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Secondary Teachers' Professional Noticing of Students' Mathematical Thinking

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of  
Philosophy

in

Mathematics and Science Education

by

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2018



The Dissertation of Raymond M. LaRochelle is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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2018

## DEDICATION

This dissertation is dedicated to three people. First, to my dissertation co-chairs, Lisa Lamb and Randy Philipp, who provided a tremendous amount of intellectual and emotional support throughout this process. Thank you for your hard work, for your patience, and for believing in me.

Second, to my partner, Javier Curiel, who listened to my ideas for countless hours and was always willing to provide feedback. In recognition for his ability to calm me down and help me get through many stressful nights, I believe he deserves an honorary PhD in mathematics education. Jokes aside, I thank you from the bottom of my heart.

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## LIST OF ABBREVIATIONS

ET	Experienced Secondary Teacher
ETL	Emerging Secondary Teacher Leader
PST	Prospective Secondary Teacher

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## PUBLICATIONS

- LaRochelle, R.**, Nickerson, S. N., Lamb, L. C. (2016). Secondary teachers' professional noticing of students' mathematical thinking. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 429). Tucson, AZ: The University of Arizona.
- Lawler, B. R., **LaRochelle, R.**, & Thompson, A. (2017). Enhancing Activities in Mathematics Methods Courses to Achieve Sociopolitical Goals. In S. Kastberg, A. Tyminski, A. Lischka, & W. Sanchez. (Eds.) (In Press). *Building Support for Scholarly Practices in Mathematics Methods*. Charlotte, NC: Information Age Publishing.
- Nickerson, S. D., Lamb, L., & **LaRochelle, R.**, (2017). Challenges in measuring secondary mathematics teachers' professional noticing of students' mathematical thinking. In E. O. Schack, M. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*. New York: Springer.
- LaRochelle, R.**, Lamb, L. C., Nickerson, S. N., (2017). Secondary teachers' professional noticing of students' proportional reasoning. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39<sup>th</sup> annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (p. 544) Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.



## ABSTRACT OF THE DISSERTATION

Secondary Teachers' Professional Noticing of Students' Mathematical Thinking

by

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Doctor of Philosophy in Mathematics and Science Education

University of California San Diego, 2018

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Researchers have ample evidence of many positive benefits for students and teachers when teachers regularly elicit and use students' ideas during instruction. However, there are many challenges associated with this style of instruction. In particular, an increase in student participation means teachers must be ready to quickly attend to important details of a student's strategy, interpret the student's mathematical understandings, and decide how to respond to the student in a way that respects and extends the student's understandings, all in a moment's notice. This in-the-moment attention, interpretation, and decision-making is called *professional noticing of students' mathematical thinking* (Jacobs, Lamb, & Philipp, 2010). In this dissertation I share two sets of findings. First, I share findings related to the development of this important expertise. In a cross-sectional analysis I compare and contrast the professional noticing expertise of three groups of secondary teachers ( $N = 72$ ): prospective teachers, experienced teachers, and

emerging teacher leaders. Results indicated that experienced secondary teachers were fairly similar to prospective secondary teachers in the attending and interpreting component-skills, and were only marginally better than prospective secondary teachers in deciding how to respond. In contrast, secondary emerging teacher leaders provided much more evidence of attending to the students' strategies, interpreting the students' understandings, and deciding how to respond based on the students' understandings. This implies a need for sustained professional development, as teaching experience alone does not appear to provide sufficient support for teachers to develop this important teaching practice. Second, I share findings related to selection of artifacts of student thinking for teacher learning. By comparing teachers' responses to 6 different artifacts of student thinking, I identify features of artifacts that increase the level of challenge associated with the artifact, and the amount of curiosity or excitement that an artifact can elicit. In particular, scalar strategies and strategies with non-integer ratios were more interesting than the alternatives, perhaps due to teachers' unfamiliarity with scalar strategies and the added complexities afforded by non-integer ratios. In addition, scalar strategies were more challenging for teachers to comprehend than unit-rate strategies.

## Chapter 1: Introduction

For the last quarter of a century, the field of mathematics education has recognized the importance of engaging students in problem solving and reasoning activities during instruction (NCTM, 1989). Researchers contend that students should have opportunities to engage with each other, the mathematics, and the teacher in a wide range of mathematical activities (Hiebert & Grouws, 2007). The teacher then must facilitate interactions among students to support their problem solving and reasoning activities (Cohen, Raudenbush, & Ball, 2003), which means the practice of teaching necessarily includes the ability to make sense of the ideas students generate in-the-moment, and use these ideas to inform instructional decisions during class to support students to move toward the mathematical goals of the lesson (Forzani, 2014; Stein, Engle, Smith, & Hughes, 2008).

Instruction that is centered on and responsive to students' ideas is a well-documented and productive method of instruction (Sowder, 2007). For one, this style of instruction can support rich learning environments (Stephan & Akyuz, 2012) and gains in student achievement (Jacobs, Franke, Carpenter, Levi, & Fennema, 2007). Additionally, when teachers learn how to respond to and build on student thinking, student thinking becomes a powerful object for teacher learning (Kazemi & Franke, 2004; Wilson & Berne, 1999). Student thinking can even support teacher learning well after professional development ends, when teachers learn how to effectively implement this style of instruction (Franke, Carpenter, Levi, & Fennema, 2001).

However, learning how to adopt an instructional style that is more responsive toward students' ideas is difficult for many reasons. Here, I list two. First, being able to make sense of and direct a classroom where students are regularly sharing ideas can be overwhelming (to say the least). Teachers are already bombarded with an immense amount of visual and auditory

stimuli in the classroom, and the development of noticing skills that allows teachers to successfully navigate this overwhelming environment can take years (Berliner, 1994). To add to this overwhelming environment activities where students discuss ideas and solve problems together, the already overwhelming environment becomes even more chaotic. Second, even without considering the intense environment, being responsive to a single child presents its own set of challenges. Children's ideas are often unpredictable, and the path for supporting students to build on their informal ideas toward the mathematical goals of the lesson is almost never straightforward (Ball, 1995; Lampert, 2001). Centering instruction on students' unpredictable ideas adds a significant complexity that does not occur when teachers lecture and students take notes.

### **Decomposing the Practice of Teaching**

In order to support teachers to be more responsive to students' ideas, teacher educators are finding ways to decompose the practice of teaching into *core practices*, and then engaging teachers in learning about, trying out, and reflecting on one's enactment of these core practices (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009; Grossman, Hammerness, & McDonald, 2009; McDonald, Kazemi, & Kavanaugh, 2013). As the name suggests, a core practice is a fundamental component of teaching that teachers engage in regularly (for a list of criteria, see Grossman, Hammerness et al., 2009). Theoretically, by learning about and practicing a thoughtfully chosen set of core practices, new teachers will be better prepared to do the work of teaching (and also be more responsive to students' ideas), rather than simply knowing about the work of teaching (Forzani, 2014). This new direction in teacher education is a response to what has been called the *problem of enactment* (Kennedy, 1999), which refers to the fact that one can know about pedagogical concepts but still be unprepared for actually doing

the work of teaching. Some examples of these component-practices of teaching include leading group discussions, eliciting and interpreting individual students' thinking, setting up and managing small group work, and providing oral and written feedback to students (Teaching Works, 2017).

In this dissertation, I aim to further the field's understanding of an important component-practice of teaching, namely *professional noticing of students' mathematical thinking* (Jacobs, Lamb, & Philipp, 2010). Many researchers have investigated the nature and development of professional noticing of students' mathematical thinking, which has proven to be a challenging expertise for teachers at all levels to develop, as well as how to support teachers to learn how to enact this important practice (Jacobs et al., 2010; Lesseig, Casey, Monson, Krupa, & Huey, 2016; Schack, Fisher, Thomas, Eisenhardt, Tassel, & Yoder, 2013). In the next section I define professional noticing of students' mathematical thinking and share my rationale for studying this expertise.

### **Professional Noticing of Students' Mathematical Thinking**

As I mentioned earlier, responsive teaching involves the teachers routinely making sense of students' contributions and quickly making decisions about how to respond, where the decision is based on the student's unique mathematical understandings (Jacobs & Empson, 2016). This practice of sense-making and decision-making has been conceptualized as *professional noticing of students' mathematical thinking*, in which a teacher: (a) attends to the details of a student's contribution, (b) interprets the student's mathematical understandings, and (c) decides how to respond on the basis of the student's understandings (Jacobs et al., 2010). These three sub-components occur simultaneously as the teacher considers a student's idea. They are also interrelated, and interdependent. For example, what a teacher attends to will

influence the teacher's interpretations, and the teacher's decision may cause the teacher to attend to more details in the work, and look confirming or disconfirming evidence. Overall, this practice highlights the improvisational nature of responsive teaching.

Professional noticing of students' mathematical thinking is one of many constructs from the field of teacher noticing (see Sherin, et al., 2011a, for a summary). For the rest of this dissertation, *noticing* will refer to an umbrella term for many noticing constructs, one of which being professional noticing of students' mathematical thinking. Professional noticing of students' mathematical thinking is different from other noticing constructs in three important ways. First, it exclusively focuses on noticing *students' mathematical thinking*, rather than other items such as noticing how teachers use multiple representations (Dreher & Kuntze, 2015), or noticing student's cultural funds of knowledge (McDuffie et al., 2014). Second, professional noticing of student's mathematical thinking includes a third component-skill, deciding how to respond, which is not always present in studies of teacher noticing. For instance, Sherin and van Es (2009) focused on teachers' selective attention and knowledge-based reasoning, which relate to the attending and interpreting component-skills, but not the deciding how to respond component-skills. Third, professional noticing of students' mathematical thinking is defined as an in-the-moment practice, occurring during a lesson whenever a teacher "notices" a student's idea. It is not a practice that happens, for example, in the planning phase of instruction (e.g. *curricular noticing*; Amador, Males, Earnest, & Dieteker, 2017).

### **The Terrain of Professional Noticing of Students' Mathematical Thinking**

When Sherin, Jacobs, and Philipp (2011b) situated teacher noticing within the field of mathematics education, they posed five questions that they felt were important for researchers to consider. I will now adapt these questions for professional noticing of students' mathematical

thinking, by replacing the phrase “teacher noticing” with “professional noticing of students’ mathematical thinking.” Thus, for this construct, researchers might ask the following:

1. “Is [professional noticing of students’ mathematical thinking] trainable?”
2. What trajectories of development related to [professional noticing of students’ mathematical thinking] exist for prospective and practicing teachers?
3. How context specific is [professional noticing of students’ mathematical thinking]?
4. How can researchers most productively study [professional noticing of students’ mathematical thinking]?
5. Why do we (or should we) study [professional noticing of students’ mathematical thinking]?” (p. 11)

Sherin et al. (2011b) posed these questions in order for researchers to fully understand the affordances and constraints that teacher noticing offers to the field. I believe these questions are useful for understanding the affordances and constraints of professional noticing as well. In my work, I will contribute to the second and third questions they posed, with a focus on secondary prospective and practicing teachers.

I frame the rest of the chapter around the second and third questions Jacobs and her colleagues (2011) posed. In particular, my first research question relates to developmental trajectories of secondary teachers’ professional noticing of students’ mathematical thinking in the domain of proportional reasoning. My second research question relates to how differences in students’ strategies afford or constrain teachers’ effectiveness at demonstrating their professional noticing of students’ mathematical thinking expertise.

## **Developmental Trajectories of Secondary Teachers’ Professional Noticing of Students’ Mathematical Thinking**

Much of the literature about professional noticing of students' mathematical thinking has investigated how prospective elementary teachers (e.g. Schack et al., 2013) and prospective secondary teachers (e.g. Lesseig et al., 2016) develop this expertise. Prospective teachers are an important group to study because for many teachers, the credential program is the only long term sustained professional development they receive in their professional careers. Additionally, it is the first professional development teachers receive. Many of these studies describe important activities for supporting teachers' professional noticing of students' mathematical thinking expertise. Additionally, as Lesseig et al. (2016) point out, they highlight how difficult it is for prospective teachers to develop a robust professional noticing expertise.

At this point in time, only Jacobs and her colleagues (2010; Jacobs, Lamb, Philipp, & Schappelle, 2011) have reported on the professional noticing of students' mathematical thinking of practicing teachers, and the long-term development of this expertise. They investigated the professional noticing expertise of three groups of experienced practicing primary teachers, who were differentiated by their number of years of participating in a professional development program about students' mathematical thinking (0 years, 2 years, and 4 years), and a group of prospective primary teachers. They found that most practicing primary teachers needed at least 4 years of sustained professional development about children's mathematical thinking to develop a robust professional noticing expertise. Additionally, they found evidence that teaching experience can support this expertise for primary teachers, but only marginally. This implies that for primary teachers, professional noticing requires sustained professional development about students' mathematical thinking. Additionally, these results suggest that in the short amount of time of a credential program, it is unlikely that prospective primary teachers will be able to develop this expertise to robust levels.



Currently, we have yet to learn how professional noticing expertise develops for practicing *secondary* teachers. It is reasonable to wonder what differences might exist between primary and secondary teachers because primary and secondary teachers have different opportunities to interact with students' mathematical thinking (Doig, Groves, Tytler, & Gough, 2005). On the one hand, primary teachers spend the day with the same 30 students but teach 5 different subjects, meaning they spend much time with individual *students'* thinking, but perhaps not with *mathematical* thinking. On the other hand, secondary teachers generally spend the day teaching the same subject but teach upwards of 150 students, which means they spend much time with *mathematical* thinking, but maybe not with individual *students'* thinking. Certainly the two types of experiences differentially support teachers' professional noticing of *students'* *mathematical* thinking, but it is unclear how. Hence, we must investigate possible developmental trajectories of this expertise for secondary teachers as well.

### **Domain Specificity: Proportional Reasoning**

In addition to the differences between secondary and primary practicing teachers, it is unclear how teachers' professional noticing of students' mathematical thinking expertise differs across content domains. Professional noticing is a domain specific expertise (Jacobs & Empson, 2016), meaning teachers' effectiveness at making sense of and deciding how to respond to students' ideas will change depending on the mathematical domain. For example, Cross (2009) documented large differences in one secondary teacher's practice, depending on the mathematical content this teacher was teaching. In her algebra class, this teacher mostly lectured, presenting formulas and demonstrating how to use those formulas, and then letting students practice on their own. In her geometry class however, this teacher had a more student-centered classroom, with lots of problem solving and discussions around the students' ideas.

Essentially, this teacher made sense of and responded to her students' ideas in very different ways depending on whether she was teaching Algebra or Geometry.

In this dissertation, I consider the domain of proportional reasoning (Lobato & Ellis, 2010). Proportional reasoning is an important mathematical domain to study for many reasons. For one, proportional reasoning is an important transition point between elementary and secondary mathematics (Lesh, Post, & Behr, 1988). Additionally, proportional reasoning spans many grade levels. It first emerges in the concepts of multiplication and division, as students learn how to create composite units and count units of units (Steffe, 1994). Later on, students continue to grapple with proportional reasoning concepts in complex mathematical domains such as linear functions, trigonometry, and probability.

Proportional reasoning is also a complex domain of mathematics to study. This domain of reasoning combines multiplicative and additive ways of reasoning (Steinthorsdottir & Sriraman, 2009), and forces the student to develop their ability to think relationally (i.e. think about one quantity in terms of another quantity; Lamon, 1993). Students must coordinate multiple quantities simultaneously (Steffe, 1994), and eventually notice multiplicative relationships between ratios as well as within ratios (Tourniaire & Pulos, 1985). Equivalence relations, division, fractions, and the eventual abstraction of ratios to rates (e.g. going from “9 miles is 3 times more than 3 hours” to “the number of miles is always three times the number of hours; Lobato & Ellis, 2010) are all concepts that emerge as students grapple with proportional relationships (Lamon, 2007). Importantly, this domain is significantly more complex than the whole number operations domain that served as the context of the Jacobs et al. (2010) study of primary teachers' professional noticing expertise. Might this complexity affect what teachers professionally notice in students' ideas? What might be characteristics of sophisticated

professional noticing skills in this complex mathematical domain? These are important questions to answer.

### **Rationale for Research Question 1**

In order to help teachers develop responsive teaching practices, we need to understand (a) characteristics of sophisticated professional noticing expertise, (b) characteristics of novice professional noticing expertise, and (c) developmental trajectories of professional noticing expertise (in particular mathematical domains). Jacobs et al. (2010) identified these characteristics and developmental trends for practicing primary teachers in the domain of whole number operations. I extend their work to secondary teachers, in the complex domain of proportional reasoning, by answering my first research question.

#### ***Research question 1.***

What differences and similarities exist in the professional noticing of students' mathematical thinking expertise in the domain of proportional reasoning among:

- a. secondary mathematics emerging teacher leaders,
- b. secondary mathematics experienced teachers, and
- c. secondary mathematics prospective teachers?

### **Context Matters: How Differences in the Context Influence What Teachers Notice**

The third question posed by Sherin et al. (2011b), “how context-specific is noticing expertise” (p. 11), has received less attention by the field than the first two questions. To say noticing expertise is context-specific implies that the context influences teachers' noticing expertise in particular ways. One can look at many contextual factors, such as the school environment (e.g. Levin, Coffey, & Hammer, 2009), content domains (e.g. Walkoe, 2015), or artifacts of student thinking (e.g. Sherin, Linsenmeier, & van Es, 2009). I focus on different

types of artifacts of student thinking, in order to investigate issues related to artifact selection for teacher learning.

### **Context Specificity: Variations in Artifacts of Student Thinking**

One important area for mathematics teacher education researchers to explore is how different artifacts of student thinking influence what teachers notice. Artifacts of student thinking are powerful and common tools that teacher educators use to support teacher learning (Little, Gearhart, Curry, & Kafka, 2003). Part of the reason they are widely used is because making sense of and responding to students' written work is an authentic activity to the practice of teaching (Brown, Collins, & Duguid, 1989; Grossman, Compton, et al., 2009; Jacobs & Philipp, 2004). Hence, teacher educators would benefit from understanding how different artifacts influence what teachers notice. In particular, we might learn what artifacts are easy or difficult to comprehend, and what artifacts might elicit curiosity or excitement from teachers.

Sherin et al. (2009) followed this line of reasoning as they looked for relationships between the student thinking exhibited in video clips and the richness of teacher discussions<sup>1</sup> about those clips. They created a framework for categorizing the video clips along 3 dimensions: number of *windows* into student thinking, *depth* of student thinking exhibited, and *clarity* of the student thinking. I will discuss these three dimensions further later. What's important to note though is that they found that particular relationships between the three dimensions for categorizing video clips supported the teachers' discussions better than other relationships. For example, when there were many windows, a low depth, and a low clarity, this meant there was lots of student thinking present but it was difficult to understand what the student did. In these

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<sup>1</sup> Sherin, Linsenmeier, and van Es's (2009) study did not directly relate to teacher noticing. However, it appears that certain video clips supported teachers to spend more effort making sense of the students' mathematical thinking, which I argue is an important component of noticing.

cases, teachers spent much time discussing the student's thinking, because they were curious to understand what the student was doing. As another example, when teachers watched clips that exhibited high windows, high depth, and high clarity, this meant the student clearly exhibited deep reasoning skills and conceptual understandings. In these cases, teachers also spent much time discussing the student's thinking because they were impressed by the students' thinking. Hence, it appears some artifacts instilled curiosity, while other artifacts instilled excitement, and both supported the surrounding conversations in their own ways. Conversely, when an artifact had a low depth of student thinking and a high amount of clarity, which meant the video showed student thinking that was clear and did not use deeper reasoning skills, teachers did not spend time discussing the student's work, because they were not interested or excited by the student's strategy. Overall, this study provided evidence that certain artifacts are better at supporting conversations than others, and it would be helpful for the field to identify criteria for selecting artifacts.

In my work, I examine how different samples of student thinking afford or constrain teachers' professional noticing of students' mathematical thinking. Rather than focus on video artifacts, I focus on written artifacts of student thinking. My goal is to extend our knowledge of artifact selection, as well as our knowledge about the context-specificity of professional noticing expertise. For these reasons, my second research question is as follows:

***Research question 2.***

(A) What features of written artifacts of student thinking afford or constrain teachers' effectiveness at demonstrating their professional noticing of students' mathematical thinking expertise? (B) What features of written artifacts of student thinking elicit teachers' interest?

## **Chapter 2: Literature Review and Theoretical Framework**

In this chapter, I situate my work in the literature. I organize this chapter into four sections. In the first section, I describe my conceptual framework. This section includes a review of noticing, a description of the professional noticing conceptual framework, distinctions between professional noticing and other teacher noticing constructs, and how professional noticing relates to other teaching practices.

In the second section, I summarize literature about the content domain of proportional reasoning. This summary will serve as an important foundation for understanding the professional noticing of students' mathematical thinking expertise exhibited by participants in my study. Professional noticing of students' mathematical thinking is a domain specific expertise, which means the mathematical content domain plays an important role in how teachers enact this expertise.

In the third and fourth sections, I summarize literature related to my two research questions, respectively. Thus, in the third section I summarize literature related to developmental trajectories of teachers' noticing of students' mathematical thinking. In the fourth section, I summarize literature related to how the context influences what teachers notice in students' ideas. I conclude each section with a summary of how answers to my question contribute to the literature of professional noticing of students' mathematical thinking.

### **Conceptual Framework**

“Clearly one notices all the time. For example, we pin notices on a noticeboard to bring things to people’s attention so that they will notice them” (Mason, 2011, p. 35). In a colloquial sense, to notice means to pay attention to a particular feature, object, or event. Mason’s example illustrates this meaning, as the noticeboard attracts people’s attention, and notices direct their

attention toward specific issues or events. In this section I will argue that noticing is not just an attention-focusing skill, but a situated, sense-making practice that is tied to our knowledge, experiences, identities, and our perceptions of the surrounding activity.

### **Characteristics of Noticing**

Clearly, what we notice depends on what is available in the environment to notice. From this idea, one might assume that noticing is a fixed ability independent of our cognition, and only dependent on what is available to notice in the environment. However, since the latter half of the 20<sup>th</sup> century, much evidence has shown that noticing expertise is not fixed, as knowledge and beliefs, cultural identity, and even the individual's perception of the surrounding activity influence what one notices in a situation. In the subsections that follow, I provide evidence for each.

#### **Noticing is influenced by one's knowledge.**

Evidence our knowledge influences what we notice has been present since the latter half of the 20<sup>th</sup> century, when we began comparing differences between what experts and novices notice in different fields (National Research Council [NRC], 2000). For example, Degroot (1965, as cited in Chase & Simon, 1973) found that an important part of expert chess players' prowess lied in *how* they were seeing the placement of pieces in the chess board. This result went against Degroot's initial hypothesis, which was that expert chess players were able to see and think through more moves or predict much further ahead than novices. It turned out that experts and novices saw the same number of moves at a time, and each predicted only about 2 or 3 moves ahead. Hence, experts and novices exhibited a similar capacity for working memory. Instead, experts noticed specific *configurations* of chess pieces, while novices noticed locations of

individual chess pieces. These configurations were a result of years of experience and knowledge of strategic positions.

The idea that experts notice meaningful patterns is well documented (NRC, 2000). We know that experts in wide range of fields develop, from their experiences and knowledge, “a sensitivity to patterns of meaningful information that are not available to novices” (NRC, 2000, p. 33). In teaching as well, studies show that experts develop robust noticing skills that help them attend to more substantive details (rather than superficial details) than novices, and create accurate interpretations of the situations (Berliner, 1994). Additionally, experts develop a sensitivity for what details need attention now, and what can be ignored for later (Ericsson, 2011), which is not always present in novices.

**Noticing is influenced by one’s participation in a community.**

In addition to being dependent on knowledge and experiences, Goodwin (1994) found that what and how individuals notice is influenced by the communities in which we participate. In his work, Goodwin defined a term called *professional vision*, which refers to “socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group” (p. 606). He examined two distinct professions, lawyers and archaeologists, and found that similar discursive practices were being used to help novice practitioners see what veteran practitioners saw. In essence, being a part of a community (especially being a central member of a community) shapes one’s noticing expertise. For example, a farmer and an archaeologist will see very different things in the same plot of dirt.

In the world of teaching, similar results can be seen as teachers participate in communities such as video clubs (e.g. Sherin & van Es, 2005, 2009; van Es & Sherin, 2008). In these studies, the teachers who participated in these smaller professional communities shifted



their noticing expertise toward noticing the students' mathematical ideas in more robust ways. Hence, by becoming members of different communities, our shifting identities will influence our noticing expertise.

**Noticing is influenced by the surrounding activity.**

In a famous experiment by Simons and Chabris (1999), participants were asked to view a video where boys and girls in black and white t-shirts passed basketballs back and forth. They were asked to count the number of times basketballs were passed back and forth, to count the number of bounce passes and non-bounce passes, or to count the number of passes between black t-shirts and white t-shirts. Researchers did not tell viewers, though, that halfway through the video a man in a gorilla suit was going to walk into view of the camera and stand in the middle of the screen! One might expect these participants to notice such a strange event. However, *because the participants were focused on the activity of counting*, 56% failed to notice the unexpected event. Hence, the activity we are engaged in, and our perception of the activity, influences what we notice.

In the profession of teaching, we have evidence that teachers' perceptions of the activity in which they are engaged will influence what teachers notice and respond to (Schoenfeld, 2011). For example, Battey and Franke (2008) investigated the teaching practices of a former typing teacher who had been a part of a professional development aimed at supporting students' algebraic reasoning in the elementary grades. She believed that mathematical activities should focus on finishing problems quickly, and should involve lots of drilling. Hence, when PD leaders engaged the teachers in a rich problem solving activity, she did not engage in the same reasoning skills as her peers, and focused on finding the formula faster than her peers. Consequently, when she led the activity with her students, her perception of the activity focused

her actions on directing students to solve the problem her way, rather than letting students develop their own strategies. The authors even noted that many students had developed correct strategies for solving the problem, but were told they were doing it wrong because their strategies did not align with the teacher's way. Her perceptions of the activity, namely that the activity involves applying the formula quickly, caused her to see different things in the students' strategies than what the authors saw.

### **Summary: Characteristics of Noticing.**

In sum, what we notice is fundamentally tied to the objects in the environment, our own knowledge, beliefs, and experiences, our cultural identities, and the surrounding activity. This is true for members of any profession, including teachers. In the next section, I consider the field of teacher noticing, and discuss the different conceptualizations.

### **Teacher Noticing: An Overview.**

Teacher noticing generally involves the teacher attending to important features of a situation and making sense of what is happening (Sherin & van Es, 2005). However, researchers have used the concept of teacher noticing in various ways, sometimes making it difficult to see how all the studies on teacher noticing relate to each other (Jacobs & Spangler, 2017). Hence, in the following paragraphs I give a brief overview of how researchers have studied the phenomenon of teacher noticing.

In the field of teacher noticing, there exists several different conceptualizations (Sherin, Russ, & Colestock, 2011). One can focus solely on teachers' attending skills (e.g. Star & Strickland, 2008). Star and Strickland argued that teachers cannot make sense of videos of teaching if they do not have the ability to pay attention to specific features. Thus, attending is an important requisite skill for observing classrooms and viewing videos of teaching. In addition,

one can focus on teachers' attending *and* sense-making skills (e.g. Sherin & van Es, 2005). This is a useful conceptualization for studying teacher noticing, and captures how teachers make sense of their environment in many different scenarios, including while watching videos of their practice (Sherin & van Es, 2009), considering written artifacts of student thinking (Fernandez et al., 2012), or even in the moment of teaching (Sherin, Russ, et al., 2011). Taking this one step further, Jacobs et al. (2010) added a third component: making decisions for how to respond. In this conceptualization, Jacobs et al. focus on teachers' in-the-moment decision, rather than decisions while reflecting on practice. Alternatively, Santagata (2011) studied teacher noticing by analyzing how teachers attended to key details, interpreted what happened, and generated new knowledge for future reference. This conceptualization focuses on teachers' reflections on their practice, rather than their in-the-moment decisions. Notice that each of these conceptualizations can be used not only to capture different aspects of teachers' attention and sense-making skills, but also to capture teachers' noticing expertise in different settings, and for different purposes. For example, Santagata (2011) focused on using noticing as a reflection tool for analyzing and modifying lessons. In contrast, Jacobs et al. (2010) focused on in-the-moment noticing, when teachers have to make sense of students' ideas and make decisions about next instructional steps during the lesson.

Within the various conceptualizations, researchers can also narrow their focus to *theme-specific* noticing (Dreher & Kuntze, 2015). In other words, some researchers narrow their focus to study how teachers notice a specific idea or theme. These themes have included focusing specifically on children's mathematical thinking (Jacobs et al., 2010; Schack et al., 2013), on equitable practices (Hand, 2012), on children's funds of knowledge (McDuffie et al., 2014); or on the use of multiple representations (Dreher & Kuntze, 2015). However, choosing a specific

theme is not a requirement to studying what teachers notice. Some researchers choose not to study teacher noticing of a particular theme, which allows them to measure general shifts in what teachers notice (e.g. Sherin & van Es, 2005; Star & Strickland, 2008).

In my dissertation, I focus on professional noticing of students' mathematical thinking (Jacobs et al., 2010). This particular construct includes the three component-skills of attending, interpreting, and deciding how to respond. In addition, professional noticing of students' mathematical thinking happens in-the-moment of instruction, rather than while reflecting on instruction, and focuses on students' mathematical thinking. In the next section, I will describe this construct in more detail, and provide a rationale for selecting this construct for further study.

### **Professional Noticing of Students' Mathematical Thinking**

Professional noticing of students' mathematical thinking is a teaching practice consisting of three component-skills. When a teacher considers a student's mathematical idea during instruction, the teacher (a) attends to specific details of the student's mathematical idea, (b) interprets the student's mathematical understandings, and (c) decides how to respond to the student in a way that supports and builds on the student's mathematical understandings (Jacobs et al., 2010). With this construct, Jacobs and her colleagues intended to capture the teacher's in-the-moment cognitive processes that happen in the instant after a student shares a mathematical idea, and before the teacher responds to this mathematical idea. As such, they defined this construct as one integrated, improvisational practice, where the three component-skills are interrelated and happen simultaneously. In this section I will describe each component-skills in greater detail.

#### **Attending.**

Attending refers to the practice of paying attention to particular details. In the context of professional noticing of student's mathematical thinking, researchers are interested in whether the teacher attended to the important mathematical details of a strategy, and the specificity of this attention (Jacobs et al., 2010). This practice also relates to selective attention (Sherin & van Es, 2009), highlighting (Goodwin, 1994), and making call outs (Frederikson, Sipusic, Sherin, & Wolfe, 1998). An important part of this component-skill is being able to attend to relevant details while ignoring extraneous details (Erickson, 2011), which means knowledge about which details to search for is crucial to developing proficiency in this expertise. This knowledge is similar to knowledge about which patterns to search for, much like how experts in other professions look for useful patterns (NRC, 2000).

### **Interpreting.**

Interpreting refers to the practice of making sense of a student's mathematical ideas. Sherin and van Es (2009) refer to this as knowledge-based reasoning, in which the teacher reasons about what the child understands mathematically. As Jacobs and her colleagues (2010) point out, we cannot expect a teacher to make complete sense of a student's mathematical ideas from just one sample. However, we can still learn many things about the teacher's ability to interpret. For example, we can identify what kind of stance they take as they make sense of a student's ideas (Sherin & van Es, 2009). Is the teacher simply sharing observable details? Is the teacher jumping straight to a judgment about whether the ideas were good or bad? Or is the teacher generating an account of what kind of mathematical understandings the student might have? Ideally, we would like teachers to be cautious about their conjectures but able to generate thoughtful and specific interpretations of what mathematical understandings the student might have. In addition, Jacobs et al. (2010) note that teachers should be consistent with both what the

student did in the problem, and what the literature says about students' mathematical thinking in that particular content domain.

### **Deciding how to Respond.**

When Jacobs and her colleagues created the construct of professional noticing of students' mathematical thinking, they included the third component-skill, that of deciding how to respond, because they wanted to investigate to what extent teachers based decisions on what they interpreted from the student's mathematical understandings, and how consistent they were with the literature on children's mathematical thinking in the domain of whole number operations. In their study, Jacobs and her colleagues found that this third component-skill was the most difficult skill for teachers to develop. A large majority of experienced teachers who had not had sustained professional development around students' mathematical thinking did not base their decisions on the student's ideas. Additionally, even with two years of sustained professional development most teachers still only provided limited evidence of considering the students' mathematical understandings when making a decision about how to respond. Hence, making decisions based on the students' mathematical understandings is far from a trivial task, and takes much practice, knowledge, and effort.

The third component-skill is useful for understanding what resources teachers bring to professional development as well (Jacobs et al., 2011). Teachers are predisposed to act in certain ways (Ericsson, 2011; Jacobs, Lamb, Philipp, & Schappelle, 2011). For example, Jacobs et al. (2011) noticed that many prospective primary teachers based decisions on making the student feel more comfortable, which seemed to stem from their disposition to care for students. Because the three component-skills are interrelated, such dispositions may influence what they attend to, interpret, and how they make their decisions. Thus, this component-skill may also be

useful for understanding teachers' pedagogical dispositions when engaging with students' mathematical ideas, and help researchers see what ideas teachers are sensitive to and what sense they make of the students' ideas (Jacobs et al., 2011).

**Identifying what does and does not count as professional noticing of students' mathematical thinking.**

Much of what teachers do is related to making sense of students' mathematical thinking and deciding how to respond during instruction. Hence, I clarify what counts as professional noticing of students' mathematical thinking, and what does not. First, professional noticing of students' mathematical thinking is an in-the-moment expertise<sup>2</sup> (Jacobs et al., 2010). Therefore, I am not considering how teachers make sense of students' mathematical thinking and make decisions about how to respond during planning or reflection phases (c.f. Amador et al., 2017; Santagata, 2011). Additionally, this construct does not include the actual response of the teacher, just the intended response<sup>3</sup>. The actual response is left out intentionally for two reasons. First, it is left out because the actual response may require other skills beyond the decision-making process in order to successfully enact. Second, it is left out because if one were to include the teacher's response in the construct of professional noticing, then professional noticing would run the risk of encompassing almost everything the teacher does during a lesson. Finally, this construct includes the goal of clarifying, building on, or extending the student's mathematical thinking. Hence, this construct does not include decisions about responding for the purpose of supporting productive norms or managing the classroom, even though these goals may have an

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<sup>2</sup> I am not claiming that this construct cannot be used for other purposes. For example, Choppin (2011) used this construct to understand how teachers modified lessons. However, this is the way Jacobs and her colleagues (2010) originally defined the construct, and also how I intend to use this construct.

