is concentrated in the range of price scores between 0 and 1, out of which 4.5% are exactly at price score 0. Though it is true that high quality firms tend to bid less aggressively, only a small fraction of the bunch is attributed to the spike in quality.

To see that the bunch is a separate phenomenon, we drop all the bids with quality greater

than 99. The resulting histogram remains virtually unchanged, see Section 1.2.3, and 3.2% of bidders still choose to bid exactly the reserve price. However, if we focus on the auctions with price weight 0.55, the pattern becomes more pronounced, see Section 1.2.3.

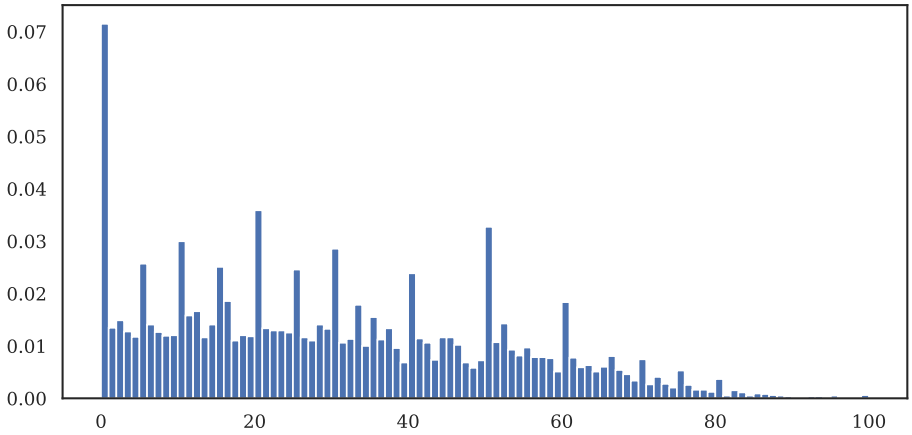

Figure 1.4: Aggregate histogram of price scores

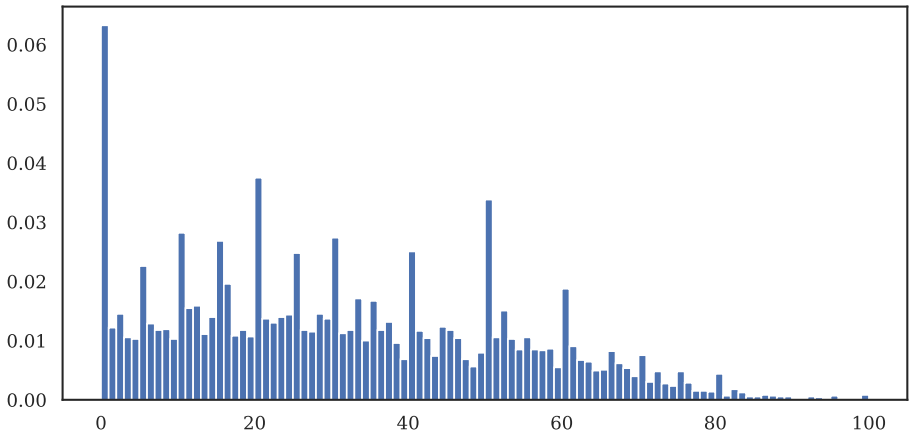

Figure 1.5: Aggregate histogram of price scores conditional on quality scores below 99

This observation indicates that there may be some strategic factor at play that makes the bidders choose the reserve price with a positive probability. In the theoretical part of the paper, we will show that this feature is indeed a hallmarc of the scoring auction.

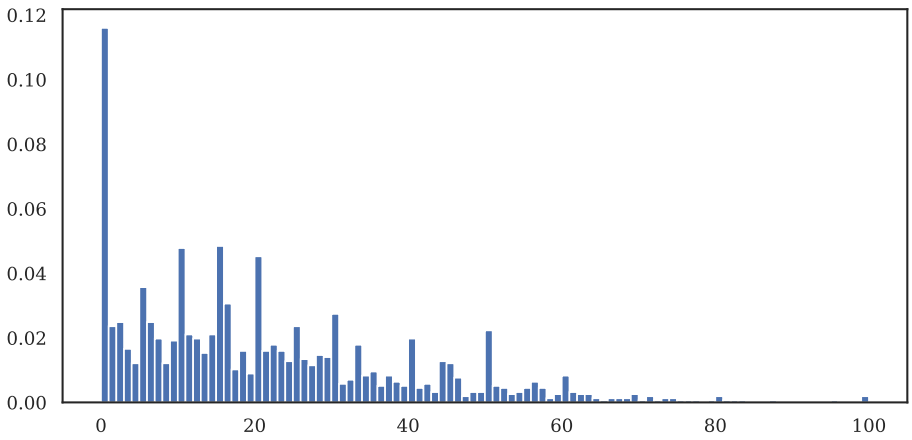

Figure 1.6: Aggregate histogram of price scores conditional on quality scores below 99 and price weight equal to 0.55

1.3 Model

A single contract is auctioned among N ex-ante identical, risk-neutral firms. Each firm simultaneously and secretly submits its bid b to the auctioneer. Each firm has quality q, which, together with the bid, is mapped into the firm's score, using a publicly known scoring rule. The firm with the highest score wins the contract at the price equal to its bid, which can not exceed a publicly known reserve price r.

A critical assumption in our model is that q is an exogenous and perfectly verifiable characteristic of the firm, observed by the auctioneer, but not the other firms. The cost c is private, and there is no technology that allows the firm to exchange it for quality or vice versa.

Assumption 1. *Each firm draws (c, q) independently, from the same distribution with a twice continuously differentiable cdf F, with full support on $[0, r] \times [\underline{q}, \overline{q}]$.*

We consider a broad class of scoring rules that we call *affine*, which nests the *quasi-linear* scoring rules studied in [Che93] and [AC08].

Definition 1. *A scoring rule is **affine** if it can be represented by a function $s(b, q)$:*

$$s(b, q) = \alpha(q) + \beta(q)(r - b). \tag{1.3}$$

Three cases of affine scoring rules are of special interest:

$$\begin{aligned}
&\textit{quasi-linear:} && \beta(q) = 1 && \Rightarrow && s(b,q) - r = \alpha(q) - b,\\
&\textit{linear:} && \beta(q) = 1,\ \alpha(q) = wq && \Rightarrow && s(b,q) - r = wq - b,\\
&\textit{log-linear:} && \alpha(q) = -r/q^w,\ \beta(q) = 1/q^w && \Rightarrow && -\log(-s(b,q)) = w\log q - \log b.
\end{aligned}$$

Assumption 2. *$\alpha(q), \beta(q)$ are twice continuously differentiable, $\alpha(q)$ is strictly increasing, and $\beta(q)$ is positive.*

Additionally, we assume that some firms will not come to the auction, and the participating firms hold identical beliefs about the turnout n.

Assumption 3. *The turnout is random and follows a discrete distribution with known probabilities p_n, where $1 \leqslant n \leqslant N$.*

We are looking for a symmetric, pure strategy BNE of this game.

1.3.1 Equilibrium Structure

We focus on the equilibrium scoring strategy, since the competition holds in the score dimension.

Our first goal is to introduce a function $\theta(c, q)$, that we will refer to as the firm's *type*, which can be used as a natural argument of the scoring strategy. We will refer to the level lines of this function as the *iso-types*.

Definition 2. *Denote the firm's **type** by $\theta(c, q)$, where*

$$\theta(c, q) = \alpha(q) + \beta(q)(r - c). \tag{1.4}$$

The type is similar to the firm's *production potential* in [Che93] and the *pseudo-type* in [AC08]. Without the reserve price, it would completely define the equilibrium behavior of the firm, due to the affine structure of the scoring rule. Indeed, if $\pi(s)$ is the equilibrium

probability of winning associated with a score s, then the firm's profit can be written as:

$$\pi(s)(b-c) = \pi(s)(\underbrace{\alpha(q) + \beta(q)(r-c)}_{\text{firm's type}} - s)/\beta(q) = \pi(s)(\theta - s)/\beta(q),$$

which proves that θ is a sufficient statistic for the optimal choice of the score. This choice, however, might be infeasible in the presence of the reserve price.

Note that the iso-types depend only on the shape of the scoring rule, in other words they are exogenous. For the linear scoring rule $s = q - 2b$ these are parallel lines, and for a log-linear $s = -b/q$ these are rays originating at $c = 0, q = 0$, see Section 1.3.1.

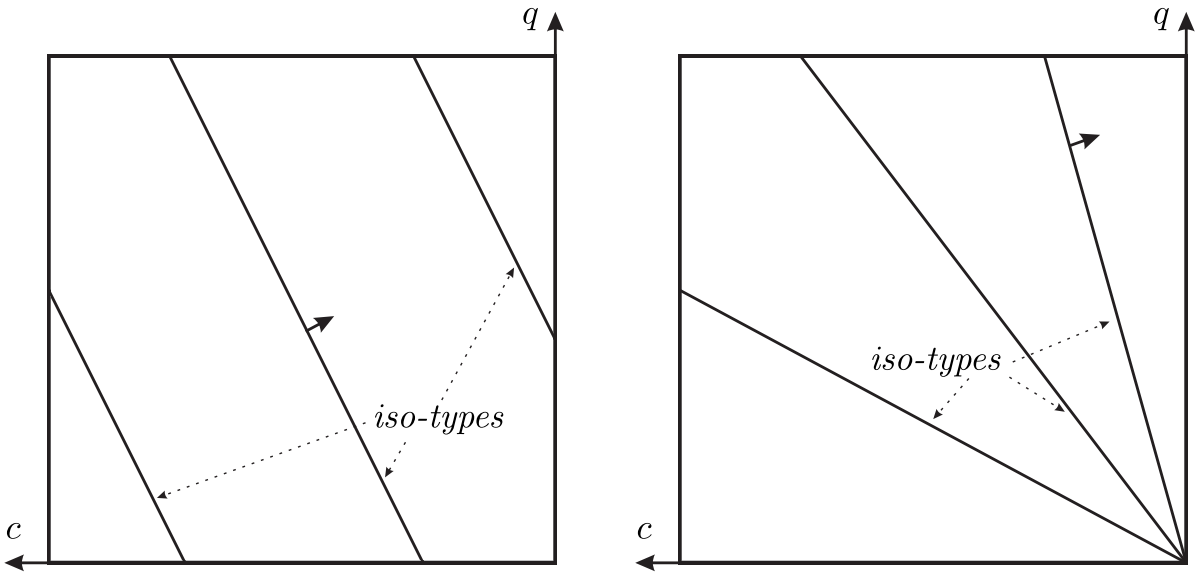

Figure 1.7: Iso-types for a linear and a log-linear scoring rule.

Our second goal is to define two scoring strategies that capture the firm's optimal choice of the score, with and without censoring imposed by the reserve price.

Definition 3. *Denote the equilibrium* ***scoring strategy*** *by* $\sigma^*(\theta, q)$.

This strategy represents the constrained choice of the score, and we will refer to its level lines as the *iso-scores*. By definition, it solves the following optimization problem:

$$\sigma^*(\theta, q) \in \arg\max_{s \geqslant \alpha(q)} \pi(s)(\theta - s). \tag{1.5}$$

The underlying bidding strategy can be easily recovered from the scoring strategy using formulas $\theta = \alpha(q) + \beta(q)(r - c)$ and $s = \alpha(q) + \beta(q)(r - b)$, which are simply accounting identities. Note that the situation when the bid is equal to r corresponds to the situation where the score is equal to $\alpha(q)$.

Definition 4. *Define* $\sigma(\theta) = \min_q \sigma^*(\theta, q)$ *as the equilibrium* ***uncensored scoring strategy****.*

This strategy represents the unconstrained choice of the score, and its level lines coincide with the iso-types. By construction, it solves the following optimization problem:

$$\sigma(\theta) \in \arg\max_s \pi(s)(\theta - s). \tag{1.6}$$

A natural conjecture is that the scoring strategy $\sigma^*(\theta, q)$ should be derived from $\sigma(\theta)$ via censoring at the $\alpha(q)$ threshold. This is not true in general, however, if we introduce a simple refinement of the equilibrium, this property can be established.

Assumption 4. $\sigma^*(\theta, q)$ *is continuous and a lower score is chosen when indifferent.*

Lemma 1. *Under Assumption 4, following equation holds:*

$$\sigma^*(\theta, q) = \max(\sigma(\theta), \alpha(q)). \tag{1.7}$$

Following equation (1.7), one can see that the iso-scores are kinked. Consequently, they are endogenous, as the position of the kink relies on the equilibrium through the $\sigma(\theta)$ function. We can nevertheless describe them qualitatively. When the score is a function of type, the iso-scores are aligned with the iso-types, but when the score is a function of quality, the iso-scores are horizontal. The set of firms that choose the reserve price, which we refer to as the *bunching region*, is outlined by the curve traced the kinks of the iso-scores, see Section 1.3.1 for a stylized illustration.

The equilibrium strategy $\sigma^*(\theta, q)$ can be derived from the equilibrium strategy $\sigma(\theta)$ using equation (1.7). Consequently, we only have to characterize the $\sigma(\theta)$ strategy in order to pin

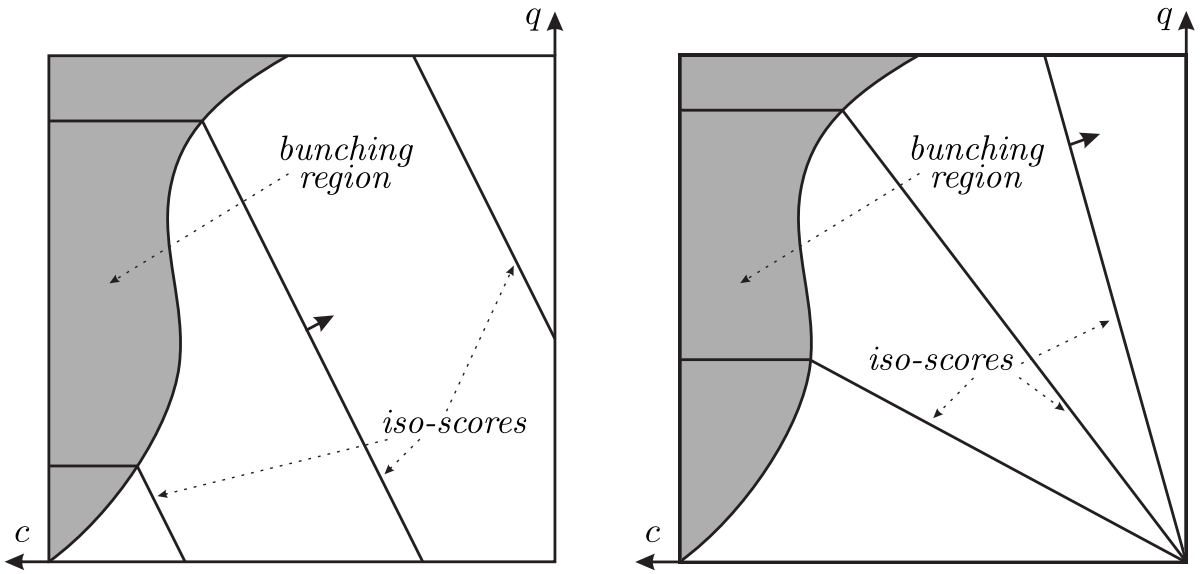

Figure 1.8: Iso-scores and the bunching region (grey) for a linear and a log-linear scoring rule.

down the equilibrium. To do that a few additional definitions and a regularity assumption are required.

Definition 5. *Denote the cumulative distribution function of* $(\theta, \alpha(q))$ *by* $\widetilde{F}$*, and its partial derivatives by* $\widetilde{F}_1'$ *and* $\widetilde{F}_2'$*. Define a function* $Z(\theta,\sigma) = (\sum p_n \widetilde{F}^{n-2}(\theta,\sigma))/(\sum p_n(n-1)\widetilde{F}^{n-2}(\theta,\sigma))$ *and a correspondence* $\delta(\theta) = \{\sigma \in [\underline{\theta},\theta] : Z(\theta,\sigma)\widetilde{F}(\theta,\sigma) - (\theta-\sigma)\widetilde{F}_2'(\theta,\sigma) = 0\}$*, where* $[\underline{\theta},\overline{\theta}]$ *is the support of* θ*.*

Assumption 5 (Regularity)**.** $\delta(\theta)$ *is a function, such that* $\delta(\theta) \geqslant \alpha(\overline{q})$ *for some* θ *in* $[\underline{\theta},\overline{\theta}]$*.*

The role of this assumption will be discussed in detail in Section 1.3.2. We are now ready to give a characterization of the equilibrium.

Proposition 1. *Under Assumptions 1-5, there exists a unique symmetric BNE in the scoring auction, with a strictly monotone uncensored scoring strategy* $\sigma(\theta)$*, that solves:*

$$\sigma(\underline{\theta}) = \underline{\theta} \quad \text{and} \quad \sigma' = \frac{(\theta-\sigma)\widetilde{F}_1'(\theta,\sigma)}{Z(\theta,\sigma)\widetilde{F}(\theta,\sigma) - (\theta-\sigma)\widetilde{F}_2'(\theta,\sigma)} \quad \text{for all types in} \quad [\underline{\theta},\overline{\theta}]. \tag{1.8}$$

To illustrate the derivation of the differential equation (1.8), assume that there are two firms and there is no uncertainty about the turnout. Observe that the equilibrium probability of winning associated with a score $\sigma(\theta)$ is equal to the cumulative distribution function

$\widetilde{F}(\theta, \alpha(q))$ evaluated at $(\theta, \sigma(\theta))$, which is also the area under the iso-score:

$$\pi(\sigma(\theta)) = Prob(\sigma^*(\hat{\theta}, \hat{q}) \leqslant \sigma(\theta)) = Prob(\sigma(\hat{\theta}) \leqslant \sigma(\theta),\ \alpha(\hat{q}) \leqslant \sigma(\theta)) = \widetilde{F}(\theta, \sigma(\theta)).$$

Indeed, if we switch to the $(\theta, \alpha(q))$ coordinates, the probability of winning will be to the left and below the (θ, σ) point. Moreover, the boundary of the bunching region will coincide with the $\sigma(\theta)$ curve, as they represent the same event when $\alpha(q)$ is equal to $\sigma(\theta)$. In these coordinates the bunching region is therefore the area between the 45 degree line and the $\sigma(\theta)$ curve. Note that while the $\sigma(\theta)$ is monotone, the boundary of the bunching region, obtained by tilting the curve to the left, might have an inflection point.

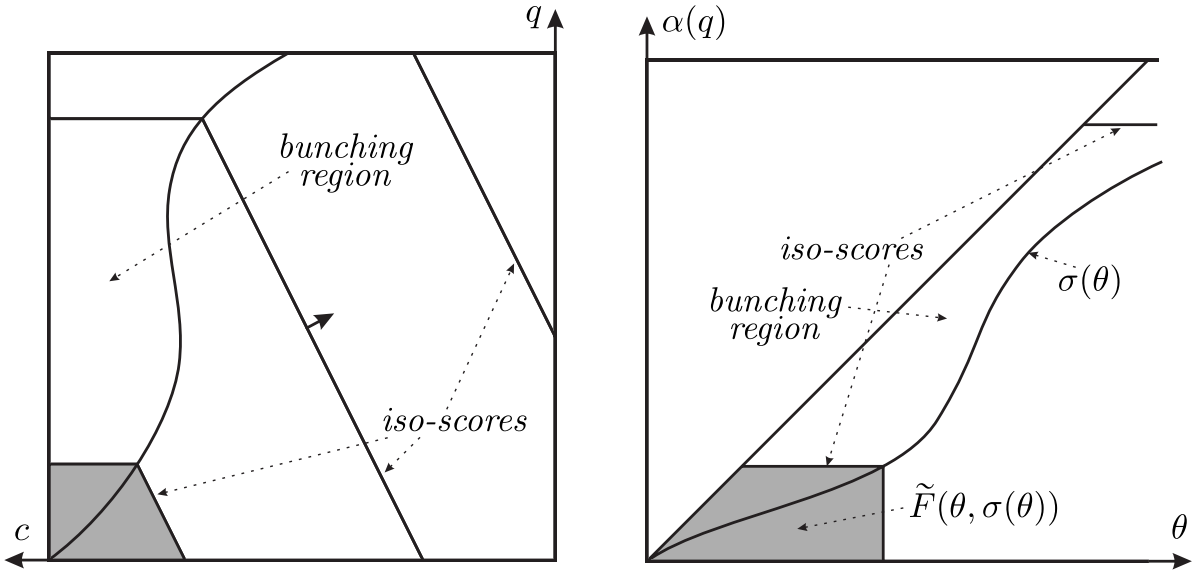

Figure 1.9: Iso-scores and the probability of winning (grey) for a linear scoring rule.

Bunching is typically considered incompatible with the symmetric equilibria in the first price auctions, as an infinitesimal deviation in the bid generates a sizable increase in the probability of winning. In the scoring auction, however, this is not the case as competition holds in the score, rather than the bid dimension. The bunch is spread across different scores due to the variation in q, and so the distribution relevant to the formation of best response is, in fact, continuous.

We can now derive the first order conditions associated with the optimal choice of the

score.

First Order Conditions (2 firms):

$$\overbrace{\widetilde{F}(\theta,\sigma)d\sigma}^{\textit{marginal cost}} \approx \overbrace{(\theta-\sigma)\cdot(\widetilde{F}_1'(\theta,\sigma)d\theta + \underbrace{\widetilde{F}_2'(\theta,\sigma)d\sigma}_{\text{bunching effect}})}^{\text{marginal benefit}}. \tag{1.9}$$

The marginal cost of raising the score above the equilibrium can be written as $d\sigma$ times the probability of winning $\widetilde{F}(\theta,\sigma)$, while the marginal benefit is the profit margin $(\sigma-\theta)$ times the marginal increase in the probability of winning. The latter comes from two populations of firms, see Section 1.3.1. The first are the firms outside the bunch, that have a marginally higher type θ, captured by the $\widetilde{F}_1'(\theta,\sigma)d\theta$ term. The second are the firms inside the bunch that have a marginally weaker constraint on the score, captured by the $\widetilde{F}_2'(\theta,\sigma)d\sigma$ term. We call the latter *bunching effect*, as it only appears in the presence of bunching.

After dividing both parts of equation (1.9) by $d\theta$, replacing $d\sigma/d\theta$ with σ', and solving for σ', we obtain a special case of the differential equation (1.8), where $Z(\theta,\sigma)=1$. We will derive it more rigorously and show that the second order conditions hold in Section 1.3.3.

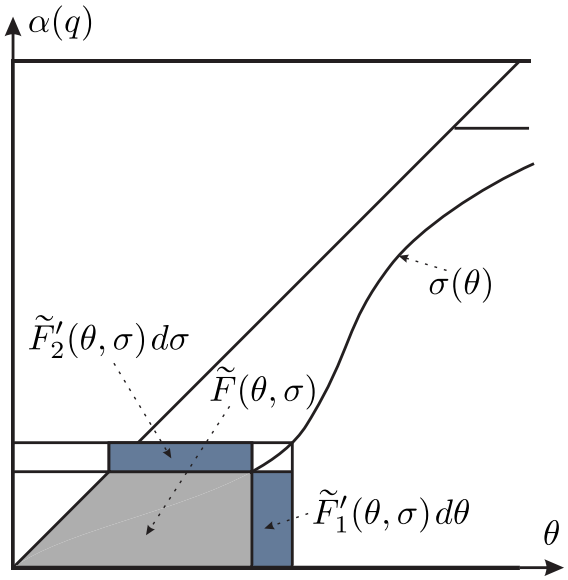

Figure 1.10: Marginal increase in the probability of winning.

While the distribution $\widetilde{F}$ itself is exogenous, the way it enters the first order conditions is clearly endogenous, as it is evaluated at (θ,σ). That is because the way expectations are

formed about the strength of the competition depends on the amount of bunching, which in turn, depends on the amount of equilibrium shading. This creates a feedback loop between the equilibrium strategy $\sigma(\theta)$ and the equilibrium probability of winning $\pi(\sigma(\theta))$, which does not happen in a classic first-price auction, or in a scoring auction with no reserve price.

While the first order conditions are fairly complex due to the equilibrium expectations about the bunch, the firm's bidding problem is simple, as all the relevant information is captured by the observed distribution of scores. The firm then behaves simply as a monopolist operating over a downward sloping demand curve $\pi(s(b,q))$, with a price ceiling r:

$$\max_{b \leqslant r} (b-c)\pi(s(b,q)), \quad s(b,q) = \alpha(q) + \beta(q)(r-b).$$

We are interested in how the monopoly price (equivalently, the optimal bid) responds to an increase in c or q, assuming that the price ceiling does not bind. The first can be considered as an increase in the monopoly's marginal cost, which implies an increase in the monopoly price since b and c are complements in the profit function. The second can be considered as a positive demand shock that shifts the demand curve upwards. While it is natural to assume that the monopoly price will increase, this is not necessarily the case. Indeed, from the inverse elasticity rule we know that the defining factor is the elasticity of demand rather than its volume. If the optimal bid does not change with a small increase in q, the iso-bid will have an inflection point, as on Section 1.3.1.

For the linear scoring rule, the shift of the demand curve can be offset by a proportional increase in the marginal cost, so that the profit margins remain the same. Consequently, the iso-bids will be parallel translations of each other, as on Section 1.3.1.

So far our analysis was for the case of two firms with no uncertainty about the turnout. To allow for random participation, we need to replace the probability of winning against a random firm $\widetilde{F}$ with the probability of winning against the strongest opponent firm. This probability is captured by a convoluted function $\sum p_n \widetilde{F}^{n-1}$, which finds its way into the first

order conditions:

First Order Conditions (general):

$$\sum_{n=1}^{N} p_n \widetilde{F}^{n-1} d\sigma \approx (\theta - \sigma) \cdot (\widetilde{F}_1' d\theta + \widetilde{F}_2' d\sigma) \cdot \sum_{n=1}^{N} p_n (n-1) \widetilde{F}^{n-2}. \tag{1.10}$$

By introducing the $Z(\theta, \sigma)$ function we are able to isolate the effect of random participation in our differential equation:

$$\frac{(\theta - \sigma) \cdot (\widetilde{F}_1' + \widetilde{F}_2' \sigma')}{\sigma' \cdot \widetilde{F}} = Z = \frac{\sum_{n=1}^{N} p_n \widetilde{F}^{n-2}}{\sum_{n=1}^{N} p_n (n-1) \widetilde{F}^{n-2}}. \tag{1.11}$$

With a turnout fixed at n, Z is simply equal to $1/(n-1)$. Finally, after solving for σ', we obtain our differential equation:

$$\sigma' = \frac{(\theta - \sigma) \widetilde{F}_1'}{Z\widetilde{F} - (\theta - \sigma) \widetilde{F}_2'}. \tag{1.12}$$

Coupled with the initial value $\sigma(\underline{\theta}) = \underline{\theta}$, this differential equation can be solved locally under standard conditions, however, the solution might not necessarily exist globally for a number of reasons. In the next section we will focus on the conditions that are sufficient for

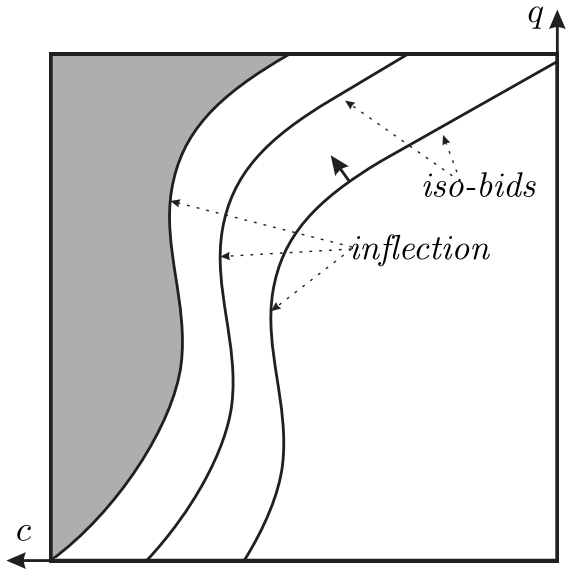

Figure 1.11: Iso-bids and bunching (grey area).

the existence of the global solution.

1.3.2 Regularity, Existence and Uniqueness

The Initial Value Problem in Proposition 3 can be interpreted as a direct instruction to finding the equilibrium. However, one should be careful when constructing the $\sigma(\theta)$ trajectory as there are two cases how it may fail.

The first case is when $Z\widetilde{F}-(\theta-\sigma)\widetilde{F}_2'$ turns into zero thus preventing us from constructing a global solution. The solution then should be sought in the wider functional space of, possibly discontinuous, monotone functions. The second case is when $\widetilde{F}_1'$ turns into zero. Then the first order condition becomes degenerate and the trajectory switches, through a kink, to a different law of motion: $(\theta-\sigma)\widetilde{F}_2' = Z\widetilde{F}$, which can be interpreted as playing against the population of firms belonging entirely to the bunch. It is even possible for the trajectory to switch between the two laws of motion multiple times. The role of Assumption 5 is to make sure that the two previously mentioned cases do not occur, so that we can focus on the most regular scenario.

Recall the definition of the $\delta(\theta)$ correspondence:

$$\delta(\theta) = \{\sigma \in [\underline{\theta}, \theta] : Z(\theta,\sigma)\widetilde{F}(\theta,\sigma) - (\theta-\sigma)\widetilde{F}_2'(\theta,\sigma) = 0\}.$$

It is non-empty for all $\theta \in [\underline{\theta}, \overline{\theta}]$ by the *Intermediate Value Theorem*, and captures the set of points in the $(\theta, \alpha(q))$ coordinates for which the denominator in the right handside of the differential equation turns into zero. If, additionally, it is a function, it serves as a lower boundary to where the $\sigma(\theta)$ trajectory can go, because any continuous strategy will by pushed upwards when approaching $\delta(\theta)$. At the same time, when approaching the 45 degree line, the $\sigma(\theta)$ trajectory will be pushed to the right. Consequently, the trajectory always remains between these two functions:

$$\delta(\theta) \leqslant \sigma(\theta) \leqslant \theta.$$

In the theory of ordinary differential equations, the $\alpha(q) = \delta(\theta)$ and $\alpha(q) = \theta$ curves are referred to as *fences*, and together they form what is called an *anti-funnel*, see Section 1.3.2. Existence and uniqueness of the trajectory that originates at $(\underline{\theta}, \underline{\theta})$ follows from the theory of fences and funnels, see [HW97] and Appendix A for more details.

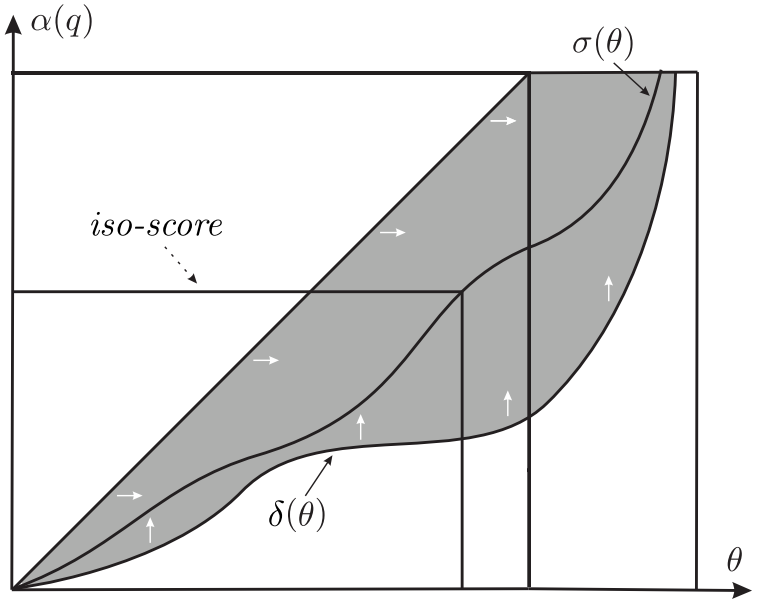

Figure 1.12: The anti-funnel (grey), the $\delta(\theta)$ and the $\sigma(\theta)$ curves.

The first part of Assumption 5 tells that $\delta(\theta)$ is a function, thus allowing us to establish the existence and uniqueness of the global solution to our differential equation. The second part tells that there exists a θ such that $\sigma(\theta) \geqslant \alpha(\overline{q})$, which means that the trajectory never passes through the point where $\widetilde{F}_1'$ turns into zero. Consequently, there is no switch in the law of motion and the differential equation correctly represents the first order conditions.

While the first order conditions are clearly necessary, it is not obvious whether they are sufficient for the equilibrium. To show this we will study a direct mechanism associated with the game in the next section.

1.3.3 Pseudo-type and the Direct Revelation Mechanism

We introduce a generalization of the *pseudo-type* in [AC08], which is a function, designed to be constant along the iso-score lines. Once the pseudo-type is defined, it can be thought of as a message in the direct mechanism, and the first order conditions can be derived using

the revelation principle.

Definition 6. *Denote the firm's* ***pseudo-type*** *by* $\rho(\theta, q)$*, where*

$$\rho(\theta, q) = \sigma^{-1}(\sigma^*(\theta, q)). \tag{1.13}$$

By construction, the pseudo-type is constant along the iso-scores and, moreover, coincides with the type when the reserve price is not binding:

$$\rho(\theta, q) = \begin{cases} \theta, & \text{if the reserve price is not binding,} \\ \alpha^{-1}(\sigma(\theta)), & \text{if the reserve price is binding.} \end{cases}$$

The intuition behind the pseudo-type is the following. Since for certain types the reserve price is binding, the mechanism pools them with higher types (but same quality) all the way up to the value of $\sigma^{-1}(\alpha(q))$, which is the only type that would have chosen the reserve price willingly. The pseudo-type is therefore the type of the firm as perceived by the mechanism in equilibrium.

Definition 7. *Denote the* ***pseudo-type distribution*** *by* $G(\rho)$ *and the* ***residual pseudo-type distribution*** *by* $\mathcal{G}(\rho)$*, where*

$$\mathcal{G}(\rho) = \sum p_k G^{k-1}(\rho). \tag{1.14}$$

The distribution of the pseudo-type plays the same role as the distribution of values in a classic first-price auction. The area under the iso-score is equal to the probability of winning against a random opponent while choosing the score $\sigma(\rho)$, see Section 1.3.3.

We can now characterize our uncensored scoring strategy $\sigma(\theta)$ as an equilibrium strategy in a direct revelation mechanism where the firm submits the pseudo-type. The optimality

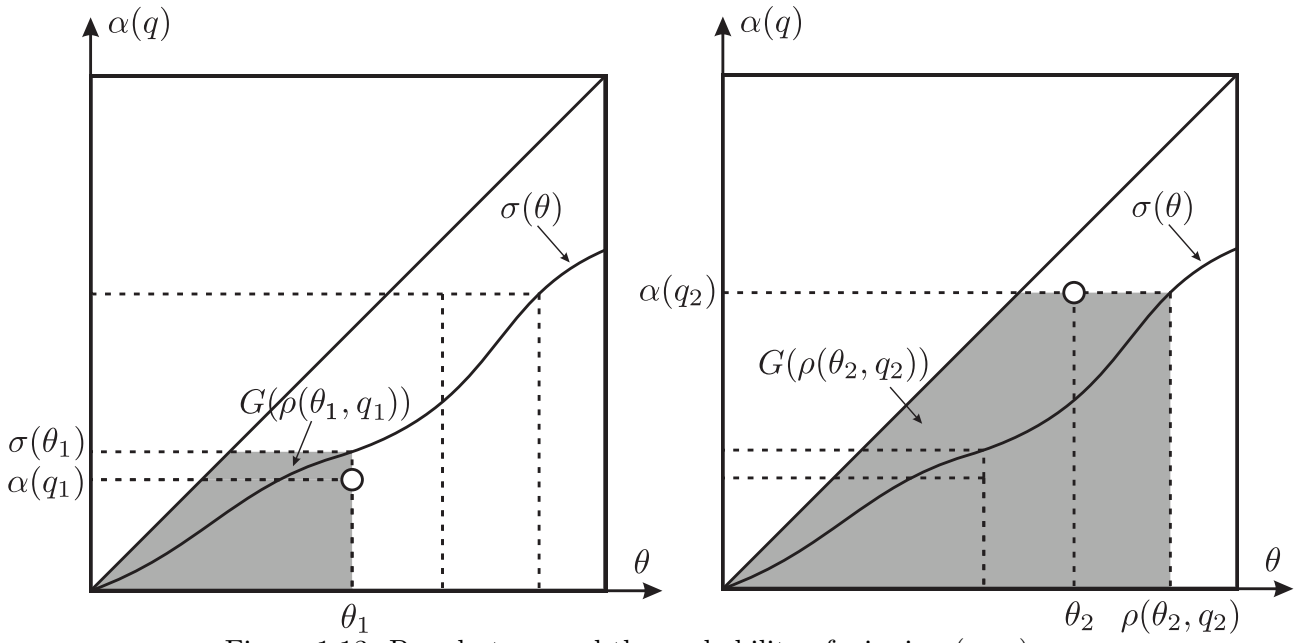

Figure 1.13: Pseudo-type and the probability of winning (grey).

conditions in this game are summarized below:

Direct Revelation Mechanism:

$$\sigma(\theta) \in \arg\max_{\rho}(\theta - \sigma(\rho))\,\mathcal{G}(\rho). \tag{1.15}$$

Indeed, the probability of winning against a single opponent by signaling ρ to the mechanism is equal to $G(\rho)$. The expected probability of winning is therefore $\mathcal{G}(\rho)$. The profit conditional on winning is equal to $b - c = (\theta - s)/\beta(q)$, where the score s is assigned based on the reported pseudo-type and therefore is equal to $\sigma(\rho)$.

Notice that the optimality conditions are exactly as in the first-price auction, if we interpret θ as value, $\sigma(\theta)$ as the bidding strategy and $\mathcal{G}$ as the value distribution. The classic first order conditions can be therefore written as:

First Order Conditions (general):

$$\overbrace{\sigma'(\rho)\cdot\mathcal{G}(\rho)}^{\text{marginal cost}} = \overbrace{(\theta-\sigma(\rho))\cdot\mathcal{G}'(\rho)}^{\text{marginal benefit}}. \tag{1.16}$$

Since $G(\theta) = \widetilde{F}(\theta, \sigma(\theta))$, the direct mechanism approach gives the same first order conditions as (1.10). But even without the first order conditions, the $\sigma(\theta)$ strategy can be characterized in terms of the pseudo-type distribution $\mathcal{G}$ by simply applying the *Envelope Theorem* to (1.15).

Proposition 2. *Under Assumptions 1-5, the equilibrium uncensored scoring strategy $\sigma(\theta)$ is a conditional expectation of the highest (among the other firms) pseudo-type below θ:*

$$\sigma(\theta) = \frac{1}{\mathcal{G}(\theta)} \int_{\underline{\theta}}^{\theta} z d\mathcal{G}(z) = \theta - \frac{1}{\mathcal{G}(\theta)} \int_{\underline{\theta}}^{\theta} \mathcal{G}(z) dz. \tag{1.17}$$

As in the classic first-price auction, the second order conditions follow from (and are equivalent to) the monotonicity of $\sigma(\theta)$, see, for example, Chapter 5 in [Kri10]. To convince ourselves that the second order conditions are indeed satisfied in our model, we evaluate the change in profit when submitting a pseudo-type $\hat{\rho}$ while having pseudo-type ρ:

$$\begin{aligned}
&(\theta - \sigma(\hat{\rho}))\mathcal{G}(\hat{\rho}) - (\theta - \sigma(\rho))\mathcal{G}(\rho) = \\
&(\hat{\rho} - \sigma(\hat{\rho}))\mathcal{G}(\hat{\rho}) + (\theta - \hat{\rho})\mathcal{G}(\hat{\rho}) - (\rho - \sigma(\rho))\mathcal{G}(\rho) - (\theta - \rho)\mathcal{G}(\rho) = \\
&\int_{\rho}^{\hat{\rho}} \mathcal{G}(z) dz + (\theta - \hat{\rho})\mathcal{G}(\hat{\rho}) - (\theta - \rho)\mathcal{G}(\rho) = \\
&\int_{\rho}^{\hat{\rho}} \mathcal{G}(z) dz - (\hat{\rho} - \rho)\mathcal{G}(\hat{\rho}) - (\rho - \theta)(\mathcal{G}(\hat{\rho}) - \mathcal{G}(\rho)) \leqslant 0.
\end{aligned}$$

The change is negative since $\rho \geqslant \theta$ by the definition of the pseudo-type.

