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Fusion rules for V_{l_2}^{s_4}

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FUSION RULES FOR $V_{L_2}^{S_4}$

A dissertation submitted in partial satisfaction
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by

Junwen Liao

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Contents

Abstract

Fusion rules for $V_{L_2}^{S_4}$

by

Junwen Liao

This dissertation determines the fusion rules for the fixed-point vertex operator algebra $V_{L_2}^{S_4}$, where L_2 denotes the rank-one even lattice satisfying $(\alpha, \alpha) = 2$. The main idea is to identify certain irreducible $V_{\mathbb{Z}\zeta}^{+}$-modules as direct sums of irreducible $V_{L_2}^{S_4}$-modules. These identifications allow us to compute the relevant entries of the S-matrix for $V_{L_2}^{S_4}$. Applying the Verlinde formula, we then obtain the complete fusion rules for $V_{L_2}^{S_4}$.

To everyone who supported me

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Chapter 1

Introduction

The classification of rational vertex operator algebras of central charge 1 can be traced back to the work of Kiritsis. In [K], Kiritsis showed that, under suitable modularity assumptions, the q-character of a rational vertex operator algebra of CFT type with central charge $c = \widetilde{c} = 1$ coincides with the character of one of the following vertex operator algebras:

$$V_L, \qquad V_L^+, \qquad V_{\mathbb{Z}\alpha}^G,$$

where L is a positive-definite even lattice of rank one, $\mathbb{Z}\alpha$ is the root lattice of type A_1, and G is a finite subgroup of $SO(3)$ isomorphic to A_4, S_4, or A_5.

The irreducible representations and fusion rules of V_L and V_L^+ have been studied by Frenkel–Lepowsky–Meurman, Abe–Dong, and Dong–Nagatomo. Subsequent work has focused on the fixed-point vertex operator algebras

$$V_{L_2}^{A_4}, \qquad V_{L_2}^{S_4}, \qquad V_{L_2}^{A_5},$$

where L_2 is the rank-one lattice satisfying $(\alpha, \alpha) = 2$. The irreducible modules of $V_{L_2}^{A_4}$ were classified in [DJ], and the fusion rules for $V_{L_2}^{A_4}$ were determined in [DJJJY].

Furthermore, the irreducible modules of $V_{L_2}^{S_4}$ were classified in [WZ].

The purpose of this dissertation is to determine the fusion rules for $V_{L_2}^{S_4}$. Determining these fusion rules describes the representation category of $V_{L_2}^{S_4}$ once the classification of its irreducible modules is known. The main strategy is to identify certain irreducible $V_{\mathbb{Z}\zeta}^{+}$-modules as direct sums of irreducible $V_{L_2}^{S_4}$-modules. These identifications allow us to compute the relevant entries of the S-matrix for $V_{L_2}^{S_4}$. We then apply the Verlinde formula to determine the complete fusion rules.

Two important tools used in this dissertation are the Verlinde formula and quantum dimensions. The Verlinde formula provides a method for computing fusion rules from the S-matrix of a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type. Quantum dimensions are used to control the possible decompositions of tensor products and to verify that the decompositions obtained are complete.

This dissertation is organized as follows. In Chapter 2, we review the necessary background on vertex operator algebras, twisted modules, lattice vertex operator algebras, orbifold theory, intertwining operators, fusion rules, and quantum dimensions. In Chapter 3, we recall the construction of the rank-one lattice vertex operator algebra V_{L_2}, the fixed-point subalgebra $V_{L_2}^{S_4}$, and the classification of its irreducible modules. We then establish the module identifications needed to compute the relevant entries of the S-matrix. Finally, we use the Verlinde formula to determine the fusion rules among all irreducible $V_{L_2}^{S_4}$-modules.

Chapter 2

Preliminaries

§ 2.1 Basics of Vertex Operator Algebras

Definition 2.1.1. A vertex algebra over $\mathbb{C}$ is a triple $(V, Y, \mathbf{1})$, where V is a vector space over $\mathbb{C}$ and

$$Y : V \to EndV[[z, z^{-1}]],$$

$$v \to Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, v_n \in EndV, n \in \mathbb{Z},$$

is a linear map, called the vertex operator map. Here $EndV[[z, z^{-1}]]$ denotes the space of formal Laurent series in z with coefficients in $EndV$. The element $\mathbf{1} \in V$ is a distinguished vector, called the vacuum vector, and the following properties are satisfied:

1. for any $u, v \in V$, we have $u_n v = 0, n \gg 0$;

2. $Y(\mathbf{1}, z) = Id_V$;

3. $Y(u, z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0} Y(u, z)\mathbf{1} = u$;

4. (Jacobi identity) for any $u, v \in V$, we have

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta(\frac{z_2 - z_1}{-z_0})Y(v, z_2)Y(u, z_1)$$
$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})Y(Y(u, z_0)v, z_2),$$

Here $n \gg 0$ means n is sufficiently large; $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is the formal δ function. We assume $(z - w)^m$ has the following expansion:

$$(z - w)^m = \sum_{n=0}^{\infty} \binom{m}{n} (-1)^n z^{m-n} w^n.$$

We define a linear map

$$D : V \longrightarrow V, \qquad v \longmapsto v_{-2}\mathbf{1}.$$

Then, by the Jacobi identity, we obtain

$$Y(Dv, z) = \frac{d}{dz}Y(v, z), \forall v \in V.$$

Lemma 2.1.2. *(Skew-symmetry) For any $u, v \in V$, we have the following identity:*

$$Y(u, z)v = e^{zD}Y(v, -z)u.$$

In particular, for any $n \in \mathbb{Z}$, we have

$$u_n v = \sum_{i=0}^{\infty} \frac{(-1)^{n+i+1}}{i!} D^i(v_{n+i}u).$$

The proof of this lemma follows from the expansion of the Jacobi identity. Since $v_n u = 0$ for sufficiently large n, the sum is finite and therefore well defined.

