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Endomorphism Algebras in Coxeter Categorifications and Harish-Chandra 2-Categories

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Endomorphism Algebras in Coxeter Categorifications and Harish-Chandra 2-Categories

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

Benjamin William West

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ABSTRACT OF THE DISSERTATION

Endomorphism Algebras in Coxeter Categorifications
and Harish-Chandra 2-Categories

by

Benjamin William West
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2020
Professor Raphaël Rouquier, Chair

Given the data of a Coxeter system (W, S), a Coxeter categorification is a 2-category in which the objects are subsets of S, the generating 1-morphisms categorify induction and restriction functors associated to parabolic subgroups, and the generating 2-morphisms impose certain coherence conditions and structural properties among the 1-morphisms. Of particular interest is the structure of the 2-homomorphism spaces of these 1-morphisms. Furthermore, given a connected, reductive, algebraic group G over an algebraically closed field k, a chosen Frobenius endomorphism $F\colon G \to G$ determines a parameter $q \in k^{\times}$, and the Weyl group of G gives rise to a Coxeter system. When this system is of rank 1, we construct by generators and relations an extension of the Coxeter categorification, independent of q, where the 2-homomorphism spaces are free modules of finite rank over the ring of Laurent polynomials with integer coefficients. An explicit description of the 2-homomorphism spaces between generating 1-morphisms is given, along with an algorithm lifting these descriptions to the 2-homomorphism spaces of arbitrary 1-morphisms. Then a nontrivial 2-functor from this 2-category is constructed into the 2-category of bimodules. Some conjectural constructions are given in the case that W has arbitrary finite rank, in particular a proposal for the endomorphism ring of the generating 1-morphism from $\emptyset$ to S that is an extension of an algebra introduced by Marin.

The dissertation of Benjamin William West is approved.

Paul Balmer

Sucharit Sarkar

Burt Totaro

Raphaël Rouquier, Committee Chair

University of California, Los Angeles

2020

To my grandfathers,

Willie Shu-Kei Kam (1928 - 2017)

and

William Ferguson West (1933 - 2019)

Contents

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Vita. Before enrolling at the University of California, Los Angeles, Benjamin West studied at the University of California, Berkeley, where he earned his bachelor's degree in pure mathematics.

1 Introduction

Suppose G is a reductive group defined over $\overline{\mathbb{F}}_p$, an algebraic closure of the finite field of p elements, with an isogeny $F\colon G \to G$ such that some power of F if a Frobenius endomorphism of G. Let G^F denote the set of fixed points, and let $K_0(\mathbb{C}G^F\text{-}\mathsf{mod})$ denote the Grothendieck group of the category of finite dimensional $\mathbb{C}G^F$-modules. In the late 1970s and early 1980s, Alvis and Curtis introduced the Alvis-Curtis duality, an involution

$$D_G\colon K_0(\mathbb{C}G^F\text{-}\mathsf{mod}) \longrightarrow K_0(\mathbb{C}G^F\text{-}\mathsf{mod})$$

given as a particular alternating sum of compositions of parabolic inductions and restrictions over F-stable parabolic subgroups of G with respect to a chosen F-stable Borel subgroup.

In 1990, Broué showed that D_G is a perfect isometry, i.e. a type of generalized induction associated to a perfect character (c.f. Definitions 1.1 and 1.4 of [1]), in all characteristics other than p. This result was a reflection of a fact of categorical flavor first conjectured by Broué, namely that D_G is induced by a self-equivalence of the bounded derived category $D^b(RG\text{-}\mathsf{mod})$ of the category of finitely generated RG-modules, where R is a complete discrete valuation ring with residue field of characteristic other than p. This conjecture was proven in slightly greater generality by Cabanes and Rickard in [2] using a coefficient system of $\mathbb{Z}[p^{-1}]G$-bimodules. The crux of the proof involves applying parabolic induction to a cuspidal module, and then determining its image in the bounded homotopy category after tensoring with various chain complexes associated to the aforementioned coefficient system. To quote the authors, an "essential ingredient" is a result of Howlett and Lehrer that parabolic induction and restriction are independent of the choice of parabolic subgroup.

Similar situations have arisen elsewhere, e.g. Chuang-Rouquier [4], and inspired by this, Dreyfus-Schmidt develops in his thesis a new categorical framework called the Coxeter complex categorification. This setting is used to categorify the Alvis-Curtis duality, as well as provides a category theoretic schema for both Harish-Chandra and Howlett-Lehrer theory.

To begin, Dreyfus-Schmidt associates to a finite Coxeter system (W, S) a family of linear, abelian categories $\mathcal{A}_I$ parametrized by the subsets of $I \subseteq S$. Among other things, for any $I \subseteq J \subseteq S$, there exist exact biadjoint functors $F_I^J\colon \mathcal{A}_I \rightleftarrows \mathcal{A}_J\colon G_I^J$, akin to the Harish-Chandra induction and restriction functors. Additionally, there are distinguished natural isomorphisms that encode categorical analogues of the standard properties of such functors, namely transitivity, independence of the choice of parabolic subgroup, and transport of structure. Dreyfus-Schmidt refers to this initial premise as a weak W-categorification, but upgrades this definition to a genuine W-categorification if the aforementioned natural isomorphisms are subject to several coherence conditions, one of which provides a notion of a Mackey decomposition like that of the usual Mackey formula for induction and restriction. By his own remark, Dreyfus-Schmidt notes that several of the coherence conditions in the definition of a W-categorification are not needed for the aim of categorifying the Alvis-Curtis duality, but would be useful in elucidating the structure of the endomorphism algebras of cuspidal objects. In fact, in this work this initial definition is enlarged. For instance, for each $I \subseteq J$, Dreyfus-Schmidt fixes an adjunction $(\epsilon_I^J, \eta_I^J) : F_I^J \dashv E_I^J$ realizing F_I^J as a left adjoint to E_I^J, but fixes no specific adjunction witnessing E_I^J as a left adjoint to F_I^J. Our extended definition does fix such an adjunction, and imposes an additional coherence condition such that the two induced maps between $\mathrm{End}(F_I^J)$ and $\mathrm{End}(E_I^J)$ by these two choices of counit-unit pairs coincide. This is not a particularly unnatural requirement, as frequently the functors F_I^J and E_I^J correspond to a generalized induction or restriction given by a symmetric algebra, and the corresponding algebra morphisms corresponding to the units and counits satisfy the same coherence conditions. In this vein, we hope to describe the endomorphism algebras of the F_I^J, and consequently those of the G_I^J once a fixed counit-unit adjunction is chosen.

Algebras similar to possible candidate endomorphism algebras have been studied for some time. A close analogue of the familiar Iwahori-Hecke algebra is the Yokonuma-Hecke algebra $\mathscr{Y}$, that is, the endomorphism algebra of the permutation representation of a Chevalley group G with respect to a chosen maximal unipotent subgroup U. In 1967, Yokonuma gave a presentation of this algebra in terms of standard generators parametrized by double coset representatives of U, and such generators satisfy the expected braid relations, as well as a slightly different quadratic relation. Some decades later, Juyumaya and Kannan gave a new presentation of the Yokonuma-Hecke algebra. After choosing a Borel subgroup and maximal torus, for each corresponding root they modify the coefficients of a linear combination of Yokonuma's standard generators with a fixed additive character of the underlying field of definition. The new quadratic relation of this nonstandard presentation then involves an idempotent which is in turn a linear combination of standard generators parametrized by the image of the corresponding coroot.

These new generators and the idempotents that appear in the quadratic relation thus generate a subalgebra of $\mathscr{Y}$, and Marin has determined a presentation for it in recent work [12]. To explain this setup, let (W, S) denote the Coxeter system for the above G, and let R denote the set of reflections in W. For ease of notation, assume that the isogeny F acts trivially on W, so that $W = W^F$. Marin then constructs an associative algebra $C_W(\underline{\tilde{q}})$ over a commutative, unital ring k, where $\underline{\tilde{q}} = (\tilde{q}_s)_{s \in S}$ is a family of parameters such that $\tilde{q}_s = \tilde{q}_t$ whenever $s, t \in S$ are conjugate. The algebra $C_W(\underline{\tilde{q}})$ is defined by generators $\{g_s\}_{s \in S}$ and $\{e_t\}_{t \in R}$ subject to some relations, two of which together impose the condition that the e_t are commuting idempotents. The coefficients in the relations only involve the parameters $\tilde{q}_s$ and the unit 1, so for our purposes specializing each $\tilde{q}_s$ to $\tilde{q}$, we may assume a simpler setting where $C_W(\underline{\tilde{q}}) =: C_W$ is defined over $\mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$. Thus given a finite subset $J \subseteq R$, one can put $e_J = \prod_{t \in J} e_t$ without ambiguity. It is a further consequence of the relations that for $J, K \subseteq R$ finite, $e_J = e_K$ whenever J and K generate the same reflection subgroup. Furthermore, C_W is freely generated as a module by the products $e_J g_w$ for $J \subseteq R$ finite, and $w \in W$. It thus follows that when W is finite, C_W has rank $m\,|W|$, where m is the number of reflection subgroups of W. In particular, the rank of C_W is independent of the field of definition of the original group G.

With this result of Marin in mind, our aim to is provide endomorphism algebras of the biadjoint functors in Dreyfus-Schmidt's W-categorification setting in such a way that their dimension is independent of the characteristic of the underlying field of definition of the associated group. To explain this in more detail, first consider a root datum $\mathcal{R} = (X^*, \Phi, X_*, \Phi^*)$ and an algebraically closed field k. Up to isomorphism, $\mathcal{R}$ uniquely determines a split reductive group (G, T), where G is a reductive algebraic group over k satisfying the usual commutator relations, and T is a split maximal torus. Let $(W(\mathcal{R}), S)$ be the associated Weyl group, which is an instance of a Coxeter group. Furthermore, G is an algebraic group with split BN-pair, and for each $J \subseteq S$, the standard parabolic subgroup P_J has the Levi decomposition $P_J = L_J \ltimes U_J$, where L_J is the standard Levi subgroup, and U_J is the unipotent complement. The Levi subgroup L_J is itself an algebraic group with split BN-pair satisfying the commutator relations, and thus has its own standard parabolic subgroups $P_I \cap L_J$ for $I \subseteq J$. In turn, $P_I \cap L_J$ has Levi decomposition

$$P_I \cap L_J = L_I \ltimes (U_I \cap L_J).$$

Additionally, that L_J has a split BN-pair is witnessed by subgroups B_J and N_J, defined by $B_J = U_{(w_0)_J} \ltimes T$ and $N_J/T = W_J$, where $U_{(w_0)_J}$ is the product of root subgroups corresponding to the positive roots with respect to J.

Now consider a general Coxeter system (W, S), a commutative ring $\tilde{R} = \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$, and $\tilde{q} \in \tilde{R}$ an indeterminate. If $M = (m_{st})$ denotes the Coxeter matrix, the Hecke algebra $\mathscr{H}_S(\tilde{q})$ associated to this Coxeter system is the $\tilde{R}$-algebra with generators T_s, for $s \in S$, subject to the relations

1. $\underbrace{T_s T_t T_s \cdots}_{m_{st}} = \underbrace{T_t T_s T_t \cdots}_{m_{st}}$,
2. $(T_s - \tilde{q})(T_s + 1) = 0$, for all $s \in S$.

To this Coxeter system (W, S) and arbitrary parameter $\tilde{q}$, there is associated a Hecke 2-category $\mathsf{Hecke}_{\tilde{R},\tilde{q}}(W)$, realized as a subcategory of the usual 2-category Bimod of bimodules. The 2-category $\mathsf{Hecke}_{\tilde{R},\tilde{q}}(W)$ has as objects the Hecke algebras $\mathscr{H}_J(\tilde{q})$ for $J \subseteq S$, morphisms generated by $(\mathscr{H}_J(\tilde{q}), \mathscr{H}_K(\tilde{q}))$- and $(\mathscr{H}_K(\tilde{q}), \mathscr{H}_J(\tilde{q}))$-bimodules $\mathscr{H}_J(\tilde{q})$ for any $K \subseteq J \subseteq S$, and the 2-morphisms are the usual bimodule homomorphisms.

However, since $(W(\mathcal{R}), S)$ is not only a Coxeter system, but arises as the Weyl group of a reductive algebraic group G, we can associate to it another 2-category, the Harish-Chandra 2-category $\mathsf{HC}(W(\mathcal{R}))$ which is given as follows. The objects of this category are the algebras kL_J, for $J \subseteq S$, where L_J is the standard Levi subgroup of G corresponding to J defined above. The morphisms are generated by the (kL_J, kL_I)-bimodules $k[L_J/(U_I \cap L_J)]$ and the (kL_I, kL_J)-bimodules $k[(U_I \cap L_J)\backslash L_J]$ for $I \subseteq J \subseteq S$. As above, the 2-morphisms are the usual bimodule homomorphisms. Note, of course, that the generating 1-morphisms are the bimodules inducing the Harish-Chandra induction and restriction functors

$$R_{L_I}^{L_J} = kL_J/(U_I \cap L_J) \otimes_{kL_I} -\colon kL_I\text{-}\mathsf{mod} \longrightarrow kL_J\text{-}\mathsf{mod}$$

and

$$^*R_{L_I}^{L_J} = k[(U_I \cap L_J)\backslash L_J] \otimes_{kL_J} -\colon kL_J\text{-}\mathsf{mod} \longrightarrow kL_I\text{-}\mathsf{mod}.$$

For the group G above, again let $F\colon G \to G$ denote a Frobenius endomorphism. The pair (G, F) determines a parameter $q \in k^\times$. Let $\mathsf{Hecke}_{k,q}(W(\mathcal{R}))$ denote the Hecke 2-category defined in the same fashion as $\mathsf{Hecke}_{\tilde{R},\tilde{q}}(W(\mathcal{R}))$ above, with k in place of $\tilde{R}$, and q in place of $\tilde{q}$. Then there is a 2-functor $\mathscr{F}$ from $\mathsf{HC}(W(\mathcal{R}))$ to $\mathsf{Hecke}_{k,q}(W(\mathcal{R}))$ via the following commutative diagrams, for $I \subseteq J \subseteq S$,

$$\begin{array}{ccc} kL_I\text{-mod} & \xrightarrow{\mathrm{Hom}(kL_I/B_I,-)} & \mathscr{H}_I(q)\text{-mod} \\ {\scriptstyle R_{L_I}^{L_J}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}_{\mathscr{H}_I(q)}^{\mathscr{H}_J(q)}} \\ kL_J\text{-mod} & \xrightarrow[\mathrm{Hom}(kL_J/B_J,-)]{} & \mathscr{H}_J(q)\text{-mod} \end{array}$$

and

$$\begin{array}{ccc} kL_J\text{-mod} & \xrightarrow{\mathrm{Hom}(kL_J/B_J,-)} & \mathscr{H}_J(q)\text{-mod} \\ {\scriptstyle ^*R_{L_I}^{L_J}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{\mathscr{H}_I(q)}^{\mathscr{H}_J(q)}} \\ kL_I\text{-mod} & \xrightarrow[\mathrm{Hom}(kL_I/B_I,-)]{} & \mathscr{H}_I(q)\text{-mod}. \end{array}$$

Furthermore, a ring morphism $\varphi\colon \tilde{R} \to k$ such that $\varphi(\tilde{q}) = q$ induces a functor $\mathscr{G}$ from $\mathsf{Hecke}_{\tilde{R},\tilde{q}}(W(\mathcal{R}))$ to $\mathsf{Hecke}_{k,q}(W(\mathcal{R}))$ via specializing $\tilde{q}$ to q. This gives the diagram

$$\begin{array}{ccc} & & \mathsf{Hecke}_{\tilde{R},\tilde{q}}(W(\mathcal{R})) \\ & & \downarrow{\scriptstyle \mathscr{G}} \\ \mathsf{HC}(W(\mathcal{R})) & \xrightarrow[\mathscr{F}]{} & \mathsf{Hecke}_{k,q}(W(\mathcal{R})). \end{array}$$

To the root datum $\mathcal{R}$, we wish to construct an $\tilde{R}$-linear 2-category $\mathscr{S}(\mathcal{R})$ yielding a diagram

$$\begin{array}{ccc} \mathscr{S}(\mathcal{R}) & \longrightarrow & \mathsf{Hecke}_{\tilde{R},\tilde{q}}(W(\mathcal{R})) \\ \downarrow & & \downarrow \mathscr{G} \\ \mathsf{HC}(W(\mathcal{R})) & \xrightarrow[\mathscr{F}]{} & \mathsf{Hecke}_{k,q}(W(\mathcal{R})) \end{array}$$

where the functor $\mathscr{S}(\mathcal{R}) \longrightarrow \mathsf{HC}(W(\mathcal{R}))$ is given by specialization.

The 2-category $\mathscr{S} := \mathscr{S}(\mathcal{R})$ should consist of the following data. To the root datum $\mathcal{R}$, there is an associated finite Coxeter system $(W(\mathcal{R}), S)$, where S is a fixed set of generators. Briefly, the 2-category $\mathscr{S}$ has objects subsets of S. The 1-morphisms are generated by the following: given subsets $I \subseteq J \subseteq S$, there is a pair of biadjoint arrows $F_I^J : I \rightleftarrows J : E_I^J$, and in cases where $w \in W$ and ${}^wI \subseteq S$, isomorphisms $\Phi_{I,w} \colon I \xrightarrow{\sim} {}^wI$. Additionally, there are 2-morphisms which encode some coherence conditions amongst the generating 1-morphisms. Precise definitions will follow in the body.

Suppose that a Borel subgroup B of G is chosen containing the torus T. Since G is split, the factorization $B = U \rtimes T$ gives a unipotent group U. Within the category $\mathsf{HC}(W(\mathcal{R}))$, there are objects $L_\emptyset = T$, $L_S = G$, and the morphism kG/U viewed as a (kG, kT)-bimodule. The 2-endomorphism algebra of this morphism is

$$\mathscr{Y}(q) := \mathrm{End}_{(kG,kT)}(kG/U) = \mathrm{End}_{kG}(kG/U)^T,$$

which can be viewed as the subalgebra of the (opposite) Yokonuma-Hecke algebra $\mathrm{End}_{kG}(kG/U)$ fixed under the conjugation action induced by T. The functor $\mathscr{F}$ sends kG/U to the $(\mathscr{H}_S(q), \mathscr{H}_\emptyset(q))$-bimodule $\mathscr{H}_S(q)$, and so $\mathscr{F}(\mathscr{Y}(q))$ is a subalgebra of $\mathrm{End}_{(\mathscr{H}_S(q),\mathscr{H}_\emptyset(q))}(\mathscr{H}_S(q)) =: \mathscr{H}$ in $\mathsf{Hecke}_{k,q}(W(\mathcal{R}))$. This endomorphism algebra $\mathscr{H}$ is the image of the analogous algebra $\tilde{\mathscr{H}} = \mathrm{End}_{(\mathscr{H}_S(\tilde{q}),\mathscr{H}_\emptyset(\tilde{q}))}(\mathscr{H}_S(\tilde{q}))$ in $\mathsf{Hecke}_{\tilde{R},\tilde{q}}(W(\mathcal{R}))$, which is sent to $\mathscr{H}$ by the functor $\mathscr{G}$. This indicates that the category $\mathscr{S}$ should contain some 2-endomorphism algebra $A_\emptyset^S$ fitting into a diagram

$$\begin{array}{ccc} A_\emptyset^S & \longrightarrow & \tilde{\mathscr{H}} \\ \downarrow & & \downarrow \\ \mathscr{Y}(q) & \longrightarrow & \mathscr{H} \end{array}$$

where each arrow is given by the application of the previously introduced functors on the level of 2-morphisms. So far, a complete construction is given only in the case where $|S| = 1$.

For a Coxeter system (W, S) arising from a Chevalley group defined over a field of q elements, the corresponding Yokonuma-Hecke algebra, introduced by Yokonuma in [14], has generators indexed by the elements of S, and others by elements of the torus, and hence has dimension dependent on q. This necessitates some algebra such as A above, which can be defined in terms of a generic parameter $\tilde{q}$, which is not possible with $\mathscr{Y}(q)$.

Lastly, a brief comment on the layout of this thesis. In Section 2 we recall some of the basic algebraic machinery, including proper definitions of the types of Hecke algebras mentioned above. Section 3 investigates some motivating examples concerning typical groups such as $SL_2(q)$ and $GL_2(q)$. For convenience, Section 4 provides a complete and slightly modified definition of Dreyfus-Schmidt's W-categorification, as this definition first appeared in [7], which is unpublished as of this writing. Section 5 defines a 2-categorical setting centered around a 2-category $\mathcal{C}'$, with a biadjoint pair of 1-morphisms, and explicitly constructs endomorphism algebras in the case of rank 1. The first main result is Theorem 5.26, which determines a description of the space of 2-morphisms between any parallel 1-morphisms. Roughly, the Mackey decomposition axiom of a W-categorification yields a decomposition of any 1-morphism as a direct sum of indecomposable 1-morphisms, of which there are only finitely many. A subspace of the space of 2-morphisms between indecomposable 1-morphisms is simply chosen, and then an algorithmic process allows one to lift these choices to a subspace of the space of 2-morphisms between arbitrary 1-morphisms. Some extensive case checking shows these potentially proper subspaces are actually the full spaces of 2-morphisms in the generated 2-category. Subsequently, in Theorem 5.28, a 2-functor is constructed from $\mathcal{C}'$ into the 2-category of bimodules, with image a nontrivial sub-2-category, showing that the 2-category $\mathcal{C}$ and a particular quotient are themselves nontrivial. Lastly, Section 6 proposes a candidate algebra for the endomorphism algebra for larger rank.

2 Background

In this section, we briefly recall some definitions and basic theorems which will be used throughout.

2.1 Groups with BN-pair

Definition 2.1. An abstract group G is said to be a group with a BN-pair if it contains subgroups $B, N \subseteq G$ such that the following conditions hold:

1. $G = \langle B, N \rangle$
2. $T := B \cap N$ is a normal subgroup of N, and the quotient group $W := N/T$ is generated by a set S of elements of order 2
3. $\dot{s}B\dot{s} \neq B$, where $s \in S$, and $\dot{s}$ denotes a representative of S in N
4. $\dot{s}Bn \subseteq B\dot{s}nB \cup BnB$ for any $s \in S$ and $n \in N$
5. $\bigcap_{n \in N} nBn^{-1} = T$.

In addition to writing $\dot{w}$ for a preimage in N of $w \in W$, we will occasionally use the notation n_w.

Definition 2.2. A group G with a BN-pair is said to have a split BN-pair if there is a normal subgroup $U \trianglelefteq B$ such that

1. For $T := B \cap N$, $B = UH$ and $U \cap T = \{1\}$. That is, $B = U \rtimes T$.
2. For any $n \in N$, $n^{-1}Un \cap B \subseteq U$.

Theorem 2.3. *(Bruhat decomposition) A group G with BN-pair has the double-coset decomposition*

$$G = \bigsqcup_{w \in W} Bn_wB.$$

Theorem 2.4. *Let G be a group with a split BN-pair such that W is finite. Let w_0 denote the longest element of W. For $w \in W$, put*

$$U_w := U \cap n_{w_0 w}^{-1} U n_{w_0 w}.$$

Any $g \in Bn_wB$ has a unique expression of form $g = bn_wu$, for $b \in B$, $w \in W$, and $u \in U_w$. Hence

$$G = \bigsqcup_{w \in W} Bn_wU_w.$$

Proofs of the above two theorems can either be found in [8] or [6], for instance.

2.2 Symmetric Algebras

Let R be a commutative, unital ring, and let A be an R-algebra. A morphism $t \in \mathrm{Hom}_R(A, R)$ is said to be a central form if $t(ab) = t(ba)$ for all $a, b \in A$. Such t induces an (A, A)-bimodule morphism

$$\hat{t}\colon A \longrightarrow \mathrm{Hom}_R(A, R) : t \mapsto \hat{t}_a$$

where $\hat{t}_a(b) = t(ab)$ for all $b \in A$. Also, for any R-module M, let $M^* := \mathrm{Hom}_R(M, R)$.

Definition 2.5. An R-algebra is said to be symmetric if A is finitely generated and projective as an R-module, and additionally if there exists a central form $t \in A^*$ such that $\hat{t}\colon A \longrightarrow A^*$ is an isomorphism of (A, A)-bimodules.

Such a form t above is called a symmetrizing form on A. In the following definitions, A and B are symmetric algebras.

Definition 2.6. An (A, B)-bimodule M is said to be exact if M is finitely generated and projective as a left A-module, and as a right B-module.

Definition 2.7. If M is an exact (A, B)-bimodule and N is an exact (B, A)-bimodule, then the pair (M, N) is said to be a selfdual pair of exact bimodules if there is an R-bilinear map $\langle\ ,\ \rangle\colon M \times N \longrightarrow R$ such that $\langle amb, n\rangle = \langle m, bna\rangle$ for all $m \in M$, $n \in N$, $a \in A$, and $b \in B$, and furthermore this map induces bimodule isomorphisms

$$M \longrightarrow N^* : m \mapsto \langle m, -\rangle \quad \text{and} \quad N \longrightarrow M^* : n \mapsto \langle -, n\rangle.$$

More details concerning symmetric algebras can be found in Chapter 2 of [5]. The theory of symmetric algebras plays a role in this paper as the group algebra RG of a finite group G over a ring R is always a symmetric algebra. The canonical symmetrizing form on RG is the projection

$$RG \to R : \sum_{g\in G} r_g g \mapsto r_e$$

sending a formal sum to the coefficient of the identity element $e \in G$.

2.3 Hecke Algebras

2.3.1 Generic Iwahori-Hecke Algebras

Let (W, S) be a finite Coxeter system with Coxeter matrix $M = (m_{st})_{s,t\in S}$. Let $\mathbf{q} := \{q_s\}_{s\in S}$ be a family of indeterminates such that $q_s = q_t$ whenever s and t are conjugate in W.

Definition 2.8. A generic Iwahori-Hecke algebra $\mathscr{H}_{\mathbf{q}}(W,S)$ associated to the Coxeter system (W,S) is the $\mathbb{Z}[q_s, q_s^{-1} : s \in S]$-algebra generated by elements $\{T_s\}_{s\in S}$ subject to the following two relations, referred to as the quadratic and braid relations, respectively:

- $(T_s - q_s)(T_s + 1) = 0$ for all $s \in S$,
- $T_sT_tT_s \cdots = T_tT_sT_t \cdots$ when $sts \cdots = tst \cdots$ in W.

For any $s \in S$, the quadratic relation may be expanded as $T_s^2 = (q_s - 1)T_s + q_s$, so that T_s is invertible in $\mathscr{H}_{\mathbf{q}}(W,S)$ with inverse $T_s^{-1} = q_s^{-1}T_s + q^{-1}(1-q)$. Furthermore, if $w \in W$ has a reduced expression $w = s_1 \dots s_r$, define $T_w := T_{s_1} \cdots T_{s_r}$. From the braid relations, Matsumoto's lemma implies that this expression is independent of the chosen reduced expression of w.

Theorem 2.9. *The set $\{T_w\}_{w\in W}$ constitute a $\mathbb{Z}[q_s, q_s^{-1} : s \in S]$-basis of $\mathscr{H}_{\mathbf{q}}(W,S)$.*

Proof. See Theorem 4.4.6 of [9]. ☐

2.3.2 Unipotent Hecke Algebras

Suppose G is a finite Chevalley group, and U is a maximal unipotent subgroup of G. Let $\chi \colon U \longrightarrow \mathbb{C}^\times$ be a linear character. The unipotent Hecke algebra $\mathscr{H}(G,U,\chi)$ is the endomorphism algebra

$$\mathscr{H}(G,U,\chi) = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\chi)).$$

Let $e_\chi = \frac{1}{|U|}\sum_{u\in U} \chi(u^{-1})u$ be the idempotent in $\mathbb{C}G$. Since $\mathrm{Ind}_U^G(\chi)$ is afforded by the $\mathbb{C}G$-module $\mathbb{C}Ge_\chi$, there is the standard isomorphism $\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\chi))^{\mathrm{op}} \simeq e_\chi\mathbb{C}Ge_\chi$ of algebras. In this way, we will often identify $\mathscr{H}(G,U,\chi)$ with $e_\chi\mathbb{C}Ge_\chi$. Of particular interest is the case where $\chi = 1_U$, the trivial character on U. In this case, the unipotent Hecke algebra $\mathscr{H}(G,U,1_U)$ is referred to as the Yokonuma-Hecke algebra.

3 Preliminary Observations and Examples

3.1 Fixed Points and Orbit Sums

Let k be a field, and let G be a group with split BN-pair. In this section, we will view (kG, kT)-bimodules as modules over $k[G \times T]$, with T acting on the right. The group ring $k[G/U]$ is then a $k[G \times T]$-module via the left and right translation actions of G and T, respectively. Let

$$\Delta(T) = \{(t, t^{-1}) \in G \times T : t \in T\}.$$

The fixed points of $k[G/U]$ under the action of the subgroup $H := (U \times \{1\}) \rtimes \Delta T$, denoted $k[G/U]^H$, determine the maps in $\mathrm{End}_{k[G\times T]}(k[G/U])$ as follows.

Proposition 3.1. *Let G be a group with split BN-pair as above. There is a bijection*

$$k[G/U]^H \longrightarrow \mathrm{End}_{k[G\times T]}(k[G/U]) : x \mapsto (U \mapsto x).$$

Proof. Let $H := (U \times \{1\}) \rtimes \Delta T$. First, there is a bijection

$$k[G/U]^H \longrightarrow \mathrm{Hom}_{kH}(k, \mathrm{Res}^{k[G\times T]}_{kH} k[G/U]) : x \mapsto \varphi_x$$

where k is the trivial kH-reprsentation, and φ_x is defined by $\varphi_x(1) = x$. This assignment is injective since a kH-map $k \to \mathrm{Res}^{k[G\times T]}_{kH} k[G/U]$ is determined by its image on 1. If φ is a kH-map, then $\varphi(1)$ is fixed under H, since $(ut, t^{-1}) \cdot \varphi(1) = \varphi((ut, t^{-1}) \cdot 1) = \varphi(1)$, and hence this assigment is surjective.

Second, there is a bijection

$$\mathrm{Hom}_{kH}(k, \mathrm{Res}^{k[G\times T]}_{kH} k[G/U]) \longrightarrow \mathrm{Hom}_{kH}(k, \mathrm{Hom}_{k[G\times T]}(k[G \times T], k[G/U]))$$

induced by the usual isomorphism of kH-modules $\mathrm{Hom}_{k[G\times T]}(k[G \times T], k[G/U]) \simeq \mathrm{Res}^{k[G\times T]}_{kH}(k[G/U])$, determined by sending a morphism f to its value $f(1)$, and conversely, sending $x \in k[G/U]$ to the $k[G \times T]$-map mapping 1 to x.

Third, the usual adjunction gives a bijection

$$\mathrm{Hom}_{kH}(k, \mathrm{Hom}_{k[G\times T]}(k[G \times T], k[G/U])) \simeq \mathrm{Hom}_{k[G\times T]}(k[G \times T] \otimes_{kH} k, k[G/U]) : \varphi \mapsto [a \otimes b \mapsto (\varphi(b))(a)].$$

Tracing through these bijections, a point $x \in k[G/U]^H$ determines a kH-map $\varphi_x \colon k \to k[G/U]$ such that

$\varphi_x(1) = x$. This corresponds to a kH-map

$$\tilde{\varphi_x}\colon k \longrightarrow \mathrm{Hom}_{k[G\times T]}(k[G\times T], k[G/U]) : 1 \mapsto (1 \mapsto x).$$

The adjunction then gives a $k[G \times T]$-map

$$\tau(\tilde{\varphi}_x)\colon k[G\times T] \otimes_{kH} k \longrightarrow k[G/U] : a \otimes b \mapsto [\tilde{\varphi}_x(b)](a).$$

Note

$$[\tilde{\varphi}_x(b)](a) = (b \cdot \tilde{\varphi}_x(1))(a) = [\tilde{\varphi}_x(1)](ab) = ab \cdot [\tilde{\varphi}_x(1)](1) = ab \cdot x.$$

Furthermore, as $k[G \times T]$-modules,

$$k[G\times T] \otimes_{kH} k = \mathrm{Ind}_{kH}^{k[G\times T]} k \simeq k[(G\times T)/H] \simeq k[G/U]$$

where the first isomorphism is given by $(g, t) \otimes 1 \mapsto (g, t)H$, and the second is given by $(g, t)H \mapsto gtU$. Identifying $k[G\times T]\otimes_{kH} k$ with $k[G/U]$, one can view $\tau(\tilde{\varphi}_x)$ as a morphism in $\mathrm{Hom}_{k[G\times T]}(k[G/U], k[G/U])$ defined by

$$\tau(\tilde{\varphi}_x)(U) = \tau(\tilde{\varphi}_x((1,1) \otimes 1) = (1,1) \cdot x = x.$$

□

Suppose now that k is a field such that $|U|$ is invertible in k. As noted before, let

$$e_U := \frac{1}{|U|} \sum_{u \in U} u \in kG.$$

There is an isomorphism of $k[G \times T]$-modules $k[G/U] \simeq kGe_U$ given by $gU \leftrightarrow ge_U$, and so

$$\mathrm{End}_{k[G\times T]}(k[G/U]) \simeq \mathrm{End}_{k[G\times T]}(kGe_U).$$

Since e_U is an idempotent in kG, there is the standard anti-isomorphism of kG-modules

$$\mathrm{End}_{kG}(kGe_U) \longrightarrow e_U kGe_U : \varphi \mapsto e_U\varphi(e_U)e_U.$$

This anti-isomorphism then sends the subalgebra $\mathrm{End}_{k[G\times T]}(kGe_U)$ of $\mathrm{End}_{kG}(kGe_U)$ to a subalgebra of the Yokonuma-Hecke algebra $e_U kGe_U$. As noted before, a fixed point $x \in K[G/U]^{(U\times\{1\})\rtimes\Delta T}$ determines a

$k[G \times T]$-endomorphism τ_x on kGe_U, where if $x = \sum_i c_i x_i U$, for $x_i \in G$, $c_i \in k$, then τ_x is determined by $\tau_x(e_U) = \sum_i c_i e_U x_i e_U$ in kGe_U. In particular, the image of $\mathrm{End}_{k[G\times T]}(kGe_U)$ in $e_U kGe_U$ is the points $e_U \tau_x(e_U) e_U$, for $x \in k[G/U]^{(U\times\{1\})\rtimes\Delta T}$.

The fixed points in $k[G/U]$ under the action of $(U \times \{1\}) \rtimes \Delta T$ are precisely the orbit sums of an element in G/U. Since $B = TU$, from the refined Bruhat decomposition it follows that if gU is a coset in $k[G/U]$, $g = hu\dot{w}v$ for unique $h \in T$, $u \in U$, $w \in W$, and $v \in U_w$, so that $gU = hu\dot{w}vU = hu\dot{w}U$. With a view towards algebraic groups, from now on assume the subgroup T is abelian.

Proposition 3.2. *Let G be a group with split BN-pair, with $B = U \rtimes T$ such that $T := B \cap N$ is abelian. The orbit of a coset $hu\dot{w}U \in G/U$ in $k[G/U]$ for $h \in T$, $u \in U$, $w \in W$ under the action of $(U \times \{1\}) \rtimes T$ is*

$$[T, \dot{w}]hU\dot{w}U := \{[t, \dot{w}]hv\dot{w}U : t \in T, v \in U\}.$$

Proof. Let $(t, t^{-1})(u', 1)$ be an arbitrary element of $(U \times \{1\}) \rtimes \Delta T$. Since T normalizes U, $u'h = hu''$ for some $u'' \in U$. Then observe

$$\begin{aligned}(t, t^{-1})(u', 1) \cdot hu\dot{w}U &= tu'hu\dot{w}Ut^{-1} = thu''u\dot{w}t^{-1}U \\ &= thu''u'(\dot{w}t^{-1}\dot{w}^{-1})\dot{w}U = t(\dot{w}t^{-1}\dot{w}^{-1})hu'''\dot{w}U \quad \text{for some } u''' \in U \\ &= [t, \dot{w}]hu'''\dot{w}U\end{aligned}$$

where the penultimate equality follows since $u''u' \in U$, and $\dot{w}t^{-1}\dot{w}^{-1} \in T$. Hence the orbit of $hu\dot{w}U$ is contained in $\{[t, \dot{w}]hu\dot{w}U : t \in T, u \in U\}$.

Conversely, suppose $[t, \dot{w}]hv\dot{w}U$ for $t \in T$, $v \in U$. As before, there exists $u' \in U$ such that $u'h = hvu^{-1}$. Then

$$(u', 1) \cdot hu\dot{w}U = u'hu\dot{w}U = hvu^{-1}u\dot{w}U = hv\dot{w}U.$$

Hence the action of $U \times \{1\}$ shows that any point of form $hv\dot{w}U$ is in the orbit of $hu\dot{w}U$. Likewise, there exists $u'' \in U$ such that $u''(\dot{w}t^{-1}\dot{w}^{-1}) = (\dot{w}t^{-1}\dot{w}^{-1})v$. The previous computation shows

$$(t, t^{-1})(hu''\dot{w}U) = [t, \dot{w}]hv\dot{w}U.$$

□

Note that since T is abelian, for any $\dot{w} \in N$ the set of commutators $[T, \dot{w}] := \{[t, \dot{w}] : t \in T\}$ is a subgroup of T. Indeed, if $a, b \in T$, then

$$[a, \dot{w}][b, \dot{w}] = a(\dot{w}a^{-1}\dot{w}^{-1})b(\dot{w}a^{-1}\dot{w}^{-1}) = ab(\dot{w}a^{-1}\dot{w}^{-1})(\dot{w}a^{-1}\dot{w}^{-1}) = ab\dot{w}a^{-1}b^{-1}\dot{w}^{-1} = [ab, \dot{w}]$$

and

$$[a, \dot{w}]^{-1} = [\dot{w}, a] = (\dot{w}a\dot{w}^{-1})a^{-1} = a^{-1}\dot{w}a\dot{w}^{-1} = [a^{-1}, \dot{w}].$$

Corollary 3.3. *Let G be a group with split BN-pair, such that $T := B \cap N$ is abelian. The $(U \times \{1\}) \rtimes \Delta T$ orbits in $k[G/U]$ are paramterized by $\bigsqcup_{w \in W} T/[T, \dot{w}]$, where a coset $[T, \dot{w}]h$ determines the orbit*

$$[T, \dot{w}]hU\dot{w}U = \{[t, \dot{w}]hu\dot{w}U : t \in T,\ u \in U\}.$$

Proof. Certainly if $[T, \dot{w}]h = [T, \dot{w}]h'$, then these cosets determine the same orbit. Conversely, suppose $[T, \dot{w}]hU\dot{w}U = [T, \dot{w'}]h'U\dot{w'}U$ are equal orbits. It follows that

$$[t, \dot{w}]hu\dot{w} = [t', \dot{w'}]h'u'\dot{w'}v$$

for some $v \in U$, and the other elements are in the obvious subgroups. These elements are in the double cosets BwB and $Bw'B$, so by the Bruhat decomposition $w = w'$.