<sup>3</sup> This point is important because this nuance risks being lost, as some researchers call it a "respond" skill rather than a "decide" skill. (c.f. Simpson & Haltiwanger, 2016).

indirect effect of supporting students' mathematical thinking. When a teacher makes a decision with a norm or management goal in mind, we do not consider this to be a decision based on the student's mathematical understandings.

### **Situating Professional Noticing of Students' Mathematical Thinking in the Practice of Teaching**

In this section, I will situate professional noticing of students' mathematical thinking in the practice of teaching. In particular, professional noticing of students' mathematical thinking is not just a fundamental component of teaching responsively, it also meets the criteria to be *core practice* of teaching (McDonald, Kazemi, & Kavanaugh, 2013; Schack et al., 2013). I first define what I mean by a practice, and in particular what I mean by a core practice and practice-based teacher education. Then, I discuss why professional noticing of students' mathematical thinking could be considered a core practice, and discuss how professional noticing fits in with other core practices of teaching.

#### **Defining a practice.**

Historically, teacher education programs have struggled to identify common language (Grossman & McDonald, 2008). To address this issue, I define what I mean by a practice and contrast this definition against other definitions of practice. When I use the term 'practice', I refer to the application of one's beliefs, knowledge, and identity for the purpose of engaging in a particular activity. In contrast, a mother might drop her child off at soccer practice, which is a regularly scheduled event where children engage in activities that support their soccer abilities. Alternatively, a lawyer with an entrepreneurial spirit might start her own legal practice. In this case, the word 'practice' refers to a business. Additionally, the word practice might refer to a profession, as in the term medical practice. My use of the word practice does not refer to these



notions though; I intend for the word to refer to the *application* of knowledge for engaging in an activity. For example, mathematicians often engage in the practice of problem-solving. Writers engage in the practice of editing. Teachers engage in the practice of facilitating group work. Hence, the practice of noticing refers to the application of knowledge to notice and make sense of what is happening in a particular situation.

### **Defining a core practice.**

Recently, a small (but growing) group of mathematics teacher education researchers has begun identifying *core practices* (also called *high-leverage practices*), such as eliciting students' ideas or leading whole class discussions, and centering teacher education on learning about and practicing these core practices (Grossman, Hammerness, et al., 2009; McDonald et al., 2013; Stein et al., 2008). By centering teacher education on core practices, their goal is to bridge the gap between theory and practice, which has been a difficult bridge to build (Cook & Brown, 1999; McDonald et al., 2013).

In order to be a core practice, the teaching practice must (a) occur with high frequency in teaching, (b) occur in different curricula or instructional approaches (c) be accessible to novices (d) allow novices to learn more about students and about teaching (e) preserve the integrity and complexity of teaching, and (f) be research-based and have the potential to improve student achievement (Grossman, Hammerness et al., 2009). Hence, authors such as McDonald et al. (2013) use the term practice-based teacher education to refer to pedagogies where program leaders select a set of core practices as learning goals for student-teachers, and base instruction around supporting student-teachers to learn and practice these core practices. In this use of the term, practice-based teacher education does not refer to teacher education programs where the credential classes are set inside K-12 classes, or teacher education programs that focus on

providing more opportunities for student-teachers to work with guide teachers in some kind of apprenticeship model (Forzani, 2014). The focus is not on whether teachers get more opportunities to practice, or to set instruction in the actual practice of teaching, but to practice these core practices of teaching.

**Professional noticing of students' mathematical thinking is a core practice.**

I will now argue that professional noticing of students' mathematical thinking is not only a practice, but a *core* practice. I do this by providing evidence that professional noticing of students' mathematical thinking satisfies each criteria provided by Grossman, Hammerness et al. (2009). As described earlier, professional noticing of students' mathematical thinking happens almost constantly when teaching responsively and centering instruction on students' ideas, regardless of the content-domain or task (Forzani, 2014; Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010). This is because in ambitious teaching, many students are sharing ideas, and thus the teacher must make sense of students' ideas quickly and productively. It follows then that professional noticing of students' mathematical thinking preserves the integrity and complexity of teaching, because making sense of students' ideas is a fundamental part of teaching responsively, and students' ideas can be quite complex (Ball, 1995). Additionally, we have evidence that professional noticing of students' mathematical thinking can be enacted and learned by novices, and supports novices to learn about students' mathematical thinking and teaching in different domains (Lesseig et al., 2016; Schack et al., 2013). Furthermore, there is evidence that improvement in professional noticing of students' mathematical thinking impacts teacher practice in positive ways (Choppin, 2011; Sherin & van Es, 2009<sup>4</sup>). Therefore, we have

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<sup>4</sup> Sherin and van Es (2009) found improvements in teachers' practice while studying teacher noticing, rather than professional noticing of students' mathematical thinking. However, there is much overlap between the two constructs.

evidence professional noticing of students' mathematical thinking satisfies each of the six criteria.

### **Relation to other teaching practices.**

When one decomposes the practice of teaching, Grossman, Compton, and colleagues (2009) warn that novices may be unsure how all the pieces fit together within the practice of teaching. Hence, I clarify how the practice of professional noticing of students' mathematical thinking fits with other practices.

The distinction I focus on is that of grain-size (Ball & Forzani, 2011). To illustrate, consider the practice of professional noticing of students' mathematical thinking and the practice of facilitating a whole class discussion (Stein et al., 2008). Professional noticing of students' mathematical thinking happens almost instantaneously as a teacher considers a student's idea. Additionally, it happens multiple, maybe even hundreds of times within a lesson where many students are sharing or where the teacher monitors the student's thinking. In contrast, facilitating a whole class discussion generally takes a significant portion of time. A whole class discussion generally happens once during a lesson, and takes much preparation, and involves many smaller practices such as eliciting students' ideas, modeling students' thinking, and even professionally noticing the students' mathematical thinking. Hence, professional noticing of students' mathematical thinking has a very small grain size, while facilitating a whole class discussion has a large grain size. Additionally, in whole class discussions teachers must make sense of students' thinking many times, thus professional noticing of students' mathematical thinking is an important component of the practice of facilitating a whole class discussion. Being a small grain size, professional noticing is a component of many larger practices.

### **Conceptual Framework Summary**

In sum, professional noticing of students' mathematical thinking is an in-the-moment teaching practice that happens in the instant after a student shares a mathematical idea and before the teacher responds. It is composed of the teacher's sense-making and decision-making skills. Additionally, being of a small grain size, it is a component-practice of larger teaching practices. In the next section, I describe the mathematical content domain that serves as the context of my study, proportional reasoning.

## **Proportional Reasoning**

### **Rationale for Studying Proportional Reasoning**

I am examining the professional noticing of students' mathematical thinking expertise in the domain of proportional reasoning because of its importance for student learning of mathematics. Lesh, Post, and Behr (1988) described proportional reasoning as a "capstone" of elementary mathematics, because it involves many conceptually difficult types of reasoning covered in elementary grades, such as second-order relationships (i.e. a relationship between two relationships), multiplicative comparisons, part-whole relationships, and composite units, among others. Additionally, Lesh and his colleagues (1988) described proportional reasoning as a "cornerstone" of higher-level mathematics, because proportional reasoning concepts are fundamental foundations for sophisticated domains such as functional reasoning, probability, and trigonometry. Karplus, Pulos, and Stage (1983) point out that proportional reasoning is an important concept in many scientific domains as well. Hence, proportional reasoning is both a topic that spans many grade levels, and an important transition point in developing one's reasoning skills (Inhelder & Piaget, 1958; Lesh et al., 1988). Thus, proportional reasoning is an important type of mathematical reasoning for teachers to support in their students.

In the sections that follow, I summarize important concepts related to proportional reasoning, for the purpose of situating teachers' professional noticing of students' mathematical thinking in the domain of proportional reasoning. In other words, I describe proportional reasoning concepts that might help me understand how teachers attend to the details of a student's solution, how teachers interpret the student's mathematical understandings, and how teachers decide how to respond in a way that supports the student's proportional reasoning skills.

I begin by describing the foundation for proportional reasoning: multiplicative reasoning. This foundation allows me to define important concepts such as ratios, equivalence, and types of multiplicative relationships between ratios. I then define what I refer to when I talk about proportional reasoning, which is actually a smaller subset of what the literature defines as proportional reasoning (Lamon, 2007). I choose this subset because I focus exclusively on missing-value tasks, so I then describe missing-value tasks and associated task-related-variables. Finally, I describe strategies and understandings students exhibit as they engage in these types of tasks, as well as possible developmental trajectories of students' mathematical thinking in these tasks. By identifying these strategies, understandings, and trajectories, I create a foundation for understanding teachers' attending, interpreting, and deciding how to respond practices.

### **Multiplicative Reasoning**

Proportional reasoning is fundamentally tied to the domain of multiplicative reasoning (Lamon, 2007; Vergnaud, 1988). Multiplicative concepts, as Vergnaud (1988) points out, encompass many related concepts, including (but not limited to) fractions, ratios, rates, rational numbers, division, multiplication, proportional reasoning, and linear functions. Additionally, Vergnaud (1988) pointed out that these multiplicative concepts are highly interrelated. Hence, I will elaborate on what I mean by multiplicative reasoning.

Multiplicative reasoning can be defined in many ways. Confrey (1994) defined multiplicative reasoning as a concept of *splitting*, where one simultaneously creates multiple copies of an original. This definition is based on ideas surrounding exponential functions. When illustrating this idea, Confrey related the concept of splitting to the creation of a tree diagram. For example, in a tree with 3 branches, one can split each branch into four more branches, thus demonstrating the concept of  $3 \times 4$ .

Alternatively, Steffe (1988, 1994) defined multiplicative reasoning as the coordination of two composite units such that one composite unit is distributed across the other composite unit. A *composite unit* is simply a unit composed of other units. This definition is based on children's counting schemes (Steffe, 1994). For example, the concept of  $2 \times 3$  can be thought of as treating the counting sequence of 1, 2, 3, as a unit, thus creating a composite unit, and distributing this counting sequence across the counting sequence of 2, in order to find the product of 6. Notice that in distributing the 3 across the 2, one is distributing the composite unit of counting to 3 across the composite unit of counting to 2.

In my work, as with others who study proportional reasoning (e.g. Singh, 2000), I draw upon Steffe's (1994) definition of multiplicative reasoning. That is, multiplicative reasoning is the formation and distribution of one composite unit across another composite unit. I take this stance because the formation of composite units highlights an important aspect of proportional reasoning, that of creating composed units (Lobato & Ellis, 2010). When thinking about multiplication, we necessarily must think about the unit each number refers to as we iterate or partition that unit (Lamon, 1993; Steffe, 1994).

### **Ratios and Multiplicative Reasoning**

With Steffe's (1994) definition of multiplicative reasoning, I can now define a ratio. A *ratio* is a multiplicative comparison between two numbers (Lobato, & Ellis, 2010). By *multiplicative comparison*, I mean that the numbers are compared to each other in terms of composite units *of the other number* (Kaput & West, 1994; Steffe, 1994). Lamon (1993) labeled this ability to see one quantity in terms of another quantity as *relative* reasoning, and claimed this was an essential component to proportional reasoning. Hence, being able to see a ratio, not just as two related numbers, but as a multiplicative comparison between the two numbers, is fundamental.

To illustrate a multiplicative comparison, consider the ratio 4:7 (where this notation is read "four to seven"). Let us multiplicatively compare 4 to 7. To compare 4 to 7 multiplicatively, we would first need to form a composite unit of 7 units. Let us call this a 7-unit. Then, we would ask the question "How many 7-units do I need to make the number 4?" Notice that we need  $\frac{4}{7}$  of a 7-unit, where  $\frac{4}{7}$  is a rational number. In other words, 4 is  $\frac{4}{7}$  of 7. Conversely, if we consider the ratio 7:4, we note that 7 is  $\frac{7}{4}$  of 4.

### **Important Multiplicative Relationships: Scalar and Functional**

Two ratios are said to be *equivalent* if the two ratios exhibit the same multiplicative comparisons. For example, 5:10 and 20:40 both exhibit the same multiplicative comparisons, because 5 is  $\frac{1}{2}$  of 10, and 20 is  $\frac{1}{2}$  of 40. Because  $\frac{1}{2} = \frac{1}{2}$ , the two multiplicative comparisons are equivalent.

Notice that when two ratios are equivalent, there are two sets of multiplicative relationships one can notice (figure 2). First, one can notice the multiplicative relationship *within* a ratio, which may sometimes be referred to as a *functional* perspective (Carney, Hughes, Brendefur, Crawford, & Totorica, 2015; Lamon, 2007; Tourniaire & Pulos, 1985). For example,

in figure 2b one can see a solution to a missing-value task that leverages the functional multiplicative ratio. This solution leverages the multiplicative relationship between 6 cookies and 2 dollars, specifically that 6 is 3 times as big as 2. In other words, this solution makes use of the functional relationship between cookies and dollars. In my work, I refer to the factor 3 as the *functional* factor. Notice that for each ratio, depending on the comparison we wish to make, there are two functional factors one can use. In figure 2b, 3 is the functional factor because 6 is 3 times as large as 2. However, one can also see that 2 is 1/3 as big as 6. Thus, 1/3 and 3 are both reflective of the multiplicative relationship within the two ratios.

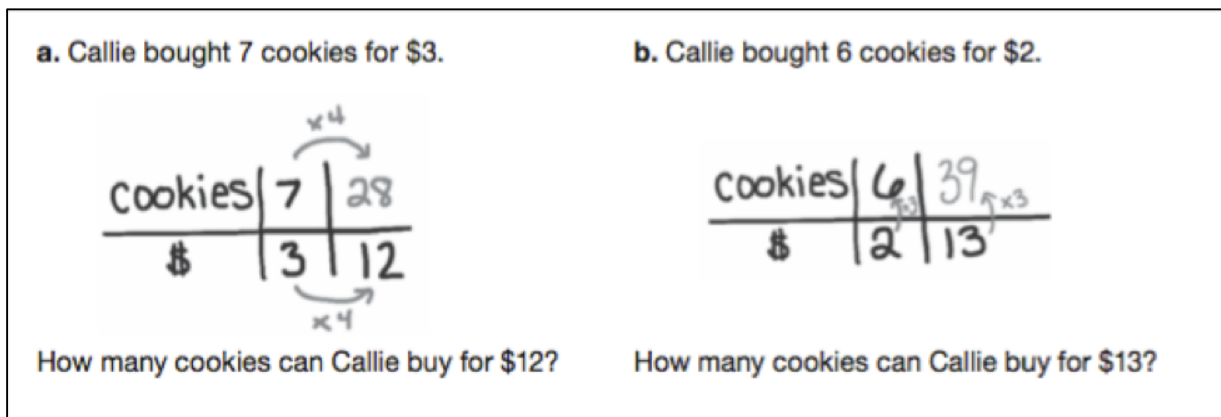


Figure 2.1. Ratio table depicting scalar and functional multiplicative relationships (Carney et al., 2015, p. 211).

Second, one can notice the multiplicative relationship *between* two ratios, which has been referred to as a *scalar* perspective (Carney et al., 2015; Lamon, 2007; Tourniaire & Pulos, 1985). For example, consider figure 2a. In this example, the solution leverages the multiplicative relationship between dollars in the first ratio and dollars in the second ratio. With equivalent ratios, the multiplicative relationship will be the same regardless of whether we compare the values of the first quantity (dollars) or of the second quantity (cookies). Thus, this solution leverages the fact that 12 dollars is 4 times as large as 3 dollars, and uses this multiplicative relationship for the cookies. Another way to say this is that the values in the second ratio are



four times as large as the values in the first ratio. In my work, I will refer to the factor 4 as the *scalar* factor. Notice that, similar to functional factors, we can identify two scalar factors here as well, because 12 is 4 times as large as 3, and 3 is  $\frac{1}{4}$  of 12. Thus 4 and  $\frac{1}{4}$  are both reflective of the multiplicative relationship between the two ratios.

### **Defining Proportional Reasoning**

Defining what does and does not count as proportional reasoning can be tricky because the content domain of proportional reasoning appears across many grade levels and in many other mathematical domains. To address this issue, Lamon (2007) offered the following definition as an umbrella definition for all studies. She proposed to define proportional reasoning as “supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products” (p. 637 – 638). In this way, Lamon (2007) attempted to include instances of multiplicative reasoning with all kinds of tasks, including tasks where products remain invariant (such as the balance beam task, Inhelder & Piaget, 1958). However, I only refer to one kind of proportional reasoning task: missing-value tasks (Kaput & West, 1994). Hence, I can significantly narrow what I mean when I discuss proportional reasoning. In my work, I define *proportional reasoning* as reasoning that supports the comprehension of scalar and functional multiplicative relationships between equivalent ratios when solving missing-value tasks. In the next section, I explain why I choose to focus on missing value tasks, and I describe variations of missing value tasks.

### **Missing-Value Tasks**

Lesh and his colleagues (1988) identified three kinds of tasks that were most common in the research base and in school curricula on proportional reasoning: (a) missing-value tasks, (b)

comparison tasks, and (c) transformation tasks. In my work I focus on teachers' professional noticing of students' solutions to missing-value tasks (Kaput & West, 1994). I focus on this kind of task for three reasons. First, there exists frameworks of student thinking and possible learning trajectories surrounding this task (Carney et al., 2015; Steinhorsdottir & Sriraman, 2009). This allows me to better summarize related concepts, select artifacts, and relate teachers' responses to the literature. Second, it is a common task, which means many teachers will be familiar with the task. In fact, it is such a common that a missing-value task even appears as one of the common core state standards: "Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were the lawns mowed?" (NGA/CCSSO, 2010, CCSS.MATH.CONTENT.6.RP.A.3.B). Finally, I focus on this task because many researchers have documented different types of strategies related to this task, which will support my own artifact selection.

There are two main ways one can modify this task. First, one might modify the context (e.g. Lamon, 1993; Tourniaire & Pulos, 1985). According to Tourniaire and Pulos's (1985) literature review, using contexts that students are familiar with better supports students' proportional reasoning than unfamiliar contexts, especially when students do not know what the quantities are. Additionally, studies have indicated that mixture problems, which involve continuous quantities, are more difficult for children to comprehend than non-mixture problems (i.e. discrete quantities).

Lamon (1993) added to the literature by comparing students' responses to four different semantic types of proportional reasoning problems: well-chunked measure, part-part-whole, associated sets, and stretcher/shrinker tasks. Well-chunked measure tasks involve contexts

where the ratio is a quantity that is well known by students. For example, speed, the ratio of distance per time, is a well known quantity. Part-part-whole problems include a ratio such that the sum of the two parts make a whole. For example, a classroom with 18 boys and 12 girls, so that the ratio of boys to girls is 18 to 12, is one such type. Associated sets are the opposite of well-chunked measures, in that an associated sets problem includes a ratio such that the ratio does not represent a well known quantity (e.g. miles per hour). For example, if 2 caterpillars eat 5 leaves, then the ratio of caterpillars to leaves is not a well-known quantity, and we call it an associated set. Finally, stretcher-shrinker problems are proportional reasoning problems where one is asked to multiplicatively compare the sizes of objects. In her study, Lamon found that the semantic type of associated sets supported students to use proportional reasoning more so than any other semantic type. When children in her study responded to the other three semantic types, they tended to use strategies that did not involve proportional reasoning.

Second, one might modify the numbers used in the task (Carney, Smith, Hughes, Brendefur, & Crawford, 2016; Steinhorsdottir & Sriraman, 2009; Tourniaire & Pulos, 1985). In particular, problems with integer ratios seem to make proportional reasoning problems much easier for students than tasks with non-integer ratio (Tourniaire & Pulos, 1985). An integer ratio is a ratio such that the multiplicative comparison involves integers. For example, the ratio of 6:2 is an integer ratio because 6 is 3 times as big as 2. However, the ratio 5:2 is not an integer ratio because 5 is 2.5 times as big as 2. According to Steinhorsdottir and Sriraman (2009), questions with integer ratios where the student iterates by a whole number are the easiest, while questions with integer ratios where the student must partition by a whole number are slightly more difficult, followed by tasks with non-integer ratios being the most difficult. Carney et al. (2016) also point out that one can modify problems based on which multiplicative relationships (i.e.

scalar and functional multiplicative relationships) have an integer ratio and which have a non-integer ratio. For example, in figure 2a the functional ratio is a non-integer ratio, while the scalar ratio is an integer ratio, while in figure 2b the functional ratio is an integer ratio while the scalar ratio is a non-integer ratio. Finally, Tourniaire and Pulos (1985) add on that one can always increase the numbers to larger numbers, say beyond 100, to increase the difficulty.

Teachers who wish to respond to and build on students' ideas need such knowledge about the different kinds of tasks and their associated difficulties. Thus, in supporting students' proportional reasoning, this section emphasizes that teachers should pay attention to the kind of context they propose as well as the selection of numbers for the next task they pose.

### **Students' Strategies in Response to Missing-Value Proportional Reasoning Tasks**

Researchers have documented five common types of strategies to proportional reasoning tasks: additive, scalar additive, scalar multiplicative, functional, and cross multiplication (Carney et al., 2015). In this section, I provide a description of each type of strategy.

An additive strategy refers to adding and subtracting equal values in order to solve (Carney et al., 2015). When a child uses an additive strategy to solve a proportional reasoning task, it is because the child has not fully comprehended the multiplicative structure of a ratio. This is also evidence that the child has not joined the two quantities into a composed unit, nor has the child made sense of the multiplicative relationship within a ratio. However, the child understands that because one value is higher, the other value should be higher too.

Scalar additive is the second kind of strategy (Carney et al., 2015). In a scalar additive strategy, the student has joined the two values of the ratio into a composed unit (Lobato & Ellis, 2010), and iterates the composed unit in order to reach the desired ratio. It is called scalar because the student is utilizing the multiplicative comparison between ratios, rather than the

functional multiplicative relationship. However, the student has not fully grasped the multiplicative relationship between ratios, because the student iterates (i.e. sums) equivalent ratios until reaching the desired ratio without predicting how many times he or she must iterate. Hence, it is scalar *additive*. This strategy also includes the ability to partition a ratio, and to add or subtract partitioned ratios to/from larger equivalent ratios.

Scalar multiplicative is the third kind of strategy (Carney et al., 2015). In a scalar multiplicative strategy, the student makes use of the between-ratio multiplicative relationship, much like a scalar additive strategy. However, unlike a scalar additive strategy, the student has comprehended the multiplicative relationship between ratios, and can leverage this to find a factor such that the product of the ratio and the factor gives the desired ratio. A student exhibiting this type of thinking may or may not have developed a concept of the functional multiplicative relationship.

Functional is the fourth kind of strategy (Carney et al., 2015). In a functional strategy, the student has comprehended the multiplicative relationship within a ratio, and can leverage this to find a factor such that the value of one quantity is the product of the factor and the value of the other quantity. Like the previous strategy, a student exhibiting this type of thinking may or may not have developed a concept of the scalar multiplicative relationship. The unit rate strategy is a common functional strategy, in which a student divides one quantity by the other in order to find the unit rate, and then multiplies the unit rate by the third quantity (Singh, 2000).

Before I move on to the final strategy, I must add that Steinhorsdottir and Sriraman (2009) found evidence that even if one develops a scalar additive, scalar multiplicative, or functional strategy, it does not mean one can use these strategies for any choice of number. They found sub-levels of sophistication in students' proportional reasoning. For example, if a student

develops a scalar additive strategy, the student at first can only use this strategy for integer ratios that only involve iteration. Later, the student develops ability to use an additive strategy to partition ratios, but only if the ratio is such that the student can partition by a whole number. Finally, the child becomes able to iterate and partition, and thus can use this strategy for non-integer ratios as well. This was true for scalar multiplicative and functional strategies as well.

The final common strategy is the standard cross-multiplication strategy (Carney et al., 2015). In a cross multiplication strategy, one converts the two ratios into fractions where one quantity is designated to be in the denominator (e.g. cookies) and the other quantity is designated to be in the numerator (e.g. dollars). A variable is used for the missing value. Then, because the ratios are equivalent, the fractions must be equivalent as well, so the student sets the two fractions equal to each other, and solves the equation. However, as much research shows, many children learn this strategy without developing an understanding of the multiplicative relationships (Cramer & Post, 1993).

At this point, according to the previous discussion of proportional reasoning strategies, one might say that proficiency in proportional reasoning would involve being able to comprehend and leverage both scalar and functional multiplicative relationships to solve missing-value tasks. Carney et al. (2015) add one more criteria for proficiency. They say proficiency should also involve being able to identify when one should use a scalar or functional multiplicative relationship to solve the problem. For example, in figure 2a, the between ratio is easier to identify. In contrast, in figure 2b, the within ratio is easier to identify. Being able to predict which ratio to use in a particular task demonstrates a sophisticated understanding of proportional reasoning.

Teachers who wish to respond to and build on students' mathematical ideas need knowledge about the different strategies students might use. With such knowledge, they can diagnose what kinds of understandings the student may or may not have. Additionally, they can predict what strategy a student might develop next, and select tasks appropriate for supporting and developing the student's proportional reasoning.

### **Summary**

Proportional reasoning is a complex and important mathematical domain. In the previous sections, I discussed foundational concepts, important definitions and objects, tasks, strategies, and relationships within the domain of proportional reasoning. To attend to key features of a student's strategy, interpret the student's understandings, and decide how to respond on the basis of the student's understanding in the domain of proportional reasoning, a teacher likely needs a basic understanding of the concepts discussed above.

### **Relevant results related to research question 1: Developmental trajectories of noticing.**

In the previous sections I identified my theoretical perspective and conceptual framework, and discussed salient concepts of students' proportional reasoning. In the rest of this chapter, I will summarize and synthesize results related to my research questions in order to situate my work in the noticing literature. This section will pertain to my first research question, which was "What differences and similarities exist in the professional noticing of students' mathematical thinking in the domain of proportional reasoning among (a) secondary mathematics emerging teacher leaders, (b) secondary mathematics experienced teachers, and (c) secondary mathematics prospective teachers?"

As explained earlier, much of the research on noticing of students' mathematical thinking (which includes professional noticing of students' mathematical thinking) has focused on

uncovering and understanding possible progressions in teachers' noticing expertise (e.g. Callejo & Zapatera, 2016; Fernandes et al., 2012; Jacobs et al., 2010; Lesseig et al., 2016; McDuffie et al., 2014; Schack et al., 2013; van Es, 2011). Through learning about these progressions, researchers have uncovered frameworks depicting different levels of noticing, activities that support teachers to improve their noticing, and indications that teachers are improving in their noticing.

At the moment, few studies of *professional noticing* of students' mathematical thinking exist. Instead, many studies look at teachers' *noticing* of students' mathematical thinking. The conceptualizations of noticing and professional noticing have much overlap though, because both involve attending to and interpreting students' mathematical ideas. Therefore, to develop a better understanding of teacher's *professional noticing* of students' mathematical thinking, in this section I will include studies on teachers' *noticing* of students' mathematical thinking.

This section will be organized into four sub-sections. In the first two sub-section I summarize and synthesize results around studies of prospective teachers' noticing of students' mathematical thinking, and then professional noticing of students' mathematical thinking. In the second sub-section I summarize and synthesize results around studies of practicing teachers' noticing of students' mathematical thinking, and finally of their professional noticing of students' mathematical thinking.

### **Prospective Teachers' Noticing of Students' Mathematical Thinking**

Most of the literature on noticing of students' mathematical thinking has focused on prospective teachers. This is an important group of participants to study not just because studying how we can prepare teachers for the profession is important, but also because for many teachers the credential program may be the only form of sustained professional development



they receive in their entire teaching career (Wilson & Berne, 1999). And, as Jacobs et al. (2010) point out, the development of all three component-skills of professional noticing expertise for most teachers requires sustained professional development, which is partially possible in longer credential programs.

Studies on prospective teachers' noticing of students' mathematical thinking has covered many domains and stances. Researchers have identified different frameworks of teachers noticing of students' mathematical thinking in the domains of derivatives (Sanchez-Matamoros et al., 2014), the transition from additive to multiplicative reasoning (Fernandez et al., 2012, 2013), early numeracy (Schack et al., 2013), algebraic thinking (Walkoe, 2015), solving equations (Lesseig et al., 2016), and young children's pattern generalization thinking (Callejo & Zapatera, 2016). These studies have shown that prospective teachers can learn to attend to specific mathematical concepts in the students' solutions, though this does not always transfer to interpreting the students' mathematical concepts.

Alternatively, van Es and Sherin (2002) analyzed prospective teachers' noticing skills in general, and assessed their stances as they noticed the students' ideas, including whether their comments were evaluative or interpretive, the degree of specificity teachers provided, or the extent to which teachers considered the student's thinking. Results indicated that the activities supported the prospective teachers to develop stances for attending to more details of the student's solutions and developing interpretive accounts of the students' understandings.

Many of these studies have shown that both secondary and primary prospective teachers can learn to attend to students' mathematical thinking, and sometimes do so relatively quickly (Callejo & Zapatera, 2016; Fernandez et al., 2012, 2013; Lesseig et al., 2016; McDuffie et al., 2014; Sanchez-Matamoros et al., 2014; Schack et al., 2013; van Es & Sherin, 2002; Walkoe,

2015). Improvements in attending skills include attending to more mathematical details (e.g. Fernandez et al., 2012), providing more detailed descriptions (e.g. Walkoe, 2015), or attending to more significant moments of mathematical instruction (e.g. van Es & Sherin, 2002).

These studies highlight many different kinds of activities that mathematics teacher educators use to support teachers' attending skills. Such activities include engaging the prospective teachers in mathematical tasks that highlight relevant concepts (Callejo & Zapatera, 2016), analyzing and discussing videos of student thinking (Fernandez et al., 2012; van Es & Sherin, 2002), interviewing students using a protocol aimed at uncovering students' mathematical conceptions (Lesseig et al., 2016), making sense of and using frameworks of student thinking (Walkoe, 2015), analyzing students' written work and discussing them (Fernandez et al., 2012), and reading theoretical papers about students' mathematical thinking in that domain (Fernandez et al., 2012; Walkoe, 2015). Overall, it appears that attending is the easiest component-skill for prospective teachers to develop.

In contrast, it appears both primary and secondary prospective teachers' ability to interpret students' mathematical understandings is more difficult to support. In particular, Callejo and Zapatera (2016) and Fernandez et al. (2012, 2013) found humble improvements in teachers' abilities to interpret student's mathematical understandings after their respective interventions. However, many other studies show that both primary and secondary prospective teachers' abilities to interpret can be improved (Lesseig et al., 2016; Sanchez-Matamoros et al., 2014; Schack et al., 2013; van Es & Sherin, 2002; Walkoe, 2015). Improvements in teachers' interpreting skills include using more specific evidence (e.g. Walkoe, 2015), moving away from an evaluative stance toward an interpretive stance (e.g. Sherin & van Es, 2005), and aligning

better with the literature of students' mathematical thinking (e.g Sanchez-Matamoros et al., 2014; Schack et al., 2013).

To support teachers' interpreting skills, it appears learning about students' mathematical thinking in that particular domain and engaging in approximations of practice where the student's thinking is central are supportive activities of teachers' ability to interpret (Grossman, Compton, et al., 2009; Lesseig et al., 2016; Sanchez-Matamoros et al., 2014; Schack et al., 2013). In contrast, simply learning about related mathematical concepts is not enough to support the development of this skill (Callejo & Zapatera, 2016). Simpson and Haltiwanger (2016) noted that many prospective secondary teachers struggled to make sense of the students' mathematical understandings simply because the student's solution route was different from their own.

### **Prospective Teachers' Professional Noticing of Students' Mathematical Thinking**

Recall that professional noticing of students' mathematical thinking includes teachers' attention and interpretation skills. Hence, it is reasonable to posit that the studies mentioned in the previous section provide many insights into the first two components of prospective teachers' professional noticing of students' mathematical thinking. In this section, then, I focus on the third component skill, deciding how to respond. I could only find three studies of prospective teachers' professional noticing of students' mathematical thinking that described some kind of developmental trajectory (Lesseig et al., 2016; Schack et al., 2013; Simpson & Haltiwanger, 2013). I will summarize their results related to this third component-skill.

Of the deciding how to respond component-skill, it is unclear what kinds of progressions there might be for prospective teachers. For instance, Lesseig et al. (2016) did not find any improvements in this component-skill in their prospective secondary teachers. However, Schack et al. (2013) did find improvements in their prospective elementary teachers' deciding how to

respond scores. Both interventions were fairly similar: both involved learning about students' mathematical thinking in their respective content domains and engaging in activities that approximated the practice of teaching, with emphasis on the practice of professional noticing of students' mathematical thinking. However, Schack and her colleagues led their primary prospective teacher participants through 5 in-class sessions related to professional noticing, while Lesseig and her colleagues led their secondary prospective teacher participants through 1 in-class session. It is highly likely then that prospective teachers' ability to decide how to respond on the basis of the students' mathematical understandings requires extended support, knowledge or frameworks of students' mathematical thinking in that domain, and several chances to practice their professional noticing of students' mathematical thinking expertise.

### **Practicing Teachers' Noticing of Students' Mathematical Thinking**

Less research exists analyzing the development of teachers' noticing of students' mathematical thinking expertise. Much of these results come from studies of practicing teachers' participation in video clubs (Sherin & van Es, 2005, 2009; van Es, 2011; van Es & Sherin, 2008). In general, pre-post analyses of these studies show that both practicing primary and practicing secondary teachers shift their attention toward students' mathematical thinking, develop stronger abilities to attend to evidence in the video, and improve their ability to generate interpretive accounts of the students' mathematical understandings based on evidence in the videos. Initially, practicing teachers seem to focus on pedagogical issues and teacher moves rather than the students' ideas. Once the foci of their noticing shifts to students' mathematical thinking though, there is even evidence that these improved noticing skills transfer to the classroom, as practicing elementary teachers shifted their pedagogy to elicit more ideas from their students and better facilitate discussions around the students' ideas (Sherin & van Es, 2009). However, these

progressions are not always straightforward; van Es and Sherin (2008) showed that teachers follow many different paths as they develop their attending and interpreting skills.