Definition 2.1.3. Let $(V, Y_V, \mathbf{1}_V)$ and $(W, Y_W, \mathbf{1}_W)$ be vertex algebras over $\mathbb{C}$. A linear map

$$f : V \longrightarrow W$$

is called a homomorphism if it satisfies

$$f(\mathbf{1}_V) = \mathbf{1}_W,$$

and

$$f(u_n v) = f(u)_n f(v), \qquad \forall n \in \mathbb{Z},\ \forall u, v \in V.$$

We call f an isomorphism from V to W if f is bijective. The vertex algebras V and W are said to be isomorphic if there exists an isomorphism from V to W. When $V = W$, an isomorphism f is called an automorphism of V.

Let W be a subspace of a vertex algebra V that contains $\mathbf{1}_V$ and is closed under all n-th products. Then W is a vertex algebra, and we call it a subalgebra of V.

We call a subspace I of V an ideal of V if it satisfies $D(I) \subseteq I$ and $u_n v \in I$ for all $u \in V$ and $v \in I$. A vertex algebra V is called simple if it contains no nontrivial ideals.

Definition 2.1.4. A vertex operator algebra over $\mathbb{C}$ is a 4-tuple $(V, Y, \mathbf{1}, \omega)$. Here $(V, Y, \mathbf{1})$ is a vertex algebra over $\mathbb{C}$, and

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

is a $\mathbb{Z}$-graded vector space decomposition such that

$$\dim V_n < \infty \quad \forall n \in \mathbb{Z}, \qquad V_n = 0 \text{ for } n \ll 0.$$

The element $\omega \in V$, called the Virasoro vector or conformal vector of V, has vertex operator given by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2} = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1},$$

and satisfies:

1. $L(-1) = D$, i.e.

$$L(-1)u = D(u) = u_{-2}\mathbf{1}, \quad \forall u \in V;$$

2. $L(0)|_{V_n} = n\,\mathrm{Id}_{V_n}$, i.e.

$$L(0)u = nu, \quad \forall u \in V_n;$$

3.

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\,\delta_{m+n,0}\, c\,\mathrm{Id}_V.$$

Here $c \in \mathbb{C}$ is a constant called the central charge of the vertex operator algebra V. The finite-dimensional subspace V_n is called the weight-n subspace with respect to $L(0)$. Elements in V_n are called weight vectors or homogeneous vectors. The weight of a homogeneous vector u is denoted by $\mathrm{wt}(u)$. Thus, $\mathrm{wt}(u) = n$ for all $u \in V_n$. For convenience, we often write V instead of $(V, Y, \mathbf{1}, \omega)$.

Lemma 2.1.5. *Let V be a vertex operator algebra, and let $v \in V_n$, $u \in V_m$ be homogeneous vectors. Then we have*

$$u_k v \in V_{m+n-k-1}.$$

Proof. The result follows by a direct computation of the coefficients appearing in the Jacobi identity. □

We call V a simple vertex operator algebra if V is simple as a vertex algebra.

Theorem 2.1.6. *Let V be a simple vertex operator algebra, and $v \in V$ is a nonzero vector. Then we have:*

$$V = \text{span}\{u_n v | u \in V, n \in \mathbb{Z}\}.$$

This theorem also follows from the expansion of the Jacobi identity.

Theorem 2.1.7. *Let V be a simple vertex operator algebra, and let $u, v \in V$ be nonzero vectors. Then, $Y(u, z)v \neq 0$.*

Definition 2.1.8. Let V and W be two vertex operator algebras. Let ω_1 be the Virasoro vector of V and ω_2 the Virasoro vector of W. A linear map $f : V \to W$ is called a homomorphism of vertex operator algebras if it is a homomorphism of vertex algebras and satisfies

$$f(\omega_1) = \omega_2.$$

If f is bijective, then it is called an isomorphism. If f is an isomorphism from V to itself, then it is called an automorphism of V.

Remark 2.1.9. Using the above notation, if $u \in V_n$, then

$$nf(u) = f(L(0)u) = f({\omega_1}_1 u) = f(\omega_1)_1 f(u) = {\omega_2}_1 f(u) = L(0)f(u).$$

Hence $f(u) \in W_n$, and therefore $f(V_n) \subseteq W_n$.

Definition 2.1.10. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. A weak module is a pair (W, Y_W). Here W is a vector space over $\mathbb{C}$ and

$$Y_W : V \to EndW[[z, z^{-1}]]$$

is a linear map given by

$$v \mapsto Y_W(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \qquad v_n \in End W,$$

and satisfies:

1. for any $u \in V$ and $v \in W$, we have $u_n v = 0, n \gg 0$;

2. $Y_W(\mathbf{1}, z) = Id_W$;

3. (Jacobi identity) for any $u, v \in V$, we have

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_W(u,z_1)Y_W(v,z_2) - z_0^{-1}\delta(\frac{z_2 - z_1}{-z_0})Y_W(v,z_2)Y_W(u,z_1)$$
$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})Y_W(Y(u,z_0)v,z_2),$$

Remark 2.1.11. Every vertex operator algebra V is naturally a weak module over itself.