For clearer notation, let n_w also denote a preimage of $w \in W$ in N, i.e., $\dot{w} = n_w$, and let w_0 denote the longest element of W. Set $U_w = U \cap n_{w_0 w}^{-1} U n_{w_0 w}$. From the factorization $U = U_{w_0 w} U_w$, one can express $v \in U$ as $v = (n_w^{-1} v' n_w)(n_{w_0 w}^{-1} v'' n_{w_0 w}) \in U_{w_0 w} U_w$, for some $v', v'' \in U$. Then

$$\begin{aligned}
[t, n_w]hun_w &= [t', n_w]h'u'n_w v \\
&= [t', n_w]h'u'n_w(n_w^{-1}v'n_w)(n_{w_0 w}^{-1}v''n_{w_0 w}) \\
&= [t', n_w]h'u'v'n_w(n_{w_0 w}^{-1}v''n_{w_0 w}).
\end{aligned}$$

Now $[t, n_w]hun_w \in Bn_w U_w$ and $[t', n_w]h'u'v'n_w(n_{w_0 w}^{-1}v''n_{w_0 w}) \in Bn_w U_w$, so by uniqueness of expression, $[t, n_w]hu = [t', n_w]h'u'v'$ in B. Since $B = T \ltimes U$, uniqueness of expression in B further implies $[t, n_w]h = [t', n_w]h'$, so that $[T, \dot{w}]h = [T, \dot{w}]h'$. ☐

To summarize, suppose G is a group with split BN-pair, such that $T := B \cap N$ is abelian, and $B = UT$. For each $w \in W$, a coset in $[T, \dot{w}]h$ in $T/[T, \dot{w}]$ determines a $(U \times \{1\}) \rtimes \Delta T$-orbit in G/U, and the sum of this orbit gives a fixed point x in $k[G/U]$. This fixed point x determines a $k[G \times T]$-endomorphism of $k[G/U]$, defined by sending the trivial coset U to x. If $x = \sum_i c_i x_i U \in k[G/U]$, for $c_i \in k$, and $x_i \in G$, this endomorphism corresponds to an element $e_U\left(\sum_i c_i x_i e_U\right) e_U$ in $e_U kGe_U$, and the subset of such elements forms a subalgebra isomorphic to $\mathrm{End}_{k[G\times T]}(k[G/U])$.

3.2 Some Examples with Small Groups

Example 3.4. Suppose q is an even prime power, and $G = SL_2(q)$, the special linear group with entries in the field $\mathbb{F}_q$ of q elements. Suppose k is a field of approriate characteristic such that q and $q-1$ are invertible in k. Let $T = \{\mathrm{diag}(a, a^{-1}) : a \in k^\times\}$, and U is the set of upper unitriangular matrices. The roots of G are $\Phi = \{\pm\alpha\}$ where $\alpha\colon T \longrightarrow \mathbb{F}_q^\times : \mathrm{diag}(a, a^{-1}) \mapsto a^2$ and the coroots are $\Phi^\vee = \{\pm\alpha^\vee\}$ where

$$\alpha^\vee : \mathbb{F}_q^\times \longrightarrow T : a \mapsto \mathrm{diag}(a, a^{-1}).$$

The Weyl group $W \simeq \mathbb{Z}/2\mathbb{Z} = \{1, s\}$, where $\dot{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\bigsqcup_{w\in W} T/[T, \dot{w}] = T/[T, 1] \sqcup T/[T, \dot{s}] = T \sqcup T/[T, \dot{s}].$$

Computation shows

$$[\mathrm{diag}(a, a^{-1}), \dot{s}] = \mathrm{diag}(a^2, a^{-2}).$$

Hence $[T, \dot{s}] = \{t^2 : t \in T\}$, and $[T, \dot{s}]$ is in bijection with the set of squares in $\mathbb{F}_q^\times$.

Since q is even, every element of $\mathbb{F}_q^\times$ is a square, and thus $[T, \dot{s}] = T$. So the $(U \times \{1\} \rtimes \Delta T$-fixed points of $k[G/U]$ are in bijection with $T \sqcup \{1\}$. If $h \in T$, the corresponding orbit is

$$[T, 1]huU : t \in T,\ u \in U\} = \{hU : t \in T, u \in U\} = \{hU\},$$

and the fixed point in $k[G/U]$ is the coset hU. This corresponds to the element $he_U = e_U he_U \in e_U kGe_U$. The coset $[T, \dot{s}] = T$ in $T/[T, \dot{s}]$ has orbit

$$\{[t, \dot{s}]u\dot{s}U : t \in T, u \in U\} = \{tu\dot{s}U : t \in T,\ u \in U\}$$

Since G is of rank 1, $U = U_s$, so we have uniqueness of expression, and the corresponding orbit sum is

$$x = \sum_{t\in T,\ u\in U} tu\dot{s}U.$$

The corresponding point in $e_U kGe_U$ is then

$$e_U \left(\sum_{t\in T,\ u\in U} tu\dot{s}e_U \right) e_U = q(q-1)e_U e_T \dot{s} e_U.$$

For $t \in T$, put $b_t = e_U t e_U$, and put $b_s = e_U e_T \dot{s} e_U$. Then the $\{b_t\}_{t\in T}$ and b_s generate a subalgebra in $e_U kGe_U$ isomorphic to $\mathrm{End}_{k[G\times T]}(k[G/U])$.

One can compute multiplication relations among these generators. To do so, recall that the Yokonuma-Hecke algebra $e_U kGe_U$ has basis $\{T_w : n \in N\}$, where $T_w = e_U \dot{w} e_U$, with relations

$$T_{\dot{s}} T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w)+1, \\ q^{-1} T_{\alpha_s^\vee(-1)sw} + q^{-1} \sum_{a\in\mathbb{F}_q^\times} T_{\alpha_s^\vee(a)w} & \text{if } \ell(sw) = \ell(w)-1. \end{cases}$$

$$T_h T_w = T_{hw}, \qquad h \in T,\ w \in W$$

and

$$T_h T_k = T_{hk}, \qquad h, k \in T.$$

In particular,

$$(e_U \dot{s} e_U)^2 = q^{-1} e_U \alpha^\vee(-1) e_U + q^{-1} e_U \dot{s} e_U \sum_{a\in\mathbb{F}_q^\times} e_U \alpha^\vee(a) e_U = q^{-1} e_U + q^{-1}(q-1) e_U \dot{s} e_T e_U$$

since $\alpha^\vee(-1) = -I_2 = I_2$ in characteristic 2, and $\alpha^\vee(a) = \mathrm{diag}(a, a^{-1})$, so that $\sum_{a\in\mathbb{F}_q^\times} \alpha^\vee(a) = \sum_{t\in T} t$.

The generators b_t for $t \in T$, and b_s then satisfy the following relations.

- $b_t b_{t'} = b_{t'} b_t = b_{tt'}$ for $t, t' \in T$.
- $b_t b_s = b_s b_t = b_s$ for $t \in T$.
- $b_s^2 = q^{-1}(q-1)\sum_{t\in T} b_t + q^{-1}(q-1) b_s$.

  Note

$$\begin{aligned} b_s^2 &= (e_U e_T \dot{s} e_U)^2 = e_T (e_U \dot{s} e_U)^2 = e_T (q^{-1} e_U + q^{-1}(q-1) e_U \dot{s} e_T e_U) \\ &= q^{-1} e_U e_T e_U + q^{-1}(q-1) e_U e_T \dot{s} e_U \\ &= q^{-1}(q-1) \sum_{t\in T} b_t + q^{-1}(q-1) b_s. \end{aligned}$$

Example 3.5. Suppose that $G = SL_2(q)$, with q an odd prime power instead. By the previous example, $[T, \dot{s}]$ is the set of squares in T and is in bijection with the squares of $\mathbb{F}_q^2$. Since q is odd, half the elements of $\mathbb{F}_q^\times$ are squares, so $T/[T,\dot{s}] \simeq \mathbb{Z}/2\mathbb{Z}$.

As before, an element $t \in T$ corresponds to an element $e_U t e_U \in e_U kG e_U$. For the coset $[T,\dot{s}]$ in $T/[T,\dot{s}]$, one gets an orbit

$$[T,\dot{s}]U\dot{s}U := \{[t,\dot{s}]u\dot{s}U : t \in T,\ u \in U\} = \{t^2 u\dot{s}U : t \in T,\ u \in U\}.$$

This corresponds to the element

$$e_U \left(\sum_{\substack{t \in T \text{ a square} \\ u \in U}} tu\dot{s}e_U \right) e_U = q \left(\sum_{t \in T \text{ a square}} t \right) e_U \dot{s} e_U.$$

The other coset is $[T,\dot{s}]h$, where h is not a square in T. The corresponding orbit is then

$$\{tu\dot{s}U : t \notin T^2,\ u \in U\},$$

and the corresponding element in $e_U kG e_U$ is

$$q \left(\sum_{t \in T \text{ a nonsquare}} t \right) e_U \dot{s} e_U.$$

Write

$$b_{s,+} = \sum_{t \in T \text{ square}} t e_U \dot{s} e_U \qquad \text{and} \qquad b_{s,-} = \sum_{t \in T \text{ nonsquare}} t e_U \dot{s} e_U.$$

Then the subalgebra of $e_U kG e_U$ isomorphic to $\mathrm{End}_{k[G\times T]}(k[G/U])$ is generated by b_t for $t \in T$, $b_{s,+}$ and $b_{s,-}$. The multiplication relations between generators depends on whether -1 is a square in $\mathbb{F}_q$, and hence on whether $q \equiv 1, 3 \pmod 4$.

In the first case, suppose $q \equiv 1 \pmod 4$, so that -1 is a square in $\mathbb{F}_q$. There are relations

- $b_t b_{t'} = b_{t'} b_t = b_{tt'}$ for $t, t' \in T$.
- $b_{s,+} b_t = b_t b_{s,+} = e_U t e_U \left(\sum_{a \in T \text{ square}} a \right) e_U \dot{s} e_U = \sum_{a \in T \text{ square}} ta e_U \dot{s} e_U = \begin{cases} b_{s,-} & \text{if } t \text{ nonsquare}, \\ b_{s,+} & \text{if } t \text{ square}. \end{cases}$
- $b_{s,-} b_t = b_t b_{s,-} = \begin{cases} b_{s,-} & \text{if } t \text{ square}, \\ b_{s,+} & \text{if } t \text{ nonsquare}. \end{cases}$
- $(b_{s,+})^2 = \frac{q^{-1}(q-1)}{2} \sum_{t \in T \text{ square}} b_t + \frac{q^{-1}(q-1)^2}{4} b_{s,+} + \frac{q^{-1}(q-1)^2}{4} b_{s,-}$.

Note

$$
\begin{aligned}
(b_{s,+})^2 &= \sum_{a\in T \text{ square}} a e_U \dot{s} e_U \cdot \sum_{b\in T \text{ square}} e_U \dot{s} e_U = \frac{q-1}{2} \sum_{a\in T \text{ square}} a(e_U\dot{s}e_U)^2 \\
&= \frac{q-1}{2} \sum_{a\in T \text{ square}} a(q^{-1}e_U\alpha^\vee(-1)e_U + q^{-1}e_U\dot{s}e_U \sum_{c\in\mathbb{F}_q^\times} e_U\alpha^\vee(c)e_U) \\
&= \frac{q-1}{2} \sum_{a\in T \text{ square}} a(q^{-1}e_U(-I_2)e_U + q^{-1}\left(\sum_{t\in T} t\right) e_U\dot{s}e_U) \\
&= \frac{q^{-1}(q-1)}{2} \sum_{a\in T \text{ square}} (-a) \left[e_U + \left(\sum_{b\in T \text{ square}} b + \sum_{c\in T \text{ nonsquare}} c \right) e_U\dot{s}e_U \right] \\
&= \frac{q^{-1}(q-1)}{2} \sum_{a\in T \text{ square}} a e_U + \frac{q^{-1}(q-1)^2}{4} \sum_{b\in T \text{ square}} b e_U\dot{s}e_U + \frac{q^{-1}(q-1)^2}{4} \sum_{c\in T \text{ nonsquare}} c e_U\dot{s}e_U \\
&= \frac{q^{-1}(q-1)}{2} \sum_{t\in T \text{ square}} b_t + \frac{q^{-1}(q-1)^2}{4} b_{s,+} + \frac{q^{-1}(q-1)^2}{4} b_{s,-}.
\end{aligned}
$$

- Similarly,

$$
(b_{s,-})^2 = \frac{q^{-1}(q-1)}{2} \sum_{t\in T \text{ nonsquare}} b_t + \frac{q^{-1}(q-1)^2}{4} b_{s,-} + \frac{q^{-1}(q-1)^2}{4} b_{s,+}.
$$

- $b_{s,-}b_{s,+} = \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{ nonsquare}} b_t + \frac{q^{-1}(q-1)}{4} b_{s,+} + \frac{q^{-1}(q-1)}{4} b_{s,-}$.

  Note

$$
\begin{aligned}
b_{s,-}b_{s,+} = b_{s,+}b_{s,-} &= \sum_{a\in T \text{ square}} e_U\dot{s}e_U \cdot \sum_{b\in T \text{ nonsquare}} b e_U\dot{s}e_U = \sum_{\substack{a\in T \text{ square} \\ b\in T \text{ nonsquare}}} ab(e_U\dot{s}e_U)^2 \\
&= \frac{q-1}{2} \sum_{b\in T \text{ nonsquare}} b(q^{-1}e_U\alpha^\vee(-1)e_U + q^{-1}e_U\dot{s}e_U \sum_{a\in\mathbb{F}_q^\times} e_U\alpha^\vee(a)e_U) \\
&= \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{ nonsquare}} (-b)e_U + \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{nonsquare}} b \sum_{t\in T} t e_U\dot{s}e_U \\
&= \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{ nonsquare}} b e_U + \frac{q^{-1}(q-1)^2}{4} \sum_{t\in T} t e_U\dot{s}e_U \\
&= \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{ nonsquare}} b_t + \frac{q^{-1}(q-1)}{4} e_U e_T \dot{s}e_U \\
&= \frac{q^{-1}(q-1)}{2} \sum_{b\in T \text{ nonsquare}} b_t + \frac{q^{-1}(q-1)}{4} b_{s,+} + \frac{q^{-1}(q-1)}{4} b_{s,-}.
\end{aligned}
$$

If $q \equiv 3 \pmod 4$, then -1 is not a square in $\mathbb{F}_q$. The last three relations change slightly, as multiplication by -1 swaps the sum of squares in T to the sum of nonsquares in T, and vice versa. The analogous relations are

- $$(b_{s,+})^2 = \frac{q^{-1}(q-1)}{2} \sum_{t \in T \text{ nonsquare}} b_t + \frac{q^{-1}(q-1)^2}{4} b_{s,+} + \frac{q^{-1}(q-1)^2}{4} b_{s,-}.$$

- $$(b_{s,-})^2 = \frac{q^{-1}(q-1)}{2} \sum_{t \in T \text{ square}} b_t + \frac{q^{-1}(q-1)^2}{4} b_{s,-} + \frac{q^{-1}(q-1)^2}{4} b_{s,+}.$$

- $$b_{s,-} b_{s,+} = \frac{q^{-1}(q-1)}{2} \sum_{b \in T \text{ square}} b_t + \frac{q^{-1}(q-1)}{4} b_{s,+} + \frac{q^{-1}(q-1)}{4} b_{s,-}$$

Example 3.6. Suppose $G = GL_2(q)$. Then $T = \{\mathrm{diag}(a,b) : a, b \in \mathbb{F}_q^\times\}$, U is the set of upper unitriangular matrices, and a representative for the nontrivial element of W is $\dot{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding coroot is

$$\alpha^\vee : \mathbb{F}_q^\times \longrightarrow T : a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Computation shows

$$[\mathrm{diag}(a,b), \dot{s}] = \begin{pmatrix} ab^{-1} & 0 \\ 0 & a^{-1}b \end{pmatrix}.$$

It follows that $[T, \dot{s}] = T \cap SL_2(q)$, and $T/[T, \dot{s}]$ is in bijection with $\mathbb{F}_q^\times$ via

$$T/[T, \dot{s}] \longrightarrow \mathbb{F}_q^\times : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} [T, \dot{s}] \mapsto ab$$

with inverse $a \mapsto \mathrm{diag}(a,1)[T, \dot{s}]$.

Let $h = \mathrm{diag}(a,1)$ be a representative in $T/[T, \dot{s}]$. The corresponding orbit is

$$\{[t, \dot{s}]hu\dot{s}U : t \in T,\ u \in U\} = \{tu\dot{s}U : t \in T,\ \det(t) = a,\ u \in U\}.$$

The corresponding element in $e_U kGe_U$ is

$$e_U \left(\sum_{\substack{\det(t)=a \\ u \in U}} tu\dot{s}e_U \right) e_U = q \sum_{\det(t)=a} te_U\dot{s}e_U.$$

So for $a \in \mathbb{F}_q^\times$, set $b_{s,a} = e_U \sum_{t \in T,\ \det(t)=a} t\dot{s}e_U$. Then the b_t, $t \in T$, and $b_{s,a}$ for $a \in \mathbb{F}_q^\times$, generate a subalgebra of $e_U kGe_U$ isomorphic to $\mathrm{End}_{k[G \times T]}(k[G/U])$.

Multiplication of these generators is given as follows.

- $b_t b_{t'} = b_{t'} b_t = b_{tt'}$ for $t, t' \in T$.

- $b_{s,a} b_t = b_t b_{s,a} = b_{s,\det(t)a}$.

  Note

$$b_t b_{s,a} = (e_U t e_U)(e_U \sum_{\det(t')=a} t' \dot{s} e_U) = e_U \sum_{\det(t')=a} tt' \dot{s} e_U = e_U \sum_{\det(r)=a \det(t)} r \dot{s} e_U = b_{s,\det(t)a}.$$

- $b_{s,a} b_{s,b} = q^{-1}(q-1)^2 b_{\dot{s},ab} + q^{-1}(q-1) \sum_{\det(t)=ab} b_t$.

  Note

$$\begin{aligned}
b_{s,a} b_{s,b} &= e_U \sum_{\det(t)=a} t \dot{s} e_U \cdot e_U \sum_{\det(t')=b} t' \dot{s} e_U \\
&= e_U \left(\sum_{\det(t)=a} t \sum_{\det(t')=b} t' \right) (e_U \dot{s} e_U)^2 \\
&= (q-1) e_U \sum_{\det(t)=ab} t [q^{-1} e_U \alpha^\vee(-1) e_U + q^{-1} e_U \dot{s} e_U \sum_{r \in \mathbb{F}_q^\times} e_U \alpha^\vee(r) e_U] \\
&= q^{-1}(q-1) e_U \sum_{\det(t)=ab} t e_U (-I_2) e_U + q^{-1}(q-1) \sum_{\det(t)=ab} t e_U \dot{s} e_U \sum_{r \in \mathbb{F}_q^\times} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} e_U \\
&= q^{-1}(q-1) e_U \sum_{\det(t)=ab} e_U + q^{-1}(q-1)^2 e_U \sum_{\det(a)=ab} t \dot{s} e_U \\
&= q^{-1}(q-1) \sum_{\det(t)=ab} b_t + q^{-1}(q-1)^2 b_{s,ab}.
\end{aligned}$$

Since $\det\colon T \to \mathbb{F}_q^\times$ is surjective, for any $a \in \mathbb{F}_q^\times$, there exists $t \in T$ such that $b_t b_{s,1} = b_{s,\det(t)} = b_{s,a}$. Hence $\mathrm{End}_{k[G \times T]}(k[G/U])$ is in fact generated by b_t for $t \in T$, and $b_{s,1}$.

4 W-Categorification

The following categorical framework of this section was first defined by Dreyfus-Schmidt in [7]. It is re-presented here with some minor modifications and additions.

4.1 Axioms of a W-categorification

Let R be a commutative, unital ring, and let $\mathcal{A}$ denote an artinian and noetherian R-linear abelian category. Let (W,S) be a finite Coxeter system. For any $I \subseteq S$, let $W_I = \langle s : s \in I \rangle$ be the parabolic subgroup in W corresponding to I. Let

$$D_I = \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in I\},$$

where ℓ denotes the length function on W. Write $D_I^{-1} = \{d^{-1} : d \in D_I\}$, and for I, J subsets of S, write $D_{IJ} = D_I \cap D_J^{-1}$. Lastly, if $I \subseteq J \subseteq S$ and $K \subseteq S$, put $W_K(I,J) = \{w \in W_K : {}^wI \subseteq J\}$, where ${}^wI$ is the conjugate wIw^{-1}. The subscript is omitted if $K = S$.

Definition 4.1. Given a Coxeter system (W,S), a weak W-categorifcation on an abelian category $\mathcal{A}$ is the data of a decomposition $\mathcal{A} = \bigoplus_{I\subseteq S} \mathcal{A}_I$, biadjoint functors $(F_I^J \colon \mathcal{A}_I \rightleftarrows \mathcal{A}_J \colon E_I^J)_{I\subseteq J\subseteq S}$, and equivalences $\Phi_{I,w}\colon \mathcal{A}_I \xrightarrow{\sim} \mathcal{A}_{{}^wI,\ w\in W(I,S)}$, with the additional natural isomorphisms,

- For all $I \subseteq J \subseteq K \subseteq S$, $\gamma_{I,J,K}\colon F_J^K F_I^J \xrightarrow{\sim} F_I^K$,
- For all $I \subseteq S$ and $w \in W_I(I,S)$, $\varphi_{I,w}\colon \Phi_{I,w} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}_I}$,
- For all $I \subseteq J \subseteq S$ and $w \in W(J,S)$, $\alpha_{I,J,w}\colon F_{{}^wI}^{{}^wJ}\Phi_{I,w} \xrightarrow{\sim} \Phi_{J,w}F_I^J$.

The above are further subject to the conditions that for all $I \subseteq S$, $w \in W(I,S)$, and $w \in W({}^vI,S)$, $\Phi_{{}^vI,w} \circ \Phi_{I,v} = \Phi_{I,wv}$, and for all $I \subseteq S$, $F_I^I = \mathrm{Id}_{\mathcal{A}_I} = E_I^I$.

Additionally, for each $I \subseteq J \subseteq S$, fix two counit-unit pairs witnessing the fact that F_I^J and E_I^J are biadjoint: put $(\epsilon_I^J, \eta_I^J)\colon F_I^J \dashv E_I^J$, and put $(\overline{\epsilon}_I^J, \overline{\eta}_I^J)\colon E_I^J \dashv F_I^J$.

These counit-unit pairs must be such that the following diagram commutes for any natural transformation $\varphi\colon F_I^J \longrightarrow F_I^J$, (the ${}^J_I$ notation below is suppressed for readability)

$$\begin{array}{ccccc}
E & \xrightarrow{\eta E} & EFE & \xrightarrow{E\varphi E} & EFE \\
{\scriptstyle E\overline{\eta}}\downarrow & & & & \downarrow{\scriptstyle E\epsilon} \\
EFE & \xrightarrow[E\varphi E]{} & EFE & \xrightarrow[\overline{\epsilon}E]{} & E
\end{array}$$

and for any natural transformation $\psi\colon E_I^J \longrightarrow E_I^J$, the following commutes:

$$\begin{array}{ccccc} F & \xrightarrow{F\eta} & FEF & \xrightarrow{F\varphi F} & FEF \\ {\scriptstyle \overline{\eta}F}\downarrow & & & & \downarrow{\scriptstyle \epsilon F} \\ FEF & \xrightarrow[F\varphi F]{} & FEF & \xrightarrow[F\overline{\epsilon}]{} & F. \end{array}$$

Remark 4.2. The final condition on the counit-unit pairs states that the usual transpose maps

$$\mathrm{Hom}(F_I^J, F_I^J) \longrightarrow \mathrm{Hom}(E_I^J, E_I^J)$$

induced by the adjunction coincide, regardless of whether the map is induced by (ϵ_I^J, η_I^J) or $(\overline{\epsilon}_I^J, \overline{\eta}_I^J)$. Hence for $\varphi \in \mathrm{Hom}(F_I^J, F_I^J)$, let $\varphi^* \in \mathrm{Hom}(E_I^J, E_I^J)$ denote the transpose natural transformation. Of course, the analogous statements are also assumed with the roles of F_I^J and E_I^J reversed. The * notation also applies to any generated biadjoint natural transformations, e.g., $F_J^K F_I^J$ is biadjoint to $E_I^J E_J^K$ via

$$\epsilon_J^K \circ F_J^K \epsilon_I^J E_J^K \colon F_J^K F_I^J E_I^J E_J^K \longrightarrow \mathrm{Id}_{\mathcal{A}_J}, \quad E_I^J \eta_J^K F_I^J \circ \eta_I^J \colon \mathrm{Id}_{\mathcal{A}_I} \longrightarrow E_I^J E_J^K F_J^K F_I^J.$$

By functoriality of the mate correspondence, in this case we have the following particular isomorphisms in a weak W-categorifcation,

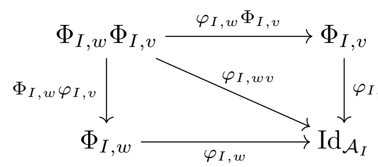
- For all $I \subseteq J \subseteq K \subseteq S$, $\gamma^*_{I,J,K}\colon E_I^K \xrightarrow{\sim} E_I^J E_J^K$
- For all $I \subseteq J \subseteq S$ and $w \in W(J,S)$, $\alpha^*_{I,J,w}\colon\ E_I^J \Phi^{-1}_{J,w} \xrightarrow{\sim} \Phi^{-1}_{I,w} E^{{}^wJ}_{{}^wI}$.

Definition 4.3. A W-categorification on $\mathcal{A}$ is a weak W-categorification that satisfies the following coherence conditions.

- For all $I \subseteq J \subseteq K \subseteq L \subseteq S$, the following diagram commutes:

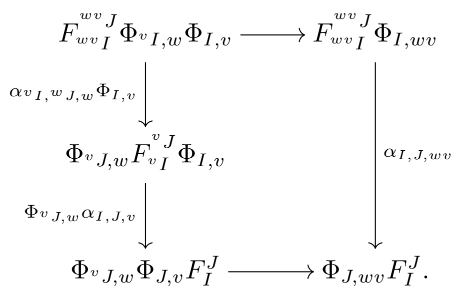
$$\begin{array}{ccc} F_K^L F_J^K F_I^J & \xrightarrow{F_K^L \gamma_{I,J,K}} & F_K^L F_I^K \\ {\scriptstyle \gamma_{J,K,L} F_I^J}\downarrow & & \downarrow{\scriptstyle \gamma_{I,J,K}} \\ F_J^L F_I^J & \xrightarrow[\gamma_{I,J,L}]{} & F_I^L \end{array}$$

- For all $I \subseteq S$, and all $v, w \in W_I(I, S)$, the following diagram commutes

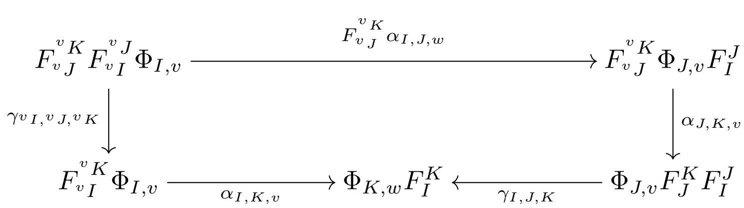


Note that since $v \in W_I$, then ${}^vI \subseteq S$ implies ${}^vI = I$, so the composition of functors is defined.

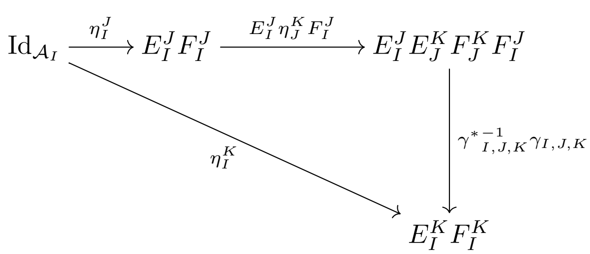
- For all $I \subseteq J \subseteq S$, $v \in W(J, S)$, and $w \in W({}^vJ, S)$, the following diagram commutes



- For all $I \subseteq J \subseteq K \subseteq S$, and all $v \in W(K, S)$, the following diagram commutes



- For all $I \subseteq J \subseteq K \subseteq S$, the following diagram commutes



The analogous statement for the other adjunction is assumed to hold as well.

- For all $I \subseteq J \subseteq K \subseteq S$, the following diagram commutes

$$\begin{array}{ccc} F_I^K E_I^K & \xrightarrow{\epsilon_I^K} & \mathrm{Id}_{\mathcal{A}_K} \\ {\scriptstyle \gamma_{I,J,K}^{-1}{\gamma^*}_{I,J,K}} \downarrow & & \uparrow {\scriptstyle \epsilon_J^K} \\ F_J^K F_I^J E_I^J E_J^K & \xrightarrow[F_J^K \epsilon_I^J E_J^K]{} & F_J^K E_J^K \end{array}$$

  The analogous statement for the other adjunction is assumed to hold as well.

- (Mackey Axiom) For all $I \subseteq J \subseteq K \subseteq S$, there is an isomorphism

$$\bigoplus_{w \in W_K \cap D_{IJ}} F^I_{{}^w J \cap I} \Phi_{J \cap {}^w I, w} E^J_{J \cap I^w} \xrightarrow{\sim} E_I^K F_J^K$$

  induced by the component maps (with the identity transformations suppressed)

$$\begin{aligned} F^I_{{}^w J \cap I} \Phi_{J \cap {}^w I, w} E^J_{J \cap I^w} &\xrightarrow{\eta_I^K} E_I^K F_I^K F^I_{{}^w J \cap I} \Phi_{J \cap {}^w I, w} E^J_{J \cap I^w} \\ &\xrightarrow{\gamma_{{}^w J \cap I, I, K}} E_I^K F^K_{{}^w J \cap I} \Phi_{J \cap I^w, w} E^J_{J \cap I^w} \\ &\xrightarrow{\alpha_{J \cap I^w, K, w}} E_I^K \Phi_{K,w} F^K_{J \cap I^w} E^J_{J \cap I^w} \\ &\xrightarrow{\varphi_{K,w}} E_I^K F^K_{J \cap I^w} E^J_{J \cap I^w} \\ &\xrightarrow{\gamma^{-1}_{J \cap I^w, J, K}} E_I^K F_J^K F^J_{J \cap I^w} E^J_{J \cap I^w} \\ &\xrightarrow{\epsilon^J_{J \cap I^w}} E_I^K F_J^K. \end{aligned}$$

Example 4.4. Suppose (W, S) is a Coxeter system of type A_1, so that $S = \{s\}$. Then a W-categorification is the data of a decomposition $\mathcal{A} = \mathcal{A}_\emptyset \oplus \mathcal{A}_S$, and pair of biadjoint functors $F : \mathcal{A}_\emptyset \rightleftarrows \mathcal{A}_S : E$ with two fixed adjunctions $(\epsilon, \eta) : F \vdash E$ and $(\overline{\epsilon}, \overline{\eta}) : E \vdash F$. There is an automorphism $\Phi_{\emptyset,s} : \mathcal{A}_\emptyset \longrightarrow \mathcal{A}_\emptyset$. Furthermore, since $\Phi_{S,s} \simeq \mathrm{Id}_{\mathcal{A}_S}$, there is an isomorphism $\alpha : F\Phi \longrightarrow F$, and the Mackey axiom implies $EF \simeq \mathrm{Id}_{\mathcal{A}_\emptyset} \oplus \Phi$.

5 Constructing a 2-category

This section contains an explicit construction of a 2-category extending that of a W-categorification in type A_1. Additional 2-morphisms e', e'', z, and $\tilde{\alpha}$ are introduced below, which are not present in Dreyfus-Schmidt's definition of a W-categorification. Moreover, a large list of explicit addition relations are given for the generating 2-morphisms, and allows one to write explicit (module) generating sets for the endomorphism algebras of the generating 1-morphisms of the 2-category.

5.1 Definitions

Put $R = \mathbb{Z}[q^{\pm 1}]$, for q an indeterminate. Let $\mathcal{C}'$ be the strict, R-linear 2-category with two objects, $\emptyset$ and S, and 1-morphisms generated by

- $F \colon \emptyset \longrightarrow S$
- $E \colon S \longrightarrow \emptyset$
- $\Phi \colon \emptyset \longrightarrow \emptyset$

and 2-morphisms generated by

- $e' \colon 1_\emptyset \longrightarrow \Phi$, $e'' \colon \Phi \longrightarrow 1_\emptyset$
- $\alpha \colon F \circ \Phi \longrightarrow F$,
- $z \colon \Phi \circ \Phi \longrightarrow 1_\emptyset$
- $1_\emptyset \xrightarrow{\eta_\emptyset} EF \xrightarrow{\epsilon_\emptyset} 1_\emptyset$
- $1_S \xrightarrow{\eta_S} FE \xrightarrow{\epsilon_S} 1_S$

where the final two bullets are fixed counit-unit adjunctions for the biadjoint 1-morphisms F and E. Impose the condition that α and z are invertible. Also, set $e = e'' \circ e'$, and $e''' = e' \circ e''$, and label the following endormophisms of 1_S by setting $e_0 = \epsilon_S \circ \eta_S$, $e_1 = \epsilon_S \circ FeE \circ \eta_S$, and $e_2 = \epsilon_S \circ \alpha E \circ Fe'E \circ \eta_S$. Furthermore, impose the condition that for any $\varphi \in \mathrm{Hom}_{\mathcal{C}'}(F, F)$, we have

$$E\epsilon_S \circ E\varphi E \circ \eta_\emptyset E = \epsilon_\emptyset E \circ E\varphi E \circ E\eta_S.$$

so that the two usual maps $\mathrm{Hom}_{\mathcal{C}'}(F, F) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(E, E)$ induced by the adjunctions coincide. It then follows that the usual maps from $\mathrm{Hom}_{\mathcal{C}'}(F, F)$ to $\mathrm{Hom}_{\mathcal{C}'}(E, E)$ are equal, as are those from $\mathrm{Hom}_{\mathcal{C}'}(E, E)$ to $\mathrm{Hom}_{\mathcal{C}'}(F, F)$.

Note also that there are two 2-morphisms

$$\mu := (\eta_\emptyset, E\alpha \circ \eta_\emptyset \Phi) \colon 1_\emptyset \oplus \Phi \longrightarrow EF$$

and

$$\tilde{\alpha} := E\epsilon_S \circ E\alpha E \circ \eta_\emptyset \Phi E \colon \Phi E \longrightarrow E.$$

Now let $\mathcal{C}$ be the strict R-linear category obtained from $\mathcal{C}'$ by inverting the 2-morphism μ, as well as α, $\tilde{\alpha}$, and z.

These arrows are subject to the following relations:

1.
$$1_\emptyset \oplus \Phi \xrightarrow{\mu} EF \xrightarrow{\epsilon_\emptyset} 1_\emptyset$$
is given by $\begin{pmatrix} q^{-1} & 0 \end{pmatrix}$. Note this is specifically the two relations $\epsilon_\emptyset \circ \eta_\emptyset = q^{-1}$, and $\epsilon_\emptyset \circ E\alpha \circ \eta_\emptyset \Phi = 0$.

2.
$$E \xrightarrow{E\eta_S} EFE \xrightarrow{\mu^{-1}E} E \oplus \Phi E \xrightarrow{\operatorname{diag}(1_E, \tilde{\alpha})} E \oplus E$$
is given by $\begin{pmatrix} q1_E \\ (-1)^\varepsilon (q1_E + q(q-1)\tilde{\alpha} \circ e'E) \end{pmatrix}$ where $\varepsilon \in \{0, 1\}$.

3.
$$F \xrightarrow{\eta_S F} FEF \xrightarrow{F\mu^{-1}} F \oplus F\Phi \xrightarrow{\operatorname{diag}(1_F, \alpha)} F \oplus F$$
is given by $\begin{pmatrix} q1_F \\ (-1)^\varepsilon (q1_F + q(q-1)\alpha \circ Fe') \end{pmatrix}$ where $\varepsilon \in \{0, 1\}$.

4.
$$1_\emptyset \xrightarrow{\eta_\emptyset} EF \xrightarrow{\tilde{\alpha}^{-1}F} \Phi EF \xrightarrow{\Phi\epsilon_\emptyset} \Phi$$
is given by $q^{-1}(1-q)e'$.

5.
$$1_\emptyset \xrightarrow{\eta_\emptyset} EF \xrightarrow{E\alpha^{-1}} EF\Phi \xrightarrow{\epsilon_\emptyset \Phi} \Phi$$
is given by $q^{-1}(1-q)e'$.

6.
$$\Phi E \xrightarrow{\tilde{\alpha}} E \xrightarrow{e'E} \Phi E \xrightarrow{\tilde{\alpha}} E$$
is given by $q^{-1}e''E + q^{-1}(q-1)eE \circ \tilde{\alpha}$.

7.
$$F\Phi \xrightarrow{\alpha} F \xrightarrow{Fe'} F\Phi \xrightarrow{\alpha} F$$
is given by $q^{-1}Fe'' + q^{-1}(q-1)Fe \circ \alpha$.

8. $\alpha \circ \alpha\Phi = q^{-1}Fz + q^{-1}(q-1)\alpha \circ Fe' \circ Fz$

9. $\tilde{\alpha} \circ \Phi\tilde{\alpha} = q^{-1}zE + q^{-1}(q-1)\tilde{\alpha} \circ e'E \circ zE$

10. $F\tilde{\alpha} \circ \alpha^{-1}E = \alpha E \circ F\tilde{\alpha}^{-1} + (q-1)(\alpha \circ Fe')E - (q-1)F(\tilde{\alpha} \circ e'E)$

11. $e''' = e\Phi = \Phi e$

12. $eEF = EFe$

13. The following diagram is commutative
$$\begin{tikzcd} F\Phi \arrow[r, "\alpha"] \arrow[d, "Fe'''"'] & F \arrow[d, "Fe"] \\ F\Phi \arrow[r, "\alpha"'] & F \end{tikzcd}$$

14. $e'' \circ e' \circ e'' = e''$

15. $e' \circ e'' \circ e' = e'$

16.
$$\begin{tikzcd} \Phi\Phi \arrow[r, "z"] & 1_\emptyset \\ \Phi \arrow[u, "\Phi e'", "e'\Phi"'] \arrow[ur, bend right, "e''"'] & \end{tikzcd}$$

17.
$$\begin{tikzcd} \Phi\Phi \arrow[d, "e''\Phi"', "\Phi e''"] & 1_\emptyset \arrow[l, "z^{-1}"'] \arrow[dl, bend left, "e'"] \\ \Phi & \end{tikzcd}$$
(These two relations also imply $\Phi e' = e'\Phi$ and $\Phi e'' = e''\Phi$.)

18. $\Phi z = z\Phi$

19. $e_0 e_1 = e_1 e_0$

20. $e_0 e_2 = e_2 e_0$

21. $e_1 e_2 = e_2 e_1$

22. $e_0 F = \big(q + (-1)^\epsilon q\big)1_F + (-1)^\epsilon q(q-1)\alpha \circ Fe'$

23. $e_1F = \big(q + (-1)^\epsilon q\big)Fe + (-1)^\epsilon q(q-1)\alpha \circ Fe'$

24. $e_2F = (-1)^\epsilon q^{-1}(q-1)Fe + \big((-1)^\epsilon + q + (-1)^\epsilon q^{-1}(q-1)^2\big)\alpha \circ Fe'$

Note also that these final three relations for e_0F, e_1F, and e_2F show that the endomorphisms of 1_S do not introduce any new endomorphisms of F.

5.2 Construction of Subspaces

Definition 5.1. Define the following R-linear subspaces, given in terms of generating sets.