Corollary 1. *Under Assumptions 1-5, the first order conditions are sufficient.*

Formula (1.17) also demonstrates that $\sigma(\theta) < \theta$ for all $\theta > \underline{\theta}$. As a result, for every level of quality except for the lowest, the threshold cost $c(q)$ for which the firm is willing to pick the reserve price lies in the interior of the support:

$$c(q) = r - \frac{\sigma^{-1}(\alpha(q)) - \alpha(q)}{\beta(q)} < r, \text{ for all } q > \underline{q}.$$

Consequently, as long as the distribution F has full support, there will be a positive mass of

bids at the reserve price, conditional on every level of quality except for the lowest.

Corollary 2. *Under Assumptions 1-5, there is bunching at every level of* $q > \underline{q}$.

Coupled with $G(\theta) = \widetilde{F}(\theta, \sigma(\theta))$ and $\mathcal{G}(\rho) = \sum p_k G^{k-1}(\rho)$, equation (1.17) pins down the evolution of both $\sigma(\theta)$ and $\mathcal{G}(\theta)$. From this system of non-linear equations, we can derive a system of differential equations which is linear in σ', G' and $\mathcal{G}'$:

$$\sigma \cdot \mathcal{G}' + \sigma' \cdot \mathcal{G} = \theta \cdot \mathcal{G}', \tag{1.18}$$

$$\mathcal{G}' = G' \cdot \sum_{k=2}^{N} (n-1) p_n G^{n-2}, \tag{1.19}$$

$$G' = \widetilde{F}_1' + \widetilde{F}_2' \cdot \sigma'. \tag{1.20}$$

By eliminating G' and $\mathcal{G}'$ we arrive to the same differential equation as in Proposition 3.

1.3.4 Welfare

In this section, we will introduce three measures of welfare generated by a firm participating in an auction: firm's profit, quality and buyer's rebate $r - b$; that will be later used to rank the counterfactual scoring rules. It will be convenient to first analyze the interim version.

Definition 8. *For a firm with type* θ *and quality* q*:*

$$\begin{aligned} \textit{interim expected profit:} \quad & P(\theta, q) = (b - c)\mathcal{G}(\rho(\theta, q)), \\ \textit{interim expected quality:} \quad & Q(\theta, q) = q\mathcal{G}(\rho(\theta, q)), \\ \textit{interim expected rebate:} \quad & R(\theta, q) = (r - b)\mathcal{G}(\rho(\theta, q)). \end{aligned}$$

By applying the envelope theorem to (1.15), we can find the profit as a solution to the boundary value problem below:

$$\frac{\partial}{\partial \theta} P(\theta, q) = \mathcal{G}\left(\rho(\theta, q)\right) / \beta(q), \quad P(\alpha(q), q) = 0.$$

This allows us to write down the interim expected profit, quality and rebate in the $(\theta, \alpha(q))$ coordinates in terms of the residual distribution of the pseudo-type and the pseudo-type itself. The total expected surplus from a single participating firm is measured by

$$(r - c) \cdot \mathcal{G}(\rho(\theta(c, q), q)) = (\theta - \alpha(q)) \cdot \mathcal{G}(\rho(\theta, q))/\beta(q),$$

which is the area of the marked region in Figure 1.14. Similarly to a classic first-price auction, this area is split between firm's profit and seller's revenue (in our case, buyer's rebate) by the $\mathcal{G}(\rho(*, q))$ curve, which captures the probability of winning having type θ.

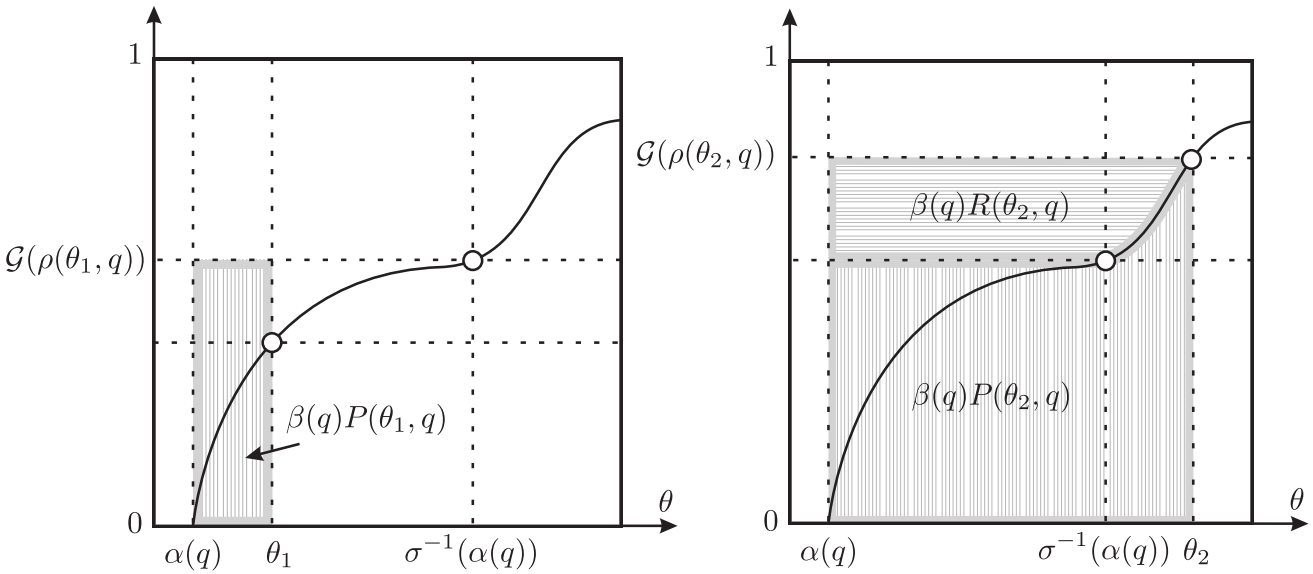

Figure 1.14: The firm's interim profit is the area under the $\mathcal{G}(\rho(*, q))$ curve, while the buyer's interim rebate is the area to the left from the $\mathcal{G}(\rho(*, q))$ curve.

Corollary 3. *The interim expected profit, quality and rebate can be computed from the residual pseudo-type distribution $\mathcal{G}(\rho)$ and the pseudo-type $\rho(\theta, q)$ using formulas below:*

$$P(\theta, q) = \int_{\alpha(q)}^{\theta} \mathcal{G}(\rho(z, q))dz/\beta(q)$$

$$Q(\theta, q) = q \cdot \mathcal{G}(\rho(\theta, q))$$

$$R(\theta, q) = (\theta - \alpha(q)) \cdot \mathcal{G}(\rho(\theta, q))/\beta(q) - P(\theta, q).$$

Had the turnout been fixed, to compute the total ex-ante quality and rebate, we would

average the interim ones with respect to the density of type and quality, and multiply by the number of participants. Due to random participation, the formulas are more tricky.

Conditional on the turnout n, the total rebate and quality extracted from firms with coordinates (θ, q) can be computed by renormalizing the interim ones by $G^{n-1}(\rho(\theta,q))/\mathcal{G}(\rho(\theta,q))$ and multiplying by n. These should be then averaged with the turnout probabilities p_n and integrated over the density of (θ, q).

Definition 9. *Define a function* $Y(\theta,q) = (\sum p_n n G^{n-1}(\rho(\theta,q)))/(\sum p_n G^{n-1}(\rho(\theta,q)))$.

Similarly to the $Z(\theta, \alpha(q))$ function, $Y(\theta, q)$ measures the extent of competition attributed to a firm with coordinates (θ, q). With fixed turnout it is simply equal to n.

Corollary 4. *The total (ex-ante) expected profit and quality can be computed from the interim ones, the* $Y(\theta,q)$ *function and the joint density of* $(\theta, \alpha(q))$ *using formulas below:*

$$\textit{total expected quality:} \quad \int\int Q(\theta,q)Y(\theta,q)\widetilde{f}(\theta,\alpha(q))d\theta d\alpha(q),$$
$$\textit{total expected rebate:} \quad \int\int R(\theta,q)Y(\theta,q)\widetilde{f}(\theta,\alpha(q))d\theta d\alpha(q).$$

1.4 Estimation

In this section, we treat our data as if there is no auction heterogeneity. We will show later in Section 1.5 that this is a reasonable approach, as long as the reserve price is a credible signal about the scale of the auction.

The theoretical analysis in Section 1.3.1 has shown that our model of scoring auctions shares important features with the classic first-price auction. Despite the fact that bidder's characteristic is two-dimensional, the core equilibrium strategy $\sigma(\theta)$ is one-dimensional, and, moreover, it is a solution to optimality conditions:

$$\sigma(\theta) \in \arg\max_{s} \pi(s)(\theta - s). \tag{1.21}$$

Since the probability of winning $\pi(s)$ is determined by the observed distribution of scores, nonparametric estimators of [GPV00] and [LPV02], which now can be considered as standard, can be easily generalized to work in our environment. However, we will make several important alterations that we explain below.

First, in order to capture the participation patterns in a symmetric model, we will estimate the strategy as a best response to the residual distribution of scores, that is the distribution of the maximum of scores among all but one bidder. Since this distribution can be thought of as generated by a composite bidder (the strongest among $n-1$ others), from the perspective of each single firm, the decision problem is as if it was competing against that composite bidder in a first-price auction. By that logic, it is absolutely justified to rely on the nonparametric techniques in [GPV00] and [LPV02] for the nonparametric estimation of the aforementioned distribution and the corresponding strategy.

Second, contrary to the classic first-price auction, it is not possible to use the estimated strategy $\sigma(\theta)$ to recover the whole distribution of (θ, q) from the observed distribution of (s, q), because only those bids that fall below the reserve price are coherent with that strategy. The bids that fall exactly at the reserve price therefore can not be used as part of the nonparametric estimation and should be dropped.

We first estimate the joint distribution of (b, q) using a bivariate kernel density estimator with boundary correction at $b = r$, advocated in [HH15]. We then transform it into the distribution of (s, q). Finally, by applying the inverse of the estimated $\sigma(\theta)$ strategy, we recover the part of the joint distribution of (θ, q) that does not map into the reserve price in equilibrium. The rest of the distribution is extrapolated from the boundary, namely, for every level of q, the density of a type that was forced to bid the reserve price will be set equal to the density of the type that chose the reserve price willingly, see Section 1.4.

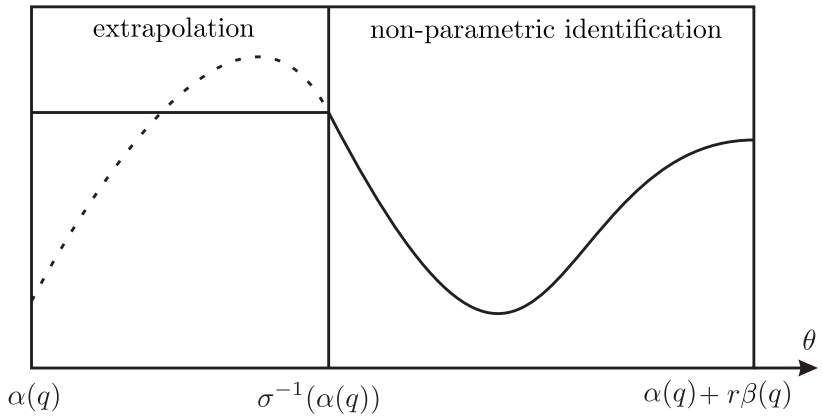

Figure 1.15: Estimation of the density of type, conditional on quality.

With the estimated distribution of (θ, q) at hand, we simulate the equilibria for the counterfactual scoring rules and calculate the expected quality and expected rebate using formulas derived in Section 1.3.4.

The rest of this section is split in three parts. In Section 1.4.1, we estimate the σ strategy. In Section 1.4.2, we recover the part of the distribution of (θ, q) that is identified. And, ultimately, in Section 1.4.3, we extrapolate the remaining part of the distribution, explain how to find the counterfactual equilibrium and compute welfare.

1.4.1 First-stage smoothing and strategy estimation

The first step in our estimation is to capture the equilibrium behavior imposed by the optimality conditions (1.21). The probability of winning $\pi(s)$ is measured by the distribution of the highest among $n-1$ firms score, which we refer to as a *residual distribution of scores*.

Definition 10. *Denote the* ***residual distribution of scores*** *by $\mathcal{H}$, and the corresponding density by $\hbar$.*

Following the logic of [GPV00], we estimate the perceived distribution $\mathcal{H}$ against which each firm is playing in our symmetric model together with its density $\hbar$.

Definition 11. *Denote the auction index by j and the bidder index (in that auction) by i.*

Definition 12. *Denote the sample of scores by $S = \{s_{ij}\}$ and its size by M.*

We apply a classic kernel density estimator to the residual scores, that is the maximal score among all but one firm, for each firm i in each auction j:

$$\widehat{\hbar}(s) = \frac{1}{M} \sum_{s_{ij} \in S} K(\frac{\max_{k \neq i} s_{kj} - s}{h_s}),$$

with the kernel K and the bandwidth b_s borrowed from [LPV02]:

$$K(u) = \frac{35}{32}(1-u^2)^3 \mathbb{I}(|u| \leqslant 1), \quad h_s = 3.16 \cdot sd(s) \cdot m^{-1/5},$$

where $sd(s)$ is the standard deviation of the score in S. The choice of this bandwidth is based on the popular statistical recipe called the 'Silverman rule', which is calibrated for normal distributions, see [Har12]. The score distributions in our data look reasonably bell-shaped, see Section 1.4.1, so we have no reason to distrust this approach.

Similarly, $\widehat{\mathcal{H}}$ can be obtained. From the practical point of view, however, we find it more natural to simply integrate the estimated density $\widehat{\hbar}$ numerically.

The estimator of [GPV00] is based on the first order conditions, and so it does not guarantee monotonicity of the strategy. However, a natural extremum estimator will have this property by construction:

$$\widehat{\sigma}(\theta) = \inf \arg\max_s \left(\widehat{\mathcal{H}}(s) \cdot (\theta - s) \right).$$

This estimator is convenient because it is aligned with our theoretical knowledge, and

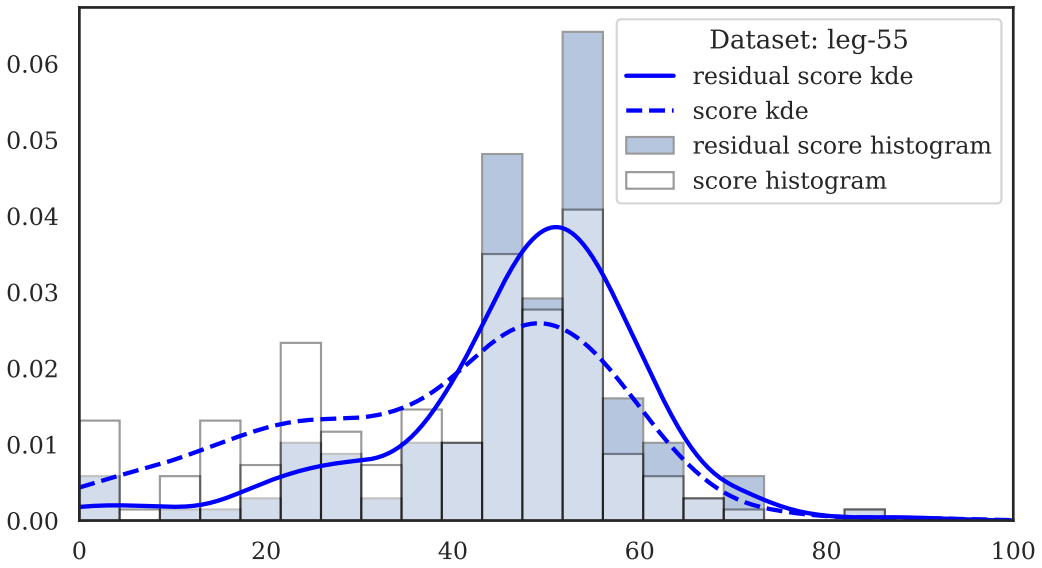

Figure 1.16: The estimators of densities of the score and residual score distributions.

because extremum estimators are among the most studied and well-understood in the literature, see [NM94] for an extensive overview.

1.4.2 Second-stage smoothing and boundary correction

The second step in our estimation is recovery of the joint distribution of (θ, q).

If we were to blindly follow the recipe in [GPV00], we would use the estimated strategy $\hat{\sigma}$ to produce a sample of estimated pairs $(\hat{\theta}_{ij}, q_{ij})$, from which the density of the distribution of (θ, q) could be obtained using a 2-dimensional kernel density estimator. The reason why we cannot do this is that only the part of the distribution of (c, q) which maps into the interior bids, i.e. above the reserve price, can be correctly identified.

Definition 13. *Denote the sample of interior bids by D_I and its size by M_I:*

$$D_I = \{(b_{ij}, q_{ij}) | b_{ij} < r\}.$$

To estimate the part of the distribution of (c, q) that is nonparametrically identified, we find it easier to change the order or actions in the [GPV00] algorithm. Namely, we will first apply a 2-dimensional kernel smoothing to D_I and only then use the estimated strategy $\hat{\sigma}$

to recover the relevant part of the distribution of (c, q). We will also put an additional effort to estimate the density at the boundary $b = r$, using *reflection method* advocated in [HH15].

Definition 14. *Denote the reflected sample of interior bids by D_{IR}:*

$$D_{IR}(\tau) = \{(b_{ij}, q_{ij}) | b_{ij} = r - \tau(\tilde{b}_{ij}, q_{ij}), (\tilde{b}_{ij}, q_{ij}) \in D_I\},$$

where

$$\tau(y, q) = y + d(q)y^2 + 0.55 \cdot d^2(q)y^3, \tag{1.22}$$

$$d(q) = \frac{\log(\varphi(h_1, q)) - \log(\psi(h_0, q))}{h_1}, \tag{1.23}$$

$$\phi(h_1, q) = \frac{1}{M_I^2} + \frac{1}{M_I h_1 h_q} \sum_{D_I} K\left(\frac{h_1 - (b_{ij} - r)}{h_1}\right) K\left(\frac{q_{ij} - q}{h_q}\right), \tag{1.24}$$

$$\psi(h_0, q) = \max\{\frac{1}{M_I^2}, \ \frac{1}{M_I h_0 h_q} \sum_{D_I} K_0\left(\frac{b_{ij} - r}{h_0}\right) K\left(\frac{q_{ij} - q}{h_q}\right)\}, \tag{1.25}$$

and K_0 is the endpoint kernel:

$$K_0(u) = (6 + 18x + 12x^2) \cdot \mathbb{I}\{-1 \leqslant u \leqslant 0\}.$$

The idea behind the reflection method is that when the data is censored at a certain threshold, we can anticipate the unobserved part of the data assuming smoothness of the original distribution.

Definition 15. *Denote the densities of (b, q), and $(\theta, \alpha(q))$ by g and $\widetilde{f}$ respectively.*

Below is a boundary corrected kernel density estimator of the joint distribution of (b, q):

$$\widehat{g}(b, q) = \frac{1}{2M_I h_b h_q} \sum_{D_I \cup D_{IR}(\tau)} K\left(\frac{b_i - b}{h_b}\right) K\left(\frac{q_i - q}{h_q}\right),$$

where the main bandwidths are taken in accordance with the 'Silverman rule':

$$h_q = 3.16 \cdot sd(q) \cdot M_I^{-1/5}, \quad h_\theta = 3.16 \cdot sd(\theta) \cdot M_I^{-1/5}$$

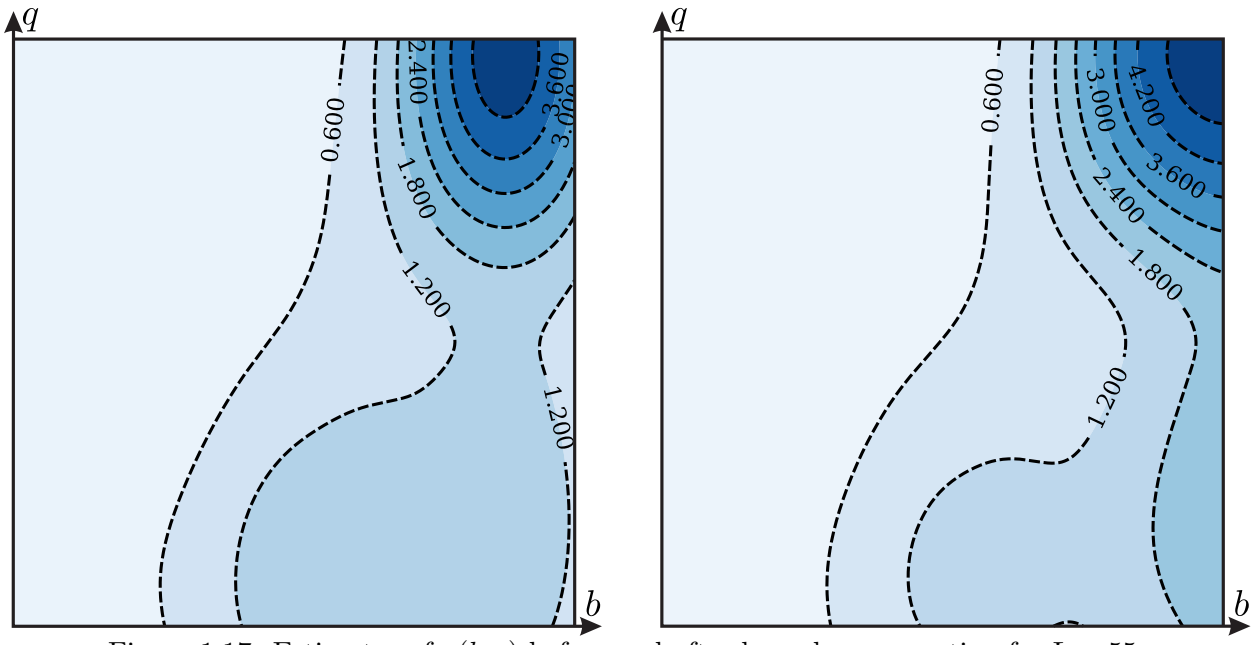

Figure 1.17: Estimator of $g(b, q)$ before and after boundary correction for Leg-55.

and the bandwidths used in the construction of τ are:

$$h_1 = h_\theta \cdot M_I^{-1/20}, \quad h_0 = 1.48 \cdot h_1.$$

The practical consequence of not doing boundary correction is that the density would unnaturally decrease near the boundary ($b = r$), as seen on the left side of Section 1.4.2. This would directly violate our knowledge of the equilibrium behavior, which predicts that firms bid essentially as if there is no reserve price and therefore the density should be smooth at the boundary.

We do not perform boundary correction for the quality dimension, on purpose, to over-smooth the spike in the neighborhood of $q = 100$, see Section 1.2.3. If the estimated distribution is not smooth enough, there is a risk of non-existence of a continuous strategy.

Finally the relevant part of the joint distribution of (θ, q) can be obtained from the estimated joint distribution of (b, q) using the standard density transformation formula:

$$\textit{density of } (\theta, q) = \begin{cases} k \cdot \frac{\sigma'(\theta)}{\beta(q)} \cdot g(r - \frac{\sigma(\theta)-\alpha(q)}{\beta(q)}, q) & , \ 0 \leqslant \frac{\sigma(\theta)-\alpha(q)}{\beta(q)} \leqslant r \\ * & , \ \text{otherwise} \end{cases},$$

where $*$ stands for the unknown value of the density in the non-identified part of the distribution, and k stands for the constant to normalize the density. We will have to find a way to fill in the missing part of the distribution, which will be discussed in the next section.

1.4.3 Extrapolation and counterfactuals.

The last thing that impedes our ability to simulate equilibria in a counterfactual scoring rule is the part of the distribution of (θ, q) that is not identified. This fundamental problem stems from the censoring of the $\sigma(\theta)$ strategy due to the presence of the reserve price.

It is important to understand at this point that multiple densities $\widetilde{f}(\theta, q)$ will be consistent with the data. We will pick one such density due to its particular simplicity. Namely, we will say that the the density at the type $\theta < \sigma^{-1}(\alpha(q))$ is equal to the density at $\sigma^{-1}(\alpha(q))$, which is the closest type for which it is identified. The equilibrium strategy serves as a boundary in the $(\theta, \alpha(q))$ coordinates to the types that choose the reserve price, therefore the level lines of the distribution are horizontal to the left of the strategy, see Section 1.4.3.

This gives us the following estimator for the whole range of $\theta \in [\alpha(q), \alpha(q) + \beta(q)r]$:

$$\widehat{\widetilde{f}}(\theta, a) = k \cdot \frac{\widehat{\sigma}'(\theta)}{\beta(\alpha^{-1}(a))\alpha'(\alpha^{-1}(a))} \cdot \widehat{g}(\min(r, r - \frac{\widehat{\sigma}(\theta) - a}{\beta(\alpha^{-1}(a))}), \alpha^{-1}(a)),$$

where k normalizes the density.

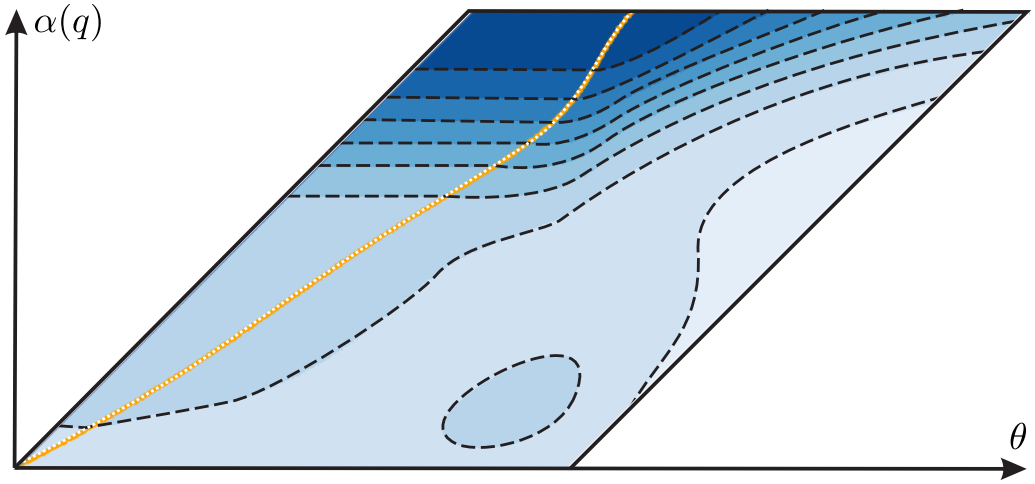

Figure 1.18: Estimator of the density of $(\theta, \alpha(q))$ after extrapolation for Leg-55.

Once the density is estimated, for a new counterfactual scoring rule, we can solve the differential equation (1.8) using any suitable numerical integration method, such as, for example, Runge-Kutta, to obtain the new $\sigma(\theta)$ strategy and the new distribution $\mathcal{G}(\rho(\theta, q))$.

There are two ways how the numerical integration may fail, see Section 1.3.2. The first is if the trajectory reaches the boundary of the anti-funnel, then a continuous strategy does not exist, however, this never happens in our data. In the hypothetical situation where it happens, increasing the smoothness of the estimated distributions should generally help. The second is if the trajectory passes through the point where $\widetilde{F}_1' = 0$. In this case, starting from the level of quality q^* at which it happened, all bidders choose the reserve price. As a result, the interim rebate for all $q > q^*$ is zero and the interim quality can be calculated according to the distribution of probabilities of winning, based on the fact that highest quality wins.

Finally, the welfare properties of the scoring rule are captured by two functions: $Q(\theta, q)$ and $R(\theta, q)$, calculated using formulas in Corollary 3. The ex-ante versions of quality and rebate are then computed using the estimated density of $(\theta, \alpha(q))$, see Corollary 4.

1.5 Heterogeneity

Typically in the literature on structural estimation, an auction model is complemented with a separate model of heterogeneity, which instructs how to prepare the data before passing it to the main estimation routine. A common approach is to represent the value (in our case, the cost) of the bidder as either a sum or a product of two components: common and idiosyncratic. The common component is then filtered out using the observed auction characteristics. Such approach was taken, for example, in [HHS03] and [Kra11].

In this paper, we take a different approach by relying on the reserve price as a control for heterogeneity. Precisely, we will assume that the bidder's cost is a product of an auction-specific component and a bidder-specific component:

$$c_{ij} = c_i \gamma_j, \quad r_j = r\gamma_j,$$

where γ_j represents the latent scale of the contract, and r_j is the reserve price in that auction. Crucially, the reserve price should follow the same scaling pattern.

Later in this section, we will show that, in this framework, the bids can be effectively homogenized by just normalizing them by the reserve price. But before that, it is important to understand why the reserve price is traditionally avoided as a control for the scale of the auction, and why our situation is different.

1.5.1 Reserve price as a control

In the classic value auctions, for which the bulk of empirical methods were developed, the reserve price is often set ad-hoc and is even sometimes entirely missing. That is because the benefits of fine-tuning the reserve price are very limited. In fact, a theoretical argument in [BR89] shows that it is more profitable to simply attract an additional bidder rather then set an optimal reserve price. Another explanation might be that costs of acquiring the statistical information needed for the optimal mechanism are prohibitively high. All of this makes the reserve price an unreliable instrument for economic analysis.

In procurement the situation is slightly different. First of all, since there is no natural upper bound to the bid (in value auctions bids are typically non-negative) the reserve price is always present, to cut the contractee's losses. Second, because the procurement contracts are often very complex and not all cost-relevant information can be conveyed in the tender documentation, the contractee might want to use the reserve price as a signaling device to help the firms estimate their costs. The reserve price then can be thought of as a publicly observed first bid. This signaling role is so important that it was, in fact, institutionalized by the Russian authorities, see Section 1.2.1

1.5.2 Homogenization of bids

Assume that, in any given auction j, the score, on top of satisfying all the assumptions of the model, has a particular shape:

$$s(b, q \mid r) = \tilde{s}(b/r, q),$$

which is true for the linear scoring formula that we have in our data. We can write down the equilibrium conditions for a representative auction with reserve price r in the Lagrangean form as in (1.26). It is easy to see that the equilibrium conditions stay unchanged if the same linear transformation $\psi_j(x) = \gamma_j x$, where $\gamma_j > 0$, is applied to all of the three variables c, b, and r:

$$\mathcal{H}\left(s(b, q \mid r)\right) \cdot (b - c) + \lambda(r - b) \to \min_{\lambda \geqslant 0} \max_{b}, \tag{1.26}$$

$$\mathcal{H}\left(s(\psi_j(b), q \mid \psi_j(r))\right) \cdot (\psi_j(b) - \psi_j(c))) + \lambda(\psi_j(r) - \psi_j(b)) \to \min_{\lambda \geqslant 0} \max_{b}. \tag{1.27}$$

Indeed, the same bidding strategy $b^*(c, q)$ is the solution to both (1.26) and (1.27), which means that the equilibrium bidding strategy in an auction that is scaled by γ_j, is also scaled by γ_j. In other words, the same equilibrium strategy $\sigma(\theta)$ can explain the behavior in multiple heterogeneous auctions as long as they are scalable to a single representative auction with a fixed reserve price, which is our model of heterogeneity.

Since in our data all the bids are divided by the reserve price inside the score, this effectively eliminates the impact of γ_j on the equilibrium behavior. Consequently, the model can be estimated as if there is no auction-specific heterogeneity.

Corollary 5. *Consider a model of heterogeneity, where for a firm i in auction j:*

$$s_j(b,q) = \tilde{s}(b/r_j, q), \quad c_{ij} = c_i\gamma_j, \quad r_j = \gamma_j r, \quad q_{ij} = q_i,$$

where (c_i, q_i) are drawn as in a symmetric model. Then the equilibrium strategies $\sigma(\theta)$ and $\sigma^*(\theta, q)$ do not depend on the scale of the auction γ_j.

While theoretically appealing, our approach to homogenization of bids has to yet prove itself successful in the data. To test it, we will randomly pick two bids (without return) from each auction and plot the scatterplot of logarithms of the two bids, and a scatterplot of the two bids normalized by the reserve price, see Section 1.5.2 for the Leg-55, Sci-55 and Sec-80 datasets.

It can be seen with a naked eye that while the log-bids indeed require additional homogenization, it is far less obvious for the normalized bids, as the reserve price absorbs a significant portion of heterogeneity.

	before normalization		*after normalization*	
Dataset	Kendall	Pearson	Kendall	Pearson
Edu-80	0.86	0.97	0.27	0.39
Edu-55	0.85	0.97	0.23	0.29
Sci-80	0.81	0.94	0.21	0.29
Sci-55	0.79	0.94	0.01	0.03
Leg-80	0.68	0.91	0.2	0.13
Leg-55	0.82	0.94	0.21	0.13
Tec-80	0.81	0.94	0.33	0.5
Tec-55	0.86	0.97	0.27	0.39
Sec-80	0.89	0.99	0.36	0.52
Sec-55	0.78	0.97	0.11	0.4

Table 1.7: Correlation in bids before and after normalization.

The quality of homogenization varies across the datasets, see Section 1.5.2, but since there is no clear rule on how much correlation among bids is acceptable, we leave the data

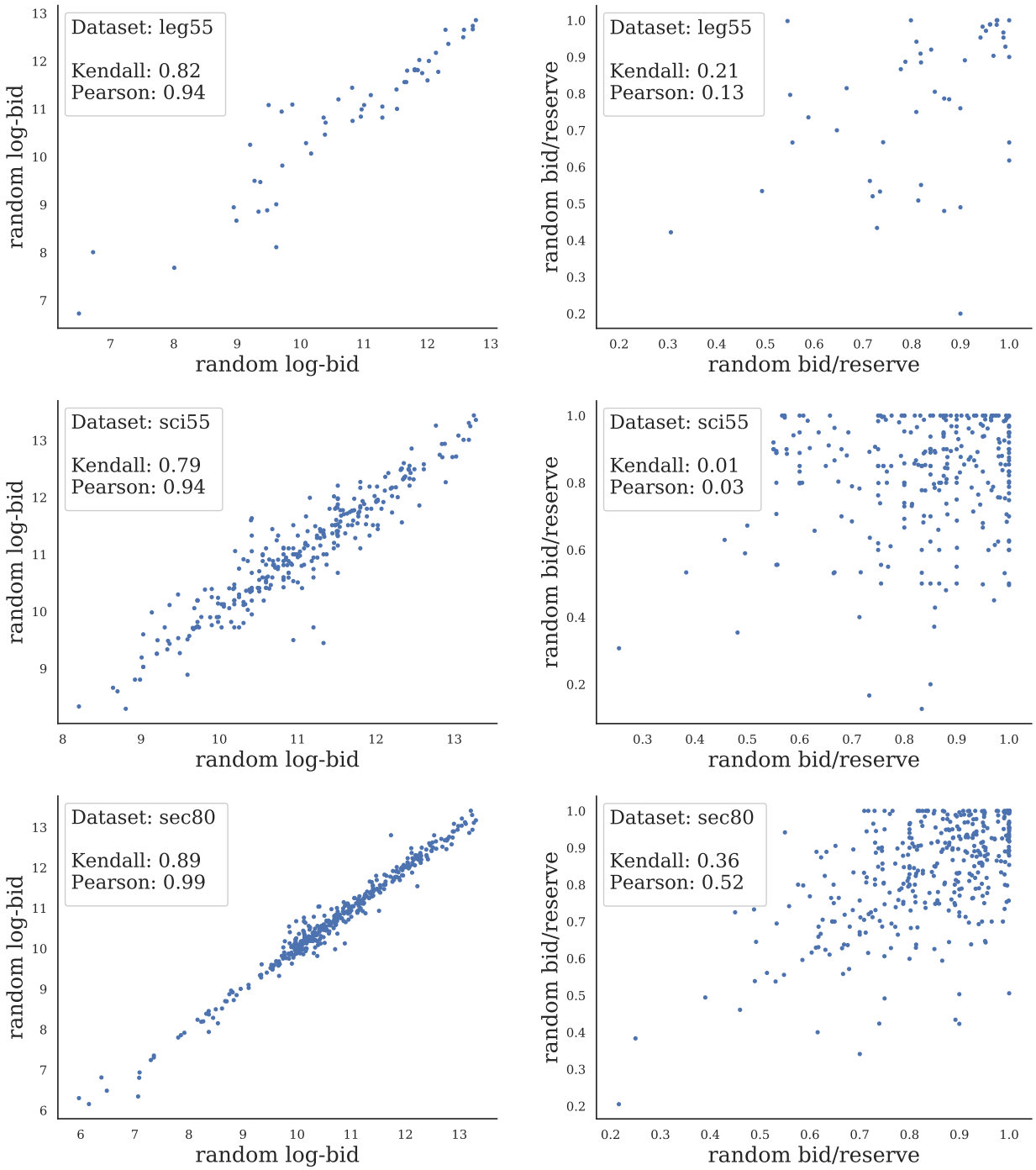

Figure 1.19: Normalization of bids.

as it is.

1.6 Empirical results

For each of the ten datasets, and both scoring designs, I have traced the corresponding frontiers in the space of expected rebate and expected quality (both measured from 0 to 100), spanned by the weight in the scoring formula. For each dataset, I produced a figure that contains these frontiers, as well as the coordinates of the default linear scoring rule and its primary competitor: the price-quality-ratio ($s(b,q) = -b/q$) scoring rule.

I will focus below on the results for the Leg-55 dataset, characterized by the diagram in Section 1.6. The diagram contains two arks, representing the welfare frontiers spanned by the two families: linear and log-linear.

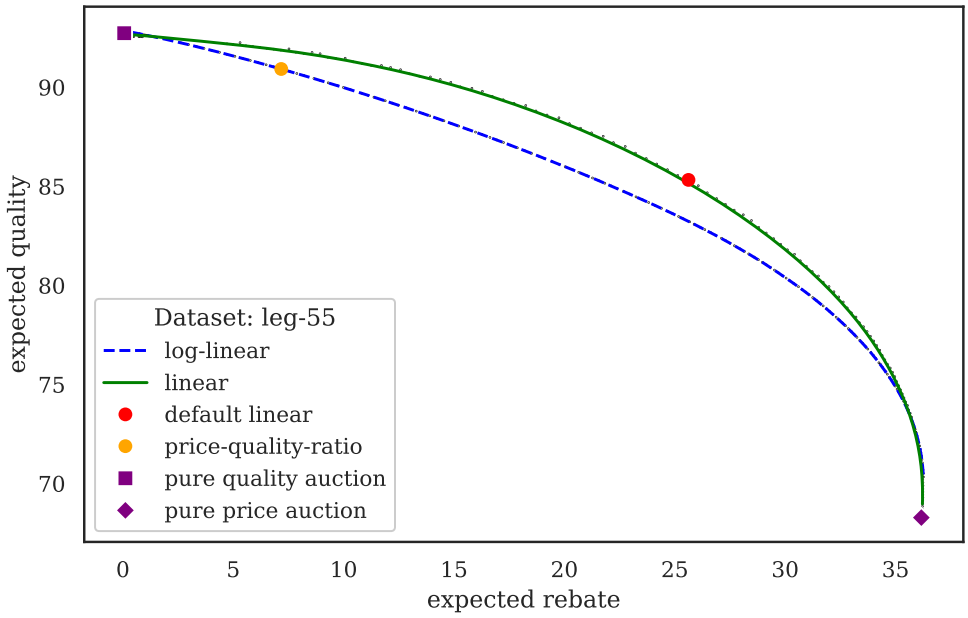

Figure 1.20: Welfare frontiers for the linear and log-linear families.