Definition 2.1.12. Let V be a vertex operator algebra over $\mathbb{C}$ and W a weak V-module. We call W admissible or $\mathbb{Z}_+$-graded if W has a vector space decomposition

$$W = \bigoplus_{n=0}^{\infty} W(n),$$

satisfying for homogeneous $u \in V, m \in \mathbb{Z}$, and $n \in \mathbb{Z}_+$

$$u_m W(n) \subseteq W(\mathrm{wt}(u) - m - 1 + n),$$

with $W(n) = 0$ if $n < 0$.

Definition 2.1.13. Let V be a vertex operator algebra and let W, M be V-modules. A linear map $f : W \longrightarrow M$ is called a V-module homomorphism if

$$f\, Y_W(u,z) = Y_M(u,z)\, f, \qquad \forall u \in V.$$

Or equivalently,

$$f(u_n v) = u_n f(v), \qquad \forall n \in \mathbb{Z},\ \forall u \in V,\ \forall v \in W.$$

Definition 2.1.14. A vertex operator algebra V is rational if any admissible V-module is a direct sum of irreducible admissible V-modules.

Definition 2.1.15. Let W be a weak V-module. We call W an ordinary V-module if it has a vector space decomposition

$$W = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda, \qquad \dim W_\lambda < \infty,$$

where

$$W_\lambda = \{w \in W\ ;\ L(0)w = \lambda w\},$$

and for any $\lambda \in \mathbb{C}$ we have

$$W_{\lambda+n} = 0 \quad \text{for } n \ll 0,\ n \in \mathbb{Z}.$$

Definition 2.1.16. A vertex operator algebra V is called C_2-cofinite if $V/C_2(V)$ is finite-dimensional, where

$$C_2(V) = \mathrm{span}_{\mathbb{C}}\{u_{-2}v \mid u, v \in V\}.$$

Definition 2.1.17. A vertex operator algebra V is said to be of CFT type if $V_n = 0$ for all $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$.

Definition 2.1.18. Let V be a vertex operator algebra, and let

$$M = \bigoplus_{n=0}^{\infty} M(n)$$

be an admissible V-module. Define

$$M' = \bigoplus_{n=0}^{\infty} M(n)^*,$$

where $M(n)^*$ denotes the dual space of $M(n)$.

Then we have a natural pairing

$$(\cdot,\cdot) : M' \times M \to \mathbb{C}, \qquad (f, w) = f(w).$$

We define a vertex operator

$$Y : V \to \mathrm{End}(M')[[z, z^{-1}]]$$

by

$$u \mapsto Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1},$$

where

$$(Y(u, z)f, w) = \left(f, Y_M\left(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1}\right)w\right)$$

for all $f \in M'$, $u \in V$, and $w \in M$. Then M' is also an admissible V-module, called the contragredient module of M.

We call V self-dual if

$$V \cong V'$$

as V-modules.

Remark 2.1.19. If V is a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type, the category of V-modules forms a modular tensor category, and the contragredient module is the dual object in the category.

Remark 2.1.20. From the expansion of

$$Y_M\left(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1}\right)w,$$

we obtain

$$(u_m f, w) = \left(f, \sum_{i\geq 0} \frac{(-1)^{\mathrm{wt}(u)}}{i!}(L(1)^i u)_{2\mathrm{wt}(u)-m-i-2}w\right)$$

for any $f \in M(n)'$, $u \in V_{\mathrm{wt}(u)}$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}_+$.

Remark 2.1.21. We call a vector $u \in V$ a primary vector if

$$L(n)u = 0$$

for all $n \geq 1$. If u is a primary vector of V, then

$$(u_m f, w) = (-1)^{\mathrm{wt}(u)}(f, u_{2\mathrm{wt}(u)-m-2}w),$$

and

$$(L(m)f, w) = (f, L(-m)w), \qquad \forall f \in M',\ \forall w \in M.$$

§ 2.2 g-Twisted Modules

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and let g be an automorphism of V of finite order T. The eigenspace decomposition of V with respect to g is given by

$$V = \bigoplus_{r=0}^{T-1} V^r,$$

where

$$V^r = \{\, v \in V \mid gv = e^{\frac{2\pi i r}{T}} v \,\}.$$

Definition 2.2.1. A *weak g-twisted V-module M* is a vector space with a linear map

$$Y_M : V \longrightarrow (End M)\{z\}$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \ \ (v_n \in End M)$$

which satisfies the following: for all $0 \leqslant r \leqslant T-1$, $u \in V^r$, $v \in V$, $w \in M$,

$$Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1},$$

$$u_l w = 0 \ for \ l \gg 0,$$

$$Y_M(\mathbf{1}, z) = Id_M,$$

$$z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1)$$
$$= z_2^{-1} \left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0) v, z_2),$$

Definition 2.2.2. A *g-twisted V-module* is a weak g-twisted V-module M that carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$, where $L(0)$ is the component operator of

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

That is, we have

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

where

$$M_\lambda = \{ w \in M \mid L(0)w = \lambda w \}.$$

Moreover, we require that $\dim M_\lambda$ is finite and that, for fixed λ, we have

$$M_{\frac{n}{T}+\lambda} = 0$$

for all sufficiently small integers n.