- $H(1_\emptyset, 1_\emptyset) := \langle 1, e\rangle \subseteq \mathrm{Hom}_\mathcal{C}(1_\emptyset, 1_\emptyset)$
- $H(1_\emptyset, \Phi) := \langle e'\rangle \subseteq \mathrm{Hom}_\mathcal{C}(1_\emptyset, \Phi)$
- $H(\Phi, 1_\emptyset) := \langle e''\rangle \subseteq \mathrm{Hom}_\mathcal{C}(\Phi, 1_\emptyset)$
- $H(\Phi, \Phi) := \langle 1, e'''\rangle \subseteq \mathrm{Hom}_\mathcal{C}(\Phi, \Phi)$
- $H(1_S, 1_S) = \langle 1_S, e_0^i e_1^j e_2^k : i, j, k \geq 0\rangle \subseteq \mathrm{Hom}_\mathcal{C}(1_S, 1_S)$
- Define $H(F, F) \subseteq \mathrm{Hom}_\mathcal{C}(F, F)$ to be the subalgebra generated by $\{1_F, Fe, \alpha \circ Fe'\}$.
- Define $H(E, E) \subseteq \mathrm{Hom}_\mathcal{C}(E, E)$ to be the image of $H(F, F)$ under the map

  $$\mathrm{Hom}_\mathcal{C}(F, F) \longrightarrow \mathrm{Hom}_\mathcal{C}(E, E) : \varphi \mapsto \; E \xrightarrow{\eta_\emptyset E} EFE \xrightarrow{E\varphi E} EFE \xrightarrow{E\epsilon_S} E$$

  Under this map 1_F corresponds to 1_E, Fe corresponds to eE, $\alpha \circ Fe$ corresponds to

  $$E\epsilon_S \circ E\alpha E \circ EFe'E \circ \eta_\emptyset E = E\epsilon_S \circ E\alpha E \circ \eta_\emptyset \Phi E \circ e'E = \tilde{\alpha} \circ e'E.$$

  So $H(E, E)$ has generating set $\{1_E, eE, \tilde{\alpha} \circ e'E\}$.
- Define $H(1_S, FE)$ to be the image of the map

  $$H(F, F) \longrightarrow \mathrm{Hom}_\mathcal{C}(1_S, FE) : \varphi \mapsto \left(1_S \xrightarrow{\eta_S} FE \xrightarrow{\varphi E} FE \right)$$

- Define $H(FE, 1_S)$ to be the image of the map

  $$H(F, F) \longrightarrow \mathrm{Hom}_\mathcal{C}(FE, 1_S) : \varphi \mapsto \left(FE \xrightarrow{\varphi E} FE \xrightarrow{\epsilon_S} 1_S \right)$$

- Define $H(FE,FE)$ to be the image of the map

$$\begin{aligned} H(F,F)\oplus H(F,F) &\to H(F,F)\oplus \mathrm{Hom}_{\mathcal{C}}(F\Phi,F) \\ &\to \mathrm{Hom}_{\mathcal{C}}(F\oplus F\Phi,F) \\ &\to \mathrm{Hom}_{\mathcal{C}}(FEF,F)\to \mathrm{Hom}_{\mathcal{C}}(FE,FE) \end{aligned}$$

given by

$$(\varphi,\psi)\mapsto(\varphi,\psi\alpha)\mapsto\begin{pmatrix}\varphi & \psi\circ\alpha\end{pmatrix}\mapsto\begin{pmatrix}\varphi & \psi\circ\alpha\end{pmatrix}\circ F\mu^{-1}\mapsto\begin{pmatrix}\varphi E & (\psi\circ\alpha)E\end{pmatrix}\circ F\mu^{-1}E\circ FE\eta_S.$$

Lemma 5.2. *Suppose X is a 1-morphism in $\mathcal{C}$. If X is not indecomposable, one can express X, up to isomorphism, as a direct sum of the indecomposable 1-morphisms $\{1_\emptyset, 1_S, \Phi, E, F, FE\}$ in a canonical way.*

Proof. If X is a product of a single generating 1-morphism, then X is equal to one of $1_\emptyset$, 1_S, Φ, E, of F, each of which is indecomposable. Otherwise, let X be a product of generating 1-morphisms of $\mathcal{C}$, $X = G_1\cdots G_d$, where $G_i\in\{\Phi,E,F\}$, with $d\geq 2$. Put $X' = G_3\cdots G_d$.

- If $G_1 = G_2 = \Phi$, we put $\Delta(X) = X'$, and $\delta'_X = zX'\colon X'\xrightarrow{\sim}\Delta(X)$.
- If $G_1 = \Phi$, and $G_2 = E$, we put $\Delta(X) = EX'$ and $\delta'_X = \tilde{\alpha}X'\colon X\xrightarrow{\sim}\Delta(X)$
- If $G_1 = E$ and $G_2 = F$, we put $\Delta(X) = X'\oplus\Phi X'$ and $\delta'_X = \mu^{-1}X'\colon X\xrightarrow{\sim}\Delta(X)$
- If $G_1 = F$ and $G_2 = \Phi$, we put $\Delta(X) = FX'$ and $\delta'_X = \alpha X'\colon X\xrightarrow{\sim}\Delta(X)$
- If $G_1 = F$, $G_2 = E$, and $G_3 = F$, we put $\Delta(X) = FG_4\cdots G_d\oplus F\Phi G_4\cdots G_d$ and

$$\delta'_X = F\mu^{-1}G_4\cdots G_d\colon X\xrightarrow{\sim}\Delta(X).$$

When these assumptions do not hold, we have $X\in\{1_\emptyset,\Phi,1_S,E,F,FE\}$, and we put $\Delta(X) = X$, and $\delta'_X = \mathrm{id}\colon X\xrightarrow{\sim}\Delta(X)$.

This exhausts all cases, and in each case X is isomorphic to a direct sum of 1-morphisms expressible as a composite of fewer non-identity generating 1-morphisms. Inductively, this process must eventually terminate as a direct sum of indecomposable 1-morphisms.

This decomposition extends to direct sums as follows. Suppose $X = \bigoplus_{i=1}^r X_i$, where each X_i is a product of generating 1-morphisms. Put $\Delta(X) := \bigoplus_{i=1}^r \Delta(X_i)$, and $\delta'_X := \delta'_{X_1}+\cdots+\delta'_{X_r}$. Inductively, define $\Delta^n(X) := \Delta(\Delta^{n-1}(X))$, and ∂^n_X by $\partial^1_X = \delta'_X$ and

$$\begin{array}{ccc} X & \xrightarrow{\partial_X^{n-1}} & \Delta^{n-1}(X) \\ & \searrow^{\partial_X^n} & \downarrow{\scriptstyle \delta'_{\Delta^{n-1}(X)}} \\ & & \Delta^n(X). \end{array}$$

Note that there exists some n such that $\Delta^n(X)$ is a direct sum all of whose summands are either $1_\emptyset$, 1_S, Φ, E, F, or FE. Since $\delta'_{X'} = 1_{X'}$ for each of these summands, it follows that for $m \geq n$, $\Delta^m(X) = \Delta^n(X)$ and $\partial_X^m = \partial_X^n$. Hence for any X which is a product of generating 1-morphisms, put $\delta_X := \partial_X^n$ for any n such that $\Delta^n(X)$ is a direct sum of irreducible 1-morphisms, which is well-defined by the previous observations.

Now suppose $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ are parallel 1-morphisms where each X_i and Y_j are indecomposable. Set

$$\bar{H}(X,Y) := \bigoplus_{i,j} H(X_i, Y_j).$$

Suppose X and Y are arbitrary 1-morphisms in $\mathcal{C}$. As noted before, there exist $n_1, n_2 \geq 0$ such that $\Delta^{n_1}(X)$ and $\Delta^{n_2}(Y)$ are (possibly direct sums) of indecomposable 1-morphisms. Let $n = \max\{n_1, n_2\}$, so that $\Delta^n(X) = \Delta^{n_1}(X)$, $\Delta^n(Y) = \Delta^{n_2}(Y)$, $\delta_X = \partial_X^n = \partial_X^{n_1}$, and $\delta_Y = \partial_Y^n = \partial_Y^{n_2}$. Then finally define $H(X,Y)$ to be the R-submodule of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ given by

$$H(X,Y) := \delta_Y^{-1} \circ \bar{H}(\Delta^n(X), \Delta^n(Y)) \circ \delta_X := \{\delta_Y^{-1} \circ A \circ \delta_X : A \in \bar{H}(\Delta^n(X), \Delta^n(Y))\}.$$

□

5.3 Stability under Vertical Composition

Lemma 5.3. *Suppose $X \in \{1_\emptyset, \Phi, 1_S, E, F, FE, EF\}$, $Y \in \{1_\emptyset, \Phi, 1_S, FE, EF\}$. If X and Y are parallel, and $f \colon X \longrightarrow Y$ is a vertical composite of generating 2-morphisms, then $f \in H(X,Y)$.*

Proof. Induct on n, the number of generating 2-morphisms in the composite f. If $n = 1$, f is simply a generating 2-morphism. By definition, one immediately has $e' \in H(1_\emptyset, \Phi)$, $e'' \in H(\Phi, 1_\emptyset)$, $\alpha \in H(F\Phi, F)$, $z \in H(\Phi \circ \Phi, 1_\emptyset)$. Also, under the various maps defined above $1_F \in H(F,F)$ maps to $\eta_\emptyset \in H(1_\emptyset, EF)$, to $\epsilon_\emptyset \in H(EF, 1_\emptyset)$, to $\eta_S \in H(1_S, FE)$, and to $\epsilon_S \in H(FE, 1_S)$. This covers all possibilities when f is a generating 2-morphism.

Suppose $n > 1$. Decompose $f = X \xrightarrow{x} X' \xrightarrow{f'} Y$, where x is a generating 2-morphism, and f' is a composite of $n-1$ generating 2-morphisms, which is in $H(X', Y)$, by induction.

- If $X = 1_\emptyset$, then $x = e'$ or $x = \eta_\emptyset$. If $x = e'$, then $X' = \Phi$, and the only possibilities for Y are

$Y = 1_\emptyset$, $Y = \Phi$, or $Y = EF$. If $Y = 1_\emptyset$, then $f' \in H(\Phi, 1_\emptyset)$. But $H(\Phi, 1_\emptyset)$ is generated by e'', and $e'' \circ e' = e \in H(1_\emptyset, 1_\emptyset)$, so that $f = f' \circ x \in H(1_\emptyset, 1_\emptyset)$. If $Y = \Phi$, then $f' \in H(\Phi, \Phi)$. Now $H(\Phi, \Phi)$ has generators 1_Φ and e'''. Precomposing with x, notice that $1_\Phi \circ e' = e' \in H(1_\emptyset, \Phi)$ and $e''' \circ e' = e' \circ e'' \circ e' = e' \in H(1_\emptyset, \Phi)$, and hence $f \in H(1_\emptyset, \Phi)$. If $Y = EF$, $f' \in H(\Phi, EF)$, so $\mu^{-1} \circ f' \in H(\Phi, 1_\emptyset \oplus \Phi)$. Denote this matrix by $\begin{pmatrix} a & b \end{pmatrix}^T$, where $a \in H(\Phi, 1_\emptyset)$, and $b \in H(\Phi, \Phi)$. Then $\mu^{-1} \circ f' \circ e' = \begin{pmatrix} a \circ e' & b \circ e' \end{pmatrix}$. Now $e'' \circ e' = e \in H(1_\emptyset, 1_\emptyset)$ and cycling b over the generators of $H(\Phi, \Phi)$ we see $1_\Phi \circ e' = e' \in H(1_\emptyset, \Phi)$ and $e''' \circ e' = e' \circ e'' \circ e' = e' \in H(1_\emptyset, \Phi)$. Hence $\mu^{-1} \circ f' \circ e' \in H(1_\emptyset, 1_\emptyset \oplus \Phi)$, so that $f' \circ e' \in H(1_\emptyset, EF)$.

If $x = \eta_\emptyset$, then $X' = EF$, and thus $Y = 1_\emptyset$, $Y = \Phi$, of $Y = EF$. If $Y = 1_\emptyset$, then $f' \in H(EF, 1_\emptyset)$. We get a diagram

$$\begin{array}{ccccc}
1_\emptyset & \xrightarrow{\eta_\emptyset} & EF & \xrightarrow{f'} & 1_\emptyset \\
\downarrow = & & \downarrow \mu^{-1} & & \downarrow = \\
1_\emptyset & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} a & b \end{pmatrix}]{} & 1_\emptyset
\end{array}$$

where $a \in H(1_\emptyset, 1_\emptyset)$ and $b \in H(\Phi, 1_\emptyset)$. Since the composite along the bottom row is simply a, by definition $f = f' \circ \eta_\emptyset \in H(1_\emptyset, 1_\emptyset)$.

If $Y = \Phi$, then $f' \in H(EF, \Phi)$. There is a similar diagram

$$\begin{array}{ccccc}
1_\emptyset & \xrightarrow{\eta_\emptyset} & EF & \xrightarrow{f'} & \Phi \\
\downarrow = & & \downarrow \mu^{-1} & & \downarrow = \\
1_\emptyset & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} a & b \end{pmatrix}]{} & \Phi
\end{array}$$

where $a \in H(1_\emptyset, \Phi)$ and $b \in H(\Phi, \Phi)$. Since the composite along the bottom row is simply a, by definition $f = f' \circ \eta_\emptyset \in H(1_\emptyset, \Phi)$.

If $Y = EF$, then $f' \in H(EF, EF)$. There is a similar diagram

$$\begin{array}{ccccc}
1_\emptyset & \xrightarrow{\eta_\emptyset} & EF & \xrightarrow{f'} & EF \\
\downarrow = & & \downarrow \mu^{-1} & & \downarrow \mu^{-1} \\
1_\emptyset & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} a & b \\ c & d \end{pmatrix}]{} & 1_\emptyset \oplus \Phi
\end{array}$$

where $a \in H(1_\emptyset, 1_\emptyset)$, $b \in H(\Phi, 1_\emptyset)$, $c \in H(1_\emptyset, \Phi)$, and $d \in H(\Phi, \Phi)$. Since the bottom horizontal composite is $\begin{pmatrix} a & c \end{pmatrix}^T \in H(1_\emptyset, 1_\emptyset \oplus \Phi)$, $f = f' \circ \eta_\emptyset \in H(1_\emptyset, EF)$.

- Suppose $X = \Phi$. Then $x = e''$, $X' = 1_\emptyset$, and the possibilities for Y are $Y = 1_\emptyset$, $Y = \Phi$, or $Y = EF$. If $Y = 1_\emptyset$, it suffices to show $\varphi \circ e'' \in H(\Phi, 1_\emptyset)$ for φ a generator in $H(1_\emptyset, 1_\emptyset)$. But $e \circ e'' = e'' \circ e' \circ e'' = e'' \in H(\Phi, 1_\emptyset)$.

  If $Y = \Phi$, it suffices to check $\varphi \circ e'' \in H(\Phi, \Phi)$ for φ a generator in $H(1_\emptyset, \Phi)$. By the defining relations, $e' \circ e'' = e''' \in H(\Phi, \Phi)$.

  If $Y = EF$, there is a diagram

$$
\begin{array}{ccccc}
\Phi & \xrightarrow{e''} & 1_\emptyset & \xrightarrow{f'} & EF \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mu^{-1}} \\
\Phi & \xrightarrow[e'']{} & 1_\emptyset & \xrightarrow[\begin{pmatrix} a \\ b \end{pmatrix}]{} & 1_\emptyset \oplus \Phi
\end{array}
$$

  where $a \in H(1_\emptyset, 1_\emptyset)$ and $b \in H(1_\emptyset, \Phi)$. Since $e \circ e'' = e'' \circ e' \circ e'' = e'' \in H(\Phi, 1_\emptyset)$ and $e' \circ e'' = e''' \in H(\Phi, \Phi)$, it follows $f = f' \circ e'' \in H(1_\emptyset, EF)$.

- Suppose $X = 1_S$. Possibilities for x are e_0, e_1, e_2, so that $X' = 1_S$, and either $Y = 1_S$ or $Y = FE$. Otherwise, $x = \eta_S$, so that $X' = FE$, and either $Y = 1_S$ or $Y = FE$.

  Assume $x = e_i$ for $i = 0, 1, 2$. If $Y = 1_S$, it suffices to check $\varphi \circ e_i \in H(1_S, 1_S)$ for $\varphi \in H(1_S, 1_S)$, but this is immediate.

  So suppose $x = e_0$, and $Y = FE$. It suffices to check $\varphi E \circ \eta_S \circ e_0 \in H(1_S, FE)$ for φ a generator in $H(1_S, FE)$. If $\varphi = 1_F$, by the defining relations we have

$$
\begin{aligned}
\eta_S \circ e_0 = e_0 FE \circ \eta_S &= [(q + (-1)^\epsilon q) 1_{FE} + (-1)^\epsilon q(q-1)(\alpha \circ Fe')E] \circ \eta_S \\
&= (q + (-1)^\epsilon q)\eta_S + (-1)^\epsilon q(q-1)(\alpha \circ Fe')E \circ \eta_S
\end{aligned}
$$

  which is in $H(1_S, FE)$, since η_S and $(\alpha \circ Fe')E$ are in $H(1_S, FE)$ by definition. If $\varphi = Fe$,

$$
\begin{aligned}
FeE \circ \eta_S \circ e_0 &= FeE \circ e_0 EF \circ \eta_S \\
&= [(q + (-1)^\epsilon q) FeE + (-1)^\epsilon q(q-1)(\alpha \circ Fe')E] \circ \eta_S
\end{aligned}
$$

which is in $H(1_S, FE)$ since $FeE \circ \eta_S$ and $(\alpha \circ Fe')E \circ \eta_S$ are generators in $H(1_S, FE)$. If $\varphi = \alpha \circ Fe'$,

$$\begin{aligned}
\alpha E \circ Fe'E \circ \eta_S \circ e_0 &= \alpha E \circ Fe'E \circ e_0 EF \circ \eta_S \\
&= \alpha E \circ [(q + (-1)^\epsilon q) Fe'E + (-1)^\epsilon q(q-1) Fe'E \circ \alpha E \circ Fe'E] \circ \eta_S \\
&= [q + (-1)^\epsilon q](\alpha \circ Fe')E \circ \eta_S + (-1)^\epsilon q(q-1)[q^{-1} Fe''E + q^{-1} FeE \circ \alpha E] \circ Fe'E \circ \eta_S \\
&= (-1)^\epsilon (q-1) FeE \circ \eta_S + [q + (-1)^\epsilon q + (-1)^\epsilon (q-1)^2](\alpha \circ Fe')E \circ \eta_S
\end{aligned}$$

which is in $H(1_S, FE)$.

Suppose $x = e_1$. As above, if $\varphi = 1_F$,

$$\begin{aligned}
\eta_S \circ e_1 &= e_1 EF \circ \eta_S \\
&= (q + (-1)^\epsilon q) FeE \circ \eta_S + (-1)^\epsilon q(q-1)(\alpha \circ Fe')E \circ \eta_S
\end{aligned}$$

which is $H(1_S, FE)$. If $\varphi = Fe$,

$$FeE \circ \eta_S \circ e_1 = (q + (-1)^\epsilon q) FeE \circ \eta_S + (-1)^\epsilon q(q-1)(\alpha \circ Fe')E \circ \eta_S$$

which is in $H(1_S, FE)$. If $\varphi = \alpha \circ Fe'$,

$$\begin{aligned}
(\alpha \circ Fe')E \circ \eta_S \circ e_1 &= (q + (-1)^\epsilon q)(\alpha \circ Fe')E \circ \eta_S + (-1)^\epsilon q(q-1)(q^{-1} Fe''E + q^{-1}(q-1) FeE \circ \alpha E) Fe'E \circ \eta_S \\
&= (-1)^\epsilon (q-1) FeE \circ \eta_S + (q + (-1)^\epsilon q + (-1)^\epsilon (q-1)^2)(\alpha \circ Fe')E \circ \eta_S
\end{aligned}$$

which is in $H(1_S, FE)$.

If $x = e_2$, the reasoning works as in the case for $x = e_1$, since $e_1 F$ and $e_2 F$ are both linear combinations of Fe and $\alpha \circ Fe'$ with coefficients in R.

Now suppose $x = \eta_S$. If $Y = 1_S$, it suffices to check $\epsilon_S \circ \varphi E \circ \eta_S \in H(1_S, 1_S)$ for φ a generating 2-morphism of $H(F, F)$. This is automatic, for as φ ranges over $\{1_F, Fe, \alpha \circ Fe'\}$, $\epsilon_S \circ \varphi E \circ \eta_S$ ranges over e_0, e_1, and e_2, respectively, all of which are in $H(1_S, 1_S)$ by definition.

If $Y = FE$, then $f' \in H(FE, FE)$, so it suffices to show

$$\Big(\varphi E \quad (\psi \circ \alpha)E\Big) \circ F\mu^{-1} E \circ FE\eta_S \circ \eta_S = \Big(\varphi E \quad (\psi \circ \alpha)E\Big) \circ F\mu^{-1} E \circ \eta_S FE \circ \eta_S$$

is in $H(1_S, FE)$ as φ and ψ range over generators of $H(F, F)$. Since $H(1_S, FE)$ is generated by 2-morphisms of form $\varphi E \circ \eta_S$ for $\varphi \in H(F, F)$, it suffices to show $\Big(\varphi E \quad (\psi \circ \alpha)E\Big) \circ F\mu^{-1} E \circ \eta_S FE$

has form ρE for $\rho \in H(F,F)$, and hence in turn it is enough to show $\begin{pmatrix} \varphi & (\psi \circ \alpha) \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F$ is in $H(F,F)$ when φ and ψ are in $H(F,F)$. First note any generator of form $\begin{pmatrix} \varphi & 0 \end{pmatrix}$ yields

$$\begin{pmatrix} \varphi & 0 \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F = \begin{pmatrix} \varphi & 0 \end{pmatrix} \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F = q\varphi.$$

So generators $\begin{pmatrix} 1_F & 0 \end{pmatrix}$, $\begin{pmatrix} Fe & 0 \end{pmatrix}$, and $\begin{pmatrix} \alpha \circ Fe' & 0 \end{pmatrix}$ correspond to $q1_F$, qFe, and $q(\alpha \circ Fe')$, all of which are in $H(F,F)$.

For generators of form $\begin{pmatrix} 0 & \psi \circ \alpha \end{pmatrix}$, note

$$\begin{aligned} \begin{pmatrix} 0 & \psi \circ \alpha \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F &= \begin{pmatrix} 0 & \psi \end{pmatrix} \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F \\ &= \begin{pmatrix} 0 & \psi \end{pmatrix} \begin{pmatrix} q1_F \\ (-1)^\epsilon (q1_F + q(q-1)\alpha \circ Fe') \end{pmatrix} \\ &= (-1)^\epsilon q\psi + q(q-1)\psi \circ \alpha \circ Fe' \end{aligned}$$

which is clearly in $H(F,F)$, by previous computations.

- If $X = E$ or $X = F$, there are no generating 2-morphisms with source E or F, so there is nothing to prove.

- If $X = FE$, then $x = \epsilon_S$, and $X' = 1_S$. The possibilities for Y are $Y = 1_S$ or $Y = FE$. If $Y = 1_S$, then $f' \in H(1_S, 1_S)$. By definition, $H(FE, 1_S)$ is generated by arrows of form $\epsilon_S \circ \varphi E$, where φ is a generating 2-morphism in $H(F,F)$. If $\psi \in H(1_S, 1_S)$, then $\psi \circ \epsilon_S = \epsilon_S \circ \psi FE$. Hence it suffices to show that ψF is a linear combination of generating 2-morphisms in $H(F,F)$, with coefficients in R. Any 2-morphism in $H(1_S, 1_S)$ is a sum of 2-morphisms of the form $e_2^i e_1^j e_0^k$, for $i,j,k \in \mathbb{Z}_{\geq 0}$. The defining relations imply that $Fe \circ Fe = Fe$, $Fe \circ \alpha \circ Fe' = \alpha \circ Fe' \circ Fe = \alpha \circ Fe'$, and

$$(\alpha \circ Fe') \circ (\alpha \circ Fe') = (q^{-1}Fe'' + q^{-1}(q-1)Fe \circ \alpha)Fe' = q^{-1}Fe + q^{-1}(q-1)\alpha \circ Fe',$$

so that $\{1_F, Fe, \alpha \circ Fe'\}$ generates $H(F,F)$ as a module. Since e_0F, e_1F, and e_2F are all linear combinations of 1_F, Fe, and $\alpha \circ Fe'$, with coefficients in R, it follows inductively that

$$e_2^i e_1^j e_0^k F = (e_2F)^i \circ (e_1F)^j \circ (e_0F)^k$$

is also a linear combination of $\{1_F, Fe, \alpha \circ Fe'\}$ with coefficients in $R[q^{\pm 1}]$, as required.

If $Y = FE$, then $f' \in H(1_S, FE)$. Since $H(1_S, FE)$ is generated by 2-morphisms of form $\rho E \circ \eta_S$, it suffices to show that $\rho E \circ \eta_S \circ \epsilon_S = \rho E \circ \epsilon_S FE \circ FE\eta_S$ is in $H(FE, FE)$ when $\rho \in H(F, F)$. Based on the form of generators of $H(FE, FE)$, it is sufficient to show $\rho E \circ \epsilon_S FE$ has form $\begin{pmatrix} \varphi E & (\psi \circ \alpha)E \end{pmatrix} \circ F\mu^{-1}E$ for $\varphi, \psi \in H(F, F)$, and in turn it is enough to show $\rho \circ \epsilon_S F$ has form $\begin{pmatrix} \varphi & \psi \circ \alpha \end{pmatrix} \circ F\mu^{-1}$, or equivalently, $\rho \circ \epsilon_S F \circ F\mu = \begin{pmatrix} \varphi & \psi \circ \alpha \end{pmatrix}$. But

$$\begin{aligned} \rho \circ \epsilon_S F \circ F\mu &= \rho \circ \epsilon_S F \circ \begin{pmatrix} F\eta_\emptyset & FE\alpha \circ F\eta_\emptyset \Phi \end{pmatrix} \\ &= \begin{pmatrix} \rho & \rho \circ \epsilon_S F \circ FE\alpha \circ F\eta_\emptyset \Phi \end{pmatrix} \\ &= \begin{pmatrix} \rho & \rho \circ \alpha \circ \epsilon_S F\Phi \circ F\eta_\emptyset \Phi \end{pmatrix} = \begin{pmatrix} \rho & \rho \circ \alpha \end{pmatrix}, \end{aligned}$$

so one can take $\rho = \psi = \varphi$.

- Suppose $X = EF$, so $x = \epsilon_\emptyset$, and $X' = 1_\emptyset$. The possibilities for Y are $Y = 1_\emptyset$, $Y = \Phi$, or $Y = EF$. If $Y = 1_\emptyset$, then $f' \in H(1_\emptyset, 1_\emptyset)$. There is a diagram

$$\begin{array}{ccccc} EF & \xrightarrow{\epsilon_\emptyset} & 1_\emptyset & \xrightarrow{f'} & 1_\emptyset \\ {\scriptstyle \mu^{-1}}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} q^{-1} & 0 \end{pmatrix}]{} & 1_\emptyset & \xrightarrow[f']{} & 1_\emptyset \end{array}$$

which clearly shows $f = f' \circ \epsilon \in (EF, 1_\emptyset)$. The same argument shows that $f = f' \circ \epsilon_\emptyset \in H(EF, Y)$ for the other possibilities of Y as well.

□

Lemma 5.4. *The composition map*

$$H(Y, Z) \times H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) : (g, f) \mapsto g \circ f$$

takes values in $H(X, Z)$ when X, Y, and Z are parallel indecomposable 1-morphisms in $\{1_\emptyset, \Phi, 1_S, E, F, FE\}$.

Proof. First, consider the parallel 1-morphisms $1_\emptyset$ and Φ from $\emptyset$ to $\emptyset$. There are eight cases.

- Suppose $(X, Y, Z) = (1_\emptyset, 1_\emptyset, 1_\emptyset)$. Since $e^2 = e$, it is clear $H(1_\emptyset, 1_\emptyset)$ is closed under composition.
- Suppose $(X, Y, Z) = (1_\emptyset, 1_\emptyset, \Phi)$. Since $e' \circ e = e' \circ e'' \circ e' = e'$, the composition map takes values in $H(1_\emptyset, \Phi)$.
- Suppose $(X, Y, Z) = (1_\emptyset, \Phi, 1_\emptyset)$. Since $e'' \circ e' = e$, the composition map takes values in $H(1_\emptyset, 1_\emptyset)$.

- Suppose $(X, Y, Z) = (\Phi, 1_\emptyset, 1_\emptyset)$. Since $e \circ e'' = e'' \circ e' \circ e'' = e''$, the composition map takes values in $H(\Phi, 1_\emptyset)$.

- Suppose $(X, Y, Z) = (1_\emptyset, \Phi, \Phi)$. Since $e''' \circ e' = e' \circ e'' \circ e' = e'$, the composition map takes values in $H(1_\emptyset, \Phi)$.

- Suppose $(X, Y, Z) = (\Phi, 1_\emptyset, \Phi)$. Since $e' \circ e'' = e'''$, the composition map takes values in $H(\Phi, \Phi)$.

- Suppose $(X, Y, Z) = (\Phi, \Phi, 1_\emptyset)$. Since $e'' \circ e''' = e'' \circ e' \circ e'' = e''$, the composition map takes values in $H(\Phi, 1_\emptyset)$.

- Suppose $(X, Y, Z) = (\Phi, \Phi, \Phi)$. Since $e''' \circ e''' = e' \circ e'' \circ e' \circ e'' = e' \circ e'' = e'''$, the composition map takes values in $H(\Phi, \Phi)$.

Second, the only indecomposable 1-morphism $\emptyset \longrightarrow S$ is F. Consider the following pairwise compositions of the nonidentity generators of $H(F, F)$:

- $Fe \circ Fe = Fe^2 = Fe$.

- $Fe \circ \alpha \circ Fe' = \alpha \circ Fe''' \circ Fe' = \alpha \circ F(e''' \circ e') = \alpha \circ Fe'$

- $\alpha \circ Fe' \circ Fe = \alpha \circ F(e' \circ e) = \alpha \circ Fe'$

- $$\begin{aligned}\alpha \circ Fe' \circ \alpha \circ Fe' &= (q^{-1}Fe'' + q^{-1}(q-1)Fe \circ \alpha) \circ Fe' \\ &= q^{-1}Fe'' \circ Fe' + q^{-1}(q-1)Fe \circ \alpha \circ Fe' \\ &= q^{-1}Fe + q^{-1}(q-1)\alpha \circ Fe'\end{aligned}$$

These give the following multiplication table on the nonidentity generators of $H(F, F)$, and from this it

$\circ$	Fe	$\alpha \circ Fe'$
Fe	Fe	$\alpha \circ Fe'$
$\alpha \circ Fe'$	$\alpha \circ Fe'$	$q^{-1}Fe + q^{-1}(q-1)\alpha \circ Fe'$

is clear that $H(F, F)$ is closed under composition.

Similarly, the only indecomposable 1-morphism $S \longrightarrow \emptyset$ is E. Again pairwise composition of nonidentity generators of $H(E, E)$ gives

- $eE \circ eE = e^2E = eE$

- $eE \circ \tilde{\alpha} \circ e'E = \tilde{\alpha} \circ e\Phi E \circ e'E = \tilde{\alpha} \circ e'''E \circ e'E = \tilde{\alpha} \circ (e''' \circ e')E = \tilde{\alpha} \circ e'E$

- $\tilde{\alpha} \circ e'E \circ eE = \tilde{\alpha} \circ e'E$

- $$\begin{aligned}\tilde{\alpha} \circ e'E \circ \tilde{\alpha} \circ e'E &= (q^{-1}e''E + q^{-1}(q-1)eE \circ \tilde{\alpha}) \circ e'E \\ &= q^{-1}(e'' \circ e')E + q^{-1}(q-1)eE \circ \tilde{\alpha} \circ e'E \\ &= q^{-1}eE + q^{-1}(q-1)\tilde{\alpha} \circ e'E\end{aligned}$$

yielding a multiplication table of nonidentity generators for $H(E, E)$, and from this it follows that $H(E, E)$ is closed under composition.

$\circ$	eE	$\tilde{\alpha} \circ e'E$
eE	eE	$\tilde{\alpha} \circ e'E$
$\tilde{\alpha} \circ e'E$	$\tilde{\alpha} \circ e'E$	$q^{-1}eE + q^{-1}(q-1)\tilde{\alpha} \circ e'E$

Fourth, the only indecomposable 1-morphisms $S \longrightarrow S$ are 1_S and FE.

- Suppose $(X, Y, Z) = (1_S, 1_S, 1_S)$. By definition, $H(1_S, 1_S)$ is defined as the R-submodule of $\mathrm{Hom}_{\mathcal{C}}(1_S, 1_S)$ generated by $\{e_0^i, e_1^j, e_2^k : i, j, k \geq 0\}$. Since $e_0e_1 = e_1e_0$, $e_0e_2 = e_2e_0$, and $e_1e_2 = e_2e_1$, $H(1_S, 1_S)$ may also be described as the commutative R-subalgebra generated by $\{1_S, e_0, e_1, e_2\}$, and so $H(1_S, 1_S)$ is closed under composition.

- Suppose $(X, Y, Z) = (1_S, 1_S, FE)$. An arbitrary composite has form $\varphi E \circ \eta_S \circ e_i$ for $i = 0, 1, 2$, and $\varphi \in H(F, F)$. In Lemma 5.3, arrows of this form are proven to be in $H(1_S, FE)$.

- Suppose $(X, Y, Z) = (1_S, FE, 1_S)$. An arbitrary composite $1_S \longrightarrow FE \longrightarrow 1_S$ has form

$$\epsilon_S \circ \varphi E \circ \psi E \circ \eta_S = \epsilon_S \circ \rho E \circ \eta_S$$

  for $\rho = \varphi \circ \psi \in H(F, F)$. As ρ cycles over the three generators of $H(F, F)$, the above composite is e_0, e_1, or e_2, all of which are in $H(1_S, 1_S)$. Hence composition on $H(FE, 1_S) \times H(1_S, FE)$ takes values in $H(1_S, 1_S)$.

- Suppose $(X, Y, Z) = (FE, 1_S, 1_S)$. An arbitrary composite $FE \longrightarrow 1_S \longrightarrow 1_S$ has form $\psi \circ \epsilon_S \circ \varphi E$ for $\psi \in H(1_S, 1_S)$ and $\varphi \in H(F, F)$. Note

$$\psi \circ \epsilon_S \circ \varphi E = \epsilon_S \circ \psi FE \circ \varphi E = \epsilon_S \circ (\psi F \circ \varphi)E$$

  and hence will be in $H(FE, 1_S)$ is $\psi F \circ \varphi \in H(F, F)$. Since $\varphi \in H(F, F)$, it is enough to show $\psi F \in H(F, F)$ when $\psi \in H(1_S, 1_S)$, and this was already shown in Lemma 5.3.

- Suppose $(X,Y,Z) = (FE,FE,FE)$. To show $H(FE,FE)$ is closed under composition, first note the defining relation

$$\begin{pmatrix} 1_E & 0 \\ 0 & \tilde{\alpha} \end{pmatrix} \circ \mu^{-1}E \circ E\eta_S = \begin{pmatrix} q1_E \\ (-1)^\epsilon(q1_E + q(q-1)\tilde{\alpha} \circ e'E) \end{pmatrix}$$

it follows that

$$F\mu^{-1}E \circ FE\eta_S = \begin{pmatrix} q1_{FE} \\ (-1)^\epsilon(qF\tilde{\alpha}^{-1} + q(q-1)Fe'E) \end{pmatrix}.$$

Then the generators for $H(FE,FE)$ have explicit form

$$q\varphi E + (-1)^\epsilon\big(q\psi E \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)\psi E \circ (\alpha \circ Fe')E\big),$$

where $\varphi, \psi \in H(F,F)$. Fixing $\varphi = 0$ and letting ψ range over $H(F,F)$, and vice versa, gives generators $q1_{FE}$, $qFeE$, $q(\alpha \circ Fe')E$, $(-1)^\epsilon\big(q\alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)(\alpha \circ Fe')E\big)$,

$$(-1)^\epsilon\big(qFeE \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)FeE \circ (\alpha \circ Fe')E\big),$$

and

$$(-1)^\epsilon\big(q(\alpha \circ Fe')E \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)(\alpha \circ Fe')E\big).$$

Since q is invertible, these generators may be replaced with the R-module generating set

$$\{1_{FE}, FeE, (\alpha \circ Fe')E, \alpha E \circ F\tilde{\alpha}^{-1}, FeE \circ \alpha E \circ F\tilde{\alpha}^{-1}, (\alpha \circ Fe')E \circ \alpha E \circ F\tilde{\alpha}^{-1}\}.$$

We check that the composites $\alpha E \circ F\tilde{\alpha}^{-1} \circ FeE$, $\alpha E \circ F\tilde{\alpha}^{-1} \circ (\alpha \circ Fe')E$, and $\alpha E \circ F\tilde{\alpha}^{-1} \circ \alpha E \circ F\tilde{\alpha}^{-1}$ are again in $H(FE,FE)$. The other possible pairwise composites are either clearly in $H(FE,FE)$, or a quick consequence of these three. For the first two, note

$$\alpha E \circ F\tilde{\alpha}^{-1} \circ FeE = \alpha E \circ F\Phi eE \circ F\tilde{\alpha}^{-1} = FeE \circ \alpha E \circ F\tilde{\alpha}^{-1}$$

and

$$\begin{aligned} \alpha E \circ F\tilde{\alpha}^{-1} \circ \alpha E \circ Fe'E &= \alpha E \circ \alpha\Phi E \circ F\Phi\tilde{\alpha}^{-1} \circ Fe'E \\ &= \alpha E \circ \alpha\Phi E \circ Fe'\Phi E \circ F\tilde{\alpha}^{-1} \\ &= \alpha E \circ \alpha\Phi E \circ F\Phi e'E \circ F\tilde{\alpha}^{-1} \\ &= (\alpha \circ Fe')E \circ \alpha E \circ F\tilde{\alpha}^{-1}, \end{aligned}$$

both of which are in the generating set. Also,

$$\begin{aligned}
&\alpha E\circ F\tilde{\alpha}^{-1}\circ\alpha E\circ F\tilde{\alpha}^{-1}=\alpha E\circ\alpha\Phi E\circ F\Phi\tilde{\alpha}^{-1}\circ F\tilde{\alpha}^{-1}=(\alpha\circ\alpha\Phi)E\circ F(\Phi\tilde{\alpha}^{-1}\circ\tilde{\alpha}^{-1})\\
&=\big(q^{-1}Fz+q^{-1}(q-1)\alpha\circ Fe'\circ Fz\big)E\circ F\big(qz^{-1}E+(1-q)\Phi\tilde{\alpha}^{-1}\circ e'E\big)\\
&=1_{FE}+q^{-1}(1-q)FzE\circ F\Phi\tilde{\alpha}^{-1}\circ e'E+(q-1)(\alpha\circ Fe')E-q^{-1}(q-1)^2(\alpha\circ Fe')E\circ FzE\circ F\Phi\tilde{\alpha}^{-1}\circ e'E\\
&=1_{FE}+q^{-1}(1-q)FzE\circ F\Phi\tilde{\alpha}^{-1}\circ e'E+(q-1)(\alpha\circ Fe')E-q^{-1}(q-1)^2FeE\circ\alpha E\circ F\tilde{\alpha}^{-1}.
\end{aligned}$$

The only term which is not immediately in $H(FE,FE)$ is $FzE\circ F\Phi\tilde{\alpha}^{-1}\circ e'E$. From the defining relations, and some of their immediate consequences, this term can be rewritten as

$$FzE\circ F\Phi\tilde{\alpha}^{-1}\circ e'E=FzE\circ Fe'\Phi E\circ F\tilde{\alpha}^{-1}=F(z\circ e'\Phi)E\circ F\tilde{\alpha}^{-1}=Fe''E\circ F\tilde{\alpha}^{-1}=qF(\tilde{\alpha}\circ e'E)+(1-q)FeE.$$

Since FeE is a generator of $H(FE,FE)$, it follows $(\alpha E\circ F\tilde{\alpha}^{-1})^2\in H(FE,FE)$ if and only if $F(\tilde{\alpha}\circ e'E)\in H(FE,FE)$. To see this,

$$\begin{aligned}
&(\alpha\circ Fe')E\circ\alpha E\circ F\tilde{\alpha}^{-1}+q^{-1}(q-1)FeE-q^{-1}(q-1)FeE\circ\alpha E\circ F\tilde{\alpha}^{-1}\\
&=\big(q^{-1}Fe''E+q^{-1}(q-1)FeE\circ\alpha E\big)\circ F\tilde{\alpha}^{-1}+q^{-1}(q-1)FeE-q^{-1}(q-1)FeE\circ\alpha E\circ F\tilde{\alpha}^{-1}\\
&=q^{-1}Fe''E\circ F\tilde{\alpha}^{-1}+q^{-1}(q-1)FeE\\
&=\big(q^{-1}Fe''E+q^{-1}(q-1)FeE\circ F\tilde{\alpha}\big)\circ F\tilde{\alpha}^{-1}\\
&=(F\tilde{\alpha}\circ Fe'E\circ F\tilde{\alpha})\circ F\tilde{\alpha}^{-1}\\
&=F(\tilde{\alpha}\circ e'E)
\end{aligned}$$

so that $F(\tilde{\alpha}\circ e'E)\in H(FE,FE)$. This shows that $\alpha E\circ F\tilde{\alpha}^{-1}\circ\alpha E\circ F\tilde{\alpha}^{-1}$ is an R-linear combination of the module generators, so is in $H(FE,FE)$, and hence $H(FE,FE)$ is closed under composition.

