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Weak Limit sets of Differential Equations

Morris W. Hirsch

Dedicated to Professor Victor L. Shapiro

ABSTRACT. Let E be a Banach space. We investigate the limit set $\mathcal{L}[u]$ and the weak limit set $\mathcal{L}_w[u]$ of solutions $u : \mathbf{R}_+ \rightarrow E$ to differential equations $Tu = f$ where T is the composition of n first order differential operators of the form $a_k D + b_k I$. Here D is differentiation, I is the identity; the coefficients $a_k, b_k : \mathbf{R}_+ \rightarrow \mathbf{R}$ are continuous and nonvanishing with $\int |a_k/b_k| = \infty$; and $f : \mathbf{R}_+ \rightarrow E$ is continuous.

Theorem Assume $n = 1$ and $a_1/b_1 > 0$. If K is a nonempty closed convex set such that $\text{dist}(f(t)/b_1(t), K) \rightarrow 0$, then $\mathcal{L}[u] \subset K$.

For the following results, assume the b_k are bounded and bounded away from zero.

Theorem $\mathcal{L}[u] \subset \mathcal{L}_w[u] \subset M\overline{\text{co}}(\mathcal{L}_w[f])$, where $M \subset \mathbf{R}$ is a certain compact interval associated to T .

Let B denote the set of maps $g : \mathbf{R}_+ \rightarrow E$ such that for every continuous linear functional L on E , the composition $L \circ g : \mathbf{R}_+ \rightarrow \mathbf{R}$ has compact (possibly empty) limit set.

Theorem If $f \in B$, then $u \in B$.

Applications are made to machine learning, nonlinear autonomous equations, and real eigenfunctions of the operator T .

1. Introduction

A fundamental problem in dynamical systems is to obtain information on the limit sets of solutions to differential equations. Generally we have to deal with nonlinear equations $dx/dt = F(t, x)$. But sometimes information can be got about a particular solution $u(t)$ from the observation that it satisfies the first order linear nonhomogeneous equation $du/dt = f(t)$ where $f(t) = F(t, u(t))$, provided we know something about F and have some *a priori* knowledge of u . In fact a situation of this kind, described next, motivated the present paper. Too simple to be called an "application" of our results, this example is presented first as motivation for the general results.

The basic question attacked in this paper is to obtain information on the limiting behavior of a solution to an n th order nonautonomous, linear, nonhomogeneous differential equation in a Banach space E of the form:

$$\begin{aligned} (1) \quad Tu &= f, \\ (2) \quad T &= (a_n D + b_n I) \circ \cdots \circ (a_1 D + b_1 I) \end{aligned}$$

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where D is the differentiation operator and I the identity. The unknown function u and the driving function f are maps from $[t_0, \infty)$ to E . We always assume the coefficients are continuous nonvanishing maps $a_k, b_k : [t_0, \infty) \rightarrow \mathbf{R}$ such that $\int |a_k/b_k| = \infty$. In particular, we would like to obtain information on the limit set $\mathcal{L}[u]$, and the weak limit set $\mathcal{L}_w[u]$, in terms of f and the coefficients.

Set $\sigma_i = \text{sgn}(a_i/b_i)$ and $\sigma = \sigma_1 \cdots \sigma_n$. Define $J_i \subset \mathbf{R}$ to be the closed interval $\mathcal{L}[b_i]$, and define the interval

$$J = J_1 \cdots J_n = \{s_1 \cdots s_n : s_k \in J_k, k = 1, \dots, n\}$$

When $0 \notin J$ we define J^{-1} to be the interval consisting of the reciprocals of numbers in J , and set $J_1/J = J_1 J^{-1}$.

Define B_E to be the set of continuous functions $g : [t_0, \infty) \rightarrow E$ such that for each $L \in E^*$ (the dual space of E), $\mathcal{L}[L \circ g]$ is bounded (perhaps empty).

The closed convex hull of a set $A \subset E$ is denoted by $\overline{\text{co}}(A)$.

Here is a special case of Theorem 4.7:

THEOREM 1.1. *Assume each J_i is a finite interval (or a point) not containing zero. Let $v : [t_0, \infty) \rightarrow E$ be a solution to the differential equation $Tv = f$. Then:*

(a): $\mathcal{L}_w[v] \subset \sigma J^{-1} \overline{\text{co}}(\mathcal{L}_w[f])$.

(b): If $f \in B_E$ then $v \in B_E$.

After discussing the original motivating problem we present generalities on limit sets in Banach spaces in Section 2. Limit sets to solutions of differential equations in $\mathbf{R}$ are treated in Section 3. Equations in arbitrary Banach spaces are covered in Section 4, together with an application to nonlinear dynamical systems.

J. Muldowney has kindly pointed out that the higher order equations considered in Section 3 include the classical disconjugate equations (Pólya [6], Coppel [2]); and that some of the results of that section are similar to Proposition 3.2 of Muldowney [5].

