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### **Publication Date**

2009-07-24

# Exceptional groups, symmetric spaces and applications

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May 19, 2009

## Abstract

In this article we provide a detailed description of a technique to obtain a simple parametrization for different exceptional Lie groups, such as  $G_2$ ,  $F_4$  and  $E_6$ , based on their fibration structure. For the compact case, we construct a realization which is a generalization of the Euler angles for  $SU(2)$ , while for the non compact version of  $G_{2(2)}/SO(4)$  we compute the Iwasawa decomposition. This allows us to obtain not only an explicit expression for the Haar measure on the group manifold, but also for the cosets  $G_2/SO(4)$ ,  $G_2/SU(3)$ ,  $F_4/Spin(9)$ ,  $E_6/F_4$  and  $G_{2(2)}/SO(4)$  that we used to find the concrete realization of the general element of the group. Moreover, as a by-product, in the simplest case of  $G_2/SO(4)$ , we have been able to compute an Einstein metric and the vielbein.

The relevance of these results in physics is discussed.

## 1 INTRODUCTION

In this article we describe our technique to analyze the structure of exceptional Lie groups, which is based on constructing a generalized Euler parametrization by starting from a suitable fibration. We review our results on  $G_2$  [1, 2],  $F_4$  [3] and  $E_6$  [4]. We also provide some new insights on the geometry of the non compact versions of these groups, by using the Iwasawa decomposition, and in particular we apply it to  $G_{2(2)}$ . Our method allows us to explicitly calculate the Haar measure for the group manifold, and, as it is compatible with the fibration used to compute it, it naturally provides a metric for the corresponding coset as well.

The layout of this paper is as follows. In section 2 we recall some of the basic facts about Lie groups and Lie algebras, that we need later. In section 3 we explain in detail how the generalized Euler parametrization is defined and we study some toy model to exemplify it. Then in the following sections we apply it to different exceptional Lie groups. In section 4 we construct  $G_2$  in two different ways as a fibration, first with  $SU(3)$  as a fiber and then

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with  $SO(4)$  as a fiber. In section 5 we determine the  $Spin(9)$  Euler angles for  $F_4$ , which we then use in section 6 to obtain the  $F_4$  Euler angles for  $E_6$ . Finally, in section 7 we introduce the Iwasawa decomposition for the non compact version of the Lie groups, which we then apply to  $G_{2(2)}$  in section 8.

Since we are able to get an explicit expression for the Haar measure on the group manifold, the most immediate application of our results is the possibility of evaluating integrals [5]. Until now the only available method to compute some of them was to use the invariance properties of the Haar measure, but knowing its explicit form gives an analytic way to calculate many of them directly.

In physics exceptional Lie groups appear naturally as the symmetry (gauge) groups of field theories which are low energy limits of certain heterotic string models [6]. Besides from being relevant for string phenomenology, these theories are interesting by themselves, e.g.  $E_6$  as a candidate for the symmetry group in a grand unified theory of high energy physics [7] and  $G_2$  as a possible example of a non confining gauge theory [8]. While the local properties of a field theory are determined exclusively at the level of the corresponding Lie algebra, in order to obtain non-perturbative results it is necessary to make use of the full global structure of the Lie group, because of the need for evaluating integrals on the group manifold. Being able to solve them analytically has drastically reduced the computer power required to run a lattice simulation. For instance our expressions for  $G_2$  are the base for the Montecarlo analysis presented in [9].

Moreover, our technique can also be applied to the noncompact versions of the Lie groups, such as  $G_{2(2)}$ ,  $F_{4(4)}$ ,  $E_{6(6)}$  or  $E_{7(7)}$ . In this case, another parametrization is the Iwasawa decomposition. As its construction uses a nilpotent subalgebra, it is particularly simple and is therefore very useful. In physics these groups represent the U-duality of supergravity theories in different dimensions.

One of the most interesting features of our method is that it is based on identifying a suitable subgroup and in studying the corresponding fibration. As a consequence it automatically yields an explicit expression for the coset space as well as for its metric, measure and vielbein, since the geometry on the group induces a geometry on the base. In the case of the maximal compact subgroups of noncompact exceptional Lie groups, e.g.  $SO(4)$  for  $G_{2(2)}$  or  $SU(8)$  for  $E_{7(7)}$ , these symmetric spaces turn out to be Einstein spaces. Being solutions of Einstein equations, they are relevant by themselves for general relativity.

In supergravity some of these cosets are interpreted as the scalar fields of the associated sigma model [10]. Moreover, they can represent the charge orbits of black holes when the attractor mechanism is studied [11] and they also appear as the moduli spaces for black holes. In [12] they are used to investigate the deep connection between black holes properties, duality and supergravity.

As an example, the coset space  $G_{2(2)}/SO(4)$  studied in section 8 is relevant for black ring solutions in 5-dimensional supergravity [13].

Finally, these symmetric spaces can be used to describe the entanglement of qubits and qutrits in information theory [14].

## 2 GENERAL SETTINGS

Because of their importance for the rest of the chapter and in order to set our conventions, we recall here some basic facts about semisimple Lie groups (see [15]).

### 2.1 Lie algebras from Lie groups

A Lie group  $G$  is a group which is also a differential manifold and for which the group structure and the differential structure are compatible. This means that the two basic group operations, the product and the inversion, are required to be differentiable maps with respect to the differential structure. The dimension of the group is the dimension of  $G$  as a manifold. Here we consider only finite dimensional groups. In this case the differentiability of the inverse map is a consequence of the differentiability of the product map and of the implicit function theorem. We use the symbol  $e$  for the unit element, which therefore identifies a particular point on  $G$ .

For any  $g \in G$  we can define two maps:

$$\begin{aligned} L_g : G &\longrightarrow G, \quad h \mapsto gh, \\ R_g : G &\longrightarrow G, \quad h \mapsto hg, \end{aligned}$$

called the left and the right translation respectively. Note that with respect to the composition product,  $L_g$  and  $R_{g'}$  commute. They define a left and a right action of the group on itself:

$$\begin{aligned} L : G \times G &\longrightarrow G; \quad (g, h) \mapsto L_g(h), \\ R : G \times G &\longrightarrow G; \quad (g, h) \mapsto R_g(h). \end{aligned}$$

Note that  $L_g$  and  $R_g$  are not homomorphisms. A homomorphism associated to these actions is:

$$\phi_g : G \longrightarrow G; \quad h \mapsto R_{g^{-1}}L_g h = ghg^{-1}. \quad (1)$$

Differentiating the  $L_g$  map at the identity, we have

$$(dL_g)_e : T_e G \longrightarrow T_g G.$$

This operation associates to each vector  $\xi \in T_e G$  a non vanishing vector field  $X_\xi$

$$X_\xi : G \longrightarrow TG; \quad g \mapsto (dL_g)_e(\xi) \in T_g G,$$

which is well defined globally. Note that  $X_\xi(e) = \xi$ . In this way, given a basis  $\{\tau_1, \dots, \tau_n\}$  of  $T_e G$ , at each point  $g$  we can obtain a set of vector fields which determine a basis for  $T_g G$ . This shows that the tangent bundle of  $G$  is trivial.

An important property of the field  $X_\xi$  is that it is  $L_g$ -invariant (*left invariant*). This means  $(L_g)_* X_\xi = X_\xi$ . Viceversa, given a left invariant vector field  $V$ , it can be verified that  $V(e) \in T_e G$  and  $V = X_{V(e)}$ . Thus, the left invariant vector fields form a finite dimensional vector space  $\mathcal{X}^L(G) \simeq T_e G$ . Moreover,  $\mathcal{X}^L(G)$  is closed under the Lie bracket of vector fields:

$$[X, Y] \in \mathcal{X}^L(G), \quad \text{for all } X, Y \in \mathcal{X}^L(G) \quad ([X, Y] = \mathcal{L}_X Y),$$

where  $\mathcal{L}_X$  is the Lie derivative along  $X$ . Thus,

$$\mathfrak{g} \equiv \text{Lie}(G) := \{\mathcal{X}^L(G), [, ]\}$$

defines an algebra: the *Lie algebra associated to  $G$* . The Lie product has the properties of being antisymmetric and of satisfying the Jacobi identity.

## 2.2 Adjoint representations and the Killing form

A powerful way to “describe” the structure of a group is by means of its representations. A representation of a group  $G$  on a vector space  $V$  (real or complex) is a homomorphism  $r : G \rightarrow \text{Aut}(V)$ , where  $\text{Aut}(V)$  is the group of automorphisms of  $V$  with the composition as product. A representation is irreducible if  $V$  does not admit any proper invariant subspaces, and it is faithful if  $\text{Ker}(r) = e$ . In a similar way, a representation of a Lie algebra on  $V$  is a homomorphism  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ , where  $\text{End}(V)$  is the Lie algebra of endomorphisms of  $V$  with the bracket of operators as Lie product. Noting that  $\text{End}(V) = \text{Lie}(\text{Aut}(V))$  and identifying  $\text{Lie}(G)$  with  $T_e G$ , it can be seen that a representation of the algebra can be obtained from a representation of the group by differentiation:  $\rho = dr_e$ .

Among the representations of a group, an example which can be constructed in a natural way is the Adjoint. It is the representation over the Lie algebra  $V = \mathfrak{g}$  obtained in the following way through the homomorphism  $\phi_g$  introduced above. For any fixed  $g$ , we define the map:

$$Ad_g : T_e G \rightarrow T_e G; \quad Ad_g := (d\phi_g)$$

where  $d$  is the differential of  $\phi_g$  at the identity. Then the *Adjoint representation* of the group is defined by:

$$Ad : G \rightarrow \text{Aut}(T_e G); \quad g \mapsto Ad_g. \quad (2)$$

Differentiating at the identity yields the *adjoint representation* of the Lie algebra

$$ad : \mathfrak{g} \rightarrow \text{End}(T_e G); \quad a \mapsto ad_a, \quad (3)$$

where  $ad_a(b) = [a, b]$  for all  $b \in \mathfrak{g}$ .

Next, from the adjoint representation of the algebra, the Killing form on  $\mathfrak{g}$  can be constructed as follows:

$$K : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}; \quad (a, b) \mapsto K(a, b) := \text{Tr}(ad_a ad_b), \quad (4)$$

where  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  is the field of  $\mathfrak{g}$ . The Killing product is symmetric and *ad*-invariant, which means:

$$K(ad_a(b), c) + K(b, ad_a(c)) = 0.$$

This defines a symmetric two form over  $T_e G$  which in turn, using the left translation, induces a symmetric two form over the whole group:

$$K_G : G \rightarrow T^*G \otimes T^*G; \quad g \mapsto L_{g^{-1}}^* K, \quad (5)$$

the pullback of  $K$  under  $L_{g^{-1}}$ . In general the Killing form is degenerate. It is obviously left invariant. If we choose a basis  $\{\tau_i\}$  for  $\mathfrak{g}$  and define the corresponding structure constants

$f_{ij}{}^k$  as  $[\tau_i, \tau_j] = \sum_{k=1}^n f_{ij}{}^k \tau_k$ , then the Killing form turns out to be  $K_{ij} = K(\tau_i, \tau_j) = \sum_{l,m} f_{il}{}^m f_{jm}{}^l$ . The  $ad$ -invariance implies that the covariant tensor  $f_{ijk} := f_{ij}{}^l K_{lk}$  is totally antisymmetric. In the basis  $\{\mu^i\}$  which is canonically dual to  $\tau_j$ , (i.e.  $\mu^i(\tau_j) = \delta_j^i$ ), the Killing form takes the particularly simple expression  $K = \sum_{ij} K_{ij} \mu^i \otimes \mu^j$ . Finally, an important role is played by the Cartan 1-form. It is a Lie algebra valued form defined as  $J := \sum_i (L_g)_*(\tau_i \mu^i) = \sum_i J^i X_{\tau_i}$  and it can be used to rewrite the Killing form as  $K_G = K(J, J) = \sum_{ij} J^i \otimes J^j K_{ij}$ .

### 2.3 Simple Lie algebras classification

Starting from a finite dimensional Lie group, the associated Lie algebra can be easily determined. Being a linear space, it is much easier to analyze than the group itself. There is a very interesting class of Lie algebras, which are completely classified: the semisimple Lie algebras. A *semisimple Lie algebra* is a Lie algebra of dimension higher than 1, which does not admit any Abelian proper ideals. If it does not contain any proper ideal at all, it is called a *simple Lie algebra*. It can be shown that any semisimple Lie algebra can be written as a direct sum of simple algebras in a unique manner (up to isomorphisms). An important result is that a Lie algebra is semisimple if and only if the corresponding Killing form is non degenerate.

From the definition, it follows that for a semisimple algebra  $\text{Ker}(ad) = 0$ , so that the adjoint representation is faithful. The Lie algebra can then be identified with its adjoint representation. This allows a classification of all complex (finite dimensional) simple Lie algebras by performing a classification of their adjoint representation. As the main ingredients will be used later, let us recall the main steps.

Any simple Lie algebra contains a unique (up to isomorphisms) Cartan subalgebra, a maximal Abelian  $\mathfrak{h} \subset \mathfrak{g}$  subalgebra such that for each  $h \in \mathfrak{h}$ ,  $ad_h$  is diagonalizable. Then  $r = \dim(\mathfrak{h})$  is called the rank of  $\mathfrak{g}$ . All such operators  $ad_h$  are simultaneously diagonalizable and their eigenvalues are called the roots  $\alpha \in \mathfrak{h}^*$  of the algebra:

$$ad_h(\lambda_\alpha) = \alpha(h)\lambda_\alpha, \quad 0 \neq \lambda_\alpha \in \mathfrak{g}.$$

Since  $\mathfrak{g}$  is finite dimensional, the set of all roots  $Root(\mathfrak{g})$  is finite. If  $\Lambda_\alpha$  is the eigenspace of  $\alpha$  then 0 is a root,  $\Lambda_0 = \mathfrak{h}$ , and  $\mathfrak{g} = \bigoplus_{\alpha \in Root(\mathfrak{g})} \Lambda_\alpha$ , with the properties that  $[\lambda_\alpha, \lambda_\beta] \in \Lambda_{\alpha+\beta}$ , and that it vanishes if  $\alpha + \beta$  is not a root.