- Suppose $(X,Y,Z)=(1_S,FE,FE)$. An arbitrary composite $1_S\longrightarrow FE\longrightarrow FE$ has form

$$\psi\circ\varphi E\circ\eta_S$$

where $\varphi\in H(F,F)$, and $\psi\in H(FE,FE)$. From the explicit generators of $H(FE,FE)$ in the previous case, one has that if x is a generator of $H(F,F)$, then $xE\in H(FE,FE)$. Since $H(FE,FE)$ is closed under composition, we have $\psi\circ\varphi E\in H(FE,FE)$. By Lemma 5.3, it is further shown that whenever $f\in H(FE,FE)$, then $f\circ\eta_S\in H(1_S,FE)$. Hence composition on $H(FE,FE)\times H(1_S,FE)$ takes values in $H(1_S,FE)$.

- Suppose $(X,Y,Z) = (FE, 1_S, FE)$. An arbitrary composite $FE \longrightarrow 1_S \longrightarrow FE$ has form

$$\psi E \circ \eta_S \circ \epsilon_S \circ \varphi E$$

for $\varphi, \psi \in H(F,F)$. However, in Lemma 5.3, it is shown that $\psi E \circ \eta_S \circ \epsilon_S \in H(FE, FE)$ whenever $\psi \in H(F,F)$. As noted previously, $\varphi E \in H(FE, FE)$ when $\varphi \in H(F,F)$, so their composite is in $H(FE, FE)$ as $H(FE, FE)$ is closed under composition.

- Suppose $(X,Y,Z) = (FE, FE, 1_S)$. A composite $FE \longrightarrow FE \longrightarrow 1_S$ has form $\epsilon_S \circ \rho E \circ \gamma$ where $\gamma \in H(FE, FE)$. Assuming γ is a generator of $H(FE, FE)$, this composite has form

$$\epsilon_S \circ \rho E \circ \left(q\varphi E + (-1)^{\epsilon}\big(q\psi E \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)\psi E \circ (\alpha \circ Fe')E\big)\right)$$

for some $\rho, \varphi, \psi \in H(F,F)$. To be in $H(FE, 1_S)$, the above composite must be an R-linear combination of terms of form $\epsilon_S \circ \sigma E$ for $\sigma \in H(F,F)$. Only the middle term $\epsilon_S \circ \rho E \circ \psi E \circ \alpha E \circ F\tilde{\alpha}^{-1}$ is not immediately of this form. From the relation $F\tilde{\alpha} \circ \alpha^{-1}E = \alpha E \circ F\tilde{\alpha}^{-1} + (q-1)(\alpha \circ Fe')E - (q-1)F(\tilde{\alpha} \circ e'E)$, we have

$$\epsilon_S \circ \rho E \circ \alpha E \circ F\tilde{\alpha}^{-1} = \epsilon_S \circ \rho E \circ \big(F\tilde{\alpha} \circ \alpha^{-1}E + (1-q)(\alpha \circ Fe')E + (q-1)F(\tilde{\alpha} \circ e'E)\big).$$

The middle term has the desired form, so we check the other two. For the first, note

$$\begin{aligned}\epsilon_S \circ \rho E \circ F\tilde{\alpha} \circ \alpha^{-1}E &= \epsilon_S \circ \rho E \circ FE\epsilon_S \circ FE\alpha E \circ F\eta_{\emptyset}\Phi E \circ \alpha^{-1}E \\ &= \epsilon_S \circ \epsilon_S FE \circ FE\alpha E \circ F\eta_{\emptyset}\Phi E \circ \rho\Phi E \circ \alpha^{-1}E \\ &= \epsilon_S \circ (\epsilon_S F \circ FE\alpha \circ F\eta_{\emptyset}\Phi \circ \rho\Phi \circ \alpha^{-1})E,\end{aligned}$$

and furthermore,

$$\epsilon_S F \circ FE\alpha \circ F\eta_{\emptyset}\Phi \circ \rho\Phi \circ \alpha^{-1} = \alpha \circ \epsilon_S F\Phi \circ F\eta_{\emptyset}\Phi \circ \rho\Phi \circ \alpha^{-1} = \alpha \circ \rho\Phi \circ \alpha^{-1}.$$

Hence $\epsilon_S \circ \epsilon_S FE \circ FE\alpha E \circ F\eta_{\emptyset}\Phi E \circ \rho\Phi E \circ \alpha^{-1}E$ will be a sum of terms of form $\epsilon_S \circ \sigma E$ if $\alpha \circ \rho\Phi \circ \alpha^{-1} \in H(F,F)$ for any $\rho \in H(F,F)$. Checking on generators, indeed

$$\alpha \circ Fe\Phi \circ \alpha^{-1} = \alpha \circ F\Phi e \circ \alpha^{-1} = Fe \circ \alpha \circ \alpha^{-1} = Fe \in H(F,F)$$

and

$$\alpha \circ \alpha\Phi \circ Fe'\Phi \circ \alpha^{-1} = \alpha \circ \alpha\Phi \circ F\Phi e' \circ \alpha^{-1} = \alpha \circ \alpha\Phi \circ \alpha^{-1}\Phi \circ Fe' = \alpha \circ Fe' \in H(F, F).$$

For the other term,

$$\begin{aligned}
\epsilon_S \circ \rho E \circ F\tilde{\alpha} \circ Fe'E &= \epsilon_S \circ \rho E \circ FE\epsilon_S \circ FE\alpha E \circ F\eta_\emptyset \Phi E \circ Fe'E \\
&= \epsilon_S \circ \epsilon_S FE \circ FE\alpha E \circ F\eta_\emptyset \Phi E \circ Fe'E \circ \rho E \\
&= \epsilon_S \circ \alpha E \circ \epsilon_S F\Phi E \circ F\eta_\emptyset \Phi E \circ Fe'E \circ \rho E \\
&= \epsilon_S \circ (\alpha \circ Fe')E \circ \rho E.
\end{aligned}$$

For $\rho \in H(F, F)$, $\epsilon_S \circ (\alpha \circ Fe')E \circ \rho E$ will be an R-linear combination of terms of form $\epsilon_S \circ \sigma E$ for $\sigma \in H(F, F)$. Thus the composition map on $H(FE, 1_S) \times H(FE, FE)$ takes values in $H(FE, 1_S)$.

$\square$

Remark 5.5. Observe that the composition map $H(X, Y) \times H(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$ takes values in $H(X, Z)$. For given $f \in H(X, Y)$ and $g \in H(Y, Z)$, the composite $g \circ f$ is given by pasting the diagrams

$$\begin{CD}
X @>f>> Y @>g>> Z \\
@V{\delta_X}VV @V{\delta_Y}VV @VV{\delta_Z}V \\
\bigoplus_i X_i @>>{(a_{ji})}> \bigoplus_j Y_j @>>{(b_{kj})}> \bigoplus_k Z_k.
\end{CD}$$

where $\bigoplus_i X_i$ is the canonical decomposition of X, etc. Writing $A = (a_{ji})$ and $B = (b_{kj})$, then the bottom row is given by the matrix $C = BA$, which has components $c_{rs} = \sum_\ell b_{r\ell} \circ a_{\ell s}$. Then $c_{rs} \in H(X_s, Z_r)$ as $b_{r\ell} \circ a_{\ell s} \in H(X_s, Z_r)$ since the composite map $H(Y_\ell, Z_r) \times H(X_s, Y_\ell) \longrightarrow H(X_s, Z_r)$ is already known to take values in $H(X_s, Z_r)$ when X_s, Y_ℓ, and Z_r are indecomposable, by Lemma 5.4.

5.4 Stability under Horizontal Composition

5.4.1 Right Horizontal Composition

Suppose $f \in H(X, Y)$, and $X, Y : \emptyset \longrightarrow \emptyset$. The only indecomposable endomorphisms of $\emptyset$ are $1_\emptyset$ and Φ. Hence there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ 1_\emptyset^{\oplus n_1} \oplus \Phi^{\oplus n_2} & \xrightarrow[(a_{ji})]{} & 1_\emptyset^{\oplus m_1} \oplus \Phi^{\oplus m_2} \end{array}$$

for some $n_1, n_2, m_1, m_2 \geq 0$, and (a_{ji}) is a matrix with components in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$. This in turn gives another diagram

$$\begin{array}{ccc} X\Phi & \xrightarrow{\quad f\Phi \quad} & Y\Phi \\ {\scriptstyle \delta_X \Phi}\downarrow & & \downarrow{\scriptstyle \delta_Y \Phi} \\ \Phi^{\oplus n_1} \oplus \Phi\Phi^{\oplus n_2} & \xrightarrow{(a_{ji}\Phi)} & \Phi^{\oplus m_1} \oplus \Phi\Phi^{\oplus m_2}. \end{array}$$

The bottom row of the above diagram does not consist of indecomposables. Extending the diagram using the prescribed algorithm yields

$$\begin{array}{ccc} X\Phi & \xrightarrow{\quad f\Phi \quad} & Y\Phi \\ {\scriptstyle \delta_X \Phi}\downarrow & & \downarrow{\scriptstyle \delta_Y \Phi} \\ \Phi^{\oplus n_1} \oplus \Phi\Phi^{\oplus n_2} & \xrightarrow{(a_{ji}\Phi)} & \Phi^{\oplus m_1} \oplus \Phi\Phi^{\oplus m_2} \\ {\scriptstyle \mathrm{diag}(1_\Phi^{\oplus n_1}, z^{\oplus n_2})}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(1_\Phi^{\oplus m_1}, z^{\oplus m_2})} \\ \Phi^{\oplus n_1} \oplus 1_\emptyset^{\oplus n_2} & \longrightarrow & \Phi^{\oplus m_1} \oplus 1_\emptyset^{\oplus m_2}. \end{array}$$

The left and right vertical composites in the above diagram are the prescribed decompositions for $X\Phi$ and $Y\Phi$ into indecomposables. So for $f\Phi$ to be an arrow in $H(X\Phi, Y\Phi)$, necessarily the bottom arrow must have components in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$. This amounts to checking four cases.

If $a_{ji} \in H(1_\emptyset, 1_\emptyset)$, consider

$$\Phi \xrightarrow{1} \Phi \xrightarrow{a_{ji}\Phi} \Phi \xrightarrow{1} \Phi.$$

Since $1_\Phi \in H(\Phi, \Phi)$ and $e\Phi = e''' \in H(\Phi, \Phi)$, the composite is in $H(\Phi, \Phi)$.

If $a_{ji} \in H(1_\emptyset, \Phi)$, consider

$$\Phi \xrightarrow{1} \Phi \xrightarrow{a_{ji}\Phi} \Phi\Phi \xrightarrow{z} 1_\emptyset.$$

Since $z \circ e'\Phi = e'' \in H(\Phi, 1_\emptyset)$, the composite is in $H(\Phi, 1_\emptyset)$.

If $a_{ji} \in H(\Phi, 1_\emptyset)$, consider

$$1_\emptyset \xrightarrow{z^{-1}} \Phi\Phi \xrightarrow{a_{ji}\Phi} \Phi \xrightarrow{1} \Phi.$$

Since $e''\Phi \circ z^{-1} = e' \in H(1_\emptyset, \Phi)$, the composite is in $H(1_\emptyset, \Phi)$.

If $a_{ji} \in H(\Phi, \Phi)$, consider

$$1_\emptyset \xrightarrow{z^{-1}} \Phi\Phi \xrightarrow{a_{ji}\Phi} \Phi\Phi \xrightarrow{z} 1_\emptyset.$$

Since $1_\emptyset \in H(1_\emptyset, 1_\emptyset)$ and $z \circ e'''\Phi \circ z^{-1} = z \circ e'\Phi \circ e''\Phi \circ z^{-1} = e'' \circ e' = e \in H(1_\emptyset, 1_\emptyset)$, the composite is in $H(1_\emptyset, 1_\emptyset)$.

Proposition 5.6. *Suppose $X, Y \colon \emptyset \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X\Phi, Y\Phi) : f \mapsto f\Phi$$

takes values in $H(X\Phi, Y\Phi)$.

In the same setting, appending E yields

$$\begin{array}{ccc}
XE & \xrightarrow{\quad fE \quad} & YE \\
\Big\downarrow{\scriptstyle \delta_X E} & & \Big\downarrow{\scriptstyle \delta_Y E} \\
E^{\oplus n_1} \oplus \Phi E^{\oplus n_2} & \xrightarrow{(a_{ji}E)} & E^{\oplus m_1} \oplus \Phi E^{\oplus m_2} \\
\Big\downarrow{\scriptstyle \mathrm{diag}(1_E^{\oplus n_1}, \tilde{\alpha}^{\oplus n_2})} & & \Big\downarrow{\scriptstyle \mathrm{diag}(1_E^{\oplus m_1}, \tilde{\alpha}^{\oplus m_2})} \\
E^{\oplus n_1} \oplus E^{\oplus n_2} & \xrightarrow[A]{} & E^{\oplus m_1} \oplus E^{\oplus m_2}
\end{array}$$

The left and right vertical composites are the decomposition maps δ_{XE} and δ_{YE}, respectively, so if the bottom arrow has components in $H(E, E)$, it follows that $fE \in H(XE, YE)$.

If $a_{ji} \in H(1_\emptyset, 1_\emptyset)$, the corresponding component in A is given by $1_E \circ a_{ji}E \circ 1_E = a_{ji}E$. Since $1_E \in H(E, E)$ and $eE \in H(E, E)$, the corresponding component in A is in $H(E, E)$.

If $a_{ji} \in H(1_\emptyset, \Phi)$, the corresponding component in A is given by $\tilde{\alpha} \circ a_{ji}E \circ 1_E$. Since $\tilde{\alpha} \circ e'E \in H(E, E)$, the corresponding component in A is in $H(E, E)$.

If $a_{ji} \in H(\Phi, 1_\emptyset)$, the corresponding component in A is given by $1_E \circ a_{ji}E \circ \tilde{\alpha}^{-1}$. As consequences of the defining relations, $e''E \circ \tilde{\alpha}^{-1} = q\tilde{\alpha}e'E + (1-q)eE \in H(E, E)$, so the corresponding component in A is in $H(E, E)$.

If $a_{ji} \in H(\Phi, \Phi)$, the corresponding component in A is given by $\tilde{\alpha} \circ a_{ji}E \circ \tilde{\alpha}^{-1}$. But

$$\tilde{\alpha} \circ e'''E \circ \tilde{\alpha}^{-1} = \tilde{\alpha} \circ e'E \circ e''E \circ \tilde{\alpha}^{-1} \in H(E, E),$$

since both $\tilde{\alpha} \circ e'E$ and $e''E \circ \tilde{\alpha}^{-1}$ are in $H(E, E)$, which is closed under composition. So the corresponding component in A is in $H(E, E)$ as well.

Hence we have the following.

Proposition 5.7. *Suppose $X, Y : \emptyset \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(XE, YE) : f \mapsto fE$$

takes values in $H(XE, YE)$.

Now suppose $X, Y : \emptyset \longrightarrow S$, and $f \in H(X, Y)$. The only indecomposable 1-morphism from $\emptyset$ to S is F, hence there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ F^{\oplus n} & \xrightarrow[(a_{ji})]{} & F^{\oplus m} \end{array}$$

for some $n, m \geq 0$. Appending Φ on the right, and extending the diagram yields

$$\begin{array}{ccc} X\Phi & \xrightarrow{f\Phi} & Y\Phi \\ {\scriptstyle \delta_X \Phi}\downarrow & & \downarrow{\scriptstyle \delta_Y \Phi} \\ F\Phi^{\oplus n} & \xrightarrow[(a_{ji}\Phi)]{} & F\Phi^{\oplus m} \\ {\scriptstyle \alpha^{\oplus n}}\downarrow & & \downarrow{\scriptstyle \alpha^{\oplus m}} \\ F^{\oplus n} & \longrightarrow & F^{\oplus m}. \end{array}$$

The vertical composites are the decomposition maps $\delta_{X\Phi}$ and $\delta_{Y\Phi}$, so it will follow that $f\Phi \in H(X\Phi, Y\Phi)$ if $\alpha \circ a_{ji}\Phi \circ \alpha^{-1} \in H(F, F)$ for $a_{ji} \in H(F, F)$. Cycling over the generators of $H(F, F)$, note $\alpha \circ 1_{F\Phi} \circ \alpha^{-1} = 1_F \in H(F, F)$. Also, $\alpha \circ Fe\Phi \circ \alpha^{-1} = \alpha \circ F\Phi e \circ \alpha^{-1} = Fe \in H(F, F)$. Finally,

$$\begin{aligned} \alpha \circ (\alpha \circ Fe')\Phi \circ \alpha^{-1} &= \alpha \circ \alpha\Phi \circ Fe'\Phi \circ \alpha^{-1} \\ &= [q^{-1}Fz + q^{-1}(q-1)\alpha \circ Fe' \circ Fz] \circ Fe'\Phi \circ \alpha^{-1} \\ &= q^{-1}Fe'' \circ \alpha^{-1} + q^{-1}(q-1)\alpha \circ Fe' \circ Fe'' \circ \alpha^{-1} \in H(F, F) \end{aligned}$$

since $Fe'' \circ \alpha^{-1} = q\alpha \circ Fe' + (1-q)Fe \in H(F, F)$, and $H(F, F)$ is closed under composition.

Proposition 5.8. *Suppose $X, Y : \emptyset \longrightarrow S$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X\Phi, Y\Phi) : f \mapsto f\Phi$$

takes values in $H(X\Phi, Y\Phi)$.

Similarly, one can append E on the right, yielding a diagram

$$\begin{array}{ccc} XE & \xrightarrow{fE} & YE \\ {\scriptstyle \delta_X E}\downarrow & & \downarrow{\scriptstyle \delta_Y E} \\ FE^{\oplus n} & \xrightarrow[(a_{ji}E)]{} & FE^{\oplus m}. \end{array}$$

Now $\delta_X E = \delta_{XE}$, $\delta_Y E = \delta_{YE}$, and FE remains indecomposable, so showing that $fE \in H(XE, YE)$ reduces to showing that the map $H(F, F) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(FE, FE)$ given by $f \mapsto fE$ takes values in $H(FE, FE)$. Observe that under the definition of $H(FE, FE)$, an element $(\varphi, 0) \in H(F, F) \oplus H(F, F)$ maps to

$$\begin{pmatrix} \varphi E & 0 \end{pmatrix} \circ F\mu^{-1}E \circ FE\eta_S = \begin{pmatrix} \varphi E & 0 \end{pmatrix} \circ \begin{pmatrix} q1_{FE} \\ (-1)^{\epsilon}(F\tilde{\alpha}^{-1} + q(q-1)Fe'E) \end{pmatrix} = q\varphi E.$$

Since q is invertible, it follows that $\varphi E \in H(FE, FE)$ whenever $\varphi \in H(F, F)$.

Proposition 5.9. *Suppose* $X, Y : \emptyset \longrightarrow S$ *are parallel* 1*-morphisms in* $\mathcal{C}$*. Then the map*

$$H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(XE, YE) : f \mapsto fE$$

takes values in $H(XE, YE)$.

Now suppose $X, Y : S \longrightarrow \emptyset$, and $f \in H(X, Y)$. The only indecomposable 1-morphism from S to $\emptyset$ is E, hence there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ E^{\oplus n} & \xrightarrow[(a_{ji})]{} & E^{\oplus m} \end{array}$$

for some $m, n \geq 1$, and $a_{ji} \in H(E, E)$. In turn,

$$\begin{array}{ccc} XF & \xrightarrow{fF} & YF \\ {\scriptstyle \delta_X F}\downarrow & & \downarrow{\scriptstyle \delta_Y F} \\ EF^{\oplus n} & \xrightarrow[(a_{ji}F)]{} & EF^{\oplus m} \\ {\scriptstyle \mu^{-1}}\downarrow & & \downarrow{\scriptstyle \mu^{-1}} \\ (1_\emptyset \oplus \Phi)^{\oplus n} & \longrightarrow & (1_\emptyset \oplus \Phi)^{\oplus m} \end{array}$$

Hence one needs to check that $\mu^{-1} \circ a_{ji}F \circ \mu \in H(1_\emptyset \oplus \Phi, 1_\emptyset \oplus \Phi)$ when $a_{ji} \in H(E, E)$, in order for

$fF \in H(XF, YF)$. Checking on generators of $H(E,E)$, we have the following three diagrams

$$\begin{array}{ccc} EF & \xrightarrow{1_{EF}} & EF \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu}\uparrow \\ 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}]{} & 1_\emptyset \oplus \Phi \end{array} \qquad \begin{array}{ccc} EF & \xrightarrow{eEF} & EF \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu}\uparrow \\ 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix}e & 0\\ 0 & e'''\end{pmatrix}]{} & 1_\emptyset \oplus \Phi \end{array} \qquad \begin{array}{ccc} EF & \xrightarrow{(\tilde{\alpha}\circ e'E)F} & EF \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu}\uparrow \\ 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix}0 & q^{-1}e''\\ e' & q^{-1}(q-1)e'''\end{pmatrix}]{} & 1_\emptyset \oplus \Phi. \end{array}$$

This commutativity of the first diagram is clear. For the second, note

$$eEF \circ \mu = \begin{pmatrix} eEF \circ \eta_\emptyset & eEF \circ E\alpha \circ \eta_\emptyset \Phi \end{pmatrix} = \begin{pmatrix} \eta_\emptyset \circ e & E\alpha \circ \eta_\emptyset \Phi \circ e''' \end{pmatrix} = \mu \circ \begin{pmatrix} e & 0 \\ 0 & e''' \end{pmatrix}$$

since $eEF \circ E\alpha \circ \eta_\emptyset \Phi = EFe \circ E\alpha \circ \eta_\emptyset \Phi = E\alpha \circ EFe''' \circ \eta_\emptyset \Phi$. For the third, first note

$$\mu \circ \begin{pmatrix} 0 & q^{-1}e'' \\ e' & q^{-1}(q-1)e''' \end{pmatrix} = \begin{pmatrix} E\alpha \circ \eta_\emptyset \Phi \circ e' & q^{-1}\eta_\emptyset \circ e'' + q^{-1}(q-1)E\alpha \circ \eta_\emptyset \Phi \circ e''' \end{pmatrix}$$

and

$$(\tilde{\alpha} \circ e'E)F \circ \mu = \begin{pmatrix} \tilde{\alpha}F \circ e'EF \circ \eta_\emptyset & \tilde{\alpha}F \circ e'EF \circ E\alpha \circ \eta_\emptyset \Phi \end{pmatrix}.$$

Comparing components, note

$$\tilde{\alpha}F \circ e'EF \circ \eta_\emptyset = \tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' = E\alpha \circ \eta_\emptyset \Phi \circ e'.$$

For the second component,

$$\begin{aligned} \tilde{\alpha}F \circ e'EF \circ \tilde{\alpha}F \circ \Phi\eta_\emptyset &= [q^{-1}e''EF + q^{-1}(q-1)eEF \circ \tilde{\alpha}F] \circ \Phi\eta_\emptyset \\ &= q^{-1}\eta_\emptyset \circ e'' + q^{-1}(q-1)EFe \circ E\alpha \circ \eta_\emptyset \Phi \\ &= q^{-1}\eta_\emptyset \circ e'' + q^{-1}(q-1)E\alpha \circ EFe''' \circ \eta_\emptyset \Phi \\ &= q^{-1}\eta_\emptyset \circ e'' + q^{-1}(q-1)E\alpha \circ \eta_\emptyset \Phi \circ e'''. \end{aligned}$$

Together, these give the following.

Proposition 5.10. *Suppose $X, Y \colon S \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(XF, YF) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(XF, YF) : f \mapsto fF$$

takes values in $H(XF, YF)$.

Now suppose $X, Y : S \longrightarrow S$ are 1-morphisms in $\mathcal{C}$, and $f \in H(X,Y)$. The only indecomposable 1-morphisms on S are 1_S and FE, hence there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ 1_S^{\oplus n_1} \oplus FE^{\oplus n_2} & \xrightarrow[(a_{ji})]{} & 1_S^{\oplus m_1} \oplus FE^{\oplus m_2} \end{array}$$

for some $n_1, n_2, m_1, m_2 \geq 0$, and the a_{ji} are arrows in one of $H(1_S, 1_S)$, $H(1_S, FE)$, $H(FE, 1_S)$, or $H(FE, FE)$. Appending F to the right, and applying the decomposition algorithm to the bottom row yields

$$\begin{array}{ccc} XF & \xrightarrow{\quad fF \quad} & YF \\ {\scriptstyle \delta_X F}\downarrow & & \downarrow{\scriptstyle \delta_Y F} \\ F^{\oplus n_1} \oplus FEF^{\oplus n_2} & \xrightarrow{(a_{ji}F)} & F^{\oplus m_1} \oplus FEF^{\oplus m_2} \\ {\scriptstyle \mathrm{diag}(1_F^{\oplus n_1}, F\mu^{-1 \oplus n_2})}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(1_F^{\oplus m_1}, F\mu^{-1 \oplus m_2})} \\ F^{\oplus n_1} \oplus (F \oplus F\Phi)^{\oplus n_2} & & F^{\oplus m_1} \oplus (F \oplus F\Phi)^{\oplus m_2} \\ {\scriptstyle \mathrm{diag}(1_F^{\oplus n_1}, 1_F^{\oplus n_2}, \alpha^{\oplus n_2})}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(1_F^{\oplus m_1}, 1_F^{\oplus m_2}, \alpha^{\oplus m_2})} \\ F^{\oplus n_1} \oplus F^{\oplus n_2} \oplus F^{\oplus n_2} & \xrightarrow[A]{} & F^{\oplus m_1} \oplus F^{\oplus m_2} \oplus F^{\oplus m_2}. \end{array}$$

As before, the left and right vertical composites are the decompositions δ_{XF} and δ_{YF}. For each of the four choices for a_{ji}, there are several subdiagrams which must be investigated.

If $a_{ji} \in H(1_S, 1_S)$, the corresponding component in A is given simply given by $a_{ji}A$. This amounts to checking that $a_{ji}F \in H(F,F)$ for $a_{ji} \in H(1_S, 1_S)$, which is clear from the defining relations for e_0F, e_1F, and e_2F.

If $a_{ji} \in H(1_S, FE)$, the corresponding component in A is given

$$\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ a_{ji}F.$$

Recall that $H(1_S, FE)$ is generated by arrows of the form $\varphi E \circ \eta_S$ for $\varphi \in H(F,F)$. If $\varphi = 1_F$, defining relations imply

$$\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F = \begin{pmatrix} q1_F \\ (-1)^{\epsilon}(q1_F + q(q-1)\alpha \circ Fe' \end{pmatrix} \in H(F, F \oplus F).$$

If $\varphi = Fe$,

$$\begin{aligned}
\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ FeEF \circ \eta_S F &= \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ FEFe \circ \eta_S F \\
&= \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F \circ Fe \\
&= \begin{pmatrix} qFe \\ (-1)^\epsilon (qFe + q(q-1)\alpha \circ Fe' \circ Fe) \end{pmatrix} \\
&= \begin{pmatrix} qFe \\ (-1)^\epsilon (qFe + q(q-1)\alpha \circ Fe') \end{pmatrix} \in H(F, F \oplus F).
\end{aligned}$$

If $\varphi = \alpha \circ Fe'$,

$$\begin{aligned}
\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ \alpha EF \circ Fe'EF \circ \eta_S F &= \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha\Phi \end{pmatrix} \begin{pmatrix} Fe' & 0 \\ 0 & Fe'\Phi \end{pmatrix} \circ F\mu^{-1} \circ \eta_S F \\
&= \begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha\Phi \end{pmatrix} \begin{pmatrix} Fe' & 0 \\ 0 & Fe'\Phi \end{pmatrix} \begin{pmatrix} q1_F \\ (-1)^\epsilon (q\alpha^{-1} + q(q-1)Fe') \end{pmatrix}.
\end{aligned}$$

Multiplying these out, the first component is $q\alpha \circ Fe' \in H(F, F)$. The second is

$$\begin{aligned}
&(-1)^\epsilon (q\alpha \circ \alpha\Phi \circ Fe'\Phi \circ \alpha^{-1} + q(q-1)\alpha \circ \alpha\Phi \circ Fe'\Phi \circ Fe') \\
&= (-1)^\epsilon (\alpha \circ \alpha\Phi \circ F\Phi e' \circ \alpha^{-1} + q(q-1)\alpha \circ \alpha\Phi \circ F\Phi e' \circ Fe') \\
&= (-1)^\epsilon (\alpha \circ \alpha\Phi \circ \alpha^{-1}\Phi \circ Fe' + q(q-1)\alpha \circ Fe' \circ \alpha \circ Fe') \\
&= (-1)^\epsilon (\alpha \circ Fe' + q(q-1)\alpha \circ Fe' \circ \alpha \circ Fe') \in H(F, F).
\end{aligned}$$

If $a_{ji} \in H(FE, 1_S)$, the corresponding component in A is

$$a_{ji} F \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

Recall $H(FE, 1_S)$ is generated by arrows of form $\epsilon_S \circ \varphi E$ for φ a generator of $H(F, F)$. First, observe that for $\varphi \in H(F, F)$, we have $\varphi\Phi \circ \alpha^{-1} = \alpha^{-1} \circ \varphi$. This is obvious if $\varphi = 1_F$. If $\varphi = Fe$, then $Fe\Phi \circ \alpha^{-1} = F\Phi e \circ \alpha^{-1} = \alpha^{-1} \circ Fe$. If $\varphi = \alpha \circ Fe'$,

$$(\alpha \circ Fe')\Phi \circ \alpha^{-1} = \alpha\Phi \circ Fe'\Phi \circ \alpha^{-1} = \alpha\Phi \circ F\Phi e' \circ \alpha^{-1} = \alpha\Phi \circ \alpha^{-1}\Phi \circ Fe' = \alpha^{-1} \circ \alpha \circ Fe'.$$

Then

$$\epsilon_S F \circ \varphi EF \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \epsilon_S F \circ F\mu \circ \begin{pmatrix} \varphi & 0 \\ 0 & \varphi\Phi \end{pmatrix} \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$
$$= \epsilon_S F \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}$$
$$= \begin{pmatrix} 1_F & 1_F \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} = \begin{pmatrix} \varphi & \varphi \end{pmatrix}.$$

So the corresponding components in A are in $H(F \oplus F, F)$.

Lastly, suppose $a_{ji} \in H(FE, FE)$. The corresponding components of A are given by

$$F \oplus F \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}} F \oplus F\Phi \xrightarrow{F\mu} FEF \xrightarrow{a_{ji}F} FEF \xrightarrow{F\mu^{-1}} F \oplus F\Phi \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}} F \oplus F.$$

By the definition of $H(FE, FE)$, the generators have form

$$\begin{pmatrix} \varphi E & (\psi \circ \alpha)E \end{pmatrix} \circ F\mu^{-1}E \circ FE\eta_S = \begin{pmatrix} \varphi E & (\psi \circ \alpha)E \end{pmatrix} \begin{pmatrix} q1_{FE} \\ (-1)^{\epsilon}(qF\tilde{\alpha}^{-1} + q(q-1)Fe'E) \end{pmatrix}$$
$$= q\varphi E + (-1)^{\epsilon}\Big(q(\psi \circ \alpha)E \circ F\tilde{\alpha}^{-1} + q(q-1)\psi E \circ \alpha E \circ Fe'E \Big)$$

where φ and ψ are in $H(F, F)$. If $\psi = 0$, the generators have form $q\varphi E$. Then

$$\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ q\varphi EF \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} q\varphi & 0 \\ 0 & q\varphi F \end{pmatrix} \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} q\varphi & 0 \\ 0 & q\alpha \circ \varphi\Phi \circ \alpha^{-1} \end{pmatrix} = \begin{pmatrix} q\varphi & 0 \\ 0 & q\varphi \end{pmatrix}$$

which has components in $H(F, F)$. If $\varphi = 0$, the a_{ji} has form

$$(-1)^{\epsilon}\Big(q\psi E \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)\psi E \circ (\alpha \circ Fe')E \Big).$$

From the previous computation, the component corresponding to the summand $\psi E \circ (\alpha \circ Fe')E$ is in $H(F, F)$ since the summand has form gE for some $g \in H(F, F)$. Also, since the corresponding component in A is given by conjugation by $\mathrm{diag}(1_F, \alpha) \circ F\mu^{-1}$, it is sufficient that

$$\begin{pmatrix} 1_F & 0 \\ 0 & \alpha \end{pmatrix} \circ F\mu^{-1} \circ (\alpha E \circ F\tilde{\alpha}^{-1})F \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

has components in $H(F, F)$, as ψE already has the form considered previously. In fact, there is a commutative diagram

$$\begin{array}{ccccc}
FEF & \xrightarrow{F\tilde{\alpha}^{-1}F} & F\Phi EF & \xrightarrow{\alpha EF} & FEF \\
\Big\downarrow{\scriptstyle F\mu^{-1}} & & & & \Big\downarrow{\scriptstyle F\mu^{-1}} \\
F \oplus F\Phi & & & & F \oplus F\Phi \\
\Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} & & & & \Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} \\
F \oplus F & & \xrightarrow[\begin{pmatrix} (1-q)\alpha \circ Fe' & 1_F \\ 1_F + (q-1)\alpha \circ F' & 0 \end{pmatrix}]{} & & F \oplus F.
\end{array}$$

To see this, first observe that

$$F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} F\eta_\emptyset & FE\alpha \circ F\eta_\emptyset \Phi \circ \alpha^{-1} \end{pmatrix}$$

the counterclockwise composite $F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} (1-q)\alpha \circ Fe' \\ 1_F + (q-1)\alpha \circ F' \end{pmatrix}$ is given by (*)

$$(1-q)F\tilde{\alpha}F \circ Fe'EF \circ F\eta_\emptyset + F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\eta_\emptyset + (q-1)F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\tilde{\alpha}F \circ Fe'EF \circ F\eta_\emptyset.$$

We rewrite some of the composites following the factors $F\eta_\emptyset$ using the defining relations. Note

$$\begin{aligned}
&F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\tilde{\alpha}F \circ \alpha^{-1}EF = F\tilde{\alpha}F \circ F\Phi\tilde{\alpha}F \circ \alpha^{-1}\Phi EF \circ \alpha^{-1}EF \\
&= F(q^{-1}zE + q^{-1}(q-1)\tilde{\alpha} \circ e'E \circ zE)F \circ (qFz^{-1} + (1-q)\alpha^{-1}\Phi \circ Fe')EF \\
&= 1_{FEF} + (q-1)F\tilde{\alpha}F \circ Fe'EF + q^{-1}(1-q)FzEF \circ \alpha^{-1}\Phi EF \circ Fe'EF + \dots \\
&\dots - q^{-1}(1-q)^2 F\tilde{\alpha}F \circ Fe'EF \circ FzEF \circ \alpha^{-1}\Phi EF \circ Fe'EF \\
&= 1_{FEF} + (q-1)F\tilde{\alpha}F \circ Fe'EF + q^{-1}(1-q)Fe''EF \circ F\alpha^{-1}EF - q^{-1}(1-q)^2 F\tilde{\alpha}F \circ \alpha^{-1}EF \circ FeEF
\end{aligned}$$

and

$$\begin{aligned}
F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\tilde{\alpha}F \circ Fe'EF &= F\tilde{\alpha}F \circ F\Phi\tilde{\alpha}F \circ \alpha^{-1}\Phi EF \circ Fe'EF \\
&= F(q^{-1}zE + q^{-1}(q-1)\tilde{\alpha} \circ e'E \circ zE)F \circ \alpha^{-1}\Phi EF \circ Fe'EF \\
&= q^{-1}FzEF \circ \alpha^{-1}\Phi EF \circ Fe'EF + q^{-1}(q-1)F\tilde{\alpha}F \circ F\Phi e''EF \circ \alpha^{-1}\Phi EF \circ Fe'EF \\
&= q^{-1}FzEF \circ F\Phi e'EF \circ \alpha^{-1}EF + q^{-1}(q-1)F\tilde{\alpha}F \circ \alpha^{-1}EF \circ Fe''EF \circ Fe'EF \\
&= q^{-1}Fe''EF \circ \alpha^{-1}EF + q^{-1}(q-1)F\tilde{\alpha}F \circ \alpha^{-1}EF \circ FeEF.
\end{aligned}$$

Hence the equation (*) above is simply given by $F\eta_\emptyset$, which is the first component of $F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$.

For the second component, $F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\mu \circ \begin{pmatrix} 1_F & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1_F \\ 0 \end{pmatrix}$ is given by

$$F\tilde{\alpha}F \circ \alpha^{-1}EF \circ F\eta_\emptyset = F\tilde{\alpha}F \circ F\Phi\eta_\emptyset \circ \alpha^{-1} = FE\alpha \circ F\eta_\emptyset\Phi \circ \alpha^{-1}.$$

So finally, we have the following.