Definition 2.2.3. An *admissible g-twisted V-module* $M = \oplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)$ is a $\frac{1}{T}\mathbb{Z}_+$-graded weak g-twisted module such that $u_m M(n) \subseteq M(\mathrm{wt}(u) - m - 1 + n)$ for homogeneous $u \in V$ and $m, n \in \frac{1}{T}\mathbb{Z}$.

When $g = \mathrm{id}_V$, the above notions reduce to the usual notions of weak, ordinary, and admissible V-modules.

Definition 2.2.4. A vertex operator algebra V is called g-rational if every admissible g-twisted V-module is a direct sum of irreducible admissible g-twisted V-modules.

Theorem 2.2.5. [DLM1] *Let V be a g-rational vertex operator algebra. Then the following hold:*

1. *Every simple admissible g-twisted V-module is an g-twisted V-module.*

2. *V has only finitely many isomorphism classes of simple admissible g-twisted modules.*

§ 2.3 Heisenberg Vertex Operator Algebras and Lattice Vertex Operator Algebras

Definition 2.3.1. Let H be a finite-dimensional vector space over $\mathbb{C}$, and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on H. We regard H as an abelian Lie algebra, namely,

$$[h_1, h_2] = 0, \qquad h_1, h_2 \in H.$$

We construct its Heisenberg Lie algebra

$$\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K, \tag{2.3.1}$$

with Lie bracket relations

$$[\alpha(m), \beta(n)] = m(\alpha, \beta)\delta_{m+n,0}K, \qquad [\alpha(m), K] = 0, \tag{2.3.2}$$

for all $\alpha, \beta \in H$ and $m, n \in \mathbb{Z}$, where

$$\alpha(m) = \alpha \otimes t^m.$$

For any $\lambda \in H$, we set $M(1,\lambda) = U(\hat{H})/J_\lambda$. $U(\hat{H})$ is the universal enveloping algebra of $\hat{H}$ and J_λ is the left ideal of $U(\hat{H})$ generated by :

$$\alpha(n), \alpha \in H, n > 0; \alpha(0) - (\alpha, \lambda); K - 1. \tag{2.3.3}$$

Clearly, $M(1,\lambda)$ is a $\hat{H}$-module.

Lemma 2.3.2. *We write* $e^\lambda = 1 + J_\lambda = [1] \in M(1,\lambda)$. *Then,*

$$M(1,\lambda) = \quad \mathrm{span}\{h_1(-n_1)\cdots h_t(-n_t)e^\lambda | h_i \in H, \forall i, n_1 \geq \cdots \geq n_t > 0\} \tag{2.3.4}$$

$$(2.3.5) \qquad = \qquad S(H \otimes t^{-1}\mathbb{C}[t^{-1}])e^{\lambda} = \mathbb{C}[h(-n)|h \in B, n > 0]e^{\lambda},$$

Here, $S(V)$ is the symmetric algebra of the vector space V over $\mathbb{C}$, and B is a basis of H.

We denote $M(1,0)$ by $M(1)$, $e^0 = 1$ and

$$M(1) \cong \mathbb{C}[h(-n)|h \in B, n > 0],$$

as a vector space.

Theorem 2.3.3. $(M(1,0), Y, \mathbf{1}, \omega)$ *is a vertex operator algebra. Its Virasoro vector* $\omega = \frac{1}{2}\sum_{i=1}^{d} \alpha_i(-1)^2\mathbf{1}$ *where* $\alpha_1, \cdots, \alpha_d$ *form an orthonormal basis of* H*. Its vertex operator* Y *is defined as*

$$Y(h_1(-n_1-1)\cdots h_t(-n_t-1)\mathbf{1}, z) =: \partial^{n_1}h_1(z)\cdots\partial^{n_t}h_t(z) :,$$

$$h_1, \cdots, h_t \in H, n_1, \cdots, n_t \in \mathbb{Z}_+.$$

Here, $\partial^n = \frac{1}{n!}(\frac{d}{dz})^n, n \geq 0, h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}, h(n) = h \otimes t^n$*. This vertex operator algebra is called the Heisenberg vertex operator algebra.*

Remark 2.3.4. $: \cdots :$ in the above definition is called normal ordering and it is defined in the following way:

$$: \alpha(m)\beta(n) := \begin{cases} \alpha(m)\beta(n), & \text{if } m \leq n; \\ \beta(n)\alpha(m), & \text{if } m \geq n; \end{cases}$$

When $n = m$, it follows that $[\alpha(m), \beta(n)] = 0$. Therefore, $\alpha(m)\beta(n) = \beta(n)\alpha(m)$. The normal ordering has no ambiguity when $m = n$.

For $\alpha_1(m_1)\cdots\alpha_n(m_n)$, its normal ordering is defined as

$$: \alpha_1(m_1) \cdots \alpha_n(m_n) := \alpha_{\delta(1)}(m_{\delta(1)}) \cdots \alpha_{\delta(n)}(m_{\delta(n)})$$

where $\delta \in S_n$ and $m_{\delta(n)} \geq \cdots \geq m_{\delta(1)}$.