1.1. A Design Problem in Machine Learning. The following problem came up in dynamical systems models of machine learning (see Baird and Hirsch [4], Hirsch [3]). Suppose we are designing a system, e.g., a neural network for pattern recognition, involving a differential equation in Euclidean n -space $\mathbf{R}^n$ which we write in the not uncommon form

$$(3) \quad \frac{dx}{dt} = -x + F(x).$$

There is a certain attractor $A \subset \mathbf{R}^n$ for the flow generated by F that we would like to reach in the following way: start in some given initial state s , add a *teaching signal* I to Equation (3), and run the initial value problem:

$$(4) \quad \frac{dy}{dt} = -y + F(y) + I,$$

in the hope that $y(t)$ will end up in the basin of attraction of A at some time t_1 . If we then set I to 0, the trajectory of Equation (3) starting at $y(t_1)$ will proceed towards A .

How should I , which could depend on t , be chosen? We want I to be simple and robust—not sensitive to mathematical details—so that it can be implemented by

a machine, or for biological modeling, by some more primitive part of the nervous system.

In general this problem is quite difficult to solve in terms of a formula for F ; basins of attractions are notoriously difficult to locate. (See e. g. Chiang *et al.* [1] for a contribution and references to this problem.)

But we do not need to find a general solution. F does not arise from nature; we have considerable freedom in designing it, subject to the constraints of our network. And a successful solution of the mathematical problem for a class of differential equations might suggest better ways to design networks.

If this idea is to succeed, some part of the basin of A needs to be easy to locate. We assume the basin contains the closed δ -neighborhood $N_\delta(K)$, $\delta > 0$ of some convex set K which is known to us. Moreover we assume δ is a uniform upper bound for $\|F(x)\|$. While these are drastic assumptions, it is easy to find plausible network equations admitting attractors that satisfy them (e. g. Baird and Hirsch [4], Hirsch [3]). We then have the following result, where the distance from z to K is denoted by $\text{dist}(z, K)$.

THEOREM 1.2. *Assume the function $I : [0, \infty) \rightarrow \mathbf{R}^n$ is such that*

$$\lim_{t \rightarrow \infty} \text{dist}(I(t), K) = 0.$$

If $x(t)$, $0 \leq t < \infty$ is a solution to Equation (4), then

$$\lim_{t \rightarrow \infty} \text{dist}(x(t), N_\delta(K)) = 0.$$

PROOF. Fix the solution x and set $f(t) = F(x(t)) + I(t)$. Then x is a solution to

$$(5) \quad \frac{dx}{dt} = -x + f(t).$$

Now

$$\text{dist}(f(t), K) \leq \|F(x(t))\| + \text{dist}(I(t), K) \leq \delta + \text{dist}(I(t), K).$$

Therefore

$$\lim_{t \rightarrow \infty} \text{dist}(f(t), N_\delta(K)) = 0.$$

Define

$$V : \mathbf{R}^n \rightarrow \mathbf{R}, \quad z \mapsto \text{dist}(z, N_\delta(K)).$$

Then V is a convex Lipschitz map, and $\lim_{t \rightarrow \infty} V(f(t)) = 0$.

LEMMA 1.3.

$$\limsup_{h \rightarrow 0} \frac{V(x(t+h)) - V(x(t))}{h} \leq -V(x(t)) + V(f(t)).$$

To prove the lemma, we use Equation (5) to obtain a Taylor estimate, abbreviating $x = x(t)$, $f = f(t)$:

$$x(t+h) = x + hx'(t) + o(h) = (1-h)x + hf(t) + o(h).$$

Using the Lipschitz property of V and convexity, we get:

$$(6) \quad \frac{V(x(t+h)) - V(x)}{h} = \frac{V((1-h)x + hf) + o(h) - V(x)}{h}$$

$$(7) \quad \leq \frac{(1-h)V(x) + hV(f) - V(x) + o(h)}{h}$$

$$(8) \quad = -V(x) + V(f) + \frac{o(h)}{h}.$$

Letting $h \rightarrow 0$ proves the lemma.

Now fix $\epsilon > 0$ and choose $t_0 \geq 0$ so large that $V(f(t)) < \epsilon$ if $t \geq t_1$. Then the lemma and the Comparison Principle (Proposition 3.1 below) imply $V(t_1+s) \leq v(s)$ for all $s \geq 0$, where $v(s)$ solves the initial value problem

$$\frac{dv}{dt} = -v + \epsilon, \quad v(0) = V(t_1).$$

Thus

$$(9) \quad V(t_1 + s) \leq (1 - e^{-s})\epsilon + e^{-s}V(t_1),$$

proving that $\limsup_{t \rightarrow \infty} V(t) \leq \epsilon$. Since this holds for all $\epsilon > 0$, the proof of Theorem 1.2 is complete. $\square$

2. Limit sets of curves in Banach spaces

Let E denote a real Banach space, with dual space E^* . The topological vector space consisting of E with its weak topology is denoted by E_w . Recall that a subset $X \subset E$ is weakly compact (i.e., compact in E_w) if and only if X is bounded and weakly closed. In particular, every bounded closed convex subset of E is weakly compact.