Proposition 5.11. *Suppose $X, Y \colon S \longrightarrow S$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(XF, YF) : f \mapsto fF$$

takes values in $H(XF, YF)$.

All the previous propositions can be collected more succinctly.

Proposition 5.12. *If X and Y are parallel 1-morphisms in $\mathcal{C}$, and Z is any appropriate 1-morphism, then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(XZ, YZ) : f \mapsto fZ$$

takes values in $H(XZ, YZ)$.

5.4.2 Left Horizontal Composition

We can also compose generating 1-morphisms on the right without leaving the candidate spaces of form $H(X,Y)$. Suppose $X, Y \colon \emptyset \longrightarrow \emptyset$. The decomposition maps δ_X and δ_Y only ever act on the two left-most factors, and any 1-morphism on $\emptyset$ has one of the following forms: $1_\emptyset$, Φ, $\Phi\Phi X'$, $\Phi E X'$, or EFX' for some appropriate 1-morphism X'. Then δ_X will be a product of matrices with components $1_{1_\emptyset}$, 1_Φ, zX', $\tilde{\alpha}X'$, or $\mu^{-1}X'$. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ \bigoplus_i X_i & \xrightarrow[(a_{ji})]{} & \bigoplus_j Y_j \end{array}$$

where X_i and Y_j are indecomposable 1-morphisms on $\emptyset$.

Postcomposing X and Y with Φ still gives 1-morphisms on $\emptyset$, hence $\delta_{\Phi X}$ is still a matrix composite with the same components. There is an extended diagram

$$\begin{array}{ccc}
\bigoplus_i \Phi X_i & \xrightarrow{(\Phi a_{ji})} & \bigoplus_j \Phi Y_j \\
\Phi\delta_X \uparrow & & \downarrow \Phi\delta_Y^{-1} \\
\Phi X & \xrightarrow{\Phi f} & \Phi Y \\
\delta_{\Phi X}^{-1} \uparrow & & \downarrow \delta_{\Phi Y} \\
\bigoplus_r (\Phi X)_r & \xrightarrow[A]{} & \bigoplus_s (\Phi Y)_s .
\end{array}$$

As before, $\delta_{\Phi X}^{-1}$ is a matrix composite with components of form $1_{1_\emptyset}$, 1_Φ, $z^{-1}X'$, $\mu X'$, or $\tilde{\alpha}^{-1}X'$, and $\delta_{\Phi Y}$ has components $1_{1_\emptyset}$, 1_Φ, zY', $\mu^{-1}Y'$, and $\tilde{\alpha}Y'$. Also, $\Phi\delta_X$ has components of form $\Phi zX'$, $\Phi\mu^{-1}X'$, $\Phi\tilde{\alpha}$, and $\Phi\delta_Y^{-1}$ has components of form $\Phi z^{-1}Y'$, $\Phi\mu Y'$, $\Phi\tilde{\alpha}^{-1}Y'$. Lastly, the possibilities for Φa_{ji} are 1_Φ, $1_{\Phi\Phi}$, Φe, $\Phi e'$, $\Phi e''$ and $\Phi e'''$. If each of these listed components is the appropriate space $H(W, Z)$, then the components of A will be in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$, so that $\Phi f \in H(\Phi X, \Phi Y)$. As seen previously, we can append any necessary X' or Y' on the right, so it is sufficient to show the following.

Lemma 5.13. *There are the following memberships.*

1. $z^{-1} \in H(1_\emptyset, \Phi\Phi)$
2. $z \in H(\Phi\Phi, 1_\emptyset)$
3. $\mu \in H(1_\emptyset \oplus \Phi, EF)$
4. $\mu^{-1} \in H(EF, 1_\emptyset \oplus \Phi)$
5. $\tilde{\alpha} \in H(\Phi E, E)$,
6. $\tilde{\alpha}^{-1} \in H(E, \Phi E)$
7. $\Phi z \in H(\Phi\Phi\Phi, \Phi)$
8. $\Phi z^{-1} \in H(\Phi, \Phi\Phi\Phi)$
9. $\Phi\mu \in H(\Phi \oplus \Phi\Phi, \Phi EF)$
10. $\Phi\mu^{-1} \in H(\Phi EF, \Phi \oplus \Phi\Phi)$
11. $\Phi\tilde{\alpha} \in H(\Phi\Phi E, \Phi E)$
12. $\Phi\tilde{\alpha}^{-1} \in H(\Phi E, \Phi\Phi E)$
13. $\Phi e \in H(\Phi, \Phi)$

14. $\Phi e' \in H(\Phi, \Phi)$

15. $\Phi e'' \in H(\Phi\Phi, \Phi)$

16. $\Phi e''' \in H(\Phi\Phi, \Phi\Phi)$

Proof. The first six claims are immediate from the definitions. For the others, note that $\Phi z \in H(\Phi\Phi\Phi, \Phi)$ since $z\Phi \in H(\Phi\Phi\Phi, \Phi)$, and $z\Phi = \Phi z$. The arguments applies to Φz^{-1} since $\Phi z^{-1} = z^{-1}\Phi$.

We have the following commutative diagram

$$
\begin{array}{ccc}
\Phi \oplus \Phi\Phi & \xrightarrow{\Phi\mu} & \Phi EF \\
 & & \downarrow{\scriptstyle \tilde{\alpha}F} \\
\begin{pmatrix} 1_\Phi & 0 \\ 0 & z \end{pmatrix} \Big\downarrow & & EF \\
 & & \downarrow{\scriptstyle \mu^{-1}} \\
\Phi \oplus 1_\emptyset & \xrightarrow[\begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix}]{} & 1_\emptyset \oplus \Phi.
\end{array}
$$

To see this, note

$$\mu \circ \begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix} \begin{pmatrix} 1_\Phi & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} E\alpha \circ \eta_\emptyset \Phi & q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha \circ \eta_\emptyset \Phi \circ e' \circ z \end{pmatrix}.$$

On the other hand, $\tilde{\alpha}F \circ \Phi\mu = \begin{pmatrix} E\alpha \circ \eta_\emptyset \Phi & \tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_\emptyset \Phi \end{pmatrix}$. However,

$$
\begin{aligned}
\tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_\emptyset \Phi &= \tilde{\alpha}F \circ \Phi\tilde{\alpha}F \circ \Phi\Phi\eta_\emptyset \\
&= (q^{-1}zEF + q^{-1}(q-1)\tilde{\alpha}F \circ e'EF \circ zEF) \circ \Phi\Phi\eta_\emptyset \\
&= q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' \circ z \\
&= q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha \circ \eta_\emptyset \Phi \circ e' \circ z.
\end{aligned}
$$

Hence $\Phi\mu \in H(\Phi \oplus \Phi\Phi, \Phi EF)$.

Now $\Phi\mu^{-1} \in H(\Phi EF, \Phi \oplus \Phi\Phi)$ since the following diagram commutes:

$$\begin{array}{ccc}
\Phi EF & \xrightarrow{\Phi\mu^{-1}} & \Phi \oplus \Phi\Phi \\
\downarrow{\scriptstyle \tilde{\alpha}F} & & \\
EF & & \downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}} \\
\downarrow{\scriptstyle \mu^{-1}} & & \\
1_{\emptyset} \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_{\emptyset}.
\end{array}$$

This is equivalent to the commutativity of

$$\begin{array}{ccc}
\Phi EF & \xleftarrow{\Phi\mu} & \Phi \oplus \Phi\Phi \\
\downarrow{\scriptstyle \tilde{\alpha}F} & & \\
EF & & \uparrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}} \\
\uparrow{\scriptstyle \mu} & & \\
1_{\emptyset} \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_{\emptyset}.
\end{array}$$

Calculating,

$$\tilde{\alpha}F \circ \Phi\mu \circ \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \circ \begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} (1-q)\tilde{\alpha}F \circ \Phi\eta_{\emptyset} \circ e' + q\tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_{\emptyset}\Phi \circ z^{-1} & \tilde{\alpha}F \circ \Phi\eta_{\emptyset} \end{pmatrix}.$$

The first component simplifies as

$$\begin{aligned}
&(1-q)\tilde{\alpha}F \circ \Phi\eta_{\emptyset} \circ e' + q\tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_{\emptyset}\Phi \circ z^{-1} \\
&= (1-q)\tilde{\alpha}F \circ \Phi\eta_{\emptyset} \circ e' + q(\tilde{\alpha} \circ \Phi\tilde{\alpha})F \circ \eta_{\emptyset}\Phi\Phi \circ z^{-1} \\
&= (1-q)\tilde{\alpha}F \circ \Phi\eta_{\emptyset} \circ e' + q(q^{-1}zEF + q^{-1}(q-1)\tilde{\alpha} \circ e'EF \circ zEF) \circ z^{-1}EF \circ \eta_{\emptyset} \\
&= \eta_{\emptyset}.
\end{aligned}$$

The second component is simply $\tilde{\alpha}F \circ \Phi\eta_{\emptyset} = E\alpha \circ \eta_{\emptyset}\Phi$, so the above matrix is that of μ. Hence $\Phi\mu^{-1} \in H(\Phi EF, \Phi \oplus \Phi\Phi)$.

For $\Phi\tilde{\alpha} \in H(\Phi\Phi E, \Phi E)$, the following must be an arrow in $H(E, E)$,

$$\begin{array}{ccc} \Phi\Phi E & \xrightarrow{\Phi\tilde{\alpha}} & \Phi E \\ {\scriptstyle z^{-1}E}\uparrow & & \downarrow{\scriptstyle \tilde{\alpha}} \\ E & & E \end{array}$$

which is the case since $\tilde{\alpha} \circ \Phi\tilde{\alpha} \circ z^{-1}E = q^{-1}1_E + q^{-1}(q-1)\tilde{\alpha} \circ e'E \in H(E, E)$.

Similarly, $\Phi\tilde{\alpha}^{-1} \in H(\Phi E, \Phi\Phi E)$ if the following composite is an arrow in $H(E, E)$,

$$\begin{array}{ccc} \Phi E & \xrightarrow{\Phi\tilde{\alpha}^{-1}} & \Phi\Phi E \\ {\scriptstyle \tilde{\alpha}^{-1}}\uparrow & & \downarrow{\scriptstyle zE} \\ E & & E. \end{array}$$

Indeed,

$$\begin{aligned} zE \circ (\Phi\tilde{\alpha}^{-1} \circ \tilde{\alpha}^{-1}) &= q^{-1}1_E + (1-q)zE \circ \Phi\tilde{\alpha}^{-1} \circ e'E \\ &= q^{-1}1_E + (1-q)zE \circ e'\Phi E \circ \tilde{\alpha}^{-1} \\ &= q^{-1}1_E + (1-q)e''E \circ \tilde{\alpha}^{-1} \\ &= q^{-1}1_E + (1-q)(q\tilde{\alpha} \circ e'E + (1-q)eE) \in H(E, E). \end{aligned}$$

The last four relations are clear since we can rewrite the arrows as $\Phi e = e\Phi$, $\Phi e' = e'\Phi$, $\Phi e'' = e''\Phi$, and $\Phi e''' = \Phi e\Phi = e\Phi\Phi$. □

Proposition 5.14. *Suppose $X, Y\colon \emptyset \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Phi X, \Phi Y) : f \mapsto \Phi f$$

takes values in $H(\Phi X, \Phi Y)$.

In the same situation, we can postcompose with F to yield a commutative diagram

$$\begin{array}{ccc} \bigoplus_i FX_i & \xrightarrow{(Fa_{ji})} & \bigoplus_j FY_j \\ {\scriptstyle F\delta_X}\uparrow & & \downarrow{\scriptstyle F\delta_Y^{-1}} \\ FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \delta_{FX}^{-1}}\uparrow & & \downarrow{\scriptstyle \delta_{FY}} \\ \oplus_r (FX)_r & \xrightarrow[A]{} & \bigoplus_s (FY)_s. \end{array}$$

Now FX and FY are 1-morphisms $\emptyset \longrightarrow S$, and any such 1-morphism is of the form F, $F\Phi X'$, or

$FEFX'$ for some X'. It follows that the decomposition arrow δ_{FY} consists of matrices with components 1_F, $\alpha X'$, or $F\mu^{-1}X'$, and δ_{FX}^{-1} consists of matrices with the inverse components 1_F, $\alpha^{-1}X'$, and $F\mu X'$. Similarly to the previous case, $F\delta_Y$ consists of matrices with components 1_F, $Fz^{-1}X'$, $F\tilde{\alpha}^{-1}X'$, and $F\mu X'$, and $F\delta_X^{-1}$ consists of inverse components 1_F, FzX', $F\tilde{\alpha}X'$ and $F\mu^{-1}X'$. The components of the Fa_{ji} are 1_F, $1_{F\Phi}$, Fe, Fe', Fe'', or Fe'''. If each of these listed components is the appropriate space $H(W,Z)$, then the components of A will be in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$, so that $Ff \in H(FX, FY)$. As before, it is sufficient to show the following.

Lemma 5.15. *There are the following memberships.*

1. $Fz \in H(F\Phi\Phi, F)$
2. $Fz^{-1} \in H(F, F\Phi\Phi)$
3. $F\mu \in H(F \oplus F\Phi, FEF)$
4. $F\mu^{-1} \in H(FEF, F \oplus F\Phi)$
5. $F\tilde{\alpha} \in H(F\Phi E, FE)$
6. $F\tilde{\alpha}^{-1} \in H(F\Phi E, FE)$
7. $Fe' \in H(F, F\Phi)$
8. $Fe'' \in H(F\Phi, F)$

Proof. By definition, $Fz \in H(F\Phi\Phi, F)$ if $Fz \circ \alpha^{-1}\Phi \circ \alpha^{-1} \in H(F, F)$. By the defining relations,

$$\begin{aligned} Fz \circ \alpha^{-1}\Phi \circ \alpha^{-1} &= Fz(qFz^{-1} + (1-q)\alpha^{-1}\Phi \circ Fe') \\ &= q1_F + (1-q)Fz \circ F\Phi e' \circ \alpha^{-1} \\ &= q1_F + (1-q)Fe'' \circ \alpha^{-1} \\ &= q1_F + (1-q)(q\alpha \circ Fe' + (1-q)Fe) \in H(F,F). \end{aligned}$$

Similarly, $Fz^{-1} \in H(F, F\Phi\Phi)$ if $\alpha \circ \alpha\Phi \circ Fz^{-1} \in H(F, F)$. By the relations,

$$\begin{aligned} \alpha \circ \alpha\Phi \circ Fz^{-1} &= (q^{-1}Fz + q^{-1}(q-1)\alpha \circ Fe' \circ Fz) \circ Fz^{-1} \\ &= q^{-1}1_F + q^{-1}(q-1)\alpha \circ Fe' \in H(F,F). \end{aligned}$$

For $F\mu$, observe

$$\begin{array}{ccc}
F \oplus F\Phi & \xrightarrow{F\mu} & FEF \\
\Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} & & \Big\downarrow{\scriptstyle F\mu^{-1}} \\
 & & F \oplus F\Phi \\
 & & \Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} \\
F \oplus F & \xrightarrow[\mathrm{diag}(1_F,1_F)]{} & F \oplus F
\end{array}$$

Since $\mathrm{diag}(1_F, 1_F) \in H(F \oplus F, F \oplus F)$, $F\mu \in H(F \oplus F\Phi, FEF)$.

Similarly, since the following diagram commutes,

$$\begin{array}{ccc}
FEF & \xrightarrow{F\mu^{-1}} & F \oplus F\Phi \\
\Big\downarrow{\scriptstyle F\mu^{-1}} & & \\
F \oplus F\Phi & & \Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} \\
\Big\downarrow{\scriptstyle \mathrm{diag}(1_F,\alpha)} & & \\
F \oplus F & \xrightarrow[\mathrm{diag}(1_F,1_F)]{} & F \oplus F
\end{array}$$

indeed $F\mu^{-1} \in H(FEF, F \oplus F\Phi)$.

One has $F\tilde{\alpha} \in H(F\Phi E, FE)$ if $F\tilde{\alpha} \circ \alpha^{-1}E \in H(FE, FE)$. By the defining relations,

$$F\tilde{\alpha} \circ \alpha^{-1}E = \alpha E \circ F\tilde{\alpha}^{-1} + (q-1)(\alpha \circ Fe')E - (q-1)F(\tilde{\alpha} \circ e'E).$$

From the prior results, since $\alpha \circ Fe' \in H(F, F)$, then $(\alpha \circ Fe')E \in H(FE, FE)$. By the definition of $H(FE, FE)$, the image of $\begin{pmatrix} 0 & 1_F \end{pmatrix}$ in $H(FE, FE)$ is given by

$$(-1)^{\epsilon}(q\alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)(\alpha \circ Fe')E)$$

so that $\alpha E \circ F\tilde{\alpha}^{-1} \in H(FE, FE)$. Third, the image of $\begin{pmatrix} 0 & \alpha \circ Fe' \end{pmatrix}$ in $H(FE, FE)$ can be computed as

$$\begin{aligned}
&(-1)^{\epsilon}(q(\alpha \circ Fe')E \circ \alpha E \circ F\tilde{\alpha}^{-1} + q(q-1)(\alpha \circ Fe')E \circ \alpha E \circ Fe'E) \\
&= (-1)\big(q(q^{-1}e''E + q^{-1}(q-1)FeE \circ \alpha E)F\tilde{\alpha}^{-1} + q(q-1)(q^{-1}Fe''E + q^{-1}(q-1)FeE \circ \alpha E)Fe'E\big) \\
&= (-1)^{\epsilon}\big(qF(\tilde{\alpha} \circ e'E) + (1-q)FeE + (q-1)FeE \circ \alpha E \circ F\tilde{\alpha}^{-1} + (q-1)FeE + (q-1)^2(\alpha \circ Fe')E\big) \\
&= (-1)^{\epsilon}\big(qF(\tilde{\alpha} \circ e'E) + (q-1)FeE \circ \alpha E \circ F\tilde{\alpha}^{-1} + (q-1)^2(\alpha \circ Fe')E\big).
\end{aligned}$$

Since $Fe \in H(F, F)$, $FeE \in H(FE, FE)$, and so $FeE \circ \alpha E \circ F\tilde{\alpha}^{-1} \in H(FE, FE)$. This implies $F(\tilde{\alpha} \circ e'E) \in H(FE, FE)$, so that $F\tilde{\alpha} \circ \alpha^{-1}E \in H(FE, FE)$, and thus $F\tilde{\alpha} \in H(F\Phi E, FE)$.

Also, $Fe' \in H(F, F\Phi)$ if $\alpha \circ Fe' \in H(F,F)$, which is indeed the case by the definition of $H(F,F)$. Lastly, $Fe'' \in H(F\Phi, F)$ if $Fe'' \circ \alpha^{-1} \in H(F,F)$, and this is the case since

$$Fe'' \circ \alpha^{-1} = q(\alpha \circ Fe') + (1-q)Fe \in H(F,F).$$

□

Proposition 5.16. *Suppose $X, Y : \emptyset \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(FX, FY) : f \mapsto Ff$$

takes values in $H(FX, FY)$.

Now suppose that $X, Y : \emptyset \longrightarrow S$ are parallel arrows in $\mathcal{C}$. Any such 1-morphisms must have form F, $F\Phi X'$, or $FEFX'$ for some appropriate 1-morphism X'. There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ \bigoplus_i X_i & \xrightarrow[(a_{ji})]{} & \bigoplus_j Y_j \end{array}$$

where δ_X and δ_Y consist of matrices with components of form 1_F, $\alpha X'$ or $F\mu^{-1}X'$. The only indecomposable arrow $\emptyset \longrightarrow S$ is F, so the components a_{ji} are in $H(F,F)$. There is a larger commutative diagram

$$\begin{array}{ccc} \bigoplus_i EX_i & \xrightarrow{(Ea_{ji})} & \bigoplus_j EY_j \\ \uparrow{\scriptstyle E\delta_X} & & \downarrow{\scriptstyle E\delta_Y^{-1}} \\ EX & \xrightarrow{Ef} & EY \\ \uparrow{\scriptstyle \delta_{EX}^{-1}} & & \downarrow{\scriptstyle \delta_{EY}} \\ \bigoplus_r (EX)_r & \xrightarrow[A]{} & \bigoplus_s (EY)_s. \end{array}$$

Now EX and EY are 1-morphisms on $\emptyset$, so as seen before, δ_{EY} is a product of matrices with components of form $1_{1_\emptyset}$, 1_Φ, zX', $\tilde{\alpha}X'$, or $\mu^{-1}X'$, and δ_{EX}^{-1} will have inverse components. Also, $E\delta_X$ is a product of matrices with components of form 1_E, $E\alpha X'$, and $EF\mu^{-1}X'$, and $E\delta_Y^{-1}$ will have inverse components. As before, the components of A will be in the appropriate candidate spaces if the following memberships hold.

Lemma 5.17. *There are the following memberships.*

1. $E\alpha \in H(EF\Phi, EF)$

2. $E\alpha^{-1} \in H(EF, EF\Phi)$

3. $EF\mu \in H(EF \oplus EF\Phi, EFEF)$

4. $EF\mu^{-1} \in H(EFEF, EF \oplus EF\Phi)$

5. $EFe \in H(EF, EF)$

6. $E(\alpha \circ Fe') \in H(EF, EF)$.

Proof.

1. The following diagram commutes

$$\begin{array}{ccc}
EF\Phi & \xrightarrow{\quad E\alpha \quad} & EF \\
\uparrow {\scriptstyle \mu\Phi} & & \\
\Phi \oplus \Phi\Phi & & \uparrow {\scriptstyle \mu} \\
{\scriptstyle \mathrm{diag}(1,z)} \downarrow & & \\
\Phi \oplus 1_\emptyset & \xrightarrow[\begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix}]{} & 1_\emptyset \oplus \Phi.
\end{array}$$

To see this, note

$$E\alpha \circ \mu\Phi = \begin{pmatrix} E\alpha \circ \eta_\emptyset\Phi & E\alpha \circ E\alpha\Phi \circ \eta_\emptyset\Phi\Phi \end{pmatrix}.$$

In the other direction,

$$\begin{aligned}
\mu \circ \begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} &= \begin{pmatrix} \eta_\emptyset & E\alpha \circ \eta_\emptyset\Phi \end{pmatrix} \begin{pmatrix} 0 & q^{-1}z \\ 1 & q^{-1}(q-1)e' \circ z \end{pmatrix} \\
&= \begin{pmatrix} E\alpha \circ \eta_\emptyset\Phi & q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha \circ \eta_\emptyset\Phi \circ e' \circ z \end{pmatrix}.
\end{aligned}$$

The first entries of these matrices are equal, so it remains to check the second entry. By the generating relations, note

$$\begin{aligned}
E(\alpha \circ \alpha\Phi) \circ \eta_\emptyset\Phi\Phi &= E(q^{-1}Fz + q^{-1}(q-1) \circ \alpha \circ Fe' \circ Fz) \circ \eta_\emptyset\Phi\Phi \\
&= q^{-1}EFz \circ \eta_\emptyset\Phi\Phi + q^{-1}(q-1)E\alpha \circ EFe' \circ EFz \circ \eta_\emptyset\Phi\Phi \\
&= q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha \circ \eta_\emptyset\Phi \circ e' \circ z
\end{aligned}$$

so that the second entries are equal. Hence

$$\begin{array}{ccc}
EF\Phi & \xrightarrow{\quad E\alpha \quad} & EF \\
\Big\downarrow{\scriptstyle \mu^{-1}\Phi} & & \\
\Phi \oplus \Phi\Phi & & \Big\downarrow{\scriptstyle \mu^{-1}} \\
\Big\downarrow{\scriptstyle \mathrm{diag}(1,z)} & & \\
\Phi \oplus 1_\emptyset & \xrightarrow[\begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix}]{} & 1_\emptyset \oplus \Phi
\end{array}$$

commutes, so by definition, $E\alpha \in H(EF\Phi, EF)$.

2. Note $E\alpha^{-1} \in H(EF, EF\Phi)$ if the following diagram commutes:

$$\begin{array}{ccc}
EF & \xrightarrow{\quad E\alpha^{-1} \quad} & EF\Phi \\
 & & \Big\downarrow{\scriptstyle \mu^{-1}\Phi} \\
\Big\downarrow{\scriptstyle \mu^{-1}} & & \Phi \oplus \Phi\Phi \\
 & & \Big\downarrow{\scriptstyle \mathrm{diag}(1,z)} \\
1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_\emptyset
\end{array}$$

Equivalently, one must show that

$$\begin{array}{ccc}
EF & \xleftarrow{\quad E\alpha \quad} & EF\Phi \\
 & & \Big\uparrow{\scriptstyle \mu\Phi} \\
\Big\uparrow{\scriptstyle \mu} & & \Phi \oplus \Phi\Phi \\
 & & \Big\uparrow{\scriptstyle \mathrm{diag}(1,z^{-1})} \\
1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_\emptyset
\end{array}$$

commutes. Computing the counter-clockwise composite yields

$$E\alpha \circ \mu\Phi \circ \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix} = \Big((1-q)E\alpha \circ \eta_\emptyset\Phi \circ e' + qE\alpha \circ E\alpha\Phi \circ \eta_\emptyset\Phi\Phi \circ z^{-1} \quad E\alpha \circ \eta_\emptyset\Phi \Big).$$

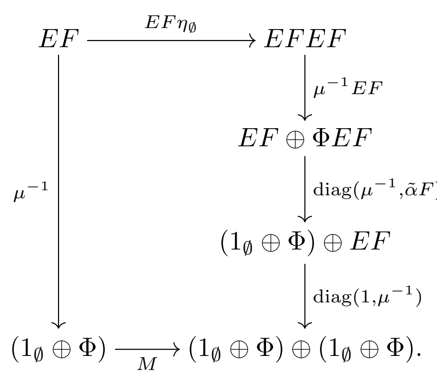
Using the previous computation, the first entry of this matrix simplifies as

$$(1-q)E\alpha \circ \eta_{\emptyset}\Phi \circ e' + [(q-1)E\alpha \circ \eta_{\emptyset}\Phi \circ e' + \eta_{\emptyset}] \circ z \circ z^{-1} = \eta_{\emptyset}$$

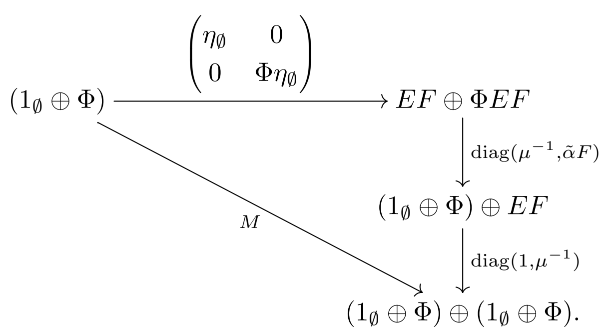
and hence the entire composite is equal to $\begin{pmatrix} \eta_{\emptyset} & E\alpha \circ \eta_{\emptyset}\Phi \end{pmatrix} = \mu$.

3. Showing $EF\mu \in H(EF \oplus EF\Phi, EFEF)$ is equivalent to showing that $EF\eta_{\emptyset} \in H(EF, EFEF)$ and $EFE\alpha \circ EF\eta_{\emptyset}\Phi \in H(EF\Phi, EFEF)$. However, if $EF\eta_{\emptyset} \in H(EF, EFEF)$, then $EF\eta_{\emptyset}\Phi \in H(EF\Phi, EFEF\Phi)$, so it is sufficient to show $EFE\alpha \in H(EFEF\Phi, EFEF)$ to conclude that $EFE\alpha \circ EF\eta_{\emptyset}\Phi \in H(EF\Phi, EFEF)$.

   First, $EF\eta_{\emptyset} \in H(EF, EFEF)$ if there is a matrix M with components in $H(1_{\emptyset}, 1_{\emptyset})$, $H(1_{\emptyset}, \Phi)$, $H(\Phi, 1_{\emptyset})$, or $H(\Phi, \Phi)$ such that the following diagram commutes



   This diagram may be simplified to



   However, the components of the rightmost matrices are already in their respective candidate morphism spaces, so it is enough to show $\eta_{\emptyset} \in H(1_{\emptyset}, EF)$, and $\Phi\eta_{\emptyset} \in H(\Phi, \Phi EF)$. First, $\eta_{\emptyset} \in H(1_{\emptyset}, EF)$ since the following clearly commutes

$$\begin{array}{ccc} 1_\emptyset & \xrightarrow{\eta_\emptyset} & EF \\ {\scriptstyle 1_{1_\emptyset}}\downarrow & & \downarrow{\scriptstyle \mu^{-1}} \\ 1_\emptyset & \xrightarrow[\begin{pmatrix}1\\0\end{pmatrix}]{} & 1_\emptyset \oplus \Phi. \end{array}$$

Secondly, the following commutes

$$\begin{array}{ccc} \Phi & \xrightarrow{\Phi\eta_\emptyset} & \Phi EF \\ & & \downarrow{\scriptstyle \tilde{\alpha}\Phi} \\ {\scriptstyle 1_\Phi}\downarrow & & EF \\ & & \downarrow{\scriptstyle \mu^{-1}} \\ \Phi & \xrightarrow[\begin{pmatrix}0\\1\end{pmatrix}]{} & 1_\emptyset \oplus \Phi \end{array}$$

since

$$\mu \circ \begin{pmatrix}0\\1\end{pmatrix} = E\alpha \circ \eta_\emptyset\Phi = \tilde{\alpha}F \circ \Phi\eta_\emptyset.$$

Furthermore, $EFE\alpha \in H(EFEF\Phi, EFEF)$ if there is a matrix M with components in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$ such that the following diagram commutes

$$\begin{array}{ccc} EFEF\Phi & \xrightarrow{EFE\alpha} & EFEF \\ {\scriptstyle \mu^{-1}}\downarrow & & \downarrow{\scriptstyle \mu^{-1}} \\ EF\Phi \oplus \Phi EF\Phi & & EF \oplus \Phi EF \\ {\scriptstyle \mathrm{diag}(\mu^{-1}\Phi, \tilde{\alpha}F\Phi)}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(\mu^{-1}, \tilde{\alpha}F} \\ (\Phi \oplus \Phi\Phi) \oplus EF\Phi & & (1_\emptyset \oplus \Phi) \oplus EF \\ {\scriptstyle \mathrm{diag}(1_\Phi, z, \mu^{-1}\Phi)}\downarrow & & \\ \Phi \oplus 1_\emptyset \oplus \Phi \oplus \Phi\Phi & & \downarrow{\scriptstyle \mathrm{diag}(1, \mu^{-1})} \\ {\scriptstyle \mathrm{diag}(1,1,1,z)}\downarrow & & \\ \Phi \oplus 1_\emptyset \oplus \Phi \oplus 1_\emptyset & \xrightarrow[M]{} & 1_\emptyset \oplus \Phi \oplus 1_\emptyset \oplus \Phi. \end{array}$$

This diagram can be extended and rewritten as

$$\begin{array}{ccc}
EFEF\Phi & \xrightarrow{\quad EFE\alpha \quad} & EFEF \\
\downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mu^{-1}} \\
EF\Phi \oplus \Phi EF\Phi & \xrightarrow{\begin{pmatrix} E\alpha & 0 \\ 0 & \Phi E\alpha \end{pmatrix}} & EF \oplus \Phi EF \\
\downarrow{\scriptstyle \mathrm{diag}(1,\tilde{\alpha}F\Phi)} & & \downarrow{\scriptstyle \mathrm{diag}(1,\tilde{\alpha}F} \\
EF\Phi \oplus EF\Phi & \xrightarrow{\begin{pmatrix} E\alpha & 0 \\ 0 & E\alpha \end{pmatrix}} & EF \oplus EF \\
\downarrow{\scriptstyle \mathrm{diag}(\mu^{-1}\Phi,\mu^{-1}\Phi)} & & \\
\Phi \oplus \Phi\Phi \oplus \Phi \oplus \Phi\Phi & & \downarrow{\scriptstyle \mathrm{diag}(\mu^{-1},\mu^{-1})} \\
\downarrow{\scriptstyle \mathrm{diag}(1,z,1,z)} & & \\
\Phi \oplus 1_\emptyset \oplus \Phi \oplus 1_\emptyset & \xrightarrow[M]{\qquad} & 1_\emptyset \oplus \Phi \oplus 1_\emptyset \oplus \Phi.
\end{array}$$

So M is a product of diagonal matrices, and the nonzero blocks are given by

$$\mu^{-1} \circ E\alpha \circ \mu\Phi \circ \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix}$$

as seen in the proof that $E\alpha \in H(EF\Phi, EF)$. Altogether, this shows $EF\mu \in H(EF \oplus EF\Phi, EFEF)$.

4. It is the case that $EF\mu^{-1} \in H(EFEF, EF \oplus EF\Phi)$ if there is a matrix M with components in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$ such that the following diagram commutes

$$\begin{array}{ccc}
EFEF & \xrightarrow{\quad EF\mu^{-1} \quad} & EF \oplus EF\Phi \\
\downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mathrm{diag}(\mu^{-1},\mu^{-1}\Phi)} \\
EF \oplus \Phi EF & \xrightarrow{\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \Phi\mu^{-1} \end{pmatrix}} & (1_\emptyset \oplus \Phi) \oplus (\Phi \oplus \Phi\Phi) \\
\downarrow{\scriptstyle \mathrm{diag}(\mu^{-1},\tilde{\alpha}F)} & & \\
(1_\emptyset \oplus \Phi) \oplus EF & & \downarrow{\scriptstyle \mathrm{diag}(1,1,1,z)} \\
\downarrow{\scriptstyle \mathrm{diag}(1,1,\mu^{-1})} & & \\
(1_\emptyset \oplus \Phi) \oplus (1_\emptyset \oplus \Phi) & \xrightarrow[M]{\qquad} & (1_\emptyset \oplus \Phi) \oplus (\Phi \oplus 1_\emptyset).
\end{array}$$

The upper square clearly commutes, and solving for M, if it exists, the only components which are not obviously in $H(1_\emptyset, 1_\emptyset)$, $H(1_\emptyset, \Phi)$, $H(\Phi, 1_\emptyset)$, or $H(\Phi, \Phi)$ are given by the composite morphism

$$\begin{pmatrix} 1_\Phi & 0 \\ 0 & z \end{pmatrix} \circ \Phi\mu^{-1} \circ \tilde{\alpha}^{-1}F \circ \mu \colon 1_\emptyset \oplus \Phi \longrightarrow \Phi \oplus 1_\emptyset.$$

The following also commutes

$$\begin{array}{ccc}
\Phi EF & \xrightarrow{\Phi\mu^{-1}} & \Phi \oplus \Phi\Phi \\
\uparrow{\scriptstyle \tilde{\alpha}^{-1}F} & & \\
EF & & \downarrow{\scriptstyle \begin{pmatrix} 1_\Phi & 0 \\ 0 & z \end{pmatrix}} \\
\uparrow{\scriptstyle \mu} & & \\
1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_\emptyset.
\end{array}$$

To see this, note the left vertical composite $\tilde{\alpha}^{-1}F \circ \mu$ is given by

$$\begin{pmatrix} \tilde{\alpha}^{-1}F \circ \eta_\emptyset & \tilde{\alpha}^{-1}F \circ E\alpha \circ \eta_\emptyset \Phi \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}^{-1}F \circ \eta_\emptyset & \tilde{\alpha}^{-1}F \circ \tilde{\alpha}F \circ \Phi\eta_\emptyset \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}^{-1}F \circ \eta_\emptyset & \Phi\eta_\emptyset \end{pmatrix}$$

whereas

$$\Phi\mu \circ \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} (1-q)\Phi\eta_\emptyset \circ e' + q\Phi E\alpha \circ \Phi\eta_\emptyset\Phi \circ z^{-1} & \Phi\eta_\emptyset \end{pmatrix}.$$

However, note

$$\begin{aligned}
&(1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q\tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_\emptyset\Phi \circ z^{-1} \\
&= (1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q\tilde{\alpha}F \circ \Phi\tilde{\alpha}F \circ \Phi\tilde{\alpha}F \circ \Phi\Phi\eta_\emptyset \circ z^{-1} \\
&= (1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q(q^{-1}zEF + q^{-1}(q-1)(\tilde{\alpha} \circ e'E)F \circ zEF) \circ z^{-1}EF \circ \eta_\emptyset \\
&= (1-q)\tilde{\alpha}F \circ e'EF \circ \eta_\emptyset + \eta_\emptyset + (q-1)\tilde{\alpha}F \circ e'EF \circ \eta_\emptyset \\
&= \eta_\emptyset
\end{aligned}$$

so that $\tilde{\alpha}^{-1}F \circ \eta_\emptyset = (1-q)\Phi\eta_\emptyset \circ e' + q\Phi E\alpha \circ \Phi\eta_\emptyset\Phi \circ z^{-1}$.

5. Since the diagram

$$\begin{array}{ccc}
EF & \xrightarrow{EFe} & EF \\
\downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mu^{-1}} \\
1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} e & 0 \\ 0 & e''' \end{pmatrix}]{} & 1_\emptyset \oplus \Phi
\end{array}$$

commutes, $EFe \in H(EF, EF)$.

6. Since it has already been shown that $E\alpha \in H(EF\Phi, EF)$, it is enough to show $EFe' \in H(EF, EF\Phi)$

to conclude $E(\alpha \circ Fe') = E\alpha \circ EFe' \in H(EF, EF)$. Since the following diagram commutes,

$$\begin{array}{ccc}
EF & \xrightarrow{EFe'} & EF\Phi \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu\Phi} \\
1_\emptyset \oplus \Phi & \xrightarrow{\begin{pmatrix} e' & 0 \\ 0 & \Phi e' \end{pmatrix}} & \Phi \oplus \Phi\Phi \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(1,z)} \\
1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} e' & 0 \\ 0 & e'' \end{pmatrix}]{} & \Phi \oplus 1_\emptyset
\end{array}$$

indeed $EFe' \in H(EF, EF\Phi)$.

$\square$

Proposition 5.18. *Suppose $X, Y : \emptyset \longrightarrow S$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(EX, EY) : f \mapsto Ef$$

takes values in $H(EX, EY)$.

Suppose $X, Y : S \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Any such 1-morphism must have form E, $\Phi EX'$, EFX', or $\Phi\Phi X'$ for some appropriate X'. Let $f \in H(X,Y)$, so there is a diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\
\bigoplus_i X_i & \xrightarrow[(a_{ji})]{} & \bigoplus_j Y_j
\end{array}$$

where X_i and Y_j are indecomposable 1-morphisms $S \longrightarrow \emptyset$. The only such arrow is E, so for each component we have $a_{ji} \in H(E, E)$. Postcomposing with Φ yields a commutative diagram

$$\begin{array}{ccc}
\bigoplus_i \Phi X_i & \xrightarrow{(\Phi a_{ji})} & \bigoplus_j \Phi Y_j \\
{\scriptstyle \Phi\delta_X}\uparrow & & \downarrow{\scriptstyle \Phi\delta_Y^{-1}} \\
\Phi X & \xrightarrow{\Phi f} & \Phi Y \\
{\scriptstyle \delta_{\Phi X}^{-1}}\uparrow & & \downarrow{\scriptstyle \delta_{\Phi Y}} \\
\bigoplus_r (\Phi X)_r & \xrightarrow[A]{} & \bigoplus_s (\Phi Y)_s
\end{array}$$

for some matrix A. Based on the form of 1-morphisms $S \longrightarrow \emptyset$, $\Phi\delta_X$ is a product of matrices with components of form $1_{\Phi E}$, $\Phi\tilde{\alpha}X'$, $\Phi z X'$, and $\Phi\mu^{-1}X'$, and $\Phi\delta_Y^{-1}$ is a product of matrices with components the inverses of those arrows. Likewise, since ΦX and ΦY are still 1-morphisms $S \longrightarrow \emptyset$, $\delta_{\Phi Y}$ is a product of matrices with components 1_E, $\tilde{\alpha}X'$, zX', or $\mu^{-1}X'$, and $\delta_{\Phi X}^{-1}$ is a product of matrices with inverse components. If each of these components is in the appropriate candidate space, and each $\Phi a_{ji} \in H(E,E)$ for $a_{ji} \in H(E,E)$, it follows that each component of A will be in $H(E,E)$, so that $\Phi f \in H(\Phi X, \Phi Y)$.