### **Practicing Teachers' Professional Noticing of Students' Mathematical Thinking**

Thus far only Jacobs, et al. (2010) and Jacobs, et al. (2011) have studied differences in practicing teachers' *professional* noticing of students' mathematical thinking expertise. These two studies, which focus on practicing primary teachers in the domain of whole number operations, show us that teaching experience can modestly support some teachers' attending and interpreting skills. Many practicing primary teachers with years and years of teaching experience showed evidence of being able to attend to most of the details of the students' mathematical ideas. Practicing teachers generally developed a robust attending expertise after 2 years of professional development (90% attended to most of the details). Additionally, many of them were able to interpret students' understandings, but only in vague and general ways (61% provided limited evidence and 16% provided robust evidence of interpreting the students' mathematical understandings). To develop robust interpreting skills, many teachers needed 4 or more years of professional development (26% demonstrated robust interpreting skills after 2 years, while 76% demonstrated robust interpreting skills after 4 years).

On the other hand, teaching experience did not seem to support primary practicing teachers' ability to decide how to respond on the basis of the students' mathematical understandings. In fact, prior to beginning professional development 74% of the experienced primary practicing teachers did not consider the students' mathematical understandings when posing a next task (in contrast, 23% did not interpret the students' mathematical understandings, a significantly lower proportion). In contrast, a majority of the teachers with 4 or more years of sustained professional development about students' mathematical thinking demonstrated robust

evidence of considering the students' mathematical thinking when deciding how to respond. This speaks to both the challenge of developing one's professional noticing expertise as well as the need for sustained extended professional support.

### **Summary**

Overall, studies on teachers' developmental trajectories show us that each component-skill can be developed through engaging in and reflecting on activities that approximate the practice of noticing students' mathematical thinking (Grossman, Compton, et al., 2009; Lesseig et al., 2016; Schack et al., 2013). Additionally, certain component-skills take less time and support to develop than others. In particular, many studies show evidence that teachers can develop robust attending skills even in fairly short amounts of time. In contrast, studies show that the deciding how to respond component-skill takes much more time and support to develop.

Many frameworks have been created to help distinguish between teachers' noticing abilities. These frameworks are helpful when researchers study relations between teachers' noticing and teachers' practice.

In my work, I add to the literature by investigating possible developmental trends along the intersection of professional noticing of students' mathematical thinking, practicing teachers, and secondary teachers, in the domain of proportional reasoning. Very few researchers have examined the development of practicing teachers' professional noticing of students' mathematical thinking (e.g. Jacobs et al., 2010; Jacobs et al., 2011), and none have examined developmental trends for practicing secondary teachers. Hence, I will begin to uncover how this expertise develops in practicing secondary teachers.

### **Research Question 2: Identifying Features of Strategies that Afford or Constrain Teachers' Professional Noticing Expertise**

In the previous section I situated my first research question in the literature on developmental trends in teachers' professional noticing. In this section, I summarize results relevant to my second research question, “(A) *What features of written artifacts of student thinking afford or constrain teachers' effectiveness at demonstrating their professional noticing of students' mathematical thinking expertise?* (B) *What features of written artifacts of student thinking elicit teachers' interest?*”

My second research question investigates how the context influences teachers' professional noticing of students' mathematical thinking expertise. To situate my question in the literature, I organize my work in two parts. First, I summarize results related to the context specificity of noticing students' mathematical thinking. Much like for my first research question, I include results related to noticing students' mathematical thinking, this time because researchers have not studied how the context influences what teachers *professionally notice* in students' mathematical thinking. After I summarize this area of research, I situate my work in the literature on using written samples of student work to support teacher learning. I include this last section because my work adds to this literature on artifact selection.

### **Summary of Results Related to Context-Specificity of Noticing**

To understand how features of student's mathematical ideas influence a teacher's noticing expertise, researchers investigate how different features afford or constrain what a teacher notices. I list a few ways we might modify features. We might modify (a) the domain of mathematics in question, (b) the assumptions we can make about students, (c) the types of student thinking or substance of student thinking made available, or (d) the medium we use to portray student thinking, among others. If we understand how these features afford or constrain

teachers' noticing skills, then we can make more informed selections of artifacts for future investigations of teacher noticing and future professional development workshops.

It is generally assumed that the noticing of students' mathematical thinking expertise is domain specific (Jacobs & Empson, 2016), meaning a teacher notices different things and in different ways depending on the mathematical domain. Walkoe (2015) found some evidence for this when she compared prospective teachers' noticing of students' mathematical thinking expertise between the sub-domain of symbol manipulations and the sub-domain of reasoning with representations. In her study, prospective teachers demonstrated more evidence on average of considering the students' mathematical thinking when analyzing artifacts of student thinking related to reasoning with representations than with symbol manipulations. Walkoe hypothesized that perhaps the procedural nature of symbol manipulation tasks constrained her teachers' abilities to consider the students' mathematical understandings at meaningful levels, and the conceptual nature of the graphing tasks supported the teachers' abilities to notice and make sense of the students' ideas.

Goldsmith and Seago (2011) compared teachers' discussions of artifacts of student thinking as teachers looked at written artifacts of student thinking and of video artifacts of student thinking. They did not claim to find differences in the richness or depth of the discussions about student thinking. Instead, they found that when the teachers received written artifacts of student thinking, teachers had to spend time deciphering what the student did in the task before discussing the student's thinking. When discussing videos of student thinking, the teachers spent less time having to decipher what the student did, because the temporal order of the operations the student performed was available for teachers to notice in the video. Teachers could hear what the child did first, second and third. Why they did what they did may still have



been confusing, but when they did what they did was clearer to these teachers in video artifacts than written artifacts. Hence, it appears that with written artifacts, there may be the extra step in attending to the student's thinking, where the teacher must figure out what the student did first, second, third, and so on.

Finally, Sherin, Linsenmeier, and van Es (2009) looked for relationships between the richness of conversations teachers had about video clips of student thinking and features of the video clips. They created a framework for categorizing the video clips along 3 dimensions: number of *windows* into student thinking (i.e. amount of evidence available to consider about the student), *depth* of student thinking exhibited (i.e. memorized procedure vs. deep reasoning and problem solving), and *clarity* of the student thinking exhibited. By categorizing videos and then noting whether the video supported deep or shallow conversations, they were able to see which features (or patterns of features) supported deeper conversations, thus providing other researchers and mathematics teacher educators with criteria for selecting video artifacts.

For example, when there were many windows into the student's thinking, a low depth, and a low clarity, teachers spent much time discussing the student's thinking because they were curious to understand what the student was doing. As another example, when teachers watched clips that exhibited many windows into the student's thinking, a high depth of student thinking, and a high level of clarity, teachers spent much time discussing the student's thinking because they were impressed by the depth of the students' thinking. Additionally, teachers appreciated videos with a high depth of student thinking and a low level of clarity, both because they were impressed with the student's work and motivated to make sense of the work. In contrast, video artifacts with a high level of clarity and a low depth of thinking did not support conversations between teachers because they were easy for teachers to understand and not very interesting.

## **Professional Development around Students' Written Work**

In this section I describe how mathematics teacher educators and teacher leaders utilize written artifacts of student thinking. I do this to better understand how teachers interact with students' written work, and thus gain some perspective on teachers' professional noticing of students' written work.

Mathematics teacher educators and teacher leaders have been using written artifacts of students' mathematical thinking for some time now (Little, Gearhart, Curry, & Kafka, 2003). Jacobs and Philipp (2004) note that part of the appeal to using written artifacts of student thinking is because written artifacts represent authentic links to the classroom. Additionally, the focus on students' mathematical thinking is a powerful impetus for motivating teachers to learn more and shift their pedagogy in productive ways (Kazemi & Franke, 2004). Hence, it is important to learn more about supporting teacher learning with written artifacts of student thinking.

### **Questions to focus teachers' attention.**

When supporting teachers to look at student work, one major theme seems to emerge from the literature. Many researchers recommend providing teachers with opportunities to slow down and start by just making sense of what the student is doing (Goldsmith & Seago, 2011; Jacobs & Philipp, 2004; McDonald, 2001; National School Reform Faculty, 2014). As McDonald (2001) points out, one must "counter an impulse conditioned by the teacher's daily pressure to make hundreds of quick judgments – Who's got it? Who doesn't? Who's paying attention? Who isn't?" (p. 218). To counter this pressure, teachers need support in slowing down to reflect on and discuss students' work.

Many researchers have developed ways to support teachers to slow down. Goldsmith and Seago (2011) recommend focusing conversations around evidence from the written artifacts. For example, when a teacher makes a claim such as “this student doesn’t get algebra,” the facilitator might stop the teacher and ask for what evidence in the artifact indicates that the student “doesn’t get algebra.”

Jacobs and Philipp (2004) offer a similar route. Jacobs and Philipp (2004) recommend proactively focusing teachers’ attending by starting with simply asking teachers what the student did, and spending time to fully understand this question first. Then, they pose a question about what the student understands. This is similar to other recommendations, such as the collaborative assessment protocol’s recommendation to “suspend judgment” when discussing, making sense of, and speculating about the student’s work (National School Reform Faculty, 2014), Kazemi and Franke’s (2004) process of sharing strategies seen in their classrooms and pushing teachers for more details about the strategies shared, and Seidel’s (1998) recommendation to simply describe the work collaboratively, and speculate about what the child is working on. After making sense of what the student did and what the student understands, Jacobs and Philipp (2004) recommend posing another question they find valuable, that of a “question to help the teacher identify instructional “next steps” to support or extend the child’s thinking” (p. 2).

### **General recommendations.**

Overall, Little et al. (2003) make three recommendations for looking at student work. First, teachers must find a space to discuss student work together. Teaching is a demanding job, and so finding a common time and space to convene and not worry about the demands of the job is crucial for being able to look at the student’s work without quick judgments. Second, student

work must be central to the discussion. Teachers will easily steer the conversation away to something else that happened in their own classroom, as many of them are predisposed to connect ideas to what they see in their own students (Ericsson, 2011). Finally, Little et al. (2003) recommend using protocols to structure discussions. Protocols force teachers to break up the conversation in awkward ways, and Little et al. (2003) claimed this awkwardness supported teachers to steer away from their habits of judging, and to deeply consider the questions being asked in the protocol and the student work in front of them.

### **Selecting work.**

The collaborative assessment conference recommends teachers choose a few interesting pieces of student work (National School Reform Faculty, 2014). According to Hidi and Renninger, interest has been shown to influence (a) what someone pays attention to, (b) goals for pursuing something, and (c) motivation for learning. However, what pieces of student work might count as interesting to teachers when they analyze students' written solutions? From the literature, there is at least one clear aspect of written samples of student work that sparks interest in teachers. Kazemi and Franke (2004) point out that sophisticated solutions to problems seem to excite practicing elementary teachers and motivate them to fully understand the student's thinking. In their article, they quoted teachers that began to really appreciate the student work being shared, saying things like "That's wild!" "How neat!" and "Wow" (p. 218)! Hence, sophisticated strategies can intrigue teachers.

### **Summary**

In sum, there is evidence that the context can afford or constrain teachers' noticing skills in different ways (Walkoe, 2015). Additionally, researchers have found that some types of artifacts support richer discussions than others among teachers (Sherin et al., 2009). When

engaging teachers with artifacts of student thinking, a common recommendation is to support teachers to slow down and focus on attending to the details and interpreting the students' understandings prior to discussing next instructional steps (Jacobs & Philipp, 2004).

My second research question looks at how different types of student thinking influence what teachers professionally notice. With this question, I add to the field of teacher noticing knowledge about how particular features of students' strategies might afford or constrain teachers' noticing skills. This could inform future selections of artifacts; perhaps a teacher educator would like to know which artifacts are "easier" or "harder" to make sense of, or which artifacts are more interesting to teachers.

## **Chapter 3: Methods**

In this chapter I describe my methods for answering my two research questions. I organize this chapter into four main sections. First, I describe the participants I wish to study, including my rationale for each participant group. Second, I describe the strategies of student thinking I used for my survey, including the important features that might afford or constrain teachers' effectiveness at demonstrating their professional noticing expertise. Third, I describe my methods of data collection, and in the fourth section I describe my methods of data analysis.

### **Participants**

I had two goals for my dissertation. First, I intended to compare the professional noticing expertise among three types of secondary teachers: prospective teachers, practicing teachers, and *emerging teacher leaders*, a term Jacobs and her colleagues used to describe teachers with at least 4 years of sustained professional support with a focus on students' mathematical thinking. I now describe each group in detail.

#### **Prospective Teachers**

Thirty prospective teachers (PSTs) participated in my study. All PSTs were recruited from a large urban university in the southwestern region of the United States, and intended to enter a post-graduate credential program. 25 were completing their degree, and 5 had completed their degree. PSTs had a range of backgrounds, with 11 having completed a content course for mathematics teachers, 13 having completed an inquiry-oriented mathematics course, 6 having completed both kinds of courses, and 12 having completed neither kind of course.

I purposefully selected PSTs that had not yet entered a post-graduate credential program or begun student teaching. Credential programs are important systems for preparing PSTs to teach, but credential programs also differ greatly in their topics, amount of support, and

effectiveness (Darling-Hammond, Chung, and Frelow, 2002). These differences make it challenging to generalize results pertaining to PSTs who are in the process of completing or have completed a particular credential program. Hence, to increase the generalizability of results for the study, I collected data from PSTs who had not yet entered a post-graduate credential program (but intended to do so). The PST group in this study is representative of other populations of PSTs who are entering post-graduate credential programs.

### **Experienced Teachers**

Thirty practicing teachers with at least 4 years of teaching experience participated in the study. On average, they had 11.47 years of teaching experience, with a range of 4 - 24 years. Because of the large amount of experience these teachers have had in the classroom, I call them experienced teachers (ETs). 13 of the ETs taught at middle schools, and 17 taught at high schools. ETs were recruited from the southwestern region of the United States. 28 of the 30 ETs came from school districts with similar populations of students. These school districts had between 58.5% - 71.2% of the students eligible for free or reduced lunch, and 12.8% - 23.7% classified as English learners. Of the remaining two ETs, one worked in a private school and the other worked in a district where 21% of the students were eligible for free or reduced lunch and 9% classified as English language learners.

I chose to invite practicing teachers with at least 4 years of teaching experience in order to capture data about the professional noticing expertise of experienced practicing teachers. This decision is similar to the decision made by Jacobs et al. (2010), who also only collected data from practicing teachers who had at least 4 years of teaching experience. By comparing data from experienced primary teachers with data from prospective primary teachers, Jacobs and her colleagues were able to make claims about the influence of teaching experience on primary

teachers' professional noticing expertise. In particular, it appeared teaching experience marginally supported primary teachers' professional noticing expertise. However, prior to my study it was unclear whether this trend would extend to secondary teachers.

It is difficult to say much about their teaching practice based on the number of years they have taught. For example, based on anecdotal evidence it appears that teachers are no longer surprised by what happens in their classrooms after 3-5 years (Berliner, 2004). This threshold of 3-5 years appears in other forms as well; Berliner (1994) believed that after 3 - 4 years most teachers become able to (a) determine what is and is not important to notice, (b) make conscious choices and set rational goals, and (c) plan on how they will reach their goals. However, Borko (as cited in Berliner, 1994) found that some teachers never reach this level of competence, and stayed at a level where they made decisions that disregarded the context or could not decide what was or was not important to address. Therefore, I do not make any claims about these teachers' capabilities, and I only state that these teachers all have a significant amount of teaching experience. Although, it may be noted that these teachers all volunteered to complete the survey with only a small incentive, which may indicate that these teachers are more generous with their time than others, and thus may have voluntarily participated in other professional development activities at other points in their professional careers.

### **Emerging Teacher Leaders**

I call the third group of teachers in my study *emerging teacher leaders*, a term Jacobs and her colleagues (2010) used to characterize teachers who had at least four years of sustained professional development with a focus on students' mathematical thinking. The teachers in this group were recruited from a sustained professional development experience. The goal of this professional development was to support high quality teachers to improve in their practice and



become leaders of their teaching communities. All teachers come from districts categorized as “high needs” school districts, with a majority of the students in these districts eligible for free or reduced lunch (range of 51.9% - 58.5%), and a portion classified as English Learners (10.9% - 23.7%). These teachers had an average of 18 years of teaching experience, with a range of 6 - 38 years. Seven of these teachers taught at high schools, and five taught at middle schools. I would have liked to recruit more emerging teacher leaders in order to help me generalize my results, but the project only had 12 teachers who had received four years of sustained professional support about students’ mathematical thinking at the time of data collection. These teachers have been attending 10 full day workshops per year (5 during the school year and 5 during summer), or around 75 hours per year of professional support. Workshop activities included solving mathematical tasks, reading research, analyzing samples of student work, analyzing videos of students solving problems, conducting clinical interviews with students, observing each other’s lessons, and discussing issues of pedagogy and of students’ mathematical thinking.

In Jacobs and her colleagues’ (2010) study, the emerging teacher leaders had participated in professional development that included five full days of workshops per year, and involved activities such as solving mathematical tasks, reading research, analyzing samples of student work, analyzing videos of students solving problems, and creating frameworks for student strategies and student thinking that they might use in their own classrooms. Jacobs and her colleagues found that only after having four or more years of sustained professional development did a majority of the teachers demonstrate a robust professional noticing expertise. However, prior to my study it was unclear whether this trend would extend to secondary teachers.

### **Summary of Participants**

To summarize, 30 prospective teachers, 30 experienced teachers, and 12 emerging teacher leaders participated in this study. At the time of data collection, all participants were either teaching at the secondary level, or intended to teach at the secondary level. Hence, I use the term “teacher” as an umbrella term for PSTs, ETs, and emerging teacher leaders.

There are two main differences I capture with these three groups of teachers. First, PSTs had not yet begun the post-graduate credential program, nor had they begun student teaching. In contrast, both the ET group and emerging teacher leader group had all been teaching for 4 or more years, with means above 10 years of experience. This will allow me to compare the professional noticing of students’ mathematical thinking expertise of teachers who have had and who have not had teaching experience. Second, The ETs and emerging teacher leaders are similar in that they are both groups of experienced teachers. However, the emerging teacher leaders have received 4 years of sustained professional development with a focus on students’ mathematical thinking. This will allow me to compare the professional noticing expertise of experienced teachers who have and who have not received several years of sustained professional support about students’ mathematical thinking.

### **Written Artifacts of Student Thinking**

As I mentioned earlier, I had two goals for my dissertation. My first goal was to compare the professional noticing expertise among three teacher groups. My second goal was to compare the professional noticing expertise teachers exhibited as they considered different written artifacts of student thinking, so that I could identify features of written artifacts of student thinking that might afford or constrain teachers’ effectiveness at demonstrating their professional noticing expertise. I now describe the six strategies that appeared in the survey, and how they

are different. After describing the strategies, I identify salient features of the strategies that distinguish them from each other.

### **Students' Strategies for Solving the Proportional Reasoning Tasks**

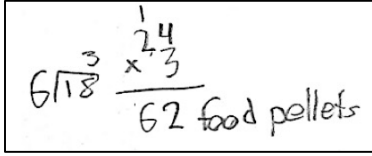
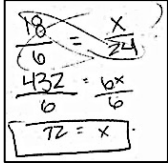
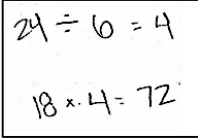
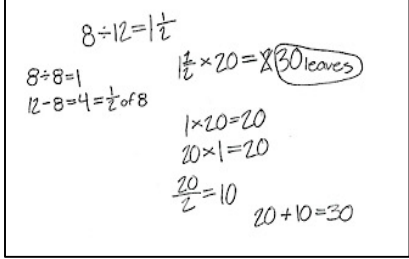
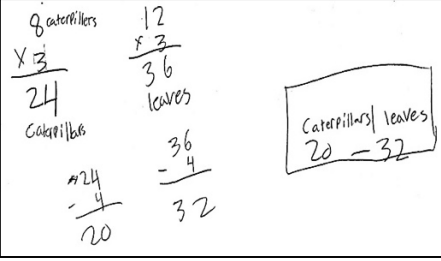
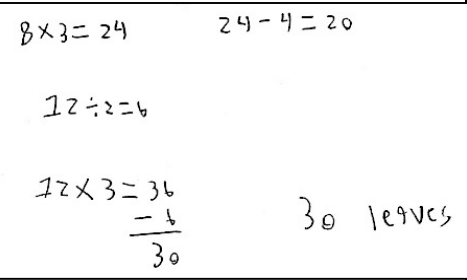
In this section I describe the six strategies that I used for the survey. First, I narrowed my focus to (a) written artifacts of student thinking, (b) in the domain of proportional reasoning, and (c) to missing-value tasks, which is a common proportional reasoning task (Kaput & West, 1994). I do this in order to control for several variables. First, I focus on written artifacts of student thinking because evidence exists that teachers engage in different noticing practices when analyzing video artifacts versus written artifacts (Goldsmith & Seago, 2011). Second, I focus on proportional reasoning, and specifically missing value tasks because evidence exists that the nature of the mathematical task can support or constrain teachers' noticing skills (Walkoe, 2015). By controlling for these variables, I can attribute differences in teachers' noticing skills to the types of student thinking exhibited in the samples of student work.

The strategies can be seen in Table 1. In the survey there were two missing-value tasks, and three students' strategies for solving each task. To select the strategies, I first collected approximately 600 pieces of student work from 6<sup>th</sup> - 8<sup>th</sup> grade classrooms located in the southwestern region of the United States. Then I selected six pieces of student work based on differences described in the literature.

It is important to note that the descriptions of the students' strategies cannot be confirmed with the students. I was not present when the strategies were collected, and I did not collect names to follow up with students after selecting the 6 strategies for my study. Instead, these descriptions are based on the evidence shown in the student work, and on the common

proportional reasoning strategies and understandings as identified in the literature (e.g. Carney, Hughes, Brendefur, Crawford, & Totorica, 2015).

Table 3.1. Artifacts, Strategies, Tasks, & Descriptions.

Strategy	Student's Work	Description
<b>Task: Each day, 6 mice eat 18 food pellets. How many food pellets do 24 mice eat?</b>		
A		Student divides 18 pellets by 6 to find a unit rate of 3 pellets per mouse. Student multiplies unit rate by 24 mice. Student makes a mistake when multiplying 24x3.
B		Student uses the traditional cross-multiplication strategy, setting up pellets in the numerator and mice in the denominator of each fraction. Student manipulates equation correctly.
C		Student divides 24 mice by 6 mice to find the number of groups of 6 mice in 24. There are 4 groups of 6 mice, so there should be 4 groups of 18 pellets.
<b>Task: Each day, 8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?</b>		
D		Student divides 12 leaves by 8 caterpillars, to find each caterpillar eats 1.5 leaves. The student divides in a nonstandard way, and makes a syntax error when writing $8 \div 12 = 1 \frac{1}{2}$ . The student then multiplies 1.5 leaves x 20 caterpillars, using the distributive property to help with the multiplication.
E		Student multiplies both 8 caterpillars and 12 leaves by 3, apparently scaling up the ratio to 24 caterpillars and 36 leaves. The student subtracts 4 from both quantities to reach 20 caterpillars, and arrives at 32 leaves. The student invokes additive reasoning for the final operation, and is not correct.
F		Student multiplies 8 caterpillars by 3 to get 24 caterpillars, then subtracts 4 to get 20 caterpillars. Student must have recognized that 4 caterpillars is half of 8 caterpillars, because next the student finds half of 12 leaves (6 leaves). The student then multiplies 12 leaves by 3, and subtracts the extra “half group of leaves” to get 30 leaves

**Task 1.**

The first task is “Each day, 6 mice eat 18 food pellets. How many food pellets do 24 mice eat?” Notice that in this task, 18 is three times as large as 6, and 24 is four times as large as 6. Hence, the functional and scalar multiplicative relationships are both integer multiplicative relationships (Carney et al., 2015). This is important for me to share because non-integer multiplicative relationships are more challenging for students to work with (Tourniaire & Pulos, 1985).

**Strategy A.**

This student first divides 18 by 6 and finds the quotient 3. Mathematically, 3 represents the unit rate of 3 pellets/mouse, which means each mouse eats 3 pellets of food. Then the student multiplies 24 by 3, which is conceptually correct because there are 24 mice and each eat 3 food pellets. However, the student gets an incorrect answer of 62, because the student makes a mistake while multiplying 24 by 3 by forgetting to add the carried 1 ten to the other 6 tens.

**Strategy B.**

This student first sets up an equation of  $18/6 = x/24$ , where  $x$  represents the missing value. The student then performs a cross multiplication manipulation, as indicated by the two ovals circling the diagonal numbers. In the next line the student writes  $432 = 6x$ , and divides both sides by 6. The student finds that  $72 = x$ . When setting up, the student could have thought about setting the two rates of consumption equal to each other (i.e. 18 pellets eaten by 6 mice equals some number of pellets eaten by 24 mice), and drew upon their conceptual understanding of proportions, rates, fractions, and equations to solve the problem. However, this is not likely. Students tend to use this procedure with little understanding attached to the procedure, except for knowing how to set up the equation (Cramer & Post, 1993). Hence it is unclear based on this

work what the student understands about proportional reasoning. The student performs all calculations correctly though and gets a correct answer.

### **Strategy C.**

This student starts by dividing 24 by 6, and gets the quotient of 4. This is related to a scalar multiplicative strategy, in which the student finds the number of times bigger one quantity is than another (which, in this case, is 4). The student then multiplies 18 by 4 and gets 72, which is conceptually correct because there are 4 times as many mice so there must be 4 times as many pellets. The student performs all calculations correctly.

### **Task 2.**

The second task is “Each day, 8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?” Notice that in this task, 12 is  $1\frac{1}{2}$  times as large as 8, and 20 is  $2\frac{1}{2}$  times as large as 8. Hence, the functional and scalar multiplicative relationships are both non-integer multiplicative relationships (Carney et al., 2015). Therefore, this task exhibits multiplicative relationships that are more challenging for students to work with than the multiplicative relationships in task 1.

### **Strategy D.**

For strategy D, the student writes  $8 \div 12 = 1\frac{1}{2}$ , and below that on the left side writes  $8 \div 8 = 1$ , and  $12 - 8 = 4 = \frac{1}{2}$  of 8. Subtracting 8 from 12 is consistent with a repeated subtraction meaning of division for the computation  $12 \div 8$ , in which one counts the total number of 8's one can subtract from the dividend, 12 (Carpenter, Fennema, Franke, Levi, & Empson, 1999). The student sees that one 8 fits into 12, and there is a remainder of 4. The student also recognizes that the remainder of 4 is half of the divisor (8), so the quotient of  $12 \div 8$  must be  $1\frac{1}{2}$ . This work indicates the student did not divide 8 by 12 as the first line signifies, but

rather 12 by 8. Hence, the student makes a syntax error when writing  $8 \div 12 = 1\frac{1}{2}$ . This work is consistent with the unit rate strategy, as the student divides 12 leaves among the 8 caterpillars and obtains  $1\frac{1}{2}$ , which is the unit rate of  $1\frac{1}{2}$  leaves per caterpillar.

The student then multiplies  $1\frac{1}{2} \times 20 = 30$ , and below that on the right side writes  $1 \times 20 = 20$ ,  $20 \times 1 = 20$ ,  $\frac{20}{2} = 10$ , and  $20 + 10 = 30$ . In these computations the student uses the distributive property, by splitting up the  $1\frac{1}{2} \times 20$  into  $1 \times 20$  and  $\frac{1}{2} \times 20$ , and summing the partial products at the end.

### **Strategy E.**

This student starts by multiplying  $8 \times 3 = 24$ . There are erased markings beneath the 3 that show the student first multiplied  $8 \times 4$ , but erased the 4 and wrote 3, which indicates the student was trying to get close to 20 caterpillars. The student also labels the 24 as 24 caterpillars. The order for the rest of the work is unclear; the student could have multiplied  $12 \times 3 = 36$  next, or subtracted  $24 - 4 = 20$  next. Either way, the upper half of the student's work is consistent with a scalar multiplicative strategy, in which the student multiplies both values in the 8 to 12 ratio by the same number in order to get the larger ratio of 24 caterpillars and 36 leaves (Carney et al., 2015). However, in the lower half the student reasons additively by subtracting 4 from both 24 and 36, which is not consistent with a scalar multiplicative strategy, and is not conceptually correct. The student's final answer is that 20 caterpillars eat 32 leaves.

### **Strategy F.**

This student writes  $8 \times 3 = 24$  and  $24 - 4 = 20$ . Mathematically, this represents the action of scaling up the 8 caterpillars by a factor of 3 and then taking away 4 caterpillars. Next this student writes  $12 \div 2 = 6$ , which represents half of a group of 12 leaves. This indicates the

student was thinking in terms of halves of groups, and in particular that the extra 4 caterpillars they subtracted from 24 caterpillars was half of a group of 8 caterpillars, so they knew they had to take away half of a group of 6 leaves eventually. Even though the computation  $8 \div 2 = 4$  is not written, there is sufficient evidence that this student recognized this relationship. The student then writes  $12 \times 3 = 36 - 6 = 30$  leaves, scaling up the 12 leaves by a factor of 3 and then subtracting the half of a group of 12 leaves away. This is consistent with a scalar multiplicative strategy and a scalar additive strategy (which are both conceptually correct strategies) because the student multiplies both values in the ratio by the same factor, and then subtracts a partitioned ratio from the larger ratio (Carney et al., 2015).

### **Differences among strategies.**

To differentiate among the strategies, I focus on 6 features: (a) strategy type, (b) integer/non-integer ratios, (c) exhibits non-standard calculation strategies, (d) conceptually correct/incorrect, (e) correct/incorrect calculations, and (f) all steps present/missing steps. “Strategy type” refers to the type of strategy the student used according to the literature on proportional reasoning (Carney et al., 2015; Lobato & Ellis, 2010). This includes unit rate strategies (students A & D), cross multiplication strategies (student B), and scaling strategies (students C, E, and F). Students C, E, and F each employ a different scaling strategy. Student C scales up the original ratio by finding the exact number of groups. Student E scales up the original ratio but overshoots the target value, and compensates by subtracting equal amounts from both values. Student F scales up the original ratio and also overshoots the target value, but this student compensates by subtracting proportional amounts from both values.

The feature “integer/non-integer ratios” refers to whether the student grappled with integer ratios or non-integer ratios, where non-integer ratios have the potential to make the



strategy more complex. Strategies A and C utilized integer ratios, because they grappled with an integer functional multiplicative relationship and an integer scalar multiplicative relationship, respectively. In contrast, strategies D and F grappled with the non-integer functional multiplicative relationship and the non-integer scalar multiplicative relationship, respectively. For strategies B and E, I did not identify whether they grappled with integer or non-integer multiplicative relationships. Strategy B used a cross-multiplication strategy, which utilizes neither the scalar nor the functional multiplicative relationships. Strategy E contains additive reasoning, which means this student did not totally make sense of the non-integer scalar multiplicative relationship in task 2.

“Correct/incorrect calculations” refers to whether the student performed all calculations correctly or not. The only students that did not perform all calculations correctly are students A and D. In strategy A, the student makes a mistake while multiplying 24 by 3, and forgets to add the carried 1. In strategy D, the student makes a syntax error, which is an error in the way the student writes a calculation (as opposed to a conceptual error). “Standard/non-standard calculations” refers to whether the strategy includes any non-standard computational strategies. The only student who included non-standard computation strategies was student D.

“Conceptually correct/incorrect” refers to whether the strategy exhibits a correct conceptual understanding of the proportional relationships in the problem. Student E is the only student who exhibits an incorrect understanding of proportional reasoning. (I did not apply this code to strategy B because many students use this strategy without a conceptual understanding of proportional reasoning.) Finally, “all steps present/missing steps” refers to whether we have to infer some of the steps the students made, or whether all the computations appear to be present. Strategy F is the only strategy that is missing steps.

In table 2, I summarize the features of each strategy. For the strategy type, I simply labeled the type of strategy the student used. For the other 5 features, I looked for whether there was evidence the strategy exhibited that feature or not. For example, strategy D exhibits non-standard calculations, while the others do not.

Table 3.2. Features of strategies.

Strategy	Strategy Type	Integer Ratios?	Correct Calc's?	Standard Calc's?	Correct Concept?	All Steps?
<b>A</b>	Unit Rate	Yes	No	Yes	Yes	Yes
<b>B</b>	Cross-Multiplication		Yes	Yes		Yes
<b>C</b>	Scale Up	Yes	Yes	Yes	Yes	Yes
<b>D</b>	Unit Rate	No	No	No	Yes	Yes
<b>E</b>	Scale Up, Subtract Equal Parts		Yes	Yes	No	Yes
<b>F</b>	Scale Up, Subtract Prop. Parts	No	Yes	Yes	Yes	No

### **Methods of data collection: Professional noticing survey**

To collect data on teachers' professional noticing expertise, I administered a survey to all 72 secondary teachers in my study. The survey is composed of three parts. The first two parts are based on the written assessments used by Jacobs et al. (2010), which collects data about the teachers' professional noticing of students' mathematical thinking expertise. In the third part, I aimed to collect additional data about the personal connections teachers may have created while considering the students' strategies.

### **Parts 1 and 2: Survey of Teachers' Professional Noticing**

In each of parts 1 and 2, teachers first solved a proportional reasoning task. Then, teachers considered three students' strategies to the task. In part 1, teachers responded to task 1 and considered strategies A, B, and C. In part 2, teachers responded to task 2 and considered strategies D, E, and F. After considering each set of strategies, teachers responded to the three professional noticing prompts used by Jacobs et al. (2010). The prompts were:

1. Please describe in detail what each student did in response to this problem. (I recognize that you do not know who this student is, and did not get a chance to talk to them - Please just do the best you can.)
2. What did you learn about these three students' understandings?
3. Pretend that you are the teacher of these students. What problem(s) might you pose next, and why? (I am interested in how you think about selecting problems, but I do not believe that there is ever a best problem, and I recognize that as the teacher of this student you would have more information to inform your selection.)

The first prompt assessed teachers' attending skills, as teachers were asked to describe what they thought the student did. The second prompt assessed teachers' interpreting skills, as teachers were asked to describe what they thought the three students understood. The third prompt assessed teachers' deciding how to respond skills. Similar to Jacobs et al. (2010), the space provided for responding to the attending prompt was labeled with each student's name (i.e. student A, student B, student C), in order to remind teachers to describe each student's strategy. Additionally, the space provided for responding to the deciding how to respond prompt was split into two parts with one part labeled "Problem(s)," and the other part was labeled "Rationale(s)," in order to remind teachers to provide a problem and a rationale.