From the  $ad$ -invariance it follows that  $K(\lambda_\alpha, \lambda_\beta) = 0$  if  $\alpha + \beta \neq 0$ . It can also be shown that if  $\alpha$  is a non vanishing root, then  $k\alpha$  is a root if and only if  $k = 0, \pm 1$  and  $\dim(\Lambda_\alpha) = 1$ . If  $K_C$  is the restriction of the Killing form to the Cartan subalgebra, it follows that  $K_C$  is non degenerate and therefore it defines a natural isomorphism between  $\mathfrak{h}$  and  $\mathfrak{h}^*$ , as well as a bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$  in an obvious way. It also follows that  $Root(\mathfrak{g})$  is real, in the sense that it contains a basis for  $\mathfrak{h}^*$ , such that the remaining roots are real combinations of the basis elements and that it is possible to consistently define the  $r$ -dimensional real space  $\mathfrak{h}_{\mathbb{R}}^* = \langle Root(\mathfrak{g}) \rangle_{\mathbb{R}}$ . Up to a multiplicative constant,  $(\cdot, \cdot)$  defines a Euclidean scalar product on  $\mathfrak{h}_{\mathbb{R}}^*$ . The main result we need in this context is:

**The Cartan Theorem:** If  $\alpha$  and  $\beta$  are two non vanishing roots, then  $n_{\alpha\beta} := 2 \frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbb{Z}$  and  $\beta - n_{\alpha\beta}\alpha$  is also a root (Weyl reflection).

This strongly constrains the relations among the roots, because if  $|\alpha|$  and  $\theta_{\alpha\beta}$  are respectively the norm and the angle between two roots defined by the Euclidean scalar product, then

$$\frac{|\beta|^2}{|\alpha|^2} = \frac{n_{\alpha\beta}}{n_{\beta\alpha}}, \quad \cos^2 \theta_{\alpha\beta} = \frac{1}{4} n_{\alpha\beta} n_{\beta\alpha}.$$

At this point it is clear that all the information on the algebra is contained in the root system. A simple root system  $SR$  is defined as a basis of the root space, such that all the remaining roots are combinations of  $SR$  with integer coefficients of the same sign. Such a system always exists, even though in general it is not unique, and it decomposes the root space into a positive and a negative part:  $Root(\mathfrak{g}) = R^+ \oplus R^-$ . Given a simple root system  $SR = \{\alpha_1, \dots, \alpha_r\}$ , the numbers  $n_{ij}$  associated to it by the Cartan Theorem must be all non positive if  $i \neq j$ , while for  $i = j$  it has to be  $n_{ii} = 2$ . Moreover, either  $|n_{ij}|$  or  $|n_{ji}|$  is always 1 if  $i \neq j$ . These numbers characterize  $SR$  completely (up to obvious equivalences) and define the the Cartan matrix  $C_{ij} = n_{ij}$ , which has the properties:  $C_{ii} = 2$ ,  $C_{ij} \leq 0$  and  $C_{ij} \neq 0$  if and only if  $C_{ji} \neq 0$ ,  $i \neq j$ . To classify all the simple Lie algebras, it is therefore enough to classify all the  $SR$  systems, or, equivalently, all the Cartan matrices compatible with them. This is done graphically by means of the Dynkin diagrams: A dot  $\circ$  is associated to each of the  $r$  simple roots. Two roots are then connected by  $N_{ij} = n_{ij}n_{ji}$  lines with a  $>$  indicating the direction from the longer root to the shorter one. Simple algebras correspond to a connected Dynkin diagram. It turns out that the admissible Dynkin diagrams can be classified into four classical series:  $A_r, B_r, C_r, D_r$ ,  $r$  being the rank of the corresponding algebras, plus five exceptional cases:  $G_2, F_4, E_6, E_7, E_8$ . The corresponding Dynkin diagrams can be found for example in [15].

Next, the real forms of each of these Lie algebras can be classified by identifying the generators which also span a real algebra (i.e. which admit real structure constants). In particular, every simple algebra has a compact form, the real algebra over which the Killing form is negative definite. The corresponding Lie group is compact. All the real forms are classified and are described for example in [16].

## 2.4 Lie groups from Lie algebras

As we have seen in the previous sections, from a Lie group it is easy to obtain the associated Lie algebra by simple differentiation. Less trivial is the issue of recovering the group from the algebra. This is indeed the main argument of the remaining sections. Here we are simply going to recall some properties of a key tool, the exponential map:

$$\exp : Lie(G) \longrightarrow G; X \mapsto g_X(1)$$

where  $g_X(t)$  is the integral curve on  $G$  associated to the left invariant vector field  $X$ , with  $g_X(0) = e$ . Its main properties are

- $\exp(0) = e$ ;
- $\exp(X + Y) = \exp(X)\exp(Y)$  if  $[X, Y] = 0$ ;

- $\exp$  is differentiable and

$$d\exp_0 : T_0\text{Lie}(G) \longrightarrow T_e G$$

realizes the natural isomorphism between  $\text{Lie}(G)$  and  $T_e G$ ;

- $\exp$  is a local diffeomorphism between an open neighborhood of  $0 \in \text{Lie}(G)$  and an open neighborhood of  $e \in G$ .

In general the exponential map is not surjective, however it generates the whole group by starting from the algebra. For matrix groups it is easy to show that

$$\exp(X) = e^X := \sum_{n=0}^{\infty} \frac{1}{n!} X^n.$$

As we are going to work with finite representations, this is our case. Given a matrix realization of the group in a suitable parametrization  $g(x_1, \dots, x_n)$ , the expression for the Cartan 1-form is:

$$J = g^{-1} dg = \sum_i J^i \tau_i, \quad (6)$$

where  $\{\tau_i\}$  is a basis for the Lie algebra. In physics the 1-forms  $J^i$  are also often called the left-invariant currents. They will play a central role in our construction.

The main problem is now to find suitable parameterizations of the group, which on the one hand should be able to capture the whole group, but on the other hand should still remain manageable from a practical point of view, i.e. suitable for concrete physical applications. This means that we need not only to explicitly individuate the elements of the group, but also to specify the complete range for the parameters, and to compute explicitly the significant quantities such as for example the left invariant currents, the invariant measure and the Killing form.

## 3 CONSTRUCTION OF COMPACT LIE GROUPS

### 3.1 A toy model

We start by illustrating the main ideas of our strategy in the simplest possible example, the construction of the  $SU(2)$  group, the set of all unitary matrices with unitary determinant. The associated Lie algebra  $\mathfrak{su}(2)$  is generated by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (7)$$

which, after multiplication by  $i$ , indeed constitute a basis for the space of  $2 \times 2$  anti Hermitian matrices. It is a well known fact that the generic element of  $SU(2)$  can be expressed in the form:

$$g = e^{i\phi \frac{\sigma_3}{2}} e^{i\theta \frac{\sigma_1}{2}} e^{i\psi \frac{\sigma_3}{2}}, \quad (8)$$



where  $\phi \in [0, 2\pi]$ ,  $\theta \in [0, \pi]$ ,  $\psi \in [0, 4\pi]$  are called the *Euler angles* for  $SU(2)$ . Let us first recall the definition of the Euler angles traditionally used in classical mechanics to describe the motion of a spin. Choose a Cartesian frame  $(x, y, z)$  and model the spin as a rod of length  $L$ , with an end fixed in the origin and the other one in the starting position  $\vec{L} \equiv (0, 0, L)$ . The top of the spin can be moved to a generic position in the following way:

- First, we rotate the system by an angle  $\alpha$  around the  $z$  axis. Accordingly, the  $x$  axis will be rotated by  $\alpha$  to a new axis  $x'$  in the  $x - y$  plane, and similarly for the  $y$  axis.
- Then we rotate the system by an angle  $\beta$  around the axis  $x'$ . The  $z$  axis will be rotated by  $\beta$  to a new axis  $z''$  in the  $y' - z$  plane.
- Finally, we rotate the system by an angle  $\gamma$  around the  $z''$  axis.

Essentially, these movements represent the inclination of the spin with respect to the vertical axis, the rotation around the vertical axis and the rotation around its proper axis. To describe these operations mathematically, we notice that a rotation  $R_{\hat{n}}(\theta)$  by an angle  $\theta$  around an (oriented) axis specified by a unit vector  $\hat{n} \equiv (n_x, n_y, n_z)$  can be written as:

$$R_{\hat{n}}(\theta) = e^{\theta(n_x\tau_1 + n_y\tau_2 + n_z\tau_3)}, \quad (9)$$

where

$$\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

are the generators of the infinitesimal rotations. Thus, the generic final position of the top of the spin will be:

$$\vec{L}' = e^{\gamma\tau_3''} e^{\beta\tau_1'} e^{\alpha\tau_3}, \quad (11)$$

where  $\tau_1'$  and  $\tau_3''$  are the generators of the rotations around the  $x'$  and  $z''$  axis respectively:

$$\tau_1' = \cos \alpha\tau_1 + \sin \alpha\tau_2 = e^{\alpha\tau_3}\tau_1 e^{-\alpha\tau_3}, \quad (12)$$

$$\tau_3'' = \cos \beta\tau_3 - \sin \beta\tau_2' = e^{\beta\tau_1'}\tau_3 e^{-\beta\tau_1'}. \quad (13)$$

By remembering that:

$$e^{e^A B e^{-A}} = e^A e^B e^{-A}$$

and substituting this in (11), we find

$$\vec{L}' = e^{\alpha\tau_3} e^{\beta\tau_1} e^{\gamma\tau_3} \vec{L}. \quad (14)$$

From this construction it is clear that we can set the range of the Euler angles to be for example  $\alpha, \gamma \in [0, 2\pi]$ ,  $\beta \in [0, \pi]$ . This is very similar to (8) and it is tempting to identify  $\phi, \theta$  and  $\psi$  with  $\alpha, \beta$  and  $\gamma$ , respectively. However, for  $SU(2)$  we have  $\psi \in [0, 4\pi]$  and not  $[0, 2\pi]$ , which is a consequence of the fact that  $SU(2)$  is a double cover of  $SO(3)$  and provides a spin  $\frac{1}{2}$  representation.

Let us now look at the structure of the construction (8). We have identified a maximal subgroup  $U(1)[\phi] = e^{i\phi\frac{\sigma_3}{2}}$ . Its Lie algebra is obviously a subalgebra of  $\mathfrak{su}(2)$ . Then we have added a second generator  $\tau_1$  which does not belong to the subalgebra, and after observing that all the remaining generators can be obtained by commuting  $\tau_1$  with the subalgebra, we have acted on  $e^{i\theta\frac{\sigma_1}{2}}$  with the subgroup both from the left and from the right:

$$g = U(1)[\phi]e^{i\theta\frac{\sigma_1}{2}}U(1)[\psi]. \quad (15)$$

This provides the structure of the generic element of the group, but more information is still needed in order to determine the minimal range for the parameters.

For completeness, let us look at the geometric properties of the group and use them to identify the parameters. It is known that the group  $SU(2)$  is geometrically equivalent to a three-sphere  $S^3$ , and admits a Hopf fibration structure with fiber  $S^1$  over the base  $S^2 \simeq \mathbb{CP}^1$ . To see this, note that by definition the generic  $U(2)$  element can be written in the form

$$g = \begin{pmatrix} u_1 & w_1 \\ u_2 & w_2 \end{pmatrix}$$

where

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

determine an orthonormal basis for  $\mathbb{C}^2$ . After imposing the condition  $\det g = 1$  we find that it becomes

$$g = \begin{pmatrix} u_1 & -u_1^* \\ u_2 & u_2^* \end{pmatrix}$$

where  $|u_1|^2 + |u_2|^2 = 1$ . Setting  $u_1 = x + iy$  and  $u_2 = t + iz$  we see the correspondence with  $S^3$ . As we have remarked in the previous section, since  $SU(2)$  is a real compact form, it is naturally endowed with an invariant metric given by the Killing product. Suitably normalized, this is  $ds^2 = -\frac{1}{2}\text{Tr}(g^{-1}dg \otimes g^{-1}dg)$ , so that we find

$$ds^2 = \frac{1}{2}\text{Tr}(dg^\dagger \otimes dg) = (dx^2 + dy^2 + dt^2 + dz^2)|_{x^2+y^2+t^2+z^2=1} \quad (16)$$

which is the usual round metric on the sphere  $S^3$ . Therefore, to determine the ranges for the parameters in (8), we can compute the associated metric, identify it with the round metric and choose the range in such a way that it covers the whole  $S^3$ . From (8) we get:

$$ds^2 = \frac{1}{4}(d\phi^2 + d\theta^2 + d\psi^2 + 2\cos\theta d\phi d\psi), \quad (17)$$

which can be obtained from (16) by setting

$$u_1 = \cos\frac{\theta}{2}e^{\frac{i}{2}\epsilon_1(\phi+\psi)+i\alpha_1}, \quad u_2 = \sin\frac{\theta}{2}e^{\frac{i}{2}\epsilon_2(\phi-\psi)+i\alpha_2}, \quad (18)$$

where  $\epsilon_i$  are signs and  $\phi_i$  constant phases. We do not need to determine these quantities to find the ranges. Indeed, for any fixed value of these parameters, to cover  $S^3$  we need to take

$$\frac{1}{2}(\phi + \psi) \in [0, 2\pi], \quad \frac{1}{2}(\phi - \psi) \in [0, 2\pi], \quad \theta \in [0, \pi] \quad (19)$$

which are equivalent to the ones we announced after (8).

In this very simple case all the phases and signs can be determined by observing that:

$$e^{i\phi\frac{\sigma_3}{2}} e^{i\theta\frac{\sigma_1}{2}} e^{i\psi\frac{\sigma_3}{2}} = \begin{pmatrix} e^{\frac{i}{2}(\phi+\psi)} \cos \frac{\theta}{2} & ie^{\frac{i}{2}(\phi-\psi)} \sin \frac{\theta}{2} \\ ie^{-\frac{i}{2}(\phi-\psi)} \sin \frac{\theta}{2} & e^{-\frac{i}{2}(\phi+\psi)} \cos \frac{\theta}{2} \end{pmatrix}, \quad (20)$$

which yields  $\epsilon_1 = 1$ ,  $\epsilon_2 = -1$ ,  $\phi_1 = 0$  and  $\phi_2 = -\frac{\pi}{2}$ .