Several observations can be made.

(a) The arcs meet at the end points.

(b) The arcs are decreasing.

(c) The default linear rule is to the bottom-right from the price-over-quality rule.

(d) The linear design arc lies above the log-linear one.

All four observations are true for each of our datasets, see Figure 1.21. We will now explain the relevance of these findings for the main economic question addressed in the current paper.

Observation (a) should not be a surprise since the pure price auction and the pure quality auction can be considered as polar in the spectrum of scoring auctions produced by varying the weight in the scoring formula. The figure then confirms our basic intuition that for the weight approaching 0 or ∞ the two scoring designs become indistinguishable in terms of welfare.

Observation (b) is also anticipated, since the weight in the scoring formula is an intuitive control for the trade-off between quality and rebate. The figure confirms that no two members of the same scoring family can be ranked without the knowledge of designer's preferences. In other words, we can only hope to achieve an unambiguous improvement by switching between the linear and log-linear designs.

Observation (c) demonstrates the failure of the traditional approach to mechanism choice, when a default linear design is compared to a single member of the log-linear design. Indeed, a switch to price-over-quality would increase expected quality by approximately 10 points, but would decrease the expected rebate by approximately 30 points. Whether this is a beneficial exchange depends completely on how the contractee values quality in monetary terms, which can not be inferred from the auction data.

To quantify our findings, we compare the default linear scoring rule to the quality-equivalent member of the log-linear family. This can be interpreted as switching to the price-over-quality scoring rule, with a specially calibrated measure of quality to guarantee the same expected quality. Our results are summarized in Section 1.6, and can also be inferred from the frontiers in Figure 1.21.

Finally, observation (d) is the main qualitative result in our paper. It appears that, for every member of the log-linear family, a member of the linear family dominates it in both welfare dimensions. In other words, even without knowing the designer's preferences for quality and rebate, we can tell that his utility, as long as it is a nondecreasing function of

dataset name	switch to price-over-quality	switch to log-linear with same expected quality
Edu-80	–26.42 % rebate and +11.99 % quality	-0.90 % rebate
Edu-55	–2.79 % rebate and +0.28 % quality	-2.10 % rebate
Sci-80	–27.59 % rebate and + 13.59 % quality	-1.16 % rebate
Sci-55	–5.90 % rebate and + 1.04 % quality	-3.12 % rebate
Leg-80	–14.50 % rebate and +4.25 % quality	-2.21 % rebate
Leg-55	–18.45 % rebate and +5.60% quality	-3.94 % rebate
Tec-80	–47.11 % rebate and +19.24 % quality	-1.31 % rebate
Tec-55	–4.11 % rebate and +0.61 % quality	-2.69 % rebate
Sec-80	–9.72 % rebate and + 6.10 % quality	-0.41 % rebate
Sec-55	–5.24 % rebate and +1.14 % quality	-1.80 % rebate

Table 1.8: Quantitative results

expected quality and rebate, is maximized at one of the linear scoring rules. Thus the linear design can be considered as superior to the log-linear one.

1.7 Conclusion

In this paper, I have investigated whether the linear scoring design is better than its main competitor, price-quality ratio, in terms of welfare, and received a strong positive answer.

My methodological contribution to the literature on scoring auctions is a class of models that are sufficiently flexible to feature both linear and log-linear scoring rules, yet they hold enough structure to produce a very tractable solution. These models are built on two core assumptions: exogenous quality and explicit reserve price. We have also developed a complete machinery for the estimation of these models and simulation of counterfactuals.

My empirical contribution is a qualitative result about the superiority of the linear scoring design. From Section 1.6, as well as from every other simulation, it appears that every possible log-linear scoring rule is dominated by some linear scoring rule. Moreover, this finding does not depend on the exact knowledge of designer's preferences over quality and rebate, which makes it easy to be interpreted as policy advice.

Implicit in this assessment is the risk-neutrality of the designer's preferences, and so it is not surprising that the linear design succeeded over the log-linear. Had the preferences been risk-averse, the log-linear scoring rule could be advantageous.

Several avenues for future research are suggested.

Endogenous quality. My analysis of the affine scoring rule can be partially extended to the situation where quality is endogenous. This model can be thought of as part of a two-stage game, where the quality is chosen at the first stage, and is considered exogenous afterwards. However, to find an equilibrium in this game, more complicated tools, such as the ones developed in [HHN16], will be required.

Discontinuous strategies. Though I focus on the most regular case of a continuous strategy, situations may arise in practice where such equilibrium will not exist. In this case, the non-linear equation (1.17) may be considered as a heuristically derived characterization of the equilibrium. In fact, there is nothing in this equation, that particularly requires continuity.

Other scoring rules. While it might seem that the next logical step is to consider a scoring rule of an arbitrary shape, as in [HHN16], the empirical application calls for an even more general model where the scoring rule additionally depends on certain statistics of other firm's bids and qualities. This model would be useful for understanding the complicated scoring rules such as the "average bid" used in Italy and the "minimal bid" used in Russia.

1.8 Appendix A

Proof of Lemma 1

Assume that for some $\hat{\theta}, \hat{q}$, it is true that $\sigma^*(\hat{\theta}, \hat{q}) \neq \max(\sigma(\hat{\theta}), \alpha(\hat{q}))$, which means:

$$\sigma(\hat{\theta}) = \sigma^*(\hat{\theta}, \underline{q}) \leqslant \alpha(\hat{q}) < \sigma^*(\hat{\theta}, \hat{q}).$$

This can only be the case if $\sigma^*(\hat{\theta}, \hat{q})$ is a local maximum of a continuous function $\pi(s)(\hat{\theta} - s)$. Moreover, this function has to stay constant all the way between $s_1 = \sigma^*(\hat{\theta}, \underline{q})$ and $s_2 = \sigma^*(\hat{\theta}, \hat{q})$, otherwise the actual score $\sigma^*(\hat{\theta}, q)$ would be discontinuous in q somewhere in the range between $\underline{q}$ and $\hat{q}$, thus contradicting the continuity assumption. Staying constant, on the other hand, contradicts the assumption that a lower score is chosen when indifferent.

Proof of Proposition 3

The first step in the proof is to establish several important properties of the $\sigma(\theta)$ strategy. Clearly, $\sigma(\theta) = \min_q \sigma^*(\theta, q)$ is continuous, because $\sigma^*(\theta, q)$ is continuous in both arguments. It is also weakly increasing, by super-modularity of $\pi(s)(\theta - s)$, see [MS94].

To see that $\sigma(\underline{\theta}) = \underline{\theta}$, note first that the actual distribution of scores $s = \alpha(q) + \beta(q)(r - b)$ is constrained by $\alpha(\underline{q}) = \underline{\theta}$ from below, as no firm is allowed to bid above the reserve price. The best (unconstrained) response $\sigma(\theta)$ therefore can not go below $\underline{\theta}$ as that would induce an upwards deviation for the lowest type. On the other hand, $\sigma(\theta)$ can not go above θ as that would guarantee a negative profit with positive probability for any type $\theta > \underline{\theta}$. Consequently, by continuity, the $\sigma(\theta)$ function at the lowest type $\underline{\theta}$ is confined to a single possible value $\underline{\theta}$.

To prove strict monotonicity, assume that the true range of the $\sigma^*(\theta, q)$ function is $[\underline{\theta}, \overline{s}]$. The $\pi(s)$ function is strictly increasing in this range by continuity of $\sigma^*(\theta, q)$ and $F(c, q)$. Therefore, the $\sigma(\theta)$ strategy is strictly increasing for all types that map into $(\underline{s}, \overline{s})$, see [MS94]. In other words, it can only be flat at two segments: one adjacent to $\underline{\theta}$ and one adjacent to $\overline{\theta}$. Typically, flat strategies are incompatible with symmetry in classic first-price auctions, however, due to the 2-dimensional firm characteristic and the wedge between σ and σ^*, eliminating such behavior requires slightly more work.

First, assume that there is a segment of types adjacent to $\underline{\theta}$ that maps into $\underline{\sigma}$. That would mean that, for all types in that segment such that $\theta > \underline{\theta}$, the actual score σ^* is equal to $\alpha(q) > \alpha(\underline{q}) = \underline{\theta}$ after truncation. Consequently, before truncation, the perceived probability of winning for all those types would be zero, which is worse than if they chose any other score above $\underline{\theta}$ but below their own type θ. Second, assume that there is a segment adjacent to $\overline{\theta}$, that maps into $\overline{s}$. That would mean that $\overline{s}$ is the maximum value of both σ and σ^*. Consequently, for a positive measure of (θ, q) the actual score would be $\overline{s}$, and, therefore there will exist at least one (θ, q) that would deviate. The $\sigma(\theta)$ strategy is therefore strictly monotone in the whole range of types.

Once we have invertibility of σ, we can construct the pseudo-type $\rho(\theta, q) = \sigma^{-1}(\sigma^*(\theta, q))$ and proceed with the direct mechanism approach in Section 1.3.3. Using the residual pseudo-type distribution $\mathcal{G}$ we can characterize $\sigma(\theta)$ as a solution to the optimality conditions below:

$$\sigma(\theta) \in \arg\max_{\rho}(\theta - \sigma(\rho))\,\mathcal{G}(\rho). \tag{1.28}$$

The necessary first order conditions can be written as a system of equations:

$$\sigma\mathcal{G}' + \sigma'\mathcal{G} = \theta\mathcal{G}', \quad Z\mathcal{G}'G = G'\mathcal{G}, \quad G' = \widetilde{F}_1' + \widetilde{F}_2'\sigma',$$

that can be uniquely solved, assuming that $\widetilde{F}_1' > 0$:

$$\sigma' = \frac{(\theta - \sigma)\widetilde{F}_1'}{ZG - (\theta - \sigma)\widetilde{F}_2'}, \quad G' = \frac{ZG\widetilde{F}_1'}{ZG - (\theta - \sigma)\widetilde{F}_2'}, \quad \mathcal{G}' = \frac{\mathcal{G}\widetilde{F}_1'}{ZG - (\theta - \sigma)\widetilde{F}_2'}.$$

Recalling that $G(\theta) = \widetilde{F}(\theta, \sigma(\theta))$, we obtain our differential equation:

$$\sigma' = \frac{(\theta - \sigma)\widetilde{F}_1'}{Z\widetilde{F} - (\theta - \sigma)\widetilde{F}_2'}.$$

The second order conditions are satisfied as $\sigma(\theta)$ strategy is a conditional expectation of the highest pseudo-type below θ, see Section 1.3.3 for a discussion. If the $\sigma(\theta)$ trajectory passes through the point where $F_1' = 0$, it will switch to a different law of motion: $(\theta - \sigma)\widetilde{F}_2' = Z\widetilde{F}$,

however, under Assumption 5 this never happens, see Section 1.3.2 for a discussion.

It remains to show that coupled with the initial condition $\sigma(\underline{\theta}) = \underline{\theta}$, the differential equation has a unique solution in the range $[\underline{\theta}, \overline{\theta}]$, for which we will employ the theory of fences and funnels from [HW97].

An *anti-funnel* consists of two functions of time (in our case θ), called upper fence and lower fence. Both functions should be continuously differentiable and such that for each point on the fence, the slope of the vector field passing through is lower than the slope of the upper fence and greater than the slope of the lower fence at that point. The obvious choice of the upper fence is $\sigma = \theta$, because the vector field has zero slope there. For the lower fence a natural candidate is the $\sigma = \delta(\theta)$ curve:

$$\delta(\theta) = \{\sigma \in [0, \theta] : Z(\theta, \sigma)\widetilde{F}(\theta, \sigma) - (\theta - \sigma)\widetilde{F}'_2(\theta, \sigma) = 0\},$$

which is well defined under Assumption 5 and the vector field has an infinite slope there.

The coordinates (θ, σ) are inconvenient for the formal proof of uniqueness and existence, because the right handside of the differential equation is formally not defined at the lower fence. But this is only a notational problem. Assume for simplicity that $\underline{\theta} = 0$ and make a change of coordinates:

$$t = (\theta + \sigma)/2, \quad x = (\sigma - \theta)/2,$$

which corresponds to a 45 degrees clockwise rotation of the vector field around the $(0, 0)$ point. The new differential equation $x' = f(t, x)$ is now well-defined at the boundary of the new anti-funnel. Moreover, since all the functions α, β, F are assumed to be twice continuously differentiable, so does $\widetilde{F}$ and $Z(\theta, \sigma)$. Therefore, $f(t, x)$ is continuously differentiable, for all points inside and on the boundary of the anti-funnel. And since the new anti-funnel is a compact, $f(t, x)$ is uniformly bounded together with its first derivatives.

To establish existence we apply *Theorem 4.7.1 (Fence Theorem)* and *Theorem 1.4.4 (Anti-Funnel Theorem: Existence)* which only require that $f(t, x)$ satisfy continuity in t and Lipshitz condition in x inside the anti-funnel, which follows from continuous differentiability

of f.

To establish uniqueness we apply *Theorem 4.7.5 (Second Uniqueness Criterion For Anti-Funnels)* which additionally requires that the anti-funnel is narrowing at the initial point (which is true in our case) and that inside the anti-funnel the following property holds:

$$\frac{\partial}{\partial x} f(t,x) \geqslant w(t), \quad \int_a^b w(t)dt > -\infty,$$

which follows from the fact that $\frac{\partial}{\partial x} f(t,x)$ is uniformly bounded there.

Proof of Proposition 4

We start from the direct mechanism derived in the proof of Proposition 3:

$$\sigma(\theta) \in \arg\max_{\rho} (\theta - \sigma(\rho))\mathcal{G}(\rho). \tag{1.29}$$

To apply the envelope argument, we use *Theorem 2* from [MS02]. The premises of this theorem are satisfied since the profit function in (1.29) is continuously differentiable in θ, and its partial derivative with respect to θ is uniformly bounded. Since σ is strictly monotone (see proof of Proposition 3), for any θ there exists a q such that $\rho(\theta, q) = \theta$, and therefore the envelope conditions can be written as:

$$(\theta - \sigma(\theta))\mathcal{G}(\theta) - (\underline{\theta} - \sigma(\underline{\theta}))\mathcal{G}(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} \mathcal{G}(z)dz. \tag{1.30}$$

Finally, since $\sigma(\underline{\theta}) = \underline{\theta}$ (see proof of Proposition 3), we obtain our formula:

$$\sigma(\theta) = \theta - \frac{1}{\mathcal{G}(\theta)} \int_{\underline{\theta}}^{\theta} \mathcal{G}(z)dz. \tag{1.31}$$

1.9 Appendix B

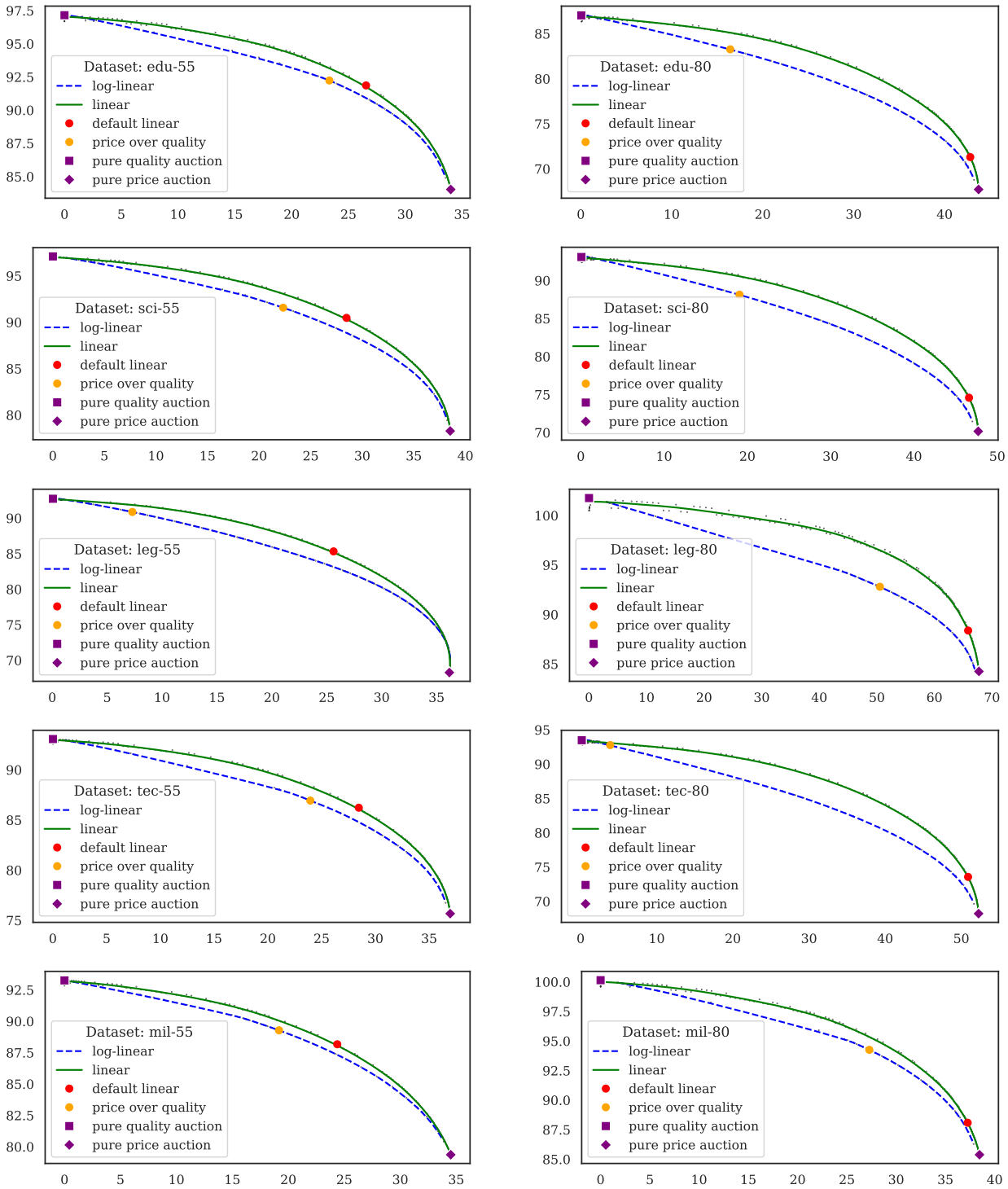

Figure 1.21: Welfare frontiers in the space of expected rebate and expected quality.

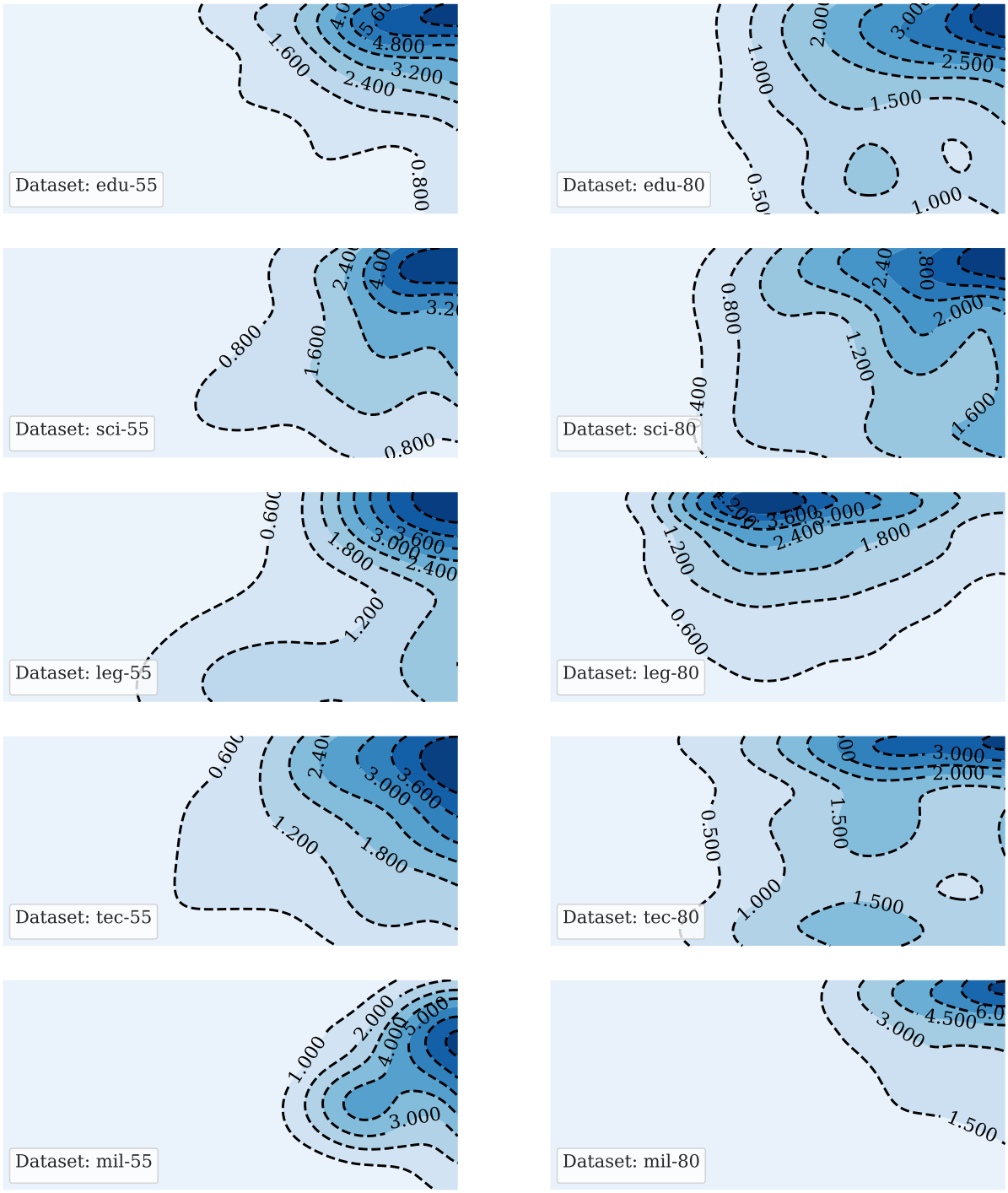

Figure 1.22: Joint distribution of bids and quality.

CHAPTER 2

Optimal Robust Divisible Public Goods

2.1 Introduction

We study *divisible* allocation of a *pure* (non-excludable) public good among n agents with quasi-linear utility. Divisibility means that the good can be allocated at different levels, in contrast with the *binary* allocation that can pick either full scale or nothing. Non-excludability suggests that everyone has an unrestricted access to the full amount produced.

Such public goods are easy to find in any modern community, the simplest example being public security, which can vary in quality and, therefore, costs of provision. We restrict attention to the special case when both the costs and the utilities are linear in production level.

We are interested in finding an optimal robust mechanism, meaning that expected welfare is maximized subject to incentive compatibility (IC), individual rationality (IR), and budget balance (BB) constraints that have to be satisfied for all realization of types, or, in other words, *ex-post*. The budget balance property indicates that the mechanism never produces deficit or surplus. This definition of robustness is in line with [HR87] and [CP16], but there are several other interpretations such as in [BM05], [BS11], or [BS12]. Instead of the ex-post IC, a dominant-strategy incentive compatibility ($DSIC$) constraint is commonly used in the literature. These two types of constraints are equivalent for direct mechanisms in private values environments like ours.

It was recently shown in [KS17] that robust mechanisms for public good, which minimize expected welfare loss among binary ones, have to take a certain threshold form. However, it is not obvious that such mechanisms should be binary in the first place. The goal of our

paper is to study mechanisms in the public goods setting, that are robustly optimal among the divisible ones.

Our contribution consists of two main results. The first main result is that, under certain additional assumptions, expected welfare loss attains minimum at a simple posted price, or, equivalently, a threshold mechanism. This finding reinforces the results in [KS17] by justifying the restriction to binary mechanisms in two steps. First, in Proposition 3, we show that a robust divisible mechanism is a lottery over simple posted prices. Second, in Proposition 4, we argue that, for any prior belief, expected welfare loss is maximized at the vertex of the lottery simplex. Our second main result is that, under certain beliefs, a significant portion of welfare is guaranteed to be lost even if the mechanism was chosen optimally among the robust ones. In Proposition 5, we derive a lower bound for the associated welfare loss, which grows linearly in the number of agents. This bound is tight for the case of two agents.

Most of the assumptions required for our results are of technical nature and barely restrain their applicability. But there is one substantial restriction, namely Assumption 8, which requires that each agent's equilibrium utility at the lowest type is equal to zero. A similar property was derived in [KS17] as a result of expected welfare loss minimization, for binary mechanisms. Due to complexity of the divisible good setting, we do not derive it, instead in Section 2.3.1, we argue that in a variety of situations it is automatically satisfied.

On the technical side, our approach is closest in spirit to [HR87], who studied robust mechanisms for bilateral trade. This problem is dual to that of a public good with two agents, and so our results can be considered as a direct extension. A distinct feature of our paper is that we consider a broad class of upper semi-continuous allocation functions, as opposed to differentiable ones in [HR87], or binary ones in [KS17]. These two classes of mechanisms typically involve different solution techniques. In contrast, we develop a unified method that suits both, which is demonstrated in the proof of Lemma 3.

The tools are novel to the mechanism design literature, and can be possibly applied to other economic environments.

2.1.1 Literature review

There are multiple opinions in the literature on what robustness exactly means. For example, in [BM05], robustness is defined as Bayesian implementation on all type spaces, which is a stronger concept than ex-post implementation. An alternative method was proposed in [BS12], which eliminated weakly dominated strategies based only on the knowledge of the support of distribution. Robustness in [BS11] means that the maximal regret from a small misspecification of beliefs is also relatively small.

Robustness in our paper implies that the incentive compatibility, individual rationality, and budget balance constraints are formulated in such a way that they do not rely on common knowledge of the distribution of types. In the context of direct mechanisms, this amounts to all the constraints being evaluated ex-post. This is the exactly same approach that was used in [HR87] to study robust bilateral trade.

In the classic papers by [MS83a] and [GH86], the optimal Bayesian (second-best) divisible mechanisms were identified for bilateral trade and pure public goods respectively. The optimal robust (third-best) divisible mechanisms were described in [HR87] and further studied in [CP16]. A systematic analysis of the second-best and third-best mechanisms for bilateral trade, which can also be thought of as a pure public good with two agents, can be found in [BKS15].

Related are numerous papers that aim at selecting a thin set of mechanisms without any particular welfare assessment. Typically, a strong solution concept like coalition-proof stability as in [Mou94] and [BH16], or a symmetry/anonymity assumption as in [Ser99] is used. Compared to these papers, we harness a relatively weak solution concept, that allows to select a rich but tractable family of asymmetric mechanisms.

The optimality of posted prices in our paper is similar to the result in [CP16], but there are a few substantial differences. On the one hand, they consider a broader class of welfare criteria captured by the notion of ex-post Pareto optimality. On the other, they insist on the interpretation of mechanisms as lotteries, which allows them to use a stronger version of ex-post *IC* and ex-post *IR* constraints. Our interpretation of mechanisms is that the

allocation is divisible, and so it is more general and consistent with the classic approach in [HR87].

Our assessment of welfare losses of robust mechanisms is similar to how revenue losses of certain simple mechanisms were studied by [HN17] in the context of monopolistic screening. Instead of directly solving for the profit, they show that a constant fraction of optimal (second-best) revenue can be attained, for all prior beliefs. Likewise, in our setting, a constant fraction of optimal (second-best) welfare is forgone by robust mechanisms, under certain beliefs.

The rest of the paper is organized as follows. We describe the setup in Section 2.2, and the preliminary results in Section 2.3. The latter consists of three parts, in which we discuss the additional assumptions, introduce the tools for studying divisible mechanisms, and prove the technical lemmas needed for the main results of the paper. The main results are in Section 3.3, which consists of three parts: the characterization of robust mechanisms in terms of lotteries over posted prices in Section 2.4.1, the proof that posted prices are optimal in Section 2.4.2, and the derivation of the welfare bounds in Section 2.4.3. We discuss our findings in Section 2.5.

2.2 Setup

A divisible public good can be produced at any level in the segment $[0, 1]$, where zero stands for no production, and the marginal cost of production is equal to $c > 0$.

Let there be n agents with quasi-linear utilities. Each agent has constant marginal valuation from the public good, which is unknown to the designer. Denote by $v = \{v_i\}$ the profile of valuations, and let it be distributed on a rectangular area $V = \{v : v_i \in [a_i, b_i]\}$, where $\sum a_i < c < \sum b_i$.

Following the tradition of mechanism design, we focus on a direct mechanism that assigns the level of production and transfers based on the reported profile of types v. We consider a divisible (as opposed to binary) mechanism, which means that for a profile of types v a

certain level of production between 0 and 1 is assigned.

Definition 16. *A **direct mechanism** μ is a mapping from V into $A \times \mathbb{R}^n$:*

$$\mu : v \mapsto (\varphi, \tau)$$

*where A is equal to $\{0,1\}$ for the **binary** and to $[0,1]$ for **divisible** case.*

A mechanism μ maps the profile of valuations v into $\varphi(v)$ - the expected allocation function, and $\tau = \{\tau_i\}_{i=1}^n$ - the vector of expected payments made by the agents. For the rest of the paper, two mechanisms are considered equivalent if their allocation function and transfers coincide.

For the reported profile $v = (v_i, v_{-i})$ and a true type $\hat{v}_i$ of agent i, his off-equilibrium utility $U_i(\hat{v}_i, v)$ is:

$$U_i(\hat{v}_i, v) = \hat{v}_i \varphi(v) - \tau_i(v)$$

For the truthfully reported profile of types $v = (v_i, v_{-i})$ the equilibrium utility $U_i(v)$ and the equilibrium budget $B(v)$ are:

$$U_i(v) = U_i(v_i, v), \quad B(v) = \sum_i \tau_i(v) - \varphi(v)c$$

Definition 17. *A mechanism μ is **robust** (with budget balance) if it satisfies:*

1. *ex post IC: $\hat{v}_i \in arg\max_{v_i} U_i(\hat{v}_i, v)$,*
2. *ex post IR: $U_i(v_i, v) \geqslant 0$,*
3. *ex post BB: $B(v) = 0$,*

for all v and $\hat{v}_i$ in the support.

A standard exercise in mechanism design is to characterize an ex-post IC mechanism as

a non-decreasing allocation function $\varphi(v)$ such that utilities and transfers satisfy:

$$U_i(v_i'', v_{-i}) - U_i(v_i', v_{-i}) = \int_{v_i'}^{v_i''} \varphi(z, v_{-i}) dz \tag{2.1}$$

$$\tau_i(v_i'', v_{-i}) - \tau_i(v_i', v_{-i}) = \int_{v_i'}^{v_i''} z d\varphi(z, v_{-i}) \tag{2.2}$$

for all v_i', v_i'', v_{-i} in the support.

Formulas (2.1) and (2.2) represent the same enveloping conditions in the quasi-linear setting, see [HR87] or [MS83a] for the discussion. A mechanism is additionally ex-post *IR*, if and only if for every agent i:

$$U_i(a_i, v_{-i}) \geqslant 0 \tag{2.3}$$

because the equilibrium utility is non-decreasing in own type. Finally, the mechanism is ex post *BB* if the total surplus is equal to the sum of equilibrium utilities:

$$B(v) = (\sum v_i - c)\varphi(v) - \sum U_i(v_i, v) = 0 \tag{2.4}$$

Conditions (2.1)-(2.4) will be the starting point of our analysis in the next section.

2.3 Preliminary results

Our first observation about the family of robust mechanisms is that they form a convex cone (with the origin at the autarky) in the space of mechanisms (φ, τ). That is because a convex combination of any collection of ex-post *IC*, *IR* and *BB* mechanisms also satisfies these properties, which follows from formulas (2.1)-(2.4).

For analytical convenience, we will often consider *randomized* mechanisms, that are distributions over a certain family of mechanisms. The hierarchy of robust mechanisms in our paper is therefore:

$$\textit{binary} \subset \textit{randomized binary} \subset \textit{divisible} = \textit{randomized divisible}$$

The fact that robust divisible mechanisms form a convex set, however, is of little use unless we know the minimal set of mechanisms that span the whole family through convexification, which will be the goal of Proposition 3.

To find this minimal set we need several additional assumptions that are described in Section 2.3.1, and a few technical tools and lemmas listed in Section 2.3.2 and Section 2.3.3.

2.3.1 Additional assumptions

Here we discuss in detail the four assumptions used in the paper.

Assumption 6. *A mechanism is* ***non-wasteful*** *if* $sup_v \varphi(v) = 1$.

This assumption was used in [CP16] and is merely a normalization of the mechanism. Clearly, any robust mechanism such that $sup_V \varphi(v) \in (0, 1)$ can be appropriately scaled to be non-wasteful, which would also be an improvement from the social point of view.

Assumption 7. *The allocation function* $\varphi(v)$ *is* ***upper-semicontinuous*** *on* V *if:*

$$\limsup_{v \to v_0} \varphi(v) = \varphi(v_0)$$

This assumption was used in [DMR17] in the context of binary mechanisms and is yet another normalization. For non-decreasing functions it is equivalent to *right-continuity* in each coordinate v_i separately, or in the vector of coordinates v as a whole. The role of this assumptions is also discussed in Section 3.3 in the definition of a simple posted price.

Assumption 8. *For every* i *there exists a* ***pivotal type*** $\xi(v_{-i})$ *such that* $U_i(\xi, v_{-i}) = 0$.

This assumption requires that for any realization of types v_{-i} there exist a type of agent i that is just indifferent to participate, and, crucially, this pivotal type should not depend on the actual type of agent i. Though seemingly artificial, this is a fairly natural property of the mechanism. In the context of binary mechanism, it was derived from a variety of different assumptions in [KS17]. For divisible mechanisms it can be derived from one of the following assumptions on the primitives of the model:

3.1 *Largest type space*: $V = \mathbb{R}^n$,

3.2 *Veto power*: $a_i + \sum_{j \neq i} b_j < c$, for all i,

3.3 *Cost sharing*: $a_i = 0$ and $\tau_i(v) \geqslant 0$, for all i.

If the public good can be disliked by an agent, then it is natural to pick the largest possible type space $V = \mathbb{R}^n$. We can think of Assumption 3.1 as part of the robustness concept, that requires the mechanism to work for distributions with unknown supports. It then follows that the pivotal type of agent i is $\xi(v_{-i}) = c - \sum_{j \neq i} v_j$.

A slightly weaker version of Assumption 3.1 is Assumption 3.2, that does not require negative valuations. It posits that, for any agent, there exists a type that will make the public good inefficient no matter what the types of other agents are. That means that every agent has a power to veto the public good by announcing just below his pivotal type $\xi(v_{-i}) = c - \sum_{j \neq i} v_j$, which is closely related to the *instance of full veto power* assumption in [KS17].

Finally, if the setting implies that the valuations are nonnegative, that is $a_i = 0$, the restriction to non-positive transfers as in Assumption 3.3 guarantees that a situation where somebody who does not care about the public good earns free money by just stepping into the mechanism would not occur. The pivotal type in this situation is $\xi_i(v_{-i}) = \max\{c - \sum_{j \neq i} v_j, a_i\}$.

Assumption 9. *$a_i = 0$, $b_i = b < c$, for all i.*

This assumption tells that the support is an equilateral cube with bounds such that nobody can have marginal value above marginal costs, and the lowest marginal value exactly zero. We use this assumption is far from necessary for our results, but we invoke it for the welfare loss bounds in Proposition 4 to have a concise analytical form.

2.3.2 Technical tools

In order to formulate and derive the main results of the paper we need to introduce some technical tools, such as *n-th mixed difference operator* and a *distribution function*, adapted

from Chapter 3 in the [Shi16] textbook. These are necessary for the construction of a random variable out of its conjectured *CDF*, which will be at the core of Proposition 3. In the context of binary mechanisms such as studied in [KS17], these tools are redundant since no randomization takes place.

Consider an n-dimensional (not necessarily equilateral) cube $C \subset V$ with left-open and right-closed boundaries:

$$C = \{v \in V \mid v_i \in (\tilde{a}_i, \tilde{b}_i],\ \forall i\}$$

and let $\mathcal{C}$ be the set of all such cubes C in V. The following definition generalizes the concept of a mixed n-th derivative to functions that do not have to be differentiable.

Definition 18. *For a cube $C \in \mathcal{C}$ the n-th mixed difference operator $D_C^{1,\dots,n}$ is defined as:*

$$D_C^{1,\dots,n}\varphi = D^1_{(\tilde{a}_1,\tilde{b}_1]}(\dots(D^n_{(\tilde{a}_n,\tilde{b}_n]}\varphi(v)))$$
$$D^k_{(\tilde{a}_k,\tilde{b}_k]}\varphi(v) = \varphi(\tilde{b}_k, v_{-k}) - \varphi(\tilde{a}_k, v_{-k})$$

Note that each separate coordinate-wise difference is an operator acting on the space of functions on V, but the n-th mixed difference maps an allocation function into a constant and thus is a functional. If $\varphi(v)$ is a *CDF* of a random variable, then $D_C^{1,\dots,n}\varphi$ measures the probability of that random variable landing inside the cube C.

One useful property of $D_C^{1,\dots,n}$ is that for two adjacent (sharing the same face) cubes C_1 and C_2 the sum of the two corresponding functionals is equal to the functional that of the merged cube $C_1 \cup C_2$:

$$D^{1,\dots,n}_{C_1 \cup C_2}\varphi = D^{1,\dots,n}_{C_1}\varphi + D^{1,\dots,n}_{C_2}\varphi. \tag{2.5}$$

In the case of random variables this means that the probability of landing in either of the two cubes is the same as the sum probabilities of landing in each cube separately.

The following definition captures the idea of the cumulative distribution function of a random variable, adapted to a special case when the domain is a bounded subset of the Euclidean space such as V.

Definition 19. *A function φ on V is a distribution function if and only if:*

1. $\inf_{v\in V}\varphi(v)=0$, $\sup_{v\in V}\varphi(v)=1$,
2. *it is right-continuous in the profile of types v,*
3. *the n-th mixed difference $D_C^{1,\dots,n}$ is non-negative for any $C\in\mathcal{C}$.*

Note that for the special case of a non-decreasing allocation function $\varphi(v)$, the second property is equivalent to upper-semicontinuity. The third property is equivalent to non-wastefulness, because there is always a region in V where production is inefficient and therefore $\varphi(v)=0$.

2.3.3 Technical lemmas

Let E be the part of the type space, where production of public good is efficient or break-even, and its complement $\bar{E}$ - where production is strictly inefficient:

$$E=\{v\in V\mid \sum_i v_i\geqslant c\},\quad \bar{E}=\{v\in V\mid \sum_i v_i<c\}$$

The following lemma takes advantage of Assumption 8 and obtains a concise analytical transcription of the budget balance property, as an integral equation with the allocation function as an unknown.

Lemma 2. *Under Assumption 3, if a mechanism is ex post IR, IC, BB then:*

$$(\sum_i^n v_i-c)\varphi(v)-\sum_{i=1}^n\int_{\xi_i(v_{-i})}^{v_i}\varphi(z,v_{-i})dz=0, \tag{2.6}$$

where $\xi_i(v_{-i})$ is the pivotal type, and $v\in E$.

Proof. Consider the budget surplus as in formula (2.4). By formula (2.1), agent i's ex post utility can be represented as an integral over the allocation function $\varphi(\cdot,v_{-i})$ with the bounds of integration $\xi_i(v_{-i})$ and v_i plus the utility at the boundary, that must be equal to zero. □

The following lemma further transforms the budget balance condition into a property of the mixed difference operator. This property will be the bedrock for the construction of the random variable in the main section.

Lemma 3. *Under Assumption 2, 3 if a mechanism is ex post IR, IC, BB then the n-th mixed difference $D_C^{1,\dots,n}\varphi$ is nonnegative for any cube $C \in \mathcal{C}$, and zero if additionally the cube fully belongs to either E or $\bar{E}$.*

Proof. Note first that $D_C^{1,\dots,n}\varphi = 0$ for any cube $C \subset \bar{E}$, because production in that region is inefficient and therefore $\varphi(v) = 0$.