Participants were given as much time as they needed to respond to the prompts (See Appendix A for the Professional Noticing Survey). After teachers completed part 1, they were allowed to start part 2, but once they started part 2 they were not allowed to go back and change their answers to the prompts in part 1. Similarly, teachers could not go back to part 2 once they started part 3. I restricted teachers in this way because professional noticing of students' mathematical thinking is supposed to be an in-the-moment expertise, so while I gave teachers as

much time as they wanted, I did not want them to return to previous sections after considering other students' responses.

### **Part 3: Survey of Personal Connections Teachers Potentially Made**

In the third part, teachers revisited the 6 strategies they saw in parts 1 and 2 and responded to two new prompts:

1. Out of all the student's solutions<sup>5</sup>, is there a student you would like to talk to further? If yes, which student would you like to talk to, and why?
2. Out of all the student's solutions, would you be interested in discussing a particular solution with other teachers? If yes, which solution would you like to discuss with other teachers, and why?

Both of these prompts assessed whether particular strategies elicited a personal connection from teachers in some way, and in what ways the teachers wanted to interact further with the student or the strategy. Interest can be a powerful motivational tool for engagement with an artifact or a surrounding discussion, so I used these questions to capture which strategies were more "interesting" to teachers (Hidi & Renninger, 2006; Kazemi & Franke, 2004; Sherin et al., 2009).

### **Methods of data analysis**

All responses were blinded by a colleague who was not a part of the project. Analysis happened in two rounds, one round per research question. Next, I describe each round in detail.

#### **Analysis for Question 1**

For my first round of analysis my goal was to differentiate the professional noticing expertise among the three groups of teachers in my study. To do so, I considered the

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<sup>5</sup> I used the word "solution" in these questions because in a pilot study multiple teachers expressed confusion about the term "strategy."

professional noticing prompts in parts 1 and 2, and followed Jacobs et al.'s (2010) coding process. In this process I scored responses to the professional noticing prompts based on the amount of evidence that the teacher considered the students' mathematical thinking. Each response received one of three scores: lack of evidence, limited evidence, or robust evidence of considering the student's mathematical ideas. By comparing how many teachers received each score I was able to compare the professional noticing expertise among the three teacher groups in my study.

### **Attending responses.**

According to Jacobs et al. (2010), "responses demonstrating evidence of attention to children's strategies were phrased in many ways, but they all tracked the entire strategy with substantial detail about the mathematically important aspects of that strategy" (p. 183). In other words, evidence of attending involves (a) describing most of the details about the strategy, and (b) tracking the entire strategy, or describing the gist of the strategy.

### ***Details of the strategy.***

To determine whether teachers described most of the details about the strategy, I followed Jacobs and her colleagues' (2010) steps for coding the attending responses. First, I identified the mathematical details of each student's strategy. For strategy A of figure 4, the details included:

- The student divides 18 food pellets by 6 mice.
- Mathematically, 3 represents the unit rate of 3 food pellets eaten by each mouse.
- The student student multiplies 3 food pellets/mouse by 24 mice.
- The student gets an incorrect answer of 62 pellets.

For strategy B of figure 4, the details included:

- The student sets up a proportion/equation, 18 food pellets to 6 mice is the same as  $x$  pellets to 24 mice.
- The student cross multiplies, i.e. sets cross products equal to each other, so  $18 \cdot 24 = 6 \cdot x$ , or  $432 = 6x$ .
- The student divides both 432 and  $6x$  by 6.
- The student gets an answer of  $x = 72$ .

For strategy C of figure 4, the details included:

- The student divides 24 mice by 6 mice.
- Mathematically, the 4 represents the number of groups of mice, or the number of times bigger the new amount of mice is than the old amount of mice.
- The student multiplies 4 times 18 pellets.
- The student gets a correct answer of 72.

For strategy D of figure 4, the details included:

- The student divides 12 by 8.
- To divide 12 by 8, the student first divides 8 by 8, then finds the remainder, then notices the remainder is half of 8.
- Mathematically,  $1 \frac{1}{2}$  represents the number of leaves each caterpillar eats.
- Student multiplies  $1 \frac{1}{2}$  by 20.
- Student performs multiplication by using the distributive property, first multiplying 1 by 20, then finding  $\frac{1}{2}$  of 20, and adding the resulting products (i.e. 20 and 10) together.
- Student obtains a correct answer of 30 leaves.

For strategy E of figure 4, the details included:

- The student multiplies 8 caterpillars by 3.
- Mathematically, 3 represents the number of groups of 8 caterpillars, or the “number of times bigger”.
- Multiplying by 3 is an attempt to get close to 20 caterpillars.
- The student multiplies 12 leaves by 3.
- The student subtracts 4 caterpillars from 24 caterpillars.
- The student subtracts 4 leaves from 36 leaves.
- The student gets an incorrect answer of 20 caterpillars eat 32 leaves.

For strategy F of figure 4, the details included:

- The student multiplies 8 caterpillars by 3.
- Mathematically, 3 represents the number of groups of 8 caterpillars, or the “number of times bigger”.
- Multiplying by 3 is an attempt to get close to 20 caterpillars.
- Student subtracts 4 caterpillars from 24 caterpillars.
- Mathematically, the group of 4 caterpillars is half of a group of 8 caterpillars.
- The student divides 12 leaves by 2.
- The student multiplies 12 leaves by 3.
- The student subtracts 6 leaves from 36 leaves.
- The student gets a correct answer of 30 leaves.

After identifying the mathematical details of each solution, I looked for evidence in each teacher’s attending response for whether the teacher attended to that detail or not. I recorded each detail that the teacher attended to, and counted the number of details for each strategy.

Thus, each teacher had 6 numbers (one number per strategy), where each number represented the

number of details the teacher provided evidence of attending to for that strategy. A little more than 20% of the data was double coded by another researcher, and interrater reliability was 93.3%.

***Converting details into scores.***

I then converted each number into a “robust,” “limited,” or “lack” score. The conversions for each strategy can be seen in table 3. In the table, the column represents the strategy, the row represents the score, and the number in the cell represents the number of details the teacher needed to attend to in order to earn that score.

Table 3.3. Conversions of "number of details" to codes for each strategy.

	<b>A</b>	<b>B</b>	<b>C</b>	<b>D</b>	<b>E</b>	<b>F</b>
<b>Robust</b>	3, 4	3, 4	3, 4	4, 5, 6	5, 6, 7	6, 7, 8, 9
<b>Limited</b>	2	2	2	2, 3	3, 4	4, 5
<b>Lack</b>	0, 1	0, 1	0, 1	0, 1	0, 1, 2	0, 1, 2, 3

These codes differ from those of Jacobs et al. (2010) because I used three levels of evidence (lack of evidence, limited evidence, or robust evidence), whereas Jacobs et al. had only two (lack of evidence or evidence of attending to details). At this point, each participant had 6 attending scores, three scores for each strategy in part 1 and three scores for each strategy in part 2.

***Gist of the strategy.***

Next, I looked for whether the responses that received a “robust” score attended to the gist of the strategy. Earlier I pointed out that Jacobs et al. (2010) noted that “responses demonstrating evidence of attention to children’s strategies... *all tracked the entire strategy*” (p. 183, emphasis added). In other words, responses demonstrating robust evidence should identify the entire strategy and not just the individual pieces in isolation; the teacher should be able to attend to how the pieces fit together, via the key underlying reasoning of the strategy. I refer to the key underlying reasoning of the strategy as the *gist* of the strategy, and I coded for the gist of



the strategy by looking for evidence that the respondent described mathematical connections among the operations, meanings for operations, and/or meanings for the numbers. Hence, for responses that were currently identified as “robust evidence,” I looked for whether the teacher identified connections between or meanings of the different operations in a way that was consistent with the student’s work and with the literature about proportional reasoning.

To illustrate, consider the following description of strategy D that did not identify connections between or meanings of the different operations, even though the teacher attended to most of the details:

Student D began by dividing 8 by 8 and obtained 1, then subtracted 8 from 12 and got the answer of 4 and took 4 as half of 8. Then proceeded by dividing 8 by 12 and obtained  $1 \frac{1}{2}$ . Then multiplied  $1 \frac{1}{2}$  by 20 and obtained 30 leaves. It seems that the student tried another method, the student multiplied 1 times 20 and obtained 20. Then multiplied 20 times 1 and obtained 20. Then divided  $20/2$  and got the answer of 10. Then divided 20 by 10 and got the answer of 30.

Notice that this teacher correctly identified each operation that the student wrote down.

However, this teacher did not describe how the subtraction of 8 from 12, the division of 8 by 8, and the fact that 4 is half of 8 are all connected to the division of 12 by 8. In fact, the teacher claims the student divided 8 by 12 *after* writing those operations, which is evidence that they did not see 12 divided by 8 in those operations. The teacher also did not identify the use of the distributive property when multiplying, nor did the teacher identify the meaning of  $1 \frac{1}{2}$  as the unit rate, nor how any of the operations related to the context. Hence, this teacher did not identify the gist of the strategy, and only identified the pieces in isolation.

In total, 23 of the 225 “robust evidence” responses did not identify the gist of the strategy, and so were moved down to a “limited evidence” score. Specific evidence of getting the gist for each strategy can be seen in table 4. A little more than 20% of the robust evidence responses was double coded by another researcher, and interrater reliability was 91.8%.

Table 3.4. Gist of the strategy for each strategy.

Strategy	Gist of Strategy
A	Teacher describes the unit rate concept.
B	Teacher describes the act of setting up and solving an equation, and/or cross multiplication.
C	Teacher describes the scaling up concept.
D	Teacher describes the unit rate concept, and/or the student's non-standard computation strategies (division of 12 by 8 by splitting 12 into two parts, and multiplication of $1\frac{1}{2}$ by 20 by using the distributive property).
E	Teacher describes the scaling up concept and the student's additive reasoning.
F	Teacher describes the scaling up concept, the partitioning of 8 caterpillars and 12 leaves in half (i.e. 4 caterpillars and 6 leaves), and the subtraction of the partitioned ratio.

***Assigning an overall score.***

Finally, I gave an overall attending score for each teacher in each part, using the three scores for each strategy in that part. If all three scores were the same (e.g. three scores of limited evidence, or three scores of robust evidence) then the participant received that overall attending score for that part. If two of the three scores were the same, and the third score was one away (e.g. two limited and one robust, or two lack and one limited), then the participant received the score that was most received as their overall score. If all three scores were different (e.g. one robust, one limited, one lack) then the participant received an overall score of “limited” for that part.

If two of the three scores were the same, and the third score was two away (e.g. two scores of robust and one lack, or two lack's and one robust), then I sought more information to select an overall score for that participant. In part 1, 15.3% of the participants needed this extra information, and in part 2, 13.8% needed this extra information. To select an overall score, I looked at the total number of details the teacher attended to in that part. In the first part, the total sum was a number between 0 to 12, so I decided that the threshold would be a total score of a 5 for a limited, and a total score of a 7 for a robust. In the second part, the total sum was a number

between 0 to 22, and I decided that the threshold would be a total score of a 7 for limited, and a 16 for robust. In this way each participant received one attending score for each part of the survey, which amounted to two attending scores for each participant.

### **Interpreting responses.**

To score the interpreting responses, I followed Jacobs and her colleagues' (2010) methods, and scored each interpreting response based on the amount of evidence that the teacher interpreted the student's mathematical understandings. Robust evidence can be seen when the teacher identified understandings that were consistent with the student's strategy, were consistent with the literature about proportional reasoning, included specific details, differentiated among the three students, and described the understandings in a nuanced and careful way. An example of a response that provided limited evidence might be one where they described some understandings that were consistent with the students' strategies, but in a vague or over-simplified way. In contrast, a response provided no evidence when the claims were so vague that they could be attributed to almost any strategy, if the response did not consider the student's understandings, or if the response was inconsistent with the student's work. Each participant received one interpreting score for each part, which amounted to two scores in total. A little more than 20% of the data was double coded by another researcher, and interrater reliability was 86.6%.

### **Deciding-how-to-respond responses.**

I scored deciding-how-to-respond responses based on the extent to which the teacher considered the students' mathematical understandings when deciding how to respond (Jacobs et al., 2010). To measure this extent, I had to analyze both the problem posed by the teacher and the rationale the teacher provided. In particular, I looked for evidence that the teacher's decision and

rationale were consistent with the students' strategy, perhaps by building on particular pieces of the student's strategy or potential understandings, or by anticipating how the student might respond to the posed problem (where the anticipated response was consistent with what the student already did). Teachers did not have to cite details from their attending or interpreting responses to support their decisions in order to earn a high score. Instead, they had to give some indication that they were considering the students' understandings in order to receive a higher score. Specific details, differentiated problems and/or rationales for each student, nuanced phrasings and ideas, and consistency with the student's work and with the proportional reasoning literature were strong indications that the teacher had considered the students' mathematical understandings when deciding how to respond. Conversely, forcing particular strategies or posing new problems based on the teacher's own goals for instruction indicated that the teacher was not considering the students' mathematical understandings. With these criteria in mind I scored responses based on whether they provided a lack of evidence, limited evidence, or robust evidence of considering the students' understandings when deciding how to respond. Each teacher received one deciding how to respond score for each part, which amounted to two scores in total. A little more than 20% of the data was double coded by another researcher, and interrater reliability was 80%.

### **Answering question 1.**

My intention was to compare and contrast the professional noticing expertise of the three teacher groups. To do so, I first converted all teachers' scores into numerical values. I converted "robust" scores to 2, "limited" scores to 1, and "lack" scores to 0. Then, I gave each teacher one average attending score, one average interpreting score, and one average deciding how to respond score. This provided me with stable measures of each teacher's professional noticing

skills. 98.1% of the pairs of component-skill scores (e.g. a teacher's part 1 attending score and part 2 attending score) were within one level from each other (e.g. if they received a 0 in attending in part 1, then they received either a 0 or a 1 in attending in part 2). Hence, the professional noticing measure appeared to be fairly stable already.

After calculating an overall attending score, interpreting score, and deciding how to respond score for each teacher, I calculated average scores for each teacher group. Comparing the averages of each teacher group in each component-skill provided me with an answer my first research question. I also decided to conduct t-tests for significance, even though I have a small sample of emerging teacher leaders. I conducted 2-sample t-tests on adjacent pairs of groups (PSTs vs. ETs, ETs vs. ETLs) to test if the means of the adjacent pairs of groups were significantly different. I hypothesized that more experience with students' mathematical thinking would support more sophisticated professional noticing skills; hence, the t-tests were 1-tailed. Specifically, I hypothesized that ETLs would have more sophisticated professional noticing skills than ETs, and ETs would be more sophisticated than PSTs. I used t-tests with unequal variance because the variances of the different groups differed by a factor larger than 1.5. I also used a Bonferroni correction because I conducted 6 tests in total, which lowered my level of significance to  $\alpha = 0.0083$ .

In addition to quantitative tests, I identified qualitative trends for each component-skill within each score. This allowed me to find indicators of "lack of" scores, "limited" scores, and "robust" scores for each component-skill, which may highlight important steps for improving one's professional noticing expertise (Jacobs et al., 2010). In addition, this allowed me to further differentiate among the teacher groups. For example, some "no evidence" scores were different from other "no evidence" scores.

## **Analysis for Question 2**

For the second round of analysis my goal was to identify features of strategies that influenced (and how they influenced) what teachers noticed. In particular, I focus on two ways strategies can influence what teachers notice, either by challenging teachers, or intriguing teachers. To identify features that challenged or intrigued teachers, I considered teachers' responses to the attending prompts in parts 1 and 2, and the two prompts that elicited data about personal connections in part 3, and looked for evidence of whether teachers identified the gist of a strategy, and whether teachers created personal connections to a strategy. By comparing these results in conjunction with the features of the strategies (see table 2), I was able to identify which features made the strategy more challenging for teachers and which features made the strategy more intriguing or attractive to teachers. In the next two sub-sections, I discuss how I coded for attending to the gist of the strategy, and how I coded for creating a personal connection to a strategy. In the last sub-section, I share how I used these codes to answer my research question.

### **Gist of the strategy.**

As a reminder, the gist of the strategy refers to the key underlying reasoning of the strategy, which gives the strategy meaning and connects the different operations to each other. By identifying the gist of the strategy, teachers provide evidence that they understand the essence of the student's strategy, in a way that is consistent with the student's work and with the literature about proportional reasoning, even if they didn't describe all of the details in the student's work. My assumption is that when fewer teachers get the gist of a particular strategy, this indicates that this strategy is more challenging for teachers to professionally notice.

In the first round of analysis I coded for attending to the gist of the strategy by looking for whether the teacher identified connections between the different operations, meanings for the

operations, or meanings for the numbers, in a way that was consistent with the work shown and with the literature on proportional reasoning. However, I only did so for the responses that had received an initial score of “robust.” Hence, I revisited teachers’ responses to the attending prompts in parts 1 and 2 and extended my analysis to the responses that received a score of limited evidence and lack of evidence. That way, I was able to identify who had attended to the essence of the strategy, even if they had not attended to most of the details of the strategy. Occasionally there was additional evidence of attending to the gist of the strategy in the interpreting and deciding how to respond responses. More than 20% of the limited evidence and lack of evidence scores were double coded, and interrater reliability was 86.6%.

### **Personal connections.**

I wanted to identify who had created personal connections with one or more strategies, and what kinds of connections they created. I looked for evidence that teachers created personal connections because I believe these personal connections have much potential to influence the depth at which teachers engage with students’ ideas, because intrigue is a powerful tool for motivating engagement (Hidi & Renninger, 2006; Kazemi & Franke, 2004; M. Sherin et al., 2009). To identify the personal connections teachers created when making sense of the different strategies, I draw upon Sherin and Russ’s (2014) framework of interpretive frames. Interpretive frames are “structures that describe the ways in which a teacher’s selective attention both grows out of and informs a teacher’s knowledge-based reasoning, and vice-versa” (p. 3). Essentially, interpretive frames are *ways* of making sense of our surroundings.

### ***Interpretive frames.***

To illustrate the notion of an interpretive frame, consider the following examples from Sherin and Russ’s (2014) study. In their study, teachers watched videos of different classroom

events, and were asked to describe what they noticed. In one instance, a teacher experienced a desire to stop a couple students from incessantly tapping their desks: “I wanted to touch them [and say] ‘Stop’” (p. 8)! This teacher made sense of the video by experiencing a personal connection to the video, putting herself in the perspective of the teacher. In another instance, a teacher watched a student struggle to explain why they were confused, and the teacher identified a pedagogical principle in that moment: “Sometimes, as a teacher, it’s hard when you go over to a student and ask them, ‘What don’t you understand,’ and they’re trying to tell you, but they don’t even know what they don’t understand” (p. 10). In this instance, the teacher’s pedagogical principle played an important role in the teacher’s noticing skills. In another instance, a teacher made sense of a video by evaluating the students’ actions: “These kids are really good at participating.” Notably, each of these instances highlight different *ways* teachers made sense of the videos. In the first example, an emotional reaction emerged that framed how the teacher made sense of that instant in the video. In the second example, the pedagogical principle framed what the teacher noticed. In the third example, the teacher’s values framed how the teacher made sense of the video. Hence, the different interpretive frames define different ways teachers make sense of their surroundings.

***Relationship between interpretive frames and personal connections.***

One important interpretive frame teachers create to make sense of their surroundings is by experiencing a personal connection to what they notice, which Sherin and Russ (2014) call a *personal frame*. In essence, the emotional connection frames what they notice. Within personal frames, Sherin and Russ identified two sub-frames that teachers create. Either teachers created an affective sub-frame, which means they experienced an affective reaction to what they noticed, or they created a perspective sub-frame, which means they put themselves in the perspective of



the teacher and described what they wanted to do as the teacher in the video. With respect to my study which uses written artifacts of student thinking instead of videos of classrooms, I found evidence of teachers creating personal frames when they described an affective reaction to a student's strategy (or a feature of the strategy), or when they described a desire to interact further with a student's strategy.

***Coding for affective frames.***

For the first type of personal connection, that of experiencing an affective reaction, I looked for moments when teachers spontaneously conveyed emotion. I found evidence of emotional reactions in teachers' responses to the attending and interpreting prompts in parts 1 and 2, and in teachers' responses to the two personal connection prompts in part 3. In total, 31.9% of the teachers spontaneously described an emotional reaction for at least one strategy, and some described an emotional reaction to multiple strategies, even though I never explicitly asked teachers about their emotions.

I only coded for positive emotional reactions (e.g. "look at the awesome way this kid multiplied by 2.5!", or "I am curious... how this student might of [*sic*] thought about this problem"). Evidence for negative emotional reactions was difficult to discern, mostly because instead of explicitly conveying a negative emotional reaction (e.g. disappointment, sadness, concern), many teachers described the student's strategy as having a low understanding, or that the student should have done something else instead. There was not enough evidence in such responses to decide whether the teacher was experiencing a negative emotional reaction or if the teacher was stating their ideas in a neutral way. In contrast, positive emotional reactions were much easier to identify. For each participant, I coded for whether or not they experienced a

positive emotional reaction to the strategy. More than 20% of the data was double coded by another researcher, and interrater reliability was 93.3%.

### ***Coding for perspective frames.***

For the second type of personal connection, that of wanting to interact with the student or the strategy in some way, I looked for moments when a teacher wanted to talk to or about a particular student further. Most of the evidence for this type of personal connection appeared in teachers' responses to the two personal connection prompts in part 3, which explicitly ask the teacher if they would like to talk to or about a student. However, sometimes teachers spontaneously mentioned they wanted to talk to a student in the attending or interpreting prompts in parts 1 and 2<sup>6</sup>, so I looked there as well. Additionally, some of these teachers provided evidence of wanting to talk to or about multiple students. I coded for instances when a teacher chose to talk to (or about) a particular student, as well as what they wanted to talk about. Responses fell into four main categories: (a) learn more about the strategy, (b) help the student, (c) share the strategy, either in class or with others, and (d) discuss mathematics related to the strategy. The fourth category emerged because teachers were creative in the ideas they wanted to discuss with other teachers, and these ideas did not fit well into one of the other three categories. In total, 90.8% of perspective frames fell into categories A, B, and C, and only 9.2% fell into category D. More than 20% of the data was double coded, and interrater reliability was 91.1%.

### **Answering research question 2.**

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<sup>6</sup> I did not consider teachers' responses to the deciding how to respond prompts because for those prompts I explicitly asked teachers to select a next task for students, which means I could not distinguish between a teacher wanting to select a particular task for a student or a teacher only selecting a new task because I asked them to do so. In contrast, in the prompts in part 3 teachers were allowed to decide whether or not they wanted to talk to or about a student.

Finally, after coding for whether teachers got the gist of a strategy and whether teachers created personal connections with a strategy, I counted the number of times this happened for each artifact. This gave me a sense of (a) which strategies were more or less challenging for teachers to understand, and (b) which strategies were more or less “intriguing” to teachers. For each code, I also looked at qualitative trends, which allowed me to identify particular features of the strategy that made the gist of the strategy more challenging for teachers to identify, as well as which features of the strategy may have made the strategy more interesting, and why. By comparing these results among the 6 strategies (where each strategy had their own distinct features), I was able to determine what features were challenging and what features were intriguing. For example, I was able to identify whether a conceptual error was more interesting than a computational error, whether unit rate strategies were more challenging than scalar strategies, or whether strategies grappling with integer ratios were more exciting than strategies grappling with non-integer ratios. I share and discuss these results in chapter 5.

## **Chapter 4: Results and Discussion for Research Question 1**

In this chapter I present the results for my first research question, and discuss these results in the context of teacher education. This chapter will be split into three sections. In the first section, I characterize the professional noticing expertise I saw across all participants, without distinguishing among the teacher groups. My focus for this section is on illustrating what the three scores (robust evidence, limited evidence, and lack of evidence) mean, describing the variety of responses within each category, and overall characterizing the different levels of evidence for each component-skill. In the second section, I distinguish among the three teacher groups, and compare and contrast the professional noticing expertise of each group. In the third section, I discuss what these findings imply for future research and teacher education. As a reminder, my first research question was:

1. What differences and similarities exist in the professional noticing of students' mathematical thinking expertise among three groups of teachers: (a) prospective secondary mathematics teachers, (b) practicing secondary mathematics teachers, and (c) emerging secondary mathematics teacher leaders?

### **Characterizations of Lack of, Limited, and Robust Evidence of Professional Noticing Expertise**

In this section I describe the three levels of professional noticing expertise (i.e. robust evidence, limited evidence, and a lack of evidence) for each of the three component-skills of professional noticing (i.e. attending, interpreting, and deciding how to respond). I split this section into three sub-sections, one sub-section per component-practice. First I describe the different levels of attending to the details of the strategy, followed by the different levels of

interpreting students' understandings, and finally the different levels of deciding how to respond based on students' understandings.

### **Attending to the Details of the Strategy**

As a reminder, attending refers to (a) the ability to pay attention to important details of a student's strategy, and to (b) get the gist of the strategy. I first describe responses that provided robust evidence of attending to the mathematically significant details of the students' strategies, followed by limited evidence, and finally a lack of evidence. For each level of evidence, I describe in detail a representative response illustrating that level of evidence, and share qualitative trends and common features of responses in that category. The percentages of teachers receiving each score for each part can be seen in table 1. As a reminder, part 1 refers to strategies A, B, and C, which included a simpler unit rate strategy, the cross-multiplication strategy, and a simpler scalar strategy, and part 2 refers to strategies D, E, and F, which included a more complex unit rate strategy, a scalar strategy with additive reasoning, and a more complex scalar strategy.

Table 4.1. Percentages of teachers providing robust, limited, or lack of evidence of attending to the strategy for each part.

N = 72	<b>Part 1</b>	<b>Part 2</b>
<b>Robust evidence</b>	44.4%	43.1%
<b>Limited evidence</b>	40.3%	31.9%
<b>Lack of evidence</b>	15.3%	25%

Each representative response is a teacher's description of student F's strategy (Fig. 1). As a reminder, student F first multiplied 8 caterpillars by 3 to get 24 caterpillars, and then subtracted 4 caterpillars from 24 caterpillars to get 20 caterpillars. Mathematically, the 4 caterpillars that were taken away is half of a group of 8 caterpillars, and although the student does not write down this relationship, it is probable that the student was thinking about half

groups of caterpillars and leaves because the student next divides 12 leaves in half to get 6 leaves. The student next follows the same process for the 12 leaves, first multiplying 12 leaves by 3, and then subtracting the extra half group of leaves (6 leaves) from the 36 leaves, getting the correct answer of 30 leaves.

Handwritten student work showing a multi-step strategy to solve a problem. The work includes:

- $8 \times 3 = 24$
- $24 - 4 = 20$
- $12 \div 2 = 6$
- $12 \times 3 = 36$
- A subtraction of 6 from 36 to get 30:

$$\begin{array}{r} 36 \\ - 6 \\ \hline 30 \end{array}$$

The final answer is written as "30 leaves".

Figure 4.1. Student F’s strategy to solve the task “If 8 caterpillars eat 12 leaves each day, how many leaves would 20 caterpillars eat?”

**Robust evidence of attending to the details of the strategy.**

The following description is representative of other responses that provided robust evidence of attending to the details of the students’ strategies.

**Teacher 1:** Student F began by multiplying the 8 caterpillars by 3 to get 24. Likely because 24 is the closest multiple of 8 to 20. They then subtracted  $24 - 4 = 20$ , likely to see how far from their target [20 caterpillars] their answer is [24 caterpillars]. They then divided 12 by 2, most likely because they realized that 4 is  $\frac{1}{2}$  of 8. They then multiplied 12 leaves by 3 (same 3 from step 1) and subtracted 6 from their answer to get 30 leaves.

This teacher provided specific descriptions of the operations the student made, including the particular numbers the student used and the units associated with the numbers. The teacher also identified the mathematical reasons behind some of the operations the student performed (e.g. “Likely because 24 is the closest multiple of 8 to 20”, and “likely because they realized that 4 is  $\frac{1}{2}$  of 8”), which provides evidence that the teacher got the gist of the strategy as well.

***Features of robust evidence attending responses.***

Responses that provided robust evidence of attending to the details of the students' strategies were all fairly similar to Teacher 1's response. They described a majority of the details I originally identified, provided specific and thorough descriptions of the students' strategies, and attended to mathematical meanings of operations and numbers or mathematical connections between the different operations. For part 1 (strategies A-C), 44.4% of the teachers received a robust score, and for part 2 (Strategies D-F) 43.1% of the teachers received a robust score, which means there were similar numbers of responses providing robust evidence of attending to the details of the students' strategies between the two parts.

**Limited evidence of attending to the details of the students' strategies.**

The following description is representative of other responses that provided limited evidence for attending to the details of the students' strategies.

**Teacher 2:** First, scaled the number of leaves and caterpillars by the same factor. Then subtracted from the number of caterpillars to achieve the desired amount. Then (I have to speculate because I cannot fully follow the student's reasoning) found a proportional amount and subtracted the number of leaves by 6 for the solution.

This teacher provided evidence of attending to some of the details of the student's strategy, such as that student F scaled up the number of leaves (12) and caterpillars (8) by the same factor, and that the student subtracted from the caterpillar product to achieve the target number of caterpillars (20). In addition, they attended to the idea of scaling up the ratio, and that the student subtracted the number of leaves by 6. However, the teacher did not describe other details, such as the idea of getting close to the target number, the fact that 4 is half of 8, the division of 12 by 2, or that the student got a correct solution. In total, they provided evidence of attending to 5 of the 9 details. Additionally, the teachers' descriptions are vague and do not include many of the computations. An important standard of evidence of attending to the details

is if a reader could recreate the student's strategy based on the teacher's description, which is challenging based on this description alone. Overall, this teacher received a score of *limited* evidence.

***Features of limited evidence attending responses.***

Many of the responses that provided limited evidence of attending to the details were similar to Teacher 2's response. They described some of the details, but missed other details, and did not fully get the gist of the strategy. Some responses attended to many of the details, but did not provide evidence of getting the gist of the strategy. For example, one teacher described student F's work like so: "Student F multiplied 8 caterpillars by 3 and got the answer of 24. Then subtracted 4 from 24 and got the answer of 20. Then the student divided the 12 leaves by 2 and got the answer of 6. He/she then multiplied 12 by 3 and obtained 36, then subtracted the answer 6 and obtained the answer of 30 leaves." Notice that this teacher accurately described *all* of the operations that the student wrote down. However, the teacher did not describe the gist of the strategy because they did not describe connections between the operations, or meanings for the operations. Hence, this response (and others like it) received a score of limited evidence.

Alternatively, some responses that provided limited evidence simply did not describe enough details to be considered robust evidence, even though they clearly got the gist of the strategy. For example, one teacher described student D's strategy as "The student found how many leaves each caterpillar ate.  $1 \rightarrow 1 \frac{1}{2}$  then [sic] multiplied by the 20 caterpillars." As a reminder, student D found a unit rate by dividing 12 leaves among 8 caterpillars, and divided in a non-standard way. Then student D multiplied 20 caterpillars by  $1 \frac{1}{2}$  leaves, and using the distributive property found the correct answer of 30 leaves. This teacher provided evidence of attending to the idea that, mathematically,  $1 \frac{1}{2}$  represents the unit rate, and that student D



multiplied  $1\frac{1}{2}$  by 20. Hence, they got the gist of the strategy. However, this teacher did not describe several details, such as *how* the student found the unit rate, the non-standard way of dividing 12 by 8, and the student's use of the distributive property when multiplying. It is unclear from this description whether or not the teacher attended to these details. Hence, responses like this one also provided limited evidence of attending to the details, even though this teacher accurately described some of the details and got the gist of the strategy.

In general, the "limited evidence" responses described some of the details but were missing others. Some provided evidence of getting the gist of the strategy, while others did not. At times teachers provided descriptions that were inconsistent with the student's work, or provided descriptions that were too vague to be counted as evidence of any details, but they shared enough strategy details to be elevated into the limited evidence category of attending. In total, 40.3% of the teachers received a score of limited evidence for part 1, and 31.9% received a score of limited evidence for part 2 (see counts in table 1).

### **Lack of evidence of attending to the details of the students' strategies.**

The following description is representative of other responses that provided a lack of evidence of attending to the details of students' strategies.

**Teacher 3:** I really can't tell. I think they found the unit rate in their head and are just multiplying 1.5 times 8 and times 12.

In this response the teacher did not identify any of the operations that the student performed. Instead, the teacher described operations that did not occur in the student's work, and mathematical ideas that were inconsistent with the work. In particular, the evidence in student F's work shows the student was thinking about scaling the ratio up, and not thinking of a unit rate. Additionally, at no point did student F multiply either 8 or 12 by 1.5. The teacher also

expressed confusion at the beginning of the description, which is strong evidence that they did not get the gist of the strategy.

***Features of lack of evidence attending response.***

Many of the responses that provided a lack of evidence of attending to the details were similar to Teacher 3's response. They did not get the gist of the strategy, and they described details that were inconsistent with the work. Others simply provided descriptions that were too vague or too short to provide evidence of attending to the details. For example, one teacher wrote "Student F multiplied both values by 3 and subtracted the 'too much' difference." Notice that this teacher's response is quite short. They only attend to 2 of the 9 details, namely that the teacher multiplied "both values" by 3, and they don't even specify which values they are talking about (I assume they mean 8 caterpillars and 12 leaves). Also, the phrase "subtracted the 'too much' difference" is too vague. It could be said that both student E and student F subtracted a 'too much' difference, but each subtracted different amounts in their respective strategies. Hence, this statement could be interpreted in at least two ways, and I argue that there are other ways to interpret this vague statement as well. Hence, this teacher did not provide evidence of attending to the details of student F's strategy. In total, 15.3% of teachers received a score of "lack" in part 1, and 25% of teachers received a score of "lack" in part 2.

**Summary of the attending responses.**

Above I described in detail the three levels of evidence for attending to the details of the students' strategies. At the "robust evidence" level, teachers described most or all of the details that I identified prior to analysis, and attended to the gist of the strategy. At the "limited evidence" level, teachers described some of the details but omitted others. When omitting other details, either the teacher simply did not describe the other details, the teacher provided vague

descriptions, or the teacher provided descriptions that were inconsistent with the student’s work. Finally, at the “lack of evidence” level, teachers’ descriptions omitted most or all of the details, again either by not describing them or by providing vague or inconsistent descriptions. Many of the “lack of evidence” responses did not provide evidence of getting the gist of the strategy.