This is a geometric technique to determine the ranges, but there is another technique which is much simpler for higher dimensional groups  $G$ . It consists in studying the maximal subgroup  $U$  of  $G$  and the quotient  $G/U$  separately. In our case we can take  $U = U(1)[\psi]$ . This is a circle with metric  $\frac{1}{4}d\psi^2$ . The range of  $\psi$  must be a period covering the whole circle, and, being

$$U(1)[\psi] = e^{i\psi\frac{\sigma_3}{2}} = \begin{pmatrix} e^{\frac{i}{2}\psi} & 0 \\ 0 & e^{-\frac{i}{2}\psi} \end{pmatrix},$$

we can take  $\psi \in [0, 4\pi]$ . The points of the quotient are parameterized by

$$H(\phi, \theta) = e^{i\phi\frac{\sigma_3}{2}} e^{i\theta\frac{\sigma_1}{2}} = \begin{pmatrix} e^{\frac{i}{2}\phi} \cos \frac{\theta}{2} & ie^{\frac{i}{2}\phi} \sin \frac{\theta}{2} \\ ie^{-\frac{i}{2}\phi} \sin \frac{\theta}{2} & e^{-\frac{i}{2}\phi} \cos \frac{\theta}{2} \end{pmatrix} \quad (21)$$

with a residual action of  $U(1)[\psi]$  on the right. For example, we see that in the quotient  $H(\phi, 0)$  degenerates to a single point and similarly for  $H(\phi, \pi)$ , because

$$\begin{aligned} H(\phi, 0) &= \begin{pmatrix} e^{\frac{i}{2}\phi} & 0 \\ 0 & e^{-\frac{i}{2}\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\phi} & 0 \\ 0 & e^{-\frac{i}{2}\phi} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ H(\phi, \pi) &= \begin{pmatrix} 0 & ie^{\frac{i}{2}\phi} \\ ie^{-\frac{i}{2}\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-\frac{i}{2}\phi} \\ e^{\frac{i}{2}\phi} & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Indeed, we can take for the quotient the representative

$$H(\phi, \pi)U(1)[- \phi] = \begin{pmatrix} \cos \frac{\theta}{2} & ie^{i\phi} \sin \frac{\theta}{2} \\ ie^{-i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (22)$$

so that as  $\phi$  and  $\theta$  vary within their ranges, this traces a two dimensional semi sphere  $x \geq 0$  in the  $(x, 0, t, z)$  space. However, the equator  $x = 0$  is contracted to a point and the semi sphere reduces to a sphere  $S^2$  of radius  $1/2$ . This is the celebrated Hopf fibration. To see this we can compute the metric on the quotient. This is not simply

$$ds_H^2 = -\frac{1}{2}\text{Tr}(H^{-1}dH \otimes H^{-1}dH),$$

because  $J_H = H^{-1}dH$  is not cotangent to the quotient, having a component which is cotangent to the fiber. Using (21) we have, indeed:

$$J_H = \frac{i}{2}(d\theta\sigma_1 + \sin\theta d\phi\sigma_2 + \cos\theta d\phi\sigma_3). \quad (23)$$

However, we can simply project out the component along the fiber, i.e. the part spanned by  $\sigma_3$ , so that

$$\tilde{J}_H := \frac{i}{2}(d\theta\sigma_1 + \sin\theta d\phi\sigma_2) \quad (24)$$

and we can recover the metric of a two-sphere of radius  $\frac{1}{2}$ :

$$ds_H^2 = -\frac{1}{2}\text{Tr}\tilde{J}_H \otimes \tilde{J}_H = \frac{1}{4}[d\theta^2 + \sin^2\theta d\phi^2] . \quad (25)$$

Changing to the complex coordinate  $z = \tan\psi e^{i\phi}$  and its complex conjugate, this metric reduces to the standard Fubini-Study metric for  $\mathbb{CP}^1$ . However, in general we can assume the ranges of the parameters for the quotient space to be unknown. Then, they can be deduced from (25) as follows: the metric becomes degenerate at  $\theta = 0, \pi$ . This is because fixing  $\theta$  and varying  $\phi$  we obtain a circle with radius  $\frac{1}{2}\sin\theta$ . Therefore, we have to restrict  $\theta$  to  $[0, \pi]$ . But we don't have such a constraint on  $\phi$  and in principle it could vary with a period which we know to be  $4\pi$  as for  $\psi$ . However, this is not the right period for the quotient. Indeed, note that  $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in U(1)[\psi]$  and it is in the center of the group, so that:

$$H(\phi, \theta) \sim H(\phi, \theta)(-I) = -H(\phi, \theta) = H(4\pi - \phi, \theta) .$$

Therefore,  $\phi \sim 4\pi - \phi$ , which means that  $0 \sim 2\pi$ , reducing the period to  $\phi \in [0, 2\pi]$ .

### 3.2 The generalized Euler construction.

Let us now generalize the previously described construction to the compact form of a generic finite dimensional simple Lie group  $G$ ,  $n = \dim G$ . In this case, our construction is not unique but is related to the choice of a maximal subgroup  $H$ . Because  $G$  is compact, the Killing product defines a scalar product ( $|\cdot\rangle$ ) on  $\mathfrak{g} = \text{Lie}(G)$  and it is convenient to choose an orthonormal basis  $\{\tau_i\}_{i=1}^n$  of  $\mathfrak{g}$ . In particular, let us assume that the first  $k := \dim H$  generators are a basis for  $\mathfrak{h} = \text{Lie}(H)$  and let us call  $\mathfrak{p}$  the subspace spanned by the remaining generators. Note that  $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ . Indeed, orthogonality and  $ad$ -invariance imply

$$([p, h]|h') = (p|[h, h']) = 0$$

for any  $p \in \mathfrak{p}$  and  $h, h' \in \mathfrak{h}$ . This means that  $G/H$  is reductive. From this, it follows that any  $g \in G$  can be written in the form

$$g = \exp a \exp b , \quad a \in \mathfrak{p} , b \in \mathfrak{h} . \quad (26)$$

For compact simple Lie groups such a parametrization is surjective, a proof can be found in [3].

Now, let's suppose we have an explicit parametrization for  $H$ , which is obviously a generalized Euler parametrization obtained inductively by choosing a maximal subgroup  $H'$  of  $H$  and proceeding in the same way. This means that we can use the parametrization to give an expression for  $\exp b$ . Now we would like to improve the expression for  $\exp a$ . To this purpose we can look for a subset of linearly free elements  $\tau_1, \dots, \tau_l \in \mathfrak{p}$  with the following properties:

- if  $V$  is the linear subspace generated by  $\tau_i$ ,  $i = 1, \dots, l$ , then  $\mathfrak{p} = Ad_H(V)$ , that is, the whole  $\mathfrak{p}$  is generated from  $V$  through the adjoint action of  $H$ ;
- $V$  is minimal, in the sense that it does not contain any proper subspaces with the previous property.

Because of simplicity, it is not hard to show that such a subspace  $V$  of  $\mathfrak{p}$  always exists. Therefore, the general element  $g$  of  $G$  can be written in the form:

$$g = \exp(\tilde{b}) \exp(v) \exp(b) , \quad b, \tilde{b} \in \mathfrak{h} , \quad v \in V . \quad (27)$$

This parametrization is obviously redundant, since in general it depends on  $2k + l \geq n$  parameters. The point is that not the whole of  $H$  is needed to generate  $V$  by the adjoint action, because  $H$  contains some subgroup  $H_o$  which generates the automorphisms of  $V$ :

$$Ad_{H_o} : V \longrightarrow V . \quad (28)$$

Then  $H_o$  must be  $r$ -dimensional, where  $r = 2k + l - n$  is the redundancy, and the generalized Euler decomposition with respect to  $H$  finally takes the form

$$G = B \exp(V) H , \quad (29)$$

where  $B := H/H_o$ . We have seen that even for the simplest case of  $SU(2)$  the automorphism group  $H_o$  is not trivial (even though it acts trivially on  $V$ ) and it coincides with  $\mathbb{Z}_2$ .

### 3.3 Determination of the range of parameters

The symbolic expression (29) means that the generic element of  $G$  can be written in the form

$$g = b e^v h, \quad b \in B, \quad v \in V, \quad h \in H, \quad (30)$$

where  $h$ ,  $v$  and  $b$  are function of  $k, l$  and  $n - l - k$  parameters respectively. Locally, they define a coordinatization for the group. However, being the parametrization surjective, the parameters can be chosen in such a way as to cover the whole group. However, because in general the group is a non trivial manifold, a surjective parametrization cannot in general be injective. A good choice for the range of the parameters is to pick a maximal open subset on which the parametrization is injective, so that its closure covers the whole group. We will call this closure *the range of parameters*. In general the determination of the range is a highly non trivial task. The aim here is to discuss two practical methods to do this.

#### 3.3.1 Geometric identification

Once the parametrization  $g[\vec{x}]$  is given, it can be used to describe the geometry of the group or of its quotient with the maximal subgroup. If such a geometry is already known by some other means, this information can be applied to determine the range of the parameters. The metric on the group can be computed by starting from the Killing metric and the

Cartan 1-form. The parametrization provides a local coordinatization, which in turns yields a local expression for the Cartan 1-form:

$$J = g^{-1} \frac{\partial g}{\partial x^J} dx^J = J^i \tau_i , \quad (31)$$

where  $\tau_i$ ,  $i = 1, 2, \dots, n$  is a basis for the Lie algebra. This defines the structure constants  $f_{ij}^k$  so that the Killing metric has components:

$$K_{ij} = -k f_{il}^m f_{jm}^l , \quad (32)$$

where  $k$  is some normalization constant. As we are working with a compact form, the metric is positive definite when  $k$  is positive. We choose the basis and  $k$  in such a way that  $K_{ij} = \delta_{ij}$ . The metric induced on the manifold is then

$$ds^2 = g_{ij} dx^i \otimes dx^j = J^l \otimes J^m \delta_{lm} . \quad (33)$$

In other words, the 1-forms  $J^l$  represent the vielbein one forms on the group. In particular they can be used to compute the invariant volume  $n$ -form

$$\omega = J^1 \wedge \dots \wedge J^n = \det(\underline{J}) dx^1 \wedge \dots \wedge dx^n \quad (34)$$

and the corresponding Haar measure

$$d\mu = |\det(\underline{J})| \prod_{I=1}^n dx^I . \quad (35)$$

From our parametrization (29), we can write the general element  $g \in G$  in the form

$$g(x_1, \dots, x_s, y_1, \dots, y_m) = p(x_1, \dots, x_s) h(y_1, \dots, y_m) \quad (36)$$

where  $h \in H$ ,  $p \in B \exp(V)$ ,  $m = \dim H$  and  $m + s = n$ . We can assume for simplicity that  $\{\tau_a\}$ ,  $a = s+1, \dots, n$  generates  $H$ . Notice that only  $H$  is a subgroup, so that  $J_h \equiv h^{-1} dh \in Lie(H)$ , whereas in general  $J_p \equiv p^{-1} dp \in Lie(G)$ . However, for the subgroup, instead of the left-invariant form we prefer to use the right-invariant form  $\tilde{J}_h \equiv dh h^{-1}$ . In this way, setting

$$J_p = \sum_{i=1}^n J_p^i \tau_i , \quad \tilde{J}_h = \sum_{i=s+1}^n \tilde{J}_h^i \tau_i , \quad (37)$$

and using orthonormality, after a simple calculation we get

$$ds^2 = \sum_{i=s+1}^n \left( J_p^i + \tilde{J}_h^i \right)^2 + \sum_{i=1}^s \left( J_p^i \right)^2 . \quad (38)$$

From this expression for the metric we can read the structure of the fibration with fiber  $H$  over  $G/H$ . Indeed, the forms  $J_p^i + \tilde{J}_h^i$ ,  $i = s+1, \dots, n$  lie on the fiber, whereas  $J_p^a$ ,  $a = 1, \dots, s$  are orthogonal to the fiber. This means that

$$d\sigma^2 = \sum_{i=1}^s \left( J_p^i \right)^2 \quad (39)$$

defines the metric on the quotient space  $G/H$ , as defined by the  $s$ -dimensional vielbein  $\hat{J}_p$  obtained from  $J_p$  by projecting out the components along the fiber. At a practical level, this decomposition not only allows us to perform the computation of the metric on the quotient space, but it also greatly simplifies the explicit calculation of the metric on the whole group. To conclude, this method can be used to provide an explicit characterization of the geometry of the group and of its quotients and if some of the geometrical structure of the group and/or of its fibration is known by any other means, by comparison we can determine the range of the parameters. We are going to see an explicit example of this procedure later.

### 3.3.2 A topological method

In general, however, the theoretical information about the group is not sufficient to determine the explicit range for the parameters. In this case, we need to introduce an alternative method which requires a minimal amount of information to work. Fortunately, such a method, which we called topological, is provided by a powerful theorem due to I.G. Macdonald which describes a simple way to compute the total volume of a compact connected simple Lie group. Let  $\mathfrak{c} \subset Lie(G)$  be a Cartan subalgebra, and  $\mathfrak{c}_{\mathbb{Z}}$  the integer lattice generated in  $\mathfrak{c}$  by a choice of simple roots (the root lattice). Then, the first geometrical ingredient is the torus  $T := \mathfrak{c}/\mathfrak{c}_{\mathbb{Z}}$ , whose dimension is  $r = \text{rank} Lie(G)$ . The second ingredient is a well known result due to Hopf [17]: the rational homology of  $G$  is equal to the rational homology of a product of odd-dimensional spheres

$$H_*(G, \mathbb{Q}) \simeq H_*\left(\prod_{i=1}^k (S^{2i+1})^{r_i}, \mathbb{Q}\right),$$

where  $r_i$  is the number of times the given sphere appears, and  $r_1 + \dots + r_k = r$ . The result of Macdonald [18] can then be stated as follows:

If we assign a Lebesgue measure  $\mu$  on a compact simple Lie group  $G$  by means of an Euclidean scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g} = Lie(G)$ , then the measure of the whole group is

$$\mu(G) = \mu_o(T) \cdot \prod_{i=1}^k Vol(S^{2i+1})^{r_i} \cdot \prod_{\alpha \in R(\mathfrak{g})} \frac{2}{|\alpha|}, \quad (40)$$

where  $R(\mathfrak{g})$  is the set of non vanishing roots,  $\mu_o$  is the Lebesgue measure on  $\mathfrak{g}$  induced by the scalar product and  $Vol(S^{2i+1}) = 2\pi^{i+1}/i!$  is the volume of the unit sphere  $S^{2i+1}$ .

On the other side, we can in principle compute the measure of the whole group, induced by the Killing scalar product, by using (35) integrated over the range of the parameters. Using (38) we get

$$d\mu = |\det(\underline{J}_p)| |\det(\underline{\tilde{J}}_h)| \prod_{I=1}^n dx^I. \quad (41)$$

Now, let's assume we have a good parametrization, which means that the one parameter subgroups spanned by the orbits  $\exp(t\tau_i)$ , where  $\{\tau_i\}$  is the basis we fixed for the Lie algebra, are subgroups embedded in  $G^1$ . Then, such orbits are compact (for a compact group) and

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<sup>1</sup>This can be accomplished for any simple group.