By a *curve* in E we mean a continuous map $g : [t_0, \infty) \rightarrow E$; unless otherwise indicated, the domain of any curve will be $[t_0, \infty)$.

The *limit set* $\mathcal{L}[g] \subset E$ of a curve g consists of all limits of sequences $g(t_k)$, $t_k \rightarrow \infty$. This set may, of course, be empty.

The *weak limit set* $\mathcal{L}_w[g]$ is the set of weak limit points of sequences $g(t_k)$, $t_k \rightarrow \infty$. Equivalently, $\mathcal{L}_w[g]$ is the set of limits of sequences $g(t_k)$, $t_k \rightarrow \infty$, in E_w . Evidently $\mathcal{L}_w[g]$ is weakly closed. It contains $\mathcal{L}[g]$, but it may be much larger.

LEMMA 2.1.

(a): Suppose the image of g lies in a compact subset $C \subset E$. Then $\mathcal{L}[g] = \mathcal{L}_w[g]$; and this set is a nonempty, compact connected subset of C .

(b): If $\mathcal{L}_w[g]$ is bounded, it is weakly compact.

(c): Suppose the image of g lies in a closed bounded convex set $K \subset E$. Then $\mathcal{L}_w[g]$ is a nonempty, weakly compact, weakly connected subset of K .

PROOF. The first statement in (a) holds because E and E_w induce the same topology on compact subsets. To prove the second statement, we note that

$$\mathcal{L}[g] = \bigcap_{t \geq t_0} \text{clos } g([t, \infty)),$$

where clos denotes closure. This exhibits $\mathcal{L}[g]$ as the intersection of a nested family of compact connected subsets of C ; a standard result in topology implies the second statement in (a). In (b) $\mathcal{L}_w[g]$ lies in some closed ball B . Since B is weakly compact

and $\mathcal{L}_w[g]$ is weakly closed, it follows that $\mathcal{L}_w[g]$ is weakly compact. For (c) we note that K is weakly compact, so reasoning as in (a) completes the proof. $\square$

EXAMPLE 2.2. Assume E is separable and infinite dimensional, and let $\{e_k\}_{k \in \mathbb{N}}$ be a dense sequence of linearly independent unit vectors. Let $h : [0, \infty) \rightarrow E$ map each natural number $k \in \mathbb{N}$ to e_k , with h affine on $[k-1, k]$. Define $g(t) = h(t)/\|h(t)\|$, $0 \leq t < \infty$. Then the image of g lies in the (boundary) unit sphere $S = \{x \in E : \|x\| = 1\}$. It is clear that $\mathcal{L}[g]$ is empty. On the other hand, $\mathcal{L}_w[g] = \{0\}$ because $g(t) \rightarrow 0$ in E_w .

EXAMPLE 2.3. Example 2.2 can be changed slightly to yield a curve h in S which converges weakly to 0, but has a unique limit point p , necessarily in S . Then $\mathcal{L}_w[h]$ is a compact connected subset of the closed unit ball B containing 0 and p . Therefore $\mathcal{L}_w[h]$ must contain a point of norm λ , for each $\lambda \in [0, 1]$.

EXAMPLE 2.4. Example 2.2 can be changed to a curve r in S which converges weakly to 0, but has precisely two limit points p, q . In this case $\mathcal{L}_w[r]$ is nonempty, weakly compact and weakly connected, while $\mathcal{L}_w[r] = \{p, q\}$, which is nonempty, compact and disconnected.

We say a closed subset $P \subset E$ *absorbs* g if P is nonempty and

$$\lim_{t \rightarrow \infty} \text{dist}(g(t), P) = 0.$$

If P absorbs g then P contains $\mathcal{L}[g]$, but it need not contain $\mathcal{L}_w[g]$.

In Example 2.2, the unit sphere S absorbs g , but it does not contain $\mathcal{L}_w[g]$. The closed unit ball B , which is convex, does contain $\mathcal{L}_w[g]$.

We say P *weakly absorbs* g if:

$$\sup L(P) \geq \overline{\lim} (L \circ g) \quad \forall L \in E^*.$$

Clearly this holds when P absorbs g . Two equivalent formulations are:

$$\inf L(P) \leq \underline{\lim} (L \circ g) \quad \forall L \in E^*;$$

and

$$L(P) \supset \mathcal{L}[L \circ g] \quad \forall L \in E^*.$$

In Example 2.2, the origin weakly absorbs g , but does not absorb g .