### **Interpreting the Students’ Understandings**

As a reminder, interpreting refers to the ability to interpret the students’ mathematical understandings in a way that is consistent with the student’s strategy and the proportional reasoning literature. In this section I describe the range of responses, representative responses, and trends in teachers’ responses, for each level of evidence (i.e. robust, limited, and lack). The percentages of teachers receiving each score for each part of the survey can be seen in table 2.

Notice that the percentages are fairly similar between the two parts of the survey.

Table 4.2. Percentages of teachers providing robust, limited, or a lack of evidence of interpreting students’ understandings.

N = 72	<b>Part 1</b>	<b>Part 2</b>
<b>Robust evidence</b>	15.3%	12.5%
<b>Limited evidence</b>	31.9%	33.3%
<b>Lack of evidence</b>	52.7%	54.2%

All representative interpreting responses will be drawn from part 2 of the survey, in which teachers were asked to interpret the mathematical understandings of students D, E, and F (see table 3).

Table 4.3. Artifacts, Task, Strategies, and Descriptions of D, E, and F.

Task: Each day, 8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?		
Artifact	Strategy	Description
D		Student divides 12 leaves by 8 caterpillars, to find each caterpillar eats 1.5 leaves (unit rate strategy). The student divides in a nonstandard way. The student then multiplies 1.5 leaves x 20 caterpillars, using the distributive property to help with the multiplication.
E		Student multiplies both 8 caterpillars and 12 leaves by 3, apparently scaling up the ratio to 24 caterpillars and 36 leaves. The student subtracts 4 from both quantities to obtain the 20 caterpillars, and arrives at 32 leaves. This last part is additive reasoning, and is not correct.
F		Student multiplies 8 caterpillars by 3 to get 24 caterpillars, then subtracts 4 to get 20 caterpillars. Student must have recognized that 4 caterpillars is half of 8 caterpillars, because next the student finds half of 12 leaves (6 leaves). The student then multiplies 12 leaves by 3, and subtracts the extra “half group of leaves” to get 30 leaves

**Robust evidence of interpreting students’ understandings.**

The following response is representative of other responses that provided robust evidence of interpreting the details of the students’ strategies:

**Teacher 4:** Student D thought about the problem functionally and found the constant of proportionality, then used that to find the answer. Student E has a confusion about what proportionality means in that they were able to correctly scale up by a factor of 3 the ratio of caterpillars and leaves but doesn't understand that you can't just subtract the numerator and denominator by the same value to retain proportionality. Student F seems to have a sophisticated take on proportionality but I'm making assumptions based on their work. Even though they didn't show where they got the 4 from in which they used to subtract from 24, I assumed they divided 8 by 2, just as they did 12 by 2 to get the 6, and so they were able to subtracted that partitioned unit of 4/6 from the 24/36 to arrive at their final answer.

Notice that this teacher differentiated among the three students, describing specific understandings of students D, E, and F. Additionally, the teacher expressed some hesitation at times, saying “Student F seems to have a sophisticated take on proportionality but I’m making assumptions based on their work...” This indicates that the teacher recognized a need for more evidence in order to confirm or refute their interpretations. Additionally, the teacher identified specific multiplicative understandings for each student, and these understandings were consistent with what the student did. Indeed, student D used a functional multiplicative relationship, student E used both multiplicative and additive reasoning, and student F was able to scale up ratios, partition ratios, and subtract partitioned ratios from larger ratios. Their description of student E’s understandings also shows they were attending to nuances in students’ understandings, because they recognized that student E has a partial understanding of proportional relationships.

***Features of robust evidence interpreting responses.***

15.3% of the responses in part 1, and 12.5% of the responses in part 2 received a score of robust evidence of interpreting the students’ understandings. Responses that provided robust evidence of interpreting the students’ understandings were similar in structure. They all differentiated among the three students, and provided several specific mathematical understandings for at least two students that were consistent with the students’ strategies. Furthermore, the mathematical understandings related to multiplicative relationships, which are fundamental relationships for proportional reasoning. For example, one teacher pointed out “Student D was able to think about a unit rate. Student D was also able to think about multiplication as repeated addition. Student E had trouble thinking about proportional reasoning as primarily a multiplicative relationship. Student F was able to coordinate the relationship

between leaves and caterpillars and that when you subtract a certain amount of caterpillars, you have to subtract a proportional amount of leaves.” Notice that every claim made by this teacher relates to a multiplicative relationship.

Even for student B, who used the standard cross multiplication strategy, all responses that provided robust evidence of interpreting students’ understandings made nuanced claims about the student’s potential understandings. For example, several teachers only claimed that the student knew how to set up two proportions in an equation, which is a careful claim because it does not over-attribute understandings to student B, and one needs a superficial conceptual understanding of ratios in order to set up the equation. Alternatively, other teachers wondered if student B might have used a memorized procedure, which is consistent with the literature on proportional reasoning (Post & Cramer, 1993). In general, the interpreting responses that provided robust evidence provided many specific and consistent claims about the students’ understandings.

#### **Limited evidence of interpreting students’ understandings.**

The following response is representative of other responses that provided limited evidence of interpreting the details of the students’ strategies:

**Teacher 5:** “I learned that student D and F both had a good understanding of ratios without doing any sort of cross multiplying. I also learned that student E had some understanding because they know that whatever they did to one side they had to do to the other which is a basic principle of math. I think with a little practice student E would be able to solve the problem too.”

This teacher made two general claims, one specific claim, and then one unsubstantiated claim.

Contrasting this response with the response that provided robust evidence, notice that there are fewer claims overall, and in particular there are fewer specific claims. The specific claim, “student E... know[s] that whatever they did to one side they had to do to the other,” provided some evidence that this teacher was interpreting the students’ understandings, as additive

reasoning stems from an understanding that one must perform the same operations on both quantities. The two general claims, “student D and F both had a good understanding of ratios,” also provided some evidence, but not as much as the specific claim. Indeed, student D’s and student F’s strategies demonstrate a higher understanding of proportional relationships than student E, but they also demonstrated different understandings from each other, which the teacher did not identify. Hence, this claim provides some evidence, but not much. The unsubstantiated claim, “with a little practice student E would be able to solve the problem too,” does not add evidence of interpreting the students’ understandings, but it also does not lower the amount of evidence the response has already accumulated. It is unsubstantiated because it is impossible to know what it would take for student E to understand how to solve this problem. For this teacher, the specific claim was an important indication that the teacher was at least partially interpreting the students’ understandings.

***Features of limited evidence responses.***

31.9% of the responses in part 1, and 33.3% of responses in part 2 received a limited evidence score. All responses that provided limited evidence of interpreting the students’ understandings made some claims about students’ understandings that were consistent with the work and the literature on proportional reasoning. Similar to teacher 5’s response, many other teachers’ responses included both specific claims and vague/oversimplified claims about the students’ understandings. Additionally, responses that provided limited evidence tended to differentiate between two of the students, but not all three students, usually by talking about two of the students as having general understandings (as teacher 5 did with student D and student F). Some responses also included unsubstantiated claims based on the work shown (e.g. “Student E appears to be trying some strategy taught to them...”), or claims related to pedagogical principles

rather than the three students' mathematical understandings (e.g. "I think the biggest lesson here is that students will digest what you are teaching them in different ways."), along with claims that were consistent with the students' strategies. Overall, there was some evidence of interpreting the students' understandings, but it was limited.

**Lack of evidence of interpreting students' understandings.**

The following response is representative of other responses that provided a lack of evidence of interpreting the details of the students' strategies:

**Teacher 6:** "These three students all did not demonstrate a higher understanding of proportional reasoning. Student D had the best method that led to the correct answer, but utilized fractions and made incomprehensible [*sic*] work. Student E and F both showed very little understanding and seemed to be scrambling to find a correct answer. Student F had the right answer, but the work could not be followed and made no sense. So I'd assume he/she had the answer and attempted to work backwards."

This teacher first gives a vague description for each of the three students' understandings, saying they "did not demonstrate a higher understanding of proportional reasoning." The participant's claim not only provides a generalization across the three students' understandings, but is inconsistent with student D's and student F's work, both of whom demonstrate reasonable understandings of proportional relationships. The teacher then claims "student D had the best method," which is a value statement with no support, and hence is not an interpretation consistent with the students' understandings or the related literature. The teacher even admits to confusion about parts of student D's work, calling it "[i]ncomprehensible". Next, the teacher claims that both E and F had very little understanding, and interprets their work as "scrambling to find a correct answer." This is inconsistent with both E and F, who both have at least some reasonable ideas supporting their strategies. In particular, both E and F are comfortable scaling up ratios by whole numbers. Additionally, F demonstrates a stronger understanding of the multiplicative nature of proportional relationships when F partitions the ratio of 8 caterpillars and



12 leaves into a smaller ratio of 4 caterpillars and 6 leaves. Hence, the interpretation that student F “had the answer and attempted to work backwards” is not supported by the evidence in the work. In total, the inconsistencies, overgeneralizations, and vague claims provide me with a *lack of evidence* that this teacher interpreted the students’ understandings.

***Features of lack of evidence interpreting responses.***

52.7% of the responses in part 1, and 54.2% of the responses in part 2 received a lack of evidence score. Many responses were similar to Teacher 6’s response, in that they made oversimplified, inconsistent, or vague claims about the students’ understandings. Whereas the limited evidence responses included one or more claims that were specific, consistent with the work, and consistent with the proportional reasoning literature, lack of evidence responses did not include claims about students’ understandings that were consistent with the students’ work.

Not all responses that provided a lack of evidence provided claims about the students’ strategies that were inconsistent with the work shown, and not all expressed confusion. Some of the interpretations were just too vague to be counted as evidence of interpreting the students’ understandings. For example, one teacher simply claimed “each student used prior knowledge and understanding of multiples to find common numbers.” It is unclear what the teacher meant by “used prior knowledge,” so this response does not provide me with evidence that this teacher was interpreting the students’ understandings.

In addition, many teachers made claims about other ideas unrelated to the students’ understandings. For example, one teacher claimed “I have insight as to how each student sets up a proportion problem so when working on future proportions I would be able to reference their preferred style of understanding.” This teacher did not describe the students’ understandings, and talked about their future plans for their own instruction of proportional reasoning. Claims

about other pedagogical topics appeared in the limited evidence responses as well, but unlike the limited evidence responses, the lack of evidence responses did not include other claims that would have given more evidence of interpreting the students' understandings. Instead, these ideas stood alone and did not provide me with evidence that the teacher was interpreting the students' understandings.

In sum, interpreting responses that scored a "lack of evidence" included claims that either were inconsistent with what the students did, were too vague or general to count as evidence, or were about ideas other than the students' mathematical understandings.

### **Summary of interpreting responses.**

In the previous section I described in detail the three levels of evidence for interpreting the students' mathematical understandings. At the "robust evidence" level, teachers described several specific mathematical understandings for at least two of the three students. At times the teacher made general claims about understandings as well, but these were still consistent with the students' work. At the "limited evidence" level, teachers' responses were a mixture of specific and consistent claims, general yet consistent claims, unsubstantiated claims, inconsistent claims, and/or claims about other pedagogical ideas. Overall, there was a balance of confirming and disconfirming evidence of interpreting the students' understandings. Finally, at the "lack of evidence" level, teachers' responses often did not differentiate among the three students, and most of the claims were either inconsistent, unsubstantiated, or too vague.

### **Deciding how to Respond on the Basis of the Students' Understandings**

As a reminder, deciding how to respond refers to deciding how to respond on the basis of the students' understandings, as exhibited in the students' strategies. In this section I will describe the range of responses teachers provided, including representative responses of each

level of evidence (i.e. robust, limited, and lack) and trends in teachers' responses. All representative responses will be drawn from part 2 of the survey, in which teachers were asked to decide how to respond to students D, E, and F (see table 3). The percentages for the number of teachers receiving each score for each part of the survey can be seen in table 4. Notice that the counts are fairly similar between the two parts of the survey.

Table 4.4. Percentages of teachers providing robust, limited, or a lack of evidence of deciding how to respond on the basis of the students' understandings.

N = 72	Part 1	Part 2
<b>Robust evidence</b>	12.5%	8.3%
<b>Limited evidence</b>	30.5%	31.9%
<b>Lack of evidence</b>	56.9%	59.7%

### **Robust evidence.**

The following response is representative of other responses that provided robust evidence of deciding how to respond based on the students' understandings:

**Teacher 7:** If 15 caterpillars eat 9 leaves, how many leaves would 25 caterpillars eat?

Rationale: I would be interested to see if Student D would be able to make sense of a unit rate in which a caterpillar eats less than one whole leaf.

Also, students E and F all used a strategy of multiplying the original amount and then adjusting by subtracting. I would want to see if they would use that strategy again. I imagine that they would double the 15 caterpillars and then try to subtract out how much 5 caterpillars eat.

If student E still made the same mistake, I would ask him/her how much 5 caterpillars eats.

Notice that this teacher provided only one problem, but the rationale showed that this problem was tailored to each of the three students. For student D, the teacher pointed out that the unit rate is now a number less than 1, and wondered if this would influence student D's strategy. This is consistent with student D's strategy, who used a unit rate to solve the previous problem, and it provides student D with an opportunity to extend their thinking. For students E and F, the

teacher anticipated that they will probably multiply the original amount and subtract, which is consistent with both students' solutions. The teacher expressed curiosity in seeing if they would use their strategies again, showing an openness to students' ideas. Additionally, partitioning into  $\frac{1}{3}$ 's might be a little more difficult than partitioning into halves, which means the teacher's choice of numbers allows student F to extend his or her thinking as well. Finally, the teacher anticipates that student E might make the same mistake, and provides a follow up question for student E. This follow up question is to consider how many leaves 5 caterpillars eat, which can be thought about in two ways. Either the student recognizes the  $\frac{1}{3}$  relationship between 5 and 15, or the student subtracts 10 from 15 to get to 5. If it's the former, then this opportunity allowed the student to correct themselves. If it's the latter, the student will probably subtract 10 from both 15 and 9, and see that 5 caterpillars eat -1 leaves, which may induce disequilibrium in the student's mind. Either way, the follow up task provides more evidence that this teacher is posing this task based on all three students' mathematical understandings. There is also an openness to the students' ideas, as the teacher says "I would be interested if..." and "I would want to see if...". Openness shows the teacher was not forcing particular strategies on the student, but providing opportunities for the student to extend their strategies. Overall, this response provides robust evidence of deciding how to respond based on the students' understandings.

***Features of robust evidence deciding responses.***

12.5% of responses in part 1 and 8.3% of responses in part 2 that provided robust evidence differentiated among the students in some way, either in the rationale (similar to Teacher 7 above) or in the problems. They tended to express an openness to students' ideas, and focused on providing opportunities rather than forcing different strategies. In addition, almost all

responses provided a specific next task, with number choices that appropriately challenged or supported students' ideas. Number choice was an important indicator for deciding how to respond on the basis of the student's understandings, because particular sets of numbers support particular strategies (Carney, Smith, Hughes, Brendefur, & Crawford, 2016). For example, one teacher used the same context as part 1, and extended the problems like so: "To student A: How much would 7 mice eat? To student B: How many mice would be needed for 24 pellets? To student C: How much would 8 mice eat?" The teacher then explained that for the first task, they wanted to test the student's understanding of the unit rate, and see if they would add 3 pellets for 1 mouse to the original 18 pellets for 6 mice, or if they would multiply 7 times 3. For the second task, the teacher was curious if the student had flexibility in setting up the proportions, and could set up a new proportion for a missing value of mice instead of pellets. For the third task, the teacher wanted to challenge the student with a non-integer ratio scaling up the 6 to the 8, and wondered if the student would try to make sense of this relationship or try to leverage the relationship between 24 mice and 8 mice. Clearly, each number was carefully selected by the teacher with the students' strategies in mind, and the rationales for each problem were consistent with the students' strategies.

Some of the responses did not provide a next task, but asked students to do something related to the current task. For example, one teacher asked student B if they could solve the problem in a different way, other than the cross-multiplication strategy. The teacher explained: "For student B, I don't know that changing the numbers would tell me anything because the cross multiplication would still work out nice, I would want to see them solve it a different way so I could see how they are reasoning with the quantities." This teacher anticipated how student B might solve other tasks, and tried to respond in a way that would elicit understandings about the

quantities from the student. Importantly, their rationale is consistent with the literature on proportional reasoning, which shows that cross-multiplication does not signify much with respect to students' understandings of proportions (Cramer & Post, 1993). This example also shows that even when teachers were confronted with a procedural strategy, responses that provided robust evidence of deciding how to respond were able to decide on teacher moves that would allow them to learn more about the students' understandings.

In sum, responses that provided robust evidence differentiated among the students and posed tasks that would challenge the students in ways that allowed them to use and extend their strategies, or would help the teacher learn more about the students' ideas. The teachers' rationales often identified the difficulties that the students might have to overcome when solving the new tasks, anticipated how students might solve the next task, or discussed how the task allowed students to use or build on their current strategies, all in a way that was consistent with the student's work and with the literature on proportional reasoning. Teachers tended to express openness to new ideas, and did not force students to adopt new strategies. Tasks were carefully selected, and number choice was an important factor. In particular, the numbers were chosen in order for students to (a) grapple with more difficult factors, (b) to use their strategies in new ways, (c) solve for a different missing quantity (i.e. search for missing value of mice instead of missing value of pellets), or (d) consider decreasing multiplicative relationships instead of increasing relationships.

**Limited evidence.**

The following response is representative of other responses that provided limited evidence of deciding how to respond based on the students' understandings:

**Teacher 8:** For students D and F: If 4 cats eat 18 mice, then find how many mice will be eaten by 15 cats.

For student E: I will give the first problem you have given us [6 mice eat 18 food pellets, how many food pellets do 24 mice eat?].

Rationale: I think this is a similar problem. I would like to see if they use the same method to solve it.

Notice that this teacher provided two different tasks, one for the two students who obtained a correct answer and one for the student who obtained an incorrect answer. The numbers in the first task are appropriately more difficult than the task the students solved previously, because dividing 18 by 4 will result in a remainder, and scaling up the ratio from 4 to 15 will require the student to partition the ratio into  $\frac{1}{4}$ 's. However, this teacher did not point out these difficulties, nor did the teacher anticipate how students D and F would solve the problem. The teacher says "I would like to see if they use the same method to solve it." This statement indicated an openness to the students' ideas, which was also present in robust evidence responses, but unlike responses that provided robust evidence this statement does not describe what new mathematics they might see if the students used new methods, or what new methods they might see. Their only rationale for selecting this task is because it "is a similar problem."

For student E, it is unclear why they selected the task with integer ratios. Perhaps it is because student E demonstrates a lack of understanding of partitioning ratios. However, student E also demonstrates that they can find larger ratios by multiplying by whole numbers, which indicates that student E would probably solve the new task correctly without any support, and they would not grapple with the idea of partitioning ratios in the process of solving. Hence, the new task does not challenge student E, and allows student E to use a strategy in which they have already demonstrated competence. It is unlikely that student E will extend their thinking, and the teacher does not discuss what they would learn from posing this task.

In sum, the differentiated tasks and openness to students' ideas provided some evidence of deciding how to respond based on the students' understandings, but the response lacked

specific details for what the teacher hoped to learn, or how the teacher hoped to challenge these students. Additionally, the teachers' choices of numbers were supportive for two of the students (D & F), but not the third student (E).

***Features of limited evidence deciding responses.***

30.5% of the responses in part 1 and 31.9% of the responses in part 2 provided limited evidence of deciding how to respond based on the students' understandings. Responses that provided limited evidence tended to build on students' ideas in vague or general ways. Many responses selected different tasks for at least two of the students, but not the third, either by grouping two students together (as in teacher 8's response) or by ignoring the third student.

Some teachers provided a single task for all three students, and did not provide different reasons for each student. However, these responses still provided some evidence of basing the decision on the students' understandings, either by showing an openness to the students' ideas, anticipating a potential strategy, or challenging students in a way that was consistent with the literature. For example, a teacher posed the following task: "Each day, 8 caterpillars eat 12 leaves. How many leaves would 3 caterpillars eat?" For the rationale, the teacher said: "Instead of increasing the number of caterpillars, I reduced the number. If students subtracted by 5 from each (which would be 3 eat 7), I would question the reasonableness of their solution because 4 caterpillars eat 6 leaves." This task allows student F to extend his/her reasoning, but not student D, who would use the same unit rate as the previous problem. The teacher does not address this distinction between D and F. Additionally, the teacher anticipates additive reasoning by at least one student, and has a plan to follow up with such an idea. However, when the teacher says "I would question the reasonableness of their solution because 4 caterpillars eat 6 leaves," it is unclear if the teacher is providing the student with an opportunity to grapple with these



relationships or is being directive and telling the student their response is unreasonable. Hence, this response provides limited evidence of deciding how to respond on the basis of the students' understandings. In sum, when tasks appropriately challenged one of the students to extend their proportional reasoning (but not another), and there was some indication of consideration for a student's strategy, the response provided limited evidence.

Similar to the robust evidence category, there were also responses that were not tasks, but questions for the student about their strategy. For example, a teacher wanted to ask students D, E, and F about relationships between their work and the context. The teacher wrote "For example, which ones [numbers] represent leaves and which ones represent caterpillars? For each operation in their work, what is happening to the number of leaves and caterpillars?" In their rationale, they explained that asking students to relate their computations to the context often deepens students' understandings. They also conjectured that these questions may help student E to "make better sense of the numbers (s)he was manipulating." Clearly there is some evidence that this teacher is deciding how to respond based on students' understandings. However, unlike similar responses in the robust category, it was less clear what mathematical understandings this teacher wanted to support in students D, E, and F, other than a general principle that connecting ideas to the context is important. For example, if this teacher had talked about supporting student E to split up the ratio of 8 and 12, this would have given me more evidence that they were deciding how to respond based on the students' mathematical understandings. Hence, responses such as this one provided limited evidence of deciding how to respond.

In sum, responses that provided limited evidence tended to provide general reasons related to the students' understandings for posing the tasks they posed, with connections to the students' strategies that were less clear than responses providing robust evidence. Some of the

responses differentiated tasks among the students, but not all. In addition, some responses provided rationales or tasks that were a mix of being consistent and inconsistent with the student's understandings. Overall, there was a balance of confirming and disconfirming evidence of deciding how to respond based on the students' mathematical understandings.

### **Lack of evidence.**

The following response is representative of other responses that provided a lack of evidence of deciding how to respond based on the students' understandings:

**Teacher 9:** I would simply give them another similar problem with an uneven ratio, and after showing them cross multiplication, would walk them through this problem: Each day, 4 caterpillars eat 10 leaves, how many leaves would 22 caterpillars eat?

Rationale: Using cross multiplication would test their multiplication skills, but would also give them an easier visual understanding of ratios. It is also a method easier to use, and more likely to work with non-integer ratios.

This teacher says they would show the students the cross multiplication strategy, and would walk the students through the problem. This forces students to adopt a particular strategy and disregard the ones they were using, which is the opposite of the openness to students' ideas expressed in responses providing "limited evidence" and "robust evidence". Forcing a particular strategy is a clear indication that the teacher is not deciding how to respond based on the students' understandings, and is instead deciding how to respond based on what they believe is the best strategy. In addition to forcing a particular strategy, this teacher chose the cross multiplication strategy, which does not necessarily reflect or support a strong understanding of proportional reasoning (Cramer & Post, 1993).

### ***Features of lack of evidence deciding responses.***

56.9% of the responses in part 1 and 59.7% of the responses in part 2 provided a lack of evidence of deciding how to respond based on the students' understandings. In addition to

responses that forced particular strategies, there were also responses that were based on inconsistent interpretations of the students' understandings. For example, in part 2 a teacher posed the task "You take your dog on a walk once per week. How many times will you have taken your dog on a walk after two weeks?" For their rationale, they said "These students need more practice with the basics of proportional reasoning. Therefore, a less difficult problem will allow for this." This task is much simpler than the task the students just solved. The claim "these students need more practice with the basics" is inconsistent with the work shown by D, E, and F, because they all demonstrated some understanding of proportional reasoning. Many teachers in part 2 assumed the students' proportional reasoning skills were low, and thus felt the students needed easier problems, which was inconsistent with the students' strategies.

Some teachers posed problems with other pedagogical ideas in mind. For example, in part 2 a teacher responded "I would try to make the problem related to a [*sic*] everyday life problem... This way students will feel more interested and see that math is everywhere and it is important." This teacher, along with a few others, wanted to use a new context that they felt would be more interesting to students. Other teachers decided how to respond with the goal of creating a welcoming classroom environment. While these are important things to worry about, the responses themselves do not provide evidence that these teachers were deciding how to respond based on the students' mathematical understandings.

Alternatively, other teachers focused on introducing students to a new content area. For example, one teacher wanted to pose a distance - rate - time problem because "motion problems with constant speeds involve proportions, can be conceptually challenging, and are concrete." It is unclear why motion problems are appropriate next steps for these students in particular.

Responses like these appear to be focused on next mathematical topics in a curriculum, rather than these particular students' understandings.

In sum, responses that provided a lack of evidence tended to provide problems and rationales that were inconsistent with the students' work and with the literature on proportional reasoning, that focused on other pedagogical ideas or mathematical content domains, that forced students to adopt particular strategies and disregard their current strategies, or that simply did not provide enough evidence or clarity. Again, the teachers' choices of numbers at times indicated their intentions, especially when teachers chose easier numbers for students who did not need easier numbers or when teachers chose numbers that would only challenge one student and not the other.

#### **Summary of deciding how to respond responses.**

In the previous section I described in detail the three levels of evidence for deciding how to respond based on the students' mathematical understandings. At the "robust evidence" level, teachers either posed different tasks for each student, or described how the task they posed could support all three students in their mathematical understandings. The tasks and rationales were consistent with the students' strategies and with the proportional reasoning literature, and focused on providing opportunities for students to extend their understandings. At the "limited evidence" level, teachers' responses were partially consistent with the students' strategies, partially differentiated among the three students, included general reasons for the tasks posed that were related to the students, and still tended to express an openness to the students' ideas. Overall, there was a balance of confirming and disconfirming evidence of deciding how to respond based on the students' understandings. Finally, at the "lack of evidence" level, teachers' responses often did not differentiate among the three students, and most of the problems posed

and associated rationales were either inconsistent with the students' strategies, forced students to adopt the teacher's preferred strategy, focused on other mathematical or pedagogical ideas, or were too vague to interpret meaning.

### **Similarities and Differences of the Professional Noticing Expertise Among the Three Teacher Groups**

In the previous section, I described the range of responses to the professional noticing prompts, focusing on each component-practice, and describing each level of evidence within each component-practice with illustrative examples. Those descriptions will act as the foundation for the comparisons I make in this section. As a reminder, my goal is to compare and contrast the professional noticing expertise of secondary mathematics prospective teachers (PSTs), experienced teachers (ETs), and emerging teacher leaders (ETLs). This section is organized into three subsections. In each subsection I compare and contrast one of the component-practices (i.e. attending, interpreting, and deciding how to respond) among the three groups of teachers.

In order to make these comparisons, I first converted each teacher's scores to numerical values (Robust = 2, Limited = 1, and Lack = 0). Then, I gave each teacher an average score for each component-practice. Hence, each teacher received an average score of either 0.0, 0.5, 1.0, 1.5, or 2.0, representing the amount of evidence that teacher provided of considering the students' mathematical thinking when responding to the professional noticing prompts. I then used these scores to calculate averages for each group, and conducted 1-tailed t-tests between ETs and ETLs, and between PSTs and ETs. I display my results in tables 5, 6, and 7, and graphs of the results in figures 2, 3, and 4. In the tables and graphs, the column "robust" represents

average scores of 1.5 or 2.0, the column “limited” represents an average score of 1.0, and the column “lack of” represents average scores of 0.0 and 0.5.

**Comparing the Attending Skills of PSTs, ETs, and ETLs.**

The percentages of each teacher group demonstrating particular amounts of evidence of attending to the details can be seen in table 5, and in figure 2. Notice that the prospective and experienced teachers have fairly similar percentages of teachers receiving each scores.

Additionally, both groups tended to have more teachers providing evidence of attending to the details than not providing evidence of attending to the details. In contrast, emerging teacher leaders clearly provided much more evidence of attending to the details. None of the emerging teacher leaders received an average score of 0 or 0.5, which is because all emerging teacher leaders provided at least limited evidence of attending to the details on both part 1 and part 2.

ETLs had an average score of 1.79, ETs had an average score of 1.10, and PSTs had an average score of 1.15. Standard deviations are reported in parentheses.

Table 4.5. Percentages of teachers demonstrating particular levels of evidence of attending to the details of the students’ strategies, and averages and standard deviations of each teacher group.

	<b>Lack of Evidence (0.0, 0.5)</b>	<b>Limited Evidence (1.0)</b>	<b>Robust Evidence (1.5, 2.0)</b>	<b>Average / (Std. Dev.)</b>
<b>PSTs</b>	26.6 %	20 %	53.3 %	1.15 (0.62)
<b>ETs</b>	36.6 %	16.6 %	46.6 %	1.10 (0.72)
<b>ETLs</b>	0 %	8.3 %	91.6 %	1.79 (0.32)

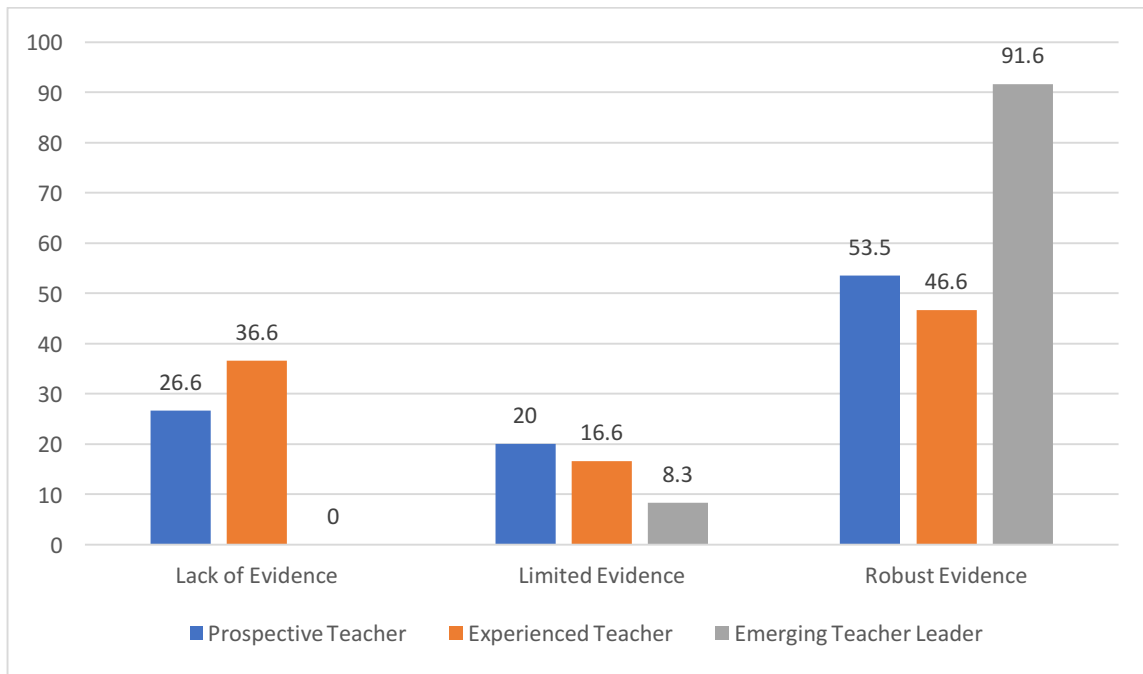


Figure 4.2. Percentages of “lack”, “limited”, and “robust” scores per teacher group in attending.

As a reminder, I conducted 1-tailed 2-sample t-tests of unequal variance on the adjacent pairs (i.e. ETLs vs. ETs, and ETs vs. PSTs), with a Bonferroni correction of  $\alpha = 0.0083$ . The 1-tailed t-tests showed that ETLs scored significantly higher than ETs at the  $\alpha = 0.0083$  level,  $t(11.3) = 0.248$ ,  $p < 0.0001$ ,  $d = 1.09^7$ , even though I had a small sample size of ETLs. Small sample sizes make it more difficult to obtain significant results, which means the difference between ETLs and ETs is relatively large. This test for significance provides more evidence that sustained professional development supports teachers’ attending skills. In contrast, ETs did not score significantly higher than PSTs at the 0.0083 level,  $t(53.1) = -0.015$ ,  $p = 0.426$ ,  $d = -0.07$ , even with both groups having a larger sample sizes than the ETL group.

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<sup>7</sup> Due to the different sample sizes of ETLs and ETs, Hedge’s ‘g’ was used to calculate effect sizes between these two groups for all three professional noticing component-skills. ETs and PSTs had the same sample sizes, so Cohen’s ‘d’ was used for those comparisons.

Looking at the types of responses provided by each teacher group, some other similarities and differences arose. First, in each teacher group there were many responses that accurately described the students' strategies, including the gist of the strategy. However, ETLs differed from ETs and PSTs in that they had a significantly higher proportion of teachers providing such descriptions. ETs and PSTs were similar to each other because they had similar proportions of teachers providing such descriptions. Second, each group of teachers had at least a couple responses that could not figure out what the student did, or made claims about the students' strategies that were inconsistent with the student's work. ETLs had the fewest number of such responses, with just 5.5% of all responses (12 teachers x 6 strategies = 72 responses) not getting the gist of the strategy and expressing confusion, and just 2 of those responses making inconsistent claims. In contrast 24.4% of the responses given by PSTs (30 teachers x 6 strategies = 180 responses), and 18.8% of the responses given by ETs (30 teachers x 6 strategies = 180 responses) admitted they did not understand the student's strategy and/or made claims that were inconsistent with the student's work. Overall, PSTs and ETs were similar to each other in many ways when describing what the students did. In both groups there was a large proportion of thorough and accurate descriptions of students' actions, and a small (but still significant) proportion of short descriptions omitting details, of inconsistent descriptions, and of teachers who did not get the gist of the strategy.