Lemma 2.3.5.

$$: \alpha(m)\beta(n) := \begin{cases} \alpha(m)\beta(n), & \text{if } m < 0 \\ \beta(n)\alpha(m), & \text{if } m \geq 0; \end{cases}$$

is an equivalent definition of normal ordering.

Proof. Case 1: $m \geq 0, n \geq 0$. Then we have $[\alpha(m), \beta(n)] = 0$. It implies that $\alpha(m)\beta(n) = \beta(n)\alpha(m)$. Therefore, the two definitions agree with each other.

Case 2: $m \geq 0, n < 0$. Clearly, $m > n$. The two definitions agree.

Case 3: $m < 0, n \geq 0$. Clearly, $m < n$. The two definitions are equal.

Case 4: $m < 0, n < 0$. We have $[\alpha(m), \beta(n)] = 0$, and $\alpha(m), \beta(n)$ commute. Therefore, the two definitions are equivalent. □

Theorem 2.3.6. [LL] *For each $\lambda \in H$, $M(1, \lambda)$ is an irreducible $M(1)$-module as a vertex operator algebra module. Every irreducible $M(1)$-module is isomorphic to $M(1, \lambda)$ for some $\lambda \in H$.*

From this theorem, we see that $M(1)$ has infinitely many irreducible modules up to isomorphism. Hence, $M(1)$ is neither rational nor C_2-cofinite.

In this dissertation, we focus on the case where H has rank one. In other words,

$$H = \mathbb{C}\alpha.$$

We now construct the lattice vertex operator algebra associated with an even lattice based on the Heisenberg vertex operator algebra.

Definition 2.3.7. A lattice L is a finite-rank free abelian group. If L is equipped with an integer-valued, nondegenerate symmetric bilinear form satisfying

$$(\alpha, \alpha) \in 2\mathbb{Z}$$

for all $\alpha \in L$, then L is called an even lattice.

Let L be an even lattice of rank d, and let

$$H = \mathbb{C} \otimes_{\mathbb{Z}} L$$

be the tensor product over $\mathbb{Z}$. Then H can be viewed as a vector space over $\mathbb{C}$ of dimension d. The nondegenerate bilinear form on L extends naturally to a nondegenerate bilinear form on H.

Let $\{\alpha_1, \cdots, \alpha_d\}$ be a basis of L. Then it is also a basis of H. Since the bilinear form is nondegenerate, there exists a dual basis

$$\{\alpha^1, \cdots, \alpha^d\}$$

such that

$$(\alpha_i, \alpha^j) = \delta_{i,j}, \qquad \forall i, j.$$

Define

$$L^\circ = \mathbb{Z}\text{-span}\{\alpha^1, \cdots, \alpha^d\}.$$

Then L° is also a lattice satisfying $L \subseteq L^\circ$, and we call L° the dual lattice of L. In fact, we have

$$L^{\circ} = \{\lambda \in H\ ;\ (\lambda, \alpha) \in \mathbb{Z},\ \forall \alpha \in L\}.$$

Lemma 2.3.8. *The quotient group L°/L is a finite group.*

Let L be an even lattice with positive-definite bilinear form, and let $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. Let $M(1)$ be the Heisenberg vertex operator algebra associated with H. We can always find a bimultiplicative map $\epsilon : L \times L \longrightarrow \{1, -1\}$ that satisfies:

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}, \forall \alpha, \beta \in L.$$

For example, for $\alpha = \sum_{i=1}^{d} a_i \alpha_i$, $\beta = \sum_{j=1}^{d} b_j \alpha_j$, we can set

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} (-1)^{(\alpha_i,\alpha_j)}, \text{if } i < j; \\ (-1)^{\frac{(\alpha_i,\alpha_i)}{2}}, \text{if } i = j; \\ 1, \text{otherwise.} \end{cases}$$

We have:

$$\epsilon(\alpha, \beta) = \Pi_{i<j}(-1)^{a_i b_j (\alpha_i,\alpha_j)} \Pi_{i=1}^{d} (-1)^{a_i b_i \frac{(\alpha_i,\alpha_i)}{2}}.$$

and $\forall \alpha \in L$, $\epsilon(\alpha, \alpha) = (-1)^{\frac{(\alpha,\alpha)}{2}}$.

Define $\mathbb{C}^{\epsilon}[L] = \bigoplus_{\alpha \in L} \mathbb{C}e^{\alpha}$ as the $\epsilon-$ twisted group algebra of L, and set

$$V_L = M(1) \otimes \mathbb{C}^{\epsilon}[L].$$

Here, the tensor product is taken over $\mathbb{C}$ as vector spaces.