$\exp(t\tau_i)$  is periodic in  $T$ . The point is that if we choose correctly the range  $\mathcal{R}$  for the parameters, then

$$\mu(G) = \int_{\mathcal{R}} d\mu. \quad (42)$$

A “suitable” range means that it covers each point of  $G$  exactly once, up to a subset of vanishing measure. Let us look at the measure weight  $f := |\det(\underline{J})|$ . In general it will depend explicitly on the parameter but not on all the parameters. For each parameter which does not appear in  $f$  we choose its period as a range. Let us call  $\bar{\mathcal{R}}$  the range for the remaining parameters. Then,  $\bar{\mathcal{R}}$  has a boundary defined by  $f = 0$ . This equation in general provides a splitting of the space into infinitely many fundamental regions, which, however, turn out to be all equivalent for our purposes. With such a choice  $\mathcal{R}_o$  for the range, we are sure that its image under our parametrization map  $g : \mathcal{R}_o \rightarrow G$  describes a closed  $n$ -dimensional variety on  $G$ . As  $G$  is connected,  $g(\mathcal{R}_o)$  has to cover  $G$  an integer number  $m$  of times

$$m = \frac{1}{\mu(G)} \int_{\mathcal{R}_o} d\mu. \quad (43)$$

If  $m > 1$  it means that it exist an automorphism group  $\Gamma : \mathcal{R}_o \rightarrow \mathcal{R}_o$  of order  $m$ , such that  $g(\Gamma x) = g(x)$ . In this case we restrict the range to

$$\mathcal{R} = \mathcal{R}_o/\Gamma. \quad (44)$$

This is, indeed, what we have done at the end of section 3.1. Let us now illustrate our procedure for some exceptional examples.

## 4 GENERALIZED EULER ANGLES FOR $G_2$

### 4.1 The Lie algebra

The exceptional Lie group  $G_2$  can be realized as the automorphism group of the octonionic algebra [19, 23]. Instead of providing a theoretical proof of this fact, we explicitly construct such a group starting for its Lie algebra.

The octonionic algebra  $\mathbb{O}$  is the eight dimensional real vector field generated by a real unit  $e_0 \equiv 1$  and seven imaginary units  $e_i, i = 1, \dots, 7$ . It is endowed with a distributive but non associative product described by the relations:

$$\begin{aligned} e_0 \cdot a &= a \cdot e_0 = a \quad \forall a \in \mathbb{O}, \\ e_i^2 &= -e_0, \quad e_i \cdot e_j = -e_j \cdot e_i, \quad 1 \leq i < j \leq 7 \end{aligned}$$

and the Fano diagram, see fig. 4.1. Each oriented line can be thought as an oriented circle, on which three distinct imaginary roots  $e_i, e_j, e_k$  are lying, the products of which are  $e_i \cdot e_j = \pm e_k$ . Here the sign is positive if and only if the triple  $\{e_i, e_j, e_k\}$  follows the orientation of the arrow. For example,  $e_1 \cdot e_3 = -e_2$  and  $e_1 \cdot e_2 = e_3$ . Notice that each circle generates a quaternionic subalgebra. An automorphism of the algebra is an invertible linear map:

$$A : \mathbb{O} \longrightarrow \mathbb{O}$$



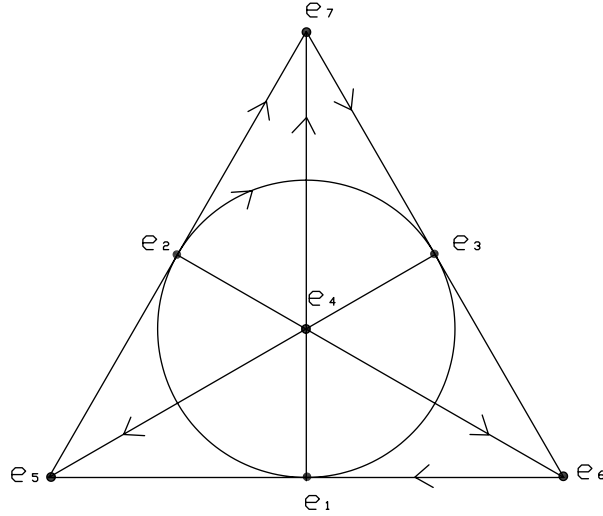


Figure 1: *The Fano diagram.*

satisfying

$$A(a \cdot b) = A(a) \cdot A(b), \quad a, b \in \mathbb{O}.$$

The set of all automorphisms is a group with respect to the composition product, and is indeed a Lie group. From this it follows immediately that its Lie algebra is the set of derivations  $\mathfrak{D}(\mathbb{O})$ , the linear operators

$$B : \mathbb{O} \longrightarrow \mathbb{O} \tag{45}$$

satisfying

$$B(a \cdot b) = B(a) \cdot b + a \cdot B(b), \quad a, b \in \mathbb{O}, \tag{46}$$

and with the commutator as Lie product. Note that  $B(1) = 0$  for all  $B \in \mathfrak{D}(\mathbb{O})$ , so that we can look for a matrix representation of  $\mathfrak{D}(\mathbb{O})$  on the real space spanned by the imaginary units. This will give the smallest fundamental representation of  $G_2$ , the **7** representation. Imposing the condition (46), with the help of a computer, we find a set of 14 linearly independent matrices:

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$



It is easy to check that these matrices do, indeed, define a Lie algebra with the commutator product, and that  $\mathbb{R}^7$  is irreducible under their action, so that they realize an irreducible representation. A rank two Cartan subalgebra is generated by  $C_5, C_{11}$  and in the adjoint representation it is easy to compute all roots which turn out to coincide with the roots of  $G_2$ , as expected (see for example [1]).

## 4.2 Two Euler parameterizations

We can now realize two distinct Euler parameterizations for  $G_2$ , based on different choices of the maximal subgroup  $H$ . The first one is based on  $H = SU(3)$ , [2], and the second one on  $H = SO(4)$ , [1]. While for the first one it is possible to apply the geometrical method, for the second one the topological method is necessary. We are going to call them the  $SU(3)$ -Euler parametrization and the  $SO(4)$ -Euler parametrization respectively.

### 4.2.1 The $SU(3)$ -Euler parametrization

Among the automorphisms of the octonions, we can look at the subgroup which leaves an imaginary unit fixed. This is a subgroup of  $G_2$  and will be contained in the  $SO(6)$  group which rotates the remaining six imaginary units. Indeed, it turns out to be an  $SU(3)$  group. We can see it immediately from our matrices: the first row and column of the first eight matrices vanish, so that they leave  $e_1$  fixed. They generate a subalgebra, and in the adjoint representation it can be verified that the roots match with  $SU(3)$ . It acts transitively on the subset of imaginary units orthogonal to  $e_1$ , defining a six dimensional sphere  $S^6$ , so that  $G_2/SU(3) \simeq S^3$ .

We then choose  $\{C_i\}$ ,  $i = 1, 2, \dots, 8$  as generators for  $H$ , so that  $\{C_a\}$ ,  $a = 9, \dots, 14$  generate  $\mathfrak{p}$ . To identify  $V$  (see (29)) we note that  $C_9$  generates the whole  $\mathfrak{p}$  under the action of  $H = SU(3)$ , and, therefore,  $V = \mathbb{R}C_9$ . Finally, note that the subalgebra of  $H$  commuting with  $C_9$  is the  $\mathfrak{su}(2)$  algebra generated by  $\{C_i\}$ ,  $i = 1, 2, 3$ . Thus  $B = SU(3)/SU(2)$ .

As a first step we need to construct the  $SU(3)$  subgroup  $H$ . We could proceed in the same way, but as the construction of  $SU(3)$  is well known, we limit ourselves here only to the final result (see [20, 21, 2]):

$$H[x_1, \dots, x_8] = e^{x_1 C_3} e^{x_2 C_2} e^{x_3 C_3} e^{x_4 C_5} e^{\sqrt{3} x_5 C_8} e^{x C_3} e^{x C_2} e^{x_8 C_3}, \quad (47)$$

with range

$$\begin{aligned} x_1 &\in [0, \pi], & x_2 &\in \left[0, \frac{\pi}{2}\right], & x_3 &\in [0, \pi], & x_4 &\in \left[0, \frac{\pi}{2}\right], \\ x_5 &\in [0, 2\pi], & x_6 &\in [0, 2\pi], & x_7 &\in \left[0, \frac{\pi}{2}\right], & x_8 &\in [0, \pi]. \end{aligned} \quad (48)$$

We just want to remark that (47) has the structure of (29) with

$$B = SO(3) = SU(2)/\mathbb{Z}_2, \quad V = \mathbb{R}C_5, \quad H = U(2). \quad (49)$$

Then, our  $SU(3)$ -Euler parametrization is

$$g[x_1, \dots, x_{14}] = e^{x_1 C_3} e^{x_2 C_2} e^{x_3 C_3} e^{\frac{\sqrt{3}}{2} x_4 C_8} e^{x_5 C_5} e^{\frac{\sqrt{3}}{2} x_6 C_9} H[x_7, \dots, x_{14}], \quad (50)$$

where we need to determine the range for  $x_1, \dots, x_6$ , whereas the remaining parameters have the range of  $SU(3)$ . To this aim, we will use the information

$$G_2/SU(3) \simeq S^3.$$

From

$$p[x_1, \dots, x_6] = e^{x_1 C_3} e^{x_2 C_2} e^{x_3 C_3} e^{\frac{\sqrt{3}}{2} x_4 C_8} e^{x_5 C_5} e^{\frac{\sqrt{3}}{2} x_6 C_9} \quad (51)$$

we can compute  $J_p = p^{-1} dp$  and then the metric (39) induced on the quotient. By a direct computation, we get

$$\frac{4}{3} d\sigma^2 = dx_6^2 + \sin^2 x_6 \left\{ dx_5^2 + \cos^2 x_5 dx_4^2 + \sin^2 x_5 \left[ s_1^2 + s_2^2 + \left( s_3 + \frac{1}{2} dx_4 \right)^2 \right] \right\} \quad (52)$$

where

$$\begin{aligned} s_1 &= -\sin(2x_2) \cos(2x_3) dx_1 + \sin(2x_3) dx_2 \\ s_2 &= \sin(2x_2) \sin(2x_3) dx_1 + \cos(2x_3) dx_2 \\ s_3 &= \cos(2x_2) dx_1 + dx_3. \end{aligned} \quad (53)$$

We recognize this as the metric of a round six sphere  $S^6$  of radius  $\sqrt{3}/2$ , with coordinates  $(x_6, \vec{X})$ , where  $x_6$  is an azimuthal coordinate,  $x \in [0, \pi]$ , and  $\vec{X}$  cover a five sphere embedded in  $\mathbb{C}^3$  via

$$\begin{aligned} \vec{X} &= (z_1, z_2, z_3) = \left( \cos x_5 e^{ix_4}, \sin x_5 \cos x_2 e^{i(x_1+x_3+\frac{x_4}{2})}, \sin x_5 \sin x_2 e^{i(x_1-x_3-\frac{x_4}{2})} \right), \\ x_1 &\in [0, \pi], \quad x_2 \in \left[0, \frac{\pi}{2}\right], \quad x_3 \in [0, 2\pi], \quad x_4 \in [0, 2\pi], \quad x_5 \in \left[0, \frac{\pi}{2}\right]. \end{aligned}$$

Computing the metric  $ds_{S^5}^2 = |dz_1|^2 + |dz_2|^2 + |dz_3|^2$  in these coordinates we find

$$\frac{4}{3} d\sigma^2 = dx_6^2 + \sin^2 x_6 \{ ds_{S^5}^2 \}. \quad (54)$$

This completes our identification for the range of the parameters.

#### 4.2.2 The $SO(4)$ -Euler parametrization

The maximal subgroup  $SO(4)$  can be singled out as follows. We know that  $1, e_1, e_2, e_3$  generate a quaternionic subalgebra  $\mathbb{H}$ . We look at the subgroup  $H$  which leaves this subalgebra invariant. This will be generated by block diagonal matrices of the form  $\{3 \times 3\} \times \{4 \times 4\}$ , which turn out to be the matrices  $C_i$ ,  $i = 1, 2, 3, 8, 9, 10$ . Indeed,  $C_1, C_2, C_3$  generate an  $\mathfrak{su}(2)$  subalgebra, which leaves each element of  $\mathbb{H}$  invariant. Let us call this group  $SU(2)_I$ . Then  $C_8, C_9, C_{10}$  span a second  $SU(2)$  group ( $SU(2)_{II}$ ), the action of which, when restricted to  $e_1, e_2, e_3$ , generates the automorphisms of  $\mathbb{H}$ . Notice that the two subgroups commute. We can now define the surjective homomorphism:

$$\begin{aligned} \phi : SU(2)_I \times SU(2)_{II} &\longrightarrow H \\ (a, b) &\longmapsto ab. \end{aligned}$$

Observe that  $\text{Ker}\phi$  is the  $\mathbb{Z}_2$  subgroup generated by the element  $(\exp(\pi C_1), \exp(\sqrt{3}\pi C_8)) = (z, z)$ , with  $z = \text{diag}\{I_3, -I_4\}$ ,  $I_n$  being the  $n \times n$  identity matrix. Thus, we can finally obtain  $SO(4)$  as:

$$H \equiv SU(2)_I \times SU(2)_{II}/\mathbb{Z}_2 = SO(4). \quad (55)$$

Its Euler parametrization can be constructed very easily by starting from the one for  $SU(2)_I$  and  $SU(2)_{II}$ : we get

$$H(x_1, \dots, x_6) = e^{x_1 C_3} e^{x_2 C_2} e^{x_3 C_3} e^{\sqrt{3}x_4 C_8} e^{\sqrt{3}x_5 C_9} e^{\sqrt{3}x_6 C_8}, \quad (56)$$

where the range is:

$$\begin{aligned} x_1 &\in [0, 2\pi], & x_2 &\in [0, \pi/2], & x_3 &\in [0, \pi] \\ x_4 &\in [0, \pi], & x_5 &\in [0, \pi/2], & x_6 &\in [0, \pi]. \end{aligned} \quad (57)$$

We also know that  $C_5, C_{11}$  generate a Cartan subalgebra, not contained in  $\text{Lie}(H)$ . The action of  $H$  on this Cartan subalgebra generates the complement of  $\text{Lie}(H)$ , so that we can take  $V = \mathbb{R}C_5 \oplus \mathbb{R}C_{11}$ . Finally, because of

$$\dim B = \dim G_2 - \dim H - \dim V = 6 = \dim H,$$

we expect for the subgroup  $H_o$  of  $H$  which commute with  $\exp V$  to be a finite group. This means that the  $SO(4)$ -Euler parametrization will take the form

$$g[x_1, \dots, x_{14}] = H(x_1, \dots, x_6) e^{\sqrt{3}x_7 C_{11} + x_8 C_5} H(x_9, \dots, x_{14}), \quad (58)$$

where  $x_9, \dots, x_{14}$  will span the whole  $SO(4)$ , whereas the range of the first six parameters will be restricted by the action of  $H_o$ .

Before determining  $H_o$ , we remark that in this case the quotient manifold  $M = G_2/SO(4)$  is known to be the eight-dimensional variety of the quaternionic subalgebras of  $\mathbb{O}$ . Unfortunately, we cannot use this information as we did before for the  $SU(3)$  Euler angles, because an invariant metric on  $M$  (independent from the one we can compute by group theoretical arguments) is not known, so we have to revert to the topological method instead.