Let do the same for $C \subset E$. If the allocation function is smooth in E, differentiate both sides of equation (2.6) with respect to every variable:

$$\sum_i \left(\frac{\partial^{n-1}}{\partial v_1 \dots \partial \hat{v}_i \dots \partial v_n} \frac{\partial}{\partial v_i}(v_i \varphi(v_i, v_{-i})) - c \frac{\partial^n}{\partial v_1 \dots \partial v_n}\varphi(v) \right) - \\ - \sum_i \frac{\partial^{n-1}}{\partial v_1 \dots \partial \hat{v}_i \dots \partial v_n} \left(\frac{\partial}{\partial v_i} \int_{c-\sum_{j\neq i} v_j}^{v_i} \varphi(\xi, v_{-i}) d\xi \right) = 0$$

after cancelling out the terms sides and dropping $(\sum_i v_i - c)$ we get:

$$\frac{\partial^n}{\partial v_1 \dots \partial v_n}\varphi(v) = \ 0 \tag{2.7}$$

that should hold for all $v \in E$. Integrating over the area of the cube $C \subset E$ we then obtain the necessary equation. See Appendix for the general proof with upper-semicontinuous functions.

Finally, let's show that $D_C^{1,\dots,n}\varphi \geqslant 0$ for any cube $C \subset V$. Using formula (2.5) and the previous result we can replace a cube C that crosses the hyperplane $\sum v_i = c$ with another cube C' (see Figure 2.1 for a graphical example) such that:

$$D_C^{1,\dots,n} = D_{C'}^{1,\dots,n}, \quad C' = \{v \in V \mid v_i \in (\tilde{a}_i, \eta_i], \ \forall i\}, \quad \eta_i = \min\{c - \sum_{j\neq i} \tilde{a}_j, \tilde{b}_i\}.$$

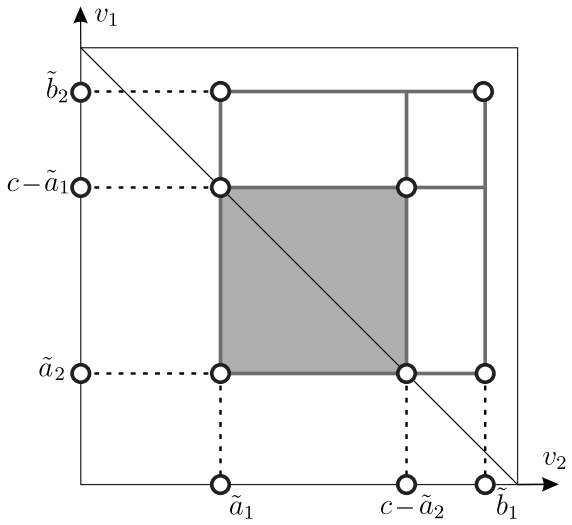

Figure 2.1: Cubes $C = [\tilde{a}_1, \tilde{b}_1) \times [\tilde{a}_2, \tilde{b}_2)$ and $C' = [\tilde{a}_1, c - \tilde{a}_2) \times [\tilde{a}_2, c - \tilde{a}_1)$ for the case $n = 2, V = [0, 1]^2$ and $c = 1$.

For the new cube we can establish the necessary property:

$$D_{C'}^{1,\dots,n} = \varphi(\eta) - \sum_i \varphi(c - \sum_{j \neq i} \eta_j, \eta_{-i}) \cdot \mathbb{I}(\tilde{b}_i \geqslant c - \sum_{j \neq i} \tilde{a}_i) \geqslant B(v) = 0$$

which follows from formula (2.6) and monotonicity of $\varphi(v)$. □

2.4 Main results

To state the main results of the paper, we need to formally define a special robust mechanism that we refer to as a *simple posted price* throughout the paper. Fix a vector of prices $\rho = \{\rho_i\}_{i=1}^n$ such that $\sum \rho_i \geqslant c$.

Definition 20. *A mechanism $\mu^{sp}(\rho) = (\varphi^{sp}(\rho), \tau^{sp}(\rho))$ is a **simple posted price** if:*

$$\varphi^{sp}(\rho, v) = \prod_i \mathbb{I}(v_i \geqslant \rho_i), \quad \tau^{sp}(\rho, v) = \rho \cdot \varphi^{sp}(\rho, v)$$

For any given ρ, the allocation function $\varphi^{sp}(\rho, v)$ is simply an indicator function with a rectangular support. If the support were strictly above the hyperplane $\sum_i v_i = c$ as in Figure 2.2 (left), then the mechanism would generate budget surplus, and if it only touches

the hyperplane as in Figure 2.2 (right), then it is exactly budget balanced.

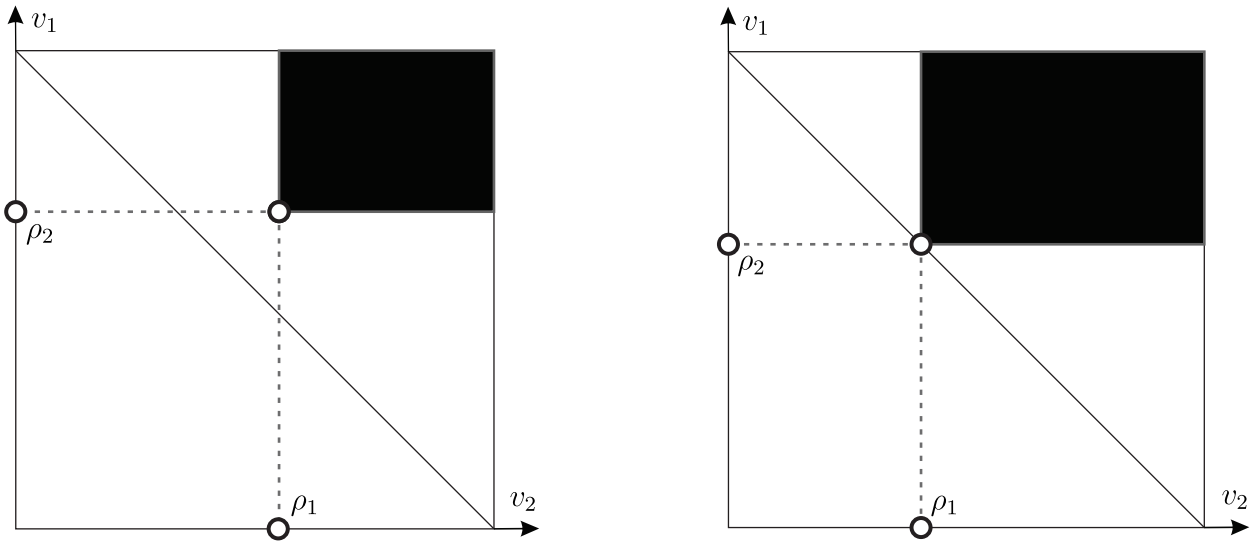

Figure 2.2: Allocation function of a *simple posted price* mechanism that runs budget surplus (left figure) and the one that is exactly budget balanced (right figure).

Our definition of the *simple posted price* is in line with the definition of a *posted price* in [CP16], but we use the word *simple* to stress that there are many more robust mechanisms that can be referred to as *posted prices*. In the context of binary outcomes, this broader class of mechanisms is referred to as *threshold mechanisms* in [KS17].

2.4.1 Characterization

Proposition 3. *There is a one-to-one correspondence between the family of robust divisible mechanisms satisfying Assumptions 1,2,3 and distributions over simple posted prices. The allocation function of such mechanism coincides with the respective cumulative distribution function.*

Proof. It's fairly easy to see that the allocation function of a single posted price μ_p is the *CDF* of a corresponding lottery with a single outcome ρ. Same property holds for any lottery or distribution over simple posted prices:

$$Prob(\rho : \rho_i \leqslant v_i \text{ for all } i) = \varphi(v).$$

To show the converse we will adapt a theorem in [Shi16] that states that there is a one-to-one correspondence between measures on V and the distribution functions as defined in Section 2.3.2. Because under Assumption 8, the transfers are uniquely defined by the allocation function $\varphi(v)$ of a robust mechanism, we only have to show that $\varphi(v)$ is actually a distribution function, or precisely, it satisfies the three characteristic properties in its definition. The first two properties follow immediately from Assumption 6 and Assumption 7, while the last property follows from Lemma 3. This establishes that $\varphi(v)$ is the *CDF* of some distribution over vectors $\rho \in V$ that are not yet restricted to the $\sum \rho_i = c$ hyperplane.

The last piece of the puzzle is the additional property in Lemma 3, that states that for any cube C that lies completely above or completely below that hyperplane, the $D_C^{1,\dots,n}$ operator is equal to zero. This means that the probability of a posted price p falling outside of the $\sum \rho_i = c$ hyperplane is zero, in other words they are all budget balanced. □

Consider a functional $\mathcal{F}$ that maps an allocation function of the mechanism into a scalar. $\mathcal{F}$ is affine if $\mathcal{F} = k + \mathcal{L}$, where k is a constant and $\mathcal{L}$ is linear:

$$\mathcal{L}(\varphi_1 + \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2), \quad \mathcal{L}(\lambda\varphi_1) = \lambda\mathcal{L}(\varphi).$$

The following statement follows immediately from the fact that the family of robust mechanisms is a convex hull of simple posted prices.

Corollary 6. *Any affine functional $\mathcal{F}$ attains minimum over the family of robust divisible mechanisms satisfying Assumptions 1,2,3 at a simple posted price, for any belief F.*

2.4.2 Optimality

Denote the welfare at the first best mechanism and the welfare at mechanism μ as:

$$W_{fb}(v) = (\sum v_i - c) \cdot \mathbb{I}[\sum v_i \geqslant c], \quad W_\mu(v) = (\sum v_i - c) \cdot \varphi_\mu(v),$$

Then denote the welfare loss function:

$$WL_\mu(v) = W_{fb}(v) - W_\mu(v)$$

The welfare loss works as an non-linear operator on the space of functions over V. It maps an allocation function $\varphi_\mu(v)$ of mechanism μ into a function $WL_\mu(v)$.

We can now define our main welfare criterion.

Definition 21. *The **expected welfare loss** $EWL_{F,\mu}$ is a function of a mechanism μ:*

$$EWL_{F,\mu} = \int WL_\mu(v)dF(v)$$

where F is the belief about the distribution of types.

Technically speaking, expected welfare loss is a non-linear functional acting on the space of allocation functions $\varphi_\mu(v)$. We will later show that the first one satisfies a certain strong property that we define below:

Definition 22. *A mechanism μ is **robustly optimal** for some prior beliefs F, if it minimizes expected welfare loss over a family of robust divisible mechanisms.*

Proposition 4. *For any prior belief F, a robustly optimal (among those satisfying Assumptions 1,2,3) mechanism is either a simple posted price, or a lottery over simple posted prices that yield the same expected welfare loss.*

Proof. By Corollary 6, it is sufficient to verify that, for any prior belief F, expected welfare loss of a robust mechanism μ is an affine functional acting on the space of robust allocation functions. This follows directly from formulas (2.2):

$$EWL_{F,\mu} = \int W_{fb}(v)dF(v) - \int W_\mu dF(v) =$$
$$= k(F) - \int \sum_i U_i(v)dF(v) = k(F) - \sum_i \int \int_{a_i}^{v_i} \varphi(z, v_{-i})dzdF(v)$$

where $k(F)$ is the first best expected utility. □

2.4.3 Welfare bounds

To derive the welfare bounds we introduce another supplementary welfare criterion.

Definition 23. *The **maximal welfare loss** MWL_μ is a function of a mechanism μ:*

$$MWL_\mu = \sup_v WL_\mu(v)$$

We will refer to the beliefs that maximize expected welfare loss at the corresponding optimal robust divisible mechanism as the *worst case beliefs*.

Proposition 5. *Expected welfare loss of a robustly optimal (among those satisfying Assumptions 1,2,3,4) mechanism at the worst case beliefs is bounded by:*

$$\frac{1}{4} \cdot (nb - c) \leqslant \sup_F \min_\mu EWL_{F,\mu} \leqslant \frac{n-1}{n} n^{\frac{-1}{n-1}} \cdot (nb - c)$$

Proof. To derive the lower bound, assume than n is even and split the agents in two equal groups: A, B. Fix a small $\varepsilon > 0$ and let F_ε be distributed on two tiny regions in V, as shown on Figure 2.3 for the special case of two agents. Precisely, let with probability 0.5

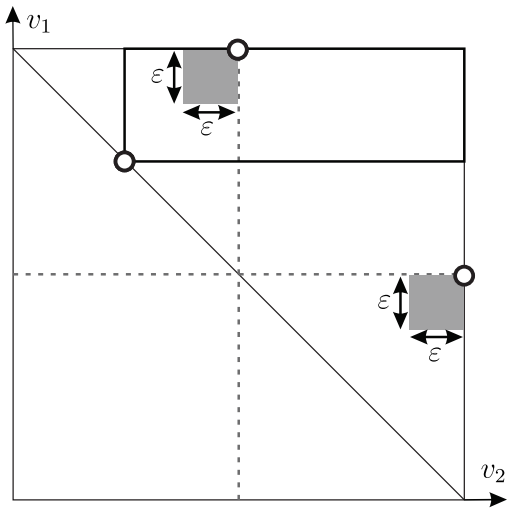

Figure 2.3: Distribution F_ε for $n = 2, V = [0,1]^2$ and $c = 1$.

the valuations for agents in group A be iid uniformly on the segment $[c/n, c/n - \varepsilon]$ and

the valuations for agents in group B be iid uniformly on the segment $[b, b-\varepsilon]$. Let with probability 0.5 the valuations be distributed similarly but with A and B switched places.

Since by Proposition 4, the optimal robust divisible mechanism is a posted price, it captures only one of the two regions and thus:

$$\lim_{\varepsilon\to 0}\min_{\mu} EWL_{F_\varepsilon,\mu} = 0.5\cdot(\frac{n}{2}(b+\frac{c}{n})-c) = \frac{1}{4}(nb-c)$$

If n is odd, let one agent always have the maximal valuation b and then repeat the procedure for the remaining $n-1$ agents with costs $c-b$ to get the exact same number. This completes the construction of the lower bound.

To derive the upper bound, notice that by the standard max-min property:

$$\sup_{F}\min_{\mu} EWL_{F,\mu} \leqslant \min_{\mu}\sup_{F} EWL_{F,\mu} = \min_{\mu} MWL_{\mu}.$$

Next, consider a uniform distribution of posted prices on a special set $S\subset\mathbb{R}^n$:

$$S = \{\textstyle\sum_i \rho_i = c,\ c-\sum_{j\neq i} v_j \leqslant \rho_i \leqslant b_i\}.$$

A corresponding divisible mechanism μ^* is robust, but it might not satisfy Assumption 8 because prices can be negative. The allocation function of this mechanism and the maximal welfare loss are easily calculated:

$$\varphi^*(v) = \frac{(\sum v_i - c)^{n-1}}{(nb-c)^{n-1}},\quad MWL_{\mu^*} = \frac{n-1}{n}n^{\frac{-1}{n-1}}\cdot(nb-c)$$

Notice that this mechanism can be modified to satisfy Assumption 8 without increasing the maximal welfare loss. To do this it is sufficient to replace every lottery outcome ρ' that falls outside of the domain V with a lottery ρ'' on the boundary of V such that $\rho''_i \geqslant \rho'_i$. Intuitively, when a lottery offers negative prices to some of the agents, we replace them with zero and use the net profit to provide discounts to other agents. This completes the construction of the upper bound. □

From the Theorem above, we can immediately derive the following property.

Corollary 7. *The lower bound in Proposition 5 is tight for two agents.*

$$n = 2: \quad \sup_F \min_\mu EWL_{F,\mu} = \frac{1}{4}(nb - c).$$

Corollary 8. *For certain beliefs, the expected welfare loss of a robust divisible mechanism is as big as* $1/2$ *of the expected welfare at the optimal Bayesian mechanism.*

Proof. Observe that under the sequence of beliefs F_ε defined in Proposition 5, the agents' types are perfectly correlated in the limit. This gives the designer ability to verify their reports in equilibrium and thus implement the first best allocation. Clearly, for ε arbitrarily close to zero, the welfare loss can be made arbitrarily small.

Since the total welfare under the beliefs F_ε is arbitrarily close to $1/2$ of the maximal welfare, the result follows from Proposition 5. □

2.5 Discussion

Divisible mechanisms have a potential to perform better than binary ones, but at the same time are significantly harder to study. Unless some sort of reduction is made, constrained optimization over non-decreasing allocation functions does not have a clear solution. One such reduction is a characterization of divisible mechanisms in terms of lotteries over binary.

In this paper we pointed out the assumptions under which such reduction is valid in our linear setting. Once we show that robust mechanisms can be represented as randomizations over simple posted prices, optimization becomes extremely simple. Clearly, linearity of utilities and costs is critical to the validity of our results. An interesting avenue of research would be to identify optimal mechanisms without these strong assumptions.

We do not explicitly solve robustly optimal mechanism for any possible prior distribution of beliefs, but we show that such mechanism has to take the shape of a simple posted price.

This gives foundation to a commonly used assumption of a binary allocation.

It is easy to see that under certain beliefs the expected welfare losses can be as small as zero. It therefore makes sense to try to identify the highest loss at a robustly optimal mechanism, over all possible beliefs. For the special case of two agents we have identified this loss to be exactly equal to $\frac{1}{4}(nb-c)$, that is, 25% of the maximal possible welfare, or 50% of the second-best welfare in this economy. For the multi-agent setting we show that the loss keeps growing at least proportionally to the size of the economy.

The question of whether there is a steady upper bound to the welfare losses is open, since we only provide a weak bound of $\frac{n-1}{n} n^{\frac{-1}{n-1}} \cdot (nb-c)$ that converges to $(nb-c)$ - the maximal possible welfare in the economy.

2.6 Appendix C

Proof of Lemma 2.

To fill in the gap in the proof of Lemma 3 we will show that equation (2.6) implies that $D_C^{1,\dots,n}\varphi = 0$ for any cube $C \in \mathcal{C}$ that fully belongs to E, as long as the allocation function $\varphi(v)$ is non-decreasing and right-continuous is every variable.

The first step is to transform the equation (2.6) into a differential equation in an appropriate sense (see [Eva10] textbook for 'weak differentiation' of generalized functions):

$$\int \dots \int \varphi(v) \left(\frac{\partial^n}{\partial v_1 \dots \partial v_n} \psi(v) \right) dv_1 \dots dv_n = 0 \tag{2.8}$$

which holds for any smooth function $\psi(v)$ with support in E.

Denote the following differentiation and integration operators:

$$\frac{\partial^n}{\partial v} = \frac{\partial^n}{\partial v_1 \dots \partial v_n}, \quad \frac{\partial^{n-1}}{\partial v_{-i}} = \frac{\partial^{n-1}}{\partial v_1 \dots \partial v_{i-1} \partial v_{i+1} \dots \partial v_n}$$
$$\int dv = \int \dots \int dv_1 \dots dv_n, \quad \int dv_{-i} = \int \dots \int dv_1 \dots dv_{i-1} dv_{i+1} \dots dv_n$$

Because $\psi(v)$ spans the same space of test functions as $(\sum_i v_i - c)\psi(v)$, let's write the desired equation as $J = 0$, where

$$J = \int \varphi(v)\frac{\partial^n}{\partial v}[(\sum_i v_i - c)\psi(v)]dv = \int \varphi(v)[(\sum_i v_i - c)\frac{\partial^n}{\partial v}\psi(v) + \sum_i \frac{\partial^{n-1}}{\partial v_{-i}}\psi(v)]dv.$$

Substituting $(\sum_i v_i - c)\varphi(v)$ from formula (2.6) and rearranging the order of integration we get the following:

$$J = \sum_i \int (A_i(v_{-i}) + B_i(v_{-i}))dv_{-i}$$

$$A_i(v_{-i}) = \int (\int_{c-\sum_{j\neq i} v_j}^{v_i} \varphi(z, v_{-i})dz)\frac{\partial^n}{\partial v}\psi(v)dv_i, \quad B_i(v_{-i}) = \int \varphi(v_i, v_{-i})\frac{\partial^{n-1}}{\partial v_{-i}}\psi(v)dv_i$$

then, applying integration by parts we get

$$A_i(v_{-i}) + B_i(v_{-i}) = [(\int_{c-\sum_{j\neq i} v_j}^{v_i} \varphi(z, v_{-i})dz)\frac{\partial^{n-1}}{\partial v_{-i}}\psi(v)]\Big|_{\underline{v}_i(v_{-i})}^{\overline{v}_i(v_{-i})}$$

where $\underline{v}_i(v_{-i}), \overline{v}_i(v_{-i})$ are boundaries of the support of $\psi(v)$ for a fixed level of v_{-i}. Finally, because $\psi(v)$ is a smooth test function, it has zero value at the boundary together with all of its derivatives, therefore $J = 0$ and the first step is complete.

The second step is to transform (2.8) into:

$$D_C^{1,\dots,n}\varphi = 0$$

that should hold for any cube $C \in \mathcal{C}$ such that $C \subset E$. To show that we will evaluate (2.8) at an indicator function

$$\hat{\psi}(v) = \mathbb{I}(v \in C), \quad C = \{v \in V \mid v_i \in (\tilde{a}_i, \tilde{b}_i],\ \forall i\}.$$

Note that since $\hat{\psi}(v)$ is not smooth, we can not do it directly.

Pick any smooth, symmetric kernel function $K(v)$ with support in $[0, 1]^n$ and construct

a sequence of smooth approximations to $\varphi(v)$:

$$\psi_n(v) = \int K_n(\eta)\hat{\psi}(v-\eta)d\eta, \quad K_n(\eta) = K(n\eta)$$

By construction this function is smooth with support in E (if n is big enough), and it's n-th mixed derivative is equal to zero everywhere except for the right-neighborhoods of the vertices of cube C. Because in these neighborhoods $\psi(v)$ is an indicator function, differentiating $\psi_n(v)$ simply recovers the smoothing kernel with alternating signs:

$$\frac{\partial^n}{\partial v}\psi_n(v) = \begin{cases} K_n(v-\tilde{b}), & \tilde{b}_j \leqslant v_j \leqslant \tilde{b}_j + 1/n \text{ for all } j \\ -K_n(v_i - \tilde{a}_i, v_{-i} - \tilde{b}_{-i}), & \tilde{a}_i \leqslant v_i \leqslant \tilde{a}_i + 1/n, \tilde{b}_j \leqslant v_j \leqslant \tilde{b}_j + 1/n \text{ for all } j \neq i \\ \dots & \\ (-1)^n K_n(v-\tilde{a}), & \tilde{a}_j \leqslant v_j \leqslant \tilde{a}_j + 1/n \text{ for all } j \\ 0, & \text{otherwise} \end{cases}$$

Finally, because the function $\varphi(v)$ is right-continuous in $(v_1, \dots, v_n)$ jointly, the integration over the smoothing kernel approximates the value at the vertices of the cube, with alternating signs. The whole integral is therefore an approximation to the sum of the values of $\varphi(v)$ at the vertices of the cube, with alternating signs:

$$\int \varphi(v)\frac{\partial^n}{\partial v}\psi_n(v)dv = D_C^{1,\dots,n}\varphi + o(1) \quad \Rightarrow \quad D_C^{1,\dots,n}\varphi = 0$$

CHAPTER 3

Robust Mechanisms of Exchange

3.1 Introduction

The design of efficient trading mechanisms for exchange economies is one of the central problems in economics. In a complete information environment, one way to achieve efficiency is to post Walrasian prices and let agents buy and sell freely. The problem is harder if agents have private information, especially when values are interdependent, as in the case of asset trade. If agents trade using a double auction, which mimics the protocol on centralized stock, bond, or commodity exchanges, the equilibria are sensitive to the information structure and typically inefficient, even with many traders (see [RW12]). Indeed, in this setting, no feasible mechanism that guarantees a near-efficient allocation in a prior-free way is known.

In a private information environment, the prior-free approach that we adopt guarantees robustness of the mechanisms in the following sense (see [Wil87], [BM05], [CE07]). Bayesian approach presumes designer's knowledge of a joint distribution of agents' payoff types, beliefs about payoff types, beliefs about beliefs etc., and builds a different mechanism for each. Prior-free approach is more demanding in that it asks for *one* mechanism that works for *all* such distributions. We require that it satisfies the incentive constraints and is near-efficient ex-post, for any realization of payoff types. This is motivated by practical considerations. The prior-free mechanism can be implemented by an algorithm, much like ones executing centralized limit order books on modern exchanges. It does not require an informed social planner.

The goal of this paper is to provide such mechanisms. It has the following three results. First, we construct a class of direct mechanisms for exchange economies with private

information and interdependent values, with no symmetry restrictions, that are ex-post individually rational, incentive compatible, and generate budget surplus for any economy size. Moreover, they result in an ex-post nearly efficient allocation, when there are many agents that are appropriately "small". Second, the level of inefficiency of the mechanisms is tight and cannot be improved by other mechanisms satisfying the constraints. Third, we show that the mechanisms can be implemented using a novel *discriminatory conditional double auction*. The auction can be run without any knowledge of either information structure or utility functions.

We motivate our results using the following stylized example (see [RW12] and Examples 2, 5 and 7). Consider an exchange economy with a single asset and two large groups of agents, say, "fundamental" and "liquidity" traders. Value of a fundamental trader depends on his signal and the average of other fundamental traders' signals.[1] How to design a trade mechanism that achieves a near-efficient allocation?

One may try a double auction: let each agent submit a demand schedule specifying how much he wants to buy or sell at any price, and have the designer choose the price that clears the market. Aside from the Bayesian formulation, this game will not lead to a near-efficient allocation, no matter how large the economy. The problem is that a single price cannot convey all the relevant public information needed to make the right choice ([RW12]). For a fundamental trader, a high price might be due to high demand by the liquidity traders, indicating a perfect selling opportunity, or due to high demand by other fundamental traders, indicating high own value. Without enough information, he will not be able to trade near-efficient quantity.

A different mechanism would elicit private information via appropriate scoring rules, much in the spirit of [CM85] and [CM88], and then use it to set the Walrasian price. The latter approach, however, works only if the designer knows the distribution of signals.[2] Fi-

[1]For the argument it is irrelevant whether the liquidity traders have private or interdependent values.

[2]Scoring rules may extract the agent's beliefs about others. However, exact knowledge of the joint distribution is needed to map such beliefs to the agent's payoff type, see [Nee04].

nally, one could try to use prior-free direct *Vickrey-Clark-Groves (VCG)* mechanisms, but in exchange environments they run a budget deficit, and so are infeasible. The mechanisms we define in this paper provide an alternative, as we explain below.

In this paper we look at exchange economies with N agents that have quasilinear utilities over a single divisible good and money. Each agent has a one-dimensional payoff type, and his utility depends on the whole profile of types (*interdependent values*). We make two important assumptions. First, the slopes of the individual demand curves are bounded. Second, the average effect of any single agent's type on marginal utilities of others converges to zero as the economy grows (*small pairwise interdependence*). In the fundamental/liquidity traders example, pairwise interdependence is of order $\frac{1}{N}$. Both assumptions formulate the notion of "small" agents in our environment, and guarantee a decentralized, competitive economy when N is large. As we show, they are sufficient and necessary for the near-efficiency in large economies.

The main result of the paper is a construction of $\sigma-$*Vickrey-Clarke-Groves mechanisms with quadratic taxes* (σVCG), which are direct mechanisms that satisfy all the constraints ex-post. Moreover, they are almost efficient ex-post, when there are many agents (see Section 3.3). For an economy of size N, the maximal distance to the efficient allocation, across all agents and type profiles, is proportional to the sum of $\frac{1}{N}$ and the bound on pairwise interdependence. The mechanisms also guarantee a vanishing deadweight loss, when pairwise interdependence is of order at most $\sqrt{\frac{1}{N}}$. More generally, σVCG provide a linear trade-off between inefficiency and budget deficit.

σVCG mechanisms are based on allocations that introduce a wedge between the market price and each agent's marginal utility, and the wedge is linear in the quantity traded, with slope σ. It can be thought of as a controlled version of demand reduction, familiar from the multi-unit auction literature ([Viv11], [ACP14]). In particular, no-one is excluded from trade with any other agents (as in [McA92], [CH98], [Tat05], [LM16], [KY16], [LM17]). Instead, relying on divisibility of the good, each agent's trade is scaled down. This makes it possible for the mechanisms to achieve vanishing inefficiency ex-post, and not just in expectation.

The second main result of the paper shows that the above positive result is tight: any mechanisms satisfying the constraints must result in efficiency losses proportional to those achieved by appropriate σVCG mechanisms, for any size of the economy. The idea is that in any incentive compatible mechanism agents must collect information rents for each inframarginal unit traded, with highest rents for the "most inframarginal" units. To prevent budget deficit, mechanism must collect enough extra funds to cover the rents. The linear wedge in our mechanism corresponds to a linear per-unit tax on each inframarginal unit traded. When the slope of the tax matches the rate of change of the information rent, funds collected cover rents ex-post with little surplus, and so cause minimal distortion. Moreover, for smaller slopes, the mechanisms trace out a "Pareto frontier" of optimal pairs of budget deficits and inefficiencies, up to a multiplicative constant.

In the paper we also construct alternative mechanisms which are based on different ways of collecting extra funds. Examples include a fixed bid-ask spread (constant per-unit wedge) or a fixed entry fee (no wedge for sufficiently large trade). We show that among such mechanisms linear wedges are strictly optimal. We also show that our results extend to the case when goods are discrete, when taxes include a quadratic part together with an appropriate bid-ask spread.

σVCG mechanisms are direct mechanisms, in which the designer knows how to pick the allocation and transfers, given reported types. The third main result of the paper is to provide an implementation using a $\sigma-$*Discriminatory Conditional Double Auction* ($\sigma DCDA$), which, in the spirit of [Wil87] or [DM00], requires no such knowledge. In the game, each player submits his inverse demand, which specifies marginal prices at which he is willing to clear the market, conditional on not just a total quantity, but on a vector of quantities that the other agents trade. The mechanism computes the market clearing quantity vector and price, as well as, for every trader, the residual demand curve that he is facing. Transfers are discriminatory: each trader pays the area under his residual demand curve, together with a tax quadratic in quantity. We show that there is an ex-post equilibrium of this game that implements the σVCG allocation. In particular, when taxes are zero, the auction implements the VCG direct mechanism – and so the efficient allocation, at a deficit.

$\sigma DCDA$ is characterized by three features, each of which relates to practical problems in market design. First, there is ample evidence that in practice market participants spend significant resources trying to identify a source of trade. Learning the source of trade from market data by high-frequency traders is a recent, well publicized example.[3] Conditioning on the whole allocated vector in a $\sigma DCDA$ is an instrument that mitigates precisely this information flow. It allows each trader to infer information about the trading behavior on the market directly as part of the trading protocol. Allowing retail investors to identify themselves as such on NYSE via Retail Liquidity Program can be interpreted as facilitating a particular kind of conditional bidding.

High-dimensionality of bids in a $\sigma DCDA$ stems from the generality of our approach, and is likely not a practical obstacle. At the extreme, when values are private, bids are conditioned only on the total quantity cleared just as in a standard double auction. We show that when utilities depend on a low-dimensional statistic of types, the equilibrium strategies are contingent on a low-dimensional statistic of the allocated vector. In the fundamental/liquidity traders example, fundametntal traders submit prices conditional on the total quantity to be cleared and the average trade of other fundamental traders, with price increasing in each variable.

Second, quadratic taxes used in a $\sigma DCDA$ are an alternative to the constant ("fixed fee") or linear ("bid-ask spread") taxes, or fees, used in practice. They are designed to make fees proportional to the price impact that the traders have, and guarantee minimal distortions while imposing no deficit ex-post. We note that one way to interpret the weaker constraint of no deficit in expectation is that it requires an informed and financially unconstrained market maker, who facilitates trade and breaks even in expectation.[4] Our mechanisms do not require such an institution and can be implemented by a computer algorithm (see Section 3.5).

[3]See e.g. the Concept Release on Equity Market Structure by the [Sec10] or, in more colorful language, [Lew14]. The rapidly growing literature on HFT is surveyed in [Jon13], [BF14] and the [Sec14].

[4]Of course, real-life market makers are not interested in efficiency but profit maximization. Near efficiency requires an additional assumption of competition between such idealized market makers.

Finally, we show that appropriate discriminatory pricing obviates any form of strategizing and endogenous demand reduction, with information rents expressed directly in terms of lower inframarginal prices. This resembles the way second price auction eliminates incentives for shading in the first price auction, in the classical auction theory. Indeed, in equilibrium agents use "revelation strategies", submitting marginal utility adjusted by the linear wedge. This makes computation of equilibrium strategies relatively easy, even in cases when linear strategies in uniform price double auction are non-analytical (see Section 3.4). While harder to quantify than efficiency gain, we consider this strategic simplicity a crucial virtue of our mechanisms.

Relationship to the Literature. Our paper contributes to the literature on strategic foundations of Walrasian Equilibrium in large exchange economies. One strand of it focuses on the question of when double auctions approximate efficiency in Bayesian environment. Important contributions include [Wil85], [PS86a], [PS86b], [KM89], [GS89], [SW89], [RSW94], [FMS07], [CS06] as well as [Yoo01], [Tat05] for double auctions with entry fees, in the case of private values; and [Hel80], [MV93], [RP06] and [Viv11] for interdependent values. In our paper, we require stronger ex-post constraints. Moreover, double auctions guarantee asymptotic efficiency only with symmetric agents (and so not in the fundamental/liquidity traders example; see [RW12]) and under additional assumptions on the distribution of signals, such as conditional independence. Finally, we stress that our mechanisms can be used for any economy size.

Aside from double auctions, [RP76], [Jac92] and [JM97] show that Walrasian mechanisms are difficult to manipulate in large replica economies. [GP92] and [MP02] identify, in the Bayesian framework, crucial information smallness requirements and [MP15] show that the benefits of ex-post deviations in the efficient mechanism are small. We rely on related smallness assumptions but build mechanisms that satisfy exact ex-post constraints, and provide an auction implementation.

In the standard auction context, under common knowledge of benefits of trade, VCG mechanisms generate budget surplus. In the private-value case the mechanism is imple-

mented by the second price auction. [Aus99], [DM00], [EM02], [PR02] and [PR05], [Aus04] and [Izm04] extend and show how to implement VCG mechanisms with interdependent values and multiple goods. With taxes set to zero, our discriminatory conditional double auction provides a new implementation of the efficient (unbalanced) VCG in the exchange setting with interdependencies. The conditional bids in our implementation are related to the conditional bids in [DM00] and [EM02]. In our case inverse demands condition on vectors of allocated quantities, while there agents condition on vectors of utility functions. Contingent demands are also used in a context of dynamic trade with private values by [SS16] and [DZ17].

Few studies have offered mechanisms for exchange environments that satisfy ex-post constraints and are almost efficient in expectation, when the economy is large. In a seminal paper, [McA92] provides a simple mechanism and an auction implementation that achieve this with independent private values and unit demands and supplies, while [HSS02], [CS08] and [LM16] deal with multiunit trade.[5,6]. [KY16] allow for interdependent values when agents are symmetric and types are conditionally independently distributed. Our framework has interdependent values with no symmetry restrictions. Crucially, σVCG mechanisms are almost efficient not just in expectation but ex-post.

The remainder of the paper is organized as follows. Section 3.2 describes the setup and establishes the lower bound on the inefficiency/budget deficit of mechanisms (Proposition 1). Section 3.3 defines the σVCG direct mechanisms and shows that they achieve inefficiency/budget deficit proportional to the lower bound (Proposition 2). Section 3.4 illustrates σVCG mechanisms in different economies and compares them to double auctions. $\sigma DCDA$ auctions are defined in Section 3.5 and Proposition 3 shows that they implement σVCG mechanisms. Other mechanisms are considered in Section 3.6, and Section 3.7 provides a discussion. The proofs and the extension to discrete setting is in the Appendix.

[5]The mechanism in [McA92] also guarantees bounded ex-post inefficiency, as the economy grows; see [Chu09] for a similar result in a multi-unit setting.

[6]See [BV03], [Seg03] for robust monopoly pricing, among others.

3.2 Setup

Payoffs. An exchange economy consists of N agents who can trade a single good, or asset, for money. Each agent i has a payoff type $s_i \in \mathbb{R}$ and has a quasilinear utility function

$$U_i(q_i, t_i, s) = u_i(q_i, s) - t_i,$$

where $q_i \in \mathbb{R}$ is the quantity of the good that agent i gets, t_i is the transfer that he makes, and $s = (s_1, ..., s_N) \in S$ is a profile of payoff types.[7] We emphasize that utilities may depend on the whole vector of types (*interdependent values*) and that we are not making any symmetry assumptions.

The set of all payoff type profiles $S \in \mathbb{R}^N$ need not be a product set, but we assume for simplicity that for every s_{-i} the projection $\{s_i | (s_i, s_{-i}) \in S\}$ is an interval.[8] Functions u_i are strictly concave in q_i and are twice continuously differentiable. We will write $q = (q_1, ..., q_N)$, $t = (t_1, ..., t_N)$ for the profiles of quantities and transfers, respectively, and $mu_i(q_i, s) = \frac{\partial}{\partial q_i} u_i(q_i, s)$, $mu_{i,j}(q_i, s) = \frac{\partial^2}{\partial q_i \partial s_j} u_i(q_i, s)$ and $mu_{i,q}(q_i, s) = \frac{\partial^2}{(\partial q_i)^2} u_i(q_i, s)$ for marginal utilities and their derivatives.

The framework is a standard exchange economy setting with a single divisible good, and money. The efficient benchmark in this setting is, of course, the *Walrasian Equilibrium (WE) allocation profile* $\{q^0(s)\}_{s \in S}$. For any vector of types s, the allocation $q^0(s)$ is defined jointly with the *Walrasian price* $p^0(s)$ by

$$\begin{aligned} mu(q_i^0(s), s) &= p^0(s), \quad \forall i \\ \sum_i q_i^0(s) &= 0. \end{aligned} \tag{3.1}$$

There is a unique WE allocation for any type profile s (see Section 3.3), and, from strict concavity of the utility functions, any other allocation results in a positive deadweight loss.

[7] Normalizing the initial endowment to zero is purely for notational convenience.

[8] This assumption is not essential for the results, but simplifies the incentivizing transfers in the mechanisms that we consider - see (3.9).

For the main results we will need additional assumptions about the utility functions.

$A1)$ $\quad |q_i^0(s)| \leqslant \overline{q}. \quad \forall i, s$

$A2)$ $\quad mu_{i,q}(q_i, s) \in [-\overline{m}_q, -\underline{m}_q] < 0. \quad \forall i, q_i, s$

$A3)$ $\quad mu_{i,i}(q_i, s) \geqslant \underline{m}_o > 0, \quad \forall i, q_i, s$

$A4)$ $\quad \frac{1}{N-1} \sum_{i \neq j} mu_{i,j}(q_i, s) \leqslant \phi_N$, with $\lim_{N \to \infty} \phi_N = 0, \quad \forall j, q_{-j}, s$

All those assumptions are substantial. Formally, Proposition 1 establishes that each of the bounds is necessary for near-efficiency in large markets. Informally, we argue below that they capture the right notion of "smallness" of agents in our environemnt, and guarantee that the economy is decentralized, or competitive, and not dominated by a single or few traders. If an agent has a large impact on the economy and can control prices, efficiency is out of reach.

Assumption $A1$ bounds the size of the efficient trade. Without this assumption, in any candidate mechanism one "'big" trader could affect the market clearing price and allocations for very many other, "small" traders, by chosing to trade either a lot or nothing.[9]

Assumption $A2$ bounds the slope of the individual demand curves, in the case when information is public. Otherwise, if all but one agents were "small", with steep demand curves, their aggregate demand curve would have a moderate slope. In other words, the aggregate of the myriad "small" agents would behave like one, leaving significant power to determine price and quantity in the hands of the single trader they face.