Moreover, since L°/L is a finite group, we can write

$$L^\circ = (L + \lambda_1) \cup \cdots \cup (L + \lambda_k).$$

For each coset $L + \lambda_i$, we define

$$V_{L+\lambda_i} = M(1) \otimes \mathbb{C}^\epsilon[L + \lambda_i].$$

Definition 2.3.9. We define some maps on vector space $V_{L+\lambda_i}$:

$$h(n) : V_{L+\lambda_i} \longrightarrow V_{L+\lambda_i}, v \otimes w \longrightarrow h(n)v \otimes w;$$

$$h(0) : V_{L+\lambda_i} \longrightarrow V_{L+\lambda_i}, v \otimes e^\alpha \longrightarrow (h, \alpha)v \otimes e^\alpha;$$

$$z^h : V_{L+\lambda_i} \longrightarrow V_{L+\lambda_i}\{z\}, v \otimes e^\alpha \longrightarrow v \otimes z^{(h,\alpha)}e^\alpha;$$

$$e_\gamma : V_{L+\lambda_i} \longrightarrow V_{L+\lambda_i}, v \otimes e^{\beta+\lambda_i} \longrightarrow \epsilon(\gamma, \beta)v \otimes e^{\gamma+\beta+\lambda_i},$$

Here, $h \in H, n \neq 0 \in \mathbb{Z}$, $\alpha \in L + \lambda_i$, $\beta, \gamma \in L$, z is a formal variable.

Definition 2.3.10. We define vertex operator $Y(u, z), u \in V_L$, over the vector space $V_{L+\lambda_i}$.

$$Y(h(-1)\mathbf{1}, z) = h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}, \forall h \in H;$$

$$E^+(\alpha, z) = exp(\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}), \forall \alpha \in L;$$

$$E^-(\alpha, z) = exp(\sum_{n<0} \frac{\alpha(n)}{n} z^{-n}), \forall \alpha \in L;$$

$$Y(e^\alpha, z) = E^-(-\alpha, z)E^+(-\alpha, z)e_\alpha z^\alpha, \forall \alpha \in L.$$

For $u = h_1(-n_1 - 1) \cdots h_s(-n_s - 1) \otimes e^\alpha \in V_L$, we set

$$Y(u, z) =: \partial^{n_1} h_1(z) \cdots \partial^{n_s} h_s(z) Y(e^\alpha, z) : .$$

Here, $: \cdots :$ is the normal ordering we defined in the construction of Heisenberg vertex operator algebra.

Theorem 2.3.11. *$(V_L, Y, 1, \omega)$ is a vertex operator algebra where $\omega = \frac{1}{2} \sum_{i=1}^d u_i(-1)^2 1 \in V_L$ is the Virasoro vector of V_L and $\{u_1, \cdots, u_d\}$ is an orthonormal basis of H.*

Theorem 2.3.12. *$V_{L+\lambda_i}$ is an irreducible V_L-module for each λ_i and $V_{L+\lambda_i} \cong V_{L+\lambda_j}$ if and only if $\lambda_i \equiv \lambda_j (mod\, L)$. Every irreducible V_L module is isomorphic to $V_{L+\lambda_i}$ for some $\lambda_i \in L^\circ$.*

Theorem 2.3.13. *V_L is a rational, simple, C_2-cofinite vertex operator algebra.*

In this dissertation, we focus on the case where L is a rank-one even lattice $L = \mathbb{Z}\alpha$ with nondegenerate bilinear form $(\alpha, \alpha) = 2$. We will denote the corresponding lattice vertex operator algebra as V_{L_2} or $V_{\mathbb{Z}\alpha}$.

From [FLM], there is an order 2 automorphism θ of V_{L_2}:

$$\theta(\alpha(-n_1) \cdots \alpha(-n_k) \otimes e^\lambda) = (-1)^k \alpha(-n_1) \cdots \alpha(-n_k) \otimes e^{-\lambda}.$$

for $n \in \mathbb{Z}_+$ and $\lambda \in L$. Let $C[H]$ be the group algebra of H with a basis $\{e^\lambda | \lambda \in H\}$. This θ can be extended to a linear isomorphism on $V_H = M(1) \otimes C[H]$. We can construct the θ-twisted V_{L_2}-module as follows:

Let $\hat{H}[-1] = H\otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be a Lie algebra, and let $\hat{H}[-1]_+ = H\otimes t^{\frac{1}{2}}\mathbb{C}[t]\oplus\mathbb{C}K$ be its subalgebra. Then, $\mathbb{C}$ can be considered as a one-dimensional module for $\hat{H}[-1]_+$ with the action

$$(\alpha \otimes t^m)\cdot 1 = 0 \text{ and } K\cdot 1 = 1 \text{ for } m \in \frac{1}{2}+\mathbb{N}.$$

Let $M(1)(\theta)$ be the induced $\hat{H}[-1]$-module. Let χ_s be a character of $L/2L$ such that $\chi_s(\alpha) = (-1)^s$ for $s = 0, 1$, and let $T_{\chi_s} = \mathbb{C}$ be the irreducible $L/2L$-module with character χ_s. Then

$$V_L^{T_s} = M(1)(\theta)\otimes T_{\chi_s}$$

is an irreducible θ-twisted V_L-module. In the rest of the dissertation we will denote $V_L^{T_0}$ by $V_L^{T_1}$ and $V_L^{T_1}$ by $V_L^{T_2}$.

§ 2.4 Orbifold Theory

From the definition of a vertex operator algebra, we see that constructing examples of vertex operator algebras is not easy. Therefore, it is useful to develop techniques for constructing new vertex operator algebras from known ones.

Orbifold theory is an important method for constructing new vertex operator algebras from existing vertex operator algebras.