Let us now proceed with the determination of  $H_o$ . This is the subgroup of  $7 \times 7$  orthogonal matrices  $A$  of  $SO(4)$ , whose adjoint action leave the Cartan subalgebra invariant:

$$AC_i A^t = C_i, \quad i = 5, 11. \quad (59)$$

A direct computation shows that it is the finite group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  generated by the idempotent matrices  $\sigma$  ( $\sigma = \sigma^{-1}$ ) and  $\eta$  ( $\eta = \eta^{-1}$ )

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad (60)$$

We need to look at the action of  $H_o$  on  $H$  to reduce the range of  $x_1, \dots, x_6$ . Starting with  $\sigma$  we see that

$$\begin{aligned} g &= H(x_1, x_2, x_3, x_4, x_5, x_6) \sigma e^V \sigma H(x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}) \\ &= H(x_1, x_2, x_3 + \frac{\pi}{2}, x_4, x_5, x_6 + \frac{\pi}{2}) e^V H(x_9 + \frac{\pi}{2}, x_{10}, x_{11}, x_{12} + \frac{\pi}{2}, x_{13}, x_{14}). \end{aligned} \quad (61)$$

This shows that we can restrict  $0 \leq a_6 < \frac{\pi}{2}$  to avoid redundancies. A similar computation can be done for the action of  $\eta$ , showing that redundancies are avoided by restricting  $a_2 \in [0, \pi/4]$ . The details can be found in [1]. So, at this time we have partially determined the ranges:

$$\begin{aligned} x_1 &\in [0, 2\pi], & x_2 &\in [0, \pi/4], & x_3 &\in [0, \pi], \\ x_4 &\in [0, \pi], & x_5 &\in [0, \pi/2], & x_6 &\in [0, \pi/2], \\ x_9 &\in [0, 2\pi], & x_{10} &\in [0, \pi/2], & x_{11} &\in [0, \pi], \\ x_{12} &\in [0, \pi], & x_{13} &\in [0, \pi/2], & x_{14} &\in [0, \pi]. \end{aligned} \quad (62)$$

To apply the topological method we must now determine the form of the invariant measure. This is easily computed using (41), (58) (and eventually the help of Mathematica):

$$d\mu = 27\sqrt{3} f(2x_7, 2x_8) \sin(2x_2) \sin(2x_5) \sin(2x_{10}) \sin(2x_{13}) \prod_{i=1}^{14} dx_i, \quad (63)$$

where

$$\begin{aligned} f(\alpha, \beta) &= \sin\left(\frac{\beta - \alpha}{2}\right) \sin\left(\frac{\beta + \alpha}{2}\right) \sin\left(\frac{\beta - 3\alpha}{2}\right) \sin\left(\frac{\beta + 3\alpha}{2}\right) \sin(\alpha) \sin(\beta) \\ &= \frac{1}{4} (\cos(\alpha) - \cos(\beta)) (\cos(3\alpha) - \cos(\beta)) \sin(\alpha) \sin(\beta). \end{aligned} \quad (64)$$

We see that for certain values of the angles  $x_2, x_5, x_7, x_8, x_{10}, x_{13}$  the measure (63) vanishes. Apart from  $x_7, x_8$ , however, this happens only on the boundary of the chosen ranges. This means that the condition of non vanishing measure determines the range for  $x_7, x_8$  through the equation

$$f(2x_7, 2x_8) > 0.$$

Note that the period of  $e^{x_{C_5}}$ , like the one for  $e^{\sqrt{x}_{C_{11}}}$ , is  $2\pi$ , so that we have to solve this equation inside the square  $[0, 2\pi] \times [0, 2\pi]$ . This provides a tiling of the square, but it is easy to see that all the regions of such a tiling are equivalent and we can pick any of them, see [1]. We fix:

$$a_7 \in [0, \pi/6], \quad 3a_7 \leq a_8 \leq \pi/2. \quad (65)$$

Our choice for the range  $R$  determines a covering  $G$  of  $G_2$ , the volume of which is easily computed to be:

$$Vol(G) = \int_R d\mu = 9\sqrt{3} \frac{\pi^8}{20}. \quad (66)$$

The final step consists in comparing this result with the expression obtained for the volume of  $G_2$  by means of Macdonald's formula (40). Indeed, the two values coincide and it holds

that  $Vol(G_2) = Vol(G)$ . We can therefore infer that with our choice for the range of the parameters, the group is covered exactly once. Instead of showing the details of this calculation here, we use the  $SU(3)$ -parametrization determined previously to compute the volume of  $G_2$  in yet another way. In that case, the measure was

$$d\mu_{G_2}^{(SU(3))} = \frac{27}{32} \sin^5 x_6 \cos x_5 \sin^3 x_5 \sin(2x_2) d\mu_{SU(3)} dx_6 dx_5 dx_4 dx_3 dx_2 dx_1, \quad (67)$$

so that

$$Vol(G_2) = 9\sqrt{3} \frac{\pi^8}{20},$$

as expected.

## 5 GENERALIZED EULER ANGLES FOR $F_4$

A simple construction for the Lie algebras of the exceptional Lie groups  $F_4$  and  $E_6$  is suggested by a theorem of Chevalley and Schafer [22] which states

**Theorem 5.1.** *The exceptional simple Lie algebra  $\mathfrak{f}_4$  of dimension 52 and rank 4 over  $K$  is the derivation algebra  $\mathfrak{D}$  of the exceptional Jordan algebra  $\mathfrak{J}$  of dimension 27 over  $K$ . The exceptional simple Lie algebra  $\mathfrak{e}_6$  of dimension 78 and rank 6 over  $K$  is the Lie algebra*

$$\mathfrak{D} + \{R_Y\}, \quad TrY = 0, \quad (68)$$

spanned by the derivations of  $\mathfrak{J}$  and the right multiplications of elements  $Y$  of trace 0.

See also [23]. To make it workable at a practical level, we have to explain here the main ingredients. For our purposes,  $K = \mathbb{R}$ . The exceptional Jordan algebra is the 27 dimensional real vector space spanned by the  $3 \times 3$  octonionic hermitian matrices endowed with the Abelian product

$$A \circ B := \frac{1}{2}(AB + BA), \quad (69)$$

that is the symmetrization of the usual matrix product.

The derivation algebra of  $\mathfrak{J}$  provides a 27 dimensional representation of the Lie algebra for  $F_4$ . However, it admits a decomposition in irreducible subspaces  $\mathbb{R}^{27} = \mathbb{R}^{26} \oplus \mathbb{R}$ , which is defined by the homomorphism:

$$\ell : \mathfrak{J} \longrightarrow \mathbb{R}, \quad A \mapsto \sum_{i=1}^3 A_{ii}. \quad (70)$$

Its kernel is a 26 dimensional invariant subspace. We could restrict ourselves to this space, but, because the 27 dimensional representation can be extended to an irreducible representation of an  $E_6$  algebra, we prefer to work with the whole space.

In order to concretely construct the representation, let us first realize an explicit isomorphism between the space of exceptional Jordan matrices and  $\mathbb{R}^{27}$ :

$$\Phi : \mathfrak{J} \longrightarrow \mathbb{R}^{27}, \quad \begin{pmatrix} a_1 & o_1 & o_2 \\ o_1^* & a_2 & o_3 \\ o_2^* & o_3^* & a_3 \end{pmatrix} \longrightarrow \begin{pmatrix} a_1 \\ \rho(o_1) \\ \rho(o_2) \\ a_2 \\ \rho(o_3) \\ a_3 \end{pmatrix}, \quad (71)$$

where  $a_i, i = 1, 2, 3$  are real numbers,  $o_i, i = 1, 2, 3$  are octonions and

$$\rho : \mathbb{O} \longrightarrow \mathbb{R}^8, \quad \sum_{i=0}^7 o^i e_i \mapsto \begin{pmatrix} o^0 \\ o^1 \\ o^2 \\ o^3 \\ o^4 \\ o^5 \\ o^6 \\ o^7 \end{pmatrix}. \quad (72)$$

In this way, the set of derivations  $\mathfrak{D}$  is mapped into the set of endomorphisms of  $\mathbb{R}^{27}$ . Indeed, choosing  $A_i = \Phi^{-1}(r_i), r_i \in \mathbb{R}^{27}$ , the identity

$$J(A \circ B) = J(A) \circ B + A \circ J(B) \quad (73)$$

provides a set of equations for the  $27 \times 27$  matrix  $M := \Phi J \Phi^{-1}$ . This linear system can be easily solved by means of a computer, yielding a set of 52 linearly independent matrices. Their explicit expressions, together with the Mathematica code generating them and their structure constants, can be found in [3]. We have chosen to normalize them with the conditions:  $-\frac{1}{6} \text{Trace}(M_I M_J) = \delta_{IJ}, I, J = 1, \dots, 52$ , and  $[M_i, M_j] = -\sum_{k=1}^3 \epsilon_{ijk} M_k$  for  $i, j \in \{1, 2, 3\}$ . Now, we need to recognize the  $26 \oplus 1$  irreducible representation. We said that this is determined by the kernel of the map  $\ell$  defined in (70). Composing it with the map  $\Phi$ , we see that  $\ker \ell \circ \Phi^{-1} = \mathbb{R} f_{27}$ , where

$$f_{27} = (e_1 + e_{18} + e_{27})/\sqrt{3}, \quad (74)$$

and  $e_a, a = 1, \dots, 27$  is the standard basis of  $\mathbb{C}^{27}$ . Indeed,  $f_{27} \in \ker M_I$  for all  $I = 1, \dots, 52$ . With respect to the new basis  $\{f_a\}_{a=1}^{27}$  for  $\mathbb{C}^{27}$

$$f_1 = (e_1 - e_{18})/\sqrt{2}, \quad (75)$$

$$f_{18} = (e_1 + e_{18} - 2e_{27})/\sqrt{6}, \quad (76)$$

$$f_{27} = (e_1 + e_{18} + e_{27})/\sqrt{3}, \quad (77)$$

$$f_a = e_a, \text{ in the other cases,} \quad (78)$$

all the matrices will have vanishing last row and column, thus explicitly evidencing the decomposition. We are going to call the resulting  $27 \times 27$  matrices  $\{c_i\}, i = 1, \dots, 52, .$  The



26 dimensional representation can then be obtained by simply deleting from each matrix the last row and the last column. However, as we have remarked previously, the  $27 \times 27$  matrices constitute the first 52 elements of the 27 dimensional fundamental irreducible representation of  $E_6$ .

Before starting with the construction of the group, let us stop momentarily to look at some properties of the algebra. Observe that with our matrices we can easily construct the 52 dimensional adjoint representation as well. Let us call  $\{C_i\}$  the corresponding matrices. We can easily check that the associated Killing form is negative definite and, indeed,  $K_{ij} \propto \delta_{ij}$ , so that we can choose the constant to fix the Euclidean metric as the invariant metric. A possible choice for a Cartan subalgebra is  $H = \mathbb{R}C_1 \oplus \mathbb{R}C_6 \oplus \mathbb{R}C_{15} \oplus \mathbb{R}C_{36}$  and the roots can then be computed by simultaneously diagonalizing the generators  $\{C_a\}$ ,  $a = 1, 6, 15, 36$ :

$$C_a \vec{v}_i = \lambda_{a,i} \vec{v}_i, \quad i = 1, \dots, 27, \quad \vec{v}_i \in \mathbb{C}^{52}.$$

The resulting vectors  $(\lambda_{1,i}, \lambda_{6,i}, \lambda_{15,i}, \lambda_{36,i})$ ,  $i = 1, \dots, 27$ , represent the roots, which, indeed, coincide with the roots of  $F_4$ , as expected, thus proving that we have obtained a realization of the compact form of the  $F_4$  Lie algebra.

To construct the corresponding group, it is useful to identify its subalgebras first. By studying the commutators, we see that the first 21 matrices generate a  $\mathfrak{so}(7)$  subalgebra, whose  $\mathfrak{so}(i)$  subalgebras, with  $i = 6, 5, 4, 3$ , are generated by the first  $i(i-1)/2$  matrices, respectively. A possible choice for the relative Cartan subalgebras is  $C_1$  for  $\mathfrak{so}(3)$ ;  $C_1, C_6$  for  $\mathfrak{so}(4)$  and  $\mathfrak{so}(5)$  and  $C_1, C_6, C_{15}$  for  $\mathfrak{so}(6)$  and  $\mathfrak{so}(7)$ . This can be used to compute the corresponding roots and to check the algebras. Adding to  $\mathfrak{so}(7)$  the matrices  $c_i$ , with  $i = 30, \dots, 36$ , we obtain a  $\mathfrak{so}(8)$  subalgebra. This is the Lie algebra associated to the  $Spin(8)$  subgroup of  $F_4$  which leaves invariant the three Jordan matrices  $J_i$ ,  $i = 1, 2, 3$ , where  $J_i$  is the matrix which has  $\{J_i\}_{ii} = 1$  as the unique non-vanishing entry. Indeed, we find that for  $i = 1, 2, 3$ ,  $\Phi(J_i)$  belongs to the kernel of the  $\mathfrak{so}(8)$  matrices.

The  $\mathfrak{so}(8)$  algebra can be extended to a  $\mathfrak{so}(9)$  subalgebra in three different ways: first, the algebra  $\mathfrak{so}(9)_1$  obtained by adding  $c_{45}, \dots, c_{52}$  to  $\mathfrak{so}(8)$ , and corresponding to the subgroup  $Spin(9)_1$  of  $F_4$  which leaves  $J_1$  invariant; second, the algebra  $\mathfrak{so}(9)_2$  obtained by adding  $c_{37}, \dots, c_{44}$  to  $\mathfrak{so}(8)$ , and corresponding to the subgroup  $Spin(9)_2$  of  $F_4$  which leaves  $J_2$  invariant; finally the  $\mathfrak{so}(9)_3$  obtained by adding  $c_{22}, \dots, c_{29}$  to  $\mathfrak{so}(8)$ , and corresponding to the subgroup  $Spin(9)_3$  of  $F_4$  which leaves  $J_3$  invariant. Again, this can be checked by applying the given matrices to  $\Phi(J_1)$ ,  $\Phi(J_2)$  and  $\Phi(J_3)$  respectively. We will use  $Spin(9)_1$  which we will refer to simply as  $Spin(9)$ .