However, ETLs' responses that exhibited confusion differed in an important way from PSTs and ETs. In particular, every ETL who expressed confusion or made an inconsistent claim admitted they were hesitant about their claims. They appeared to believe that the student was making some sense, but they could not figure out what sense the student was making. Even the two responses that made inconsistent claims began by hedging their claims, saying things like "I



am not sure what the reasoning is, but maybe...”, which indicates that the teacher was hesitant and unsure of his/her conjecture. In contrast, about one-third of the ETs and PSTs who did not get the gist of the strategy appeared quite certain that the student was the one who was confused, and not the teacher. Hence, it appears all ETLs had developed a disposition to question their interpretations, while some ETs and PSTs had not.

### **Comparing the Interpreting Skills of PSTs, PTs, and ETLs.**

The percentages of each teacher group demonstrating particular levels of evidence of interpreting the students’ mathematical understandings can be seen in table 6 and in figure 3. Overall, ETLs showed much more evidence of interpreting the students’ understandings than both ETs and PSTs. In the ETL group, only 25 % of the teachers provided an overall score of a lack of evidence (i.e. a score of 0 or 0.5). The other 9 ETLs provided either limited evidence or robust evidence of interpreting the students’ understandings on both parts 1 and 2. In contrast, 76.6 % of the PSTs and 63.3 % of the ETs provided an overall score of a lack of evidence. There were few limited evidence responses from these groups and even fewer robust evidence responses. ETLs had an average score of 1.17, ETs had an average score of 0.58, and PSTs had an average score of 0.40. Standard deviations are reported in parentheses.

Table 4.6. Percentages of teachers demonstrating particular levels of evidence of interpreting the students’ understandings, and averages and standard deviations of each teacher group.

	<b>Lack of Evidence (0.0, 0.5)</b>	<b>Limited Evidence (1.0)</b>	<b>Robust Evidence (1.5, 2.0)</b>	<b>Averages / (Std. Dev.)</b>
<b>PSTs</b>	76.6 %	16.6 %	6.6 %	0.40 (0.57)
<b>ETs</b>	63.3 %	20 %	16.6 %	0.58 (0.59)
<b>ETLs</b>	25 %	25 %	50 %	1.17 (0.77)

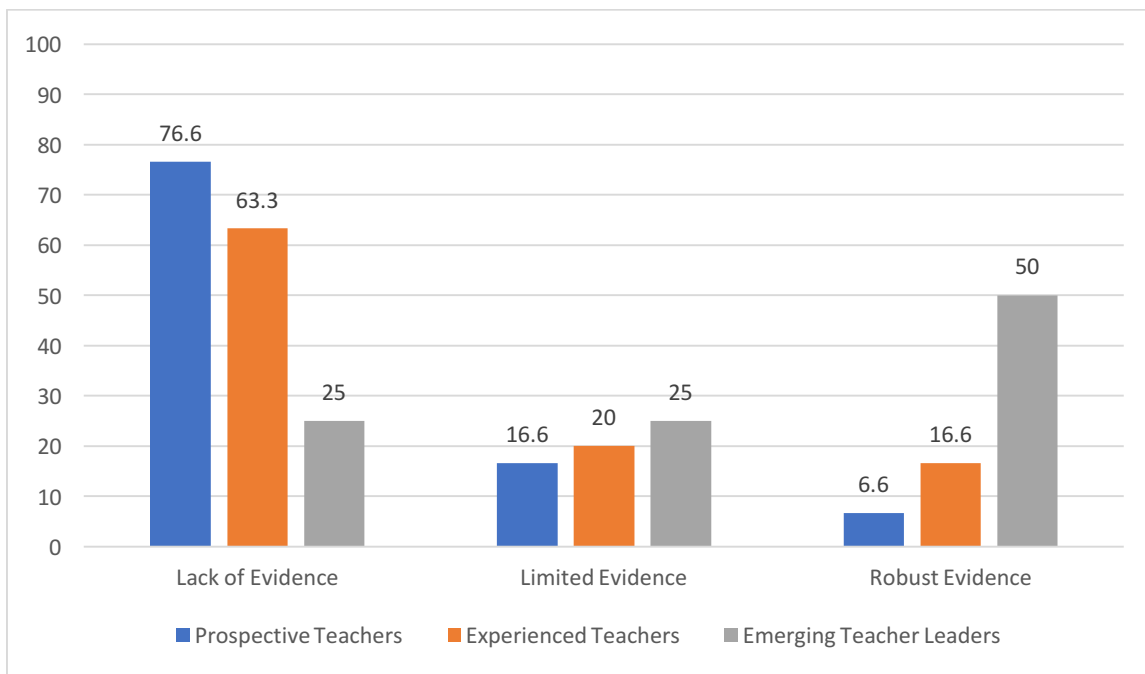


Figure 4.3. Percentages of “lack,” “limited,” and “robust” scores per teacher group in interpreting.

Again, I ran tests for significance. Differences between ETLs and ETs were not significant at the  $\alpha = 0.0083$  level,  $t(21.99) = 0.220$ ,  $p = 0.018$ ,  $d = 0.92$ , and differences between ETs and PSTs were not significant at the 0.0083 level,  $t(57.6) = 0.073$ ,  $p = 0.117$ ,  $d = 0.31$ . Even though the means between ETLs and ETs were quite large (almost one whole standard deviation away from each other, according to the effect size), the level of significance was not obtained. This is likely due to the large variance in the ETL group for this component-skill (standard of deviation was 0.77), as well as the small sample size of 12 ETLs.

The robust evidence responses were fairly similar across all groups. They all differentiated among the three students, and identified specific mathematical understandings for at least two of the three students. They also tended to discuss their ideas in a careful way, either by being careful not to over-attribute understandings to students, or by qualifying claims with words like “perhaps...”, or “the student may have understood...”, or others.

As described earlier, there were a variety of responses that provided limited and a lack of evidence. Some made claims that were inconsistent with the students' ideas, others provided vague or general statements that could have meant many different ideas, and others discussed pedagogical topics that were not related to the students' ideas. When ETLs received lower scores, it was generally because they made vague comments, rather than inconsistent statements. Inconsistent statements appeared in only 8.3% of the responses ETLs gave. In contrast, inconsistent statements appeared in 38.3% of the responses PSTs gave, and in 21.6% of the responses ETs gave. PSTs gave many more inconsistent claims than ETs. When ETLs and ETs received lower scores, it tended to be because they talked about other pedagogical topics or made vague or unclear statements.

### **Comparing the Deciding how to Respond Skills of PSTs, PTs, and ETLs**

The percentages of each teacher group demonstrating particular levels of evidence of deciding how to respond based on the students' understandings can be seen in table 7 and in figure 4. 25% of ETLs fell into the lack of evidence category, but all 25% provided limited evidence of deciding how to respond on the basis of the students' understandings for at least one part of the survey, as all ETLs had an average score of 0.5 or higher. Hence, every ETL provided at least some evidence of deciding how to respond based on the students' understanding. In addition, the 33.3% of the ETLs that provided robust evidence did so on both parts 1 and 2, which did not occur for any ET or PST. All ETs that fell in the robust evidence category provided robust evidence on one of the parts, and limited evidence on the other part. ETs provided less evidence of deciding how to respond based on the students' understandings, with 63.3% of ETs receiving an overall score of lack of evidence (i.e. a score of 0 or 0.5). PSTs provided the least amount of evidence, with 93.3% receiving an overall score of lack of

evidence, and no PST providing robust evidence on either part 1 or part 2. Unlike the previous two component-skills, there appears to be a larger difference between ETs and PSTs in the amount of evidence provided for basing decisions on the students’ mathematical ideas. ETLs had an average score of 1.21, ETs had an average score of 0.57, and PSTs had an average score of 0.20. Standard deviations are reported in parentheses.

Table 4.7. Percentages of teachers demonstrating particular levels of evidence of deciding how to respond based on the students’ mathematical understandings, and averages and standard deviations of each teacher group.

	<b>Lack of Evidence (0.0, 0.5)</b>	<b>Limited Evidence (1.0)</b>	<b>Robust Evidence (1.5, 2.0)</b>	<b>Average / (Std. Dev.)</b>
<b>PSTs</b>	93.3 %	6.6 %	0 %	0.20 (0.31)
<b>ETs</b>	63.3 %	23.3 %	13.3 %	0.57 (0.53)
<b>ETLs</b>	25 %	41.6 %	33.3 %	1.21 (0.59)

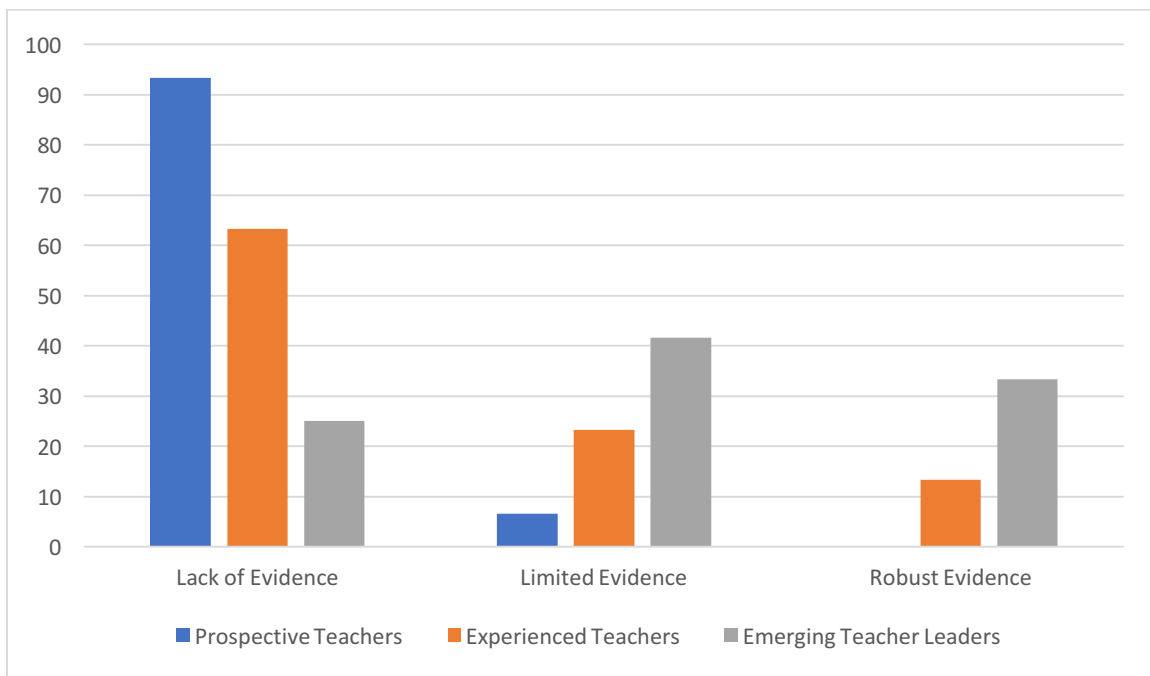


Figure 4.4. Percentages of “lack,” “limited,” and “robust” scores per teacher group in deciding how to respond.

Again, I ran tests for significance. This time, differences between ETLs and ETs were significant at the  $\alpha = 0.0083$  level,  $t(20.50) = 0.346$ ,  $p = 0.0028$ ,  $d = 1.17$ , and differences between ETs and PSTs were also significant at the 0.0083 level,  $t(35.4) = 0.235$ ,  $p = 0.0011$ ,  $d =$

0.85. These tests provide evidence that sustained professional development supports teachers' deciding how to respond skills, and that teaching experience supports this component-skill as well.

Responses that provided robust evidence of deciding how to respond based on the students' understandings were fairly similar to each other, regardless of whether the response was given by an ET or an ETL. All of the robust evidence responses either provided separate tasks for each student, or described why the task would be helpful for each student. The teachers built on students' ideas by allowing them to use their strategies and extend or challenge their ideas, and many teachers anticipated how students would respond to the tasks. Responses that provided limited evidence were also fairly similar across the three teacher groups as well. Limited evidence responses addressed the students' ideas still, but in a vague and general way. Across the three groups, there were teachers whose decisions were partially consistent with the students' strategies, and there were teachers whose decisions were vague and loosely based on the students' strategies.

In the responses that provided a lack of evidence, there were some noticeable differences. Of the four responses from ETLs that provided a lack of evidence, three (75% of ETLs' "lack of" responses) posed tasks and provided rationales that were inconsistent with what the students did, because they conjectured the students did not understand and needed a simpler problem. This means that, even though they based their decisions on an inconsistent interpretation of the students' understandings, they still based their decisions *on what they interpreted about the students*. (The fourth gave a rationale that was vague and difficult to follow.) In contrast, only 21.9% of the "lack of" responses given by ETs, and 25% of "lack of" responses given by PSTs, based decisions on inconsistent interpretations of the students' ideas. This means the other

78.1% of ETs' responses and 75% of PSTs' responses did *not* appear to be connected to what the students did to solve the task. Hence, ETs and PSTs with lower scores often based their decisions on what they believed were important mathematical topics, content areas, or contexts to consider, with little regard for the students' ideas.

### Summary of Results

Average scores and standard deviations for each teacher group in each component-skill can be seen in table 8. The table shows that the ETLs provided much more evidence of attending to the details of students' strategies than ETs, whereas the PSTs' and ETs' attending scores were almost the same. In interpreting, none of the pairwise comparisons were statistically significant, although I partially attribute the lack of significance between ETLs and ETs to the statistical power of the test (given the small n of 12 ETLs), and the large variance associated with the ETL group. Effect size between ETLs and ETs was 0.92, which means the average scores almost differed by a whole pooled standard deviation. In deciding how to respond, ETs provided more evidence than PSTs and ETLs provided much more evidence than ETs in deciding how to respond based on the students' understandings. In the final section of this chapter I discuss these findings.

Table 4.8. Average scores and standard deviations for each teacher group in each component-skill.

	<b>Attending</b>	<b>Interpreting</b>	<b>Deciding how to Respond</b>
<b>PSTs</b>	1.15 (0,62)	0.40 (0.57)	0.20 (0.31)
<b>ETs</b>	1.10 (0.72)	0.58 (0.59)	0.57* (0.53)
<b>ETLs</b>	1.79* (0.32)	1.17 (0.77)	1.21* (0.59)

\*Significant pairwise comparison at the  $\alpha = 0.0083$  level adjusted using a Bonferonni correction.

### **Discussion of PSTs', PTs', and ETLs' professional noticing of students' mathematical thinking**

In this section I discuss the results in the context of teacher education. To do so I organize this section into three subsections, where in each subsection I discuss each teacher

group's professional noticing skills in greater detail. I leverage the similarities and differences I identified in the previous sections to help me characterize the professional noticing expertise of each teacher group.

### **The Professional Noticing of Students' Mathematical Thinking Expertise of ETLs**

Clearly, ETLs provided the most evidence of considering the students' mathematical thinking when responding to the professional noticing prompts. These teachers had completed their 4<sup>th</sup> year of a professional development project that had an intense focus on students' mathematical thinking. Based on the difference in scores between ETs and ETLs, it appears the professional development project and its focus on students' mathematical thinking supported these teachers to develop their professional noticing expertise in all three component-skills.

In the attending component-skill, there appeared to be a ceiling for these teachers as most of the emerging teacher leaders provided robust evidence of attending to the students' strategies. An overwhelming majority of them provided specific, accurate, and thorough descriptions of the students' strategies.

In the interpreting component-skill, many of the emerging teacher leaders provided limited evidence or robust evidence of interpreting the students' understandings. ETLs tended to identify different understandings for each of the students. They also tended to hedge their claims with phrases such as "I think that..." and "It seems like..." and "I can't quite tell for sure but...", which implies they recognized a need for more evidence in order to make stronger claims. Many of the responses described specific understandings as well, which indicates they were able to dive deeper than surface level descriptions or general descriptions of understandings. However, there appeared to be some room for growth for some of the emerging teacher leaders. In particular, three of the emerging teacher leaders provided a lack of evidence of interpreting the

students' understandings on both parts 1 and 2. These 3 teachers' responses to the interpreting prompts were too short and vague, and/or related to other pedagogical ideas.

In the deciding how to respond component-skill, many of the ETLs provided limited or robust evidence of deciding how to respond based on the students' understanding. In particular, *every* ETL provided at least limited evidence on at least one of the professional noticing parts of the survey. Even when ETLs provided a lack of evidence of deciding how to respond based on the students' understandings, most of their decisions were still based on their interpretations of the students' ideas; their interpretations were just inconsistent with the evidence the students provided. Thus, it appears that ETLs have gained much experience, knowledge, and practice that supports them to individualize instruction for students and be responsive to their ideas, as well as a disposition to do so. In fact, in 70.8% of the responses to the deciding how to respond prompts, ETLs provided specific problems for each student, and/or specific rationales for each student. Many ETLs were able to identify specific understandings they were hoping to support, or identify specific ways students might use their current strategies in the next task.

### **The Professional Noticing of Students' Mathematical Thinking Expertise of ETs.**

ETs provided much less evidence than ETLs of considering the students' mathematical thinking when responding to the professional noticing prompts. These teachers were recruited from school districts in the southwestern region of the United States, and were not a part of the long-term sustained professional development in which the ETLs were participating. ETs had at least 4 years of teaching experience, with an average of 11.47 years of teaching experience. ETs provided a similar amount of evidence to PSTs of attending to and interpreting the students' strategies. However, ETs provided a lot more evidence of deciding how to respond based on the



students' understandings than PSTs. Based on these differences, it appears that teaching experience may support teachers' deciding how to respond skills.

The average score for ETs attending to the details of the students' strategies was 1.10. This score indicates that ETs provided robust evidence slightly more often than they provided a lack of evidence, which shows us that many ETs can thoroughly and accurately attend to the mathematically significant details of students' strategies. So, although my analysis did not provide evidence that teaching experience supports this component-skill, it appears ETs already can attend to the details of the students' strategies.

Again, no significant differences were found between ETs and PSTs; my analysis did not provide evidence that teaching experience supports teachers' interpreting skills. The ETs average interpreting score was 0.58, which was only slightly higher than the PST group's average interpreting score (0.40). This may indicate that teaching experience marginally supports teachers' interpreting skills, but more work must be done to confirm this.

These are surprising results, considering the fact that ETs have spent several years working with students and their mathematical thinking. One might reasonably expect ETs to have improved in these two component-skills, given their years of experience. I conjecture that teaching experience alone does not provide adequate support for teachers to develop their attending or interpreting skills. This is particularly true in lecture-style instruction, where teachers do not regularly elicit students' ideas and use students' ideas to inform their instruction. Teachers that lecture have considerably fewer opportunities to practice these component-skills than teachers who use students' ideas in-the-moment to decide on next steps for the lesson. Furthermore, the fast-paced nature of the classroom and overwhelming responsibilities put on the teacher may actually support teachers to focus on *quickly* attending to students' strategies, rather

than *thoroughly* attending to students' strategies. The intense demand of the profession provides little incentive for teachers to thoroughly investigate students' ideas, especially during class. If there is no external support (such as a long term, sustained professional development program), then there is little motivation for secondary teachers to be more responsive to students' ideas and thoroughly investigate their ideas and understandings, especially given how challenging it is to adopt such an instructional style at all (hence the term *ambitious* teaching; Forzani, 2014).

Given the importance of attending to students' strategies and interpreting students' understandings for providing individualized instruction, scaffold, or support, I put forth a suggestion for professional developers. As teacher educators working with secondary teachers, I agree with other researchers, and also recommend teacher educators provide opportunities for teachers to slow down and spend time during professional development attending to the mathematically significant details of the students' strategies and interpreting the students' understandings (Goldsmith & Seago, 2011; Jacobs & Philipp, 2004). Secondary teachers often do not get a chance to spend an extended amount of time with an individual student's strategy, and thus might be conditioned to make sense of the strategy quickly, and perhaps miss important details that could support the teachers' decisions for how to respond. Teachers may at first push back on this activity, because deeply considering a student's strategy is not consistent with the fast-paced reality of their classrooms. However, borrowing from the metaphor that Grossman et al. (2009) used, "if you're learning to paddle, you wouldn't practice kayaking down the rapids. You would paddle on a smooth lake to learn your strokes" (p. 2076). Students' ideas are notoriously difficult to understand, and responding to individual ideas presents loads of challenges, even without the hustle and bustle of the classroom (Ball, 1995). Perhaps by giving secondary teachers a chance to slow down and deeply consider the students' ideas on calm

waters, they too can learn to paddle (i.e. be responsive to students' ideas) on the open ocean (i.e. in the fast-paced classroom environment).

Finally, in my study ETs provided significantly more evidence of deciding how to respond based on the students' mathematical understandings than the PSTs. The ET group's average (0.57) was almost three times as large as the PST group's average (0.20), and differences were statistically significant. Additionally, there existed ETs in my study that provided robust evidence of deciding how to respond based on the students' understandings. In contrast, none of the PSTs in my study provided robust evidence of deciding how to respond based on the students' mathematical understandings. Hence, it appears that teaching experience supports some ETs to develop this skill. However, a little more than a third of the ETs did not provide any evidence of deciding how to respond based on the students' understandings for both parts 1 and 2 of the survey. Thus, teaching experience seems to act differentially on teachers in supporting them in developing this skill. Jacobs et al. (2010) noted that deciding how to respond on the basis of the students' understandings was a challenging skill for primary teachers to develop, and was not well supported by teaching experience alone. At the secondary level, the same appears to be true. Teaching experience can support some ETs to develop this skill, but not many of them. Again, it is likely that learning to paddle on calm waters (i.e. deeply consider and respond to individual students' ideas without the extra pressures and responsibilities of the classroom) will support teachers to paddle in the open ocean (be responsive to students' ideas in the fast-paced classroom).

### **The Professional Noticing of Students' Mathematical Thinking Expertise of PSTs.**

PSTs and ETs had similar scores for attending to and interpreting the students' understandings, and much less evidence of deciding how to respond based on the students'

understandings. The PSTs in my study were recruited from a large urban university in the south western region of the united states. All were either completing their undergraduate degree, or had completed their degree but had not yet begun a credential program for secondary mathematics. All PSTs intended to enter a credential program and teach secondary mathematics.

Based on these results, it appears PSTs are fairly competent at attending to the details of the students' strategies. In particular, PSTs provided robust evidence of attending to the details of the students' strategies more often than they provided a lack of evidence, with an average score of 1.15. Hence, many PSTs can attend to the mathematically significant details of the students' strategies. Student F's strategy appeared to be the most challenging strategy for all teachers to understand, and 15 of the PSTs (50%) appeared to understand what student F did and why. I conjecture that providing PSTs with opportunities to make sense of students' written strategies is both an appropriate and productive activity for PSTs. To practice making sense of students' ideas is an approximation of practice (Grossman, Compton et al., 2009), and has been used productively by other mathematics teacher educators as well (e.g. Schack et al., 2013). My results indicate that it is an appropriate because many PSTs provided evidence of being able to make sense of the students' strategies, and it is productive because many PSTs were challenged by at least one of the students' strategies. As mathematics teacher educators in post-graduate credential programs, my results indicate that we have opportunities to build on some fairly strong noticing skills that PSTs bring with them.

PSTs provided little evidence of interpreting the students' understandings, with an average score of 0.40. There were some PSTs that provided limited evidence, and there even were two PSTs that provided robust evidence of interpreting the students' understandings on both parts 1 and 2, but many of the PSTs did not provide evidence of interpreting the students'

understandings. There were even some PSTs that provided strong evidence of attending to the details of the students' strategies, but struggled to interpret the students' understandings in a way that was consistent with the students' strategies. For example, one PST made sense of student F's strategy, which challenged many teachers. This PST even recognized that the student divided 12 leaves by 2 because the student recognized that 4 caterpillars was half of a group of 8 caterpillars. However, in the interpreting response this PST claimed that all three students (D, E, and F) "weren't very familiar with the work and didn't quite understand ratios." Recognizing 8 caterpillars and 12 leaves as a group that needs to be partitioned requires a fairly deep understanding, and is a strong indication of reasoning proportionally (Carney et al., 2015; Steinhorsdottir & Sriraman, 2009). Hence, even if PSTs attend well to the details of the students' strategies, it doesn't mean they will interpret the students' strategies in a way that is consistent with the student's work. This finding means that asking PSTs to discuss what they believe the students understand could elicit a wide variety of responses from PSTs, and support rich discussions. In such discussions, there may even be some PSTs that exhibit sophisticated interpreting skills.

Finally, PSTs provided very little evidence of deciding how to respond based on the students' understandings, with an average score of 0.20. Few PSTs provided limited evidence, and no PSTs provided robust evidence. Again, this result speaks to the challenging nature of deciding how to respond based on the students' understandings (Jacobs et al., 2010). Many of the PSTs did not appear to base their decisions on the students' ideas, and instead focused on their preferred strategy for solving the problems, or a new content domain or mathematical topic, or trying to create problems that might pique students' interests.

Jacobs et al. (2011) conjectured that many of the decisions and rationales they saw could be leveraged as resources for teacher educators. I believe this is an important perspective to take for my PSTs as well. For example, while many PSTs decided how to respond by forcing students to adopt a particular strategy, they often wanted students to adopt either the cross multiplication strategy or the unit rate strategy. A discussion about which proportional reasoning strategies are helpful and productive for students could elicit many ideas from PSTs as they defend the two strategies. Such a discussion could also elicit PSTs' beliefs about what it means to deeply understand mathematics as well. A teacher educator could enrich such a discussion by introducing other viable and valuable proportional reasoning strategies as well.

As another example, many PSTs wanted students to grapple with other content domains or mathematical topics. For example, one PST wanted the three students to create a formula, which moves students into the domain of functions and algebra. This PST showed evidence of understanding relationships between proportional reasoning and algebraic thinking, which are important parts of teachers' horizon content knowledge (Ball, Thames, & Phelps, 2008). Supporting teachers to understand connections between different content domains can be productive for understanding both content domains.

## Chapter 5: Results and Discussion about Research Question 2

In this chapter I present results that pertain to my second research question, and discuss these results in the context of artifact selection for teacher education. This chapter will be split into two sections. In the first section, I share the degree to which teachers were able to capture the gist of the strategy, and identify features that challenged teachers. I also discuss these features in the context of the literature on teacher education and artifact selection. In the second section, I share which strategies elicited more interest from teachers, and identify features that intrigued teachers. I also discuss these features in the context of artifact selection. As a reminder, my second research question is:

2. *(A) What features of written artifacts of student thinking afford or constrain teachers' effectiveness at demonstrating their professional noticing of students' mathematical thinking expertise? (B) What features of written artifacts of student thinking elicit teachers' interest?*

### Challenge of Strategies

In this section I share the degree to which teachers were able to capture the gist of each strategy, and consequently which features made a strategy more challenging.

#### Illustrating Ways Teachers Did/Did Not get the Gist of the Strategy

As a reminder, the gist of the strategy refers to the key underlying reasoning of the strategy, which connects all the details together and gives meaning to the strategy. I coded for the gist of the strategy by looking for evidence that the teacher identified mathematical connections among the different operations, meanings for the operations, and/or meanings for the numbers. Hence, teachers either received a score of “evidence of getting the gist,” or “lack of evidence of getting the gist.” I illustrate these two codes with examples of teachers' responses to

the attending prompt for strategy C (figure 1), in order to give the reader a sense of what these codes mean.

In strategy C, the student is solving the problem “6 mice eat 18 food pellets each day. How many food pellets do 24 mice eat?” To solve, the student first divided 24 mice by 6 mice and gets 4, which represents the number of times larger the 24 is than 6. The student then uses this relationship, also known as the scale factor, to find how many times larger the pellets will need to be, thus multiplying 18 pellets by 4 and obtaining 72.

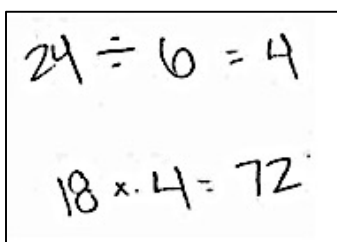

$$\begin{array}{l} 24 \div 6 = 4 \\ 18 \times 4 = 72 \end{array}$$

Figure 5.1. Student C's strategy.

#### **Evidence of capturing the gist of the strategy.**

Responses that provided evidence of getting the gist of the strategy were fairly similar in how they identified the gist of the strategy, but varied in the amount of detail they described. On one end, responses provided evidence of getting the gist as well as evidence of attending to a significant number of the details. In other words, these responses provided robust evidence of attending to the students' strategies. For example, one teacher wrote “The student divided 24 mice by 6 mice to determine how much the group of mice would grow. The 4 means it grows by 4 times so they multiplied 18 pellets by 4.” Notice that this description is thorough, accurate, identifies connections between the operations, and identifies meanings for the operations and values.

On the other end, some responses provided evidence of getting the gist but did not provide much evidence of attending to the details of the strategy. For example, one teacher



simply wrote “The student saw that 24 mice is 4 times as many.” The idea of scaling up a ratio is built on the idea of finding out how many times larger one quantity is than another quantity. Hence, by identifying the meaning of the number 4 as “[number of] times as many,” this teacher provided evidence of getting the gist of the strategy even though the response included significantly fewer details than other responses that got the gist of the strategy.

**Lack of evidence of getting the gist of the strategy.**

Responses that did not identify the gist of the strategy varied in many ways. First, some responses identified strategies that were inconsistent with the evidence provided and the proportional reasoning literature. For example, one teacher thought student C had set up an equation in their head, and solved the proportion in a “short cut” manner. They claimed “Student C is also using a proportion but with the equation set up. This is sort of doing a short cut process of setting up and solving the proportion.  $24/6 = P/18$ .” Responses like this one identified connections between operations and reasons behind the student’s work, but the connections were not well supported by the evidence in the student’s work and the literature about proportional reasoning. For instance, this explanation is based upon the student using a variable in an equation, yet there is no evidence that student C knows how to use variables to set up and solve equations. Sometimes these responses were mathematically incorrect, too. For example, one teacher thought student C used a unit rate strategy, labelling 24 as the number of pellets eaten by 6 mice and 4 as the number of pellets eaten by each mouse. This is incorrect, because 24 represents the number of mice (not pellets), and the unit rate is 3 pellets per mouse.

Second, some teachers claimed students were confused, lost, wrong, or even cheating, even when the work exhibited reasonable proportional reasoning skills. For example, one teacher wrote “Student C came to the proper answer but I don't think they fully understand what

they are solving for or how to set up the problem.” Unlike other teachers that admitted they were confused, yet still believed the student had made some sense, this teacher does not seem to recognize the student’s strategy as a strategy that makes sense and has a correct conceptual interpretation. Some responses provided more insight into what aspects of the strategy made them think the student was confused. For example, one teacher wrote “The student divided mice by mice which is incorrect. Then multiplied that number of food by food. Which is incorrect of [sic] finding the answer.” Note that the idea of dividing the same quantities does have a mathematically valid interpretation, but this teacher’s response indicated that they did not consider that. Consequently, the division operation was confusing to this teacher.

Third, some teachers were confused by what the student did, and explicitly said so. Sometimes these responses were short and did not explain what confused them. For example, one teacher wrote “I am not sure how student C came up with their plan to solve.” Other times, the teacher shared which features of the strategy confused them. For example, another teacher wrote “It is unclear to me what the student did here. The answer is correct, but I cannot figure out their reasoning. My guess as to why 24 was divided by 6 was that the proportion was set in  $24 \text{ mice} / 6 \text{ mice}$  which would be 4 mice, who consume 18 pellets so  $4 \times 18 = 72$ . I am not sure of the student's thinking here.” This teacher did not make correct sense of what the “4” represented, as the teacher believed the 4 represented 4 mice, instead of the number of times as large one ratio is than the other. When teachers explained what parts confused them, these explanations provided more insight into what features of the strategy made the strategy difficult to understand.

Finally, some teachers simply described the operations the student performed without identifying connections between the operations or meanings for the operations or values. For example, one teacher only wrote “divided 24 by 6, got 4 and multiplied 18 by 4 to get 72.”

Notice that the teacher identified each operation and value, but did not describe what the values and operations represented in the context, nor the mathematical reasons that supported this strategy. In these responses, it is unclear if the teacher actually got the gist of the strategy and did not describe it in their response, or if the teacher did not get the gist of the strategy. Overall, there was just not enough evidence.

### **Results Related to Gist of Strategy**

In this section I share how many teachers attended to the gist of each strategy. As a reminder, the six strategies can be seen in table 1, and the features of the strategies can be seen in table 2.

I organize my results about each strategy, and characterize the general trends I saw in the ways teachers did or did not provide evidence of getting the gist of the strategy for each strategy. This organization allows me to compare trends between categories of strategies in the next section, such as integer vs. non-integer strategies, procedural vs. conceptual strategies, and unit rate vs. scalar strategies. In table 3 I share results about how many teachers got the gist of the strategy for each strategy, and in table 4 I share the types of responses I saw that did not attend to the gist of the strategy.

#### **Strategy A.**

For strategy A, 89% of the teachers captured the gist of the strategy and 11% of the teachers did not. Considering the 89% that got the gist of the strategy, 76% got the gist of the strategy *and* provided robust evidence of attending to the details, which means most of the teachers who got the gist thoroughly described the strategy as well. Considering the 11% that did not capture the gist of the strategy, most of these teachers did not get the gist because they provided a list of the operations without explicitly connecting them to the student's use of the

Table 5.1. Artifacts, Strategies, Tasks, and Descriptions of strategies A - F.