Lemma 5.19. *There are the following memberships.*

1. $\Phi z \in H(\Phi\Phi\Phi, \Phi)$
2. $\Phi z^{-1} \in H(\Phi, \Phi\Phi\Phi)$
3. $\Phi\tilde{\alpha} \in H(\Phi\Phi E, \Phi E)$
4. $\Phi\tilde{\alpha}^{-1} \in H(\Phi E, \Phi\Phi E$
5. $\Phi\mu \in H(\Phi \oplus \Phi\Phi, \Phi E F)$
6. $\Phi\mu^{-1} \in H(\Phi E F, \Phi \oplus \Phi\Phi)$
7. $\Phi e E \in H(\Phi E, \Phi E)$
8. $\Phi(\tilde{\alpha} \circ e' E) \in H(\Phi E, \Phi E)$

Proof.

1. By the relations, $\Phi z = z\Phi$, so $\Phi z = z\Phi \in H(\Phi\Phi\Phi, \Phi)$.
2. That $\Phi z^{-1} \in H(\Phi, \Phi\Phi\Phi)$ as the above.
3. One has $\Phi\tilde{\alpha} \in H(\Phi\Phi E, \Phi E)$ if there exists $M \in H(E,E)$ such that the following diagram commutes

$$\begin{array}{ccc} \Phi\Phi E & \xrightarrow{\Phi\tilde{\alpha}} & \Phi E \\ {\scriptstyle zE}\downarrow & & \downarrow{\scriptstyle \tilde{\alpha}} \\ E & \xrightarrow[M]{} & E. \end{array}$$

Solving for M yields

$$\tilde{\alpha} \circ \Phi\tilde{\alpha} \circ z^{-1}E = (q^{-1}zE + q^{-1}(q-1)\tilde{\alpha} \circ e'E \circ zE) \circ z^{-1}E = q^{-1}1_E + q^{-1}(q-1)\tilde{\alpha} \circ e'E \in H(E,E).$$

4. One has $\Phi\tilde{\alpha}^{-1} \in H(\Phi E, \Phi\Phi E)$ if there exists $M \in H(E, E)$ such that the following diagram commutes

$$\begin{CD} \Phi E @>{\Phi\tilde{\alpha}^{-1}}>> \Phi\Phi E \\ @V{\tilde{\alpha}}VV @VV{zE}V \\ E @>>{M}> E. \end{CD}$$

From the defining relation for $\Phi\tilde{\alpha} \circ \tilde{\alpha}$, one can conclude $\Phi\tilde{\alpha}^{-1} \circ \tilde{\alpha}^{-1} = q^{-1}z^{-1}E + (1-q)\Phi\tilde{\alpha}^{-1} \circ e'E$. Then note

$$\begin{aligned} zE \circ (\Phi\tilde{\alpha}^{-1} \circ \tilde{\alpha}^{-1}) &= q^{-1}1_E + (1-q)zE \circ \Phi\tilde{\alpha}^{-1} \circ e'E \\ &= q^{-1}1_E + (1-q)zE \circ e'\Phi E \circ \tilde{\alpha}^{-1} \\ &= q^{-1}1_E + (1-q)e''E \circ \tilde{\alpha}^{-1} \\ &= q^{-1}1_E + (1-q)(q\tilde{\alpha} \circ e'E + (1-q)eE) \in H(E, E). \end{aligned}$$

5. Note $\Phi\mu \in H(\Phi \oplus \Phi\Phi, \Phi EF)$ since the following commutes

$$\begin{CD} \Phi \oplus \Phi\Phi @>{\Phi\mu}>> \Phi EF \\ @. @VV{\tilde{\alpha}F}V \\ @V{\mathrm{diag}(1,z)}VV EF \\ @. @VV{\mu^{-1}}V \\ \Phi \oplus 1_\emptyset @>>{\begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix}}> 1_\emptyset \oplus \Phi. \end{CD}$$

To see this, note

$$\mu \circ \begin{pmatrix} 0 & q^{-1} \\ 1 & q^{-1}(q-1)e' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} E\alpha \circ \eta_\emptyset \Phi & q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha \circ \eta_\emptyset\Phi \circ e' \circ z \end{pmatrix}$$

whereas

$$\tilde{\alpha}F \circ \Phi\mu = \begin{pmatrix} \tilde{\alpha}F \circ \Phi\eta_\emptyset & \tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_\emptyset\Phi \end{pmatrix} = \begin{pmatrix} E\alpha \circ \eta_\emptyset\Phi & \tilde{\alpha}F \circ \Phi\tilde{\alpha}F \circ \Phi\Phi\eta_\emptyset \end{pmatrix}.$$

Comparing the second entry of these matrices, note

$$\begin{aligned}(\tilde{\alpha} \circ \Phi\tilde{\alpha})F \circ \Phi\Phi\eta_\emptyset &= (q^{-1}zEF + q^{-1}(q-1)\tilde{\alpha}F \circ e'EF \circ zEF) \circ \Phi\Phi\eta_\emptyset \\ &= q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' \circ z \\ &= q^{-1}\eta_\emptyset \circ z + q^{-1}(q-1)E\alpha\Phi\eta_\emptyset\Phi \circ e' \circ z.\end{aligned}$$

6. Note $\Phi\mu^{-1} \in H(\Phi EF, \Phi \oplus \Phi\Phi)$ since the following commutes

$$\begin{array}{ccc} \Phi EF & \xrightarrow{\Phi\mu^{-1}} & \Phi \oplus \Phi\Phi \\ \downarrow{\scriptstyle \tilde{\alpha}F} & & \\ EF & & \downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}} \\ \downarrow{\scriptstyle \mu^{-1}} & & \\ 1_\emptyset \oplus \Phi & \xrightarrow[\begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix}]{} & \Phi \oplus 1_\emptyset \oplus \Phi. \end{array}$$

To see this, note

$$\tilde{\alpha}F \circ \Phi\mu \circ \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} (1-q)e' & 1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} (1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q\tilde{\alpha}F \circ \Phi E\alpha \circ \Phi\eta_\emptyset\Phi \circ z^{-1} & E\alpha \circ \eta_\emptyset\Phi \end{pmatrix}.$$

The first component simplifies as

$$\begin{aligned}&(1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q(\tilde{\alpha} \circ \Phi\tilde{\alpha})F \circ \eta_\emptyset\Phi\Phi \circ z^{-1} \\ &= (1-q)\tilde{\alpha}F \circ \Phi\eta_\emptyset \circ e' + q(q^{-1}zEF + q^{-1}(q-1)\tilde{\alpha}F \circ e'EF \circ zEF) \circ z^{-1}EF \circ \eta_\emptyset = \eta_\emptyset\end{aligned}$$

and hence the above matrix is $\begin{pmatrix} \eta_\emptyset & E\alpha \circ \eta_\emptyset\Phi \end{pmatrix} = \mu$.

7. Since $e\Phi = \Phi e$, $\Phi eE \in H(\Phi E, \Phi E)$ since

$$\tilde{\alpha} \circ \Phi eE \circ \tilde{\alpha}^{-1} = \tilde{\alpha} \circ e\Phi E \circ \tilde{\alpha}^{-1} = \tilde{\alpha} \circ \tilde{\alpha}^{-1} \circ eE = eE \in H(E, E).$$

8. It is sufficient to show $\Phi e'E \in H(\Phi E, \Phi\Phi E)$ to conclude $\Phi(\tilde{\alpha} \circ e'E) \in H(\Phi E, \Phi E)$ since it has already been shown $\Phi\tilde{\alpha} \in H(\Phi\Phi E, \Phi E)$. First, $zE \circ \Phi e'E \circ \tilde{\alpha}^{-1} = (z \circ \Phi e')E \circ \tilde{\alpha}^{-1} = e''E \circ \tilde{\alpha}^{-1}E$. Rearranging the relation $\tilde{\alpha} \circ e'E \circ \tilde{\alpha} = q^{-1}e''E + q^{-1}(q-1)eE \circ \tilde{\alpha}$ shows $e''E \circ \tilde{\alpha}^{-1} = q\tilde{\alpha} \circ e'E + (1-q)eE$, which is in $H(E, E)$. Hence $\Phi e'E \in H(\Phi E, \Phi\Phi E)$.

□

Proposition 5.20. *Suppose $X, Y\colon S \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Phi X, \Phi Y) : f \mapsto \Phi f$$

takes values in $H(\Phi X, \Phi Y)$.

We can also compose with F, yielding a commutative diagram for some matrix A

$$\begin{array}{ccc}
\bigoplus_i FX_i & \xrightarrow{(Fa_{ji})} & \bigoplus_j FY_j \\
\uparrow{\scriptstyle F\delta_X} & & \downarrow{\scriptstyle F\delta_Y^{-1}} \\
FX & \xrightarrow{Ff} & FY \\
\uparrow{\scriptstyle \delta_{FX}^{-1}} & & \downarrow{\scriptstyle \delta_{FY}} \\
\bigoplus_r (FX)_r & \xrightarrow[A]{} & \bigoplus_s (FY)_s.
\end{array}$$

Similarly to before, $F\delta_X$ is a product of matrices with components of form 1_{FE}, $F\tilde{\alpha}$, FzX', or $F\mu^{-1}X'$, and $F\delta_Y^{-1}$ is a product of matrices with components with the inverse components. Now, FX and FY are 1-morphisms $S \longrightarrow S$, and any such 1-morphism must have form 1_S, FE, $FEFX'$, or $F\Phi X'$ for some X'. It follows that δ_{FY} is a product of matrices with components 1_{1_S}, 1_{FE}, $F\mu^{-1}X'$, or $\alpha X'$, for some appropriate X', and δ_{FX}^{-1} is a product of matrices with components with the inverse components. Lastly, for A to have components in the appropriate candidate spaces, on also requires $FeE \in H(FE, FE)$, and $F(\tilde{\alpha} \circ e'E) \in H(FE, FE)$. All these components are in their corresponding candidate spaces by the proof of the case for FX and FY, when $X, Y\colon \emptyset \longrightarrow \emptyset$.

Proposition 5.21. *Suppose $X, Y\colon S \longrightarrow \emptyset$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(FX, FY) : f \mapsto Ff$$

takes values in $H(FX, FY)$.

Suppose $X, Y\colon S \longrightarrow S$ are parallel 1-morphisms in $\mathcal{C}'$. Let $f \in H(X,Y)$, so there is a diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\
\bigoplus_i X_i & \xrightarrow[(a_{ji})]{} & \bigoplus_j Y_j.
\end{array}$$

The X_i and Y_j are indecomposable 1-morphisms $S \longrightarrow S$, the only such of which are 1_S and FE, so each component a_{ji} is either in $H(1_S, 1_S)$ or $H(FE, FE)$. Postcomposing with E yields a commutative diagram for some matrix A

$$\begin{array}{ccc} \bigoplus_i EX_i & \xrightarrow{(Ea_{ji})} & \bigoplus_j EY_j \\ {\scriptstyle E\delta_X}\uparrow & & \downarrow{\scriptstyle E\delta_Y^{-1}} \\ EX & \xrightarrow{Ef} & EY \\ {\scriptstyle \delta_{EX}^{-1}}\uparrow & & \downarrow{\scriptstyle \delta_{EY}} \\ \bigoplus_r (EX)_r & \xrightarrow[A]{} & \bigoplus_s (EY)_s. \end{array}$$

Now $EX, EY : S \longrightarrow \emptyset$, and as seen before, any such 1-morphism has form E, $\Phi EX'$, EFX', or $\Phi\Phi X'$ for some appropriate X'. All the components in the matrices comprising δEX^{-1} and δ_{EY} are in their corresponding candidate space by previous cases.

On the other hand, the components of matrices comprising δ_X and δ_Y are either 1_{1_S}, 1_{FE}, $F\mu^{-1}X'$, or $\alpha X'$ for some appropriate X'. Hence $E\delta_X$ is a product of matrices whose components consist of 1_E, 1_{EFE}, $EF\mu^{-1}X'$, or $E\alpha X'$, and $E\delta_Y^{-1}$ is a product of matrices with inverse components. By previous cases, all these components are in their corresponding candidate spaces. Furthermore, for A to have components in the appropriate candidate spaces, necessarily $Ea_{ji} \in H(E, E)$ for $a_{ji} \in H(1_S, 1_S)$, and $Ea_{ji} \in H(EFE, EFE)$ for $a_{ji} \in H(FE, FE)$. To this end, there is the following lemma.

Lemma 5.22.

1. *If $a_{ji} \in H(1_S, 1_S)$, then $Ea_{ji} \in H(E, E)$.*

2. *If $a_{ji} \in H(FE, FE)$, then $Ea_{ji} \in H(EFE, EFE)$.*

Proof. Recall that $H(1_S, 1_S)$ is generated by 1_{1_S}, $e_0 = \epsilon_S \circ \eta_S$, $e_1 = \epsilon_S \circ Fe \circ \eta_S$, and $e_2 = \epsilon_S \circ (\alpha \circ Fe')E \circ \eta_S$. Clearly $E1_{1_S} = 1_E \in H(E, E)$. For $Ee_0 = E\epsilon_S \circ E\eta_S$, note that commutativity of

$$\begin{array}{ccc} E & \xrightarrow{E\eta_S} & EFE \\ & \searrow & \downarrow{\scriptstyle \mu^{-1}E} \\ & {\scriptstyle \begin{pmatrix} q1_E \\ (-1)^{\epsilon}(q1_E + q(q-1)\tilde{\alpha} \circ e'E) \end{pmatrix}} & E \oplus \Phi E \\ & & \downarrow{\scriptstyle \mathrm{diag}(1, \tilde{\alpha})} \\ & & E \oplus E \end{array}$$

is one of the defining relations, hence $E\eta_S \in H(E, EFE)$. Furthermore, the following commutes

$$\begin{array}{ccc} EFE & \xrightarrow{E\epsilon_S} & E \\ {\scriptstyle \mu^{-1}E}\downarrow & & \\ E\oplus\Phi E & \nearrow{\scriptstyle \begin{pmatrix}1_E & 1_E\end{pmatrix}} & \\ {\scriptstyle \mathrm{diag}(1,\tilde{\alpha})}\downarrow & & \\ E\oplus E & & \end{array}$$

since

$$\begin{aligned} E\epsilon_S \circ \mu^{-1}E \circ \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\alpha}^{-1} \end{pmatrix} &= \begin{pmatrix} E\epsilon_S \circ \eta_\emptyset E & E\epsilon_S \circ E\alpha E \circ \eta_\emptyset \Phi E \circ \tilde{\alpha}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1_E & \tilde{\alpha} \circ \tilde{\alpha}^{-1} \end{pmatrix} = \begin{pmatrix} 1_E & 1_E \end{pmatrix}. \end{aligned}$$

Thus $E\epsilon_S \in H(EFE, E)$, and so $Ee_0 \in H(E, E)$. Since $EFe \in H(EF, EF)$, it follows $Ee_1 \in H(E, E)$. Also, it has previously been shown that $E(\alpha \circ Fe') \in H(EF, EF)$, so $Ee_2 \in H(E, E)$ as well. This proves the first claim.

For the second, recall that $H(FE, FE)$ is generated by morphisms with form $\begin{pmatrix} \varphi E & (\psi \circ \alpha)E \end{pmatrix} \circ F\mu^{-1}E \circ FE\eta_S$ for $\varphi, \psi \in H(F, F)$. By the defining relations, one has

$$F\mu^{-1}E \circ FE\eta_S = \begin{pmatrix} 1_{FE} & 0 \\ 0 & F\tilde{\alpha}^{-1} \end{pmatrix} \begin{pmatrix} q1_{FE} \\ (-1)^\epsilon(q1_{FE} + q(q-1)F(\tilde{\alpha} \circ e'E)) \end{pmatrix} = \begin{pmatrix} q1_{FE} \\ (-1)^\epsilon(qF\tilde{\alpha}^{-1} + q(q-1)Fe'E). \end{pmatrix}$$

So we can assume any morphism Ea_{ji} for $a_{ji} \in H(FE, FE)$ has form

$$\begin{pmatrix} E\varphi E & E(\psi \circ \alpha)E \end{pmatrix} \begin{pmatrix} q1_{EFE} \\ (-1)^\epsilon(qEF\tilde{\alpha}^{-1} + q(q-1)EFe'E) \end{pmatrix}.$$

However, from previous cases, it is immediate that $E\varphi E \in H(EFE, EFE)$ when $\varphi \in H(F, F)$, $E\alpha E \in H(EF\Phi E, EFE)$, and $EFe'E \in H(EFE, EF\Phi E)$. It remains to check that $EF\tilde{\alpha}^{-1} \in H(EFE, EF\Phi E)$ in order to conclude $Ea_{ji} \in H(EFE, EFE)$. Indeed, the following diagram commutes

$$\begin{array}{ccc} EFE & \xrightarrow{EF\tilde{\alpha}^{-1}} & EF\Phi E \\ {\scriptstyle \mu^{-1}E}\downarrow & & \downarrow{\scriptstyle \mu^{-1}\Phi E} \\ E\oplus\Phi E & \xrightarrow{\begin{pmatrix} \tilde{\alpha}^{-1} & 0 \\ 0 & \Phi\tilde{\alpha}^{-1} \end{pmatrix}} & \Phi E \oplus \Phi\Phi E \\ {\scriptstyle \mathrm{diag}(1,\tilde{\alpha})}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(\tilde{\alpha}, zE)} \\ E\oplus E & \xrightarrow[\begin{pmatrix} 1_E & 0 \\ 0 & zE\circ\Phi\tilde{\alpha}^{-1}\circ\tilde{\alpha}^{-1} \end{pmatrix}]{} & E\oplus E. \end{array}$$

Previous computations have shown that

$$zE \circ (\Phi\tilde{\alpha}^{-1} \circ \tilde{\alpha}^{-1}) = q1_E + (1-q)(q\tilde{\alpha} \circ e'E \circ (1-q)E \in H(E,E)$$

so that $EF\tilde{\alpha}^{-1} \in H(EFE, EF\Phi E)$.

Proposition 5.23. *Suppose $X, Y : S \longrightarrow S$ are parallel 1-morphisms in $\mathcal{C}$. Then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(EX, EY) : f \mapsto Ef$$

takes values in $H(EX, EY)$.

As before, these propositions can be collected more succinctly.

Proposition 5.24. *If X and Y are parallel 1-morphisms in $\mathcal{C}$, and Z is any appropriate 1-morphism, then the map*

$$H(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(ZX, ZY) : f \mapsto Zf$$

takes values in $H(ZX, ZY)$.

□

5.5 Closure

Corollary 5.25. *Suppose $f\colon X \longrightarrow Y$ is either a generating 2-morphism in $\mathcal{C}$, or one of α^{-1}, $\tilde{\alpha}^{-1}$, μ^{-1}, or z^{-1}. Then for any 1-morphisms A and B such that AXB and AYB are defined, the 2-morphism AfB is an element of $H(AXB, AYB)$.*

Proof. If f is any of the aforementioned 2-morphisms, then in all cases $f \in H(X,Y)$. Then AfB is the image of the composite

$$H(X,Y) \longrightarrow H(XB, YB) \longrightarrow H(AXB, AYB) : f \mapsto fB \mapsto AfB.$$

□

Theorem 5.26. *For any parallel 1-morphisms X and Y in $\mathcal{C}$, $H(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$.*

Proof. Pick $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Observe that f is a composite of 2-morphisms of form AxB, where x is a generating 2-morphism, or one of α^{-1}, $\tilde{\alpha}^{-1}$, μ^{-1}, or z^{-1}. Induct on the number of such factors. If $f = AxB$, the previous corollary shows $f = AxB \in H(X,Y)$. If f is a composite of more than 1 such factor, write

$f = f' \circ AxB$, for a nontrivial 2-morphism $x\colon C \longrightarrow C'$. By the corollary, $AxB \in H(X, AC'B)$, and since f' is a composite of fewer factors, the induction hypothesis implies $f' \in H(AC'B, Y)$. Since the candidate spaces are closed under composition, $f \in H(X, Y)$. □

5.6 A Functor into the 2-Category of Bimodules

Let Bimod denote the usual 2-category of bimodules, with 0-morphisms rings, 1-morphisms bimodules, and 2-morphisms bimodule homomorphisms. Let $G = SL_2(q)$ and let k be a field of appropriate characteristic such that q and $q-1$ are invertible in k. Let B denote the standard Borel subgroup of G, and write $B = UT$ be the Levi decomposition, for T a maximal torus and U the unipotent radical. Let e_U and e_T denote the idempotents in kG corresponding to U and T, respectively. Let (W, Π) be the corresponding Weyl group of rank 1, with simple reflection s. Let $\pi\colon N \longrightarrow W$ be the canonical projection onto W, and let n_w denote a preimage in N of $w \in W$.

Definition 5.27. Define a 2-functor $\mathscr{F}: \mathcal{C}' \longrightarrow \mathsf{Bimod}$ as follows. On 0-morphisms, put $\mathscr{F}(\emptyset) = kT$, and $\mathscr{F}(S) = kG$. On generating 1-morphisms, put with the obvious bimodule structures,

- $\mathscr{F}(1_\emptyset) = kT$
- $\mathscr{F}(1_S) = kG$
- $\mathscr{F}(F) = kGe_U$
- $\mathscr{F}(E) = e_U kG$
- $\mathscr{F}(\Phi) = {}_s kT$. As a kT-bimodule, the left action $\cdot$ on ${}_s kT$ is given by $t' \cdot t = ({}^{\dot{s}}t')t = \dot{s}t'\dot{s}^{-1}t$. The right kT-action is the usual multiplication.

On generating 2-morphisms, set

- $\mathscr{F}(e')\colon kT \longrightarrow {}_s kT : 1 \mapsto e_T$
- $\mathscr{F}(e'')\colon {}_s kT \longrightarrow kT : 1 \mapsto e_T$
- $\mathscr{F}(\alpha) = kGe_U \otimes_{kT} {}_s kT \longrightarrow kGe_U : e_U \otimes 1 \mapsto e_U \dot{s} e_U$
- $\mathscr{F}(z) = {}_s kT \otimes_{kT} {}_s kT \longrightarrow kT : a \otimes b \mapsto ({}^{\dot{s}}a)b$
- $\mathscr{F}(\eta_\emptyset)\colon kT \longrightarrow e_U kG \otimes_{kG} kGe_U : 1 \mapsto q^{-1}(e_U \otimes e_U)$
- $\mathscr{F}(\eta_S)\colon kG \to kGe_U \otimes_{kT} e_U kG : 1 \mapsto \sum_{g \in G/B} g e_U \otimes e_U g^{-1}$, where g ranges over a complete set of coset representatives in G/B.

- $\mathscr{F}(\epsilon_S)\colon kGe_U \otimes_{kT} e_U kG \longrightarrow kT : e_U \otimes e_U \mapsto qe_U$

- $\mathscr{F}(\epsilon_\emptyset)\colon e_U kG \otimes_{kG} kGe_U \longrightarrow kT : e_U n_{\dot{w}} e_U \mapsto \begin{cases} n_w & \text{if } \pi(n_w) = 1, \\ 0 & \text{otherwise.} \end{cases}$.

For the last definition of $\mathscr{F}(\epsilon_\emptyset)$, we have identified $e_U kG \otimes_{kG} kGe_U \simeq e_U kGe_U$, and as a kT-bimodule map, $\mathscr{F}(\epsilon_\emptyset)$ is completely determined by its images on elements of form $e_U n_w e_U$, by the Bruhat decomposition. Note also that the definition of $\mathscr{F}$ on the adjunctions $(\epsilon_S, \eta_\emptyset) : F \dashv E$ and $(\epsilon_\emptyset, \eta_S) : E \dashv F$ follows from the standard adjunctions on self-dual pairs of exact bimodules, as found on p. 158 in [5], and hence still satisfy the triangle relations in Bimod.

To reduce notation, if M is an (A, B)-bimodule, no distinction will be made between $A \otimes_A M$ and M, or $M \otimes_B B$ and M.

Theorem 5.28. *The 2-functor $\mathscr{F} : \mathcal{C}' \longrightarrow$ Bimod induces a 2-functor $\mathcal{C} \longrightarrow$ Bimod, also denoted by $\mathscr{F}$.*

Three main properties need to be verified. First, that the morphisms $\mathscr{F}(\mu)$, $\mathscr{F}(\alpha)$, $\tilde{\alpha}$, and μ are invertible in Bimod, second, that $\mathscr{F}$ preserves the defining relations of $\mathcal{C}$, and third, that the standard maps induced by the adjunctions on $\mathrm{Hom}_{\mathsf{Bimod}}(\mathscr{F}(F), \mathscr{F}(F)) \longrightarrow \mathrm{Hom}_{\mathsf{Bimod}}(\mathscr{F}(E), \mathscr{F}(E))$, etc., coincide.

Proposition 5.29. *The morphisms $\mathscr{F}(\mu)$, $\mathscr{F}(\alpha)$, $\mathscr{F}(\tilde{\alpha})$, and $\mathscr{F}(z)$ are invertible in Bimod, with the following inverses. One has*

- $\mathscr{F}(\mu)^{-1}\colon e_U kGe_U \longrightarrow kT \oplus {}_s kT : e_U t e_U \mapsto (qt, 0),\ e_U \dot{s} t e_U \mapsto (0, qt)$

- $\mathscr{F}(\alpha)^{-1}\colon kGe_U \longrightarrow kGe_U \otimes_{kT} {}_s kT : ge_U \mapsto ge_U \xi_0 \otimes 1$

- $\mathscr{F}(\tilde{\alpha})^{-1}\colon e_U kG \longrightarrow {}_s kT \otimes_{kT} e_U kG : e_U g \mapsto 1 \otimes \xi_0 e_U g$

- $\mathscr{F}(z)^{-1}\colon kT \longrightarrow {}_s kT \otimes_{kT} {}_s kT\colon t \mapsto 1 \otimes t.$

Proof. First observe that $\mathscr{F}(\mu)$, $\mathscr{F}(\alpha)$, $\mathscr{F}(\tilde{\alpha})$ and $\mathscr{F}(z)$ are invertible in Bimod. Beginning with $\mathscr{F}(\mu)$, let kT have ordered basis $(t_1, \ldots, t_{q-1})$ and ${}_s kT$ have ordered basis $(t'_1, \ldots, t'_{q-1})$. By the definition of μ, for $t_i \in kT$,

$$\mathscr{F}(\mu)(t_i) = \mathscr{F}(\eta_\emptyset)(t_i) = q^{-1} e_U t_i e_U,$$

and for $t'_i \in {}_s kT$,

$$\begin{aligned}\mathscr{F}(\mu)(t'_i) &= \mathscr{F}(E\alpha \circ \eta_\emptyset \Phi)(t'_i) = \mathscr{F}(E\alpha)(q^{-1} e_U \otimes e_U \otimes t'_i) \\ &= q^{-1} e_U \otimes \mathscr{F}(\alpha)(e_U \otimes 1) t'_i = q^{-1} e_U \otimes e_U \dot{s} e_U t'_i = q^{-1} e_U \dot{s} t'_i e_U.\end{aligned}$$

Since $G = B \sqcup U\dot{s}B$, $e_U kG e_U$ has a k-basis $\{e_U t_i e_U, e_U \dot{s} t_i e_U\}_{i=1}^{q-1}$, and an explicit inverse $e_U kG e_U \to kT \oplus {}_s kT$ for $\mathscr{F}(\mu)$ is defined on this basis as

$$e_U t_i e_U \mapsto (qt_i, 0), \qquad e_U \dot{s} t_i e_U \mapsto (0, qt_i).$$

By Theorem 2.3 of [10], recall that if (W, Π) is a Coxeter system for a group G with BN-pair with parabolic subgroup $P = UL$, then the following holds.

Theorem 5.30. *Let $J \subseteq \Pi$ and $w \in W$ be such that $K = wJ \subseteq \Pi$. Then there is a linear isomorphism*

$$\phi \colon RGe_{U_K} \longrightarrow RGe_{U_J} : \xi \mapsto \xi e_{U_K} w e_{U_J}$$

satisfying $\phi(ge_{U_K} t) = g\phi(e_{U_K})t^w$ for all $g \in RG$ and $t \in RL$. The inverse is given by right multiplication by suitable $\xi_0 \in e_{U_J} RGe_{U_K}$, as there exists such ξ_0 satisfying $\xi_0 e_{U_K} w e_{U_J} = e_{U_J}$ and $e_{U_K} w e_{U_J} \xi_0 = e_{U_K}$.

In our case, with $K = J = \emptyset$, $U_J = U_K = U$, and $w = \dot{s}$, so that $\xi_0 e_U \dot{s} e_U = e_U = e_U \dot{s} e_U \xi_0$. Additionally, $\phi^{-1} \colon kGe_U \to kGe_U : \xi \mapsto \xi\xi_0$ is a (kG, kT)-bimodule map, and for $t \in T$,

$$t\xi_0 = te_U\xi_0 = e_U t\xi_0 = \phi^{-1}(e_U t) = \phi^{-1}(e_U)t^{\dot{s}^{-1}} = e_U\xi_0 t^{\dot{s}^{-1}} = \xi_0 t^{\dot{s}^{-1}}.$$

So define

$$\beta \colon kGe_U \longrightarrow kGe_U \otimes_{kT} {}_s kT : ge_U \mapsto ge_U\xi_0 \otimes 1.$$

Then β is clearly a left kG-module map, and is also a right kT-module map as from the above commutativity relation,

$$\begin{aligned}\beta(e_U t) = \beta(te_U) = te_U\xi_0 \otimes 1 = e_U t\xi_0 \otimes 1 = e_U\xi_0\, {}^{\dot{s}}t \otimes 1 = e_U\xi_0 \otimes {}^{\dot{s}}t \cdot 1 = e_U\xi_0 \otimes {}^{\dot{s}^{-1}}\,{}^{\dot{s}}t \\ = e_U\xi_0 \otimes t = (e_U\xi_0 \otimes 1) \cdot t = \beta(e_U)t.\end{aligned}$$

Also,

$$\beta \circ \mathscr{F}(\alpha)(e_U \otimes 1) = \beta(e_U \dot{s} e_U) = e_U \dot{s} e_U \xi_0 \otimes 1 = e_U \otimes 1$$

and

$$\mathscr{F}(\alpha)\beta(e_U) = \mathscr{F}(\alpha)(e_U\xi_0 \otimes 1) = \xi_0 e_U \dot{s} e_U = e_U,$$

so that β is the inverse of $\mathscr{F}(\alpha)$. Similarly,

$$\mathscr{F}(\tilde{\alpha})\colon\ {}_s kT \otimes_{kT} e_U kG \longrightarrow e_U kG : 1 \otimes e_U \mapsto e_U \dot{s} e_U$$

has inverse given by

$$\tilde{\beta}\colon e_U kG \longrightarrow {}_s kT \otimes_{kT} e_U kG : e_U g \mapsto 1 \otimes \xi_0 e_U g.$$

Lastly, define

$$\zeta\colon kT \longrightarrow {}_s kT \otimes_{kT} {}_s kT : t \mapsto 1 \otimes t.$$

This is easily checked to be a kT-bimodule map, and the inverse to $\mathscr{F}(z)$. □

Proposition 5.31. *The defining relations of the category $\mathcal{C}$ are preserved by $\mathscr{F}$.*

Proof.

1. For the first relation, consider $\mathscr{F}(\epsilon_\emptyset) \circ \mathscr{F}(\eta_\emptyset)\colon kT \longrightarrow kT$. One a generator $1 \in kT$,

$$\mathscr{F}(\epsilon_\emptyset) \circ \mathscr{F}(\eta_\emptyset)(1) = \mathscr{F}(\epsilon_\emptyset)(q^{-1} e_U \otimes e_U) = q^{-1}$$

and

$$\mathscr{F}(\epsilon_\emptyset) \circ \mathscr{F}(E\alpha) \circ \mathscr{F}(\eta_\emptyset \Phi)(1) = \mathscr{F}(\epsilon_\emptyset) \circ \mathscr{F}(E\alpha)(q^{-1} e_U \otimes e_U \otimes 1) = \mathscr{F}(\epsilon_\emptyset)(q^{-1} e_U \otimes e_U \dot{s} e_U) = 0.$$

2. For the second relation, first note that since $G = B \sqcup U\dot{s}B$, one has $G/B = \{B, u\dot{s}B\}_{u \in U}$. Then the relative Casimir element in $kGe_U \otimes_{kT} e_U kG$ is

$$\sum_{g \in G/B} g e_U \otimes e_U g^{-1} = (e_U \otimes e_U) + \sum_{u \in U} u\dot{s} e_U \otimes e_U \dot{s}^{-1} u^{-1}.$$

The composite

$$E \xrightarrow{\mathscr{F}(E\eta_S)} EFE \xrightarrow{\mathscr{F}(\mu^{-1}E)} E \oplus \Phi E \xrightarrow{\operatorname{diag}(\mathscr{F}(1_E), \mathscr{F}(\tilde{\alpha}))} E \oplus E$$

is given by $\begin{pmatrix} q\mathscr{F}(1_E) \\ (-1)^\varepsilon (q\mathscr{F}(1_E) + q(q-1)\mathscr{F}(\tilde{\alpha} \circ e'E)) \end{pmatrix}$ where ε is determined by $\dot{s}^2 = (-1)^\varepsilon$. Explicitly, first observe

$$
\begin{aligned}
\mathscr{F}(\mu^{-1}E)\circ\mathscr{F}(E\eta_S)(e_U) &= \mathscr{F}(\mu^{-1}E)\left(e_U\otimes\sum_{g\in G/B} ge_U\otimes e_Ug^{-1}\right)\\
&= \mathscr{F}(\mu^{-1}E)\left(e_U\otimes e_U\otimes e_U+\sum_{u\in U} e_U\otimes u\dot{s}e_U\otimes e_U\dot{s}^{-1}u^{-1}\right)\\
&= \mathscr{F}(\mu^{-1}E\left(e_U\otimes e_U+\sum_{u\in U} e_U\dot{s}e_U\otimes e_U\dot{s}^{-1}u^{-1}\right)\\
&= \begin{pmatrix} q\\ 0\end{pmatrix}\otimes e_U+\begin{pmatrix} 0\\ q\end{pmatrix}\otimes qe_U\dot{s}^{-1}e_U\\
&= \begin{pmatrix} qe_U\\ q^2e_U\dot{s}^{-1}e_U\end{pmatrix}.
\end{aligned}
$$

Applying $\operatorname{diag}(\mathscr{F}(1_E),\mathscr{F}(\tilde{\alpha}))$ yields $\begin{pmatrix} qe_U\\ q^2e_U\dot{s}e_U\dot{s}^{-1}e_U\end{pmatrix}$. The first component is thus given by $q\mathscr{F}(1_E)$. For the second component, note

$$
q^2e_U\dot{s}e_U\dot{s}^{-1}e_U = q^2(-1)^{\varepsilon}(e_U\dot{s}e_U)^2 = (-1)^{\varepsilon}(qe_U+q(q-1)e_U\dot{s}e_Te_U)
$$

and the arrow

$$
e_UkG \xrightarrow{\mathscr{F}(e'E)} {}_skT\otimes_{kT} e_UkG \xrightarrow{\mathscr{F}(\tilde{\alpha})} e_UkG
$$

corresponds to $e_U\mapsto e_T\otimes e_U\mapsto e_Te_U\dot{s}e_U = e_U\dot{s}e_Te_U$. Hence the second component is given by $(-1)^{\varepsilon}(q\mathscr{F}(1_E)+q(q-1)\mathscr{F}(\tilde{\alpha}\circ e'E))$.

3. For the third relation,

$$
F \xrightarrow{\mathscr{F}(\eta_S F)} FEF \xrightarrow{\mathscr{F}(F\mu^{-1})} F\oplus F\Phi \xrightarrow{\operatorname{diag}(\mathscr{F}(1_F),\mathscr{F}(\alpha))} F\oplus F
$$

is given by $\begin{pmatrix} q\mathscr{F}(1_F) \\ (-1)^\varepsilon(q\mathscr{F}(1_F)+q(q-1)\mathscr{F}(\alpha\circ Fe')) \end{pmatrix}$. Note

$$\begin{aligned}
\mathscr{F}(F\mu^{-1})\circ\mathscr{F}(\eta_S F)(e_U) &= \mathscr{F}\mu^{-1}\left(\sum_{g\in G/B} ge_U\otimes e_U g^{-1}\otimes e_U\right) \\
&= \mathscr{F}(F\mu^{-1})\left(\sum_{u\in U} u\dot{s}e_U\otimes e_U\dot{s}^{-1}u^{-1}\otimes e_U + e_U\otimes e_U\otimes e_U\right) \\
&= \mathscr{F}(F\mu^{-1})\left((-1)^\varepsilon qe_U\dot{s}e_U\otimes e_U\dot{s}e_U + e_U\otimes e_U\right) \\
&= (-1)^\varepsilon qe_U\dot{s}e_U\otimes\begin{pmatrix}0\\ q\end{pmatrix} + e_U\otimes\begin{pmatrix}q\\ 0\end{pmatrix} \\
&= \begin{pmatrix} qe_U \\ (-1)^\varepsilon q^2 e_U\dot{s}e_U\end{pmatrix}.
\end{aligned}$$

Applying $\operatorname{diag}(\mathscr{F}(1_F),\mathscr{F}(\alpha))$ yields $\begin{pmatrix} qe_U \\ (-1)^\varepsilon q^2 e_U\dot{s}e_U\dot{s}e_U\end{pmatrix}$. The first component is then given by $q\mathscr{F}(1_F)$. For the second, as computed before, $(-1)^\varepsilon q^2(e_U\dot{s}e_U)^2 = (-1)^\varepsilon(qe_U+q(q-1)e_U\dot{s}e_Te_U)$, viewed as elements of kGe_U, instead of e_UkG. The arrow

$$kGe_U \xrightarrow{\mathscr{F}(Fe')} kGe_U\otimes_{kT}{}_skT \xrightarrow{\mathscr{F}(\alpha)} kGe_U$$

is the map $e_U\mapsto e_Ue_Te_U\mapsto e_U\dot{s}e_Te_U$, thus verifying the second component.