Let V be a vertex operator algebra, and let $\mathrm{Aut}(V)$ denote the automorphism group of V. Let G be a subgroup of $\mathrm{Aut}(V)$. We denote by V^G the fixed-point subspace of V under the action of G. That is,

(2.4.1) $$V^G = \{v \in V \mid \delta(v) = v,\ \forall \delta \in G\}.$$

By the definition of a homomorphism of vertex operator algebras, we know that $\mathbf{1}, \omega \in V^G$. V^G is closed under all n-th products:

$$\delta(u_n v) = \delta(u)_n \delta(v) = u_n v, \qquad \forall u, v \in V^G,\ \forall \delta \in G.$$

Hence, V^G is a subalgebra of V. By choosing different subgroups G, we can construct different vertex operator algebras from a given vertex operator algebra V.

Following [DRX], in the rest of this dissertation, we assume that the vertex operator algebra V satisfies the following conditions:

1. $V = \bigoplus_{n \geq 0} V_n$ is a simple vertex operator algebra of CFT type.

2. G is a finite automorphism group of V and V^G is a vertex operator algebra of CFT type.

3. V^G is rational and C_2-cofinite.

4. The conformal weight of any g-twisted V-module for $g \in G$ is positive except for V itself.

By [ADJR, HKL, ABD], if V and G satisfy the above conditions, V is C_2-cofinite and g-rational for all $g \in G$.

Let V be a vertex operator algebra, and let g be an automorphism of V of order T, and let $M = \bigoplus_{n \in \frac{\mathbb{Z}_+}{T}} M_{\lambda+n}$ be a g-twisted V-module. We define the trace function associated with a homogeneous element $v \in V$ by:

$$Z_M(v, q) = \mathrm{tr}_M o(v) q^{L(0)-\frac{c}{24}} = q^{\lambda-\frac{c}{24}} \sum_{n \in \frac{\mathbb{Z}_+}{T}} \mathrm{tr}_{M_{\lambda+n}} o(v) q^n.$$

where $o(v) = v_{wt(v)-1}$. If $|q| < 1$ and V is C_2-cofinite, then $Z_M(v,q)$ converges to a holomorphic function. In particular, if $v = \mathbf{1}$, then

$$Z_M(\mathbf{1},q) = q^{\lambda-\frac{c}{24}} \sum_{n\in\frac{\mathbb{Z}_+}{T}} \dim M_{\lambda+n}\, q^n$$

is called the q-character of M. We will also use $Z_M(v,\tau)$ to denote $Z_M(v,q)$, where τ lies in the upper half-plane of $\mathbb{C}$ and $q = e^{2\pi i\tau}$.

By [DLM2] and [Z], $Z_M(v,\tau)$ satisfies the modular invariance property:

$$Z_{M_i}(v, \frac{-1}{\tau}) = \tau^{wt[v]} \sum_{j=0}^{d} S_{i,j} Z_{M_j}(v,\tau)$$

where M_i is an irreducible V-module and M_j ranges over a complete list of irreducible V-modules up to isomorphism. The numbers $S_{i,j}$ form a matrix called the *S-matrix*, and this matrix is independent of the choice of v. In [Z], Zhu introduced a new vertex operator algebra $(V, Y[\cdot,z], \mathbf{1}, \tilde{\omega})$ associated with V. Here

$$\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1},$$

and

$$Y[v,z] = Y(v, e^z - 1)e^{z\mathrm{wt}(v)} = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1}. \tag{2.4.2}$$

The notation $\mathrm{wt}[v]$ in the above equation denotes the weight of v with respect to the new conformal vector $\tilde{\omega}$.

Let $g, h \in \mathrm{Aut}(V)$ be automorphisms of finite order. Let (M, Y_M) be a g-twisted weak V-module. Then there exists an $h^{-1}gh$-twisted weak V-module $(M\circ h, Y_{M\circ h})$, where

$M \circ h \cong M$ as vector spaces and

$$Y_{M\circ h}(v, z) = Y_M(hv, z).$$

If g and h commute, then h acts on the set of g-twisted V-modules. Let $S(g)$ denote the set of equivalence classes of irreducible g-twisted V-modules, and define

$$S(g, h) = \{\, M \in S(g) \mid M \circ h \cong M \text{ as } g\text{-twisted } V\text{-modules} \,\}.$$

Then, for any $M \in S(g, h)$, there exists a g-twisted V-module isomorphism

$$\psi(h) : M \circ h \longrightarrow M.$$

This linear map is unique up to a scalar, and when h is the identity automorphism, $\psi(1)$ is the identity map.

Let M be a g-twisted V-module, and define

$$G_M = \{\, h \in G \mid M \circ h \cong M \,\}.$$

The assignment $h \mapsto \psi(h)$ defines a projective representation of G_M on M such that

$$\psi(h)\, Y_M(v, z)\, \psi(h)^{-1} = Y_M(hv, z).$$

Then M is a $\mathbb{C}^{\alpha_M}[G_M]$-module, where $\mathbb{C}^{\alpha_M}[G_M]$ is a twisted group algebra and α_M is a 2-cocycle.

Let Λ_{G_M,α_M} denote the set of irreducible characters of $\mathbb{C}^{\alpha_M}[G_M]$. Then we have

$$M = \bigoplus_{\lambda\in\Lambda_{G_M,\alpha_M}} W_\lambda \otimes M_\lambda,$$

where W_λ is the irreducible $\mathbb{C}^{\alpha_M}[G_M]$-module corresponding to the irreducible character

λ, and M_λ is the corresponding multiplicity space.