We note that $A2$ is violated in a model with discrete allocations. Even though the allocations can be convexified using lotteries, marginal utility of each agent drops discontinuously at discrete units. Indeed, as before, no matter how many agents in the economy, the aggregate demand curve may be vertical, for prices between the common grid of marginal utilities. In Appendix G we show, however, that both Proposition 1 and Proposition 2 extend to the case of discrete allocations, when Assumption $A2$ is replaced by one that bounds the change

[9]The assumption is needed only bound efficiency losses: without it, mechanisms we construct satisfy all other constraints, budget surplus in particular (see Proposition 2).

in marginal utilities for consecutive units. When units traded are small, the mechanisms we propose achieve robust asymptotic efficiency; when units are large, no mechanism can achieve that.

Finally, assumptions $A3$ and $A4$ jointly mean that as the economy grows, the average impact of each single agent on marginal utilities of others becomes relatively insignificant.[10] Otherwise, no matter the economy size, a fixed small group of traders could monopolize the relevant information about the common-value component of the asset. Each of those "informationally large" traders has a large impact on the price and the incentive problem is similar as if only they were present. Near-efficiency is out of reach (see [MS83b]). We will refer to $A4$ as the assumption of *small pairwise interdependence.*

We note that $A3$ and $A4$ do not restrict the models we consider to small perturbations of the private value model. Agents' utilities may be virtually determined by public shocks (see Examples 1-3). The only restriction is that the information about those public signals is dispersed.

Working with economies of fixed size N, we will be using the following counterpart of $A4$:

$$A4^*) \qquad \tfrac{1}{N-1}\textstyle\sum_{i\neq j} mu_{i,j}(q_i,s) \leqslant \phi_N, \text{ with } \phi_N < \tfrac{1}{8}\tfrac{\underline{m}_q}{\overline{m}_q}\underline{m}_o, \ \ \forall j, q_{-j}, s$$

In particular, we allow for the single crossing assumption to be violated, and so $j's$ signal to be more relevant to some other trader, $mu_{i,j}(q_i,s) > m_{j,j}(q_j,s)$. However, $A4^*$ is related to strict single crossing, and will play the same role of guaranteing strict monotonicity of allocations in our mechanisms. It is weaker, in that it bounds only the average effect of $j's$ signal on others, but the bound, which is the impact on own utilities $\underline{m}_o$, is scaled down by $\frac{\underline{m}_q}{\overline{m}_q} < 1$. The bound is scaled down even further, because our mechanisms are efficient not for the original, but for the perturbed utilities (see proof of Proposition 2 for details).

[10] Assumption $A3$ normalizes the signals. Without it, assumption $A4$ has no bite and can be assumed in any family of models, given a straightforward change of variables.

Example 1. *(Fundamental Value Model, [Viv11], [RW12]) Utilities are*

$$u_i(q_i, s) = (\alpha s_i + \beta \overline{s})q_i - \frac{\mu}{2}q_i^2,$$

for some constants $\alpha, \mu > 0$ and $\beta \geqslant 0$, where $\overline{s}$ is the arithmetic average of the types. Each agent is uncertain about the value of the first infinitesimal unit of the good or the intercept of the linear marginal utility function. The value is a weighted average of common and idiosyncratic shocks, $\overline{s}$ and s_i. It may reflect both the expected cash flow of the asset, common to all the agents that are partially informed about it, and private hedging needs. We have $mu_{i,j}(q_i, s) = \frac{\beta}{N}$ for any $j \neq i$, and so the small pairwise interdependence assumption A4 is satisfied; for fixed N, A4 holds as long as $\alpha > \frac{7\beta}{N}$. The efficient Walrasian Equilibrium allocation is*

$$q_i^0(s) = \frac{\alpha}{\mu}(s_i - \overline{s}). \tag{3.2}$$

Example 2. *(Group Model, [RW12]) Suppose agents are divided into two groups of the same size, and the utilities are*

$$u_i(q_i, t_i, s) = (\alpha_G s_i + \beta_G \overline{s}^G)q_i - \frac{\mu}{2}q_i^2, \quad \forall i \in G, G = 1, 2$$

where $\overline{s}^1$ and $\overline{s}^2$ are the average types in each group, $\alpha_1, \alpha_2, \mu > 0$ and $\beta_1, \beta_2 \geqslant 0$. Compared to Example 1, beside the idiosyncratic shocks now there are two shocks, $\overline{s}^1, \overline{s}^2$, common to the respective group and irrelevant to the other. Agents are trading either for fundamental or liquidity reasons, and values are correlated within a group. The efficient Walrasian Equilibrium allocation is

$$q_i^0(s) = \frac{\alpha s_i + \beta \overline{s}^G - (\alpha + \beta)\overline{s}}{\mu}. \quad \forall i \in G, G = 1, 2 \tag{3.3}$$

Example 3. *(Fundamental Value Model with Heterogenous Traders) There are two groups of agents, "small" and "big", and utilities are*

$$u_i(q_i, s) = (\alpha s_i + \beta \overline{s})q_i - \frac{\mu_G}{2}q_i^2, \quad \forall i \in G, G = S, B$$

where $\mu_S = \overline{\mu}, \mu_B = \underline{\mu}, \overline{\mu} > \underline{\mu}$. *As in the Fundamental Value Model, all the agents are partially and symmetrically informed about the fundamental value of the risky asset, but they differ now in their capacity to hold large position, with big agents relatively risk neutral (or bearing large cost of staying away from their ideal position). The efficient Walrasian Equilibrium allocation is*

$$q_i^0(s) = \frac{\alpha s_i + \beta \overline{s} - p^\sigma(s)}{\mu_G}, \quad \forall i \in G, G = S, B \tag{3.4}$$

A natural source of interdependence in preferences is informational, when agent's type corresponds to a signal informative of all agents' values of the asset. We have used this interpretation in the examples above, and will continue to do it throughout the paper. This informational interpretation can be further justified via Bayesian models. Indeed, suppose each agent i has a private state space Θ_i and a continuously differentiable utility function $v_i(q_i, \theta_i)$ as well as a prior $\delta_i \in \Delta(\Theta_i \times S_1 \times ... \times S_N)$. If one defines

$$u_i(q_i, s) = \mathbb{E}_{\delta_i}[v_i(q_i, \theta_i)|s],$$

then the correlations of signal vectors s and value θ_i translate to utilities u_i with interdependent values.

In particular, utility functions in the examples above are expected utilities in the model in which each agent i has linear-quadratic utilities,

$$v_i(q_i, \theta_i) = \theta_i q_i - \frac{\mu}{2} q_i^2, \tag{3.5}$$

and believes that $(\theta_i, s_1, ..., s_n)$ has a joint normal distribution, where the linear parameters $w_j, j \leqslant N$, are such that

$$\mathbb{E}_{\delta_i}[\theta_i|s_1, ..., s_N] = \sum\nolimits_j w_j s_j, \tag{3.6}$$

from the Projection Theorem. Put otherwise, a fixed linear-quadratic utility function with parameters $w_j, j \leqslant N$ may be justified, for example, by any normal prior beliefs for which

(3.6) holds. We stress here that *many* beliefs may give rise to the *same* utilities u_i; we discuss this Bayesian justification for the interdependent preferences in the context of robustness in Section 3.7 (see also Example 6).

Our notion of small pairwise interdependence is closely related to the informational smallness used in the context of such Bayesian models (see [GP92], [MP02] and [MP15]). If Θ_i and $\frac{\partial mv_j}{\partial \theta_i}$ are bounded for all i, j, then u_i satisfy $A4$ precisely when[11,12]

$$A4^B) \qquad \tfrac{1}{N-1}\textstyle\sum_{i\neq j}\left|\frac{\partial \delta_i(\theta_i|s)}{\partial s_j}\right| \leqslant \phi_N^B, \text{ with } \lim_{N\to\infty}\phi_N^B = 0 \quad \forall j, \theta_{-j}, s$$

In other words, given a Bayesian justification, bound on pairwise interdependence $A4$ is equivalent to any single agent j having a small average impact on beliefs of others about the payoff relevant state.

Mechanisms. Given a profile of utility functions $(u_1, ..., u_N)$, a *(direct) mechanism* in our setting is $\{(q(s), t(s))\}_{s\in S}$, where $\{q(s)\}_{s\in S}$ is the *allocation profile* and $\{t(s)\}_{s\in S}$ the *transfers profile*, $q(s), t(s) \in \mathbb{R}^N$. Below we list the constraints and the objective that we want mechanisms to satisfy. A direct mechanism satisfies

- Market Clearing, if

$$MC) \qquad \textstyle\sum_i q_i(s) = 0, \quad \forall s$$

- $\delta-$Budget Surplus, for $\delta \geqslant 0$, if

$$\delta - BS) \qquad \textstyle\sum_i t_i(s) \geqslant -\delta, \quad \forall s$$

- Individual Rationality, if

$$IR) \qquad U_i(q_i(s), t_i(s), s) \geqslant U_i(0, 0, s), \quad \forall i, s$$

[11] We identify a distribution δ_i with its *cdf*.

[12] The main difference is that [MP02] require the bound in $A4^B$ to hold not for every s_{-j} but only for s_{-j} with probability $1 - \phi_N^B$. The strengthening in our paper is dictated by the stronger, ex-post version of incentive compatibility.

- Incentive Compatibility, if

$$IC) \qquad U_i(q_i(s), t_i(s), s) \geqslant U_i(q_i(s'_i, s_{-i}), t_i(s'_i, s_{-i}), s), \ \ \forall i, s, s'_i$$

- ε-Efficiency, if

$$\varepsilon - Eff) \qquad |q_i(s) - q_i^0(s)| \leqslant \varepsilon. \ \ \forall i, s$$

Finally, consider a family of mechanisms $\{(q(s), t(s))\}_{s \in S}$ for any number of players N and any utility profiles $(u_1, ..., u_N)$. For a function $f : \mathbb{N} \to \mathbb{R}_+$, we say that the family is *robustly asymptotically f-Efficient* if each mechanism satisfies the constraints MC, BS, IR, IC and $f_N - Eff$.

Market clearing is a standard feasibility constraint. Budget surplus is similar, as we want mechanisms to run without an outside source of money; positive δ allows for some slack. Individual rationality requires participation in a mechanism to be voluntary, and incentive compatibility means that truthful reporting of own type is optimal.

Crucially, all the constraints and the objective are required to be satisfied ex-post, for any vector of types. While this is natural in the case of market clearing, those are strong requirements in the case of IR, IC, BS and Efficiency, relative to their Bayesian counterparts. All are motivated by practical considerations. Specifically, the benefits of the ex-post constraints are as follows.

First, ex-post IR and IC imply Bayesian IR and IC for any type space describing agents' beliefs – about types, about beliefs of others, etc. (see e.g. [BM05]). Thus, ex-post IR and IC guarantee that the mechanism is *robust with respect to the misspecification of agents' beliefs*. For practical purposes, this means that the designer does not need to know what agents believe about others, when designing a mechanism.

Second, ex-post IR and IC also guarantee that agents have ex-post *no regret* about their allocation. This means that a mechanism will work in a weak contractual environment, when each agent can walk away from the transaction (or change their action) at any stage, even after observing the full allocation vector and possibly inferring types of other agents. This

shares many of the features of a standard spot market. Relatedly, ex-post constraints nullify any benefits of (inefficient) spying on other agents to infer their types ([BV02]).

Third, ex-post BS and (near) Efficiency guarantee that BS and (near) Efficiency will be satisfied for any distribution of types. Thus a mechanism is *robust with respect to the misspecification of type distribution,* and so the mechanism designer need not know it. Intuitively, Bayesian formulation requires an idealized – informed and financially unconstrained – market maker that facilitates trade. If market maker knows the distribution of types, is risk neutral and faces no liquidity constraints, he could broker the trade, while breaking even and achieving near efficiency in expectation.[13] Given the volume of trade and the number of assets traded on centralized exchanges, however, the informational and liquidity requirements of such risk-neutral brokering are staggering. In contrast, an ex-post BS and (near) Efficient mechanisms can be implemented by a computer algorithm (see Section 3.5). What we have in mind is a counterpart of an algorithm that executes centralized limit order book on modern exchanges.

We note that, similarly as in most of the literature (see e.g., [McA92], [KY16]), our notion of efficiency does not penalize for budget surpluses. One interpretation is that extra money need not be flushed down the drain, but can be used elsewhere. On the other hand, as long as marginal utilities are bounded, spelling out $\varepsilon-$Efficiency directly in terms of differences in allocations and not utilities is clearly without loss of generality.[14] Moreover, ε-Efficient mechanisms generate deadweight losses of order $\varepsilon^2 N$.[15]

Theorem 4. *Suppose that* $|mu_i(0,s)| \leqslant m$, *for all* i, s. *If a mechanism* $\{(q(s),t(s))\}_{s\in S}$ *is*

[13]Of course, real-life market makers are not interested in efficiency but profit maximization. Near efficiency requires an additional assumption of competition between such idealized market makers.

[14]It trivially follows from Proposition 1 below that without a bound on marginal utilities no mechanism that satisfies all other constraints can avoid infinite inefficiencies, spelled out in terms of utilities.

[15]The second part of the Lemma follows from $A2$ and

$$\left|\sum_{i=1}^N u_i(q_i(s),s) - \sum_{i=1}^N u_i(q_i^0(s),s)\right| = \left|\int_{q_i^0(s)}^{q_i(s)} \left(mu_i(x,s) - p^0(s)\right) dx\right|.$$

ε-Efficient and satisfies MC, then for every s

$$\begin{aligned}\left|u_i(q_i(s),s)-u_i(q_i^0(s),s,)\right| &\leqslant \varepsilon m,\\ \left|\sum_{i=1}^N u_i(q_i(s),s)-\sum_{i=1}^N u_i(q_i^0(s),s)\right| &\leqslant \varepsilon^2\frac{N\overline{m}_q}{2}.\end{aligned}$$

The following proposition sets the stage for the main results in the following sections.

Proposition 1. *There are constants $C_1, C_2 > 0$ such that for every $\varepsilon < \overline{q}/2$ and economy of size $N > 2$ that satisfies $A1, A2, A3$ and $A4^*$, the IR, IC, MC and $\varepsilon-$Efficient mechanisms violate $\delta - BS$, where*

$$\delta = N\left[C_1(\frac{1}{N}+\phi_N)-C_2\varepsilon\right]. \tag{3.7}$$

The result implies that, for a fixed economy size, no mechanism can guarantee robust implementation of the efficient Walrasian Equilibrium allocation. Versions of such impossibility in an exchange environment are well known, and go back to the seminal contribution by [MS83b]. However, the proposition also provides lower bounds for the extent to which the constraints may be violated, which will be the benchmark for the positive result in Proposition 2. The exact values of C_1 and C_2, which depend on the bounds in assumptions $A1-A4^*$, are in Appendix A. We postpone interpreting the terms in the bound on the budget deficit until we establish in Proposition 2 that this bound is tight.

On one extreme, full efficiency ($\varepsilon = 0$) implies budget deficit at least of order $C_1(1+N\phi_N)$. The deficit is uniformly bounded for every economy size both in the private value environment ($\phi_N = 0$), as well as when the interdependence is of order $\frac{1}{N}$. This need not mean that the subsidies required for efficiency are small; in particular, C_1 is proportional to $\overline{q}^2$.

On the other extreme, one might "decentralize" the problem of patching up the deficit and have the traders themselves provide the necessary funds. The result shows that requiring no budget deficit ($\delta = 0$) will compromise efficiency by at least a term of order $\frac{1}{N}+\phi_N$. The term vanishes as the economy grows. Moreover, the proposition traces out a linear curve for the intermediate cases, when both efficiency and budget surplus are violated.

3.3 Vickrey-Clarke-Groves Mechanisms with $\sigma-$Quadratic Taxes

Proposition 1 says that no mechanism can achieve efficiency loss ε and budget deficit δ below the linear curve (3.7). In this section, in the main result of the paper, we construct a class of mechanisms that achieve this linear bound (up to a multiplicative constant), for every economy. This is simplest in the case of full efficiency ($\varepsilon = 0$), when conditions (3.1) pin down both the allocation and the incentivizing tansfers. Other cases ($\varepsilon > 0$) require a judicious choice of the allocation function, in every economy, so that the accompanying incentivizing transfers achieve deficit proportional to (3.7).

Our results rely on the following family of mechanisms. We define the allocations and transfers separately.

Definition 1. ***(VCG with σ-quadratic taxes: allocation)*** *Fix a profile of utility functions $(u_1, ..., u_N), \sigma \geqslant 0$ and a vector of types s. The allocation $q^\sigma(s)$ and price $p^\sigma(s)$ in a* Vickrey-Clarke-Groves with $\sigma-$quadratic taxes (σVCG) mechanism *are defined jointly via*

$$\begin{aligned} mu(q_i^\sigma(s), s) &= p^\sigma(s) + \sigma \times q_i^\sigma(s), \quad \forall i \\ \sum_i q_i^\sigma(s) &= 0. \end{aligned} \tag{3.8}$$

One may think of the σVCG allocation and price as the standard efficient Walrasian Equilibrium allocation and Walrasian price, but for distorted utilities

$$\widetilde{u}_i(q_i, s) = u_i(q_i, s) - \sigma \frac{q_i^2}{2}.$$

The second term is a "tax", which is quadratic in the quantity traded. Of course, $0VCG$ price is the Walrasian price, and a $0VCG$ allocation is the efficient WE allocation. Strictly positive σ, however, results in insufficient trade and compromises efficiency: a marginal utility of a buyer ($q_i^\sigma(s) > 0$) is strictly greater than a marginal utility of a seller ($q_i^\sigma(s) < 0$). This is related to demand reduction in the multi-unit auctions (see e.g. [ACP14]; see Example 4).

Given continuous differentiability and strict concavity of the utility functions, for any $\sigma \geqslant 0$, p and $i \leqslant N$ there is at most one $q_i(p,s)$ that satisfies

$$mu_i(q_i(p,s),s) = p + \sigma \times q_i(p,s),$$

and the functions $q_i(\cdot,s)$ are continuous and strictly decreasing in their domains. This means that there is at most one p for which $\sum_{i=1}^{N} q_i(p,s) = 0$, which establishes uniqueness of $(p^\sigma(s), q^\sigma(s))$, for any $\sigma \geqslant 0$ and s. Existence follows from $A2$.

Definition 2. ***(VCG with σ-quadratic taxes: transfers)*** *Fix a profile of utility functions $(u_1,...,u_N), \sigma \geqslant 0$ and a vector of types s. The transfers $t^\sigma(s)$ in a* Vickrey-Clarke-Groves with $\sigma-$quadratic taxes (σVCG) mechanism *are defined as*

$$t_i^\sigma(s) = \int_0^{q_i^\sigma(s)} p^\sigma\left(s_i(x), s_{-i}\right) dx + \frac{\sigma}{2} q_i^\sigma(s)^2, \tag{3.9}$$

where $\{q^\sigma(s)\}_{s\in S}$ and $\{p^\sigma(s)\}_{s\in S}$ are the σVCG allocations and prices, and for any agent i and quantity x between 0 and $q_i^\sigma(s)$ $s_i(x)$ is the signal such that[16]

$$x = q_i^\sigma\left(s_i(x), s_{-i}\right), \tag{3.10}$$

$$s_i(x) = \begin{cases} \inf S_i \text{ if } q_i^\sigma(s_i', s_{-i}) > x \text{ for all } s_i', \\ \sup S_i \text{ if } q_i^\sigma(s_i', s_{-i}) < x \text{ for all } s_i'. \end{cases}$$

Transfers by any agent i consist of a discriminatory part and a tax. Tax is quadratic in own quantity traded, or, alternatively, is a linear per-unit tax. It is a positive contribution by everyone. Discriminatory part is the integral over per-unit prices for each inframarginal unit traded, where price for unit x is the σVCG price under a counterfactual report $s_i(x)$ that results in i trading exactly x. Thus, it is simply the integral under the residual demand curve that i is facing.

In the case of private values, at any type profile, the discriminatory transfers by agent

[16] The second and the third clause in the definition are relevant only in the case when S_i is finite. It follows form strict monotonicity of $q_i^\sigma(s)$ in s_i, established in the proof of Proposition 2, that $s_i(x)$ is a function.

i are the integral under the aggregate linear tax adjusted marginal utilities of agents other than i (see Figure 1). With interdependent values the transfers are more discriminatory: say, a lower report by a buyer, which results in a lower quantity he trades, also decreases marginal utilities of other agents, and so further depresses the prices for inframarginal units (see Figure 2).

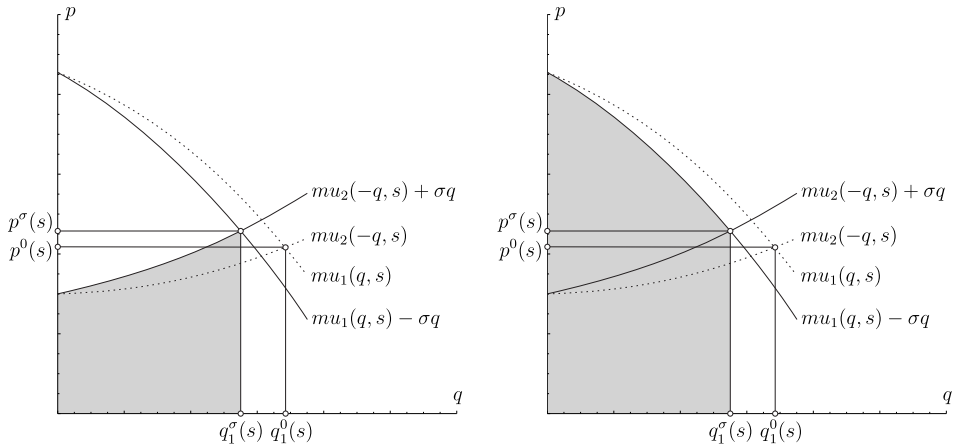

Figure 3.1: σVCG allocation and discriminatory transfers in a two person economy with private values, for $\sigma > 0$. Dotted lines are the marginal utility curves, and solid lines adjust for the linear per unit tax. Shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).

The following is the main result of the paper.[17]

Proposition 2. *Suppose that assumptions $A1-A4$ hold. For appropriate $D>0$ and slopes σ_N, the family of $\sigma_N VCG$ mechanisms is robustly asymptotically $D(\frac{1}{N}+\phi_N)-$Efficient.*

More precisely, there are constants D_1, D_2 and D_3 such that for every slope $\sigma < \underline{m}_q/4$ and economy of size $N>2$ that satisfies $A1, A2, A3, A4^*$, the σVCG mechanism satisfies $MC, IR, IC, \delta-BS$ and $\varepsilon-$Efficiency, where

$$\delta = N\left[D_1(\frac{1}{N}+\phi_N) - D_2\sigma\right]_+, \tag{3.11}$$

$$\varepsilon = D_3\sigma.$$

[17]See Appendix B for the explicit values of the constants. The "$[\cdot]_+$" refers to the positive part of a number, $[x]_+ = \max\{0,x\}$.

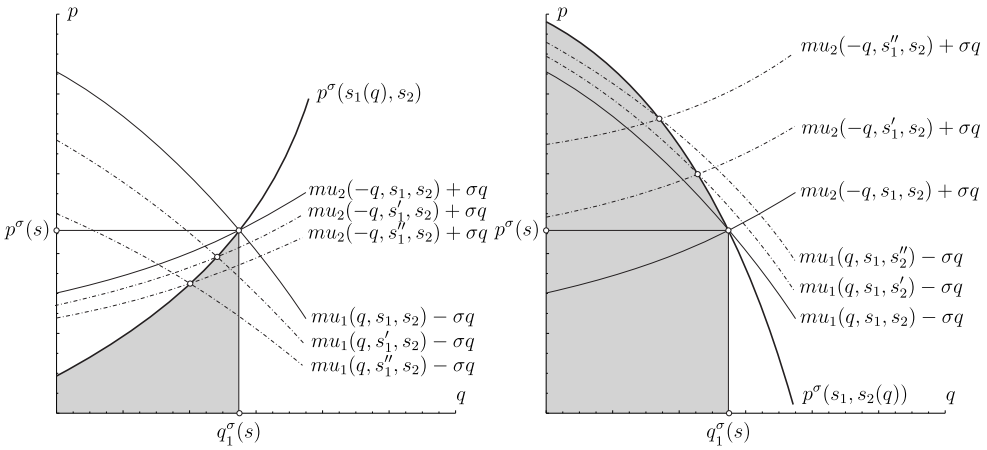

Figure 3.2: σVCG allocation and discriminatory transfers in a two person economy with interdependent values, for $\sigma > 0$. Thin lines are the linear tax adjusted marginal utility curves for the realized type profile (solid), and lower types for the buyer on the left panel and higher types for the seller on the right panel (dotted). Thick solid lines are the residual demand curves and shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).

Moreover, when $\sigma \geqslant \sigma_N$ the σVCG mechanism satisfies MC, IR, IC and BS in every economy of size $N > 2$ that satisfies $A2, A3$ and $A4^*$ only.

The proposition says that for any number of agents N and any vector of utility functions, the appropriate $\sigma_N VCG$ mechanism, with slope σ_N of order $\frac{1}{N} + \phi_N$, satisfies all the ex-post constraints. Moreover, it provides an explicit uniform (or "worst case") bound, across all agents and states, on how much the implemented allocation differs from the optimal one, for any economy size N. It follows from Proposition 1 that the bound is tight, and any mechanism that satisfies the constraints will generate a proportional efficiency loss, for any N. The result is more general: for smaller slopes, $\sigma < \sigma_N$, σVCG mechanisms achieve the optimal tradeoff between budget deficit and inefficiency, as established in Proposition 1. In particular, the efficient VCG mechanism runs a budget deficit of order $D_1(1 + N\phi_N)$.

In case of large economies, as long as the assumption $A4$ of small pairwise interdependence is met, the $\sigma_N VCG$ allocations approximate the efficient ones, and inefficiency vanishes at a rate $\frac{1}{N} + \phi_N$. In particular, as long as the interdependence parameter ϕ_N is of order smaller than $\sqrt{1/N}$, as in Examples 1-3, Lemma 4 implies that $\sigma_N VCG$ mechanisms guarantee

vanishing deadweight loss, uniformly across all states:

$$\left|\sum_{i=1}^{N} u_i(q_i^{\sigma_N}(s), s) - \sum_{i=1}^{N} u_i(q_i^0(s), s)\right| \leqslant D_3^2 \sigma_N^2 \frac{N\overline{m}_q}{2} = O(\frac{1}{N} + \phi_N + N\phi_N^2).$$

Let us sketch the intuition behind the results. As regards the incentive compatibility of the mechanisms, it relies on the logic that dates back to [Vic61]. In our case, fix a type vector s and a buyer i, $q_i^\sigma(s) > 0$. In order to achieve incentive compatibility, the price that i pays for every $x'th$ inframarginal unit of the good, $x < q_i^\sigma(s)$, must equal his value for this unit had he reported the type that makes him pivotal. Agent i is pivotal for $x'th$ unit of the good precisely when he reports the type $s_i(x)$ such that he gets exactly x, $q_i^\sigma(s_i(x), s_{-i}) = x$, and so his value for it equals $mu_i(x, (s_i(x), s_{-i}))$. Given the definition of the σVCG allocation, this marginal utility equals $p^\sigma(s_i(x), s_{-i}) + \sigma x$. Integrating over such per-unit payments gives rise to the σVCG transfers. Monotonicity of the allocation follows from the argument below.

The accounting of efficiency losses due to strictly positive slope σ is as follows. We show in the proof that the difference between the Walrasian price and the σVCG price is proportional to σ. By definition, σVCG distorts marginal utilities from σVCG price proportionally to σq_i, and so the overall wedge between the implemented and efficient marginal utilities is of order σ, given $A1$. Assumption $A2$ then implies that the wedge between the quantities is also proportional to σ.

Establishing budget surplus of the $\sigma_N VCG$ mechanism is the centerpiece of the result. As the first step, let us consider the efficient VCG mechanism and argue that in a large economy, under assumptions $A1 - A4$, the per-player deficit is small ex-post, for any profile of signals.

Each buyer i pays the integral of prices, and the price is increasing in units purchased (for sellers: the price is decreasing in units sold). Since only the price for the marginal unit equals the Walrasian price p^0, the per-player deficit is proportional to the speed with which it decreases when buyer i decreases demand, $\frac{\partial p^0}{\partial q_i}$. Now, the effect of $i's$ type on the Walrasian

price, $\frac{\partial p^0}{\partial s_i}$, is the average of its effect on marginal utilities of the agents, $mu_{j,i}$, weighted by the slope of their individual demand curves, $mu_{j,q}$. Given $A2$, the weights can be ignored, and so the effect is of order $\frac{1}{N}mu_{i,i}+\frac{N-1}{N}\phi_N$, or simply $\frac{1}{N}mu_{i,i}+\phi_N$. Assumption $A4^*$ implies that the price changes slower than $i's$ marginal utility, which results in strict monotonicity, $\frac{\partial q_i}{\partial s_i} \gtrsim mu_{i,i} - \frac{\partial p^0}{\partial q_i} > 0$. In sum, the effect of $i's$ quantity traded on the Walrasian price, equal to $\frac{\partial p^0}{\partial s_i}/\frac{\partial q_i}{\partial s_i}$, is of order $\frac{1}{N}\underline{m}_o+\phi_N$, using $A3$. Small pairwise interdependence, $A4$, implies that this effect is small in a large economy.

As an aside, one may consider also a mechanism that naively implements the Walrasian price-posting allocation: for any profile of types, the allocation is the efficient one, and agents pay the uniform per-unit Walrasian price. Proof of Proposition 2 implies that truthful reporting is almost ex-post IC. This follows immediately from the fact that the price impact $\frac{\partial p^0}{\partial q_i}$ vanishes in a large economy. This, however, does not imply that one can use this Walrasian price-posting mechanism to design a mechanism that is fully IC, and, in particular, does not imply that there is a "nearby" nontruthful ex-post equilibrium in the game.[18]

Second, we argue that for the right choice of a slope σ_N, $\sigma_N VCG$ mechanism allows to elimininate the budget deficit, while introducing minimal distortion of the allocation ex-post, for any profile of signals. The idea is simple. In the efficient allocation, buyer i that ends up with $q_i(s)$ units "gets a discount" of order $(\frac{1}{N}+\phi_N)x$ for the $q_i(s)-x$'th unit he purchases, contributing to the deficit. Setting quadratic tax $(\frac{1}{N}+\phi_N)q_i^2(s)$ is equivalent to setting a per-unit tax of $(\frac{1}{N}+\phi_N)x$ on every inframarginal $q_i(s)-x$'th unit that he buys. In other words, each trader pays in taxes exactly what he gains via information rents, resulting in budget surplus with minimal distortions. This argument shows that budget deficit will be eliminated even when the quantities traded are unbounded, as stated in the proposition.

[18] In an ex-post equilibrium, for fixed $i's$ type s_i the fixed report $\widetilde{s}_i$ must remain ex-post optimal for any profile s_{-i}. But this generically cannot be the case, when crosspartials mu_{ij} are nonzero. For example, one may consider utilities $u_i(q_i,s) = \varphi_i(s)\,q_i - \frac{q_i^2}{2}$, for which the Walrasian price, $p^0(s) = \overline{\varphi(s)}$ and the allocation, $q_i^0(s) = \varphi_i(s) - \overline{\varphi(s)}$, can be computed explicitly. To act optimally, trader i's fixed report must shade his trade proportionally to his price impact, which in the Walrasian mechanism depends on the effect of his signal on the marginal utilities of others. But this is not possible if $i's$ price impact depends on the (reported) types of the opponents. In a sense, mechanism must use finer instruments, which allow to condition $i's$ allocation and transfer on the type vector of others in a precise way. This conditioning is a central feature of our mechanisms, made explicit in the implementation.

As we noted in Section 3.2, the results extend to environemnts with discrete allocations, as long as the units traded are small (see Appendix G). The problem with discrete allocations, when all traders' utilities may lie on a fixed grid with step Δ, is as follows. When a buyer reports a lower signal and decreases his demand, the price might have to drop by the grid step Δ before any of the other traders picks up his demand to clear the market, no matter how big the economy. In a sense, a buyer gets both a linear discount, as above, as well as a fixed discount of size Δ for each inframarginal unit he trades. Following the same logic as above, adding a constant per-unit tax of Δ eliminates budget deficit with minimal distortions (see Section 3.6 for such fixed per-unit taxes, or "bid-ask" spreads).

3.4 Examples

In the following, we derive σVCG mechanisms for the examples from Section 3.2. For each example, we also look at information structures with normal beliefs that can be used to justify the utilities (see Section 3.2) and compare the σVCG mechanisms to the Bayesian equilibria in standard double auctions.

Example 4. *Consider the Fundamental Value Model from Example 1. For any σ, the σVCG price and allocation are*

$$\begin{aligned} p^{\sigma}(s) &= (\alpha+\beta)\overline{s}, \\ q_i^{\sigma}(s) &= \frac{\alpha}{\mu+\sigma}(s_i-\overline{s}). \end{aligned} \tag{3.12}$$

Using Proposition 2, the slope that guarantees BS equals

$$\sigma_N = \frac{\mu(\alpha+\beta)}{(N-2)\alpha-\beta}, \tag{3.13}$$

which is positive, given A4 (see Example 1). For $\sigma=\sigma_N$ the mechanism boils down to*

$$q_i^{\sigma_N}(s) = \frac{(N-2)\alpha-\beta}{\mu(N-1)}(s_i-\overline{s}), \quad t_i^{\sigma_N}(s) = p^{\sigma_N}(s)\times q_i^{\sigma_N}(s), \tag{3.14}$$

and so satisfies budget balance. This follows from the fact that in this example $\frac{\partial p^{\sigma_N}(s_i(x),s_{-i})}{\partial x} = \sigma_N$*, as can be verified from (3.12).*

Consider now a Bayesian framework as in [Viv11] and [RW12]. Each agent i *has a linear-quadratic utility function* $v_i(\theta_i, q_i)$ *as in (3.5), and agents believe that their values* $(\theta_1, ..., \theta_N)$ *are jointly normally distributed with mean zero, variances* 1 *and covariances* $\rho \geqslant 0$*. Each agent* i *observes a signal* $s_i = \theta_i + \varepsilon_i$*, with noise* $\varepsilon_i \sim N(0, \zeta)$*,* $\zeta > 0$*, independent of all other variables. One may think of private value* $\theta_i = \theta + \theta_i^{id}$ *as consisting of a common shock* $\theta \sim N(0, \rho)$ *and an idiosyncratic shock* $\theta_i^{id} \sim N(0, 1-\rho)$*.*

Applying Proposition 2 from [RW12] to this information structure, we verify in Appendix C that

$$\mathbb{E}[v_i(\theta_i, q_i)|s] = \left(\frac{1-\rho}{1-\rho+\zeta} s_i + \frac{N\rho\zeta}{(1-\rho+\zeta)\,(1+(N-1)\rho)} \bar{s} \right) q_i - \frac{\mu}{2} q_i^2, \tag{3.15}$$

and the linear Bayesian equilibrium in double auction results in the identical allocation and transfers as the $\sigma_N VCG$ *mechanism given by (3.14).*

The main takeaway is that in the Fundamental Value Model the most efficient $\sigma_N VCG$ mechanism – with the lowest slope σ_N still consistent with BS – agrees with, and so can be implemented by a double auction. Since a demand schedule in a double auction allows an agent to condition his demand on own signal and the equilibrium price, given that the price is privately revealing (i.e., carries the same onformation as the type vector, [RW12]), agents are effectively choosing ex-post optimal price-quantity pairs. Moreover, in this example, the equilibrium demand reduction and the positive slope σ_N of $\sigma_N VCG$ mechanisms have the same effect on allocations and transfers. In particular, it follows that this Bayesian equilibrium remains an equilibrium for any information structure justifying payoffs as in Example 1.

Example 5. *Consider the Group Model from Example 2, for the simplest case of symmetric agents, with* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$ *(see [RW12]). For any* σ *the* σVCG *allocation and* σVCG

price satisfy

$$p^{\sigma}(s) = (\alpha+\beta)\overline{s},$$
$$q_i^{\sigma}(s) = \frac{\alpha s_i + \beta \overline{s}^G - (\alpha+\beta)\overline{s}}{\mu+\sigma}. \quad \forall i \in G, G = 1,2$$

The slope that guarantees BS equals (Proposition 2)

$$\sigma_N = \frac{\mu(\alpha+\beta)}{(N-2)\alpha},$$

and just as in Example 4 it is easy to verify that the corresponding transfers are $t_i^{\sigma_N}(s) = p^{\sigma_N}(s) \times q_i^{\sigma_N}(s)$, *and so satisfy budget balance.*

Suppose now agents have utility functions $v_i(\theta_i, q_i)$ *as in (3.5) and believe that their values* $(\theta_1, ..., \theta_N)$ *are jointly normally distributed with mean zero, variances* 1 *and covariances* $\rho \geqslant 0$ *for the agents only in the same group. Thus, private value* $\theta_i = \theta_G + \theta_i^{id}$ *consists of a shock* $\theta_G \sim N(0,\rho)$, $i \in G$, *common to own group, and an idiosyncratic shock* $\theta_i^{id} \sim N(0, 1-\rho)$. *As before, each agent* i *observes a signal* $s_i = \theta_i + \varepsilon_i$, $\varepsilon_i \sim N(0,\zeta)$. *Just as in Example 4, it follows that the expected utilities conditional on a profile of signals take the form of utilities in the Fundamental Value Model, for own subgroup: for any* $i \in G, G = 1,2,$

$$\mathbb{E}[v_i(\theta_i, q_i)|s] = \left(\frac{1-\rho}{1-\rho+\zeta} s_i + \frac{(N/2)\,\rho\zeta}{(1-\rho+\zeta)\,(1+(N/2-1)\rho)} \overline{s}^G \right) q_i - \frac{\mu}{2} q_i^2. \tag{3.16}$$

Applying Proposition 2 from [RW12] to this information structure, we verify in Appendix C that in the linear Bayesian equilibrium in double auction the equilibrium price and allocation are

$$p^{l,N}(s) = \frac{c_s^N}{1-c_p^N}\overline{s}, \quad q_i^{l,N}(s) = \frac{c_s^N}{\mu}\left(\frac{\frac{N-2}{N-1} - c_p^N}{1-c_p^N} \right)(s_i - \overline{s}), \tag{3.17}$$

for constants c_s^N, c_p^N *that depend on* ζ, ρ, N *such that*

$$\lim_{N\to\infty} c_s^N = \frac{2-\rho}{2-\rho+2\zeta}, \quad \lim_{N\to\infty} c_p^N = \frac{2\zeta}{2-\rho+2\zeta}.$$

Allocations and transfers in the two mechanisms now differ. Figure 3 shows ratios of expected utilities in each mechanism, for different correlations of values ρ within each group, noise variances ζ and economy sizes N. In particular, when economy grows expected utilities converge to efficiency only in the case of $\sigma_N VCG$:

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}[v_i(\theta_i,q_i^{\sigma_N}(s))|s] &= \lim_{N\to\infty}\mathbb{E}\left[q_i^{\sigma_N}(s)\left(\mathbb{E}[\theta_i|s]-\frac{\mu}{2}q_i^{\sigma_N}(s)\right)\right] \\ &= \mathbb{E}\left[q_i^0(s)\left(\mathbb{E}[\theta_i|s]-\frac{\mu}{2}q_i^0(s)\right)\right] = \frac{\mu}{2}\mathbb{E}\left[q_i^0(s)^2\right] = \frac{1}{\mu}\left(\frac{\rho}{4}+\frac{(1-\rho)^2}{2(1-\rho+\zeta)}\right), \\ \lim_{N\to\infty}\mathbb{E}\left[v_i(\theta_i,q_i^{l,N}(s))|s\right] &= \frac{\mu}{2}\mathbb{E}\left[q_i^{l,\infty}(s)^2\right] = \frac{1}{\mu}\left(\frac{(2-\rho)^2}{4(2-\rho+2\zeta)}\right) < \frac{1}{\mu}\left(\frac{\rho}{4}+\frac{(1-\rho)^2}{2(1-\rho+\zeta)}\right),\end{aligned} \tag{3.18}$$

for every $i \in G, G = 1, 2$, where replacing the order of limits under the expectation may be justified by the Dominated Convergence Theorem, and the fourth equality is justified by $\mathbb{E}[q_i^0(s)(\mathbb{E}[\theta_i|s] - \mu q_i^0(s))] = 0$.