LEMMA 2.5. *The following statements (i), (ii), (iii) are equivalent:*

- (i): *The image of g lies in a compact subset of E .*
- (ii): *$\mathcal{L}[g]$ is a nonempty compact set that absorbs g .*
- (iii): *Some nonempty compact set absorbs g .*

When the image of g lies in a locally compact set (e.g., $E = \mathbb{R}^n$), then these are equivalent to:

- (iv): *$\mathcal{L}[g]$ is a nonempty compact set.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. If (iii) holds then it is easy to see that the closure of the image of g is compact, proving (i). Now assume the image of g lies in a locally compact set X and $\mathcal{L}[g]$ is a nonempty compact set. Then there exists a compact set $N \subset X$ which is a neighborhood of $\mathcal{L}[g]$ in X , so that $\mathcal{L}[g]$ lies in the relative interior N^0 of N in X . Since the compact set $\partial N = N \setminus N^0$ is disjoint from $\mathcal{L}[g]$, it follows that there exists t_1 such that $g(t) \notin \partial N$ for all $t > t_1$. Because $\mathcal{L}[g]$ is a nonempty set in N , this implies $g(t) \in N$ for all $t \geq t_1$, and this yields (iv). $\square$

LEMMA 2.6. *If P weakly absorbs g , then $\mathcal{L}_w[g] \subset \overline{\text{co}}(P)$.*

PROOF. Set $K = \overline{\text{co}}(P)$. Evidently $N_\epsilon(K)$ weakly absorbs g for every $\epsilon > 0$. It follows that if $H \subset E$ is any supporting hyperplane for the convex set $N_\epsilon(K)$, then $\mathcal{L}_w[g]$ and K lie on the same side of H . The hyperplane separation theorem therefore implies $\mathcal{L}_w[g] \subset N_\epsilon(K)$. Since this holds for every $\epsilon > 0$, the lemma is proved. $\square$

The *weakly absorbing hull* $\mathcal{A}[g]$ of g is the smallest set that weakly absorbs g , namely

$$\mathcal{A}[g] = \{x \in E : Lx \in \mathcal{L}[L \circ g] \forall L \in E^*\}.$$

This set is convex and weakly closed, and is contained in every convex weakly closed set that weakly absorbs g .

PROPOSITION 2.7. *Let g be a curve in E . Then:*

(a): $\mathcal{L}_w[g] \subset \mathcal{A}[g]$.

(b): *If the image of g lies in a compact set, then $\mathcal{A}[g] \subset \overline{\text{co}}(\mathcal{L}[g])$.*

PROOF. Lemma 2.6 implies (a). For (a) we note that compactness implies

$$\mathcal{L}[L \circ g] \subset L(\mathcal{L}[g]) \subset L(\overline{\text{co}}(\mathcal{L}[g])).$$

Therefore $x \in \mathcal{A}[g]$ implies $Lx \in L(\overline{\text{co}}(\mathcal{L}[g]))$ for all $L \in E^*$, and the hyperplane separation theorem shows $x \in \overline{\text{co}}(\mathcal{L}[g])$. $\square$

3. Differential Equations in $\mathbf{R}$

For notational convenience we adjoin endpoints $\pm\infty$ to the real line $\mathbf{R}$, obtaining the *extended line* $\overline{\mathbf{R}} = [-\infty, \infty]$. For any function $g : [t_0, \infty) \rightarrow \mathbf{R}$ we allow only finite values for $Dg(t) = g'(t)$, but the following inferior and superior limits are taken in $\overline{\mathbf{R}}$:

$$\begin{aligned} \overline{\lim} g &= \limsup_{t \rightarrow \infty} g(t), \\ \underline{\lim} g &= \liminf_{t \rightarrow \infty} g(t), \\ \overline{D}g(t) &= \limsup_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}, \\ \underline{D}g(t) &= \liminf_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}, \\ Dg(t) &= g'(t). \end{aligned}$$

We define $\mathcal{L}[g]$ to be the closed interval $[\underline{\lim} g, \overline{\lim} g] \subset \overline{\mathbf{R}}$.

We have already used a special case of the following well-known result:

PROPOSITION 3.1 (COMPARISON PRINCIPLE). *If continuous functions $x, y : [t_0, \infty) \rightarrow \mathbf{R}$ satisfy $x(t_0) \leq y(t_0)$ and $\overline{D}x \leq \underline{D}y$, then $x \leq y$.*

Now let $u, f : [t_0, \infty) \rightarrow \mathbf{R}$ be continuous functions.

LEMMA 3.2.

(a): *If $\overline{D}u \leq -u + f$, then $\overline{\lim} u \leq \overline{\lim} f$.*

(b): *If $\underline{D}u \geq -u + f$, then $\underline{\lim} u \geq \underline{\lim} f$.*

(c): *Suppose $Du = -u + f$. Then $\mathcal{L}[u] \subset \mathcal{L}[f]$.*

PROOF. The proof of (a) is similar to that of Theorem 1.1. Part (b) follows from (a) by considering $-u$. Part (c)(i) follows from (a) and (b). $\square$

We shall also need the following more subtle fact:

LEMMA 3.3. Assume $Du = u + f$. Then:

- (a): Either $\overline{\lim} u \leq \overline{\lim}(-f)$, or else $u(t) \rightarrow \infty$.
- (b): Either $\underline{\lim} u \geq \underline{\lim}(-f)$, or else $u(t) \rightarrow -\infty$.
- (c): If $|u(t)| \not\rightarrow \infty$, then $\mathcal{L}[u] \subset \mathcal{L}[-f] = -\mathcal{L}[f]$.
- (d): If $f(t) \rightarrow \pm\infty$ then $u(t) \rightarrow \pm\infty$, respectively.