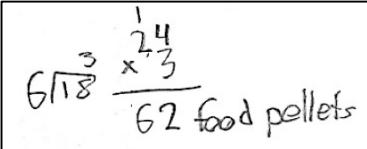
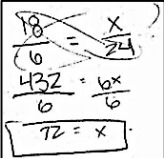
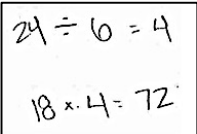
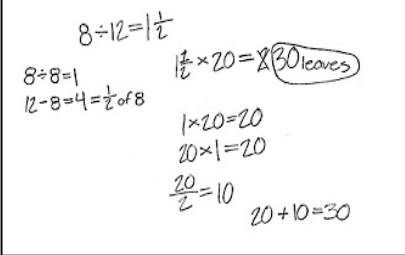
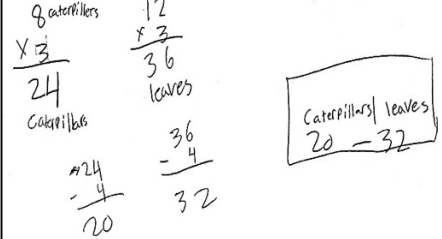
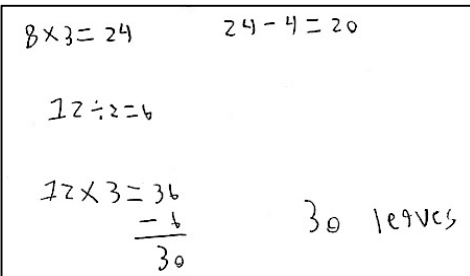
Artifact	Strategy	Description
<b>Task: Each day, 6 mice eat 18 food pellets. How many food pellets do 24 mice eat?</b>		
<b>A</b>		<p>Student divides 18 pellets by 6 to find a unit rate of 3 pellets per mouse. Student multiplies unit rate by 24 mice. Student makes a mistake when multiplying <math>24 \times 3</math>.</p>
<b>B</b>		<p>Student uses the traditional cross-multiplication strategy, setting up pellets in the numerator and mice in the denominator of each fraction. Student manipulates equation correctly.</p>
<b>C</b>		<p>Student divides 24 mice by 6 mice to find the number of groups of 6 mice in 24. There are 4 groups of 6 mice, so there should be 4 groups of 18 pellets.</p>
<b>Task: Each day, 8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?</b>		
<b>D</b>		<p>Student divides 12 leaves by 8 caterpillars, to find each caterpillar eats 1.5 leaves. The student divides in a nonstandard way, and makes a syntax error when writing <math>8 \div 12 = 1 \frac{1}{2}</math>. The student then multiplies 1.5 leaves x 20 caterpillars, using the distributive property to help with the multiplication.</p>
<b>E</b>		<p>Student multiplies both 8 caterpillars and 12 leaves by 3, apparently scaling up the ratio to 24 caterpillars and 36 leaves. The student subtracts 4 from both quantities to reach 20 caterpillars, and arrives at 32 leaves. The student invokes additive reasoning for the final operation, and is not correct.</p>
<b>F</b>		<p>Student multiplies 8 caterpillars by 3 to get 24 caterpillars, then subtracts 4 to get 20 caterpillars. Student must have recognized that 4 caterpillars is half of 8 caterpillars, because next the student finds half of 12 leaves (6 leaves). The student then multiplies 12 leaves by 3, and subtracts the extra “half group of leaves” to get 30 leaves</p>

Table 5.2 Features of Strategies

Artifact	Strategy Type	Integer Ratios?	Correct Calc's?	Standard Calc's?	Correct Concept?	All Steps?
<b>A</b>	Unit Rate	Yes	No	Yes	Yes	Yes
<b>B</b>	Cross-Multiplication	N/A	Yes	Yes	N/A	Yes
<b>C</b>	Scale Up	Yes	Yes	Yes	Yes	Yes
<b>D</b>	Unit Rate	No	No	No	Yes	Yes
<b>E</b>	Scale Up, Subtract Equal Parts	N/A	Yes	Yes	No	Yes
<b>F</b>	Scale Up, Subtract Prop. Parts	No	Yes	Yes	Yes	No

Table 5.3. Percentages of teachers who provided evidence/lack of evidence of capturing the gist of the strategy.

	Part 1 Integer Ratios			Part 2 Non-Integer Ratios		
	<b>A</b> Unit Rate, Calc. Err	<b>B</b> Cross - Multiply	<b>C</b> Scalar	<b>D</b> Unit Rate, Creative Calc's, Syntax Err	<b>E</b> Scalar w/ Additive Reasoning	<b>F</b> Scalar w/ Partitioned Ratio, Missing Step
N = 72						
<b>Evidence</b>	89%	97%	63%	83%	69%	57%
<b>Lack of Evidence</b>	11%	3%	38%	17%	31%	43%

Table 5.4. Features of responses that did not capture the gist.

Instead of capturing gist, teacher...	<b>A</b> <i>n</i> = 8	<b>B</b> <i>n</i> = 2	<b>C</b> <i>n</i> = 27	<b>D</b> <i>n</i> = 12	<b>E</b> <i>n</i> = 21	<b>F</b> <i>n</i> = 31
...identified other strategy			3			2
...believed student was wrong	3	1	11	4		7
...admitted confusion		1	6	3	11	21
...listed operations with no connections/meanings	5		7	5	11	1

unit rate, but some of them (3 teachers, or 4% of all teachers) believed the student was confused.

These interpretations that the student was confused were not related to the calculation error the student made when multiplying 3 by 24. Instead, these teachers believed the student either had no understanding or performed the wrong calculations.

### Strategy B.

For strategy B, almost every teacher (97%) got the gist of the strategy. Just one teacher admitted they had no idea what the student did, and one teacher thought the student performed the wrong calculations. Less than a third of the teachers (29%) provided robust evidence of attending to the details, which means teachers tended to identify the gist and then not describe many of the details of the strategy.

### **Strategy C.**

For strategy C, 63% of the teachers got the gist of the strategy, and almost 2/3 of these teachers also provided robust evidence of attending to the details. Looking at those who did not capture the gist of the strategy, which was 38% of the teachers, the reasons fell into various categories, as indicated in Table 3. 24% (17 teachers) were either confused or believed the student was confused, and more than half of these teachers cited the division of 24 mice by 6 mice as their evidence for why the student was confused, or their reason for why they were confused. It seems the idea of “number of times larger” was difficult for teachers to identify.

### **Strategy D.**

For strategy D, 83% of the teachers provided evidence of getting the gist of the strategy, and a little more than half of these teachers also provided robust evidence of attending to the details. Looking at the other category, 17% of the teachers did not attend to the gist of the strategy, for various reasons (table 3). The four teachers (6% of the teachers) who believed the student was wrong cited the fact the student wrote  $8 \div 12 = 1 \frac{1}{2}$  as evidence that the student was conceptually confused (as opposed to claiming the student made a simple syntax error), and some cited the “extra work” as evidence of confusion as well. The three teachers (4%) who were confused by the student’s strategy each talked about how the work was difficult to follow. Said one teacher, “I see the one and a half being used in the same vertical relationship in both

fractions. I am not sure what all the rest is. I can understand what the student did because they wrote clearly but am lost after that.” Notice that this teacher said they could identify the operations the student performed, but did not see reasons for the different operations or connections among them. There were nine operations in total appearing in the student’s work, and it appears the connections among these operations challenged these teachers when looking for the key underlying mathematical reasoning.

### **Strategy E.**

For strategy E, 69% of the teachers captured the gist of the strategy, and a little less than  $\frac{3}{4}$  of these teachers also provided robust evidence of attending to the details, which means many were attending the details in addition to capturing the gist of the strategy. Of the teachers who captured the gist of the strategy, most of them recognized that the student was scaling up the ratio and applying additive reasoning to adjust the numbers, but some teachers identified a slightly different (but complementary) idea in the student’s work. Specifically, 14% of the teachers (about  $\frac{1}{5}$  of those who got the gist) attended to the notion of performing the same operations on both quantities, which is consistent with the proportional reasoning literature as well (e.g. Carney et al., 2015).

Considering the other category, 31% of the teachers did not capture the gist of the strategy. Half of these teachers (11 teachers) were confused by the strategy, while the other half provided a list of operations without connecting them to the scaling and additive strategies the student used. Looking at what confused the teachers, 9 of the 11 identified the student’s use of the numbers 3 and 4 as confusing to them. Many of them explicitly wondered where the student got these numbers, saying things like “I do not get where they are getting the 3 and 4.” Some even tried to come up with possible reasons for why the student used the 3 and the 4. For

example, one teacher wrote “The student subtracted 4 because 4 is the difference between 12 and 8?” Another wondered, “Student E seemed to get the number 3 from somewhere. Perhaps they were thinking of number of hours in each day which is 24 divided by 8 caterpillars?” It appears the numbers 3 and 4, which were values not found in the story problem, made the gist of the strategy more challenging for these teachers to get.

### **Strategy F.**

For strategy F, 57% of the teachers got the gist of the strategy, and a little more than  $\frac{3}{4}$  of these teachers provided robust evidence of attending to the details of the strategy. This means most of the teachers who got the gist of the strategy provided thorough descriptions of the student’s strategy. Looking at the other category, we see that 43% of teachers did not get the gist of the strategy. Most of these teachers did not get the gist because they were confused about what the student did (table 3), or believed the student was confused. For this strategy, all of the 28 teachers who expressed confusion or thought the student was confused did not describe the idea that a group of 4 caterpillars is half of a group of 8 caterpillars, or that 4 caterpillars eat 6 leaves. It appears the partitioning concept was a challenging but important concept for teachers to attend to in order for them to be able to capture the gist of the strategy. It was also the missing step in the student’s work. I conjecture that the combination of being a new concept (i.e. partitioning a ratio was not present in any other strategies) with being a missing step (i.e. “4 is half of 8” is not explicit in the student’s work) made this detail more difficult to identify. Similar to the descriptions of strategy E, some of the teachers also shared that they did not know where the student got the 3 from, how the student knew to subtract by 4, or why the student divided by 2, which were all values not present in the story problem.



## Comparisons Among Strategies: Features that Influence Teachers' Ability to Capture the Gist

There are three comparisons that I want to make. First, I compare between the procedural cross multiplication strategy and the other strategies, which are based on proportional reasoning concepts. I make this comparison because procedural strategies present their own unique challenges when thinking about building on or responding to such students (Walkoe, 2015), and I want to unpack what potential challenges procedural strategies might present. Second, I compare unit rate strategies and scalar strategies, because these two types of strategies utilize different multiplicative relationships (Carney et al., 2015). Third, I compare strategies for which students grappled with integer ratios and the strategies for which students grappled with non-integer ratios, because the non-integer ratios afforded students opportunities to make the work much more complex, such as the use of non-standard computation strategies of strategy D and the partitioning concept in strategy F.

### Procedural vs. conceptual strategies.

Almost all teachers (97%) were able to describe the gist of strategy B, which was the cross multiplication strategy (see Table 5). In contrast, every other strategy, which were all based on proportional reasoning concepts, had fewer teachers identifying the gist of the strategy (57% - 89%). These results indicate that the cross-multiplication strategy was the easiest strategy for teachers to get the gist of the strategy. I unpack what might have supported the procedural strategy to be the least challenging for teachers after the next section.

Table 5.5. Comparing degree of getting gist for procedural vs. unit rate & scalar strategies

N = 72	<b>Procedure (Strategy B)</b>	<b>Unit Rate &amp; Scalar (Strategies A, C, D, E, &amp; F)</b>
<b>Evidence</b>	97%	57% - 89%
<b>Lack of Evidence</b>	3%	11% - 43%

### **Unit rate vs. scalar strategies.**

A lower percentage of teachers were able to capture the gist of the three scalar strategies (57%-69%, see table 6) compared with the unit rate strategies (83%-89%), with strategy F being the most challenging of all (57% captured the gist). These results indicate that the scalar multiplicative relationship was a more challenging multiplicative relationship for teachers to grapple with, regardless of whether the student grappled with an integer ratio or non-integer ratio. In particular, in both strategy A and strategy C, the student divides one value by a second value, and then multiplies the quotient by the third value given in the story problem. Strategy A even includes a small mistake, which are sometimes challenging to identify. However, far more teachers got the gist of strategy A than strategy C, which indicates the underlying concept of strategy C was more challenging to identify. In fact, the teachers who expressed confusion or believed the student was confused often wondered about the meaning of the “4” in the student’s work, which is a key part of the scalar strategy. This gives more evidence that the scaling up concept was difficult for teachers to identify. I discuss what might support the scalar strategies to be more challenging in the next section.

Table 5.6. Comparing degree of getting gist for unit rate vs. scalar strategies.

N = 72	<b>Unit Rate (Strategies A &amp; D)</b>	<b>Scalar (Strategies C, E &amp; F)</b>
<b>Evidence</b>	83% - 89%	57% - 69%
<b>Lack of Evidence</b>	11% - 17%	31% - 43%

### **Familiarity.**

There are many factors that could support teachers to be able to capture the gist of a strategy. For example, a strategy could make use of simple concepts, have fewer steps, or have

an accompanying written explanation from the student. However, I believe the challenge associated with the cross-multiplication, unit rate, and scalar strategies are mostly a function of familiarity.

***Familiarity of the cross-multiplication strategy.***

The cross-multiplication strategy makes use of several deep concepts, like the concept of equality, of variables, of equivalent fractions, and of rates and/or scale factors. In addition, there are several steps associated with it (and in particular there are more steps in strategy B than in strategies A and C). Also, the student did not explain the steps or label numbers or variables. Hence, the cross-multiplication does not seem to be an easy strategy by being conceptually simple (it is not), by containing fewer steps (it contains more than strategies A and B), or by including a written explanation (it does not). I conjecture that teachers were more successful at capturing the gist of the cross multiplication strategy because the strategy was *familiar* to many teachers. Many teachers even described the cross multiplication strategy as “standard,” “traditional,” or “usual,” which provides more evidence that this strategy was a familiar strategy to teachers.

In addition, I did not code for whether teachers identified the concept of rates or of equivalent fractions when I coded whether teachers got the gist of the strategy; teachers only had to identify the act of setting up and solving an equation and/or the use of cross multiplication. (Very few teachers would have gotten the gist of the strategy if I had included these additional criteria.) I looked only for these types of descriptions because many students do not think about rates or equivalent fractions when performing the cross multiplication procedure (Cramer & Post, 1993). Hence, I believe the proceduralization of this strategy by our culture has supported

its familiarity among teachers, because teachers only had to identify a few surface-level characteristics for the teacher to know what the student was doing.

***Familiarity of the unit rate concept.***

My results indicate that the unit rate concept was also easier for teachers to identify than the scalar concept. This is in opposition with research that says students struggle more to utilize the unit rate concept than the scalar concept<sup>8</sup> (Lamon, 1993; Steinhorsdottir & Sriraman, 2009). Hence, one might reasonably hypothesize that the unit rate concept would be harder for teachers as well, but this was not the case. Additionally, some of the unit rate and scalar strategies were similar on the surface, and still produced such large differences in number of teachers getting the gist. Specifically, both strategy A and strategy C began with division of one quantity by a second quantity, and then multiplied the quotient by the third quantity in the story problem. I conjecture that the unit rate concept might be more familiar to teachers than the scalar concept, because the unit rate concept appears in many other content domains, such as linear functions (the slope is often given as a rate in story problems), calculus (derivatives), and even in equal sharing tasks. Consequently, teachers may see the unit rate idea more often than the scalar idea, and thus be more familiar with the concept.

***Other potential influences of familiarity.***

One might wonder if the familiarity influenced teachers' professional noticing in other ways as well. Recall that for the cross multiplication strategy, only 21 teachers out of 70 teachers who got the gist of the strategy also provided robust evidence of attending to the details. In contrast, for every other strategy more than half of the teachers who got the gist of the strategy also provided robust evidence of attending to the details. It is possible that the familiarity of the

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<sup>8</sup> This result has been contested by other researchers (e.g. Carney et al., 2016).

cross multiplication strategy contributed to the low attending scores; one might reasonably conjecture that teachers did not feel a need to describe other aspects about the strategy because it was so familiar. However, this conjecture is not fully supported by my data, because strategy A, which is arguably another familiar strategy, had the highest number of teachers providing robust evidence of attending to the details. Hence, I believe the procedural nature of strategy B also contributed to the low attending scores. Walkoe (2015) found similar results; in her study, a group of prospective secondary teachers exhibited two different trajectories of growth in noticing skills, based on the particular mathematical domain. In particular, Walkoe claimed that the procedural nature of one of the domains constrained her teachers' abilities to exhibit more sophisticated noticing skills. In my study, the fact that teachers tended to get the gist of student B's strategy but not describe many of the details of the strategy aligns with her findings. It appears the procedural nature of the strategy inhibits teachers from thoroughly describing the strategy, even when they get the gist of the strategy.

#### **Integer vs. non-integer ratios.**

There are smaller but consistent differences when comparing strategies for which students grappled with integer ratios vs. strategies for which students grappled with non-integer ratios (see table 7). As a reminder, I do not include strategy B because strategy B did not include work related to the scalar or functional multiplicative relationship, and I do not include strategy E because strategy E included additive reasoning, which deals with neither an integer nor a non-integer multiplicative relationship. Within the unit rate strategies, strategy D was slightly more challenging (83%) than strategy A (89%). Within the scalar strategies, strategy F (57%) was slightly more challenging than strategy C (63%). Although the differences between integer and

non-integer ratios are not as striking as those identified in the previous section, they are consistent across both unit-rate strategies and scalar strategies.

Table 5.7. Comparing degree of getting gist for integer vs. non-integer strategies

N = 72	Unit Rate		Scalar	
	A Integer	D Non-Integer	C Integer	F Non-Integer
<b>Evidence</b>	89%	83%	63%	57%
<b>Lack of Evidence</b>	11%	17%	38%	43%

However, there are important features that emerged as a result of analyzing the non-integer ratios. For the unit rate strategies, the non-integer ratios afforded student D opportunities to create non-standard (and potentially challenging-to-comprehend) computation strategies. The student would not have created these non-standard computation strategies if the task included only integer ratios. The student exhibited creative strategies for dividing and multiplying by working with fractions, mixed numbers, and parts of a whole.

For the scalar strategies, there were two potential challenges that emerged from the non-integer ratios. First, the student in strategy C divided the 24 mice by 6 mice, whereas in strategy F the students did not find the multiplier from 8 to 20 through division, and instead multiplied both 8 and 12 by the same number to get close to 20. Hence, the scale factor in strategy C (4 times larger) was made explicit in the work, while the scale factor for strategy F (2.5 times larger) was not. As a consequence of using a multiplier in strategy F, some teachers expressed confusion about where the numbers 3 and 4 came from, and why they multiplied 8 by 3 and subtracted 4. (This was especially true for strategy E, where 9 out of 11 teachers who were confused by the student's strategy explicitly wondered where the student got the 3 and the 4.) Second, the non-integer ratios allowed student F to grapple with two distinct but important

proportional reasoning concepts, that of scaling up a ratio and of partitioning a ratio (Lobato & Ellis, 2010). The partitioning concept in particular (recognizing that 4 is half of 8 caterpillars so we need half of a group of 12 leaves) was often lost on the teachers who did not capture the gist of the strategy. This step was also not made explicit in the student's work. Strategy F was clearly the most confusing to teachers, as 21 teachers explicitly said they did not understand what the student did, and 7 believed the student was confused. With integer ratios, such complexities would not have been present and thus the lack of complexity likely supported the teachers to be able to capture the gist of these strategies at higher rates than strategies with non-integer ratios.

Before I move on, I note an important caveat related to strategy E. Strategy E was less challenging than strategy C for teachers to get the gist of the strategy, even though strategy E included more computations, a conceptual error, and the numbers 3 and 4, which were not found in the context of the story problem and were clearly challenging to some teachers. I conjecture strategy E may have been slightly less challenging for teachers to get the gist of the strategy because there was a feature of strategy E that I believe made the strategy more familiar to teachers. Specifically, the non-integer ratios supported student E to use a combination of multiplicative and additive reasoning, and consequently the student performed the same operations on both quantities. The idea of performing the same operations on both quantities appears in many other content domains (e.g. solving equations, finding equivalent fractions) which may have increased the familiarity of this strategy for teachers. As a reminder, about 1/5 of the responses that provided evidence of getting the gist of the strategy focused on the idea that the student performed the same operations on both quantities.

**Summary of features that make a strategy more or less challenging.**

Clearly some strategies are more challenging than other strategies. Based on these results, it appears that the strategy type was a factor for influencing the difficulty of the strategy. Scalar strategies were more challenging for teachers to identify than unit rate strategies, and the procedural cross multiplication strategy was the easiest strategy for teachers to identify. I believe this is an indication that the cross multiplication strategy is more familiar to teachers than unit rate or scalar strategies, and the unit rate concept is more familiar to teachers than the scalar concept.

The strategies with non-integer ratios provided the students with opportunities to make their work more complicated for teachers to interpret. For example, the non-integer ratio afforded student D to creatively compute the division of 12 by 8 and the multiplication of  $1\frac{1}{2}$  by 20, which added many other operations to the work as scratch work. Other features that were challenging included the fact that student E and student F used the numbers 3 and 4 rather than the scalar factor of 2.5, and the fact that student F used both scaling up and partitioning concepts. Hence, the non-integer ratios afforded opportunities for students to make their work more complex and difficult for teachers to follow, although these features did not increase the level of challenge as much as the difference between scalar and functional multiplicative relationships.

When considering which artifacts to select in the context of teacher education, one factor one might consider is whether a particular artifact exhibits a student's strategy that is challenging to unpack. For example, Sherin, Linsenmeier, and van Es (2009) noticed that the clarity of a video artifact could influence the amount of discussion among teachers about the artifact. At times, teachers appreciated the challenge of understanding a student's strategy, even if the strategy did not exhibit much conceptual depth. At other times, teachers appreciated a student's strategy even if it was clear (and not very challenging to unpack), because the strategy exhibited



a large amount of conceptual depth. Based on my results, I conjectured that the familiarity of particular concepts had a large impact on the degree of challenge associated with the strategy. The complexity of the strategy also influenced the degree of challenge of a strategy, but not as much as the distinction between whether a strategy included unit rate or scalar concepts.

### **Intrigue of Strategies**

In this section I describe which strategies were more intriguing to teachers and which were less intriguing, and consequently which features made a strategy more interesting. I start by describing the different ways teachers created personal connections with the strategies, as exhibited by the personal frames they created.

#### **Illustrating Ways Teachers Did/Did Not Create Personal Frames**

I coded for personal frames by investigating two types. First, I looked for evidence that teachers were experiencing an emotional reaction to what they were noticing (i.e. creating an affective frame). Second, I looked for evidence that teachers were putting themselves in the perspective of the teacher, and imagining what they would do if they could interact further with the student or strategy<sup>9</sup> (i.e. creating a perspective frame). For affective frames, I only coded for instances when teachers felt a positive emotional reaction. For perspective frames, I coded for whether the teacher wanted to learn something about the student/strategy, wanted to help the student, wanted to share the strategy, or wanted to discuss other topics related to the strategy. To illustrate these codes, I share representative examples of each of these codes in teachers' responses as teachers considered strategy F (figure 2).

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<sup>9</sup> As a reminder, I did not count teachers' responses to the deciding how to respond prompt as evidence of creating a perspective frame, because I wanted to capture moments when teachers voluntarily created a perspective frame.

$8 \times 3 = 24$        $24 - 4 = 20$   
 $12 \div 2 = 6$   
 $12 \times 3 = 36$   
 $\quad - 6$   
 $\quad \hline$ 
 $\quad 30$       30 leaves

Figure 5.2. Student F's strategy for solving the task, "8 caterpillars eat 12 leaves. How many leaves do 20 caterpillars eat?"

**Affective frames.**

Because I sought to identify features of strategies that had the potential to elicit teachers' interest (and thus presumably support their desire to learn more), I focused on positive emotions, such as surprise, curiosity, excitement, and appreciation. Evidence for such emotions emerged from explicit descriptions of such emotions, either in teachers' attending responses, interpreting responses, or the responses in part 3 when teachers were asked if they wanted to interact with a strategy further. There were several ways teachers could display such emotions. For example, one teacher explicitly said "I liked the approach student F used and I could see good proportional reasoning in his method." Expressing that one likes a strategy was clear evidence the teacher reacted positively toward the strategy. Another teacher, referring to student F, wrote "this student is a genius." Effusive comments like this indicated that the teacher felt excitement about the strategy, and thus provided evidence of a positive emotional reaction. Alternatively, some teachers felt surprise or curiosity toward a strategy, which also demonstrated excitement. For example, one teacher wrote about student F "I really want to know where the student is getting

that 3 from.” Clearly this teacher felt a desire to understand more about the strategy, which supported their curiosity (and some excitement) for the strategy.

### **Perspective frames.**

Perspective frames related to the different ways teachers wanted to interact with the student or student’s strategy, of which there were 4 main types of interactions. Either the teacher wanted to (a) learn more about the student’s strategy, (b) help the student, (c) share the strategy with others, or (d) discuss other topics related to the strategy.

#### ***Learn more about the student’s strategy.***

At times teachers wanted to interact with the student’s strategy (by either talking to the student or discussing the strategy with other teachers), because they wanted to learn more about the student’s strategy. Some of the teachers who wanted to learn more also admitted to being confused about the strategy. For example, one teacher wanted to talk to student F, saying “I would like to know where the thought to divide 12 by 2, then subtract the quotient from 36, came from.” Hence, confusion sometimes caused teachers to want to learn more about the strategy, and this teacher identified specific operations that confused him/her. In addition, there were teachers who had captured the gist of the strategy, or who had believed the student was wrong, but wanted to learn more about the student’s strategy anyway. For example, one teacher wanted to talk to student F, and wrote “To see if my theory is correct about his/her reasoning about the problem.” Sometimes, teachers wanted to learn more about the student’s thinking to confirm their interpretations. In this category as well were teachers who wanted to ask other teachers for help making sense of student F’s strategy. For example, one teacher wrote “I would want other teachers [*sic*] input on how this student came to the right answer.”

#### ***Help the student.***

Another common type of interaction teachers wanted to have was to help the student, or discuss ways to help the student. In general, teachers who wanted to help the student assumed the student had either a lack of understanding or a partial understanding, and described how they would help the student. In other words, no teacher who wanted to help a student believed the student had a strong understanding. To illustrate this code, I provide an example of a teacher wanting to help student E, who exhibited a conceptual misunderstanding in his/her work. For example, one teacher wrote “Our job is to help students learn. Student E started with multiplying # of caterpillars and leaves by the same number, then subtracted 4 from both numbers. I want to help this student understand the multiplicative property of proportional reasoning.” In this example, the teacher seems to imply that the student is not fully understanding “the multiplicative property of proportional reasoning,” and wants to help the student understand this property.

These types of responses occurred for other students as well; they were not limited to students who made a mistake or exhibited a conceptual misunderstanding (i.e. strategies A, D, and E). For example, one teacher wrote, “[Students] D/E/F all had trouble understanding how to deal with the mixed number  $8/12$ . I would teach all 3 ‘ $8/12 = 20/x$ .’ F had not much a clue.” Notice that this teacher assumed students D, E, and F were confused, even though student D and student F exhibited more sophisticated proportional reasoning skills, and wanted to interact with the students by helping them learn a new technique.

***Share the strategy with others.***

The third common type of interaction teachers wanted to have was to share the strategy with others, whether it was with the rest of the class or with other teachers. In general, teachers wanted to share the strategy because they saw an idea in the strategy that they valued, or they

were excited by the strategy. For example, one teacher wrote, “These last two solutions [i.e. strategy E and strategy F] would make for a wonderful class discussion because there was obviously a lot of thinking that went into both of them.” Clearly this teacher appreciated the mathematical thinking in student F’s work (and student E’s work), and wanted to share the strategy with the class. The teacher even described the potential discussion as “wonderful.” As another example, one teacher wanted to share the strategy with other teachers and wrote “I would want to say, ‘look at the awesome way this kid multiplied by 2.5, look at the proportional reasoning!’” Many times, when teachers wanted to share a strategy, there was a certain amount of excitement or appreciation associated with the strategy. Other times, there was less excitement, but the teacher still showed that they valued the strategy. For example, one teacher wanted to share student F’s strategy because “This method for solving is unconventional, yet correct. I think it is important that teachers see multiple representations and methods for solving so we can share these with students so that they all may gain a deeper understanding of the topic.” Even though the teacher did not express excitement for the strategy, they still provided evidence of valuing the strategy, and wanted to share the strategy for that reason.

***Discuss other topics related to the strategy.***

Finally, sometimes teachers wanted to discuss other topics, where the topics emerged from considering the strategy. In contrast to instances where teachers wanted to share the strategy, the teachers giving these responses did not appear to express excitement or that they valued a particular aspect of the strategy, and instead wanted to discuss other topics related to the strategy. For example, one teacher wanted to share student F’s strategy with other teachers, and wrote “Did another math teacher tutor this student? If yes what's the approach?” This is different from learning about the strategy itself, or discussing how to help student F, or wanting to show

off student F's strategy to other teachers. Instead, this teacher is curious about instruction that supports such a strategy. Another teacher wrote "I think that not all teachers would be able to understand student F's work. It would be interesting to help other teachers to follow student F's work and to talk about how this student could explain his/her reasoning to others and help improve the proportional reasoning skills of other students." This teacher, rather than wanting to discuss how to help the student, wanted to help other teachers think about proportional reasoning instruction. Both examples I shared thus far relate to instruction, but there were many kinds of topics teachers wanted to discuss, not just topics related to instruction. For example, one teacher simply wanted to know what other teachers thought about student F.

### **Results Related to Personal Frames**

In this section I share results related to the personal connections teachers created with the various strategies. I organize my results in a similar way to the previous section, by sharing the personal frames teachers created for each strategy. Afterwards, I compare trends between categories of strategies, such as integer vs. non-integer strategies, procedural vs. conceptual strategies, and unit rate vs. scalar strategies. In table 8 I share results about how many teachers created an affective frame and how many teachers created a perspective frame<sup>10</sup> for each strategy. The student work exhibiting the strategies and the descriptions of the strategies can be seen in table 5.1 of this chapter. Percentages and Counts for teachers creating particular personal frames can be seen in table 8.

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<sup>10</sup> Percentages and counts will not add up to the totals in the last row, because a teacher could create different types of personal frame in different responses at different points during the survey. Percentages and counts in the last row will not add up to 100%, because a teacher could create multiple personal frames for different strategies.

Table 5.8. Percentages and Counts for teachers creating particular personal frames.

N = 72	A	B	C	D	E	F
<b>Experienced a positive emotional reaction</b>	0% (0)	3% (2)	6% (4)	11% (8)	7% (5)	22% (16)
<b>Want to learn about strategy</b>	3% (2)	4% (3)	14% (10)	18% (13)	22% (16)	46% (33)
<b>Want to help student</b>	4% (3)	1% (1)	4% (3)	6% (4)	38% (27)	8% (6)
<b>Want to share strategy</b>	0% (0)	4% (3)	7% (5)	4% (3)	6% (4)	18% (13)
<b>Want to discuss other topics</b>	1% (1)	6% (4)	1% (1)	7% (5)	1% (1)	8% (6)
<b>Total creating a personal frame</b>	8% (6)	17% (12)	26% (19)	39% (28)	63% (45)	78% (56)

### Strategy A.

Notice that no teacher felt a positive emotional reaction to strategy A, which was a unit rate strategy with integer ratios. This may indicate that the unit rate strategy was a more familiar strategy to teachers, because it did not appear to be a surprising, interesting, or exciting strategy to any teacher. The unit rate concept is a useful concept in many other mathematical domains such as algebraic functions and derivatives, so it is reasonable to believe this strategy was familiar to many teachers.

This strategy elicited very few personal connections as well. Only two teachers wanted to learn more about the strategy. Three teachers wanted to help the student, and two of them focused on helping the student fix their calculation mistake, while the third thought student A had a poor understanding. The last teacher wanted to compare and contrast strategy A with strategy B with other teachers, and discuss what it meant to deeply understand proportional

reasoning. This is an idea that emerged several times in other strategies as well, and it seemed to stem from teachers' recognition that strategy B may not exhibit strong proportional reasoning.

### **Strategy B.**

Strategy B, which was the cross multiplication strategy, also elicited little emotion from teachers. The two teachers who got excited about strategy B both valued the strategy, calling it clear, concise, and even beautiful. In a similar line of reasoning, three teachers wanted to share the strategy with others, indicating that they valued the strategy and wanted other students or teachers to see the strategy.

The rest of the teachers wanted to talk to or about the student because they were unsure if the cross multiplication strategy exhibited strong proportional reasoning or not. Three teachers wanted to learn about student B in order to see if the student really understood proportional reasoning or just followed a memorized procedure. One teacher wanted to talk with other teachers about how to help student B develop an understanding beyond just using a memorized procedure. And in particular, four of the teachers wanted to discuss the cross multiplication strategy with other teachers, and talk about what does and does not count as strong proportional reasoning.

### **Strategy C.**

Notice that more teachers wanted to interact with student C and more teachers exhibited a positive emotional reaction toward student C than in students A and B. Ten teachers wanted to learn about student C's strategy, and five teachers wanted to share the student's strategy with other teachers or with the class. Some of the teachers wanted to share the strategy because they thought it was a neat strategy, while others felt it was not a common strategy and it would be helpful for teachers or for other students to learn about uncommon or different strategies. Three



teachers wanted to help student C, and all three of these teachers assumed the student had a poor understanding, and was confused or was not thinking about the quantities in the story problem. The last teacher, who wanted to discuss other topics, wanted to think more about the strategy itself, and discuss what would happen if the scalar factor was not a whole number.

### **Strategy D.**

Strategy D had the second highest number of teachers who exhibited a positive emotional response, and almost all of the emotional responses were reactions to the ways the student divided and multiplied. The non-standard computations were discussed by many teachers in many ways, and seemed to be the “stars” of this student’s strategy for teachers. For example, almost all of the teachers who wanted to learn more about the student expressed their curiosity toward the “extra work” that came after finding the unit rate of leaves per caterpillar and multiplying the unit rate by 20 caterpillars. In addition, when teachers wanted to show off the strategy to other teachers or to the class, it was because they enjoyed the way the student used the distributive property, or the way the student saw parts of a whole when dividing 12 by 8. Even the teachers who wanted to discuss other topics seemed to focus on the non-standard algorithms. Two teachers wanted to discuss with others how a teacher might support other students to understand these computation strategies, and two of the teachers wanted to see what other teachers thought of the computation strategies.

Otherwise, there were similar personal connections for strategy D as with previous strategies. Four teachers wanted to help student D, and three of these teachers believed the student had a poor understanding of proportional reasoning. This is an idea that emerged for other strategies that exhibited correct understandings as well; some teachers believed students held problematic ideas when the evidence in their work indicated the students had correct

understandings. The fourth teacher who wanted to help the student just wanted to correct the student's syntax error of writing  $8 \div 12$  instead of  $12 \div 8$ . Also, there was one more teacher who wanted to discuss other topics, and this teacher wanted to compare student D's strategy with student B's strategy, and discuss with other teachers what it means to have a deep understanding of proportional reasoning. This is an idea that teachers brought up for strategies A and B as well.