Finally, recall that if  $\mathfrak{p}$  is the linear complement of  $\mathfrak{so}(9)$  in  $F_4$ , from  $ad$ -invariance and orthogonality it follows that:

$$[\mathfrak{so}(9), \mathfrak{p}] \subset \mathfrak{p}, \quad (79)$$

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{so}(9). \quad (80)$$

## 5.1 The generalized $Spin(9)$ -Euler construction.

### 5.1.1 The maximal subgroup

In this section we start with the construction of the Euler parametrization for  $F_4$ , based on its maximal subgroup  $H = Spin(9)$ . In particular, out of the three  $Spin(9)$  subgroups we have been able to identify previously, we pick  $Spin(9)_1$ . Then, its complementary subalgebra  $\mathfrak{p}$  is the 16 dimensional real vector space generated by the matrices  $c_i$ , with  $i = 22, \dots, 29, 37, \dots, 44$ . A look at the structure constants shows that as subspace  $V$  (see (29)) we can take any 1-dimensional subspace of  $\mathfrak{p}$ . We choose  $c_{22}$  as the generator for  $V$ . Since  $\dim G - \dim H - \dim V = 21$  we expect for  $H_o$  to be a  $Spin(7)$  subgroup of  $Spin(9)$ . To check that this is true, let us first recall that first 21 matrices generate an  $so(7)$  algebra. We are now able to construct a new set of 21 generators  $\{\tilde{c}_i\}$ ,  $i = 1, \dots, 21$ , which commute with  $c_{22}$  and which have the same structure constants as the  $\{c_i\}$ . To this end we start with the  $so(8)$  subalgebra generated by  $\{c_I\}$ ,  $I = 1, \dots, 21, 30, \dots, 36$ . Then the matrices  $\{c_\alpha\}$ ,  $\alpha = 30, \dots, 36$ , generate the whole  $so(7)$  algebra through:

$$c_{\frac{k(k-1)}{2}+i+1} = [c_{30+i}, c_{30+k}], \quad k = 1, \dots, 6, \quad i = 0, \dots, k-1. \quad (81)$$

Notice that for  $a, b \in \{22, \dots, 29\}$  the commutator  $[c_a, c_b]$  is a combination of four elements of  $so(8)$ , all having the same commutator with  $c_{22}$ . With this in mind, let us define

$$\tilde{c}_{30+i} := -[c_{22}, c_{23+i}], \quad i = 0, \dots, 7, \quad (82)$$

and then

$$\tilde{c}_{\frac{k(k-1)}{2}+i+1} = [\tilde{c}_{30+i}, \tilde{c}_{30+k}], \quad k = 1, \dots, 6, \quad i = 0, \dots, k-1. \quad (83)$$

Thus, the matrices  $\{\tilde{c}_I\}$ ,  $I = 1, \dots, 21, 30, \dots, 36$  have exactly the same structure constants as the  $\{c_I\}$  and  $[\tilde{c}_i, c_{22}] = 0$  for  $i = 1, \dots, 21$ . This is the  $\mathfrak{so}(7)$  we were looking for, let us call it  $\mathfrak{h}_o$ , so that  $H_o = \exp(\mathfrak{h}_o)$ . In order to apply (29) we need to have an explicit expression for  $H$  first. This can be done by applying to it the same method we are using for  $F_4$ , i.e. by constructing its Euler parametrization with  $SO(8)$  as a maximal subgroup, which in turn can be constructed from its  $SO(7)$  maximal subgroup and so on, with an inductive procedure. To avoid annoying repetitions to the reader, we limit ourselves here to the final expression for  $H$ :

$$\begin{aligned} Spin(9)[x_1, \dots, x_{36}] = & e^{x_1 c_3} e^{x_2 c_{16}} e^{x_3 c_{15}} e^{x_4 c_{35}} e^{x_5 c_5} e^{x_6 c_1} e^{x_7 c_{30}} e^{x_8 c_{45}} e^{x_9 c_3} e^{x_{10} c_{16}} e^{x_{11} c_{15}} \\ & e^{x_{12} c_{35}} e^{x_{13} c_5} e^{x_{14} c_1} e^{x_{15} c_{30}} e^{x_{16} c_3} e^{x_{17} c_5} e^{x_{18} c_4} e^{x_{19} c_7} e^{x_{20} c_{11}} e^{x_{21} c_{16}} \\ & e^{x_{22} c_3} e^{x_{23} c_5} e^{x_{24} c_4} e^{x_{25} c_7} e^{x_{26} c_{11}} e^{x_{27} c_3} e^{x_{28} c_5} e^{x_{29} c_4} e^{x_{30} c_7} e^{x_{31} c_3} e^{x_{32} c_5} \\ & e^{x_{33} c_4} e^{x_{34} c_3} e^{x_{35} c_2} e^{x_{36} c_3}, \end{aligned} \quad (84)$$

with ranges

$$\begin{aligned} x_i & \in [0, 2\pi], & i & = 1, 2, 3, 9, 10, 11, 16, 22, 27, 31, 34, \\ x_i & \in [0, \pi], & i & = 4, 8, 12, 17, 21, 23, 26, 28, 30, 32, 33, 35, \\ x_i & \in [-\frac{\pi}{2}, \frac{\pi}{2}], & i & = 5, 13, 18, 19, 20, 24, 25, 29, \end{aligned}$$

$$\begin{aligned}
x_i &\in [0, \frac{\pi}{2}], & i = 6, 7, 14, 15, \\
x_{36} &\in [0, 4\pi],
\end{aligned} \tag{85}$$

and measure

$$\begin{aligned}
d\mu_{Spin(9)}[x_1, \dots, x_{36}] &= \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8 \\
&\quad \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15} \\
&\quad \sin x_{17} \cos^2 x_{18} \cos^3 x_{19} \cos^4 x_{20} \sin^5 x_{21} \\
&\quad \sin x_{23} \cos^2 x_{24} \cos^3 x_{25} \sin^4 x_{26} \\
&\quad \sin x_{28} \cos^2 x_{29} \sin^3 x_{30} \sin x_{32} \sin^2 x_{33} \sin x_{35} \prod_{i=1}^{36} dx_i. \tag{86}
\end{aligned}$$

### 5.1.2 The whole $F_4$

To construct the quotient  $B$  we need to identify the subgroup  $SO(7)$  in  $H$ . We have seen that this group is generated by the matrices  $\{\tilde{c}_i\}$ ,  $i = 1, \dots, 21$ . We have also seen that these matrices satisfy the same commutation relation as the matrices  $\{c_i\}$ . Thus, if we are able to extend this to the whole  $\mathfrak{so}(9)$  algebra, we can use the same expression (84) with the matrices  $\{\tilde{c}_i\}$  instead. Luckily, we see that it is enough to add the matrices

$$\begin{aligned}
\tilde{c}_i &= c_{i+8}, & i = 22, \dots, 28, \\
\tilde{c}_i &= c_{i+16}, & i = 29, \dots, 36,
\end{aligned} \tag{87}$$

to obtain the desired set  $\{\tilde{c}_a\}$ ,  $a = 1, \dots, 36$  generating the whole  $Spin(9)$  group. Because the last 21 exponentials in (84) generate the  $SO(7) = H_o$  group, we get

$$\begin{aligned}
B[x_1, \dots, x_{15}] &= e^{x_1 \tilde{c}_3} e^{x_2 \tilde{c}_{16}} e^{x_3 \tilde{c}_{15}} e^{x_4 \tilde{c}_{35}} e^{x_5 \tilde{c}_5} e^{x_6 \tilde{c}_1} e^{x_7 \tilde{c}_{30}} e^{x_8 \tilde{c}_{45}} e^{x_9 \tilde{c}_3} e^{x_{10} \tilde{c}_{16}} e^{x_{11} \tilde{c}_{15}} \\
&\quad e^{x_{12} \tilde{c}_{35}} e^{x_{13} \tilde{c}_5} e^{x_{14} \tilde{c}_1} e^{x_{15} \tilde{c}_{30}}. \tag{88}
\end{aligned}$$

Therefore, the resulting Euler parametrization of  $F_4$  is:

$$F_4[x_1, \dots, x_{52}] = B[x_1, \dots, x_{15}] e^{x_{16} c_{22}} Spin(9)[x_{17}, \dots, x_{52}]. \tag{89}$$

Here, the range for the parameters  $x_1, \dots, x_{16}$  remains to be determined, while the other ranges are the ones for  $Spin(9)$ . We need to apply the topological method. To this purpose, we need to compute  $\det(J_p)$  as in (41). This computation is quite involved, and it requires some technical trick to be performed. We refer to [3] for the details. The resulting measure is:

$$d\mu_{F_4}[x_1, \dots, x_{52}] = d\mu_o[x_1, \dots, x_{16}] d\mu_{Spin(9)}[x_{17}, \dots, x_{52}], \tag{90}$$

$$\begin{aligned}
d\mu_o[x_1, \dots, x_{16}] &= 2^7 \cos^7 \frac{x_{16}}{2} \sin^{15} \frac{x_{16}}{2} \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8 \cdot \\
&\quad \cdot \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15} \prod_{i=1}^{16} dx_i. \tag{91}
\end{aligned}$$

From this we can select the ranges along the lines explained in section 3.3.2. Note that the exponentials are trigonometric functions of  $x_i/2$  with periods  $4\pi$ , so that we should take the range  $x_i \in [0, 4\pi]$  for  $i = 1, 2, 3$  and  $i = 9, 10, 11$ . However, for all the  $\tilde{c}_i \in so(7)$  we have that  $e^{2\pi\tilde{c}_i}$  commute with  $\tilde{c}_j$  and with  $c_{22}$ , so that it can be reabsorbed in the  $Spin(9)$  factor of  $F_4$  and these periods can be reduced to  $[0, 2\pi]$ . The ranges determined by the topological method are then

$$\begin{aligned} x_1 \in [0, 2\pi], \quad x_2 \in [0, 2\pi], \quad x_3 \in [0, 2\pi], \quad x_4 \in [0, \pi], \\ x_5 \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad x_6 \in [0, \frac{\pi}{2}], \quad x_7 \in [0, \frac{\pi}{2}], \quad x_8 \in [0, \pi], \\ x_9 \in [0, 2\pi], \quad x_{10} \in [0, 2\pi], \quad x_{11} \in [0, 2\pi], \quad x_{12} \in [0, \pi], \\ x_{13} \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad x_{14} \in [0, \frac{\pi}{2}], \quad x_{15} \in [0, \frac{\pi}{2}], \quad x_{16} \in [0, \pi]. \end{aligned} \quad (92)$$

This choice of the range covers the whole group at least once. Let us call  $M$  the corresponding homological cycle. Integrating the measure on the full range we obtain

$$\mu(M) = \frac{2^{26} \cdot \pi^{28}}{3^7 \cdot 5^4 \cdot 7^2 \cdot 11}. \quad (93)$$

To be sure that we covered  $F_4$  exactly once, we must compute the volume of the group by means of the Macdonald formula. Its Betty numbers were computed in [24]. For  $F_4$  there are four free generators for the rational homology, corresponding to four spheres having dimensions

$$d_1 = 3, \quad d_2 = 11, \quad d_3 = 15, \quad d_4 = 23. \quad (94)$$

They contribute to the volume with a term

$$Vol(S^3)Vol(S^{11})Vol(S^{15})Vol(S^{21}) = 2\pi^2 2^{\frac{\pi^5}{4!}} 2^{\frac{\pi^7}{6!}} 2^{\frac{\pi^{11}}{10!}}. \quad (95)$$

The simple roots are [15]

$$r_1 = L_2 - L_3 \quad (96)$$

$$r_2 = L_3 - L_4 \quad (97)$$

$$r_3 = L_4 \quad (98)$$

$$r_4 = \frac{L_1 - L_2 - L_3 - L_4}{2} \quad (99)$$

where  $L_i$ ,  $i = 1, \dots, 4$  is an orthonormal base for the dual Cartan algebra. The volume of the fundamental region representing the torus is then  $1/2$ . Finally, there are 48 non vanishing roots, 24 of with length 1, and 24 with length  $\sqrt{2}$ . We have determined them explicitly and, as expected, they correspond to the ones just presented, with  $L_i = e_i$ , the canonical basis of  $\mathbb{R}^4$ . The resulting contribution is the term:

$$\prod_{\alpha \in R(F_4)} \frac{2}{|\alpha|} = 2^{48} (\sqrt{2})^{24}. \quad (100)$$

The volume of the group is then

$$Vol(F_4) = \frac{2^{26} \cdot \pi^{28}}{3^7 \cdot 5^4 \cdot 7^2 \cdot 11} . \quad (101)$$

We conclude that the range we have determined covers the group exactly a single time.

## 6 THE $F_4$ -EULER ANGLES FOR $E_6$

As the construction of the Euler parametrization for  $E_6$  is very similar to the one for  $F_4$ , we are going to be very short and refer to [4] for more details. We can use the theorem of Chevalley and Schafer previously cited to extend the representation of the  $F_4$  algebra to the 27 irreducible representation of the whole  $E_6$  algebra, by simply adding the the matrices representing the action of  $R_Y$ . We only need to associate a  $27 \times 27$  matrix  $M(A)$  to each  $A \in \mathfrak{J}$ , in such the way that, if  $v \in \mathbb{R}^{27}$ , then

$$M(A)v = \Phi(A \circ \Phi^{-1}(v)). \quad (102)$$

The set of traceless Jordan matrices being 26-dimensional, this adds 26 new generators, which complete the  $F_4$  algebra to the 78-dimensional  $E_6$ -algebra. However, by computing the Killing form we can easily check that this is not the compact form with signature  $(52, 26)$ . It is instead the non compact form  $E_{6(-26)}$ . Fortunately, we can obtain the compact form by multiplying the 26 generators we have added by  $i$ . In this way, the algebra remains real and the representation becomes complex, and now  $V = \mathbb{C}^{27}$ . We haverealized these matrices with Mathematica and in the basis

$$\begin{pmatrix} a_1 & o_1 & o_2 \\ o_1^* & a_2 & o_3 \\ o_2^* & o_3^* & -a_1 - a_2 \end{pmatrix} \quad (103)$$

for the traceless Jordan matrices. They can be found in [4].

As the next step, we now need to choose a maximal compact subgroup. It is convenient to select the largest one, which we know to be  $H = F_4$ , in our case the group generated by the firsts 52 matrices. Its linear complement  $\mathfrak{p}$  (in the algebra) contains two preferred elements associated to the two diagonal matrices (103) with  $a_1 = 1, a_2 = 0$  and  $a_1 = 0, a_2 = 1$ , respectively. Following the order dictated by the map  $\Phi$ , after orthonormalization w.r.t. the product  $(J|J') = \text{Trace}(J \circ J')$  in  $\mathfrak{J}$ , these will correspond to the matrices  $c_{53}$  and  $c_{70}$  respectively. This is indeed the expression we used in [4] to do the computer calculations. There, we have found convenient a posteriori to recombine these two matrices in the new generators

$$\begin{aligned} \tilde{c}_{53} &= \frac{1}{2}c_{53} + \frac{\sqrt{3}}{2}c_{70}, \\ \tilde{c}_{70} &= -\frac{\sqrt{3}}{2}c_{53} + \frac{1}{2}c_{70}. \end{aligned}$$

These, added to the four matrices previously considered for the Cartan subalgebra of  $F_4$ , generate a Cartan subalgebra of  $E_6$  and the corresponding roots are exactly the ones described, for example, in [15], with  $L_i$  replaced by the elements  $e_i$  of the standard basis of  $\mathbb{R}^6$ .