PROOF. To prove (a), assume $\overline{\lim} u > \overline{\lim}(-f)$. There exist t_1, β and $\epsilon > 0$ such that for all $t \geq t_1$:

$$u(t) > \beta + \epsilon, \quad f(t) > -\beta,$$

so

$$u'(t) = u(t) + f(t) > \epsilon,$$

whence $u(t) \rightarrow \infty$. The proofs of (b) and (d) are similar, and (c) follows from (a) and (d). $\square$

3.1. Nonconstant coefficients. We fix some interval $[t_0, \infty)$ and consider differential operators acting on functions $u : [t_0, \infty) \rightarrow \mathbf{R}$. A first order operator is *allowable* if it has the form $aD + bI$, where D is differentiation, I is the identity operator, and $a, b : [t_0, \infty) \rightarrow \mathbf{R}$, satisfy the following two conditions:

HYPOTHESIS 3.4.

- (i): $a(t) \neq 0, b(t) \neq 0$ for all $t \in [t_0, \infty)$.
- (ii): $\int_{t_0}^{\infty} \frac{|b(t)|}{|a(t)|} dt = \infty$.

An n th order differential operator T is allowable if it is the composition of n allowable first order operators. An *allowable differential equation* is one of the form $Tu = f$ where T is an allowable operator and $f : [t_0, \infty) \rightarrow \mathbf{R}$ is continuous.

Consider an allowable first order equation:

$$(10) \quad a \frac{dv}{dt} + bv = f$$

PROPOSITION 3.5. Suppose $a/b > 0$, and let $v : [t_0, \infty) \rightarrow \mathbf{R}$ be a solution to Equation (10). Then:

- (a): $\mathcal{L}[v] \subset \mathcal{L}[f/b]$.
- (b): If $f(t)/b(t) \rightarrow c \in \overline{\mathbf{R}}$ then $v(t) \rightarrow c$.

PROOF. Equation (10) is equivalent to

$$(11) \quad \frac{a}{b} \frac{dv}{dt} = -v + \frac{f}{b}.$$

We simplify further by replacing the time variable t with

$$(12) \quad s = \int_{t_0}^t \frac{|b(s)|}{|a(s)|} dt,$$

noting that s runs from 0 to ∞ by Hypothesis 3.4 (ii). We express t as a function $t(s)$, then set $w(s) = v(t(s))$ and $h(s) = f(t(s))/b(t(s))$. Now Equation (11) is equivalent to:

$$\frac{dw}{ds} = -w + h.$$

The proof of (a) is completed by applying Lemma 3.2(c), and (b) follows. $\square$

PROPOSITION 3.6. *Assume $a/b < 0$ and let $v : [t_0, \infty) \rightarrow \mathbf{R}$ be a solution to Equation (10). Then:*

- (a): *Either $\overline{\lim} v \leq \overline{\lim} (-f/b)$, or else $v(t) \rightarrow \infty$.*
- (b): *Either $\underline{\lim} v \geq \underline{\lim} (-f/b)$, or else $v(t) \rightarrow -\infty$.*
- (c): *If $f(t)/b(t) \rightarrow \pm\infty$, then $v(t) \rightarrow \pm\infty$, respectively.*

PROOF. Making the change of variable (12), we obtain the equivalent equation

$$\frac{dw}{ds} = w + h,$$

where $h(s) = f(t(s))/b(t(s))$, $0 \leq s < \infty$. The proof is completed by applying Lemma 3.3. $\square$

In some cases there is a convenient estimate for the limit set of v :

LEMMA 3.7. *Let $v : [t_0, \infty) \rightarrow \mathbf{R}$ be a solution to Equation (10). Assume $\mathcal{L}[b]$ is a compact interval J not containing 0. Set $\sigma = \text{sgn}(a/b)$. Then:*

$$\mathcal{L}[v] \subset \sigma \mathcal{L}[f]/\mathcal{L}[b].$$

PROOF. From Propositions 3.5 and 3.6 we see that $\mathcal{L}[v] \subset \sigma \mathcal{L}[f/b]$, and the assumptions on $\mathcal{L}[b]$ imply $\mathcal{L}[f/b] \subset \mathcal{L}[f]/\mathcal{L}[b]$. $\square$

Define $\mathbf{B} = \mathbf{B}([t_0, \infty), \mathbf{R})$ to be the set of continuous functions $g : [t_0, \infty) \rightarrow \mathbf{R}$ whose limit sets are bounded (possibly empty). Thus $g \in \mathbf{B}$ provided $\mathcal{L}[g]$ is a compact interval (possibly a point), or $|g(t)| \rightarrow \infty$. Roughly speaking, $g(t)$ cannot traverse arbitrarily large intervals infinitely often.