When the economy is large, in a σVCG mechanism agents can trade across groups to exploit the difference between the realized group shocks, θ_A and θ_E, based on the arbitrarily precise estimates $\overline{s}^A$ and $\overline{s}^E$. This is captured by the first term in the limiting expected utility in (3.18). The second term represents the benefit of trading away the idiosyncratic part of agent i's value, θ_i^{id}, given the idiosyncratic part of his signal, $\theta_i^{id} + \varepsilon_i$. In the Bayesian equilibrium of the double auction price shifts each agent's estimate of his value in step. Consequently, trade of each agent i is based solely on the information about θ_i provided by his private signal s_i. In particular, large gains from intergroup trade are not realized. For example, large noise ζ of private signals may render trade in a double auction nearly useless without wiping out the gains from trade in a σVCG mechanism.

Example 6. *In the Fundamental Value Model with Heterogenous Traders from Example 3, for any σ the σVCG allocations and prices satisfy*

$$\begin{aligned} q_i^\sigma(s) &= \frac{\alpha s_i + \beta\overline{s} - p^\sigma(s)}{\mu_G + \sigma}, \quad \forall i \in G, G = S, B \\ p^\sigma(s) &= \alpha\left(\gamma_S\overline{s}^S + \gamma_B\overline{s}^B\right) + \beta\overline{s}, \end{aligned}$$

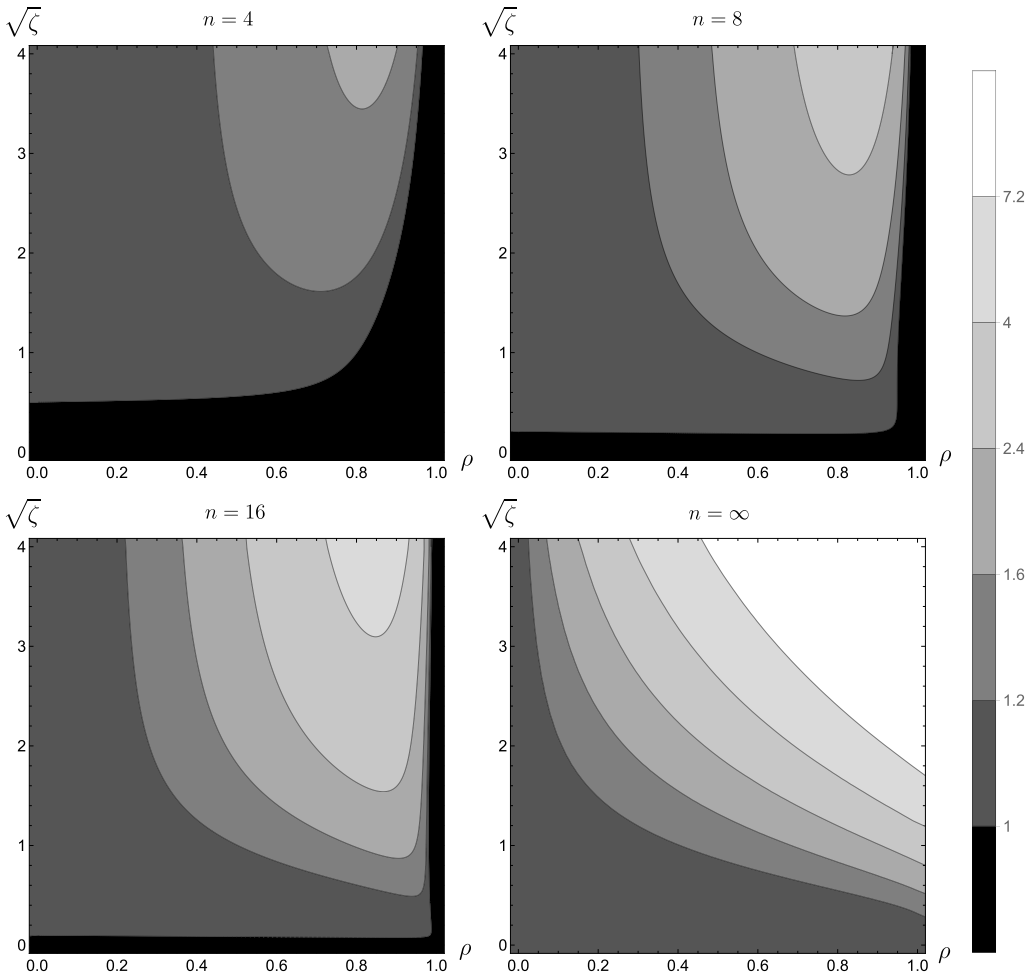

Figure 3.3: Ratios of $\sigma_N VCG$ mechanism to double auction expected utilities in a Group Model, for different correlations within group ρ, signal noise variances ζ and economy sizes N.

where $\overline{s}^S$ *and* $\overline{s}^B$ *are average signals in each group and* $\gamma_G = \frac{\sigma+\mu_{-G}}{2\sigma+\mu_B+\mu_S}$, $G = S, B$. *Compared to the Fundamental Value Model, now big agents trade more aggressively on their information and thus have a larger price impact. Given the allocations and prices, we compute the transfers as in the previous examples (see Appendix C).*

While the same Bayesian model as in Example 4 can be used to justify the utility function, there are many more. Specifically, suppose the agents have the utilities $v_i(\theta_i, q_i)$ *as in (3.5), and believe that, for* $i \in G, G = S, B$,

$$\theta_i = \theta + \theta_G + \theta_i^{id},$$

$$s_i = \theta_i + \varepsilon_G + \varepsilon_i,$$

where all random variables $\theta, \theta_G, \theta_i^{id}, \varepsilon_G, \varepsilon_i^{id}$, *for* $G = S, B$ *and* $i \leqslant N$, *are independently normally distributed with variances* $\sigma_\theta^2, \sigma_{\theta_G}^2, \sigma_{\theta_{id}}^2, \sigma_{\varepsilon_G}^2, \sigma_{\varepsilon_{id}}^2$. *In other words, there is a common schock* θ *to everyone's value, there are two separate group value shocks* θ_S, θ_B *and idiosyncratic value schocks* θ_i^{id}. *Likewise, noise consists of a group noise component* $\varepsilon_G, G = S, B$, *and idiosyncratic noise* ε_i. *Intuitively, agents in the same group care about similar aspects of the asset and also observe signals from similar sources, giving rise to positively correlated noises.*[19]

With no group value and noise shocks, we recover the model from Example 4 as a special case. However, for fixed parameters α *and* β *in utility function there is a continuum of Bayesian models from this class, with variances solving the system of linear equations (see Appendix C):*

$$\sigma_{\theta_{id}}^2 = \frac{\alpha\sigma_{\varepsilon_{id}}^2}{1-\alpha}, \quad \sigma_{\theta_G}^2 = \frac{\alpha\sigma_{\varepsilon_G}^2}{1-\alpha}, \quad \sigma_\theta^2 = \frac{\beta\left(2\sigma_{\varepsilon_{id}}^2 + N\sigma_{\varepsilon_G}^2\right)}{2N(1-\alpha)(1-\alpha-\beta)}. \tag{3.19}$$

For example, increasing common group noise $\sigma_{\varepsilon_G}^2$, *which decreases the attractiveness of signals from own group, is compensated by the increase in correlation* $\sigma_{\theta_G}^2$ *of values within a group.*

This continuum of models gives rise to a continuum of different linear Bayesian equilibria in double auctions. Moreover, for a fixed vector of parameters that solve (3.19), generically an analytical characterization of the equilibrium is unavailable (see Appendix C). However, we may argue indirectly as follows. The same linear equilibria result in the same linear market clearing price $p^l(s) = \gamma_S \overline{s}^S + \gamma_B \overline{s}^B$, *for appropriate* γ_S, γ_B, *and the same price impact* $\frac{\partial p^l}{\partial q_i} = d_i \geqslant 0$, *for any agent* i. *Agent* i*'s demand* $q_i^l(s_i, p)$, *given signal* s_i *and price* p^l, *satisfies*

$$q_i^l(s_i, p) = \frac{E\left[\theta_i | s_i, p^l\right] - p^l}{\mu_G + d_i}, \tag{3.20}$$

[19] Other Bayesian models are available. We assumed that agents that share a group value shock and have positively correlated noises also share the degree of risk aversion. The two splits could disagree, or there could be more groups with correlated values and signals. Likewise, variances of either group value shocks or idiosyncratic value components could differ (see Section 3.7).

for $i \in G, G = S, B$. As we show in the Appendix C, the expectation function $E\left[\theta_i | s_i, p^l\right]$ can agree for at most countably many models.

Despite the differences in risk aversion among the agents, in the model with uncorrelated noises ($\sigma^2_{\varepsilon_G} = \sigma^2_{\theta_G} = 0$) expected utilities converge to the efficient ones ([MV16], see also [Hel80]). This, however, is the only model featuring asymptotic efficiency. It follows from equation (3.20) that with vanishing price impacts, as long as $\mu_B \neq \mu_S$, the market clearing price $p^l(s) = \gamma_S \overline{s}^S + \gamma_B \overline{s}^B$ may not assign equal weights to the two group averages, $\gamma_S \neq \gamma_B$. As long as noises are positively correlated within a group ($\sigma^2_{\varepsilon_G}, \sigma^2_{\theta_G} > 0$), neither $\overline{s}^S$ nor $\overline{s}^B$, and thus also not the price converges in probability to the average of all signals $\overline{s}$. It follows that $E\left[\theta_i | s_i, p^l\right] \neq E\left[\theta_i | s_i, \overline{s}\right]$, and so, from (3.20), allocations do not converge to the efficient ones.

The example shows how a continuum of Bayesian models underlying a fixed utility profile may i) feature a continuum of Bayesian equilibria in double auctions that are sensitive to the details of the underlying information structure, and which ii) (generically) lack analytical characterization, and iii) (generically) do not yield asymptotic expected efficiency. All this contrasts with uniqueness, simplicity and asymptotic efficiency of the $\sigma_N VCG$ mechanism.

3.5 Implementation

σVCG mechanisms are direct mechanisms: agents report their types to the mechanism designer who enforces the reallocation of good and payments. While robustness strips him of the responsibility to know the information structure, he must know the utility functions. In this section we present a way to implement σVCG mechanisms in a way that does not require any such knowledge.

We will make the following two assumptions (see Section 3.7):

$$INF) \qquad S_i = \mathbb{R}, \quad \forall i$$

$$INJ) \qquad \inf_{q,s} \det \left[\; mu_{i,j}(q_i, s) \; \right]_{i,j \leqslant N} > 0.$$

For the linear-quadratic utility functions, the assumption INJ needs to be checked only at a fixed pair $q= \varnothing, s = \varnothing$, and is easily verified in each of the examples from Section 3.3.

Fix slope $\sigma \geqslant 0$ and consider the following auction format.

Definition 3. *($\sigma-$Discriminatory Conditional Double Auction, $\sigma DCDA$)*

Actions. *Each agent i chooses a continuous* conditional inverse demand correspondence

$$d_i : \mathbb{R}^{N-1} \twoheadrightarrow \mathbb{R},$$

where $d_i(q_{-i})$ is the set of marginal prices at which i is willing to clear the market and purchase $\sum_{j\neq i} q_j$, when each other agent $j \neq i$ is allocated q_j.

Allocation. *Mechanism finds the price-allocation pair (p, q) that clears the market:*

$$p \in d_i(q_{-i}), \quad \forall i \tag{3.21}$$

Transfers. *A continuous function $p_i : \mathbb{R} \to \mathbb{R}$ is a* residual demand curve for i, *$i \leqslant N$, if for every q_i there is q_{-i} such that the price-allocation pair $(p_i(q_i), q_i, q_{-i})$ clears the market. Transfers by i are defined as*

$$t_i = \int_0^{q_i} p_i(x)dx + \frac{\sigma}{2}q_i^2,$$

for the residual demand curve p_i.

If for the submitted conditional inverse demands there is no unique residual demand curve for every $i \leqslant N$, and so no unique market clearing (p, q), then the allocation and transfers are zero.

The first step in running a $\sigma DCDA$, which is finding the price-allocation vector, requires solving a system of N inclusions (3.21) in $N + 1$ unknowns (p, q), together with the market clearing condition. Aside from the added difficulty in allowing for set-valued demands discussed below, this generalizes the fixed point problem of finding the market clearing price in a double auction. It boils down to it when inverse demands condition only on the total quantity to be cleared.

Again, much like in a double auction, reports by all agents other than i determine the residual demand curve that i is facing. It is the menu of (marginal) price-quantity pairs (p, q) that i is choosing from, which is pinned down by market clearing and $N-1$ demands submitted by other agents. One term in the transfers is a tax quadratic in the quantity traded, paid by every agent. The other term is the discriminatory payment: an integral over all the inframarginal units under the residual demand curve that i is facing.[20]

We note that the discriminatory nature of payments refers to the trader paying different prices for each unit he trades. Different discriminatory payments are used in practice, such as, for example, when a trader pays the area under his own submitted demand curve. This is analogous to "paying own bid" in the first price auction. Traders paying the area under the residual curve they face, in our auction, is analogous to the payments in the second price auction.

Consider the following bids in the auction. For any $\sigma \geqslant 0$, a σVCG *conditional inverse demand* strategy in a $\sigma DCDA$ for agent $i \leqslant N$ is defined as

$$d_i(s_i)(q_{-i}) = \left\{ mu_i(-\sum\nolimits_{j \neq i} q_j, s_i, s_{-i}) + \sigma \sum\nolimits_{j \neq i} q_j \,\middle|\, q^{\sigma}_{-i}(s_i, s_{-i}) = q_{-i} \right\}.$$

In the case of $\sigma = 0$, the $0VCG$ conditional inverse demands are reports of own marginal utilities, evaluated at the efficient allocation. In the case of private values, the conditioning is vacuous, and the agents are simply reporting their marginal utilities $\{ mu_i(q_i, s_i) |\, q_i \in \mathbb{R}\}$.[21] This is in analogy with the bids in the second price auction, in the standard auction setting. When the values are interdependent, conditioning on the efficient allocation is a way to provide the agents with additional information about the types of other agents. Such information is necessary for the efficient trade. Finally, in the case of strictly positive slope

[20] Given little restrictions on bids, they need not determine unique allocation or transfers, in which case the economy reverts to autarky. This is a standard proviso from double auctions when multiple market clearing prices exist. Alternatives such as picking an allocation or residual demand curves according to some pre-specified order would do just as well. Of course, we verify in the proof of the following Proposition that the multiplicity does not arise in equilibrium.

[21] Strictly speaking, for q_i that do not arise in any efficient allocation consistent with s_i, the agent is reporting $\varnothing$. The difference does not play any role.

$\sigma > 0$, the agents report their modified, tax adjusted marginal utilities.

The following Example solves for the σVCG conditional inverse demands in case of linear utilities.

Example 7. *In case of linear quadratic utilities, as in the previous examples, σVCG allocations satisfy*

$$q_i^\sigma(s) = \frac{\sum_j w_{ij} s_j - p^\sigma(s)}{\mu_i + \sigma},$$

where $w_{ij} = mu_{i,j}(0,0)$. This implies

$$\begin{aligned} p^\sigma(s) \cdot \sum_j \widetilde{w}_{ij} &= \left(s_i - \sum_j \widetilde{w}_{ij}(\mu_j + \sigma) q_j^\sigma(s) \right) \\ &= \left(s_i - \sum_{j \neq i} (\widetilde{w}_{ij}(\mu_j + \sigma) - \widetilde{w}_{ii}(\mu_i + \sigma)) q_j^\sigma(s) \right), \end{aligned} \tag{3.22}$$

where $\left[\widetilde{w}_{ij}\right]_{i,j \leqslant N}$ is the inverse of the matrix $\left[w_{ij}\right]_{i,j \leqslant N}$.

In the Fundamental Value Model, matrix inversion yields σVCG conditional inverse demand functions

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) \left[s_i + \frac{\mu + \sigma}{\alpha} \sum_{j \neq i} q_j(s) \right],$$

which depend only on the total quantity an agent must clear.

In contrast, in the Group Model from Example 2, in the symmetric case when $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, σVCG conditional inverse demand functions are

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) s_i + \frac{\mu + \sigma}{\alpha} \left[\left(\alpha + \beta \frac{N-2}{N} \right) \sum_{j \neq i} q_j + \beta \overline{q}^{own} \right],$$

Marginal price at which an agent is willing to clear the market is increasing in own signal and in the quantity $\sum_{j \neq i} q_j$ that he must clear, due to decreasing marginal utility of absorbing more units. Unlike in a standard inverse demand function, however, each agent conditions his price also on the average trade by the agents in his group. For a fixed own type and total quantity to be cleared, higher demand by own group indicates higher own group shock and so value of the good to the agent, thus increasing the price.

Similarly, in the Fundamental Value Model with Heterogenous Traders, σVCG condi-

tional inverse demand by a big agent i is (and similarly for small agents):

$$d_i(s_i)(q_{-i}) = (\alpha+\beta)\left[s_i + \frac{\mu+\sigma}{\alpha}\sum\nolimits_{j\neq i} q_j(s)\right] + \frac{\beta\,(\mu_S-\mu_B)}{2\alpha}\overline{q}^S.$$

While small agents have equally precise information as big agents, they trade less aggressively. Thus, keeping the total demand by all other agents fixed, high demand by small agents reveals high value of the good and so pushes up the price demanded to clear the market.

The following is the second main result of the paper. Recall that an ex-post equilibrium is the game theoretic counterpart of ex-post incentive compatibility. It means that each agent's strategy is optimal at every type profile of other agents, and so irrespectively of the underlying information structure (see [BM05]).

Proposition 3. *Suppose that assumptions $INF, INJ, A2$ and $A4^*$ hold. Fix $\sigma \geqslant 0$ and consider a game induced by the $\sigma DCDA$. The profile of σVCG conditional inverse demand strategies constitutes an ex-post equilibrium that results in transfers and allocation as in the σVCG mechanism.*

The idea behind the result is as follows. Assumptions INF and INJ are sufficient for the function from signal to marginal utility profiles, evaluated at any allocation, to be globally invertible. Crucially, given the definition of σVCG, this implies that the function from signal profiles to σVCG allocation-price pairs is invertible, for any $\sigma \geqslant 0$. But this, intuitively, establishes that the informational content of either is the same. More precisely, when each agent i reports the σVCG conditional inverse demand $d_i(s_i)$, it is equivalent to reporting the set of σVCG allocation-price pairs consistent with own signal s_i. Injectivity implies that the intersection of the reports across all the agents is the unique σVCG price-allocation $(p^\sigma(s), q^\sigma(s))$. Uniqueness of residual demand curves follows analogously, and ex-post optimality follows from IC of the σVCG mechanism.

The conditional inverse demands in our implementation are related to the conditional bids in [DM00], [EM02], in an auction setting. In each paper they serve the same purpose of allowing to express own demand as a function of types of other agents, and a fixed point has

to be solved to determine the alocation. In [DM00] as well as [EM02] bids are conditioned on vectors of opponents' utility *functions,* where a utility function assigns utility to each allocation a trader could get. In our setting such vectors of functions would be elements in $\left(\mathbb{R}^{\mathbb{R}}\right)^{N-1}$.[22] Our inverse demands are conditioned on vectors of opponents' allocations, and so elements of $\mathbb{R}^{N-1}$. We show that the allocation vectors carry enough information to condition on. The bids are thus simpler; formally, the set of all the available bids has a smaller dimension.[23]

The extent to which σVCG conditional inverse demands are complicated reflects the relatively unrestricted nature of the primitives in our framework. On the other hand, as Example 7 shows, when utilities are simple, and depend only on low-dimensional statistic of the types of others, the inverse demands are also simple. This is not a coincidence:

Theorem 5. *Fix $n \geqslant 0$ and suppose that for every i there exist functions $w_i : S_{-i} \to \mathbb{R}^n$ and $\widetilde{u}_i : \mathbb{R}^{n+2} \to \mathbb{R}$ such that*

$$u_i(\cdot, s_i, s_{-i}) = \widetilde{u}_i(\cdot, s_i, w_i(s_{-i})).$$

Then there is a function $\widetilde{w}_i : \mathbb{R}^N \to \mathbb{R}^n$ such that σVCG conditional inverse demand satisfies

$$d_i(s_i)(q_{-i}) = \left\{x | x = m\widetilde{u}_i(-\sum\nolimits_{j\neq i} q_j, s_i, \widetilde{w}_i(q_{-i}, x)) + \sigma \sum\nolimits_{j\neq i} q_j\right\}, \quad \forall i \tag{3.23}$$

If $d_i(s_i)$ is a function, then there is $\widetilde{w}_i : \mathbb{R}^{N-1} \to \mathbb{R}^n$ such that

$$d_i(s_i)(q_{-i}) = m\widetilde{u}_i(-\sum\nolimits_{j\neq i} q_j, s_i, \widetilde{w}_i(q_{-i})) + \sigma \sum\nolimits_{j\neq i} q_j, \quad \forall i \tag{3.24}$$

Another potential complication is that a σVCG conditional inverse demand may be a

[22] [DM00], [EM02] consider discrete, heterogenous goods, and so in those papers an allocation is a partition of available goods, not a quantity vector.

[23] The set of available bids in [DM00] and [EM02], or what they condition on, can be reduced to only those that occur in equilibrium. The number of equilibrium bids is determined by the set of traders' signals. From the implementation perspective, however, this requires designer's knowledge of traders' utility functions, which is precisely the kind of knowledge that the implementation is supposed to do away with.

correspondence. Intuitively, with a lot of interdependence, high shocks of others can move the marginal utilities of all the agents in step, not affecting the allocation. Consequently, an agent may report several marginal utilities consistent with a given allocation. This is a familiar phenomenon from the rational expectations or the double auction literature. Too much interdependence may imply demand upward sloping in price, resulting in the nonexistence of equilibrium (e.g., [Kyl89]). The following lemma shows a sufficient condition (necessary for linear utilities) for those issues not to arise in our setting.

Consider the following assumption:

$$FUN) \qquad \text{The sum of every row in } \left[mu_{i,j}(q_i, s)\right]^{-1}_{i,j \leqslant N} \text{ is nonzero,} \quad \forall q, s.$$

Theorem 6. *Fix $\sigma \geqslant 0$. If FUN is satisfied, then σVCG conditional inverse demands are functions. In case of linear utilities, the converse is also true.*

Below we present two sufficient conditions for FUN. The first one requires, roughly, that there is not too much interdependence in preferences. Surprisingly, the second sufficient condition is *equicommonality* ([RW12]), satisfied when the average level of interdependence across the agents is constant, no matter its level. It is satisfied in all the examples considered in this paper. In the Bayesian framework, with normal beliefs and equal quadratic coefficients, the condition is sufficient for the existence of a symmetric linear equilibrium.

Theorem 7. *Either of the following conditions is sufficient for FUN. For every i, j, q_i, q_j, s*

$$i) \qquad mu_{i,i}(q_i, s) \geqslant 2N \cdot mu_{j,i}(q_j, s),$$
$$ii) \qquad \sum_k mu_{i,k}(q_i, s) = \sum_k mu_{j,k}(q_j, s).$$

3.6 Other Mechanisms

One way to interpret a σVCG mechanism is this. Agents are charged with taxes that are quadratic in the quantity they trade. Then, assuming that the taxes are internalized by the agents and so their utilities properly adjusted, σVCG mechanisms are the *(generalized)*

VCG mechanisms that implement the efficient allocation. More precisely, fix a *tax* $\tau : \mathbb{R}_+ \to \mathbb{R}_+$, which is twice continuously differentiable, increasing and convex. A mechanism $\{(q(s), t(s))\}_{s\in S}$ is a *VCG mechanism with taxes* τ (τVCG) if

$$\begin{aligned} q_i(s) &= \widetilde{q}_i(s), \\ t_i(s) &= \widetilde{t}_i(s) + \tau(|\widetilde{q}_i(s)|), \end{aligned}$$

where $\{(\widetilde{q}(s), \widetilde{t}(s))\}_{s\in S}$ is an IR, IC, MC mechanism implementing the efficient allocation for utility functions $\widetilde{u}_i$, $i \leqslant N$,

$$\widetilde{u}_i(q_i, s) = u_i(q_i, s) - \tau(|q_i|).$$

A VCG mechanism with taxes satisfy IR, IC and MC for every choice of tax function. Taxes allow to strike a balance between efficiency and budget surplus: Trivial taxes result in the efficient VCG mechanism and budget deficits. Larger taxes distort efficiency of the allocation, relative to the underlying utilities, but mitigate budget deficit. The σVCG mechanisms strike an optimal balance between efficiency and budget surplus (Propositions 1 and 2). In this section we show that they are, roughly, uniquely optimal in this sense.

Consider the following two examples.

Example 8. ***(Linear tax: Bid-ask spread)*** *Fix $\delta \geqslant 0$ and consider a linear tax $\tau(q) = \delta\,|q|$. For any type profile s the allocation $q(s)$ of a VCG mechanism with such taxes is determined jointly with bid-ask prices $p_b(s), p_a(s)$ such that $p_a(s) = p_b(s) + 2\delta$, markets clear and*

$$\begin{aligned} mu_i(q_i(s), s) &= p_a(s), \quad \forall i \text{ such that } q_i(s) > 0 \\ mu_i(q_i(s), s) &= p_b(s). \quad \forall i \text{ such that } q_i(s) < 0 \end{aligned} \tag{3.25}$$

Transfers $t(s)$ that guarantee incentive compatibility and individual rationality are

$$t_i(s) = T_i(s_{-i}) + \begin{cases} \int_0^{q_i(s)} p_a\left(s_i(x), s_{-i}\right) dx & \text{if } q_i(s) > 0, \\ \int_0^{q_i(s)} p_b\left(s_i(x), s_{-i}\right) dx & \text{if } q_i(s) < 0, \end{cases}$$

and $t_i(s) = T_i(s_{-i})$ *if* $q_i(s) = 0$*, where the* s_i *function is defined as in (3.10) and* $T_i(s_{-i}) \leqslant 0$.

In other words, the allocation is as if market determined the market clearing bid-ask price pair with a fixed spread of 2δ, each agent got all the relevant information and was free to trade at those prices. The discriminatory Vickrey transfers are then chosen to guarantee incentive compatibility and individual rationality.

Example 9. ***(Entry Fee tax)*** *Fix* $\phi \geqslant 0$ *and consider the entry fee tax* $\tau(q) = \phi \times \mathbf{1}_{q \neq 0}$*. For any type profile* s*, the allocation* $q(s)$ *of the VCG mechanism with such taxes is determined jointly with a price* $p(s)$ *such that markets clear and*

$$\begin{aligned} mu_i(q_i(s), s) = p(s), \quad \forall i \text{ such that } q_i(s) \neq 0 \\ q_i(s) \neq 0 \text{ iff } \int_0^{q_i(s)} mu_i(x, s)dx - p(s)q_i(s) \geqslant \phi. \end{aligned} \tag{3.26}$$

The transfers $t(s)$ *that guarantee incentive compatibility are*

$$t_i(s) = T_i(s_{-i}) + \sigma + \int_0^{q_i(s)} p\left(s_i(x), s_{-i}\right) dx, \quad \text{if } q_i(s) \neq 0$$

and $t_i(s) = T_i(s_{-i})$ *if* $q_i(s) = 0$*, where the* s_i *function is defined in (3.10) and* $T_i(s_{-i}) \leqslant 0$.

Intuitively, now the allocation is as if market determined the market clearing price, each agent got all the relevant information and then could decide whether to pay a fixed entry fee ϕ in order to trade at this price.

Proposition 4. *Suppose assumptions* $A1 - A4$ *hold. If* $\tau_N VCG$ *mechanisms are robustly asymptotically* $E_1(\frac{1}{N} + \phi_N)-$*Efficient, for some* $E_1 > 0$*, then there is* $E_2 > 0$ *such that for every* $\underline{q} > 0$

$$\tau_N'(q) - \tau_N'(0) \geqslant E_2 \left(\frac{1}{N} + \phi_N\right) \times q, \quad \text{for all } q \in [\underline{q}, \overline{q}], N \geqslant N(\underline{q}). \tag{3.27}$$

The result shows that if some mechanisms converge to efficiency at the optimal rate, then taxes must be at least quadratic in quantity, as in σVCG mechanisms. In this sense,

quadratic taxes create strictly minimal distortions. In particular, VCG mechanisms with entry fees or bid-ask spreads perform strictly worse.

To gain intuition, let us focus on the difference between quadratic taxes on the one hand and entry fees and bid-ask spreads on the other. When many agents have average signals and are not willing to pay a fee, or their marginal utilities fall within a spread, the few trading ones have large price impact. For example, a small fixed fee that a buyer pays results in a steep decrease in prices for his last inframarginal units before other agents enter the market. Thus, the fee is dwarfed by the decrease in prices for all but the last inframarginal units, and so it may not patch up the budget. Similarly, a per-unit fee ϕ decreases the price for almost all inframarginal units by ϕ, which is compounded with the decrease in prices at signals when most agents trade; while in this case budget can be balanced, it requires large fee, and so large distortion.

3.7 Discussion

We would like to discuss and interpret now some of the assumptions behind the results, and point to the possible extensions.

Weaker Assumptions. Assumptions on the utilities can be weakened to fit specific applications: i) Assumptions $A2$ and $A3$ can be qualified to hold only when $mu_i(q_i, s) \in [\min_{i \leqslant N} mu_i(0, s), \max_{i \leqslant N} mu_i(0, s)]$. In particular, this allows environments with nonnegative marginal utilities at any allocation. ii) With large economies, when σVCG allocations are nearly efficient, assumptions $A4$ and $A4^*$, can be weakened to hold only locally, at allocations q_{-j} such that all $mu_i(q_i, s)$, $i \neq j$, are close. iii) When in assumption $A4^*$ the average interdependence $\frac{1}{N-1} \sum_{i \neq j} mu_{i,j}(q_i, s)$ is replaced by $\max_{i \neq j} mu_{i,j}(q_i, s)$, then the scaling constant $\frac{m_q}{\overline{m}_q}$ can be dropped. For large N, the other scaling factor $\frac{1}{8}$ can approximate $\frac{1}{2}$ (see proofs of Propositions 1 and 2).

Multidimensional Signals. Without additional restrictions, our results cannot be extended to environments with multidimensional signals. [JVM06] have shown that with multidimensional signals and interdependence, typically no nontrivial allocation can be ex-

post incentive compatible. There are, nevertheless, several cases when such analysis is not doomed. For example, results in [EM02] and [JVM08] suggest that our analysis may be extended to the case of multidimensional signals as long as marginal utilities are linear functions of signals (see also [Bik06]). In case of private value environments, the impossibility has no bite, and results in [BCL06] suggest the right notion of monotonicity. Detailed analysis of each extension is beyond the scope of this paper.

Tightness. Efficiency losses in Proposition 2 are tight in that there is no other class of mechanisms that converges to efficiency at a faster rate as the economy grows (Proposition 1). While a tight rate of convergence is a standard optimality criterion in the asymptotic worst case analysis, we may strengthen the result as follows. Assume that the marginal utilities decrease linearly in quantity,

$$A2') \qquad mu_{i,q}(q_i, s) = m_q < 0, \quad \forall i, q_i, s$$

and consider the following alternative measure of ex-post efficiency loss:

$$\varepsilon - qEff) \qquad \left|q_i(s) - q_i^0(s)\right| \leqslant \varepsilon |q_i^0(s)|. \quad \forall s$$

Proposition 5. *Suppose assumptions* $A1, A2', A3$ *and* $A4$ *hold.* $\sigma_N^q VCG$ *mechanisms, with* $\sigma_N^q = m_q(\frac{1}{N-1} + \frac{\phi_N}{\underline{m}_o})$*, are robustly asymptotically* $(\frac{1}{N-1} + \frac{\phi_N}{\underline{m}_o})-qEfficient$*. Moreover, for any* $\gamma < 1$*, no robustly asymptotically* $\gamma(\frac{1}{N-1} + \frac{\phi_N}{\underline{m}_o})-qEfficient$ *mechanisms exist.*

Signals as Types. One source of interdependence in preferences is informational, when agent's type corresponds to a signal informative of each agent's value. In a Bayesian model, interdependencies in expected utilities, conditional on a profile of signals, arise naturally when either signals or values are correlated (see Section 3.2). It seems like this informational motivation of the interdependence does not square with the prior-free approach of this paper. We note that, despite this objection, one may still view σVCG mechanisms as particularly simple Bayesian mechanisms that attain asymptotic efficiency.

However, the problem is only apparent. It is true that ex-post utilities and a fixed distri-

bution of correlated signals and values pin down expected utilities with interdependencies, but the reverse is not true. Since many distributions may give rise to one expected utility function, knowing only the latter assumes strictly less than the Bayesian framework. The following example makes the point stark. Suppose agent i has a linear ex-post utility function $v_i(q_i, \theta_i)$ as in (3.5) and fix a linear expected utility function

$$u_i(q_i, s) = q_i \sum w_{ij} s_j - \frac{\mu}{2} q_i^2,$$

for some $\mu, w_{ij}, j \leqslant N$. Prior-free analysis requires knowledge of u_i only, while a Bayesian game complements u_i with a fixed distribution $\delta_S \in \Delta(S)$. We may assume that δ_S is normal.

Theorem 8. *For any normal distribution $\delta_S \in \Delta(S)$, there is a normal distribution $\delta_{\Theta_i \times S} \in \Delta(\Theta_i \times S)$ such that*

$$u_i(q_i, s) = \mathbb{E}_{\delta_{\Theta_i \times S}}[v_i(q_i, \theta_i)|s].$$

The proof is the statement of the Projection Theorem read as a system of N linear equations in N variables, $Cov(s, \theta_i)$. In other words, expected utilities put no restrictions whatsoever on the distribution of signals δ_S.

Equilibrium Uniqueness. Ex-post implementation implies that, in every information structure, the game induced by a σVCG mechanism has a truthtelling equilibrium. In the paper, we are not requiring that this equilibrium is unique. The following is a direct consequence of Theorem 1 in [BM09]. In case of linear utilities, with $mu_{i,i}(q_i, s)$ normalized to 1, $i \leqslant N$, truthtelling is the unique equilibrium of a σVCG mechanism in every information structure precisely when the matrix

$$\begin{bmatrix} 0 & |w_{1,2}| & \dots & |w_{1,N}| \\ |w_{1,2}| & 0 & & \\ \dots & & \dots & \\ |w_{1,N}| & & & 0 \end{bmatrix}$$

with $w_{ij} = mu_{i,j}(q_i, s)$, has the largest eigenvalue $\lambda < 1$. Moreover, the condition is also

necessary for the equilibrium uniqueness under *any* mechanism ([BM09], Section 3).[24] In other words, whether ex-post near-efficient implementation can be strengthened to guarantee equilibrium uniqueness in every information structure is not a mechanism design problem, and the answer depends solely on the properties of an exchange economy.

Conditions for implementation. Assumptions INF and INJ in Section 3.5 are sufficient for the function from the type to the marginal utility profiles (at any allocation) to be globally invertible.[25] Given the definition of σVCG mechanisms, this guarantees that the mapping from type profiles to endogenous σVCG allocation and price profiles is globally invertible as well. The conditions help achieve a sensible implementation, but are not necessary. If they are violated, a profile of reported σVCG conditional inverse demands may result in multiple profiles of market clearing price-allocation pairs and transfers. In such event, instead of shutting down trade, a mechanism may allow another stage with "all-out" communication: Have each agent report his type, and then agents pick the allocation and transfer profile, with the unanimous choice implemented. While such communication requires a daunting amount of coordination (see [DM00]), there seems to be little alternative if there is no simpler sufficient statistic for the information carried by the types.

3.8 Appendix D

A Proof of Proposition 1

Fix N even, and consider the following utility functions:

$$u_i(q_i, t_i, s) = ((\underline{m}_o - \phi_N)\, s_i + \phi_N \sum_j s_j) q_i - \frac{\overline{m}_q}{2} q_i^2, \quad \forall i, q_i, s \tag{3.28}$$

[24] Since we are not insisting on responsive allocation functions and strict ex-post incentive compatibility, we may not use Theorem 2 in [BM09]. However, the direct analysis in Section 3 of their paper applied to our linear setting implies that if the above condition is violated, then there is an agent i and his beliefs over the types of his opponents such that all i's types are indistinguishable.

[25] There are alternative sufficient conditions for the global invertibility (see (38) in [DM00], chapter 6 in [KP12]).

with $\phi_N < \frac{1}{8}\frac{\overline{m}_q}{\overline{\overline{m}}_q}\underline{m}_o$, so that $A4^*$ is satisfied, for which the efficient allocation and price are defined as (see Example 1)

$$q_i^0(s) = \frac{\underline{m}_o - \phi_N}{\overline{m}_q}(s_i - \overline{s}), \quad p^0(s) = (\underline{m}_o + \phi_N(N-1))\,\overline{s}. \tag{3.29}$$

Let the signal space S be such that $A1$ is satisfied. Note that for every s_{-i} the projections $\{s_i | (s_i, s_{-i}) \in S\}$ are closed intervals.

In this environment, the derivative of the Walrasian price with respect to WE quantity allocated to player 1 satisfies

$$\begin{aligned}\frac{dp^0\left(s_1^0\left(q_1\right), s_{-1}\right)}{dq_1} &= \frac{\frac{\partial p^0\left(s_1^0(q_1), s_{-1}\right)}{\partial s_1}}{\frac{\partial q_1^0(s_1^0(q_1), s_{-1})}{\partial s_1}} = \frac{(\underline{m}_o + \phi_N(N-1))}{\frac{1}{\overline{\overline{m}}_q}\left(\underline{m}_o(N-1) - \phi_N(N-1)\right)} \\ &= \frac{\overline{m}_q}{\underline{m}_o - \phi_N}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right) =: p_q^0,\end{aligned} \tag{3.30}$$

where the function $s_1^0(x)$ is defined so that $q_1^0(s_1^0(x), s_{-1}) = x$, for every x.