In any case, it is easy to check that  $\tilde{c}_{53}, \tilde{c}_{70}$  can be taken as generators of  $V$ . Obviously, they commute. To realize the Euler parametrization, we note that the redundancy is now 28-dimensional, so that we expect to find a 28 dimensional subgroup  $H_o$  of  $H$  which commutes with  $V$ . In fact, this happens to be the  $SO(8)$  subgroup generated by the first 28 matrices  $\{c_i\}$ ,  $i = 1, 2, \dots, 28$ . We can then write

$$E_6[x_1, \dots, x_{78}] = B_{E_6}[x_1, \dots, x_{24}]e^{x_{25}c_{53}+x_{26}c_{70}}F_4[x_{27}, \dots, x_{78}] ,$$

with  $B_{E_6} = F_4/SO(28)$  and  $F_4$  as in the previous section. This means that in particular

$$B_{E_6}[x_1, \dots, x_{24}] = B[x_1, \dots, x_{15}]e^{x_{16}c_{22}}B_9[x_{17}, \dots, x_{23}]e^{x_{24}c_{37}} , \quad (104)$$

where  $B$  is given by (88) and

$$B_9[x_1, \dots, x_7] = e^{x_1\tilde{c}_3}e^{x_2\tilde{c}_{16}}e^{x_3\tilde{c}_{15}}e^{x_4\tilde{c}_{35}}e^{x_5\tilde{c}_5}e^{x_6\tilde{c}_1}e^{x_7\tilde{c}_{30}} .$$

We can now compute the associated invariant measure. The calculation is quite involved and details can be found in [4]. Here we give the final result only:

$$\begin{aligned} d\mu_{E_6} &= 2^7 \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8 \cdot \\ &\quad \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15} \cos^{15} \frac{x_{16}}{2} \sin^7 \frac{x_{16}}{2} \cdot \\ &\quad \sin x_{20} \cos x_{21} \cos x_{22} \sin^2 x_{22} \cos^2 x_{23} \sin^4 x_{23} \sin^7 x_{24} \cdot \\ &\quad \sin^8 x_{25} \sin^8 \left( \frac{\sqrt{3}}{2} x_{26} + \frac{x_{25}}{2} \right) \sin^8 \left( \frac{\sqrt{3}}{2} x_{26} - \frac{x_{25}}{2} \right) \cdot \\ &\quad d\mu_{F_4}[x_{27}, \dots, x_{78}] \prod_{i=1}^{26} dx_i . \end{aligned} \quad (105)$$

Proceeding as for  $F_4$ , from this measure we can determine the range  $R$  for the parameters:

$$\begin{aligned} x_1 &\in [0, 2\pi] , & x_2 &\in [0, 2\pi] , & x_3 &\in [0, 2\pi] , & x_4 &\in [0, \pi] , \\ x_5 &\in [-\frac{\pi}{2}, \frac{\pi}{2}] , & x_6 &\in [0, \frac{\pi}{2}] , & x_7 &\in [0, \frac{\pi}{2}] , & x_8 &\in [0, \pi] , \\ x_9 &\in [0, 2\pi] , & x_{10} &\in [0, 2\pi] , & x_{11} &\in [0, 2\pi] , & x_{12} &\in [0, \pi] , \\ x_{13} &\in [-\frac{\pi}{2}, \frac{\pi}{2}] , & x_{14} &\in [0, \frac{\pi}{2}] , & x_{15} &\in [0, \frac{\pi}{2}] , & x_{16} &\in [0, \pi] , \\ x_{17} &\in [0, 2\pi] , & x_{18} &\in [0, 2\pi] , & x_{19} &\in [0, 2\pi] , & x_{20} &\in [0, \pi] , \\ x_{21} &\in [-\frac{\pi}{2}, \frac{\pi}{2}] , & x_{22} &\in [0, \frac{\pi}{2}] , & x_{23} &\in [0, \frac{\pi}{2}] , & x_{24} &\in [0, \pi] , \\ x_{25} &\in [0, \frac{\pi}{2}] , & -\frac{x_{25}}{\sqrt{3}} &\leq x_{26} \leq \frac{x_{25}}{\sqrt{3}} , \end{aligned} \quad (106)$$

and  $x_j$ ,  $j = 27, \dots, 78$ , chosen to cover the whole  $F_4$  group. This choice defines a 78 dimensional closed cycle  $W$  having volume

$$Vol(W) = \int_R d\mu_{E_6} = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^5 \cdot 7^3 \cdot 11} . \quad (107)$$

To complete the work we need to check that this is indeed the volume of  $E_6$  as given by the Macdonald formula. The rational homology of  $E_6$  is  $H_*(E_6) = H_*(\prod_{i=1}^6 S^{d_i})$ , with ([24])

$$d_1 = 3, d_2 = 9, d_3 = 11, d_4 = 15, d_5 = 17, d_6 = 23 . \quad (108)$$

$E_6$  is simply laced, with simple roots

$$r_1 = L_1 + L_2 \quad (109)$$

$$r_2 = L_2 - L_1 \quad (110)$$

$$r_3 = L_3 - L_2 \quad (111)$$

$$r_4 = L_4 - L_3 \quad (112)$$

$$r_5 = L_5 - L_4 \quad (113)$$

$$r_6 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2} \quad (114)$$

where  $L_i, i = 1, \dots, 6$  is an orthogonal basis for the dual of the Cartan algebra. The volume of the torus associated to it is then  $\frac{2}{L}$ . As a check for the algebra, we have explicitly verified that the 72 roots of the algebra coincide with the roots of  $E_6$ , each one having length  $\sqrt{2}$ . They have, indeed, the structure given in [15], with  $L_i = e_i$ , the canonical basis of  $\mathbb{R}^6$ . The Macdonald formula then provides the result

$$Vol(E_6) = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^5 \cdot 7^3 \cdot 11}, \quad (115)$$

which concludes our check.

## 7 CONSTRUCTION OF NON COMPACT SPLIT FORMS AND THEIR COSET MANIFOLDS

Up to now we have considered compact groups only. However, as discussed in the introduction, it is important to be able to concretely realize non compact groups also. A particular class is given by the split forms, for which a particularly suitable technique is the Iwasawa decomposition, that we are going to discuss in this section. In order to clearly show the advantage of such method, in the next section we are going to compare the construction of a non compact form obtained by analytic continuation of a compact one with the direct Iwasawa construction.

### 7.1 Analytic continuation of the generalized Euler angles

A first way to realize a split by starting from the compact one is the following. Suppose we have realized a Euler parametrization of the compact group  $G$  with respect to a maximal subgroup  $H$ , say

$$G[x_1, \dots, x_p; y_1, \dots, y_r; z_1, \dots, z_m] = B[x_1, \dots, x_p]e^{V[y_1, \dots, y_r]}H[z_1, \dots, z_m], \quad p+r+m = n.$$

This is based on the orthogonal decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ . We know that  $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$  and  $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ . Let us now suppose that the further condition  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$  is satisfied. This condition is also called *symmetry*. It is a non trivial condition and it requires  $\mathfrak{h}$  to be a maximal subalgebra. Indeed, suppose we start with such a decomposition and we fix a subgroup  $H' \subset H$ . This determines the new orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h}' + \mathfrak{p}' = \mathfrak{h}' + \mathfrak{p} + \mathfrak{p}'',$$

with  $\mathfrak{p}'' = \mathfrak{p}' \cap \mathfrak{h}$ . Thus,

$$[\mathfrak{p}, \mathfrak{p}''] \subset \mathfrak{p} \subset \mathfrak{p}'$$

violates symmetry.

On the other hand we can easily see that symmetry is satisfied by all examples we considered, and indeed this happens to be true for all simple Lie groups [16]. Therefore, we can go from the compact form to the non compact form corresponding to the given maximal subgroup, simply by the *Weyl unitary trick* [16, 15]:

$$\mathfrak{p} \longmapsto i\mathfrak{p},$$

$i$  being the imaginary unit. Thus, the Euler parametrization of the split form is given by

$$G_{\text{split}}[x_1, \dots, x_p; y_1, \dots, y_r; z_1, \dots, z_m] = G[ix_1, \dots, ix_p; iy_1, \dots, iy_r; z_1, \dots, z_m].$$

## 7.2 The Iwasawa decomposition

While the Euler decomposition is particularly suitable for realizing compact Lie groups, for the non compact split forms a much simpler realization is provided by the Iwasawa decomposition[25]. It is based on the Cartan decomposition relative to a maximal subgroup  $H$ ,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ , where  $\mathfrak{h}$  is the Lie algebra associated to  $K$  and  $\mathfrak{s}$  its linear complement. The Cartan decomposition requires the existence of a linear involution  $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$  such that restricted to  $\mathfrak{s}$  the quadratic form

$$B : \mathfrak{s} \times \mathfrak{s} \longrightarrow \mathbb{R}, \quad (a, b) \longmapsto B(a, b) = K(a, \theta(b))$$

is positive definite.

Recall that the Killing form on the compact form is negative definite. Starting from our orthogonal decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$  we see that the map

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (a, b) \mapsto (a, -b), \quad \forall (a, b) \in \mathfrak{p} \oplus \mathfrak{h},$$

satisfies the required conditions so that we can identify  $\mathfrak{p}$  with  $\mathfrak{s}$ .

The next step consists in selecting a Cartan subalgebra of  $\mathfrak{p}$ . We call it  $\mathfrak{a}$  and  $A$  the group it generates. Being a Cartan subalgebra, the adjoint action of  $\mathfrak{a}$  is diagonalizable and we can associate to it a complete set of positive roots. From the basic properties of the root spaces, we know that the corresponding eigenmatrices generate a nilpotent subalgebra  $\mathfrak{n}$ . The Iwasawa decomposition states that the non compact form of  $G$  associated to the maximal subgroup  $H$  can be realized as

$$G = HAN = e^{\mathfrak{h}} e^{\mathfrak{a}} e^{\mathfrak{n}}. \tag{116}$$



Note that since  $H$  is a compact group, we can use our Euler parametrization to describe it. Then all new information is contained in the non compact quotient  $G/H$ . Before investigating how this can be described, let us make some further comments on the comparison between the Euler and Iwasawa constructions. On one hand, there doesn't exist any compact counterpart of the Iwasawa construction, but surely there are many other possibilities, as for example the exponential map itself. In this case, the big advantage of the Euler construction is that involves only parametric angles, which appear in the expression for the group elements in a trigonometric form. Computationally, this is not immediately an advantage because of the difficulties in handling trigonometric simplifications with Mathematica. Indeed, at some steps direct manipulations of the expressions by hand has been necessary and in fact much simpler than direct computer computing. However, the true advantage arises when the explicit range of the parameters has to be established. For this purpose, as we have seen, the periodicity of the trigonometric expressions provides a quite direct way to determine such ranges, whereas for a generic parametrization this would require the solution of some transcendental equations, which can be handled only numerically.

On the other hand, when we work with a non compact form, the difficult problem of determining the explicit range for the parameters is restricted to the compact subgroup only. Therefore, we need to use the trigonometric expressions for the compact subgroup only, while it is now possible to use a simpler realization for the non compact part, possibly much easier to handle. Such a realization can be provided exactly by the Iwasawa decomposition, where only Abelian or nilpotent matrices appear in the non compact part. In particular, from the structure of the root spaces the nilpotency of the non compact part will be at most the rank  $r$  of the group, so that we expect for the non compact part to appear polynomial terms of degree at most  $r$ , instead of trigonometric expressions (or hyperbolic after the Weyl trick).

### 7.3 The coset manifold

Let us now look at the construction of the non compact quotient  $G/K$ . We need to compute the induced metric (39) starting from the Iwasawa expression. Obviously, we can proceed exactly as for the compact case. However, following the tradition, we have written the decomposition taking  $H$  as a left factor instead of a right factor, so that it will be convenient here to exchange left invariant forms with right invariant form. This does not change the substance, being the Killing form bi invariant. Now, let us introduce the one form

$$\begin{aligned} J_G^R &= dG G^{-1} = HAdN N^{-1}A^{-1}H^{-1} + HdA A^{-1}H^{-1} + dH H^{-1} \\ &\equiv HAJ_NA^{-1}H^{-1} + HJ_AH^{-1} + J_H. \end{aligned}$$

To compute the metric of  $M = G/H$ , we need to eliminate the components of  $J_G^R$  along the fibers ( $\mathfrak{h}$ ), so as to define the reduced form  $J'_G$ , giving the metric

$$d\sigma^2 = \kappa \text{Tr}(J'_G \otimes J'_G),$$

where  $\kappa$  is a normalization constant. Let us study the structure of this metric. First, notice that the term  $J_H$ , which appears in  $J_G^R$ , is projected out to obtain  $J'_G$ . Moreover, the adjoint

action of  $H$  commutes with the projection, because it respects the direct decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ . Therefore, if we define

$$J_p := \pi(AJ_N A^{-1}),$$

where  $\pi$  is the projection out from the fibers, then

$$d\sigma^2 = \kappa \text{Tr}(J_p \otimes J_p) + \kappa \text{Tr}(J_A \otimes J_A).$$

Let us remark that  $J_p$  is orthogonal to  $J_A$ . Indeed,  $J_A$  is easily computed, because  $A = \exp(\sum_{i=1}^r y_i H_i)$  defines an Abelian group. Here  $H_i$  identifies an orthonormal basis (with respect to the product  $\kappa \text{Tr}$ ) for the Cartan subspace, so that  $J_A = \sum_{i=1}^r dy_i H_i$  and

$$\kappa \text{Tr}(J_A \otimes J_A) = \sum_{i=1}^r dy_i^2.$$

On the other side,  $J_p$  can be easily determined from the simple properties of  $N$ . Recall that the generators of  $N$  are the positive root matrices  $R_l$ ,  $l = 1, \dots, m := (n-r)/2$ , so that two such matrices commute if the sum of the corresponding roots is not a root, otherwise the commutator is proportional to the matrix associated to the resulting root. Now,  $N(x_1, \dots, x_m) = e^{\sum_{i=1}^m x_i R_i}$ , and

$$J_N = \sum_{i=1}^m n^i(\vec{x}) R_i, \quad n^i(\vec{x}) = \sum_{j=1}^m n^i_j(\vec{x}) dx_j, \quad (117)$$

where the  $n^i_j(x_1, \dots, x_m)$  are all polynomials in the  $x_i$ . Now, the  $R_i$  are eigenmatrices for the action of  $A$  and, therefore, we have

$$A R_i A^{-1} = e^{\sum_{a=1}^r r_{i,a} y_a} R_i, \quad (118)$$

where  $\vec{r}_i = (r_{i,1}, \dots, r_{i,r})$  are the positive roots whose components are the eigenvalues  $r_{i,a}$ ,  $a = 1, \dots, r$  of  $H_a$  with eigenvector  $R_i$ . Thus,

$$A J_N A^{-1} = \sum_{i=1}^m e^{\sum_{a=1}^r r_{i,a} y_a} n^i(\vec{x}) R_i, \quad (119)$$

and to obtain  $J_p$  we only need to take the projection on  $\mathfrak{p}$ . The metric on the quotient is then

$$d\sigma^2 = \sum_{i=1}^r dy_i^2 + \sum_{i=1}^r e^i \otimes e^i, \quad (120)$$

$$e^i = \kappa \text{Tr}[A J_N A^{-1} P_i], \quad (121)$$

where  $P_i$ , together with  $H_a$  realize an orthonormal basis of  $\mathfrak{p}$  with respect to the product  $(a|b) = \kappa \text{Tr}(ab)$ .