The following result shows that $\mathbf{B}$ is closed under solutions to certain kinds of linear differential equations:

PROPOSITION 3.8. *Assume $f/b \in \mathbf{B}$ and let $v : [t_0, \infty) \rightarrow \mathbf{R}$ be a solution to Equation (10). Then: $v \in \mathbf{B}$.*

PROOF. This is a consequence of Propositions 3.5 and 3.6. $\square$

3.2. Higher Order Equations. Consider now an allowable n th order linear differential operator T of the form:

$$(13) \quad T = T_n \circ \cdots \circ T_1,$$

$$(14) \quad T_i = a_i D + b_i I, \quad i = 1, \dots, n$$

Suppose $n \geq 2$. Given a function $f : [t_0, \infty) \rightarrow \mathbf{R}$, we recursively define the domain of T to be the set of differentiable functions $v : [t_0, \infty) \rightarrow \mathbf{R}$ such that $T_1 v$ is in the domain of $T_n \circ \cdots \circ T_2$. In particular, every C^n function is in the domain of T . In what follows, the notation Tv implies the tacit assumption that v is in the domain of T .

THEOREM 3.9. *Let v be a solution to the allowable differential equation $Tv = f$. Then:*

(a): Assume each interval $\mathcal{L}[b_i]$ is finite and does not contain zero. Set $\sigma_i = \text{sgn}(a_i/b_i)$ and $\sigma = \sigma_1 \cdots \sigma_n$. Then:

$$\mathcal{L}[v] \subset \sigma \mathcal{L}[f] / \mathcal{L}[b_1] \cdots \mathcal{L}[b_n].$$

(b): Assume

$$f/b_n \dots b_i \in B, \quad i = 1, \dots, n.$$

Then $v \in B$. In particular, $v \in C$ provided the $\mathcal{L}[b_i]$ are as in (a) and $f \in B$.

PROOF. The proof is by induction on n , the case $n = 1$ being covered by Propositions 3.5, 3.6 and 3.8. Assume $n \geq 2$ and define the allowable differential operator

$$S = T_n \circ \cdots \circ T_2.$$

The equation

$$(15) \quad Sw = f$$

is allowable and has the solution w where

$$(16) \quad (a_1 D + b_1)v = w.$$

By the induction hypothesis applied to Equation (15), in case (a),

$$(17) \quad \mathcal{L}[w] \subset \tau \mathcal{L}[f] / \mathcal{L}[b_n] \cdots \mathcal{L}[b_2]$$

where $\tau = \sigma_1 \cdots \sigma_{n-1}$. And $w \in B$ in case (b).

Now v is a solution to Equation (16), or equivalently, to

$$(18) \quad \left(\frac{a_1}{b_1} D + I\right)v = \frac{w}{b_1}.$$

so $v \in B$ by Proposition 3.8. In case (a), Lemma 3.7 shows

$$\mathcal{L}[v] \subset \sigma_1 \mathcal{L}[w/b_1] \subset \sigma_1 \mathcal{L}[w] / \mathcal{L}[b_1].$$

Therefore Equation (17) implies

$$(19) \quad \mathcal{L}[v] \subset \frac{\sigma_1 \tau \mathcal{L}[f] / \mathcal{L}[b_2] \cdots \mathcal{L}[b_n]}{\mathcal{L}[b_1]}$$

$$(20) \quad = \sigma \mathcal{L}[f] / \mathcal{L}[b_1] \cdots \mathcal{L}[b_n].$$

In case (b), Proposition 3.8 shows $v \in B$. The induction is complete. $\square$

4. Differential Equations in Banach Spaces

Now we extend the preceding results to Banach spaces. The notion of an allowable first order differential operator $aD + bI$ on functions $u : [t_0, \infty) \rightarrow E$ is formally the same as before, namely $a, b : [t_0, \infty) \rightarrow \mathbf{R}$ satisfy Hypothesis 3.4. A composition $T = T_n \circ \cdots \circ T_1$ of allowable first order operators is also called allowable. A differential equation of the form $Tu = f$ is allowable if T is an allowable operator and $f : [t_0, \infty) \rightarrow \mathbf{R}$ is continuous.

We first treat allowable first order equations

$$(21) \quad a \frac{dw}{dt} + bw = f.$$

THEOREM 4.1. Let w be a solution to Equation (21), and assume $a/b > 0$.

(a): If $V : E \rightarrow \mathbf{R}$ is convex and locally Lipschitz, then

$$\overline{\lim}(V \circ w) \leq \overline{\lim}(V \circ f/b).$$

(b): For all $L \in E^*$,

$$\mathcal{L}[L \circ w] \subset \mathcal{L}[L \circ (f/b)].$$

PROOF. We can make the change of variable (12), and replace f by f/b . Thus it suffices to consider the special case:

$$\frac{dw}{dt} = -w + f.$$

To prove (a), imitate the proof of Theorem 1.1 to obtain

$$\overline{D}(V \circ w) \leq \overline{D}(V \circ f).$$

Now apply Theorem 3.2(a).