Consider the signal profile s^*, with

$$\begin{aligned} s_i^* &= \widetilde{s} > 0, \quad q_i^0(s^*) = \frac{N-1}{N}\overline{q}, \quad \forall i \leqslant N/2 \\ s_i^* &= -\widetilde{s} < 0, \quad q_i^0(s^*) = -\frac{N-1}{N}\overline{q}. \quad \forall i > N/2 \end{aligned}$$

The signal profile is chosen in such a way that for the function $s_1(x)$ defined as in (3.10), $\left(s_1(x), s_{-1}^*\right) \in S$, for every $x \in [0, q_1^0(s^*)]$.[26]

Fix $\varepsilon > 0$ and a mechanism that satisfies IR, IC, MC and is $\varepsilon-$Efficient. The transfers

[26] When, say, agent $1's$ signal is changed to $-\frac{1}{N-1}$, we have that

$$\begin{aligned} &q_1^0(-\frac{1}{N-1}, s_{-1}^*) = 0, \\ &q_i^0(-\frac{1}{N-1}, s_{-1}^*) = \overline{q}, \quad \forall i, j \leqslant N/2 \\ &|q_i^0(-\frac{1}{N-1}, s_{-1}^*)| < \overline{q}. \quad \forall i, j > N/2 \end{aligned}$$

by player 1 satisfy[27]

$$\begin{aligned} t_1(s^*) &\leqslant \int_0^{q_1(s^*)} mu_1(x, s_1(x), s^*_{-1})dx \leqslant \int_0^{q_1(s^*)} mu_1(x, s^0_1(x+\varepsilon), s^*_{-1})dx \\ &= \int_0^{q_1(s^*)} \left[mu_1(x+\varepsilon, s^0_1(x+\varepsilon), s^*_{-1}) + \varepsilon \underline{m}_q\right] dx \\ &= \int_0^{q_1(s^*)} \left[p^0(s^0_1(x+\varepsilon), s^*_{-1}) + \varepsilon \overline{m}_q\right] dx, \end{aligned}$$

where the first inequality follows from local IC and IR at $s_1(0)$, and the second one follows from $\varepsilon-$Efficiency. Consequently,

$$\begin{aligned} t_1(s^*) &\leqslant \int_0^{q_1(s^*)} \left[p^0(s^0_1(x+\varepsilon), s^*_{-1}) + \varepsilon \overline{m}_q\right] dx \\ &= \int_0^{q_1(s^*)} \left[p^0(s^*) + p^0_q \times (x + \varepsilon - q^0_1(s^*)) + \varepsilon \overline{m}_q\right] dx \\ &\leqslant q_1(s^*) \times \left[p^0(s^*) + p^0_q \times \left(\frac{q^0_1(s^*)+\varepsilon}{2} + \varepsilon - q^0_1(s^*)\right) + \varepsilon \overline{m}_q\right] \\ &\leqslant \overline{q}\varepsilon\overline{m}_q \left(1 + \frac{3}{2}\frac{p^0_q}{\overline{m}_q}\right) - \left(\frac{N-1}{N}\overline{q} - \varepsilon\right)\frac{3}{2}\frac{N-1}{N}\overline{q}p^0_q \\ &\leqslant C_2 \times \varepsilon - C_1\left(\frac{1}{N} + \phi_N\right), \end{aligned}$$

where, using $\varepsilon < \frac{\overline{q}}{2}$, $N > 2$ and $\frac{p^0_q}{\overline{m}_q} < \frac{5}{7}$, from $A4^*$,

$$C_1 = \overline{q}^2 \frac{1}{6}\frac{\overline{m}_q}{\max\{1, \underline{m}_o\}}, \quad C_2 = \overline{q}\frac{29}{14}\overline{m}_q.$$

The same bound holds for the transfers by any buyer $i \leqslant \frac{N}{2}$. Analogous argument shows that the same bound on transfers holds for any seller $i > \frac{N}{2}$. This establishes the proof.

B Proof of Proposition 2

Fix a slope $\sigma \geqslant 0$ and a σVCG mechanism. By construction, the mechanism satisfies MC.

[27]Note that the allocation $q_1(\cdot, s^*_{-1})$ must be weakly increasing, from IC. If $q_1(\cdot, s^*_{-1}) = q'_1$ is constant over a range of signals $[\underline{s}, \overline{s}]$, then $s_1(q'_1)$ can be defined as any selection from $[\underline{s}, \overline{s}]$.

1. σVCG mechanism satisfies IR and IC

Note that the σVCG transfers satisfy

$$t_i^\sigma(s) = \widetilde{t}_i\left(q_i^\sigma(s), s_{-i}\right)$$

for appropriate functions $\widetilde{t}_i$, $i \leqslant N$. Fix a type vector s and an agent i and consider the allocation $q_i^\sigma(\cdot, s_{-i})$ as a function of $i's$ signal. Local IC is equivalent to

$$\begin{aligned} mu_i\left(q_i^\sigma(s_i, s_{-i}), (s_i, s_{-i})\right) \frac{\partial q_i^\sigma(s_i, s_{-i})}{\partial s_i} &= \frac{\partial \widetilde{t}_i\left(q_i^\sigma(s_i, s_{-i}), s_{-i}\right)}{\partial q_i} \frac{\partial q_i^\sigma(s_i, s_{-i})}{\partial s_i}, \\ \frac{\partial \widetilde{t}_i\left(q_i^\sigma(s_i, s_{-i}), s_{-i}\right)}{\partial q_i} &= mu_i\left(q_i^\sigma(s_i, s_{-i}), (s_i, s_{-i})\right), \end{aligned}$$

and so for any $s_i' < s_i$

$$\begin{aligned} t_i^\sigma(s) = \widetilde{t}_i\left(q_i^\sigma(s), s_{-i}\right) &= t_i^\sigma(s_i', s_{-i}) + \int_{q_i^\sigma(s_i', s_{-i})}^{q_i^\sigma(s)} \frac{\partial \widetilde{t}_i(x, s_{-i})}{\partial q_i} dx \\ &= t_i^\sigma(s_i', s_{-i}) + \int_{q_i^\sigma(s_i', s_{-i})}^{q_i^\sigma(s)} mu_i\left(x, (s_i(x), s_{-i})\right) dx \\ &= t_i^\sigma(s_i', s_{-i}) + \int_{q_i^\sigma(s_i', s_{-i})}^{q_i^\sigma(s)} \left[p^\sigma\left(s_i(x), s_{-i}\right) + \sigma x\right] dx, \end{aligned}$$

which is precisely the definition of σVCG tranfers. This implies local IC.

Let us establish monotonicity of the allocation, $\frac{\partial q_i^\sigma(s)}{\partial s_i} \geqslant 0$, for all i and s. From the first order condition we have for any j and i

$$mu_{j,q}(q_i^\sigma(s), s) \frac{\partial q_j^\sigma(s)}{\partial s_i} + mu_{j,i}(q_j^\sigma(s), s) = \frac{\partial p^\sigma(s)}{\partial s_i} + \sigma \frac{\partial q_j^\sigma(s)}{\partial s_i}. \tag{3.31}$$

With $i = j$ this establishes that

$$\frac{\partial q_i^\sigma(s)}{\partial s_i} = \frac{mu_{i,i}(q_i^\sigma(s), s) - \frac{\partial p^\sigma(s)}{\partial s_i}}{\sigma - mu_{i,q}(q_i^\sigma(s), s)}. \tag{3.32}$$

On the other hand, since market clears for all s', it follows that

$$0 = \sum_j \frac{\partial q_j^\sigma(s)}{\partial s_i} = \sum_j \frac{mu_{j,i}(q_j^\sigma(s), s) - \frac{\partial p^\sigma(s)}{\partial s_i}}{\sigma - mu_{j,q}(q_j^\sigma(s), s)},$$

and so

$$\frac{\partial p^\sigma(s)}{\partial s_i} = \frac{\sum_j mu_{j,i}(q_j^\sigma(s), s) \times (\sigma - mu_{j,q}(q_j^\sigma(s), s))^{-1}}{\sum_j (\sigma - mu_{j,q}(q_j^\sigma(s), s))^{-1}}. \tag{3.33}$$

Since the ratio of any two weights $(\sigma - mu_{j,q}(q_j^\sigma(s), s))^{-1}$ in formula (3.33) is at most $\frac{\overline{m}_q}{\underline{m}_q}$, formulas (3.32) and (3.33), together with $A2$ and $A4^*$ establish monotonicity. Given local IC and $mu_{i,i}(q_i, s) > 0$ this implies full IC. By construction, IR is satisfied in each of the three (exclusive) cases: for the signal s_i under which $q_i^\sigma(s_i, s_{-i}) = 0$, signal $\inf S_i$ when $q_i^\sigma(S_i, s_{-i}) > 0$ and $\sup S_i$ when $q_i^\sigma(S_i, s_{-i}) < 0$. IC implies then IR at all signals, in each of the three cases.

2. σVCG mechanism satisfies $\delta - BS$, for δ as in (3.11), and appropriate constants D_1, D_2.

Fix s, i and q_i and let us write

$$y_j = \left(\sigma - \frac{\partial mu_j(q_j^\sigma(s_i(q_i), s_{-i}), s_i(q_i), s_{-i})}{\partial q_j}\right)^{-1}, \quad \widetilde{y}_j = \frac{y_j}{\sum_{k=1}^N y_k}, \tag{3.34}$$

with $s_i(x)$ as in (3.10). Using formulas (3.32) and (3.33) we may bound the sensitivity of

$\sigma-$Walrasian price with respect to $i's$ allocation:

$$\frac{dp^{\sigma}\left(s_i\left(q_i\right),s_{-i}\right)}{dq_i}=\frac{\frac{\partial p^{\sigma}(s_i(q_i),s_{-i})}{\partial s_i}}{\frac{\partial q_i^{\sigma}(s_i(q_i),s_{-i})}{\partial s_i}}=\frac{\sum_j mu_{j,i}(q_j^{\sigma}(s),s)\widetilde{y}_j}{\left(mu_{i,i}(q_i^{\sigma}(s),s)-\sum_j mu_{j,i}(q_j^{\sigma}(s),s)\widetilde{y}_j\right)y_i} \tag{3.35}$$

$$=\frac{mu_{i,i}(q_i^{\sigma}(s),s)}{mu_{i,i}(q_i^{\sigma}(s),s)\sum_{j\neq i}y_j-\sum_{j\neq i}mu_{j,i}(q_j^{\sigma}(s),s)y_j}$$

$$+\frac{\sum_{j\neq i}mu_{j,i}(q_j^{\sigma}(s),s)\widetilde{y}_j}{\left(mu_{i,i}(q_i^{\sigma}(s),s)\sum_{j\neq i}\widetilde{y}_j-\sum_{j\neq i}mu_{j,i}(q_j^{\sigma}(s),s)\widetilde{y}_j\right)y_i}$$

$$\leqslant\frac{\underline{m}_o}{\underline{m}_o\frac{N-1}{\sigma+\overline{m}_q}-\frac{N-1}{\sigma+\underline{m}_q}\phi_N}+\frac{\frac{N-1}{\sigma+\underline{m}_q}\phi_N}{\left(\underline{m}_o\frac{N-1}{\sigma+\overline{m}_q}-\frac{N-1}{\sigma+\underline{m}_q}\phi_N\right)\frac{1}{\sigma+\overline{m}_q}}$$

$$\leqslant\frac{\underline{m}_o}{\left(\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right)\frac{N-1}{\sigma+\overline{m}_q}}+\frac{\frac{\overline{m}_q}{\underline{m}_q}\phi_N}{\left(\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right)\frac{1}{\sigma+\overline{m}_q}}=\frac{\sigma+\overline{m}_q}{\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N}\left(\frac{\underline{m}_o}{N-1}+\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right).$$

Given this bound the transfers satisfy

$$t_i^{\sigma}\left(s\right)=\int_0^{q_i^{\sigma}(s)}\left[p^{\sigma}\left(s_i\left(x\right),s_{-i}\right)+\sigma x\right]dx \tag{3.36}$$

$$\geqslant\int_0^{q_i^{\sigma}(s)}\left[p^{\sigma}\left(s\right)-\frac{\sigma+\overline{m}_q}{\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N}\left(\frac{\underline{m}_o}{N-1}+\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right)\left(q_i^{\sigma}(s)-x\right)+\sigma x\right]dx$$

$$=p^{\sigma}\left(s\right)q_i\left(s\right)+\frac{q_i^{\sigma}\left(s\right)^2}{2}\left[\sigma-\frac{\sigma+\overline{m}_q}{\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N}\left(\frac{\underline{m}_o}{N-1}+\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right)\right].$$

This implies that BS will be satisfied as long as

$$\sigma\geqslant\sigma_N:=\sigma_N:=\frac{\overline{m}_q}{\frac{N-2}{N-1}\underline{m}_o-2\frac{\overline{m}_q}{\underline{m}_q}\phi_N}\left(\frac{\underline{m}_o}{N-1}+\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right).$$

Moreover, given assumption $A4^*$ and $N>2$ we have $\frac{1}{\underline{m}_o-\frac{\overline{m}_q}{\underline{m}_q}\phi_N}\left(\frac{\underline{m}_o}{N-1}+\frac{\overline{m}_q}{\underline{m}_q}\phi_N\right)<\frac{5}{7}$, which

implies

$$t_i^\sigma(s) \geqslant p^\sigma(s)\, q_i(s) + \frac{q_i^\sigma(s)^2}{2}\left[\frac{2}{7}\sigma - \frac{8}{7}\frac{\overline{m}_q^2}{\underline{m}_o \underline{m}_q}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right)\right] \tag{3.37}$$
$$\geqslant p^\sigma(s)\, q_i(s) + \left[D_2 \times \sigma - D_1\left(\frac{1}{N} + \phi_N\right)\right]_-,$$

where

$$D_1 = \overline{q}^2 \frac{6}{7}\frac{\overline{m}_q^2}{\underline{m}_q \min\{\underline{m}_o, 1\}}, \quad D_2 = \frac{\overline{q}^2}{7}.$$

Adding up the bounds (3.37) over all the agents establishes the proof of part 2.

3. σVCG mechanism satisfies $\varepsilon - Eff$, for ε as in (3.11), and appropriate constants D_3.

Fix a $\sigma-$Walrasian Equilibrium mechanism $\{(q^\sigma(s), t^\sigma(s))\}_{s\in S}$ and s. First, we claim that for the Walrasian price $p^0(s)$,

$$p^0(s) \in \left[\min_i mu_i(q_i^\sigma(s), s), \max_i mu_i(q_i^\sigma(s), s)\right].$$

This is because the allocation $q^\sigma(s)$ clears the market. If, say, $p^0(s)$ was above $\max_i mu_i(q_i^\sigma(s), s)$ and so each agent i got less than $q_i^\sigma(s)$, market would not clear.

This in turn implies that

$$\begin{aligned}\max_i \left|q_i^\sigma(s) - q_i^0(s)\right| &\leqslant \frac{1}{\underline{m}_q}\max_i \left|mu_i(q_i^\sigma(s), s) - p^0(s)\right| \\ &\leqslant \frac{1}{\underline{m}_q}\left(\max_i mu_i(q_i^\sigma(s), s) - \min_i mu_i(q_i^\sigma(s), s)\right) \leqslant \sigma\frac{1}{\underline{m}_q} 2 \max_i |q_i^\sigma(s)| \\ &\leqslant \sigma\frac{1}{\underline{m}_q} 2\left(\overline{q} + \max_i \left|q_i^\sigma(s) - q_i^0(s)\right|\right),\end{aligned}$$

and so, as long as $\sigma < \underline{m}_q/4$,

$$\max_i \left|q_i^\sigma(s) - q_i^0(s)\right| \leqslant \frac{2\sigma}{\underline{m}_q}\frac{\overline{q}}{1 - \frac{2\sigma}{\underline{m}_q}} \leqslant D_3 \times \sigma,$$

with

$$D_3 = \frac{4\overline{q}}{\underline{m}_q}.$$

This concludes the proof of the second part of the proposition. The first part of the proposition follows, with

$$D = \frac{D_1 D_3}{D_2} = 16\overline{q}\frac{\overline{m}_q^2}{\underline{m}_o \underline{m}_q^2}.$$

C Proofs for Section 3.4

Example 4. Proposition 2 (and the preceding discussion) in [RW12] implies that (using original notation)

$$\begin{aligned} \alpha = c_s &= \frac{1-\rho}{1-\rho+\zeta}, \\ \frac{\beta}{\alpha+\beta} = c_p &= \frac{\left(2-\frac{N-2}{N-1}\right)\rho}{1-\frac{N-2}{N-1}+\rho}\frac{\zeta}{1-\rho+\zeta} = \frac{N\rho\zeta}{(1-\rho+\zeta)\left(1+(N-1)\rho\right)}, \end{aligned}$$

and so

$$\beta = \frac{\frac{N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)}\frac{1-\rho}{1-\rho+\zeta}}{\frac{(1-\rho+\zeta)(1+(N-1)\rho)-N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)}} = \frac{N\rho\zeta}{(1-\rho+\zeta)(1+\zeta+(N-1)\rho)}.$$

The submitted inverse demands are

$$\widetilde{p}(s_i)\left(\textstyle\sum_{j\neq i} q_j\right) = \frac{c_s}{1-c_p}s_i + \frac{\mu}{\frac{N-2}{N-1}-c_p}\sum_{j\neq i} q_j = (\alpha+\beta)\left[s_i + \frac{\mu(N-1)}{\alpha(N-2)-\beta}\sum_{j\neq i} q_j\right],$$

whereas the market clearing price $p^l(s)$ and allocation $q_i^l(s)$, for any s, satisfy

$$\begin{aligned} p^l(s) &= \frac{c_s}{1-c_p}\overline{s} = (\alpha+\beta)\overline{s}, \\ q_i^l(s) &= \frac{c_s}{\mu}\left(\frac{\frac{N-2}{N-1}-c_p}{1-c_p}\right)(s_i-\overline{s}) = \frac{(N-2)\alpha-\beta}{\mu(N-1)}\left(s_i-\overline{s}\right), \end{aligned}$$

which means that the transfers and the allocation is the same as in the $\sigma_N VCG$ mechanism (see (3.14)).

Example 5. Proposition 2 in [RW12] implies that for any s the equilibrium price $p^l(s)$ equals $\frac{c_s}{1-c_p}\overline{s}$, and the linear equilibrium has the form (3.17) for constants (using the original notation):

$$c_s = \frac{1-\frac{N-2}{2(N-1)}\rho}{1-\frac{N-2}{2(N-1)}\rho+\zeta},$$
$$c_p = \frac{\frac{N}{N-1}\frac{N-2}{2(N-1)}\rho}{\frac{1}{N-1}+\frac{N-2}{2(N-1)}\rho}\frac{\zeta}{1-\frac{N-2}{2(N-1)}\rho+\zeta} = \frac{\frac{N(N-2)}{2(N-1)}\rho}{1+\frac{N-2}{2}\rho}\frac{\zeta}{1-\frac{N-2}{2(N-1)}\rho+\zeta}.$$

Example 6. 1. With slope σ_N from Proposition 2 the transfers for $i \in G, G = S, B$, become:

$$\begin{aligned} t_i(s) &= \int_0^{q_i(s)} \left[p^{\sigma_N}(s_i(x), s_{-i}) + \sigma_N x\right] dx = p^{\sigma}(s) q_i(s) \\ &+ \frac{q_i(s)^2}{2}\left[\sigma_N - \frac{\sigma_N+\mu_G}{\alpha(N-2\gamma_G)}(\beta+2\alpha\gamma_G)\right] \\ &= p^{\sigma}(s) q_i(s) + \frac{q_i(s)^2}{2}\left[\sigma_N - \frac{\sigma_N+\mu_G}{\alpha(N-2\frac{\sigma_N+\mu_{-G}}{2\sigma_N+\mu_B+\mu_S})}\left(\beta+2\alpha\frac{\sigma_N+\mu_{-G}}{2\sigma_N+\mu_B+\mu_S}\right)\right]. \end{aligned}$$

2. Given a Bayesian model, the parameters $\sigma^2_\theta, \sigma^2_{\theta_G}, \sigma^2_{\theta_{id}}, \sigma^2_{\varepsilon_G}, \sigma^2_{\varepsilon_{id}}$ that give rise to utilities as in Example (3) are, from the Projection Theorem,

$$\begin{bmatrix} \alpha \\ \beta/2 \\ \beta/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - Var(\varepsilon_S+\varepsilon_i, \overline{s}^S, \overline{s}^B)^{-1}_{3\times 3} \cdot Cov(\varepsilon_S+\varepsilon_i, (\varepsilon_S+\varepsilon_i, \overline{s}^S, \overline{s}^B)^T)_{3\times 1},$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = Var(\varepsilon_S+\varepsilon_i, \overline{s}^S, \overline{s}^B)_{3\times 3} \cdot \begin{bmatrix} 1-\alpha \\ -\beta/2 \\ -\beta/2 \end{bmatrix} - Cov(\varepsilon_S+\varepsilon_i, (\varepsilon_S+\varepsilon_i, \overline{s}^S, \overline{s}^B)^T)_{3\times 1}$$

$$= \begin{bmatrix} A & B & \sigma^2_\theta \\ B & B & \sigma^2_\theta \\ \sigma^2_\theta & \sigma^2_\theta & B \end{bmatrix} \cdot \begin{bmatrix} 1-\alpha \\ -\beta/2 \\ -\beta/2 \end{bmatrix} - \begin{bmatrix} \sigma^2_{\varepsilon_G}+\sigma^2_{\varepsilon_{id}} \\ \sigma^2_{\varepsilon_G}+\frac{2}{N}\sigma^2_{\varepsilon_{id}} \\ 0 \end{bmatrix},$$

for $i \in S$ and similarly for $i \in B$, where

$$A = \sigma_\theta^2 + \sigma_{\theta_G}^2 + \sigma_{\theta_{id}}^2 + \sigma_{\varepsilon_G}^2 + \sigma_{\varepsilon_{id}}^2,$$
$$B = \sigma_\theta^2 + \sigma_{\theta_G}^2 + \sigma_{\varepsilon_G}^2 + \frac{\sigma_{\theta_{id}}^2 + \sigma_{\varepsilon_{id}}^2}{N/2}.$$

It is readily verified that the solution to the above system of linear equations is given by (3.19).

3. For a fixed set of parameters $\sigma_\theta^2, \sigma_{\theta_G}^2, \sigma_{\theta_{id}}^2, \sigma_{\varepsilon_G}^2, \sigma_{\varepsilon_{id}}^2$ consider a linear Bayesian equilibrium in double auction of the form

$$q_i^l(s_i)(p) = a_G s_i - b_G p, \quad \forall i \in G, G = S, B,$$

and so the equilibrium price p^l is given by

$$p^l(s) = \frac{a_S}{b_S + b_B}\overline{s}^S + \frac{a_B}{b_S + b_B}\overline{s}^B. \tag{3.38}$$

From the first order condition, demand schedules are given by equation (3.20), for

$$d_i = \frac{1}{\frac{N-2}{2}b_G + \frac{N}{2}b_{-G}}. \quad \forall i \in G, G = S, B \tag{3.39}$$

From the Projection Theorem, we compute $E\left[\theta_i | s_i, p\right]$, for $i \in G$, as

$$E\left[\theta_i | s_i, p^l\right] = \widetilde{a}_G s_i - \widetilde{b}_G p^l, \tag{3.40}$$

where

$$\begin{bmatrix} \widetilde{a}_G \\ -\widetilde{b}_G \end{bmatrix} = Var(s_i, p^l)^{-1}_{2\times 2} \cdot Cov(\theta_i, (s_i, p^l)^T)_{2\times 1},$$

$$Var((s_i, p^l))_{2\times 2} =$$

$$= \begin{bmatrix} \sigma^2_\theta + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} + \sigma^2_{\varepsilon_G} + \sigma^2_{\varepsilon_{id}} & \sigma^2_\theta + \frac{a_G}{b_G+b_{-G}}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}}+\sigma^2_{\varepsilon_{id}}\right)}{N}\right) \\ \sigma^2_\theta + \frac{a_G}{b_G+b_{-G}}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}}+\sigma^2_{\varepsilon_{id}}\right)}{N}\right) & \sigma^2_\theta + \frac{a^2_G+a^2_{-G}}{(b_G+b_{-G})^2}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}}+\sigma^2_{\varepsilon_{id}}\right)}{N}\right) \end{bmatrix},$$

$$Cov(\theta_i, (s_i, p^l)^T)_{2\times 1} = \begin{bmatrix} \sigma^2_\theta + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} \\ \sigma^2_\theta + \frac{a_G}{b_G+b_{-G}}\left(\sigma^2_{\theta_G} + \frac{2}{N}\sigma^2_{\theta_{id}}\right) \end{bmatrix}.$$

Substituting (3.19) into these formulas, we get

$$\begin{aligned} \tilde{a}_G &= \frac{2(\alpha-1)\alpha\left(a^2_G(n-2)\sigma_{\varepsilon_{id}} + a^2_{-G}n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\right)}{2(\alpha-1)\left(a^2_G(n-2)\sigma_{\varepsilon_{id}} + a^2_{-G}n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\right) + \beta(n-2)\sigma_{\varepsilon_{id}}(a_G-a_{-G})^2} \\ &+ \frac{\beta(a_G-a_{-G})(\alpha a_G(n-2)\sigma_{\varepsilon_{id}} + a_{-G}(-\alpha n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G}) + n\sigma_{\varepsilon_G} + 2\sigma_{\varepsilon_{id}}))}{2(\alpha-1)\left(a^2_G(n-2)\sigma_{\varepsilon_{id}} + a^2_{-G}n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\right) + \beta(n-2)\sigma_{\varepsilon_{id}}(a_G-a_{-G})^2}, \\ \tilde{b}_G &= \frac{(\alpha-1)\beta(b_G+b_{-G})(a_G(n-2)\sigma_{\varepsilon_{id}} + a_{-G}n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G}))}{2(\alpha-1)\left(a^2_G(n-2)\sigma_{\varepsilon_{id}} + a^2_{-G}n(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\right) + \beta(n-2)\sigma_{\varepsilon_{id}}(a_G-a_{-G})^2}. \end{aligned}$$

Equilibrium is thus characterized by the solutions a_S, a_B, b_S, b_B of the system of equations

$$a_G = \frac{\widetilde{a}_G(a_G, a_{-G})}{\mu_G + d_G(b_G, b_{-G})}, \quad b_G = \frac{\widetilde{b}_G(a_G, a_{-G}, b_G+b_{-G})+1}{\mu_G + d_G(b_G, b_{-G})}. \quad G = S, B \tag{3.41}$$

Note that $a_G \neq a_{-G}$ as long as $\mu_G \neq \mu_{-G}$

$$\frac{a_G}{a_{-G}} = 1 \Rightarrow = \frac{b_G}{b_{-G}} = 1 \Rightarrow \mu_G = \mu_{-G}.$$

In the asymmetric case (3.41) is a system of four equations of third order and so generically does not admit an analytical solution (Abel-Ruffini Theorem).

4. Finally, to see that the solution to the system (3.41) depends locally on the parameters

$\sigma_{\varepsilon_{id}}$ and σ_{ε_G}, even when α, β are fixed, observe that:

$$\frac{\partial \tilde{a}_G}{\partial \sigma^2_{\varepsilon_{id}}} = \frac{(\alpha-1)\beta a_{-G}(n-2)n\sigma_{\varepsilon_G}(a_G - a_{-G})\left(2(\alpha-1)\left(a_G^2 + a_{-G}^2\right) + \beta(a_G - a_{-G})^2\right)}{\left(2(\alpha-1)\left(a_G^2(n-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 n(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right) + \beta(n-2)\sigma_{\varepsilon_{id}}(a_G - a_{-G})^2\right)^2} \neq 0,$$

$$\frac{\partial \tilde{a}_G}{\partial \sigma^2_{\varepsilon_G}} = -\frac{(\alpha-1)\beta a_{-G}(n-2)n\sigma_{\varepsilon_{id}}(a_G - a_{-G})\left(2(\alpha-1)\left(a_G^2 + a_{-G}^2\right) + \beta(a_G - a_{-G})^2\right)}{\left(2(\alpha-1)\left(a_G^2(n-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 n(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right) + \beta(n-2)\sigma_{\varepsilon_{id}}(a_G - a_{-G})^2\right)^2} \neq 0,$$

as long as $n > 2, \alpha \neq 1$, $\beta \neq 0$, $\sigma_{\varepsilon_{id}} > 0$, $\sigma_{\varepsilon_G} > 0$ and $a_G \neq a_{-G} \neq 0$, which is true when $\mu_G \neq \mu_{-G}$.

D Proofs for Section 3.5

Proof of Proposition 3. Fix $(u_1, ..., u_N)$ and $\sigma \geqslant 0$. We first establish that the mapping $\varphi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}$, where $\mathbb{B} \subset \mathbb{R}^N$ has coefficients add up to 0,

$$\varphi(s) = (p^\sigma(s), q^\sigma(s)),$$

is a diffeomorphism. For any $q \in \mathbb{R}^N$ the function $\zeta_q : \mathbb{R}^N \to \mathbb{R}^N$,

$$\zeta_q(s) = (mu_1(q_1, s), ..., mu_N(q_N, s)),$$

is a diffeomorphism. This follows from local invertibility assumption INJ and INF (Hadamard, see e.g., Theorem 6.2.4 in [KP12]). Moreover, the mapping $\xi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}$, $\xi(mu_1, ..., mu_N) = (p, q)$ such that

$$mu_i = p + \sigma q_i, \quad \forall i$$

is a diffeomorphism as well. This is because ξ satisfies

$$\xi(mu_1, ..., mu_N) = \begin{bmatrix} \frac{1}{N} \cdot e^T \\ \frac{1}{\sigma} I - \frac{1}{N} e e^T \end{bmatrix}_{(N+1)\times N} \cdot (mu_1, ..., mu_N)^T,$$

for $N \times N$ identity matrix I and $N \times 1$ unit vector e, and so is linear with rank N.

It follows that if $\varphi(s) = \varphi(s') = (p^\sigma(s), q^\sigma(s))$, then

$$s = \zeta^{-1}_{q^\sigma(s)} \circ \xi^{-1}(p^\sigma(s), q^\sigma(s)) = s'.$$

On the other hand, for any $(p, q) \in \mathbb{R} \times \mathbb{B}$ we have $(p, q) = \varphi(s)$ for

$$s = \zeta^{-1}_q \circ \xi^{-1}(p, q).$$

We may verify now that that at every s the profile of σVCG conditional inverse demands yields $(p^\sigma(s), q^\sigma(s))$ as the unique market clearing price-allocation pair. An alternative representation of a conditional inverse demand correspondence d_i is via the set $D(d_i)$ of market clearing price-allocation pairs consistent with it (see [PR02]),

$$D_i(d_i) = \bigcup\nolimits_{q_{-i}} (d_i(q_{-i}), -\sum\nolimits_{j \neq i} q_j, q_{-i}).$$

Market clearing price-allocation pairs are then represented by the intersection $\bigcap_i D_i(d_i)$. In the case of the σVCG conditional inverse demand of i at s_i we have

$$\begin{aligned} D_i(d_i(s_i)) &= \bigcup\nolimits_{q_{-i}} \bigcup\nolimits_{s_{-i}: q^\sigma_{-i}(s_i, s_{-i}) = q_{-i}} (p^\sigma(s_i, s_{-i}), q^\sigma(s_i, s_{-i})). \\ &= \bigcup\nolimits_{s_{-i}} (p^\sigma(s_i, s_{-i}), q^\sigma(s_i, s_{-i})). \end{aligned}$$

The injectivity of φ implies then that

$$\bigcap\nolimits_i D_i(d_i(s_i)) = (p^\sigma(s), q^\sigma(s)),$$

and so indeed $(p^\sigma(s), q^\sigma(s))$ is the unique market clearing price-allocation pair.

Fix s and a player i. If all the players $j \neq i$ play the equilibrium strategies, the graph G

of any residual demand curve for a player i satisfies

$$G \subseteq Proj_{P\times Q_i} \bigcap\nolimits_{j\neq i} D_j(d_j(s_j)) = Proj_{P\times Q_i} \bigcup\nolimits_{s'_i} (p^\sigma(s'_i, s_{-i}), q^\sigma(s'_i, s_{-i})).$$

In other words the unique residual demand curve p_i that i is facing is given by

$$p_i(q_i) = p^\sigma(s'_i, s_{-i}), \text{ for } s'_i \text{ such that } q_i = q_i^\sigma(s'_i, s_{-i}).$$

Given the definition of transfers, at s agent i is choosing between either i) a zero quantity and transfer pair (when he submits an incompatible conditional inverse demand function) or 2) pairs $(q_i^\sigma(s'_i, s_{-i}), t_i^\sigma(s'_i, s_{-i}))$, for some s'_i, when he submits the σVCG conditional inverse demand at s'_i. The result thus follows from IC and IR of the σVCG mechanism.

Proof of Lemma 5 Fix s_i. Given assumption INJ, the function $f_i : S_{-i} \to \mathbb{R}^N$, $f_i(s_{-i}) = (p^\sigma(s_i, s_{-i}), q^\sigma_{-i}(s_i, s_{-i}),)$ is injective. We have

$$\begin{aligned}
d_i(s_i)(q_{-i}) &= \left\{x | x = p^\sigma(s_i, s_{-i}) \text{ and } q^\sigma_{-i}(s_i, s_{-i}) = q_{-i}\right\} \\
&= \left\{x | x = mu_i(-\textstyle\sum_{j\neq i} q_j, s_i, s_{-i}) - \sigma \left|\sum_{j\neq i} q_j\right| \text{ and } q^\sigma_{-i}(s_i, s_{-i}) = q_{-i}\right\} \\
&= \left\{x | x = m\widetilde{u}_i(-\textstyle\sum_{j\neq i} q_j, s_i, w_i(s_{-i})) - \sigma \left|\sum_{j\neq i} q_j\right| \text{ and } q^\sigma_{-i}(s_i, s_{-i}) = q_{-i}\right\} \\
&= \left\{x | x = m\widetilde{u}_i(-\textstyle\sum_{j\neq i} q^\sigma_j, s_i, w_i(f_i^{-1}(q^\sigma_{-i}, x))) - \sigma \left|\sum_{j\neq i} q^\sigma_j\right| \text{ and } q^\sigma_{-i} = q_{-i}\right\},
\end{aligned}$$

and so the result follows for $\widetilde{w}_i = w_i \circ f_i^{-1}$.

If σVCG inverse demands are functions then there are no s_{-i} and s'_{-i} with $q^\sigma_{-i}(s_i, s_{-i}) = q^\sigma_{-i}(s_i, s'_{-i})$ but $p^\sigma(s_i, s_{-i}) \neq p^\sigma(s_i, s'_{-i})$. In other words, the function $g_i : S_{-i} \to \mathbb{R}^{N-1}$, $g_i(s_{-i}) = q^\sigma_{-i}(s_i, s_{-i})$ is injective. Following the steps as above establishes the proof, with $\widetilde{w}_i = w_i \circ g_i^{-1}$.

Proof of Lemma 6. Fix $\sigma \geqslant 0$ and let

$$J(s) = \left[mu_{i,j}(q_i^\sigma(s), s)\right]_{i,j\leqslant N}.$$

be the $N \times N$ Jacobian matrix of the function $\zeta(s) = (mu_1(q_1, s), ..., mu_N(q_N, s))$ for $q = q^\sigma(s)$. Totally differentiating equations (3.8) that define the σVCG allocation we get the system of N equations

$$J(s)ds = \left(\sigma I - \left[mu_{i,q}(q_i^\sigma, s)\right]_{i \leqslant N} \cdot I\right) dq + e \cdot dp$$

or equivalently, since $J(s)$ is invertible by INJ,

$$ds = J(s)^{-1}\left(\sigma I - \left[mu_{i,q}(q_i^\sigma, s)\right]_{i \leqslant N} \cdot I\right) dq + J(s)^{-1} e \cdot dp \tag{3.42}$$

where I is the $N \times N$ identity matrix and e is the unit vector.

Consider two profiles of signals (s_k, s_{-k}) and (s'_k, s'_{-k}) with $s_{-k} \neq s'_{-k}$ such that $q^\sigma(s_k, s_{-k}) = q^\sigma(s'_k, s'_{-k})$ but $p^\sigma(s_k, s_{-k}) \neq p^\sigma(s'_k, s'_{-k})$; we need to establish that $s_k \neq s'_k$. Let $\gamma(t), t \in [0, 1]$ be the straight line in $\mathbb{R}^{N+1}$ such that $\gamma(0) = (q^\sigma(s_k, s_{-k}), p^\sigma(s_k, s_{-k}))$ and $\gamma(1) = (q^\sigma(s'_k, s'_{-k}), p^\sigma(s'_k, s'_{-k}))$. Since $\varphi(s) = (p^\sigma(s), q^\sigma(s))$ is a diffeomorphism onto $\mathbb{R} \times \mathbb{B}$, where $\mathbb{B} \subset \mathbb{R}^N$ has coefficients add up to 0 (see proof of Proposition 3), $\varphi^{-1}(\gamma(t))$ is also a path in $\mathbb{R}^N$ from (s_k, s_{-k}) to (s'_k, s'_{-k}). However, it follows from (3.42) that along this path

$$ds = J(s)^{-1} e \cdot dp,$$

and so in particular, from FUN and the continuity of $J(s)^{-1}e$ in s, $ds_k \neq 0$ and does not change the sign along the whole path. It follows that $s_k = \varphi^{-1}(\gamma(0))_k \neq \varphi^{-1}(\gamma(1))_k = s'_k$.

Suppose now that utilities are linear and $(J(s)^{-1}e)_k = 0$ for some agent k. It follows that for a vector $v = J(s)^{-1}e$ we have

$$mu(0, v) = mu(0, 0) + J(0) \cdot v = mu(0, 0) + e,$$

and so for any $\sigma \geqslant 0$, from the definition of σVCG mechanism,

$$q^{\sigma}(v) = q^{\sigma}(0),$$
$$p^{\sigma}(v) = p^{\sigma}(0) + 1,$$

implying that the σVCG conditional inverse demand of agent k at $s_k = 0$ is a correspondence.

Proof of Lemma 7 i) Let matrices A and a be the diagonal and off-diagonal parts of matrix $\left[mu_{i,j}(0,0)\right]_{i,j\leqslant N}$, and let $b = \left[mu_{i,j}(0,0)\right]^{-1}_{i,j\leqslant N} - A^{-1}$. We have

$$(A^{-1} + b)(A + a) = I$$
$$A^{-1}a + bA + ba = 0$$
$$aA^{-1} + Ab(1 + aA^{-1}) = 0$$
$$Ab = -aA^{-1}(1 + aA^{-1})^{-1}$$

Since the infinity norm is both sub-multiplicative and sub-additive:

$$||Ab||_{\infty} \leqslant \frac{||aA^{-1}||_{\infty}}{1 - ||aA^{-1}||_{\infty}}.$$

Finally, by assumption, each element of the aA^{-1} matrix is less then $1/2n$, therefore $||aA^{-1}||_{\infty} < 1/2$. By the inequality above $||Ab||_{\infty} < 1$ and therefore matrix $W^{-1} = A^{-1}(I + Ab)$ has positive row sums, since A^{-1} has only positive diagonal elements.

ii) Equicommonality means that $\left[mu_{i,j}(0,0)\right]_{i,j\leqslant N} \cdot e = [\alpha]_{N\times 1}$, for unit vector e and some $\alpha \neq 0$. Left-multiplying by $\left[mu_{i,j}(0,0)\right]^{-1}_{i,j\leqslant N}$ yields the result.

E Proof of Proposition 4

Fix a sequence of tax functions $\{\tau_N\}_{N\in\mathbb{N}}$ so that the respective VCG mechanism with transfers τ_N are $D\left(\frac{1}{N} + \phi_N\right)$-Efficient, for some $D > 0$. $D\left(\frac{1}{N} + \phi_N\right)$-Efficiency and convexity implies

that taxes have derivatives uniformly bounded by $D\left(\frac{1}{N}+\phi_N\right)$. Therefore, from the Arzela-Ascoli Theorem, condition (3.27) is equivalent to pointwise convergence, for every $q \in (0, \overline{q}]$.

Suppose that the condition (3.27) is violated, and so there is $q \in (0, \overline{q}]$ and a sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ of positive numbers converging to zero such that

$$\tau'_N(q) - \tau'_N(0) \leqslant \varepsilon_N \left(\frac{1}{N}+\phi_N\right) \times q. \tag{3.43}$$

We will establish that for sufficiently large N the mechanisms violate BS.

Consider utility functions as in (3.28), large N (to be determined later) and a profile of signlas $s^* = (s_1^*, s_2^*, 0, ..., 0)$ such that $s_2^* = -s_1^*$ and

$$mu_1(q, s^*) = s_1^* \underline{m}_o + \phi_N s_2^* - \underline{m}_q q = \tau'_N(q),$$

so that $q^{\tau_N}(s^*) = (q, -q, 0, ..., 0)$. Let $a_N = \lim_{q\to 0} \tau_N(q) - \tau_N(0) \geqslant 0$ be the entry fee of the tax and, with slight abuse of notation, let $\tau'_N(0) = \lim_{q\to 0} \tau'_N(q)$ be the intial slope of the tax scheme. As in Proposition 2 the transfers satisfy

$$\begin{aligned}
t_1(s^*) &\leqslant \int_0^q p^{\tau_N}(s_1(x), s_{-1}^*)dx + \tau_N(q), \\
t_2(s^*) &\leqslant \int_0^{-q} p^{\tau_N}(s_2(x), s_{-2}^*)dx + \tau_N(q), \\
t_i(s^*) &\leqslant 0, \quad \forall i > 2,
\end{aligned}$$

where $p^{\tau_N}(s)$ are defined, analogously to σVCG prices, as the Walrasian prices for the economy with distorted utilities $\widetilde{u}_i$, and $s_1(x)$ and $s_2(x)$ are defined analogously as in (3.10):

$$\begin{aligned}
mu_i(q_i^{\tau_N}(s), s) &= p^{\tau_N}(s) + \tau'_N(q_i^{\tau_N}(s)), \quad \forall i, s \\
x &= q_i^{\tau_N}\left(s_i(x), s_{-i}^*\right). \quad i = 1, 2, \forall x \in [-q, q]
\end{aligned}$$

Below we establish that $t_1(s^*) < 0$; as the proof of $t_2(s^*) > 0$ is analogous, this will establish the result.