4. For the fourth relation, explicit computation shows that $\xi_0 = qe_U\dot{s}e_U-(q-1)e_Ue_Te_U$. Then

$$kT \xrightarrow{\mathscr{F}(\eta_\emptyset)} e_UkGe_U \xrightarrow{\mathscr{F}(\tilde{\alpha}^{-1}F)} {}_skT\otimes_{kT} e_UkGe_U \xrightarrow{\mathscr{F}(\Phi\epsilon_\emptyset)} {}_skT$$

is given by $q^{-1}(1-q)\mathscr{F}(e')$, since

$$\begin{aligned}
\mathscr{F}(\Phi\epsilon_\emptyset)\circ\mathscr{F}(\tilde{\alpha}^{-1}F)\circ\mathscr{F}(\eta_\emptyset)(1) &= \mathscr{F}(\Phi\epsilon_\emptyset)\circ\mathscr{F}(\tilde{\alpha}^{-1}F)(q^{-1}e_U\otimes e_U) \\
&= \mathscr{F}(\Phi\epsilon_\emptyset)(q^{-1}\otimes\xi_0\otimes e_U) \\
&= \mathscr{F}(\Phi\epsilon_\emptyset)(q^{-1}\otimes(qe_U\dot{s}e_U-(q-1)e_Ue_Te_U)) \\
&= q^{-1}(1-q)e_T
\end{aligned}$$

and $1\mapsto e_T$ corresponds to $\mathscr{F}(e')\colon kT\longrightarrow {}_skT$.

5. Note

$$\begin{aligned}\mathscr{F}(\epsilon_\emptyset\Phi)\circ\mathscr{F}(E\alpha^{-1})\circ\mathscr{F}(\eta_\emptyset)(1) &= \mathscr{F}(\epsilon_\emptyset\Phi)\circ\mathscr{F}(E\alpha^{-1})(q^{-1}e_U\otimes e_U)\\ &= \mathscr{F}(\epsilon_\emptyset\Phi)(q^{-1}e_U\otimes\xi_0\otimes 1)\\ &= q^{-1}(1-q)e_T\otimes 1 = q^{-1}(1-q)e_T.\end{aligned}$$

Similarly to the above, this morphism is given by $q^{-1}(1-q)\mathscr{F}(e')$.

6. Note

$$\begin{aligned}\mathscr{F}(\tilde\alpha)\circ\mathscr{F}(e'E)\circ\mathscr{F}(\tilde\alpha)(1\otimes e_U) &= \mathscr{F}(\tilde\alpha)\circ\mathscr{F}(e'E)(e_U\dot{s}e_U)\\ &= \mathscr{F}(\tilde\alpha)(e_T\otimes e_U\dot{s}e_U)\\ &= e_T(e_U\dot{s}e_U)^2 = e_T(q^{-1}e_U + q^{-1}(q-1)e_U\dot{s}e_Te_U)\\ &= q^{-1}e_Ue_Te_U + q^{-1}(q-1)e_U\dot{s}e_Te_U.\end{aligned}$$

Since $\mathscr{F}(e''E)$ is given by $1\otimes e_U\mapsto e_Ue_Te_U$ and $\mathscr{F}(eE)\circ\mathscr{F}(\tilde\alpha)$ is given by $1\otimes e_U\mapsto e_U\dot{s}e_Te_U$ as morphisms ${}_skT\otimes_{kT} e_UkG\longrightarrow e_UkG$, and so

$$\mathscr{F}(\tilde\alpha\circ e'E\circ\tilde\alpha) = q^{-1}\mathscr{F}(e''E) + q^{-1}(q-1)\mathscr{F}(eE\circ\tilde\alpha).$$

7. Similarly,

$$\begin{aligned}\alpha\circ\mathscr{F}(Fe')\circ\mathscr{F}(\alpha)(e_U\otimes 1) &= \mathscr{F}(\alpha)\circ\mathscr{F}(Fe')(e_U\dot{s}e_U)\\ &= \mathscr{F}(\alpha)(e_U\dot{s}e_U\otimes e_T)\\ &= (e_U\dot{s}e_U)^2e_T\\ &= q^{-1}e_Ue_Te_U + q^{-1}(q-1)e_U\dot{s}e_Te_U.\end{aligned}$$

Since $\mathscr{F}(Fe'')$ is given as $e_U\otimes 1\mapsto e_Ue_Te_U$ and $\mathscr{F}(Fe\circ\alpha)$ is given by $e_U\otimes 1\mapsto e_U\dot{s}e_Te_U$ as morphisms $kGe_U\otimes_{kT}{}_skT\longrightarrow kGe_U$, one has

$$\mathscr{F}(\alpha\circ Fe'\circ\alpha) = q^{-1}\mathscr{F}(Fe'') + q^{-1}(q-1)\mathscr{F}(Fe\circ\alpha).$$

8. Observe

$$\begin{aligned}\mathscr{F}(\alpha)\circ\mathscr{F}(\alpha\Phi)(e_U\otimes 1\otimes 1) &= \mathscr{F}(\alpha)(e_U\dot{s}e_U\otimes 1)\\ &= (e_U\dot{s}e_U)^2 = q^{-1}e_U + q^{-1}(q-1)e_U\dot{s}e_Te_U.\end{aligned}$$

But as morphisms $kGe_U\otimes_{kT}{}_skT\otimes_{kT}{}_skT\longrightarrow kGe_U$, $\mathscr{F}(Fz)$ is given by $e_U\otimes 1\otimes 1\mapsto e_U$, and

$$\mathscr{F}(\alpha)\circ\mathscr{F}(Fe')\circ\mathscr{F}(Fz)(e_U\otimes 1\otimes 1) = \mathscr{F}(\alpha)\circ\mathscr{F}(Fe')(e_U\otimes 1) = \mathscr{F}(\alpha)(e_U\otimes e_T) = e_U\dot{s}e_Te_U.$$

Hence

$$\mathscr{F}(\alpha \circ \alpha\Phi) = q^{-1}\mathscr{F}(Fz) + q^{-1}(q-1)\mathscr{F}(\alpha \circ Fe' \circ Fz).$$

9. Observe

$$\begin{aligned}\mathscr{F}(\tilde{\alpha}) \circ \mathscr{F}(\Phi\tilde{\alpha})(1 \otimes 1 \otimes e_U) &= \mathscr{F}(\tilde{\alpha})(1 \otimes e_U \dot{s} e_U) \\ &= (e_U \dot{s} e_U)^2 = q^{-1}e_U + q^{-1}(q-1)e_U \dot{s} e_T e_U.\end{aligned}$$

But as morphisms ${}_s kT \otimes_{kT} {}_s kT \otimes_{kT} e_U kG \longrightarrow e_U kG$, $\mathscr{F}(zE)$ is given by $1 \otimes 1 \otimes e_U \mapsto e_U$, and

$$\mathscr{F}(\tilde{\alpha}) \circ \mathscr{F}(e'E) \circ \mathscr{F}(zE)(1 \otimes 1 \otimes e_U) = \mathscr{F}(\tilde{\alpha}) \circ \mathscr{F}(e'E)(1 \otimes e_U) = \mathscr{F}(\tilde{\alpha})(e_T \otimes e_U) = e_U \dot{s} e_T e_U.$$

Hence

$$\mathscr{F}(\tilde{\alpha} \circ \Phi\tilde{\alpha}) = q^{-1}\mathscr{F}(zE) + q^{-1}(q-1)\mathscr{F}(\tilde{\alpha} \circ e'E \circ zE).$$

10. First note that as an endomorphism of $kGe_U \otimes_{kT} e_U kG$, we have

$$\begin{aligned}\mathscr{F}(F\tilde{\alpha}) \circ \mathscr{F}(\alpha^{-1}E)(e_U \otimes e_U) &= \mathscr{F}(F\tilde{\alpha})(\xi_0 \otimes 1 \otimes e_U) \\ &= \xi_0 \otimes e_U \dot{s} e_U \\ &= q e_U \dot{s} e_U \otimes e_U \dot{s} e_U - (q-1)e_U e_T e_U \otimes e_U \dot{s} e_U\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathscr{F}(\alpha E) \circ \mathscr{F}(F\tilde{\alpha}^{-1})(e_U \otimes e_U) &= \mathscr{F}(\alpha E)(e_U \otimes 1 \otimes \xi_0) \\ &= \mathscr{F}(\alpha)(e_U \otimes 1) \otimes \xi_0 \\ &= e_U \dot{s} e_U \otimes \xi_0 \\ &= q e_U \dot{s} e_U \otimes e_U \dot{s} e_U - (q-1)e_U \dot{s} e_U \otimes e_U e_T e_U\end{aligned}$$

and

$$\mathscr{F}(\alpha E) \circ \mathscr{F}(Fe'E)(e_U \otimes e_U) = \mathscr{F}(\alpha E)(e_U \otimes e_T \otimes e_U) = e_U \dot{s} e_T e_U \otimes e_U.$$

Together, these imply

$$\mathscr{F}(F\tilde{\alpha} \circ \alpha^{-1}E) = \mathscr{F}(\alpha E \circ F\tilde{\alpha}^{-1} + (q-1)(\alpha \circ Fe')E - (q-1)F(\tilde{\alpha} \circ e'E)).$$

11. The morphism $\mathscr{F}(e''') = \mathscr{F}(e'') \circ \mathscr{F}(e')$ on ${}_s kT$ is given by $1 \mapsto e_T$. Since $\mathscr{F}(e\Phi)(1) = e_T \otimes 1$ and

$e_T \otimes 1 = e_T$ under the identification $kT \otimes_{kT} {}_s kT \simeq {}_s kT$, it follows that $\mathscr{F}(e''') = \mathscr{F}(e\Phi)$.

12. As morphisms on $e_U kGe_U$, $\mathscr{F}(eEF)(e_U ge_U) = e_T \otimes e_U ge_U$ and $\mathscr{F}(EFe)(e_U ge_U) = e_U ge_U \otimes e_T$ for any $g \in G$. But $e_T \otimes e_U ge_U = e_U e_T ge_U = e_U ge_U \otimes e_T$. To see this, for any group G with split BN-pair, by the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B = \bigsqcup_{w \in W} UT\dot{w}U.$$

Writing $g = ut\dot{w}u'$ for some w, then

$$e_T e_U ge_U = e_T e_U ut\dot{w}u' e_U = e_T e_U t\dot{w}e_U = e_U t\dot{w}e_U e_T = e_U ge_U e_T$$

since e_T commutes with e_U, t and $\dot{w}$. Thus $\mathscr{F}(eEF) = \mathscr{F}(EFe)$.

13. Note

$$\begin{aligned}\mathscr{F}(Fe) \circ \mathscr{F}(\alpha)(e_U \otimes 1) &= \mathscr{F}(Fe)(e_U \dot{s} e_U) \\ &= e_U \dot{s} e_U \otimes e_T \\ &= \mathscr{F}(\alpha)(e_U \otimes e_T) \\ &= \mathscr{F}(\alpha) \circ \mathscr{F}(Fe''')(e_U \otimes 1)\end{aligned}$$

so that $\mathscr{F}(Fe) \circ \mathscr{F}(\alpha) = \mathscr{F}(\alpha) \circ \mathscr{F}(Fe''')$.

14. It is clear that $\mathscr{F}(e'' \circ e' \circ e'') = \mathscr{F}(e'')$ since e_T is an idempotent.

15. Same as above.

16. Note

$$\mathscr{F}(z) \circ \mathscr{F}(e'\Phi)(1) = \mathscr{F}(z)(e_T \otimes 1) = \dot{s} e_T \dot{s}^{-1} = e_{\dot{s}T\dot{s}^{-1}} = e_T.$$

Hence $\mathscr{F}(z \circ e'\Phi) = \mathscr{F}(e'')$. The same relation holds with $e'\Phi$ replaced with $\Phi e'$.

17. Note

$$\mathscr{F}(e''\Phi) \circ \mathscr{F}(z^{-1})(1) = \mathscr{F}(e''\Phi)(1 \otimes 1) = e_T \otimes 1 = e_T$$

and hence $\mathscr{F}(e''\Phi \circ z^{-1}) = \mathscr{F}(e')$. Again, the same relation holds with $e''\Phi$ replaced with $\Phi e''$.

18. That $\mathscr{F}(\Phi z) = \mathscr{F}(z\Phi)$ follows quickly from

$$\mathscr{F}(\Phi z)(1 \otimes 1 \otimes 1) = 1 \otimes 1 = \mathscr{F}(z\Phi)(1 \otimes 1 \otimes 1).$$

19. First note that

$$\mathscr{F}(e_0)(1) = \mathscr{F}(\epsilon_S) \circ \mathscr{F}(\eta_S)(1) = \mathscr{F}(\epsilon_S)\left(\sum_{g \in G/B} g e_U \otimes e_U g^{-1}\right) = q \sum_{g \in G/B} g e_U g^{-1}.$$

Since $\sum_{g \in G/B} g e_U g^{-1}$ is a central element in kG, $\mathscr{F}(e_0)$ acts a multiplication by a central element, and hence $\mathscr{F}(e_0) \circ \mathscr{F}(e_1) = \mathscr{F}(e_1) \circ \mathscr{F}(e_0)$.

20. By the same reasoning above, $\mathscr{F}(e_0) \circ \mathscr{F}(e_2) = \mathscr{F}(e_2) \circ \mathscr{F}(e_0)$.

21. Observe that

$$\begin{aligned}
\mathscr{F}(e_1) &= \mathscr{F}(\epsilon_S) \circ \mathscr{F}(FeE) \circ \mathscr{F}(\eta_S)(1) \\
&= \mathscr{F}(\epsilon_S) \circ \mathscr{F}(FeE)\left(\sum_{g \in G/B} g e_U \otimes e_U g^{-1}\right) \\
&= \mathscr{F}(\epsilon_S)\left(\sum_{g \in G/B} g e_U \otimes e_T \otimes e_U g^{-1}\right) \\
&= q\left(\sum_{g \in G/B} g e_T e_U g^{-1}\right) = q\left(\sum_{g \in G/B} g e_B g^{-1}\right).
\end{aligned}$$

Hence $\mathscr{F}(e_1)$ is given by multiplication by a central element in kG, hence $\mathscr{F}(e_1) \circ \mathscr{F}(e_2) = \mathscr{F}(e_2) \circ \mathscr{F}(e_1)$.

22. Note

$$\begin{aligned}
\mathscr{F}(e_0 F)(e_U) &= q \sum_{g \in G/B} g e_U g^{-1} \otimes e_U \\
&= q e_U \left(\sum_{g \in G/B} g e_U g^{-1}\right) e_U \otimes e_U \\
&= q(e_U + (-1)^\epsilon q e_U \dot{s} e_U \dot{s} e_U) \otimes e_U \\
&= (q e_U + (-1)^\epsilon q^2 (e_U \dot{s} e_U)^2) \otimes e_U \\
&= q e_U \otimes e_U + [(-1)^\epsilon q e_U + (-1)^\epsilon q(q-1) e_B \dot{s} e_B] \otimes e_U \\
&= [q + (-1)^\epsilon q] e_U + [(-1)^\epsilon q(q-1)] e_B \dot{s} e_B
\end{aligned}$$

As endomorphisms of kGe_U, $\mathscr{F}(1_F)$ is the identity, and $\mathscr{F}(\alpha \circ Fe')$ is defined by $e_U \mapsto e_U \dot{s} e_T e_U = e_B \dot{s} e_B$, and hence

$$\mathscr{F}(e_0 F) = \big(q + (-1)^\epsilon q\big)\mathscr{F}(1_F) + (-1)^\epsilon q(q-1)\mathscr{F}(\alpha \circ Fe').$$

23. Note

$$\begin{aligned}\mathscr{F}(e_U) &= q\left(\sum_{g\in G/B} g e_B g^{-1}\right)\otimes e_U\\ &= q e_U\left(\sum_{g\in G/B} g e_B g^{-1}\right) e_U\otimes e_U\\ &= q(e_B + (-1)^{\epsilon} q e_B \dot{s} e_B \dot{s} e_B)\otimes e_U\\ &= q e_B + q^2(-1)^{\epsilon}(q^{-1}(q-1)e_B\dot{s}e_B + q^{-1}e_B)\otimes e_U\\ &= [q+(-1)^{\epsilon}q]e_B + (-1)^{\epsilon}q(q-1)e_B\dot{s}e_B\end{aligned}$$

and since as an endormorphism of kGe_U, $\mathscr{F}(Fe)$ is given by $e_U \mapsto e_U e_T e_U = e_B$, it follows that

$$\mathscr{F}(e_1F) = \big(q+(-1)^{\epsilon}q\big)\mathscr{F}(Fe) + (-1)^{\epsilon}q(q-1)\mathscr{F}(\alpha\circ Fe').$$

24. Finally, note

$$\begin{aligned}e_U \mapsto q\left(\sum_{g\in G/B} g e_B\dot{s}e_B g^{-1}\right)\otimes e_U\\ &= q(e_B\dot{s}e_B + q(-1)^{\epsilon}e_B\dot{s}e_B\dot{s}e_B\dot{s}e_B)\otimes e_U\\ &= q e_B\dot{s}e_B + (-1)^{\epsilon}q[(q^{-2}(q-1)^2+q^{-1})e_B\dot{s}e_B + q^{-2}(q-1)e_B]\otimes e_U\\ &= [(-1)^{\epsilon}q^{-1}(q-1)]e_B + [(-1)^{\epsilon}+q+(-1)^{\epsilon}q^{-1}(q-1)^2]e_B\dot{s}e_B\end{aligned}$$

so that

$$\mathscr{F}(e_2F) = (-1)^{\epsilon}q^{-1}(q-1)\mathscr{F}(Fe) + \big((-1)^{\epsilon}+q+(-1)^{\epsilon}q^{-1}(q-1)^2\big)\mathscr{F}(\alpha\circ Fe').$$

□

Third, before proving the equality of the two usual maps $\mathrm{End}_{\mathsf{Bimod}}(\mathscr{F}(F)) \longrightarrow \mathrm{End}_{\mathsf{Bimod}}(\mathscr{F}(E))$ induced by the adjunctions, first recall that if M is an exact (A,B)-bimodule, the functor $M\otimes_B -$ is both left and right adjoint to $M^*\otimes_A -$ (c.f. Proposition 2.4 of [13]).

Proposition 5.32. *Suppose A and B are symmetric K-algebras, for K a field, and M is an exact (A,B)-bimodule, so that the functor $M\otimes_B -$ is left and right adjoint to $M^*\otimes_A -$, say with fixed adjunctions $(\epsilon,\eta): M\otimes_B - \dashv M^*\otimes_A -$ and $(\epsilon',\eta'): M^*\otimes_A - \dashv M\otimes_B -$. Then, writing $M\otimes_B - = \Phi$ and $M^*\otimes_A - = \Psi$,* *the induced maps* $\text{End}(\Phi) \longrightarrow \text{End}(\Psi)$ *given by*

$$\varphi \mapsto \Psi\epsilon \circ \Psi\varphi\Psi \circ \eta'\Psi \quad \textit{and} \quad \varphi \mapsto \epsilon'\Psi \circ \Psi\varphi\Psi \circ \Psi\eta$$

coincide. Analogously, the induced maps $\text{End}(\Psi) \longrightarrow \text{End}(\Phi)$ *coincide.*

Proof. Since A and B are symmetric algebras over field K, fix symmetrizing forms t_A and t_B, respectively. Let $\{a_i\}$ and $\{a'_i\}$ be dual bases for A with respect to t_A, i.e., $t_A(a_i a'_j) = \delta_{ij}$, and likewise define $\{b_i\}$ and $\{b'_i\}$. There is an isomorphism of right A-modules

$$\text{Hom}_A(M, A) \to M^* : f \mapsto t_A \circ f$$

with inverse sending $u \in M^*$ to $x \mapsto \sum_i a'_i u(a_i x)$. Note this gives $u = \sum_i t_A(a'_i u(a_i -))$ (c.f. Proposition 2.10 of [5]). There is a similar isomorphism $\text{Hom}_B(M, B) \to M^*$. On bimodules, the adjunctions are given as

- $\epsilon_A\colon M \otimes_B M^* \longrightarrow A : m \otimes \xi \mapsto \sum_i a'_i \xi(a_i m)$
- $\eta_B\colon B \longrightarrow M^* \otimes_A M : 1 \mapsto \sum_k (t_A \circ \alpha_k) \otimes m_k$
- $\epsilon'_B\colon M^* \otimes_A M \longrightarrow B : \xi \otimes m \mapsto \sum_j b'_j \xi(m b_j)$
- $\eta'_A\colon A \longrightarrow M \otimes_B M^* : 1 \mapsto \sum_\ell m_\ell \otimes (t_B \circ \beta_\ell)$

On bimodules, the triangle equation $\epsilon_A M \circ M\eta_B = 1_M$ translates to

$$m \mapsto m \otimes \sum_k (t_A \circ \alpha_k) \otimes m_k \mapsto \sum_{i,k} a'_i t_A \alpha_k(a_i m) m_k = \sum_k \alpha_k(m) m_k = m$$

and $M\epsilon'_B \circ \eta'_A M = 1_M$ translates to

$$m \mapsto \sum_\ell m_\ell \otimes (t_B \circ \beta_\ell) \otimes m \mapsto \sum_{\ell,j} m_\ell b'_j t_B(\beta_\ell(m b_j)) = \sum_\ell m_\ell \beta_\ell(m) = m.$$

Observe also that since $\alpha_k(mb_j) \in A$, we can write $\alpha_k(mb_j) = \sum_i c_i a_i$ for some $c_i \in k$. Hence

$$t_A(a'_r \alpha_k(mb_j)) = \sum_i c_i t_A(a'_r a_i) = c_r.$$

Thus $\alpha_k(mb_j) = \sum_i t_A(a'_i \alpha_k(mb_j)) a_i$. Applying $t_B(\beta_\ell(- \cdot \psi(m_k)))$, for ψ an (A, B)-endomorphism of M, yields

$$\sum_i t_A(a_i'\alpha_k(mb_j))t_B(\beta_\ell(a_i\psi(m_k))) = t_B(\beta_\ell(\alpha_k(mb_j)\psi(m_k))).$$

Suppressing the tensor product notation, observe that the following diagram commutes,

$$\begin{array}{ccccccc}
MM^*M & \xrightarrow{M\eta_B M^*M} & MM^*MM^*M & \xrightarrow{MM^*\psi M^*M} & MM^*MM^*M & \xrightarrow{MM^*\epsilon_A M} & MM^*M \\
{\scriptstyle \eta_A' M}\uparrow & & & & & & \downarrow{\scriptstyle M\epsilon_B'} \\
M & & & \xrightarrow[\psi]{\qquad\qquad\qquad\qquad} & & & M
\end{array}$$

Indeed, following the five maps up and around the top of the diagram, one has

$$\begin{aligned}
m &\mapsto \sum_\ell m_\ell \otimes (t_B \circ \beta_\ell) \otimes m \\
&\mapsto \sum_{\ell,k} m_\ell \otimes (t_A \circ \alpha_k) \otimes m_k \otimes (t_B \circ \beta_\ell) \otimes m \\
&\mapsto \sum_{k,\ell} m_\ell \otimes (t_A \circ \alpha_k) \otimes \psi(m_k) \otimes (t_B \circ \beta_\ell) \otimes m \\
&\mapsto \sum_{i,k,\ell} m_\ell \otimes (t_A \circ \alpha_k) \otimes a_i'(t_B \circ \beta_\ell)(a_i\psi(m_k))m \\
&\mapsto \sum_{i,j,k,\ell} m_\ell b_j'(t_A \circ \alpha_k)(a_i'(t_B \circ \beta_\ell)(a_i\psi(m_k))mb_j)
\end{aligned}$$

Using the equation derived prior to the diagram, this last quantity can be simplified as

$$\begin{aligned}
\sum_{i,j,k,\ell} m_\ell b_j'(t_A \circ \alpha_k)(a_i'(t_B \circ \beta_\ell)(a_i\psi(m_k))mb_j) &= \sum_{i,j,k,\ell} m_\ell b_j'(t_A(\alpha_k(a_i'mb_j)))t_B(\beta_\ell(a_i\psi(m_k))) \\
&= \sum_{i,j,k,\ell} m_\ell b_j'(t_A(a_i'\alpha_k(mb_j)))t_B(\beta_\ell(a_i\psi(m_k))) \\
&= \sum_{j,k,\ell} m_\ell b_j' t_B(\beta_\ell(\alpha_k(mb_j)\psi(m_k))).
\end{aligned}$$

Focusing on the sum only over the index k,

$$\sum_k \alpha_k(mb_j)\psi(m_k) = \psi\left(\sum_k \alpha_k(mb_j)m_k\right) = \psi(mb_j) = \psi(m)b_j$$

so this, in conjunction with the triangle equations on bimodules, shows that the above simplifies to

$$\sum_{j,\ell} m_\ell b_j' t_B(\beta_\ell(\psi(m)b_j)) = \sum_\ell m_\ell \beta_\ell(\psi(m)) = \psi(m).$$

Hence the map

$$\mathrm{Hom}(M,M) \longrightarrow \mathrm{Hom}(M^*,M^*) : \psi \mapsto \; M^* \xrightarrow{\eta_B M^*} M^*MM^* \xrightarrow{M^*\psi M^*} M^*MM^* \xrightarrow{M^*\epsilon_A} M^*$$

is inverse to

$$\mathrm{Hom}(M^*,M^*) \longrightarrow \mathrm{Hom}(M,M) : \psi \mapsto \; M \xrightarrow{\eta'_A M} MM^*M \xrightarrow{M\psi M} MM^*M \xrightarrow{M^*\epsilon'_B} M.$$

However, it is standard that

$$\mathrm{Hom}(M,M) \longrightarrow \mathrm{Hom}(M^*,M^*) : \psi \mapsto \; M^* \xrightarrow{M^*\eta'_A} M^*MM^* \xrightarrow{M^*\psi M^*} M^*MM^* \xrightarrow{\epsilon'_B M^*} M^*$$

is also an inverse, hence they are the same map. By symmetry, we also find that for any $\psi\colon M \to M$,

$$M \xrightarrow{\eta'_A M} MM^*M \xrightarrow{M\psi M} MM^*M \xrightarrow{M^*\epsilon'_B} M \;=\; M \xrightarrow{M\eta_B} MM^*M \xrightarrow{M\psi M} MM^*M \xrightarrow{\epsilon_A M} M.$$

□

Remark 5.33. With A and B symmetric K-algebras as above, suppose that (M,N) is a selfdual pair of exact bimodules. The duality gives an isomorphism $N \simeq M^*$ of (B,A)-bimodules, so Proposition 5.32 holds when M^* is replaced with any (B,A)-bimodule N such that (M,N) is a selfdual exact pair.

Example 5.34. Let G be any finite group, U a subgroup of G, and T a subgroup contained in the normalizer $N_G(U)$ of U. Then the pair of bimodules (kGe_U, e_UkG) is selfdual.

Proof. Let $A = kG$ and $B = kT$. These are both symmetric algebras with the canonical symmetrizer $\sum_{g\in G} c_g g \mapsto c_1$ sending an element of the respective group ring to the coefficient of the identity element. Then kG is a natural (A,B)-bimodule under left and right translation, and is clearly finitely generated and projective either as a left A-module or right B-module.

Define a k-linear pairing by

$$\langle ge_U, e_Ug'\rangle = \begin{cases} 1 & \text{if } e_Ug' = e_Ug^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced map $kGe_U \longrightarrow (e_UkG)^* : m \mapsto \langle m, -\rangle$ has inverse given as follows. Fix a complete set $\{g_0 = 1, g_1, \ldots, g_n\}$ of right coset representatives of U in G, so that $\{g_0^{-1}, g_1^{-1}, \ldots, g_n^{-1}\}$ is a complete set of left coset representatives. A functional $\varphi \in (e_UkG)^*$ then corresponds to the element $\sum_i \varphi(e_Ug_i)g_i^{-1}e_U \in kGe_U$, giving the inverse $(e_UkG)^* \longrightarrow kGe_U$. □

It follows now that maps $\mathrm{End}_{(kG,kT)}(kGe_U) \longrightarrow \mathrm{End}_{(kG,kT)}(kGe_U)$ induced from either pair of adjunctions coincide, and likewise for $\mathrm{End}_{(kT,kG)}(e_U kG) \longrightarrow \mathrm{End}_{(kT,kG)}(e_U kG)$, and Theorem 5.28 is proven. In particular, this shows that the category $\mathcal{C}$ is not trivial.

6 Relation to Marin's Algebra

In [12], for a given Coxeter system (W, S), Marin constructs an algebra C_W defined in terms of generators and relations extending the usual Iwahori-Hecke algebra. As noted below, if W is the Weyl group of a Chevalley group G, the Yokonuma-Hecke algebra associated to the unipotent radical of G has generators indexed by S, and others by the elements of a maximal torus. In [11], Juyumaya and Kannan introduce some new generators $\{g_s\}_{s\in S}$ for the Yokonuma-Hecke algebra, such that the quadratic relation involves an idempotent sum e_s of elements of the torus. These generators $\{g_s, e_s\}_{s\in S}$ generate a subalgebra of the Yokonuma-Hecke algebra, of which C_W is a presentation. When W is finite, C_W has finite rank dependent on the number of reflection subgroups of W, but independent of the characteristic of the ground field. Following Marin quite closely, we construct an similar algebra which contains generators which track sign changes when representatives of S may square to -1 in G, e.g., when $G = SL_2(q)$.

6.1 Constructing a Representation

Let $k = \mathbb{F}_q$, and let G a simple, simply connected Chevalley group over k. Let T denote a maximally split torus of G, B a Borel subgroup containing T, and U the unipotent radical of B.

Let Φ denote the set of roots with respect to T, and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots. Put $N = N_G(T)$, so that $W = N/T$ is the Weyl group of G with $S = \{s_\alpha : \alpha \in \Delta\}$ the set of simple reflections. Then (W, S) is a Coxeter system, and let m_{ij} denote the order of $s_{\alpha_i}s_{\alpha_j}$ in W.

Let $\pi\colon N \to W$ denote the canonical projection. The Weyl group W acts on T via $w(t) = w \cdot t = \dot{w}t\dot{w}^{-1}$, where $\dot{w} \in N$ is an element such that $\pi(\dot{w}) = w$. Recall also that for any $\alpha \in \Phi$, there exists $\dot{s}_\alpha \in N$ such that $\pi(\dot{s}_\alpha) = s_\alpha$, and a homomorphism $\varphi_\alpha\colon SL_2(k) \to G$ such that

$$\dot{s}_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha^\vee(r) = \varphi_\alpha \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

for $r \in k^\times$.

The Yokonuma-Hecke algebra $\mathscr{Y}_n(q)$ is the endomorphism algebra

$$\mathscr{Y}_n(q) = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(1_U)).$$

We can identify $\mathscr{Y}_n(q) = e_U\mathbb{C}Ge_U$, where $e_U = \frac{1}{|U|}\sum_{u\in U} u$. From the Bruhat decomposition

$$G = \bigsqcup_{n\in N} UnU$$

there is the standard basis $\{R_n : n \in N\}$, where $R_n = e_U n e_U$. If $n = \dot{s}_\alpha$, write $R_\alpha := R_{\dot{s}_\alpha}$, and if $n = \alpha^\vee(r)$, denote R_n by $H_\alpha(r)$, and define E_α as

$$E_\alpha := \sum_{r \in k^\times} H_\alpha(r)$$

for $\alpha \in \Phi$. Then the E_α pairwise commute, and $E_\alpha^2 = (q-1)E_\alpha$. Recall the following theorem of Yokonuma [14].

Theorem 6.1. *The Yokonuma-Hecke algebra $\mathscr{Y}_n(q)$ is generated as an algebra by $(R_\alpha)_{\alpha\in\Phi}$ and $(R_t)_{t\in T}$. The following relations among the generators give a presentation for $\mathscr{Y}_n(q)$.*

1. $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$
2. $\underbrace{R_\alpha R_\beta R_\alpha \cdots}_{m_{\alpha\beta}} = \underbrace{R_\beta R_\alpha R_\beta \cdots}_{m_{\alpha\beta}}$
3. $R_t R_\alpha = R_\alpha R_{\dot{s}_\alpha}(t)$ *for* $t \in T$
4. $R_u R_v = R_{uv}$ *for* $u, v \in T$.

Following Juyumaya and Kannan [11], notice that W induces an action on $\{E_\alpha\}_{\alpha\in\Phi}$ by defining

$$E_\alpha^w = \sum_{r\in k^\times} H_\gamma(r)$$

where $\gamma = w(\alpha)$. From Yokonuma's theorem, it follows that if $s = s_\alpha$, then $E_\beta R_\alpha = R_\alpha E_\beta^s$. It follows that R_α^2 commutes with all E_β. Observe

$$E_\beta R_\alpha^2 = qE_\beta H_\alpha(-1) + E_\beta R_\alpha E_\alpha = R_\alpha^2 E_\beta = qH_\alpha(-1)E_\beta + R_\alpha E_\alpha E_\beta.$$

Hence

$$\begin{aligned} H_\alpha(-1)E_\beta &= E_\beta H_\alpha(-1) + q^{-1}(E_\beta E_\alpha R_\alpha - E_\beta^s E_\alpha R_\alpha) \\ &= E_\beta H_\alpha(-1) + q^{-1}(E_\beta - E_\beta^s)E_\alpha R_\alpha. \end{aligned}$$

This gives the relation

$$H_\alpha(-1)E_\beta = E_\beta H_\alpha(-1) + q^{-1}(E_\beta - E_\beta^s)E_\alpha R_\alpha.$$

Note for $\alpha \in \Phi$, $E_\alpha = E_{-\alpha}$. We have $E_{-\alpha} = \sum_{t\in K^\times} k_{(-\alpha)^\vee(t)} = \sum_{t\in K^\times} k_{(s_\alpha(\alpha))^\vee(t)}$. But

$$\{s_\alpha(\alpha)^\vee(t) : t \in K^\times\} = \{\omega_\alpha \alpha^\vee(t)\omega_\alpha^{-1} : t \in K^\times\}.$$

Computing,

$$\omega_\alpha \alpha^\vee(t)\omega_\alpha^{-1} = \varphi_\alpha \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \varphi_\alpha \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \alpha^\vee(t^{-1}).$$

Since $t \mapsto t^{-1}$ is a bijection on $K^\times$, we get $E_{-\alpha} = E_\alpha$.

It follows that $E_\alpha E_\beta = E_\alpha E_{s_\alpha(\beta)}$ for any $\alpha, \beta \in \Phi$. If $\beta \in \{\pm\alpha\}$, this follows by the above. Otherwise, $s_\alpha(\beta)^\vee = \beta^\vee + m\alpha^\vee$ for some m. Then

$$\begin{aligned} \alpha^\vee(t)(s_\alpha\beta)^\vee(u) &= \alpha^\vee(t)(\beta^\vee + m\alpha^\vee)(u) \\ &= \alpha^\vee(t)\beta^\vee(u)\alpha^\vee(u)^m = \alpha^\vee(tu^m)\beta^\vee(u). \end{aligned}$$

So

$$E_\alpha E_{s_\alpha\beta} = \sum_{t,u \in K^\times} k_{\alpha^\vee(t)} k_{(s_\alpha\beta)^\vee(u)} = \sum_{t,u \in K^\times} k_{\alpha^\vee(tu^m)} k_{\beta^\vee(u)} = E_\alpha E_\beta$$

since $(t, u) \leftrightarrow (tu^m, u)$ is a bijection on $(K^\times)^2$.

Since the E_α commute amongst themselves, this implies if $s = s_\alpha$ is the reflection corresponding to α, then

$$(E_\beta - E_\beta^s)E_\alpha = E_\beta E_\alpha - E_{s_\alpha(\beta)}E_\alpha = 0.$$

Then the above relation simplifies to $H_\alpha(-1)E_\beta = E_\beta H_\alpha(-1)$.

Also, if $q-1$ is invertible, then setting $e_\alpha = \frac{1}{q-1}E_\alpha$ yields

$$e_\alpha^2 = \frac{1}{(q-1)^2}E_\alpha^2 = \frac{(q-1)}{(q-1)^2}E_\alpha = e_\alpha.$$

By the above, $e_\alpha e_\beta = e_\alpha e_{s_\alpha(\beta)}$. Then if $J \subseteq \mathscr{P}_f(\mathcal{R})$, and $e_J = \prod_{t \in \mathcal{R}} e_t$, then it makes sense to define $e_J = e_{W_0}$, where $W_0 = \langle J \rangle$, under the identification $e_\alpha = e_{s_\alpha}$. With this scaled generator, the quadratic relation $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$ can be rewritten as

$$R_\alpha^2 = qH_\alpha(-1) + (q-1)R_\alpha e_\alpha.$$

Drawing from the computations above, define the following algebra.

Definition 6.2. Let G be a simple, simply connected Chevalley group defined over the field $k = \mathbb{F}_q$. Fix a maximally split torus T, Borel subgroup B. Let Φ denote the corresponding set of roots, and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots, and let $\mathcal{R}$ denote the set of reflections. If $\beta \in \Phi^+$, let w_β denote the corresponding reflection in $\mathcal{R}$. If $\beta \in \Delta$, the corresponding reflection will also be denoted s_β.

Define a k-algebra A with generators $\{\tau_s\}_{s\in S}$, $\{\iota_\alpha\}_{\alpha\in\Delta}$, and $\{e_w\}_{w\in\mathcal{R}}$ subject to the following relations.

- $\tau_{s_\alpha}^2 = q\iota_\alpha + (q-1)\tau_{s_\alpha}e_{s_\alpha}$ for all $\alpha\in\Delta$
- $e_t^2 = e_t$ for all $t\in\mathcal{R}$
- $e_{t_1}e_{t_2} = e_{t_2}e_{t_1}$ for all $t_1,t_2\in\mathcal{R}$
- $e_te_{t_1} = e_te_{tt_1t^{-1}}$ for all $t,t_1\in\mathcal{R}$
- $\iota_\alpha^2 = 1$ for all $\alpha\in\Delta$
- $\underbrace{\tau_{s_{\alpha_i}}\tau_{s_{\alpha_j}}\tau_{s_{\alpha_i}}\cdots}_{m_{ij}} = \underbrace{\tau_{s_{\alpha_j}}\tau_{s_{\alpha_i}}\tau_{s_{\alpha_j}}\cdots}_{m_{ij}}$ for all $\alpha_i,\alpha_j\in\Delta$
- $\tau_{s_{\alpha_i}}e_{w_\beta} = e_{s_{\alpha_i}w_\beta s_{\alpha_i}^{-1}}\tau_{s_{\alpha_i}}$
- $\iota_{\alpha_i}\iota_{\alpha_j} = \iota_{\alpha_j}\iota_{\alpha_i}$ for all $\alpha_i,\alpha_j\in\Delta$,
- $\tau_{s_\alpha}\iota_\beta = \iota_{s_\alpha(\beta)}\tau_{s_\alpha}$
- $\iota_\alpha e_w = e_w\iota_\alpha$
- $\iota_\alpha e_{w_\alpha} = e_{w_\alpha}$

By Matsumoto's Theorem, if $w\in W$ has a reduced expression $w=s_1\cdots s_r$, define $\tau_w=\tau_{s_1}\cdots\tau_{s_r}$. Also if $\beta=\sum_i c_i\alpha_{s_i}$ is an expression of a root β in terms of simple roots, then put $\iota_\beta=\prod\iota_{\alpha_{s_i}}^{c_i}$. Also, for finite $J\subseteq\mathcal{R}$, set $e_J=\prod_{t\in J}e_t$. For $s,t\in J$, $e_se_t=e_se_te_t=e_se_{sts}e_t$, so it follows that $e_J=e_{\langle J\rangle}$, where $\langle J\rangle$ is the generated subgroup in W.

Note that any product of generators in A can be written in form $e_J\left(\prod_{i=1}^l\iota_{\alpha_i}^{\epsilon_i}\right)\tau_w$, for $J\subseteq\mathcal{R}$, $\epsilon_i\in\{0,1\}$, and $w\in W$.