Part (b) is proved by applying (a) twice: first to $V = L$ to get $\overline{\lim}(L \circ w) \leq \overline{\lim}(L \circ f)$, then to $V = -L$ to get $\underline{\lim}(L \circ w) \leq \underline{\lim}(L \circ f)$. $\square$

COROLLARY 4.2. Let w satisfy Equation (21) with $a/b > 0$. Suppose $K \subset E$ is a closed nonempty convex set that absorbs (or weakly absorbs) f/b . Then K absorbs (respectively, weakly absorbs) w . In particular:

$$(22) \quad \mathcal{L}[w] \subset \overline{\text{co}}(\mathcal{L}[f/b]).$$

PROOF. We refer to Section 2 for the definition of "absorbs". Define $V : [t_0, \infty) \rightarrow \mathbf{R}$, $V(z) = \text{dist}(z, K)$. Then V is convex and continuous, and therefore locally Lipschitz (Van Tiel [7], theorem 5.21). Notice that K absorbs a curve $x : [t_0, \infty) \rightarrow E$ if and only if $\overline{\lim}(V \circ x) = 0$. Therefore Theorem 4.1(a) implies that if K absorbs f/b , then K absorbs w ; and this implies (22). If K weakly absorbs f/b , then Theorem 4.1(b) implies K weakly absorbs w . $\square$

THEOREM 4.3. Let w satisfy Equation (21) with $a/b > 0$. If $\mathcal{L}_w[f/b]$ is bounded, then

$$\mathcal{L}[w] \subset \mathcal{L}_w[w] \subset \overline{\text{co}}(\mathcal{L}_w[f/b]).$$

PROOF. Since the first inclusion is universal, it suffices to consider weak limits. Note that $\mathcal{L}_w[f/b]$ is compact by Lemma 2.1(b). Given $L \in E^*$, consider the curve

$$v = L \circ w : [t_0, \infty) \rightarrow \mathbf{R},$$

which satisfies the equation

$$a \frac{dv}{dt} + bv = L \circ f.$$

If $p \in \mathcal{L}_w[w]$, then $Lp \in \mathcal{L}[L \circ (w)]$. Therefore Proposition 3.5 (a)(b) implies

$$(23) \quad L(\mathcal{L}_w[w]) \subset \mathcal{L}[(L \circ f)/b],$$

$$(24) \quad \subset L\mathcal{L}_w[(f/b)]$$

the last inclusion following from weak compactness of $\mathcal{L}_w[f/b]$. Thus we have shown that $\mathcal{L}_w[f/b]$ weakly absorbs $\mathcal{L}_w[w]$. The proof is completed by Lemma 2.6. $\square$

Next we consider the case $a/b < 0$.

PROPOSITION 4.4. Let $w : [t_0, \infty) \rightarrow E$ be a solution to Equation (21), and assume $a/b < 0$.

(a): Let $L \in E^*$ be such that $|Lw(t)| \not\rightarrow \infty$. Then:

$$\mathcal{L}[L \circ w] \subset -\mathcal{L}[L \circ (f/b)].$$

(b): If $\mathcal{L}_w[f/b]$ is bounded, then $\mathcal{L}_w[w] \subset -\overline{\text{co}}(\mathcal{L}_w[f/b])$.

PROOF. Part (a) is proved by applying Proposition 3.6(a)(b) to the curve $v = L \circ w : [t_0, \infty) \rightarrow \mathbf{R}$, which satisfies the equation

$$a \frac{dv}{dt} + bv = L \circ f.$$

The proof of (b) is similar to that of Theorem 4.3. □

If $K \subset E$ is convex and $J \subset \mathbf{R}$ is an interval not containing zero, then we define the set

$$K/J = \{\lambda^{-1}x : \lambda \in J, x \in K\},$$

and note that it is convex.

COROLLARY 4.5. Let $w : [t_0, \infty) \rightarrow E$ be a solution to Equation (21). Assume:

(a): $\mathcal{L}_w[f/b]$ is bounded,

(b): $\mathcal{L}[b]$ is a finite interval not containing 0.

Set $\sigma = \text{sgn}(a/b) \in \{\pm 1\}$. Then:

$$\mathcal{L}_w[w] \subset \sigma \overline{\text{co}}(\mathcal{L}_w[f])/\mathcal{L}[b].$$

PROOF. Follows from Propositions 4.3 and 4.4, because (b) implies

$$\overline{\text{co}}(\mathcal{L}_w[f/b]) \subset \overline{\text{co}}(\mathcal{L}_w[f])/\mathcal{L}[b].$$

□

4.1. Application to Autonomous Nonlinear Dynamical Systems. Let $F : E \rightarrow E$ be continuous and let $\Lambda \subset E$ be a compact invariant set for the equation

$$(25) \quad dx/dt = F(x).$$

For $\gamma \in \mathbf{R}$ define the map

$$G_\gamma : E \rightarrow E, \quad x \mapsto x + \gamma F(x).$$

THEOREM 4.6. Suppose there exists a solution $u : [t_0, \infty) \rightarrow \Lambda$ to Equation (25) that is dense in Λ (e.g., Λ is a periodic orbit). Then

$$\Lambda \subset \overline{\text{co}}(G_\gamma(\Lambda))$$

for all $\gamma \in \mathbf{R}$.