### **Strategy E.**

Strategy E, which had the conceptual error, had the largest percentage of teachers who wanted to help the student (or discuss with other teachers how to help the student; 38%). All of them wanted to help the student overcome the conceptual error the student made, and most of them talked about the student as having a low or partial understanding of proportional reasoning. It seems the conceptual error attracted many teachers to want to interact with the student or discuss the student's strategy with other teachers. Even the ones who wanted to share the strategy chose to do so because they wanted to share a common misconception with other teachers. One teacher wanted to use the clear misunderstanding as evidence to convince other teachers to teach the cross multiplication strategy.

Sixteen teachers (or 22%) wanted to learn more about the student's strategy, which is the second largest percentage of teachers that wanted to learn more about a strategy. Many of these teachers were confused, and in particular wondered about why the student decided to multiply by 3, and subtract by 4. Others had already gotten the gist of strategy E and wanted to learn more about how the student thought about the operations he/she had performed.

Five teachers exhibited some type of positive emotional reaction toward this strategy. Three of these teachers expressed that they were curious about the student's understandings, and

the other two were excited to share the strategy because it “clearly had a lot of thinking,” or “was fascinating to see.”

### **Strategy F.**

Strategy F was by far the most interesting strategy to teachers. 22% of the teachers exhibited a positive emotional reaction to this strategy, which is the largest number of teachers exhibiting affect for any strategy. Most of these teachers were impressed or excited by the strategy, calling the student a “genius,” calling the strategy “fascinating,” or calling the reasoning “wonderful.” Lots of teachers cited the way the student knew to subtract half a group of 12 as being especially exciting. In addition, 18% of the teachers wanted to share the strategy (and most of them wanted to share the strategy with other teachers), which is also the largest percentage of teachers that wanted to share a strategy with others. These teachers often wanted to share the strategy because of the “awesome” reasoning the student demonstrated. Hence, this strategy elicited much excitement from teachers. For others who experienced an affective reaction, the positive emotional reaction was a form of curiosity, as teachers wanted to know how the student reasoned to get a correct answer. A total of 46% of the teachers wanted to learn more about the student and the student’s strategy, which provides more evidence that this strategy piqued teachers’ curiosity in many ways. Overall, this strategy grabbed many teachers’ attention.

Similar to other strategies, there were also teachers who thought the student had partial or poor understanding of proportional reasoning, and thus wanted to help the student develop better proportional reasoning skills. There were also teachers who were curious about instruction that might support students to create such strategies, teachers who wanted to see what other teachers

thought about the strategy, and teachers who wondered how this strategy would apply to other proportions.

### **Comparisons Among Strategies: Identifying Features that Made a Strategy Interesting**

There are five comparisons I want to make. First, I compare between unit rate strategies and scalar strategies, because these strategies utilized different multiplicative relationships (Carney et al., 2015). Then, I compare the integer vs. non-integer ratio strategies, because the presence of a non-integer ratio afforded students with different opportunities to make their work more complex. Afterwards, I look at similarities between strategy D and strategy F, which elicited the most positive affective reactions from teachers. Then, I compare strategies that made conceptual errors vs. strategies that made computational errors. Finally, I consider the procedural cross multiplication strategy in relation to the other strategies, as there was an idea that the procedural strategy elicited that the others did not.

#### **Unit rate strategies vs. scalar strategies.**

Comparing unit-rate strategy A with scalar strategy C, we see more teachers were attracted to strategy C in every category, especially when they wanted to learn more about the student or show off the strategy to others (see table 9). Comparing unit-rate strategy D with scalar strategies E and F, we see similar results. In total, more teachers created personal connections to strategies E and F than strategy D, in several different categories. Even when teachers wanted to learn more about strategy D or share it with others, their reasons for doing so stemmed from the non-standard computations, rather than the underlying reasoning of the strategy (i.e. the unit rate concept). The fact that more teachers wanted to learn about the scalar strategies (14%, 22%, and 46% for Strategies C, E, and F, respectively) than the unit rate strategies (3% and 18%, respectively) provides additional evidence that the scalar strategies were

less familiar to teachers than the unit rate strategies. Even when teachers wanted to learn more about strategy D (18% of teachers), they tended to want to learn more about the non-standard computations rather than the unit rate strategy. It seems that the more unfamiliar a strategy is to teachers, the more intriguing the strategy is to teachers.

Table 5.9. Unit-rate strategies versus. scalar strategies.

N = 72	Task 1: Integer ratios		Task 2: Non-integer ratios		
	A Unit	C Scalar	D Unit	E Scalar	F Scalar
Experienced a positive emotional reaction	0%	6%	11%	7%	22%
Want to learn about strategy	3%	14%	18%	22%	46%
Want to help student	4%	4%	6%	38%	8%
Want to share strategy	0%	7%	4%	6%	18%
Want to discuss other topics	1%	1%	7%	1%	8%
Total percentage of teachers creating personal frame	8%	26%	39%	63%	78%

### Integer vs. non-integer ratio strategies.

Unlike results for the gist of the strategy, there was a clearer connection between the intrigue that a strategy elicited and whether a student shared a strategy that included non-integer ratios. Strategies D and F had higher percentages than strategies A and C for *every* category (Table 10). Considering the non-integer ratio strategies, each of strategy D and F had its own features that intrigued teachers, and these features would not have been present if the numbers in the task exhibited integer ratios instead. For strategy D, the non-standard computations would not have been as complicated if the student did not have to grapple with fractional parts of a whole and mixed numbers. Consequently, when teachers created a personal connection with strategy D, they almost always were intrigued by the non-standard computations, wanting to

show them off to other teachers, learn more about them, or just express that they enjoyed the computational strategies. For strategy F, the strategy itself was the most challenging part for teachers to interpret, with the missing step and the scaling up and partitioning concepts. The combination of scaling up and partitioning concepts would not have been present in a strategy with only integer ratios. These two strategies also elicited the most positive emotional reactions as well.

Table 5.10. Integer versus non-integer strategies.

N = 72	Unit Rate		Scalar	
	A Integer	D Non-integer	C Integer	F Non-integer
Experienced a positive emotional reaction	0%	11%	6%	22%
Want to learn about strategy	3%	18%	14%	46%
Want to help student	4%	6%	4%	8%
Want to share strategy	0%	4%	7%	18%
Want to discuss other topics	1%	7%	1%	8%
Total percentage of teachers creating personal frame	8%	39%	26%	78%

### Similarities between strategy D and strategy F: The ‘wow!’ factor.

Strategies D and F were similar in that both elicited much excitement from teachers, and many teachers wanted to show off the “neat” strategies to other teachers (Table 11). In particular, Strategies D and F elicited more positive emotional reactions than the other four strategies.

What elicited such excitement from teachers in strategies D and F, then?

Table 5.11. Similarities between exciting strategies.

N = 72	A Unit	C Scalar	E Scalar	D Unit	F Scalar
Experienced a positive emotional reaction	0%	6%	7%	11%	22%

In strategy D, we see that the non-standard computations for dividing 12 by 8 and for multiplying  $1\frac{1}{2}$  by 20 excited teachers. These were based on division and multiplication concepts, were novel computation strategies, and used the concepts in a mathematically correct way. The non-standard computation strategies were also quite complex, with several different computations for both computation strategies. Such a complexity was not exhibited in the other strategies; strategies A and C only contained two computations, and strategy E performed the same two computations on both values. Similarly, strategy F was based on proportional reasoning concepts, was unfamiliar to teachers, was conceptually correct, and exhibited a certain complexity in the numerous distinct operations. Based on these results, I put forth four criteria for written mathematical solution strategies that, when combined, give the strategy a “wow!” factor. They are:

- The strategy is rooted in mathematical concepts (as opposed to a traditional or procedural strategy).
- The strategy is unfamiliar/novel to teachers.
- The strategy is complex.
- The strategy is conceptually correct.

I call it the “wow!” factor based on the following teacher’s response, who figured out student F’s strategy in the middle of writing down what about the strategy confused them. The teacher wrote:

I still don't fully understand what he did. The fact that he got the correct solution (which happens with incorrect work all the time) also surprised me. I'm not sure why he decided to divide 12 by 2 and then take the solution (6) and subtract it by 36, which he got by multiplying 12 by 3. I understand why he multiplied  $12 \times 3$ , it is because he multiplied the  $8 \times 3$ . Actually... I just got it. 6 is  $\frac{1}{2}$  of 12. 4 is  $\frac{1}{2}$  of 8. Because you subtract  $\frac{1}{2}$  of 8 (4) to get 24 to get 20... you have to

subtract  $\frac{1}{2}$  of 12 (6) to get 36 to get 30. Wow... maybe I don't need to talk to the student anymore.

Notice that the teacher expressed emotion (surprise) and confusion (“I’m not sure why”) at first. Clearly this strategy was not familiar to the teacher, and the teacher was struggling to make sense of the strategy. Then, when the teacher comprehends the strategy, the teacher decided to write “Wow...” This teacher could have easily omitted the word “wow”, but the teacher chose to express their amusement. Hence, the “wow” factor refers to the potential for a strategy to elicit amusement or excitement from teachers. In my study, I believe that the non-standard computation strategies in student D’s work, and student F’s ‘scaling up with partitioning’ strategy all had a “wow” factor associated with them.

It is important to note that this set of criteria only emerged from the six strategies I used in my study; I imagine that different sets of strategies will elevate a different set of criteria. For example, I am not convinced that a strategy has to be conceptually correct in order to impress teachers; I imagine that a strategy that cleverly uses ideas in a way that is not conceptually correct could also impress teachers, or perhaps a partially correct strategy could have a beautiful idea embedded in it. Hence, I do not claim that this is the only set of criteria. Instead, I claim that these criteria align with the two strategies that spontaneously elicited the largest percentages of positive emotional reactions from teachers, and thus have potential to elicit feelings of awe from teachers when other strategies also fit these criteria.

Affect and intrigue are clearly important factors to consider for teacher education. Philipp, Thanheiser, and Clement (2002) integrated children’s mathematical thinking into a content course for prospective elementary school teachers with the goal of supporting prospective teachers to build on their care for children and develop a care for children’s mathematical thinking, which in turn could motivate the prospective elementary school teachers



to learn more mathematics and shift their beliefs (see also Philipp et al., 2007). Kazemi and Franke (2004) shared results highlighting the power of students’ strategies in eliciting teachers’ curiosity and excitement, which in turn supported a group of practicing elementary school teachers to develop a desire to engage more with students, and learn more about students’ ways of doing mathematics. Sherin et al. (2009) also found that certain types of video artifacts elicited more curiosity and excitement from teachers, and thus supported richer discussions. In my study, I put forth these criteria as one set of criteria teacher educators could use to motivate and engage secondary mathematics teachers.

**Conceptual vs. computational errors.**

Strategy A, D, and E all exhibited some kind of error. Strategies A and D exhibited computational (or syntax) errors, and strategy E exhibited additive reasoning, which is a conceptual error in proportional reasoning. 38% of the teachers wanted to help student E with their conceptual error, while only 4% of the teachers wanted to help student A, and 6% of the teachers wanted to help student D. Clearly, the conceptual error was more interesting to teachers than the computational or syntax error.

Table 5.12. Comparing strategies with calculation errors with strategies with conceptual errors.

N = 72	A Calc. Err	D Calc. Err	E Concept Err
Want to help student	4%	6%	38%

**Reflecting on the cross multiplication strategy.**

Strategy B elicited a personal connection that did not emerge for teachers when considering any other proportional reasoning strategy. In particular, 6 teachers wanted to think about what counted as evidence of strong proportional reasoning (4 who wanted to “discuss other topics” for strategy B, 1 who wanted to compare strategy A with strategy B, and 1 who wanted to

compare strategy D with strategy B). This is a huge but important question, with much potential for a rich discussion. It seemed to stem from teachers' recognition that the cross-multiplication strategy may not exhibit strong proportional reasoning skills. It is unclear if this idea would still have emerged if strategy B had been shown to teachers alone (i.e. without the other strategies for teachers to compare) or not. Either way, the traditional cross-multiplication strategy appeared to support this important question, while other strategies did not. I conjecture that procedural strategies, or strategies that are often memorized by students, can elicit these important discussions when contrasted with other conceptually-based strategies.

Sherin et al. (2009) saw teachers have in-depth discussions related to artifacts of student thinking that exhibited procedural activities by students as well. In their study, they saw instances where teachers appreciated the challenge of unpacking the students' mathematical ideas in a video of students solving routine mathematical tasks, and spent much time discussing the students' mathematical ideas. In this case, the video had a low level of clarity, and so teachers enjoyed the challenge of identifying the students' understandings. Thus, Sherin et al. identified one possible way artifacts exhibiting procedural student thinking could elicit rich discussions among teachers. My study provides evidence for another way such artifacts exhibiting procedural student thinking could support rich discussions. In particular, by supporting teachers to recognize the limitations of memorized procedures in comparison with conceptually-based strategies, teachers might become interested in discussing what it means to have a deep understanding of a particular domain of mathematics.

#### **Summary of features that make a strategy interesting.**

Clearly some strategies are more interesting to teachers than other strategies. In particular, the scalar strategies (which I conjecture were more unfamiliar to teachers) were more

interesting than the unit rate strategies. Additionally, the non-integer ratio strategies were more interesting than the integer-ratio strategies (but this is likely due to the affordances of the non-integer ratios). I conjectured that strategies have potential to elicit much excitement from teachers when they are (a) rooted in concepts, (b) unfamiliar, (c) complex, and (d) correct, as per the non-standard computation strategies in strategy D and the challenging nature of strategy F. Additionally, conceptual errors were more interesting to teachers than computational errors. Finally, I hypothesized that traditional or procedural strategies could support rich discussions about what it means to develop strong proportional reasoning skills.

## **Chapter 6: Conclusion**

In this chapter I conclude my dissertation by discussing final topics. First, I discuss the significance of my results in terms of the literature on professional noticing and teacher education. Then, I discuss the limitations of my study. I finish by presenting possible future directions that could emerge as a result of my work.

### **Significance of Results**

I identify 4 important results from my study. First, results from my first research question inform the field's understanding of secondary teachers' professional noticing expertise, and the similarities and differences among prospective teachers, experienced teachers, and emerging teacher leaders. Second, results from my first research question also provide researchers with descriptions of robust, limited, and lack of evidence of each professional noticing component-skill in the content domain of proportional reasoning, which has not been thoroughly explored in the literature on professional noticing of students' mathematical thinking. Third, in my second research question, I contribute to the field by identifying qualities of students' mathematical strategies that interest teachers, and in what ways they interest teachers, which are new directions of study. Fourth, the methodology used to investigate my second research question is a new methodology for the field to use.

### **Secondary Teachers' Professional Noticing Expertise**

For my first research question I compared and contrasted the professional noticing expertise of three groups of secondary teachers with different amounts of experience and types of experience with students' mathematical thinking. As a reminder, the three groups of teachers were prospective teachers (PSTs), all of whom intended to enter a post graduate credential program, experienced teachers (ETs), all of whom had at least 4 years of teaching experience,

and emerging teacher leaders, all of whom had at least 4 years of teaching experience *and* 4 years of sustained professional support about students' mathematical thinking. These similarities and differences are important to identify for many reasons, but in general they give the field a better understanding of the professional noticing expertise of secondary teachers.

Prior to my study these comparisons had only been made among primary teachers, and it would have been inappropriate to extrapolate results from primary teachers to secondary teachers. In particular, primary and secondary teachers have different opportunities to interact with students' mathematical thinking because their school structures are quite different (Doig, Groves, Tytler, & Gough, 2005). Primary teachers spend the day with the same 30 students but teach 5 different subjects, meaning they spend much time with individual *students'* thinking, but perhaps not with *mathematical* thinking, while secondary teachers generally spend the day teaching the same subject but teach upwards of 150 students, which means they spend much time with *mathematical* thinking, but maybe not with individual *students'* thinking. Certainly the two types of experiences differentially support teachers' professional noticing of *students'* *mathematical* thinking, but it was unclear how. Now, we have findings that inform the field on secondary teachers' professional noticing expertise.

My results indicated that long-term, sustained professional development about students' mathematical thinking supports emerging teacher leaders to develop all three professional noticing skills. The mean of the attending component-skill for emerging teacher leaders approached the ceiling of 2.0, and the means of the interpreting and deciding how to respond component-skills for emerging teacher leaders were both more than double the means for experienced teachers, even though both groups had similar amounts of teaching experience.

That said, there were still emerging teacher leaders who did not provide evidence of interpreting the students' understandings or deciding how to respond to the students' understandings. This implies two things. First, this speaks to the challenging nature of professional noticing of students' mathematical thinking. Studies have shown that professional noticing, and in particular the third component-skill of deciding how to respond, is a challenging skill to develop (e.g. Lesseig et al., 2016; Schack et al., 2013). Second, this means there is still room for growth for some of the teachers in the emerging teacher leader group. It may be interesting to explore what factors pertaining to the professional development, to the teacher, or to the teacher's environment allow teachers to develop their professional noticing expertise during the time when a teacher is a part of a professional development program (e.g. Levin, Hammer, & Coffey, 2009).

My results also indicated that both experienced and prospective secondary teachers were fairly competent at attending to the students' strategies, because both of their means were above the halfway mark of 1.0 (1.10 for ETs to 1.15 for PSTs). However, I was surprised to see that the prospective teachers and experienced teachers provided a similar amount of evidence of attending to the student's strategies. While my data does not allow me to properly investigate why this is the case, I conjectured that the intense responsibilities of secondary teachers might not provide much incentive for teachers to practice *thoroughly* investigating students' ideas, and instead support them to focus on *quickly* investigating students' ideas.

Finally, my results indicated that teaching experience seems support teachers' skills when deciding how to respond, and may marginally support teachers' interpreting skills. For these component-skills, the ETs' means were higher than the PSTs' means (though only slightly higher in the interpreting component-skill; ETs had an average score of 0.58, and PSTs had an average

score of 0.40 on interpreting; ETs had an average score of 0.57, and PSTs had an average score of 0.20 on deciding how to respond).

However, both were much lower than the halfway mark of 1.0. One could have reasonably hypothesized that because secondary teachers spend all day teaching mathematics to students, they may have had ample opportunities to develop their professional noticing of students' mathematical thinking expertise. Such a result would have been dramatically different than the results for primary teachers, which showed us teaching experience does not adequately support this expertise (Jacobs et al., 2010). However, it appears that teaching experience alone is not sufficient for supporting secondary teachers to develop their professional noticing expertise either.

### **Professional Noticing of Students' Mathematical Thinking in the Domain of Proportional Reasoning**

The second important result that emerged from my study is the set of descriptions of high and low levels of professional noticing expertise in the domain of proportional reasoning. Professional noticing is a domain-specific expertise (Jacobs & Empson, 2016), meaning teachers may exhibit different levels of ability of professional noticing depending on the content domain. Thus far, researchers have described teachers' professional noticing<sup>11</sup> expertise in the domains such as young children's pattern generalization thinking (Callejo & Zapatera, 2016), derivatives (Sanchez-Matamoros, Fernandez, & Llinares, 2014), solving algebraic equations (Lesseig et al., 2016), the transition from additive to multiplicative reasoning (Fernandez, Llinares, & Valls, 2012), and young children's early numeracy (Schack et al., 2013), among others. By describing

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<sup>11</sup> Some of these studies claim they share findings related to teachers' professional noticing of students' mathematical thinking expertise, but only share results related to the first two component-skills of attending and interpreting.

different levels of professional noticing expertise in a particular content domain, some of the authors above have even begun to create frameworks to help identify the different levels (e.g. Callejo & Zapatera, 2016). These frameworks may become useful for future studies of teacher learning, as we explore how to support teachers to develop this challenging skill. Thus far, I am not aware of any studies describing high and low levels of professional noticing of students' mathematical thinking in the domain of proportional reasoning, which has been argued to be one of the more important and foundational domains of mathematics (Lesh, Post, & Behr, 1988). Hence, my study contributes to the field by providing descriptions of high and low levels of professional noticing of students' mathematical thinking in the domain of proportional reasoning.

For the attending component-skill, high levels of attending included not only identifying the mathematical details of the strategy (e.g. the student multiplied 8 by 3), but also the multiplicative relationships between the different ratios (e.g. a group of 4 caterpillars is half of a group of 8 caterpillars; each caterpillar eats  $1\frac{1}{2}$  leaves; there are 3 food pellets per mouse). In particular, no teacher who identified the fundamental multiplicative relationships of the strategy expressed that they were confused by the strategy, and every teacher who expressed confusion about the strategy did not identify the fundamental multiplicative relationship exhibited in the strategy. This makes sense because proportional reasoning is a form of multiplicative reasoning, so by not identifying the important multiplicative relationships, the teacher cannot attend to the student's proportional reasoning strategy. On the basis of my results, researchers should themselves attend to these important details when designing future instruments and coding schemes.

For the interpreting component-skill, high levels of interpreting included the careful identification of specific understandings for each student that were consistent with the student's



strategy, and teachers who exhibited robust evidence mostly focused on identifying multiplicative relationships that students' seemed to understand.

For the deciding how to respond component-skill, high levels of deciding included differentiated tasks and/or rationales for each student, and evidence that the teacher was building on students' strategies or anticipating next strategies in a way that was consistent with the student's work. Similar to the findings of Jacobs et al. (2010), number choice was a major indication for whether teachers were considering the students' understandings when deciding how to respond. Teachers chose numbers in order to support several different ideas in students, such as (a) to grapple with more difficult factors, (b) to use their strategies in proportional reasoning tasks with more than two quantities, (c) to solve for a different missing quantity (i.e. search for missing value of mice instead of missing value of pellets), (d) to consider decreasing multiplicative relationships instead of increasing relationships, or (e) to support students to possibly identify other multiplicative relationship (i.e. instead of finding a unit rate, choose numbers that may invite students to scale the ratio up).

### **Identifying Features of Students' Strategies that may Interest Teachers.**

Artifacts of student thinking are important tools for teacher education (Little, Gearhart, Curry, & Kafka, 2003). Clearly, we want to select artifacts of student thinking that will support rich discussions among teachers, help teachers identify important aspects of student thinking, or intrigue teachers and support in them a desire to shift their instructional beliefs and practices (e.g. Kazemi & Franke, 2004). However, little is known about which artifacts of student thinking will accomplish which goals. Sherin, Linsenmeier, and van Es (2009) identified characteristics of video artifacts that supported rich discussions among teachers, and were surprised that some of their assumptions about what characteristics would support rich

discussions among teachers did not actually elicit the rich discussions they anticipated. What other surprises might we find in a search for “interesting” artifacts?

In particular, the third significant result from my study is the identification of different features of students’ strategies, and *how* these features intrigued teachers, as each feature (or set of features) intrigued teachers in different ways. For example, teachers in my study were clearly drawn toward helping students with conceptual misunderstandings more than students with computational mistakes. Additionally, some teachers were drawn toward the procedural strategy, but only because they were interested in discussing with other teachers what counted as strong proportional reasoning skills and what did not count as strong proportional reasoning skills, or more broadly what it meant to have strong proportional reasoning skills. Finally, I put forth four criteria for student strategies that I believe have the potential to excite teachers, which I amicably call the “wow” factor: (a) the strategy is conceptually based, (b) the strategy is unfamiliar or novel, (c) the strategy is complex, and (d) the strategy is correct. I conjecture that these four criteria could apply to video artifacts of student thinking as well, and not only written artifacts of student thinking. I also conjecture that these four criteria could apply to other content domains as well. Given that these criteria emerged from the six strategies I selected for my study, I wonder what other criteria we might find by having teachers consider other sets of strategies that exhibit different features.

Clearly there are many reasons a mathematics teacher educator or professional development leader might select an artifact of student thinking, and not just to inspire awe in teachers. I invite other researchers to pursue this direction of research and help the field develop robust criteria for selecting artifacts of student thinking for different purposes.

## **New Methodology for Understanding how Teachers Interact with Artifacts of Student Thinking.**

Sherin and Russ (2014) created a framework for the field to understand the different interpretive frames teachers tacitly create as they notice classroom artifacts. The fourth significant result from my study is more methodological in nature, as I provide the field with a way to use their framework to understand how certain artifacts of student thinking elicit particular interactions or ideas from teachers. In my study I collected data on teachers' personal frames, and then counted the number and types of personal frames teachers created for each artifact. In this way I was able to identify which artifacts intrigued teachers and for what reasons.

This methodology is analogous to finding out which mathematical tasks are more difficult than others. Mathematical tasks are not inherently easy or difficult; they are easy or difficult only when our students find them easy or difficult. We can only find out which tasks are more difficult by identifying which tasks cause the most confusion or wrong answers. Similarly, artifacts of student thinking are not inherently interesting; they are only interesting if teachers find them interesting.

### **Limitations**

I share two limitations to my study. First, the data I captured likely represents an upper bound for teachers' actual professional noticing skills. In the classroom teachers must attend to many stimuli, and teachers rarely get a chance to sit down and deeply consider each student's strategy. In my study, teachers had as much time as they wanted to describe each student's strategy, and did not have to attend to the added responsibilities and noise of the classroom. Hence, it is likely that the data that I captured represents the upper limit of teachers' professional

noticing skills. This may be part of the reason why experienced teachers and prospective teachers exhibited similar levels of evidence of attending to the students' strategies. Studies show that beginning teachers struggle with the complexities of a classroom, but as teachers gain more classroom experience they develop routines and patterns of noticing that allow them to competently navigate the classroom environment (Berliner, 1994). Perhaps a comparison of experienced teachers' attending skills in-the-moment of instruction with prospective teachers' attending skills in-the-moment of instruction will elevate clearer differences between the two groups' attending skills.

Second, the results pertaining to secondary emerging teacher leaders in my study are based on a small sample size,  $n = 12$ . This is quite a bit smaller than the sample size of 33 emerging teacher leaders that Jacobs et al. (2010) studied. I would have liked to recruit more emerging teacher leaders, but only 12 were available to me as these 12 had completed 4 years of professional development focused on students' mathematical thinking.

Third, at the time of data collection, 18 of the prospective teachers had completed either an inquiry oriented course, or a content course for middle and high school teachers that included opportunities to make sense of students' ideas. Hence, 60% of the prospective teachers in my study had opportunities to practice making sense of their peers' mathematical ideas or of middle or high school students' ideas. Such experiences can improve one's professional noticing expertise (e.g. Callejo & Zapatera, 2016; Lesseig et al., 2016). However, I am not aware of how prevalent these types of courses are across institutions. Hence, I caution the reader to consider this fact when generalizing the results of this study to other populations.

### **Future Directions**

There are many directions one can pursue based on my work. To conclude, I will list several questions that have emerged for me:

1. Jacobs et al. (2010) looked at three different groups of experienced K-3 teachers, based on their years of professional development about students' mathematical thinking: 0 years, 2 years, and 4+ years. With this information they were able to identify which component-skills improved more quickly and which took more time. I only looked at two groups, one of which had 4 years of professional development about students' mathematical thinking, and the other did not. Do the component-skills improve in similar ways for secondary teachers as they do for primary teachers? A longitudinal study may shed more light on this question. By following a group of teachers through a long term, sustained professional development program, one could investigate this group of teacher's professional noticing expertise at different points of time, and identify professional noticing developmental trajectories.
2. Additional research is needed to identify differences in prospective teachers' attending skills and experienced teachers' attending skills. Although both groups had similar attending scores, I suspect that there may have been differences that were challenging to pull out in this paper-pencil assessment.
3. In my study there were emerging teacher leaders who did not provide evidence of interpreting or deciding how to respond to students' understandings. What factors might have impeded such teachers from providing more evidence of interpreting students' understandings and deciding how to respond on the basis of the students' understandings?
4. There were also a small number of experienced teachers and prospective teachers who did provide evidence of interpreting students' understandings and of deciding how to

respond based on students' understandings. What internal factors or environmental factors might have supported such teachers to develop this skill?

5. The relationship between a teacher's noticing skills and mathematical knowledge for teaching is not totally clear (e.g. Thomas, Jong, Fisher, & Schack, 2017), but clearly what one notices is influenced by one's mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). In my study teachers had to interpret and respond to several different types of proportional reasoning strategies, and sometimes teachers expressed confusion about particular strategies. Hence, I wonder if the ability to anticipate possible student strategies, which relies on teachers' knowledge of content and students (a subset of mathematical knowledge for teaching), might be a predictor of professional noticing of students' mathematical thinking?
6. Identifying which strategies "interest" teachers is a new direction of research. I am curious about what kinds of results might emerge when comparing among other types of strategies, or exploring strategies in other content domains, or by thinking about what interests particular groups of teachers. I posited that a strategy will inspire awe in teachers if it is (a) conceptually based, (b) unfamiliar to the teacher, (c) complex, and (d) correct. Does this set of criteria hold up under additional scrutiny from the field?

## Appendix A: Professional Noticing Survey Items

Part 1a: Solve the following proportional reasoning task.

**Each day, 6 mice eat 18 food pellets. How many food pellets would 24 mice eat?**

*Solve the problem in a way that makes sense to you. After you solve it one way, try to solve it again in a different way.*

Solution 1:

Solution 2:

Below are three students' solutions to the proportional reasoning task. Feel free to tear this page out to help you answer parts 1b, 1c, and 1d.

**Each day, 6 mice eat 18 food pellets. How many food pellets would 24 mice eat?**

Student A

$$\begin{array}{r} 6 \overline{)18} \\ \underline{12} \phantom{0} \\ 6 \phantom{0} \\ \underline{6} \phantom{0} \\ 0 \phantom{0} \end{array} \quad \begin{array}{r} 1 \\ 24 \\ \times 3 \\ \hline 62 \text{ food pellets} \end{array}$$

Student B

$$\begin{array}{r} 18 \\ \hline 6 \end{array} = \frac{x}{24}$$
$$\frac{432}{6} = \frac{6x}{6}$$
$$72 = x$$

Student C

$$24 \div 6 = 4$$
$$18 \times 4 = 72$$



Part 1b: Describe in detail what each student did in response to the problem. Then respond to the prompts about each solution. (I recognize that you do not know these students and did not get a chance to talk to them - Please just do the best you can.)

**Student A Description:**

**Student B Description:**

**Student C Description:**

Part 1c: What did you learn about these three students' understandings?

Part 1d: Pretend you are the teacher of these students. What problem(s) might you pose next, and why? (I am interested in how you think about selecting problems, but I do not believe that there is ever a best problem, and I recognize that as the teacher of these students, you would have more information to inform your selection.)

Problem(s):

Rationale(s):

Part 2a: Solve the following proportional reasoning task.

**Each day, 8 caterpillars eat 12 leaves. How many leaves would 20 caterpillars eat?**

*Solve the problem in a way that makes sense to you. After you solve it one way, try to solve it again in a different way.*

Solution 1:

Solution 2:

Below are three students' solutions to the proportional reasoning task. Feel free to tear this page out to help you answer parts 2b, 2c, and 2d.

**Each day, 8 caterpillars eat 12 leaves. How many leaves would 20 caterpillars eat?**

Student D

$$8 \div 12 = 1\frac{1}{2}$$

$$8 \div 8 = 1$$

$$12 - 8 = 4 = \frac{1}{2} \text{ of } 8$$

$$\frac{1}{2} \times 20 = 10 \text{ (30 leaves)}$$

$$1 \times 20 = 20$$

$$20 \times 1 = 20$$

$$\frac{20}{2} = 10$$

$$20 + 10 = 30$$

Student E

8 caterpillars	12	
$\times 3$	$\times 3$	
24	36	
Caterpillars	leaves	
	36	
$\frac{24}{-4}$	$\frac{36}{-4}$	
20	32	

Caterpillars	leaves
20	32

Student F

$$8 \times 3 = 24$$

$$24 - 4 = 20$$

$$12 \div 2 = 6$$

$$17 \times 3 = 36$$

$$\frac{36}{-6}$$

$$30$$

30 leaves

Part 2b: Describe in detail what each student did in response to the problem. Then respond to the prompts about each solution. (I recognize that you do not know these students and did not get a chance to talk to them - Please just do the best you can.)

**Student D Description:**

**Student E Description:**

**Student F Description:**

Part 2c: What did you learn about these three students' understandings?

Part 2d: Pretend you are the teacher of these students. What problem(s) might you pose next, and why? (I am interested in how you think about selecting problems, but I do not believe that there is ever a best problem, and I recognize that as the teacher of these students, you would have more information to inform your selection.)

Problem(s):

Rationale(s):

Part 3: Below are all the solutions you considered. Feel free to tear this page out to help

you answer parts 3a, 3b, and 3c.

Student A

$$6 \overline{)18} \quad \begin{array}{r} 1 \\ 24 \\ \times 3 \\ \hline 62 \text{ food pellets} \end{array}$$

Student C

$$24 \div 6 = 4$$

$$18 \times 4 = 72$$

Student B

~~$$\begin{array}{r} 18 \\ \times 4 \\ \hline 72 \end{array}$$~~

$$\begin{array}{r} 432 \\ \times 6 \\ \hline 72 = x \end{array}$$

Student D

$$8 \div 12 = 1 \frac{1}{2}$$

$$8 \div 8 = 1$$

$$12 - 8 = 4 = \frac{1}{2} \text{ of } 8$$

$$\frac{1}{2} \times 20 = 10 \text{ (circled)} = 30 \text{ leaves}$$

$$1 \times 20 = 20$$

$$20 \times 1 = 20$$

$$\frac{20}{2} = 10$$

$$20 + 10 = 30$$

Student E

8 caterpillars	$\begin{array}{r} 12 \\ \times 3 \\ \hline 36 \\ \text{leaves} \end{array}$	Caterpillars   leaves 20 - 32
$\begin{array}{r} \times 3 \\ \hline 24 \end{array}$		
Caterpillars	$\begin{array}{r} 36 \\ - 4 \\ \hline 32 \end{array}$	

$$\begin{array}{r} 24 \\ - 4 \\ \hline 20 \end{array}$$

Student F

$$8 \times 3 = 24$$

$$24 - 4 = 20$$

$$12 \div 2 = 6$$

$$12 \times 3 = 36$$

$$\begin{array}{r} 36 \\ - 6 \\ \hline 30 \end{array}$$

30 leaves

Part 3a. Out of all the students' solutions, is there a student you would like to talk to further?

Circle one:    **Yes**            **No**

If yes, which student would you like to talk to, and why?

Student: \_\_\_\_\_

Explanation:

Part 3b. Out of all the students' solutions, would you be interested in discussing a particular solution with other teachers?

Circle one:    **Yes**            **No**

If yes, which solution would you like to discuss with other teachers, and why?

Student: \_\_\_\_\_

Explanation:

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