First, fix a simple root $\alpha_i\in\Delta$, and define integers n_k by the equations $s_{\alpha_i}(\alpha_k)=\alpha_k+n_k\alpha_i$ for $k=1,\ldots,l$. Observe the effect of left multiplication by τ_{α_i} on a product of form $e_J\left(\prod_{i=1}^l\iota_{\alpha_i}^{\epsilon_i}\right)\tau_w$.

Suppose $\ell(s_{\alpha_i}w)=\ell(w)+1$. Then

$$\begin{aligned}
\tau_{\alpha_i}e_J\cdot\prod_{k=1}^l\iota_{\alpha_k}^{\epsilon_k}\cdot\tau_w &= e_{s_{\alpha_i}Js_{\alpha_i}}\cdot\prod_{k=1}^l\iota_{s_{\alpha_i}(\alpha_k)}^{\epsilon_k}\cdot\tau_{s_{\alpha_i}}\tau_w\\
&= e_{s_{\alpha_i}Js_{\alpha_i}}\cdot\prod_{k=1}^l\iota_{\alpha_k+n_k\alpha_i}^{\epsilon_k}\cdot\tau_{s_{\alpha_i}w}\\
&= e_{s_{\alpha_i}Js_{\alpha_i}}\cdot\prod_{k=1}^l\iota_k^{\epsilon_k}\iota_{\alpha_i}^{n_k\epsilon_k}\cdot\tau_{s_{\alpha_i}w}\\
&= e_{s_{\alpha_i}Js_{\alpha_i}}\cdot\iota_{\alpha_i}^{\epsilon_i+\sum_{k\neq i}\epsilon_kn_k}\cdot\prod_{k\neq i}\iota_{\alpha_k}^{\epsilon_k}\cdot\tau_{s_{\alpha_i}w}.
\end{aligned}$$

Suppose now $\ell(s_{\alpha_i}w) = \ell(w) - 1$. Write $w = s_{\alpha_i}w'$ where $\ell(w') = \ell(w) - 1$. Then

$$\begin{aligned}
\tau_{s_{\alpha_i}} \cdot e_J \cdot \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_w &= e_{s_{\alpha_i} J s_{\alpha_i}} \iota_{\alpha_i}^{\epsilon_i + \sum_{k \neq i} \epsilon_k n_k} \prod_{k \neq i} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_{s_{\alpha_i}}^2 \tau_{w'} \\
&= e_{s_{\alpha_i} J s_{\alpha_i}} \iota_{\alpha_i}^{\epsilon_i + \sum_{k \neq i} \epsilon_k n_k} \prod_{k \neq i} \iota_{\alpha_k}^{\epsilon_k} (q\iota_{\alpha_i} + (q-1)\tau_{s_{\alpha_i}} e_{s_{\alpha_i}}) \tau_{w'} \\
&= q e_{s_{\alpha_i} J s_{\alpha_i}} \iota_{\alpha_i}^{\epsilon_i + \sum_{k \neq i} \epsilon_k n_k} \prod_{k \neq i} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_{w'} + (q-1) e_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\}} \iota_{\alpha_i}^{\epsilon_i + \sum_{k \neq i} \epsilon_k n_k} \prod_{k \neq i} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_w .
\end{aligned}$$

For right multiplication, observe that if $\ell(ws_{\alpha_i}) = \ell(w) + 1$, then

$$e_J \cdot \prod_{k=1}^{l} \iota_{\alpha_i}^{\epsilon_i} \cdot \tau_w \tau_{s_{\alpha_i}} = e_J \cdot \prod_{k=1}^{l} \iota_{\alpha_i}^{\epsilon_i} \cdot \tau_{ws_{\alpha_i}} .$$

If $\ell(ws_{\alpha_i}) = \ell(w) - 1$, write $w = w's_{\alpha_i}$ with $\ell(w') = \ell(w) - 1$, and assume $w(\alpha_i) = \sum_k c_k \alpha_k$. Then

$$\begin{aligned}
e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_w \tau_{s_{\alpha_i}} &= e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_{w'} \tau_{s_{\alpha_i}}^2 \\
&= e_J \prod_{k=1}^{l} \iota_{\alpha_j}^{\epsilon_j} \tau_{w'} (q\iota_{\alpha_i} + (q-1)\tau_{s_{\alpha_i}} e_{s_{\alpha_i}}) \\
&= q e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \iota_{w'(\alpha_i)} \tau_{w'} + (q-1) e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \tau_{w'} \tau_{s_{\alpha_i}} e_{s_{\alpha_i}} \\
&= q e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \iota_{w(-\alpha_i)} \tau_{ws_{\alpha_i}} + (q-1) e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} e_{ws_{\alpha_i} w^{-1}} \tau_w \\
&= q e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \prod_{k=1}^{l} \iota_{\alpha_k}^{-c_k} \tau_{ws_{\alpha_i}} + (q-1) e_{J \cup \{ws_{\alpha_i} w^{-1}\}} \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \tau_w \\
&= q e_J \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k + c_k} \cdot \tau_{ws_{\alpha_i}} + (q-1) e_{J \cup \{ws_{\alpha_i} w^{-1}\}} \prod_{k=1}^{l} \iota_{\alpha_k}^{\epsilon_k} \cdot \tau_w .
\end{aligned}$$

Now let V be a free k-module with basis

$$\left(v_{J,(\epsilon_k),w} : J \subseteq \mathcal{R},\ (\epsilon_k) \in \mathbb{F}_2^l,\ w \in W \right)$$

where we declare $v_{J,(\epsilon_k),w} = v_{K,(\epsilon'_k),w'}$ if $w = w'$, $\langle J \rangle = \langle K \rangle$, and if $\epsilon_k \neq \epsilon'_k$, then $s_{\alpha_i} \in \langle J \rangle = \langle K \rangle$.

With the same Coxeter system (W, S) as before, define the following k-linear operators on V.

Definition 6.3. Fix $\alpha_i \in \Delta$, and let integers n_k be determined by the equations $s_{\alpha_i}(\alpha_k) = \alpha_k + n_k \alpha_i$.

Define $T_{\alpha_i} := T_{s_{\alpha_i}} \in \mathrm{End}_k(V)$ by

$$\begin{cases} v_{s_{\alpha_i} J s_{\alpha_i}}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_l), s_{\alpha_i} w \\ \text{if } \ell(s_{\alpha_i} w) = \ell(w) + 1, \\ q v_{s_{\alpha_i} J s_{\alpha_i}, (\epsilon_1, \ldots, \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w} + (q-1) v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), w} \\ \text{if } \ell(s_{\alpha_i} w) = \ell(w) - 1. \end{cases}$$

Similarly define $T'_{\alpha_i} := T'_{s_{\alpha_i}} \in \mathrm{End}_k(V)$ by

$$T'_{\alpha_i}(v_{J,(\epsilon_k),w}) = \begin{cases} v_{J,(\epsilon_k), w s_{\alpha_i}} & \text{if } \ell(w s_{\alpha_i}) = \ell(w) + 1), \\ q v_{J,(\epsilon_k + c_k), w s_{\alpha_i}} + (q-1) v_{e_{J \cup \{w s_{\alpha_i} w^{-1}\}}, (\epsilon_k), w} & \text{if } \ell(w s_{\alpha_i}) = \ell(w) - 1. \end{cases}$$

where the integers c_k are determined by the equation $w(\alpha_i) = \sum_k c_k \alpha_k$.

Lemma 6.4. *For any $\alpha_i, \alpha_j \in \Delta$, $T_{\alpha_i} T'_{\alpha_j} = T'_{\alpha_j} T_{\alpha_i}$.*

Proof.

1. First suppose $\ell(s_{\alpha_i} w) = \ell(w s_{\alpha_i}) = \ell(w) + 1$.

   - Suppose also that $\ell(s_{\alpha_i} w s_{\alpha_j}) = \ell(w s_{\alpha_j}) + 1 = \ell(s_{\alpha_i} w) + 1$. Then

     $$T_{\alpha_i} T'_{\alpha_j}(v_{J,(\epsilon_k),w}) = T_{\alpha_i}(v_{J,(\epsilon_k), w s_{\alpha_j}}) = v_{s_{\alpha_i} J s_{\alpha_i}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w s_{\alpha_j}}$$

     and

     $$\begin{aligned} T'_{\alpha_j} T_{\alpha_i}(v_{J,(\epsilon_k),w}) &= T'_{\alpha_j}(v_{s_{\alpha_i} J s_{\alpha_i}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w}) \\ &= v_{s_{\alpha_i} J s_{\alpha_i}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w s_{\alpha_j}} \end{aligned}$$

     which are both equal.

   - Suppose instead $\ell(s_{\alpha_i} w s_{\alpha_j}) = \ell(s_{\alpha_i} w) - 1 = \ell(w s_{\alpha_j}) - 1$. Then

     $$\begin{aligned} &T_{\alpha_i} T'_{\alpha_j}(v_{J,(\epsilon_k),w}) = T_{\alpha_i}(v_{J,(\epsilon_k), w s_{\alpha_j}}) \\ &= q v_{s_{\alpha_i} J s_{\alpha_i}, (\epsilon_1, \ldots, \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w s_{\alpha_j}} + (q-1) v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\}, (\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), w s_{\alpha_j}}. \end{aligned}$$

Let the integers c_k be determined by the equation $(s_{\alpha_i}w)(\alpha_j) = \sum_k c_k\alpha_k$. Then

$$\begin{aligned}T'_{\alpha_j}T_{\alpha_i}(v_{J,(\epsilon_k),w}) &= T'_{\alpha_j}(v_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w})\\ &= qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1+c_1,\ldots,\epsilon_i+c_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n+c_n),s_{\alpha_i}ws_{\alpha_j}}+\\ &\quad (q-1)v_{s_{\alpha_i},Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w}.\end{aligned}$$

To see that the first terms in each computation are equal, we have to consider the discrepancy of $\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k$ and $\epsilon_i+c_i+\sum_{k\neq i}\epsilon_k n_k$. Since $\ell(ws_{\alpha_j}) = \ell(w)+1$, it follows that $w(\alpha_j)\in\Phi^+$, but $\ell(s_{\alpha_i}ws_{\alpha_j}) = \ell(ws_{\alpha_j})-1$, so that $(s_{\alpha_i}w)(\alpha_j)\in\Phi^-$. Thus $w(\alpha_j)$ is a positive root made negative by s_{α_i}, and so $w(\alpha_j)=\alpha_i$. Thus

$$\sum_k c_k\alpha_k = (s_{\alpha_i}w)(\alpha_j) = s_{\alpha_i}(\alpha_i) = -\alpha_i$$

so that $c_i=-1$, and $c_k=0$ for $k\neq i$. Hence the first terms are equal since

$$\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k = \epsilon_i-1+\sum_{k\neq i}\epsilon_k n_k$$

in $\mathbb{F}_2$.

To see that the second terms are equal, $\ell(s_{\alpha_i}ws_{\alpha_j} = \ell(w)$ and $\ell(s_{\alpha_i}w) = \ell(ws_{\alpha_j})$ together imply $s_{\alpha_i}w = ws_{\alpha_j}$. This in turn implies $s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i} = ws_{\alpha_j}s_{\alpha_j}w^{-1}s_{\alpha_i} = s_{\alpha_i}$, so that $s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\}$ and $s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\}$ clearly generate the same subgroup.

2. Suppose $\ell(s_{\alpha_i}w) = \ell(w)+1$ and $\ell(ws_{\alpha_j}) = \ell(w)-1$. It follows that necessarily $\ell(s_{\alpha_i}ws_{\alpha_j}) = \ell(w)$, for otherwise $\ell(s_{\alpha_i}ws_{\alpha_j}) = \ell(w)-2$, implying $\ell(s_{\alpha_i}w)\leq\ell(w)-1$, a contradiction. Write $w(\alpha_j) = \sum_k c_k\alpha_k$ for some $c_k\in\mathbb{Z}$. Then

$$\begin{aligned}T_{\alpha_i}T'_{\alpha_j}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(qv_{J,(\epsilon_k+c_k),ws_{\alpha_j}}+(q-1)v_{J\cup\{ws_{\alpha_j}w^{-1}\},(\epsilon_k),w})\\ &= qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1+c_1,\ldots,\epsilon_i+c_i+\sum_{k\neq i}(\epsilon_k+c_k)n_k,\ldots,\epsilon_n+c_n),s_{\alpha_i}ws_{\alpha_j}}+\\ &\quad (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w}.\end{aligned}$$

Write $s_{\alpha_i}w(\alpha_j) = \sum_k d_k\alpha_k$. Then

$$\begin{aligned}T'_{\alpha_j}T_{\alpha_i}(v_{J,(\epsilon_k),w}) &= T'_{\alpha_j}(v_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w})\\ &= qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1+d_1,\ldots,\epsilon_i+d_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n+d_n),s_{\alpha_i}ws_{\alpha_j}}+\\ &\quad (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w}.\end{aligned}$$

The final second term of each computation is identical. To check equality of the first term, note

$$\sum_k d_k\alpha_k = s_{\alpha_i}w(\alpha_j) = s_{\alpha_i}\left(\sum_k c_k\alpha_k\right) = s_{\alpha_i}(c_i\alpha_i) + \sum_{k\neq i} c_k s_{\alpha_i}(\alpha_k)$$
$$= -c_i\alpha_i + \sum_{k\neq i} c_k(\alpha_k + n_k\alpha_i) = \left(-c_i + \sum_{k\neq i} c_k n_k\right)\alpha_i + \sum_{k\neq i} c_k\alpha_k.$$

Hence $d_i = -c_i + \sum_{k\neq i} c_k n_k$ and $d_k = c_k$ for $k \neq i$. This shows $\epsilon_k + c_k = \epsilon_k + d_k$ for $k \neq i$. Comparing the ith entry as elements of $\mathbb{F}_2$,

$$\epsilon_i + c_i + \sum_{k\neq i}(\epsilon_k + c_k)n_k = \epsilon_i - c_i + \sum_{k\neq i}(\epsilon_k + c_k)n_k$$
$$= \epsilon_i - c_i + \sum_{k\neq i}\epsilon_k n_k + \sum_{k\neq i} c_k n_k$$
$$= \epsilon_i + d_i + \sum_{k\neq i}\epsilon_k n_k.$$

3. Suppose $\ell(s_{\alpha_i}w) = \ell(w) - 1$ and $\ell(ws_{\alpha_j}) = \ell(w) + 1$. Necessarily $\ell(s_{\alpha_i}ws_{\alpha_j}) = \ell(w)$. Then

$$T_{\alpha_i}T'_{\alpha_j}(v_{J,(\epsilon_k),w}) = T_{\alpha_i}(v_{J,(\epsilon_k),ws_{\alpha_j}})$$
$$= qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}ws_{\alpha_j}} +$$
$$(q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),ws_{\alpha_j}}$$

and

$$T'_{\alpha_j}T_{\alpha_i}(v_{J,(\epsilon_k),w}) = T'_{\alpha_j}(qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w} +$$
$$(q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),w})$$
$$= qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}ws_{\alpha_j}} +$$
$$(q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),ws_{\alpha_j}})$$

which are identical.

4. Suppose $\ell(s_{\alpha_i}w) = \ell(w) - 1 = \ell(ws_{\alpha_j})$.

   - Suppose $\ell(s_{\alpha_i}ws_{\alpha_j}) = \ell(ws_{\alpha_j}) - 1 = \ell(s_{\alpha_i}w) - 1$. Write $w(\alpha_j) = \sum_k c_k\alpha_k$ and

$s_{\alpha_i}w(\alpha_j) = \sum_k d_k\alpha_k$. Then

$$T_{\alpha_i}T'_{\alpha_j}(v_{J,(\epsilon_k),w}) = T_{\alpha_i}(qv_{J,(\epsilon_k+c_k),ws_{\alpha_j}} + (q-1)v_{J\cup\{ws_{\alpha_j}w^{-1}\},(\epsilon_k),w})$$
$$= q[qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1+c_1,\dots,\epsilon_i+c_i+1+\sum_{k\neq i}(\epsilon_k+c_k)n_k,\dots,\epsilon_n+c_n),s_{\alpha_i}ws_{\alpha_j}}$$
$$+ (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1+c_1,\dots,\epsilon_i+c_i+\sum_{k\neq i}(\epsilon_k+c_k)n_k,\dots,\epsilon_n+c_n),ws_{\alpha_j}}]$$
$$+ (q-1)[qv_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),s_{\alpha_i}w}$$
$$+ (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i},s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),w}].$$

Now write $s_{\alpha_i}w(\alpha_j) = \sum_k d_k\alpha_k$ for some $d_k \in \mathbb{Z}$. Then

$$T'_{\alpha_j}T_{\alpha_i}(v_{J,(\epsilon_k),w}) = T'_{\alpha_j}(qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),s_{\alpha_i}w}$$
$$+ (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{alpha_i}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),w})$$
$$= q[qv_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1+d_1,\dots,\epsilon_i+d_i+1+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n+d_n),s_{\alpha_i}ws_{\alpha_j}} +$$
$$+ (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),s_{\alpha_i}w}]$$
$$+ (q-1)[qv_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1+c_1,\dots,\epsilon_i+c_i+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n+c_n),ws_{\alpha_j}}$$
$$+ (q-1)v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}ws_{\alpha_j}w^{-1}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_kn_k,\dots,\epsilon_n),w}].$$

The first terms of each final computation are equal. As before, $d_i = -c_i + \sum_{k\neq i} c_kn_k$ and $d_k = c_k$ for $k \neq i$. Thus $\epsilon_k + d_k = \epsilon_k + c_k$ for $k \neq i$, and in $\mathbb{F}_2$ the ith entries are equal since

$$\begin{aligned}\epsilon_i + d_i + 1 + \sum_{k\neq i}\epsilon_kn_k &= \epsilon_i - c_i + \sum_{k\neq i} n_kc_k + \sum_{k\neq i}\epsilon_kn_k\\ &= \epsilon_i - c_i + 1 + \sum_{k\neq i}(\epsilon_k + c_k)n_k\\ &= \epsilon_i + c_i + 1 + \sum_{k\neq i}(\epsilon_k + c_k)n_k.\end{aligned}$$

The second term of the first computation is equal to the third term of the second computation, although the ith coordinates differ, as this coordinate corresponds to the reflection s_{α_i}, which is in the subgroup generated by $s_{\alpha_i}Js_{\alpha_i} \cup \{s_{\alpha_i}\}$, so these basis vectors are equal, regardless.
The third term of the first computation is identical to the second term of the second expression.
Lastly, the fourth terms of both computations are equal since

$$\langle s_{\alpha_i}Js_{\alpha_i}, s_{\alpha_i}, s_{\alpha_i}ws_{\alpha_j}w^{-1}s_{\alpha_i}\rangle = \langle s_{\alpha_i}Js_{\alpha_i}, s_{\alpha_i}, ws_{\alpha_j}w^{-1}\rangle.$$

- Suppose $\ell(s_{\alpha_i} w s_{\alpha_j}) = \ell(w) = \ell(s_{\alpha_i} w) + 1 = \ell(w s_{\alpha_j}) + 1$. Since $\ell(s_{\alpha_i} w) = \ell(w s_{\alpha_j})$, necessarily $s_{\alpha_i} w = w s_{\alpha_j}$. Write $w(\alpha_j) = \sum_k c_k \alpha_k$. Then

$$\begin{aligned} T_{\alpha_i} T'_{\alpha_j}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(q v_{J,(\epsilon_k + c_k), w s_{\alpha_j}} + (q-1) v_{J \cup \{w s_{\alpha_j} w^{-1}\},(\epsilon_k),w}) \\ &= q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1 + c_1, \ldots, \epsilon_i + c_i + \sum_{k \neq i}(\epsilon_k + c_k) n_k, \ldots, \epsilon_n + c_n), s_{\alpha_i} w s_{\alpha_j}} \\ &+ (q-1) q v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i} w s_{\alpha_j} w^{-1} s_{\alpha_i}\},(\epsilon_1, \ldots, \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w} \\ &+ (q-1)^2 v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i} w s_{\alpha_j} w^{-1} s_{\alpha_i}, s_{\alpha_i}\},(\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), w} \end{aligned}$$

  and

$$\begin{aligned} T'_{\alpha_j} T_{\alpha_i}(v_{J,(\epsilon_k),w}) &= T'_{\alpha_j}(q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1, \ldots, \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w} \\ &+ (q-1) v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\},(\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), w}) \\ &= q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1, \ldots, \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), s_{\alpha_i} w s_{\alpha_j}} \\ &+ (q-1) q v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\},(\epsilon_1 + c_1, \ldots, \epsilon_i + c_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n + c_n), w s_{\alpha_j}} \\ &+ (q-1)^2 v_{s_{\alpha_i} J s_{\alpha_i} \cup \{w s_{\alpha_j} w^{-1}, s_{\alpha_i}\},(\epsilon_1, \ldots, \epsilon_i + \sum_{k \neq i} \epsilon_k n_k, \ldots, \epsilon_n), w}. \end{aligned}$$

  Since $\ell(w s_{\alpha_j}) = \ell(w) - 1$, $w(\alpha_j) \in \Phi^-$, and since $\ell(s_{\alpha_i} w s_{\alpha_j}) = \ell(s_{\alpha_i} w) + 1$, $s_{\alpha_i} w(\alpha_j) \in \Phi^+$. Thus $w(\alpha_j)$ is a negative root made positive by s_{α_i}, so $w(\alpha_j) = -\alpha_i$. Hence $c_i = -1$ and $c_k = 0$ for $k \neq i$. Hence as elements of $\mathbb{F}_2$,

$$\epsilon_i + c_i + \sum_{k \neq i} (\epsilon_k + c_k) n_k = \epsilon_i - 1 + \sum_{k \neq i} \epsilon_k n_k = \epsilon_i + 1 + \sum_{k \neq i} \epsilon_k n_k$$

  which gives equality of the first terms.

  Comparing the second terms of each computation, it was noted before that $s_{\alpha_i} w = w s_{\alpha_j}$. It remains to check $\langle s_{\alpha_i} J s_{\alpha_i}, s_{\alpha_i} w s_{\alpha_j} w^{-1} s_{\alpha_i}, s_{\alpha_i} \rangle = \langle s_{\alpha_i} J s_{\alpha_i}, s_{\alpha_i} \rangle$, but this is clear since as before, $s_{\alpha_i} w = w s_{\alpha_j}$ implies $s_{\alpha_i} w s_{\alpha_j} w^{-1} s_{\alpha_i} = s_{\alpha_i}$.

  Finally, the third terms are equal since $\langle s_{\alpha_i} J s_{\alpha_i}, s_{\alpha_i} w s_{\alpha_j} w^{-1} s_{\alpha_i}, s_{\alpha_i} \rangle = \langle s_{\alpha_i} J s_{\alpha_i}, s_{\alpha_i}, w s_{\alpha_j} w^{-1} \rangle$.

So $T_{\alpha_i} T'_{\alpha_j} = T'_{\alpha_j} T_{\alpha_i}$.

□

Definition 6.5. For $K \subseteq \mathcal{R}$, define an operator $E_K \in \mathrm{End}_k(V)$ by

$$E_K(v_{J,(\epsilon_k),w}) = v_{J \cup K,(\epsilon_k),w}.$$

If $K = \{w\}$ is a singleton, write $E_K = E_s$.

Definition 6.6. For $\alpha_i \in \Delta$, define $I_{\alpha_i} \in \mathrm{End}_k(V)$ by

$$I_{\alpha_i}(v_{J,(\epsilon_k),w}) = v_{J,(\epsilon_1,\ldots,\epsilon_i+1,\ldots,\epsilon_n),w}.$$

Then clearly for $\alpha_i, \alpha_j \in \Delta$, $I_{\alpha_i} I_{\alpha_j} = I_{\alpha_j} I_{\alpha_i}$, so if $\beta = \sum_k c_k \alpha_k$, write $I_\beta = \prod_k I_{\alpha_k}^{c_k}$.

We check that these operations satisfy the relations $T_{\alpha_i}^2 = qI_{\alpha_i} + (q-1)T_{\alpha_i}E_{s_{\alpha_i}}$, $T_{\alpha_i}E_K = E_{s_{\alpha_i}Ks_{\alpha_i}}T_{\alpha_i}$, $T_{\alpha_i}I_{\alpha_j} = I_{s_{\alpha_i}(\alpha_j)}T_{\alpha_i}$, and $\underbrace{T_{\alpha_i}T_{\alpha_j}T_{\alpha_i}\cdots}_{m_{ij}} = \underbrace{T_{\alpha_j}T_{\alpha_i}T_{\alpha_j}\cdots}_{m_{ij}}$. That the E_K, T_s, and I_α satisfy the other relations analogous to those satisfied by the $e_t, \tau_s, \iota_\alpha$ in A is clear.

Lemma 6.7. *For $\alpha_i \in \Delta$ and $s_{\alpha_i} \in S$, the relation*

$$T_{\alpha_i}^2 = qI_{\alpha_i} + (q-1)T_{\alpha_i}E_{s_{\alpha_i}}$$

holds in $\mathrm{End}_k(V)$.

Proof. Suppose $\ell(s_{\alpha_i}w) = \ell(w) + 1$. Then

$$\begin{aligned} T_{\alpha_i}^2(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{s_{\alpha_i}Js_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w}) \\ &= qv_{J,(\epsilon_1,\ldots,\epsilon_i+1+2\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),w} + (q-1)v_{J\cup\{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+2\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w} \\ &= qv_{J,(\epsilon_1,\ldots,\epsilon_i+1,\ldots,\epsilon_n),w} + (q-1)v_{J\cup\{s_{\alpha_i}\},(\epsilon_k),s_{\alpha_i}w}. \end{aligned}$$

Observe

$$I_{\alpha_i}(v_{J,(\epsilon_k),w}) = v_{J,(\epsilon_1,\ldots,\epsilon_i+1,\ldots,\epsilon_n),w}.$$

Also,

$$\begin{aligned} T_{\alpha_i}E_{s_{\alpha_i}}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J\cup\{s_{\alpha_i}\},(\epsilon_k),w}) \\ &= v_{s_{\alpha_i}Js_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n),s_{\alpha_i}w} \\ &= v_{J\cup\{s_{\alpha_i}\},(\epsilon_k),s_{\alpha_i}w} \end{aligned}$$

where the last equality follows since $\langle s_{\alpha_i}Js_{\alpha_i}, s_{\alpha_i}\rangle = \langle J \cup \{s_{\alpha_i}\}\rangle$, and the discrepancy that possibly $\epsilon_i \neq \epsilon_i + \sum_{k\neq i}\epsilon_k n_k$ is irrelevant since $s_{\alpha_i} \in \langle J \cup \{s_{\alpha_i}\}\rangle$.

Suppose instead $\ell(s_{\alpha_i} w) = \ell(w) - 1$. Then

$$\begin{aligned}
T^2_{\alpha_i}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w} + (q-1) v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n)w}) \\
&= q v_{J,(\epsilon_1,\dots,\epsilon_i+1+2\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),w} \\
&+ (q-1) q v_{J\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+1+2\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w} \\
&+ (q-1)^2 v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+2\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),w} \\
&= q v_{J,(\epsilon_1,\dots,\epsilon_i+1,\dots,\epsilon_n),w} + (q-1) q v_{J\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+1,\dots,\epsilon_n),s_{\alpha_i}w} \\
&+ (q-1)^2 v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i,\dots,\epsilon_n),w}.
\end{aligned}$$

However,

$$\begin{aligned}
T_{\alpha_i} E_{s_{\alpha_i}}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J\cup\{s_{\alpha_i}\},(\epsilon_k),w}) \\
&= q v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w} \\
&+ (q-1) v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),w}.
\end{aligned}$$

The discrepancy at the ith coordinate is irrelevant since $\langle s_{\alpha_i} J s_{\alpha_i}, s_{\alpha_i}\rangle = \langle J, s_{\alpha_i}\rangle$. The claim now follows. □

Lemma 6.8. *For any $K \subseteq \mathcal{R}$ and any $\alpha_i \in \Delta$, the relations $T_{\alpha_i} E_K = E_{s_{\alpha_i} K s_{\alpha_i}} T_{\alpha_i}$ and $T'_{\alpha_i} E_K T'_{\alpha_i}$ hold in $\mathrm{End}_k(V)$.*

Proof. Suppose $\ell(s_{\alpha_i} w) = \ell(w) + 1$. Then

$$\begin{aligned}
T_{\alpha_i} E_K(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J\cup K,(\epsilon_k)w}) \\
&= v_{s_{\alpha_i}(J\cup K)s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w} \\
&= E_{s_{\alpha_i} K s_{\alpha_i}}(v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w}) \\
&= E_{s_{\alpha_i} K s_{\alpha_i}} T_{\alpha_i}(v_{J,(\epsilon_k),w}).
\end{aligned}$$

If $\ell(s_{\alpha_i} w) = \ell(w) - 1$, then

$$\begin{aligned}
T_{\alpha_i} E_K(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J\cup K,(\epsilon_k),w}) \\
&= q v_{s_{\alpha_i}(J\cup K)s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w} \\
&+ (q-1) v_{s_{\alpha_i}(J\cup K)s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),w} \\
&= E_{s_{\alpha_i} L s_{\alpha_i}}(q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\dots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),s_{\alpha_i}w}) \\
&+ E_{s_{\alpha_i} K s_{\alpha_i}}((q-1) v_{s_{\alpha_i} J s_{\alpha_i}\cup\{s_{\alpha_i}\},(\epsilon_1,\dots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\dots,\epsilon_n),w}) \\
&= E_{s_{\alpha_i} K s_{\alpha_i}} T_{\alpha_i}(v_{J,(\epsilon_k),w}).
\end{aligned}$$

That E_K commutes with T'_{α_i} is immediate. $\square$

Lemma 6.9. *For $\alpha_i, \alpha_j \in \Delta$, the relation $T_{\alpha_i} I_{\alpha_j} = I_{s_{\alpha_i}(\alpha_j)} T_{\alpha_i}$ holds in* $\mathrm{End}_k(V)$*.*

Proof. Suppose $\ell(s_{\alpha_i} w) = \ell(w) + 1$. Then

$$\begin{aligned} T_{\alpha_i} I_{\alpha_j}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J,(\epsilon_1,\ldots,\epsilon_j+1,\ldots,\epsilon_n),w}) \\ &= v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+n_j+\sum_{k\neq i}\epsilon_n),\ldots,\epsilon_j+1,\ldots,\epsilon_n),s_{\alpha_i} w}. \end{aligned}$$

On the other hand, recall $s_{\alpha_i}(\alpha_j) = \alpha_j + n_j \alpha_i$. Then

$$\begin{aligned} I_{s_{\alpha_i}(\alpha_j)} T_{\alpha_i}(v_{J,(\epsilon_k),w}) &= I_{\alpha_i + n_k \alpha_j}(v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_n, s_{\alpha_i} w}) \\ &= v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+n_j+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_j+1,\ldots,\epsilon_n),s_{\alpha_i} w} \end{aligned}$$

and the claim follows. If $\ell(s_{\alpha_i} w) = \ell(w) - 1$, then

$$\begin{aligned} T_{\alpha_i} I_{\alpha_j}(v_{J,(\epsilon_k),w}) &= T_{\alpha_i}(v_{J,(\epsilon_1,\ldots,\epsilon_j+1,\ldots,\epsilon_n) w}) \\ &= q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_j+1,\ldots,\epsilon_n),s_{\alpha_i} w} \\ &+ (q-1) v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+n_j+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_j+1,\ldots,\epsilon_n),w} \\ &= I_{\alpha_j + n_j \alpha_i}(q v_{s_{\alpha_i} J s_{\alpha_i},(\epsilon_1,\ldots,\epsilon_i+1+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_j,\ldots,\epsilon_n),s_{\alpha_i} w} \\ &+ (q-1) v_{s_{\alpha_i} J s_{\alpha_i} \cup \{s_{\alpha_i}\},(\epsilon_1,\ldots,\epsilon_i+\sum_{k\neq i}\epsilon_k n_k,\ldots,\epsilon_j,\ldots,\epsilon_n),w}) \\ &= I_{s_{\alpha_i}(\alpha_j)} T_{\alpha_i}(v_{J,(\epsilon_k),w}). \end{aligned}$$

Hence $T_{\alpha_i} I_{\alpha_j} = I_{s_{\alpha_i}(\alpha_j)} T_{\alpha_i}$, and it follows that $T_\beta I_\gamma = I_{w_\beta(\gamma)} T_\beta$ for any roots $\beta, \gamma \in \Phi$. $\square$

Lastly, we check the braid relation.

Lemma 6.10. *For $\alpha, \beta \in \Delta$, the relation*

$$\underbrace{T_\alpha T_\beta T_\alpha \cdots}_{m_{\alpha\beta}} = \underbrace{T_\beta T_\alpha T_\beta \cdots}_{m_{\alpha\beta}}$$

holds in $\mathrm{End}_k(V)$*.*

Proof. Put $u = \underbrace{s_\alpha s_\beta s_\alpha \cdots}_{m_{\alpha\beta}} = \underbrace{s_\beta s_\alpha s_\beta \cdots}_{m_{\alpha\beta}} \in W$. Observe

$$\underbrace{T_\alpha T_\beta T_\alpha \cdots}_{m_{\alpha\beta}}(v_{\emptyset,\vec{0},1}) = v_{\emptyset,\vec{0},\underbrace{s_\alpha s_\beta s_\alpha \cdots}_{m_{\alpha\beta}}} = v_{\emptyset,\vec{0},\underbrace{\beta\alpha\beta \cdots}_{m_{\alpha\beta}}} = \underbrace{T_\beta T_\alpha T_\beta \cdots}_{m_{\alpha\beta}}(v_{\emptyset,\vec{0},1}).$$

Let $w \in W$ have reduced expression $t_1 \cdots t_r$. Using the prior relations and that fact that the operators T and T' commute, (and suppressing the $m_{\alpha\beta}$ underbrace notation below where it is clear), observe

$$\begin{aligned}
\underbrace{T_\alpha T_\beta T_\alpha \cdots}_{m_{\alpha\beta}}(v_{J,(\epsilon_k),w}) &= T_\alpha T_\beta T_\alpha \cdots T'_{t_r} \cdots T'_{t_1}(v_{J,(\epsilon_k),1}) \\
&= T'_{t_r} \cdots T'_{t_1} T_\alpha T_\beta T_\alpha \cdots E_J \left(\prod_{k=1}^{n} I_{\alpha_k}^{\epsilon_k}\right)(v_{\emptyset,\vec{0},1}) \\
&= E_{uJu^{-1}} \left(\prod_{k=1}^{n} I_{u(\alpha_k)}^{\epsilon_k}\right) T'_{t_r} \cdots T'_{t_1} T_\alpha T_\beta T_\alpha \cdots (v_{\emptyset,\vec{0},1}) \\
&= E_{uJu^{-1}} \left(\prod_{k=1}^{n} I_{u(\alpha_k)}^{\epsilon_k}\right) T'_{t_r} \cdots T'_{t_1} T_\beta T_\alpha T_\beta \cdots (v_{\emptyset,\vec{0},1}) \\
&= E_{uJu^{-1}} \left(\prod_{k=1}^{n} I_{u(\alpha_k)}^{\epsilon_k}\right) T_\beta T_\alpha T_\beta \cdots T'_{t_r} \cdots T'_{t_1}(v_{\emptyset,\vec{0},1}) \\
&= E_{uJu^{-1}} \left(\prod_{k=1}^{n} I_{u(\alpha_k)}^{\epsilon_k}\right) T_\beta T_\alpha T_\beta \cdots (v_{\emptyset,\vec{0},w}) \\
&= T_\beta T_\alpha T_\beta \cdots E_J \left(\prod_{k=1}^{n} I_{\alpha_k}^{\epsilon_k}\right)(v_{\emptyset,\vec{0},w}) \\
&= \underbrace{T_\beta T_\alpha T_\beta \cdots}_{m_{\alpha\beta}}(v_{J,(\epsilon_k),w}).
\end{aligned}$$

So $\underbrace{T_\alpha T_\beta T_\alpha \cdots}_{m_{\alpha\beta}} = \underbrace{T_\beta T_\alpha T_\beta \cdots}_{m_{\alpha\beta}}$. □

Proposition 6.11. *The algebra A is a free k-module with basis*

$$\mathcal{B} = \{e_J \left(\prod_{i=1}^{n} \iota_{\alpha_i}^{\epsilon_i}\right) \tau_w : J \subseteq \mathcal{R} \text{ finite}, \ \epsilon_i \in \{0,1\} \text{ and } \epsilon_i = 0 \text{ if } s_{\alpha_i} \in \langle J \rangle, \ w \in W\}.$$

Proof. From the preceding lemmas, it follows that there is a k-algebra map

$$\varphi\colon A \to \mathrm{End}(V) : \tau_s \mapsto T_s, \ \iota_\alpha \mapsto I_\alpha, \ e_w \mapsto E_w.$$

Observe that

$$\varphi\left(e_J \left(\prod_{k=1}^{n} \iota_{\alpha_k}^{\epsilon_k}\right) \tau_w\right)(v_{\emptyset,\vec{0},1}) = E_J \left(\prod_{k=1}^{n} I_{\alpha_k}^{\epsilon_k}\right) T_w(v_{\emptyset,\vec{0},1}) = v_{J,(\epsilon_k),w},$$

so that φ is surjective onto V. Moreover, suppose $\sum \left(c_{J,(\epsilon_k),w} e_J \left(\prod_{k=1}^{n} \iota_{\alpha_k}^{\epsilon_k}\right) \tau_w\right) \in \ker\varphi$ is a k-linear combination of elements of $\mathcal{B}$, for some scalars $c_{J,(\epsilon_k),w}$. Then applying φ

$$\sum c_{J,(\epsilon_k),w} \cdot E_J \left(\prod_{k=1}^{n} I_{\alpha_k}^{\epsilon_k}\right) T_w \equiv 0$$

and evaluation at $v_{\emptyset,\vec{0},1}$ yields

$$\sum c_{J,(\epsilon_k),w} \cdot v_{J,(\epsilon_k),w} = 0$$

so that each coefficient $c_{J,(\epsilon_k),w} = 0$, and so φ is injective. $\square$

The above proposition shows that A has dimension dependent on the cardinality of W and the structure of its reflection subgroups, not on the characteristic of the field of definition of the original Chevalley group. As in the rank 1 case, this algebra should fit into a diagram of form

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{End}_{(\mathscr{H}_S(\tilde{q}),\mathscr{H}_\emptyset(\tilde{q}))}(\mathscr{H}_S(\tilde{q})) \\ \downarrow & & \downarrow \\ \mathrm{End}_{(kG,kT)}(kG/U) & \longrightarrow & \mathrm{End}_{(\mathscr{H}_S(q),\mathscr{H}_\emptyset(q))}(\mathscr{H}_S(q)). \end{array}$$

This algebra A then serves as a conjectural definition of the 2-endomorphism algebra of $F_\emptyset^S$ in the context of a W-categorification. As in Section 5.2, the counit-unit adjunctions induce corresponding conjectural definitions for the k-vector spaces of 2-homomorphisms between parallel morphisms involving $F_\emptyset^S$, $E_\emptyset^S$, $1_\emptyset$, and 1_S. However, further investigation is required to determine a conjectural definition of the endomorphism algebra of F_I^J when $\emptyset \subsetneq I \subsetneq J \subseteq S$.

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