PROOF. Fix $\gamma \neq 0$. Let u be a solution curve that is dense in Λ , and set

$$f(t) = \gamma^{-1}G_\gamma(u(t)).$$

Then u satisfies the equation

$$\frac{du}{dt} + \gamma^{-1}u = f(t).$$

By density of u , Corollary 4.2 and continuity of G_γ :

$$\begin{aligned}\Lambda &= \mathcal{L}[u] \\ &\subset \overline{\text{co}}(\mathcal{L}[\gamma f]) \\ &= \overline{\text{co}}(\mathcal{L}[G_\gamma \circ u]) \\ &= \overline{\text{co}}(G_\gamma(\mathcal{L}[u])) \\ &= \overline{\text{co}}(G_\gamma(\Lambda)).\end{aligned}$$

□

4.2. Higher order operators in Banach spaces. Define $C_E = C([t_0, \infty), E)$ to be the set of continuous functions $g : [t_0, \infty) \rightarrow E$ such that $L \circ g \in C([0, \infty), \mathbf{R})$ for each $L \in E^*$. Thus $g \in C_E$ if and only if for each $L \in E^*$, either $\mathcal{L}[L \circ g]$ is a nonempty compact interval, or $|Lg(t)| \rightarrow \infty$ and $\mathcal{L}[L \circ g]$ is empty.

For $i = 1 \dots, n$ let $a_i, b_i : [t_0, \infty) \rightarrow \mathbf{R}$ be given, set $T_i = a_i D + b_i I$, and consider the n 'th order linear differential operator

$$T = T_n \circ \dots \circ T_1$$

where D denotes differentiation and I the identity.

Define $J_i \subset \mathbf{R}$ to be the closed interval $\mathcal{L}[b_i]$, and $J = J_1 \dots J_n$. Set $\sigma_i = \text{sgn}(a_i/b_i)$ and $\sigma = \sigma_1 \dots \sigma_n$.

THEOREM 4.7. *Let $f : [t_0, \infty) \rightarrow \mathbf{R}$ be continuous, and assume each pair of functions a_i, b_i satisfy Hypothesis 3.4. Let $v : [t_0, \infty) \rightarrow E$ be a solution to the differential equation $Tv = f$.*

(a): *Assume each J_i is a finite interval not containing zero. Then:*

$$\mathcal{L}_w[v] \subset \sigma J^{-1} \overline{\text{co}}(\mathcal{L}_w[f]).$$

(b): *Assume*

$$f/b_n \dots b_i \in C_E, \quad i = 1, \dots, n.$$

Then $v \in C_E$. In particular, $v \in B_E$ provided $f \in C_E$ and the J_i are as in (a).

PROOF. The proof, similar to that of Theorem 3.9, is by induction on n , using Corollary 4.5. □

We call a real number λ an *eigenvalue* of T with *eigenfunction* $u : [t_0, \infty) \rightarrow E$ if u is not identically zero, and

$$(26) \quad Tu = \lambda u.$$

Assume the hypothesis of Theorem 4.7. Let Equation (26) hold, and consider the finite interval

$$M = [\alpha, \beta] = \sigma \lambda J^{-1} = \{\sigma \lambda s \in \mathbf{R} : s^{-1} \in J\}.$$

If $0 \in M$ then 0 is an endpoint of M . Then:

THEOREM 4.8.

(a): *If $\lambda = 0$ then $\mathcal{L}_w[u] = \{0\}$.*

(b): *If $\alpha < 0$ then either $\mathcal{L}_w[u] = \{0\}$ or else 0 is an internal point of $\overline{\text{co}}(\mathcal{L}_w[u])$.*

(c): *If $|\alpha| > 1$ then $0 \in \overline{\text{co}}(\mathcal{L}_w[u])$.*

(d): *If $|\beta| < 1$ then $\mathcal{L}_w[u]$ is unbounded.*

PROOF. Let K denote the nonempty closed convex set $\overline{\text{co}}(\mathcal{L}_w[u])$. Then

$$(27) \quad K \subset [\alpha, \beta]K$$

by Theorem 4.7 applied to the equation $Tu = f$, taking $f(t) = \lambda u(t)$. This proves (a), because $\alpha = \beta = 0$ if $\lambda = 0$. The other parts are also consequences of (27). $\square$

COROLLARY 4.9. *Let the operator T be as in Theorem 4.7. If the eigenfunction u in Equation (26) is bounded and $\overline{\text{co}}(\mathcal{L}_w[u])$ does not contain the origin, then $\lambda \neq 0$ and $0 < \alpha \leq \beta \leq 1$.*

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