## 8 REALIZING $G_{2(2)}$ AND $G_{2(2)}/SO(4)$

The non compact form  $G_{2(2)}$  is the split form of  $G_2$  associated to the maximal compact subgroup  $SO(4)$ . Referring to section 4, we know that  $SO(4)$  is generated by  $C_i$ ,  $i = 1, 2, 3, 8, 9, 10$ , and  $\mathfrak{p}$  is generated by  $C_a$ ,  $a = 4, 5, 6, 7, 11, 12, 13, 14$ . To determine the split form we could multiply  $C_a$  by the imaginary unit  $i$ . Alternatively, noting that all matrices are antisymmetric, we prefer to transform the matrices  $C_a$  into symmetric matrices. The representative matrices obtained in this way are normalized with the condition  $Tr(Q_I Q_J) = \eta_{IJ}$ , where  $\eta = \text{diag}\{-1, -1, -1, 1, 1, 1, 1, -1, -1, -1, 1, 1, 1, 1\}$ :

$$\begin{aligned}
 Q_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 Q_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} & Q_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 Q_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 Q_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
Q_9 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} & Q_{10} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
Q_{11} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_{12} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
Q_{13} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_{14} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

The matrices  $\{Q_1, Q_2, Q_3, Q_8, Q_9, Q_{10}\}$  generate the Lie algebra of  $SO(4)$ , and the elements  $Q_5$  and  $Q_{11}$  commute, and generate a non compact Cartan subalgebra contained in  $\mathfrak{p}$ .

## 8.1 Euler construction of $G_{2(2)}/SO(4)$

As we have seen, a first way to realize the non compact form is by analytic continuation. In this case it simply means that we have to substitute the matrices  $C_I$  with  $Q_I$  in (58):

$$g[x_1, \dots, x_{14}] = H(x_1, \dots, x_6) e^{\sqrt{3}x_7 Q_{11} + x_8 Q_5} H(x_9, \dots, x_{14}), \quad (122)$$

$$H(x_1, \dots, x_6) = e^{x_1 Q_3} e^{x_2 Q_2} e^{x_3 Q_3} e^{\sqrt{3}x_4 Q_8} e^{\sqrt{3}x_5 Q_9} e^{\sqrt{3}x_6 Q_8}. \quad (123)$$

We can then proceed with the computation of the invariant measure exactly as for the compact space. We are going to skip all details here and give only the final result:

$$d\mu_{G_{2(2)}} = 27\sqrt{3} f(2x_7, 2x_8) \sin(2x_2) \sin(2x_5) \sin(2x_{10}) \sin(2x_{13}) \prod_{i=1}^{14} dx_i, \quad (124)$$

where

$$f(\alpha, \beta) = \sinh\left(\frac{\beta - \alpha}{2}\right) \sinh\left(\frac{\beta + \alpha}{2}\right) \sinh\left(\frac{\beta - 3\alpha}{2}\right) \sinh\left(\frac{\beta + 3\alpha}{2}\right) \sinh(\alpha) \sinh(\beta), \quad (125)$$

from which we can determine the range for the parameters:

$$\begin{aligned}
0 \leq a_1 \leq \pi, & \quad 0 \leq a_2 \leq \frac{\pi}{2}, & \quad 0 \leq a_3 \leq \frac{\pi}{2}, \\
0 \leq a_4 \leq 2\pi, & \quad 0 \leq a_5 \leq \frac{\pi}{4}, & \quad 0 \leq a_6 \leq \pi, \\
0 \leq a_9 \leq 2\pi, & \quad 0 \leq a_{10} \leq \frac{\pi}{2}, & \quad 0 \leq a_{11} \leq \pi, \\
0 \leq a_{12} \leq \pi, & \quad 0 \leq a_{13} \leq \frac{\pi}{2}, & \quad 0 \leq a_{14} \leq \pi, \\
0 \leq a_7 \leq \infty, & \quad 3a_7 \leq a_8 \leq \infty.
\end{aligned} \tag{126}$$

Next, we can also compute the metric (39) on the quotient  $G_{2(2)}/SO(4)$ . The details are very similar to the ones in [1]. Introducing the 1-forms

$$\begin{aligned}
I_1(x, y, z) &:= \sin(2y) \cos(2z) dx - \sin(2z) dy, \\
I_2(x, y, z) &:= \sin(2y) \sin(2z) dx + \cos(2z) dy, \\
I_3(x, y, z) &:= dz + \cos(2y) dx,
\end{aligned} \tag{127}$$

we get

$$\begin{aligned}
ds_{G_{2(2)}/SO(4)}^2 &= da_8^2 + da_7^2 + [\sinh^2 a_8 \cosh^2 a_7 + \cosh^2 a_8 \sinh^2 a_7] (da_5^2 + \sin^2(2a_5) da_4^2 \\
&\quad + 3da_2^2 + 3\sin^2 2a_2 da_1^2) \\
&\quad + \frac{1}{2} \cosh(2a_8) \cosh(2a_7) \sinh^2(2a_7) \{ [I_1(a_4, a_5, a_6) + 3I_2(a_1, a_2, a_3)]^2 \\
&\quad + [I_2(a_4, a_5, a_6) - 3I_1(a_1, a_2, a_3)]^2 \} \\
&\quad + \frac{3}{4} \sinh^2(2a_7) [I_3(a_4, a_5, a_6) - I_3(a_1, a_2, a_3)]^2 \\
&\quad + \frac{1}{4} \sinh^2(2a_8) [I_3(a_4, a_5, a_6) + 3I_3(a_1, a_2, a_3)]^2.
\end{aligned} \tag{128}$$

Such a computation is already quite complicated for the  $G_2$  group, and for higher dimensional groups it quickly becomes prohibitive.

## 8.2 Iwasawa construction of $G_{2(2)}/SO(4)$

The Iwasawa parametrization is the most suitable for the computation of the metric on  $G_{2(2)}/SO(4)$ . We know that the Cartan subalgebra of  $\mathfrak{p}$  is generated by  $H_1 := C_{11}$  and  $H_2 := C_5$ . The roots of  $G_2$  can thus been computed by diagonalizing the adjoint action of  $H_i$ . We obtain

$$\begin{aligned}
r_1 &= \left(\frac{2}{\sqrt{3}}, 0\right); & r_2 &= (\sqrt{3}, 1); & r_3 &= \left(\frac{1}{\sqrt{3}}, 1\right); \\
r_4 &= (0, 2); & r_5 &= \left(-\frac{1}{\sqrt{3}}, 1\right); & r_6 &= (-\sqrt{3}, 1),
\end{aligned}$$

where we write only a choice of positive roots. The corresponding eigenmatrices (up to some normalization constants) are

$$R_1 = \sqrt{3}C_3 - C_8 + 2C_{12}; \quad (129)$$

$$R_2 = \frac{1}{\sqrt{3}}(C_1 - C_2 + C_6 - C_7) - C_9 + C_{10} + C_{13} + C_{14}; \quad (130)$$

$$R_3 = \sqrt{3}(C_1 + C_2 + C_6 + C_7) + C_9 + C_{10} - C_{13} + C_{14}; \quad (131)$$

$$R_4 = C_3 - 2C_4 + \sqrt{3}C_8; \quad (132)$$

$$R_5 = -\sqrt{3}(C_1 - C_2 + C_6 - C_7) - C_9 + C_{10} + C_{13} + C_{14}; \quad (133)$$

$$R_6 = -\frac{1}{\sqrt{3}}(C_1 + C_2 + C_6 + C_7) + C_9 + C_{10} - C_{13} + C_{14}. \quad (134)$$

If we choose to parameterize the nilpotent subgroup  $N$  as  $N(x_1, \dots, x_6) = \prod_{i=1}^6 e^{x_i R_i}$ , we get

$$n^1 = dx_1; \quad (135)$$

$$n^2 = dx_2 - 4\sqrt{3}x_1 dx_3 + 16x_1^2 dx_5 - \frac{64}{3\sqrt{3}}x_1^3 dx_6; \quad (136)$$

$$n^3 = dx_3 - \frac{8}{\sqrt{3}}x_1 dx_5 + \frac{16}{3}x_1^2 dx_6; \quad (137)$$

$$n^4 = dx_4 + 8x_3 dx_5 - \frac{8}{3}x_2 dx_6; \quad (138)$$

$$n^5 = dx_5 - \frac{4}{\sqrt{3}}x_1 dx_6; \quad (139)$$

$$n^6 = dx_6. \quad (140)$$

As we see, these are polynomials. Then

$$d\sigma^2 = dy_1^2 + dy_2^2 + \sum_{i=1}^6 e_i \otimes e_i \quad (141)$$

with

$$\begin{aligned} e^1 &= -2e^{-2y_2} \left( dx_4 + 8x_3 dx_5 - \frac{8}{3}x_2 dx_6 \right), \\ e^2 &= \frac{1}{\sqrt{3}} \left( e^{-\sqrt{3}y_1 - y_2} (dx_2 - 4\sqrt{3}x_1 dx_3 + 16x_1^2 dx_5 - \frac{64}{3\sqrt{3}}x_1^3 dx_6) - e^{\sqrt{3}y_1 - y_2} dx_6 \right) \\ &\quad + \sqrt{3} \left( e^{-\frac{1}{\sqrt{3}}y_1 - y_2} (dx_3 - \frac{8}{\sqrt{3}}x_1 dx_5 + \frac{16}{3}x_1^2 dx_6) - e^{\frac{1}{\sqrt{3}}y_1 - y_2} (dx_5 - \frac{4}{\sqrt{3}}x_1 dx_6) \right) \\ e^3 &= -\frac{1}{\sqrt{3}} \left( e^{-\sqrt{3}y_1 - y_2} (dx_2 - 4\sqrt{3}x_1 dx_3 + 16x_1^2 dx_5 - \frac{64}{3\sqrt{3}}x_1^3 dx_6) + e^{\sqrt{3}y_1 - y_2} dx_6 \right) \end{aligned}$$

$$\begin{aligned}
& +\sqrt{3}\left(e^{-\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_3-\frac{8}{\sqrt{3}}x_1dx_5+\frac{16}{3}x_1^2dx_6\right)+e^{\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_5-\frac{4}{\sqrt{3}}x_1dx_6\right)\right) \\
e^4 & = 2e^{-\frac{2}{\sqrt{3}}y_1}dx_1, \\
e^5 & = e^{-\sqrt{3}y_1-y_2}\left(dx_2-4\sqrt{3}x_1dx_3+16x_1^2dx_5-\frac{64}{3\sqrt{3}}x_1^3dx_6\right)-e^{\sqrt{3}y_1-y_2}dx_6 \\
& \quad +e^{\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_5-\frac{4}{\sqrt{3}}x_1dx_6\right)-e^{-\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_3-\frac{8}{\sqrt{3}}x_1dx_5+\frac{16}{3}x_1^2dx_6\right), \\
e^6 & = e^{-\sqrt{3}y_1-y_2}\left(dx_2-4\sqrt{3}x_1dx_3+16x_1^2dx_5-\frac{64}{3\sqrt{3}}x_1^3dx_6\right)+e^{\sqrt{3}y_1-y_2}dx_6 \\
& \quad +e^{\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_5-\frac{4}{\sqrt{3}}x_1dx_6\right)+e^{-\frac{1}{\sqrt{3}}y_1-y_2}\left(dx_3-\frac{8}{\sqrt{3}}x_1dx_5+\frac{16}{3}x_1^2dx_6\right), \\
e^7 & = dy_1, \\
e^8 & = dy_2.
\end{aligned}$$

The polynomial dependence on the variables makes the computation feasible by a computer even for higher dimensional groups.

## 9 CONCLUSIONS

We have given a detailed explanation of the methods for studying the geometry of exceptional Lie groups that we have first introduced in [1] for  $G_2$ . Indeed, here we have seen the elementary reasonings which constitute the basis of our ideas and provide a powerful tool for computing global parameterizations of Lie groups. Recall that a parametrization differs from a coordinatization in that it does not provide a diffeomorphism between the manifold and the space of parameters. However, a parametrization locally yields a coordinatization and it is global when it covers the whole group. This means that, if  $\mathcal{R}$  is the space of parameters and  $G$  the group, then the parametrization

$$p : \mathcal{R} \longrightarrow G$$

is surjective. In general, however, it cannot be injective. Indeed, in general group manifolds have a non vanishing curvature tensor and cannot be globally covered by a single chart. Nevertheless, a parametrization can be considered good when it is “minimal”, in the sense that  $\mathcal{R}$  is the closure of an open local chart. This means that the bijectivity of  $p$  is lost only on a subset of vanishing measure, which is the boundary  $\partial\mathcal{R}$  of  $\mathcal{R}$ . In this case we call the set  $\mathcal{R}$  the range of the parameters.

In general, for a finite dimensional simple Lie group the true difficulty lies not so much in constructing a global parametrization, but rather in determining the range. Here is where the idea of the generalized Euler angles comes into play, as it is particularly suitable for computing the full range of the parameters, since it allows us to express them in terms of Cartesian products.

In particular, we have seen that there are essentially two methods to determine the range. The first is geometric and is based on the detailed knowledge of the geometry of the quotient space of the group and its maximal subgroup. We have described it for the example of the  $SU(3)$ -Euler parametrization of  $G_2$ , but it can be adopted for the Euler parametrization

of any of the compact classical simple Lie groups, as for example  $SU(N)$  [20]. The second method is topological and can be used when the geometrical information on the quotient space is lacking or the geometrical method is not sufficient to fix the range, as for example for the  $SO(4)$ -Euler parametrization of  $G_2$  [1] or for the parameterizations of  $F_4$  [3] and  $E_6$  [4].

Finally, we have also considered the construction of non compact Lie groups. In that case the Iwasawa decomposition is simpler than the Euler one. In particular, we have shown how it can be applied to the non compact Lie group  $G_{2(2)}$ . The material exposed in the last section is all new.

## Acknowledgments

B.L.C. would like to thank S. Ferrara and A. Marrani for enlightening discussions. The work of B.L.C. has been supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and in part by NSF grant 10996-13607-44 PHHXM.

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