Let s_1' and s_1'' be the signals for player 1, $s_1'' \leqslant s_1' \leqslant s_1^*$, such that

$$\max_{q_3}\{u_3(q_3, s_1', s_{-i}^*) - q_3 \times p^{\tau_N}(s_1', s_{-i}^*) - a_N\} = 0, \tag{3.44}$$
$$\max_{q_3}\{u_3(q_3, s_1'', s_{-i}^*) - \int_0^{q_3}(p^{\tau_N}(s_1'', s_{-i}^*) + \tau_N'(x))dx - a_N\} = 0.$$

In other words, if the tax scheme consisted of only the entry fee a_N then agents $i > 2$ would start trading (buying from agent 2) if 1's signal drops below s_1'. Similarly, agents $i > 2$ start trading given the original tax scheme τ_N when 1's signal drops below s_1''.

Let us compute prices $p^{\tau_N}(s_1(x), s_{-1}^*)$ for the inframarginal units x, $x \in [0, q]$. For $s_1 \geqslant s_1''$ only players 1 and 2 trade nonzero quantities, and we have

$$mu_1(q_1^{\tau_N}(s_1, s_{-1}^*), s_1, s_{-1}^*) = s_1\underline{m}_o + \phi_N s_2^* - \underline{m}_q q_1^{\tau_N}(s_1, s_{-1}^*) = p^{\tau_N}(s_1, s_{-1}^*) + \tau_N'(q_1^{\tau_N}(s_1, s_{-1}^*)),$$
$$mu_1(q_2^{\tau_N}(s_1, s_{-1}^*), s_1, s_{-1}^*) = s_2^*\underline{m}_o + \phi_N s_1 + \underline{m}_q q_2^{\tau_N}(s_1, s_{-1}^*) = p^{\tau_N}(s_1, s_{-1}^*) + \tau_N'(q_2^{\tau_N}(s_1, s_{-1}^*)),$$

and so

$$\frac{\partial q_1^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1} = \frac{\underline{m}_o - \frac{\partial p^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1}}{\underline{m}_q + \tau_N''(q_1^{\tau_N}(s_1, s_{-1}^*))},$$
$$\frac{\partial q_2^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1} = \frac{\phi_N - \frac{\partial p^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1}}{\underline{m}_q + \tau_N''(q_2^{\tau_N}(s_1, s_{-1}^*))}.$$

Using MC, we thus have

$$\frac{\partial p^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1} = \frac{\frac{\underline{m}_o}{\underline{m}_q + \tau_N''(q_1^{\tau_N}(s_1, s_{-1}^*))} + \frac{\phi_N}{\underline{m}_q + \tau_N''(q_2^{\tau_N}(s_1, s_{-1}^*))}}{\frac{1}{\underline{m}_q + \tau_N''(q_1^{\tau_N}(s_1, s_{-1}^*))} + \frac{1}{\underline{m}_q + \tau_N''(q_2^{\tau_N}(s_1, s_{-1}^*))}} = \frac{\underline{m}_o + \phi_N}{2},$$

and so, for any x such that $s_1(x) \geqslant s_1''$

$$\frac{\partial p^{\tau_N}(s_1(x), s_{-1}^*)}{\partial x} = -\frac{\frac{\partial p^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1}}{\frac{\partial q_1^{\tau_N}(s_1, s_{-1}^*)}{\partial s_1}} = \frac{\frac{\underline{m}_o + \phi_N}{2}}{\frac{\underline{m}_o - \phi_N}{2\left(\underline{m}_q + \tau_N''(x)\right)}} = \left(\underline{m}_q + \tau_N''(x)\right)\frac{\underline{m}_o + \phi_N}{\underline{m}_o - \phi_N}. \tag{3.45}$$

Similarly, for $s_1 < s_1''$ and $x \geqslant 0$ such that $s_1(x) < s_1''$, when all agents trade nonzero

quantities, we have

$$\begin{aligned}
\frac{\partial p^{\tau_N}(s_1, s^*_{-1})}{\partial s_1} &= w_1(s_1)\underline{m}_o + (1 - w_1(s_1))\phi_N, \\
\frac{\partial q_1^{\tau_N}(s_1, s^*_{-1})}{\partial s_1} &= \frac{\underline{m}_o - \frac{\partial p^{\tau_N}(s_1, s^*_{-1})}{\partial s_1}}{\underline{m}_q + \tau''_N(q_1^{\tau_N}(s_1, s^*_{-1}))}, \\
\frac{\partial p^{\tau_N}(s_1(x), s^*_{-1})}{\partial x} &= -\frac{\underline{m}_q + \tau''_N(x)}{(1 - w_1(s_1(x)))\,(\underline{m}_o - \phi_N)}\left(w_1(s_1(x))\underline{m}_o + (1 - w_1(s_1(x)))\phi_N\right),
\end{aligned} \tag{3.46}$$

where

$$w_1(s_1) = \frac{\frac{1}{\underline{m}_q + \tau''_N(q_1^{\tau_N}(s_1, s^*_{-1}))}}{\frac{1}{\underline{m}_q + \tau''_N(q_1^{\tau_N}(s_1, s^*_{-1}))} + \frac{1}{\underline{m}_q + \tau''_N(q_2^{\tau_N}(s_1, s^*_{-1}))} + \frac{N-2}{\underline{m}_q + \tau''_N(q_3^{\tau_N}(s_1, s^*_{-1}))}}.$$

Since $\tau'_N(0) \geqslant 0$, τ'_N is increasing and bounded by $D\left(\frac{1}{N} + \phi_N\right)$, it follows that for $x \in [0, q]$ $\tau''_N(x)$ is bounded above by 1 on an interval of measure $q - D\left(\frac{1}{N} + \phi_N\right)$. It follows from the definition of $w_1(s_1)$ and (3.46) that there are $c, c' > 0$ such that $w_1(s_1(x))$ is bounded from below by $\frac{c'}{N}$, and so

$$\frac{\partial p^{\tau_N}(s_1(x), s^*_{-1})}{\partial x} \geqslant c \times \left(\frac{1}{N} + \phi_N\right), \tag{3.47}$$

for a subset of $x \in [0, q]$ of measure $q - D\left(\frac{1}{N} + \phi_N\right)$, for every N.

Moreover, it follows from $D\left(\frac{1}{N} + \phi_N\right)$-Efficiency of the allocation for agents $i > 2$ that

$$\tau'_N(0), p^{\tau_N}(s''_1, s^*_{-i}), p^{\tau_N}(s'_1, s^*_{-i}) = O\left(\frac{1}{N} + \phi_N\right),$$

and so, from (3.45),

$$q_1^{\tau_N}(s'_1, s^*_{-1}), q_1^{\tau_N}(s''_1, s^*_{-1}) = q - O\left(\frac{1}{N} + \phi_N\right).$$

Finally, since $\tau'_N \geqslant 0$ and $u_3(q_3, s''_1, s^*_{-i}) \leqslant u_3(q_3, s'_1, s^*_{-i})$ for every q_3, equations in (3.44) imply that

$$p^{\tau_N}(s'_1, s^*_{-i}) - p^{\tau_N}(s''_1, s^*_{-i}) \geqslant \tau'_N(0).$$

The integral of the inframarginal prices agent 1 pays is

$$\begin{aligned}
\int_0^q p^{\tau_N}(s_1(x), s_{-1}^*)dx &= I + II + III,\\
I &= q_1^{\tau_N}(s_1', s_{-1}^*) \times p^{\tau_N}(s_1', s_{-1}^*) + \int_{q_1^{\tau_N}(s_1', s_{-1}^*)}^{q} p^{\tau_N}(s_1(x), s_{-1}^*)dx,\\
II &= q_1^{\tau_N}(s_1'', s_{-1}^*) \times \left(p^{\tau_N}(s_1'', s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*)\right)\\
&\quad + \int_{q_1^{\tau_N}(s_1'', s_{-1}^*)}^{q_1^{\tau_N}(s_1', s_{-1}^*)} p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*))dx,\\
III &= \int_0^{q_1^{\tau_N}(s_1'', s_{-1}^*)} \left(p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1'', s_{-1}^*)\right)dx.
\end{aligned}$$

Below we bound from above each of the three terms I, II and III.

First, since

$$a_N = u_3(q_3', s_1', s_{-i}^*) - q_3' \times p^{\tau_N}(s_1', s_{-i}^*) = \frac{p^{\tau_N}(s_1', s_{-i}^*)^2}{2\underline{m}_q},$$

where q_3' is the maximizer, we have, for N sufficiently high

$$I < q_1^{\tau_N}(s_1', s_{-1}^*) \times p^{\tau_N}(s_1', s_{-1}^*) = \left(q - O\left(\frac{1}{N} + \phi_N\right)\right) \times p^{\tau_N}(s_1', s_{-1}^*) < a_N.$$

Second,

$$\begin{aligned}
II &= q_1^{\tau_N}(s_1', s_{-1}^*) \times \left(p^{\tau_N}(s_1'', s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*)\right)\\
&\quad + \int_{q_1^{\tau_N}(s_1'', s_{-1}^*)}^{q_1^{\tau_N}(s_1', s_{-1}^*)} p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1'', s_{-1}^*))dx\\
&= q_1^{\tau_N}(s_1', s_{-1}^*) \times \left(p^{\tau_N}(s_1'', s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*)\right) + O\left(\frac{1}{N} + \phi_N\right)^2\\
&< q_1^{\tau_N}(s_1', s_{-1}^*) \times (-\tau_N'(0)) + O\left(\frac{1}{N} + \phi_N\right)^2 = -q \times \tau_N'(0) + O\left(\frac{1}{N} + \phi_N\right)^2.
\end{aligned}$$

Finally, the bound (3.47) yields, for $c > 0$ independent of N,

$$\begin{aligned} III &= \int_0^{q_1^{\tau_N}(s_1'', s_{-1}^*)} \left(p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1'', s_{-1}^*)\right) dx \\ &< -c \times \left(q_1^{\tau_N}(s_1'', s_{-1}^*) - O\left(\frac{1}{N} + \phi_N\right)\right) \times \left(\frac{1}{N} + \phi_N\right) \\ &= -c \times q \times \left(\frac{1}{N} + \phi_N\right) + O\left(\frac{1}{N} + \phi_N\right)^2. \end{aligned}$$

The three bounds imply

$$\begin{aligned} t_1(s^*) &= I + II + III + \tau_N(q) = I + a_N + II + \tau_N'(0) \times q + III + \int_0^q \left(\tau_N'(x) - \tau_N'(0)\right) dx \\ &< -c \times q \times \left(\frac{1}{N} + \phi_N\right) + \int_0^q \left(\tau_N'(x) - \tau_N'(0)\right) dx + O\left(\frac{1}{N} + \phi_N\right)^2 \\ &< -c \times q \times \left(\frac{1}{N} + \phi_N\right) + \varepsilon_N \left(\frac{1}{N} + \phi_N\right) \times q^2 + O\left(\frac{1}{N} + \phi_N\right)^2 \\ &= q \times \left(\frac{1}{N} + \phi_N\right) \times (\varepsilon_N q - c) + O\left(\frac{1}{N} + \phi_N\right)^2, \end{aligned}$$

where the second inequality follows from assumption (3.43). It follows that for N sufficiently high the transfers $t_1(s^*)$ are negative, which establishes the proof.

F Proof of Proposition 5

In order to establish robust asymptotic $(\frac{1}{N-1} + \frac{\phi_N}{\underline{m}_o})$−qEfficiency, BS is implied by the inequality (3.36) and assumption $A2'$. Moreover, the efficiency loss equals

$$\left|q_i^\sigma(s) - q_i^0(s)\right| \leqslant \frac{1}{\underline{m}_q} \left|mu_i(q_i^\sigma(s), s) - mu_i(q_i^0(s), s)\right| = \frac{1}{\underline{m}_q} \sigma \left|q_i^\sigma(s)\right|,$$

where the last equality follows from $p^0(s) = p^\sigma(s)$, for every type profile s and $\sigma \geqslant 0$, which is implied by $A2'$.

To establish tighness, fix N and consider the same utility functions and signal profile s as in the proof of Proposition 1. Fix $\delta > 0$ small and a mechanism that satisfies IR, IC, MC

and is $\delta-$Efficient. For the function $s_1(x)$ defined as in (3.10) and $s_1^0(x)$ defined the same way but for the efficient allocation $q^0(s)$ the transfers by player 1 satisfy

$$\begin{aligned}
t_1(s) &\leqslant \int_0^{q_1(s)} mu_1(x, s_1(x), s_{-i})dx \leqslant \int_0^{q_1(s)} mu_1(x, s_1^0(x(1+\delta)+\delta), s_{-i})dx \\
&= \int_0^{q_1(s)} \left[mu_1(x(1+\delta), s_1^0(x+\delta), s_{-i}) + x\delta\underline{m}_q\right] dx \\
&= \int_0^{q_1(s)} \left[p^0(s_1^0(x+\delta), s_{-1}) + x\delta\underline{m}_q\right] dx \\
&\leqslant \int_0^{q_1(s)} \left[p^0(s_1 + s_1^0(x(1+\delta)) - s_1, s_{-1}) + x\delta\underline{m}_q\right] dx \\
&= \int_0^{q_1(s)} \left[p^0(s) + p_{q_1}^0 \times (x(1+\delta) - q_1^0(s)) + x\delta\underline{m}_q\right] \\
&\leqslant q_1(s) \times \left[p^0(s) + p_{q_1}^0 \times (\frac{q_1^0(s)(1+\delta)^2}{2} - q_1^0(s)) + \frac{q_1^0(s)(1+\delta)}{2}\delta\underline{m}_q\right] \\
&= q_1(s)p^0(s) + q_1(s)\frac{q_1^0(s)}{2}\left[(1+\delta)\delta\underline{m}_q - p_{q_1}^0(1-2\delta-\delta^2)\right],
\end{aligned}$$

where $p_{q_1}^0 = \frac{m_q}{\underline{m}_o - \phi_N}\left(\frac{m_o}{N-1} + \phi_N\right)$ is defined as in (3.30).

This implies that if $\delta < (\frac{1}{N-1} + \frac{\phi_N}{\underline{m}_o})$ then for N sufficiently large we have $t_1(s) < 0$, and, by an analogous argument, $t_2(s) < 0$. $p^0(s) = 0$ and $q_i^0(s) = 0$ for $i > 2$ together with IR imply that $\sum_{i>2} t_i(s) \leqslant 0$, and so establish that the mechanism violates BS.

G Discrete Allocations

In this section we show how our results can be extended to a setting with discrete allocations. Specifically, we assume that for the feasible allocations $q_i(s)$ are multiples of Δ for $\Delta \leqslant 1$,

$$q_i(s) \in \{... - 2\Delta, -\Delta, 0, \Delta, 2\Delta, ...\} =: \mathbb{Q}_\Delta, \quad \forall i, s$$

First, the definition of marginal utility must be adjusted to the discrete setting, with $mu_i(q_i, s) = \frac{u_i(q_i+\Delta, s) - u_i(q_i, s)}{\Delta}$, for $q_i \in \mathbb{Q}_\Delta$. Similarly, assumption $A2$ is now replaced by

$$A2^d) \qquad mu_i(q_i, s) - mu_i(q_i + \Delta, s) \in [\Delta\underline{m}_q, \Delta\overline{m}_q]. \quad \forall i, q_i \in \mathbb{Q}_\Delta, s$$

For the simplicity of the presentation, in this section we use a slightly stronger version of the small pairwise interdependence assumption, which bounds each individual interdependece (as opposed to the average, see Section 3.2).

$$A4^{d}) \qquad mu_{i,j}(q_i, s) \leqslant \phi_N, \text{ with } \lim_{N\to\infty} \phi_N = 0, \ \ \forall i \neq j, q_{-j}, s$$

$$A4^{d*}) \qquad \tfrac{1}{N-1} mu_{i,j}(q_i, s) \leqslant \phi_N, \text{ with } \phi_N < \tfrac{1}{8}\underline{m}_o, \ \ \forall i \neq j, q_{-j}, s$$

We proceed to define the versions of the σVCG mechanisms in discrete setting. Fix a slope σ and a spread 2δ, $\sigma, \delta \geqslant 0$. For any agent i, s and q_i define the tax adjusted marginal utility $\widetilde{mu}_i(q_i, s)$ as

$$\widetilde{mu}_i(q_i, s) = mu_i(q_i, s) - \sigma q_i - \delta \mathbf{1}_{q_i>0} + \delta \mathbf{1}_{q_i<0}.$$

For the taxes τ that correspond to slope σ and spread δ, the τVCG allocation $q^\tau(s)$ is defined so that

$$\begin{aligned} \widetilde{mu}_i(q_i^\tau(s) + \Delta, s) < \widetilde{mu}_j(q_j^\tau(s), s), \quad \forall i \neq j \\ \sum_i q_i^\tau(s) = 0. \end{aligned} \tag{3.48}$$

The first condition replaces now the equality of all the tax-adjusted marginal utilities. It explicitly requires that any additional trade of one unit of size Δ, from agent j to agent $i \neq j$, results in strict lowering of the sum of their tax-adjusted utilities. In other words, the allocation maximizes the sum of tax-adjusted utilities. As in the main text, when $\sigma = \delta = 0$, the allocation is the efficient Walrasian Equilibrium allocation, for discrete setting.

The τVCG transfers $t^\tau(s)$ are defined as

$$\begin{aligned} t_i^\tau(s) &= \Delta \sum_{x=\Delta}^{q_i^\tau(s)} mu_i\left(x, s_i(x), s_{-i}\right), \quad q_i^\tau(s) > 0 \\ t_i^\tau(s) &= \Delta \sum_{x=-\Delta}^{q_i^\tau(s)} -mu_i\left(x, s_i(x), s_{-i}\right), \quad q_i^\tau(s) < 0 \end{aligned} \tag{3.49}$$

where $s_i(x)$ is the pivotal signal for good x,

$$x = q_i^\tau(s_i(x), s_{-i}) \text{ and } x > q_i^\tau(s_i', s_{-i}) \text{ for } s_i' < s_i(x), \quad x > 0,$$
$$x = q_i^\tau(s_i(x), s_{-i}) \text{ and } x < q_i^\tau(s_i', s_{-i}) \text{ for } s_i' > s_i(x), \quad x < 0,$$
$$s_i(x) = \begin{cases} \inf S_i \text{ if } q_i^\tau(s_i', s_{-i}) > x \text{ for all } s_i', \\ \sup S_i \text{ if } q_i^\tau(s_i', s_{-i}) < x \text{ for all } s_i'. \end{cases}$$

The following are the analogues of Propositions 1 and 2, followed by the proofs.[28]

Proposition 6. *Suppose that assumptions $A1, A2^d, A3, A4^{d*}$ hold. There are constants $C_1^d, C_2^d, C_3^d > 0$, independent of the unit size $\Delta \leqslant 1$, such that for every $\varepsilon < \overline{q}/2$ and economy size $N > 2$, the IR, IC, MC and $\varepsilon-$Efficient mechanisms violate $\delta - BS$, where*

$$\delta = N\left[C_1^d(\frac{m_o}{N-1} + \phi_N) + C_2^d\Delta - C_3^d\varepsilon\right].$$

Proposition 7. *Suppose that assumptions $A1, A2^d, A3, A4^d$ hold. For appropriate positive constants D^d, D'^d independent of the unit size $\Delta \leqslant 0$, slopes σ_N^d and spreads $2\delta_N$ the family of the corresponding $\tau_N VCG$ mechanisms is robustly asymptotically $\left[D^d(\frac{1}{N} + \phi_N) + D'^d\Delta\right]-$Efficient.*

More precisely, there are constants $D_1^d - D_7^d$ such that for every slope $\sigma < \underline{m}_q/4$, spread $2\delta \geqslant 0$, unit size $\Delta \leqslant 1$ and economy of size $N > 2$ that satisfies $A1, A2^d, A3, A4^{d*}$, the corresponding τVCG mechanism satisfies $MC, IR, IC, \delta - BS$ and $\varepsilon-$Efficiency, where

$$\delta = N\left[\left[D_1^d(\frac{m_o}{N-1} + \phi_N) - D_2^d\sigma\right]_+ + \left[\Delta(D_3^d + D_4^d\sigma) - D_5^d\delta\right]_+\right], \tag{3.50}$$
$$\varepsilon = D_6^d\sigma + D_7^d\delta.$$

Proof of Proposition 6. The proof mirrors that of Proposition 2. Fix N even, and

[28] In the propositions and the proofs we use the standard notation, with $[x]_+ = \min\{0, x\}$, $[x]_- = \max\{0, x\}$ and $\lfloor x \rfloor$ the highest integer below x.

consider the utility functions:

$$u_i(q_i, t_i, s) = ((\underline{m}_o - \phi_N)\, s_i + \phi_N \sum_j s_j) q_i - \frac{\underline{m}_q}{2} q_i^2, \quad \forall i, q_i, s$$

and consider the signal profile s^*, such that

$$s_i^* = \widetilde{s} > 0, \quad q_i^0(s^*) \in \left[\frac{N-1}{N}\overline{q} - \Delta, \frac{N-1}{N}\overline{q} \right], \quad \forall i \leqslant N/2$$
$$s_i^* = -\widetilde{s} < 0, \quad q_i^0(s^*) = -\left[\frac{N-1}{N}\overline{q} - \Delta, \frac{N-1}{N}\overline{q} \right]. \quad \forall i > N/2$$

For those signals we have $mu_i(q_i^0(s^*), s^*) = 0$, for all i. Given the efficient allocation, it is easy to verify that for agent 1 and any positive $x \in \mathbb{Q}_\Delta$,

$$\begin{aligned}
&mu_i\left(x, s_i^0(x), s_{-i}^*\right) = \qquad (3.51)\\
&= \Delta\left(\left\lfloor \frac{x - q_i^0(s)}{\Delta(N-1)} \right\rfloor + 1\right)\underline{m}_q + \phi_N\left(x - q_i^0(s) + \Delta\left\lfloor \frac{x - q_i^0(s)}{\Delta(N-1)} \right\rfloor\right)\frac{\underline{m}_q}{\underline{m}_o - \phi_N}\\
&= \frac{\underline{m}_q}{\underline{m}_o - \phi_N}\left\{\Delta(\underline{m}_o - \phi_N) + \phi_N\left(x - q_i^0(s)\right) + \Delta\underline{m}_o \times \left\lfloor \frac{x - q_i^0(s)}{\Delta(N-1)} \right\rfloor\right\}, \quad x > q_i^0(s)\\
&= -\Delta\left(\left\lfloor \frac{q_i^0(s) - x}{\Delta(N-1)} \right\rfloor + 1\right)\underline{m}_q - \phi_N\left(z + \Delta + \Delta\left\lfloor \frac{q_i^0(s) - x}{\Delta(N-1)} \right\rfloor\right)\frac{\underline{m}_q}{\underline{m}_o - \phi_N}\\
&= -\frac{\underline{m}_q}{\underline{m}_o - \phi_N}\left\{\Delta\underline{m}_o + \phi_N\left(q_i^0(s) - x\right) + \Delta\underline{m}_o \times \left\lfloor \frac{q_i^0(s) - x}{\Delta(N-1)} \right\rfloor\right\}, \quad x \leqslant q_i^0(s)
\end{aligned}$$

Fix $\varepsilon > 0$ that is a multiple of Δ[29] and a mechanism that satisfies IR, IC, MC and is $\varepsilon-$Efficient. The transfers by player 1 satisfy

[29]Otherwise, in the rest of the proof consider $\varepsilon' > \varepsilon$ that is.

$$
\begin{aligned}
t_1(s^*) &\leqslant \Delta \sum_{x=\Delta}^{q_1(s^*)} mu_1(x, s_1(x), s^*_{-1}) \leqslant \Delta \sum_{x=\Delta}^{q_1(s^*)} mu_1(x, s^0_1(x+\varepsilon), s^*_{-1}) \\
&= \Delta \sum_{x=\Delta}^{q_1(s^*)} \left[mu_1(x+\varepsilon, s^0_1(x+\varepsilon), s^*_{-1}) + \varepsilon \underline{m}_q \right] \\
&\leqslant \varepsilon \underline{m}_q q_1(s^*) + \Delta \left(\sum_{z=q^0_1(s^*)+\Delta}^{q^0_1(s^*)+2\varepsilon} mu_1(x, s^0_1(x), s^*_{-1}) + \sum_{z=\Delta+\varepsilon}^{q^0_1(s^*)} mu_1(x, s^0_1(x), s^*_{-1}) \right) \\
&\leqslant \varepsilon \underline{m}_q q_1(s^*) + \Delta 2 \varepsilon \underline{m}_q + 4\varepsilon^2 \frac{\underline{m}_q}{\underline{m}_o - \phi_N} \left(\phi_N + \frac{\underline{m}_o}{N-1} \right) \\
&- \frac{\underline{m}_q}{\underline{m}_o - \phi_N} \left(\begin{array}{c} \Delta \left(q^0_1(s^*) - \varepsilon \right) \underline{m}_o + \frac{\left(q^0_1(s^*) - \varepsilon \right)\left(q^0_1(s^*) - \varepsilon + \Delta \right)}{2} \left(\phi_N + \frac{\underline{m}_o}{N-1} \right) \\ -\Delta^2 \underline{m}_o \frac{N-2}{2} \times \frac{q^0_1(s^*) - \varepsilon}{\Delta(N-1)} \end{array} \right) \\
&\leqslant \varepsilon \underline{m}_q \left(\overline{q} + 2\Delta + \frac{20}{7}\varepsilon \right) - \frac{\underline{m}_q}{\underline{m}_o - \phi_N} \left(\begin{array}{c} \Delta \frac{\left(q^0_1(s^*) - \varepsilon \right) \underline{m}_o}{2} \\ + \frac{\left(q^0_1(s^*) - \varepsilon \right)^2}{2} \left(\phi_N + \frac{\underline{m}_o}{N-1} \right) \end{array} \right)
\end{aligned}
$$

where the last inequality used $\frac{p^0_q}{\underline{m}_q} < \frac{5}{7}$, from $A4^{d*}$, and the second last inequality follows from (3.51) and the formula

$$
\sum_{x=1}^{q} \left\lfloor \frac{x}{N-1} \right\rfloor \geqslant \sum_{x=\Delta}^{q} \frac{x}{N-1} - \frac{N-2}{2} \times \frac{q}{N-1},
$$

with the exact equality when x is divisible by $N-1$. Consequently,

$$
t_1(s^*) \leqslant C^d_3 \times \varepsilon - C^d_1 \left(\frac{\underline{m}_o}{N-1} + \phi_N \right) - C^d_2 \times \Delta,
$$

with

$$
C^d_1 = \frac{\underline{m}_q}{\underline{m}_o} \left(\frac{\overline{q}}{6} - 1 \right)^2, \quad C^d_2 = \frac{\underline{m}_q}{12} \overline{q}, \quad C^d_3 = \underline{m}_q \left(\frac{17}{7} \overline{q} + 2 \right).
$$

The same bound holds for the transfers by any buyer $i \leqslant \frac{N}{2}$. Analogous argument shows that the same bound on transfers holds for any seller $i > \frac{N}{2}$. This establishes the proof.

Proof of Proposition 7. Fix a slope $\sigma \geqslant 0$, spread 2δ and the corresponding τVCG mechanism.

1. The proof that the τVCG mechanism satisfies MC, IR and IC is analogous to the corresponding parts of the proof of Proposition 2.

2. τVCG mechanism satisfies $\delta - BS$, for δ as in (3.50), and appropriate constants $D_1^d - D_5^d$.

Fix s, a τVCG allocation $q^\tau(s)$, agent i, and suppose that $q_i^\tau(s) > 0$. Define

$$\begin{aligned} \underline{p}^\tau(s) &= \min_i \left\{ \widetilde{mu}_i(q_i^\tau(s), s) \right\}, \\ \overline{p}^\tau(s) &= \max_i \left\{ \widetilde{mu}_i(q_i^\tau(s), s) \right\}. \end{aligned} \tag{3.52}$$

Step i). $\underline{p}^\tau(s) = \overline{p}^\tau(s)$.

Let us also first assume that the values are private, $\phi_N = 0$. From the definition of the τVCG mechanism, as well as assumption $A2^d$ agent $i's$ adjusted marginal utility $\widetilde{mu}_i(x, s_{-i}, s_i(x))$ at the pivotal type for any of his last $N-1$ units of size Δ is at most $\Delta(\overline{m}_q + \sigma)$ lower than $\widetilde{mu}_i(q_i^\tau(s), s)$. Once $i's$ allocation decreases by $\Delta(N-1)$ (all the other agents "bought" a unit of size Δ from i already), the marginal utility for the $q_i^\tau(s) - \Delta(N-1)$ unit decreases by further at most $\Delta(\overline{m}_q + \sigma)$. Consequently, with private values we have

$$\widetilde{mu}_i(x, s_{-i}, s_i(x)) \geqslant \widetilde{mu}_i(q_i^\tau(s), s) - \Delta\left(\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor + 1 \right)(\sigma + \overline{m}_q), \qquad x > 0 \tag{3.53}$$

Let us now consider a general, interdependent case, $\phi_N > 0$. In this case, as the signal for agent i decreases, so does the marginal utility of other agents. The lowest value for $s_i(q_i^\tau(s))$ satisfies, from (3.48),

$$\begin{aligned} \widetilde{mu}_i(q_i^\tau(s), s_{-i}, s_i(q_i^\tau(s))) &= \widetilde{mu}_j(q_j^\tau(s), s_{-i}, s_i(q_i^\tau(s))) - \Delta(\sigma + \overline{m}_q), \\ [s_i - s_i(q_i^\tau(s))] \times \underline{m}_o &\leqslant [s_i - s_i(q_i^\tau(s))] \times \phi_N + \Delta(\sigma + \overline{m}_q), \\ s_i(q_i^\tau(s)) &\geqslant s_i - \Delta \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}, \end{aligned} \tag{3.54}$$

and, for any $x \leqslant q_i^\tau(s)$, the pivotal signals satisfy

$$s_i(x) \geqslant s_i - \left(q_i^\tau(s) - x + \Delta + \Delta \left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor \right) \left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} \right). \quad x > 0 \tag{3.55}$$

Given those bounds on the pivotal signals and the interdependence, the adjusted marginal utilities at those types satisfy,

$$\begin{aligned} &\widetilde{mu}_i(q_i^\tau(s), s) - \widetilde{mu}_i(x, s_{-i}, s_i(x)) \\ \leqslant &\Delta \left(\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor + 1 \right) (\sigma + \overline{m}_q) + \phi_N \left(q_i^\tau(s) - x + \Delta + \Delta \left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor \right) \left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} \right) \end{aligned} \tag{3.56}$$

for $x > 0$. The formula implies that the adjusted marginal utility decreases by at most

$$\gamma := \frac{\Delta}{N-1} (\underline{m}_o - \phi_N + N \times \phi_N) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} = \Delta \left(\frac{\underline{m}_o}{N-1} + \phi_N \right) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}, \tag{3.57}$$

on average for every unit of size Δ purchased (compare with line (3.35) in the proof of Proposition **2**). Moreover, the difference $\widetilde{mu}_i(q_i^\tau(s), s) - \widetilde{mu}_i(x, s_{-i}, s_i(x))$ between the adjusted marginal utilities is always smaller than $\Delta(\sigma + \overline{m}_q) + \gamma \left(\frac{q_i^\tau(s) - x}{\Delta} + 1 \right)$. Just as in (3.36) in the proof of Proposition **2**, this implies that the transfers satisfy

$$\begin{aligned} t_i^\tau(s) &= \Delta \sum_{x=\Delta}^{q_i^\tau(s)} mu_i(x, s_i(x), s_{-i}) = \Delta \sum_{x=\Delta}^{q_i^\tau(s)} \{\widetilde{mu}_i(x, s_i(x), s_{-i}) + \sigma x + \delta\} \\ &\geqslant q_i^\tau(s) \underline{p}^\tau(s) + \Delta \sum_{x=\Delta}^{q_i^\tau(s)} \left\{ -\gamma \left(\frac{q_i^\tau(s) - x}{\Delta} + 1 \right) - \Delta(\sigma + \overline{m}_q) + \sigma x + \delta \right\} \\ &\geqslant q_i^\tau(s) \underline{p}^\tau(s) + q_i^\tau(s) [\delta - \Delta(\sigma + \overline{m}_q)] + \Delta(\Delta + 2\Delta + ... + q_i^\tau(s)) \times \left[\sigma - \frac{\gamma}{\Delta} \right] \\ &\geqslant q_i^\tau(s) \underline{p}^\tau(s) + \overline{q} [\delta - \Delta(\sigma + \overline{m}_q)]_- + \frac{\overline{q}(\overline{q} + \Delta)}{2} \left[\sigma - \left(\frac{\underline{m}_o}{N-1} + \phi_N \right) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} \right]_-. \end{aligned} \tag{3.58}$$

Step ii). Let us now consider the general case, when at the τVCG allocation the adjusted marginal utilities are not equal, $\underline{p}^\tau(s) = \overline{p}^\tau(s)$. From the definition of the τVCG allocation, the distance between $\underline{p}^\tau(s)$ and $\overline{p}^\tau(s)$ is at most $\Delta(\sigma + \overline{m}_q)$.

In this case, we need the following adjustments. First, in the private value case, the

bound (3.53) changes to

$$\widetilde{mu}_i(x, s_{-i}, s_i(x)) \geqslant \underline{p}^\tau(s) - \Delta\left(\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor + 1\right)(\sigma + \overline{m}_q). \qquad x > 0$$

Second, given that the difference bewteen $\widetilde{mu}_i(q_i^\tau(s), s)$ and $\underline{p}^\tau(s)$ is at most $\sigma + \overline{m}_q$, the bounds (3.54), (3.55) and (3.56) in the general case change to, for $x > 0$

$$\begin{aligned}
s_i(q_i^\tau(s)) &\geqslant s_i - 2\Delta\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}, \\
s_i(x) &\geqslant s_i - \left(q_i^\tau(s) - x + 2\Delta + \Delta\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor\right)\left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}\right), \\
\underline{p}^\tau(s) - \widetilde{mu}_i(x, s_{-i}, s_i(x)) &\leqslant \boldsymbol{\Delta}\left(\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor + 1\right)(\sigma + \overline{m}_q) \\
&\quad + \phi_N\left(q_i^\tau(s) - x + 2\Delta + \Delta\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor\right)\left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}\right).
\end{aligned}$$

Altogether, we have that the difference

$$\begin{aligned}
\underline{p}^\tau(s) - \widetilde{mu}_i(x, s_{-i}, s_i(x)) &\leqslant \boldsymbol{\Delta}\left(\sigma + \overline{m}_q + \phi_N\left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}\right)\right) + \gamma\left(\frac{q_i^\tau(s) - x}{\Delta} + 1\right) \\
&= \Delta\frac{(\sigma + \overline{m}_q)\,\underline{m}_o}{\underline{m}_o - \phi_N} + \gamma\left(\frac{q_i^\tau(s) - x}{\Delta} + 1\right),
\end{aligned}$$

for γ as in (3.57). Consequently, similarly as in the proof of Proposition 2, using $A4^{d*}$ we have

$$\begin{aligned}
t_i^\tau(s) &\geqslant q_i^\tau(s)\,\underline{p}^\tau(s) + \left[\delta - \Delta\frac{(\sigma + \overline{m}_q)\,\underline{m}_o}{\underline{m}_o - \phi_N}\right] q_i^\tau(s) + \left[\sigma - \frac{\gamma}{\Delta}\right]\frac{q_i^\tau(s)\,(q_i^\tau(s) + \Delta)}{2} \qquad (3.59) \\
&\geqslant q_i^\tau(s)\,\underline{p}^\tau(s) + \left[\delta - \Delta\frac{8}{7}(\sigma + \overline{m}_q)\right] q_i^\tau(s) \\
&\quad + \left[\frac{2}{7}\sigma - \frac{8}{7}\frac{\overline{m}_q}{\underline{m}_o}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right)\right]\frac{q_i^\tau(s)\,(q_i^\tau(s) + \Delta)}{2} \\
&\geqslant q_i^\tau(s)\,\underline{p}^\tau(s) + \left[D_2^d \times \sigma - D_1^d\left(\frac{\underline{m}_o}{N-1} + \phi_N\right)\right]_- + \left[D_5^d \times \delta - \Delta(D_3^d + D_4^d \times \sigma)\right]_-
\end{aligned}$$

with

$$D_1^d = \overline{q}\,(\overline{q}+1)\,\frac{4}{7}\frac{\overline{m}_q}{\underline{m}_o}, \quad D_2^d = \frac{\overline{q}\,(\overline{q}+1)}{7},$$
$$D_3^d = \frac{8\overline{qm}_q}{7}, \quad D_4^d = \frac{8\overline{q}}{7}, \quad D_5^d = \overline{q}.$$

Since the same inequality holds for every seller i, with $q_i^\sigma(s) < 0$, adding up the inequalities establishes the proof of part 2.

3. τVCG mechanism satisfies $\varepsilon - Eff$, for ε as in (3.50), and appropriate constants D_6^d, D_7^d.

Fix a τVCG mechanism $\{(q^\tau(s), t^\tau(s))\}_{s\in S}$ and s. First, just as in the proof of Proposition 2,

$$\max_i mu_i(q_i^0(s), s) \leqslant \max_i mu_i(q_i^\tau(s), s) + \delta,$$
$$\min_i mu_i(q_i^0(s), s) \geqslant \min_i mu_i(q_i^\tau(s), s) - \delta,$$

This in turn implies that

$$\begin{aligned}\max_i \left|q_i^\tau(s) - q_i^0(s)\right| &\leqslant \frac{1}{\underline{m}_q}\max_i \left|mu_i(q_i^\tau(s), s) - mu_i(q_i^0(s), s)\right| \\ &\leqslant \frac{1}{\underline{m}_q}\left(\max_i mu_i(q_i^\tau(s), s) - \min_i mu_i(q_i^\tau(s), s) + \delta\right) \\ &\leqslant \frac{1}{\underline{m}_q}\left(2\sigma \max_i |q_i^\tau(s)| + 3\delta\right) \leqslant \frac{1}{\underline{m}_q}\left(2\sigma\left(\overline{q} + \max_i \left|q_i^\tau(s) - q_i^0(s)\right|\right) + 3\delta\right)\end{aligned}$$

and so, as long as $\sigma < \underline{m}_q/4$,

$$\max_i \left|q_i^\tau(s) - q_i^0(s)\right| \leqslant \frac{2\sigma\overline{q} + 3\delta}{\underline{m}_q}\frac{1}{1 - \frac{2\sigma}{\underline{m}_q}} \leqslant D_6^d \times \sigma + D_7^d \times \delta,$$

with

$$D_6^d = \frac{4\overline{q}}{\underline{m}_q}, \quad D_7^d = \frac{6}{\underline{m}_q}.$$

This concludes the proof of the second part of the proposition. The first part of the proposition follows from the second, with

$$D^d = D_6^d \frac{D_1^d}{D_2^d} + \Delta D_7^d \frac{D_4^d}{D_5^d} \frac{D_1^d}{D_2^d}, \quad D'^d = D_7^d \frac{D_3^d}{D_5^d}.$$

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