Theorem 2.4.1. [DRX] *With the notation above, the following statements hold:*

- *(1) M_λ is nonzero for each $\lambda \in \Lambda_{G_M,\alpha_M}$.*
- *(2) Each M_λ is an irreducible V^{G_M}-module.*
- *(3) M_λ and M_γ are equivalent as V^{G_M}-modules if and only if $\lambda = \gamma$.*

Let $S = \bigcup_{g\in G} S(g)$. Then G acts on S, and $M \cong M \circ h$ as a V^G-module for any $M \in S$ and $h \in G$.

Theorem 2.4.2. [DRX] *Let $g, h \in G$. Let M be an irreducible g-twisted V-module and let N be an irreducible h-twisted V-module. Assume that M and N are not in the same G-orbit. Then:*

- *(1) Each M_λ is an irreducible V^G-module.*
- *(2) For any $\lambda \in \Lambda_{G_M,\alpha_M}$ and $\mu \in \Lambda_{G_N,\alpha_N}$, the irreducible V^G-module M_λ and N_μ are inequivalent.*

Theorem 2.4.3. [DRX] *Any irreducible V^G-module is isomorphic to M_λ for some $\lambda \in \Lambda_{G_M,\alpha_M}$ and some irreducible g-twisted module M.*

§ 2.5 Intertwining Operators and Fusion Rules

Definition 2.5.1. Let V be a vertex operator algebra, and let (W_i, Y_{W_i}) $(i = 1, 2, 3)$ be V-modules. An intertwining operator of type

$$\begin{pmatrix} W_3 \\ W_1 \ \ W_2 \end{pmatrix}$$

is a linear map

$$I(\cdot, z) : W_1 \otimes W_2 \longrightarrow W_3\{z\}$$

such that for each $v \in W_1$,

$$I(v, z) = \sum_{n \in \mathbb{C}} v_n z^{-n-1},$$

where $v_n \in \mathrm{Hom}(W_2, W_3)$, satisfying the following conditions:

1. For fixed $n \in \mathbb{C}$ and $u \in W_2$,

$$v_{n+k} u = 0$$

   for sufficiently large integers k;

2. (Jacobi identity) for any $a \in V$, $v \in W_1$, and $u \in W_2$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{W_3}(a, z_1) I(v, z_2) u - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) I(v, z_2) Y_{W_2}(a, z_1) u$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{W_1}(a, z_0)v, z_2)u;$$

3. ($L(-1)$-derivative property)

$$\frac{d}{dz} I(v, z) = I(L(-1)v, z).$$

Let $I_V\begin{pmatrix} W_3 \\ W_1 \, W_2 \end{pmatrix}$ be the vector space of all intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 \, W_2 \end{pmatrix}$. The fusion rule $N_{W_1,\, W_2}^{W_3}$ is the dimension of the vector space $I_V\begin{pmatrix} W_3 \\ W_1 \, W_2 \end{pmatrix}$.

Theorem 2.5.2. [FHL] *Let V be a vertex operator algebra and W_1, W_2, W_3 three V-modules. Then, we have*

$$I_V\begin{pmatrix} W_3 \\ W_1\, W_2 \end{pmatrix} \cong I_V\begin{pmatrix} W_3 \\ W_2\, W_1 \end{pmatrix} \cong I_V\begin{pmatrix} (W_2)' \\ W_1\, (W_3)' \end{pmatrix}$$

Definition 2.5.3. Let V be a vertex operator algebra, and let W_1, W_2 be two V-modules. A pair (W, I), where

$$I \in I_V\binom{W}{W_1 \;\; W_2},$$

is called a tensor product of W_1 and W_2 if the following universal property holds:

For any V-module M and any intertwining operator

$$\mathcal{Y} \in I_V\binom{M}{W_1 \;\; W_2},$$

there exists a unique V-module homomorphism

$$f : W \to M$$

such that

$$\mathcal{Y} = f \circ I.$$

We denote the tensor product (W, I) by

$$W_1 \boxtimes_V W_2.$$

The tensor product exists if V is rational and C_2-cofinite. When W^1 and W^2 are two irreducible V-modules, we have

$$W_1 \boxtimes_V W_2 = \bigoplus_W N_{W_1, W_2}^{W}\, W,$$

where W runs over the set of equivalence classes of irreducible V-modules, and N_{W_1, W_2}^{W} is a finite nonnegative integer. From [H] and [V], we have the following theorem:

Theorem 2.5.4. *(Verlinde formula) Let V be a rational and C_2-cofinite simple vertex operator algebra of CFT type and assume $V \cong V'$. Let $S = \left(S_{i,j}\right)_{i,j=0}^{d}$ be the S-matrix as defined above. Then*

(1) $\left(S^{-1}\right)_{i,j} = S_{i,j'} = S_{i',j}$, and $S_{i',j'} = S_{i,j}$;

(2) S is symmetric and $S^2 = \left(\delta_{i,j'}\right)$;

(3) $N_{i,j}^{k} = \sum_{s=0}^{d} \frac{S_{i,s} S_{j,s} S_{s,k}^{-1}}{S_{0,s}}$.

Remark 2.5.5. In fact, if V is a rational, C_2-cofinite, simple, self-dual vertex operator algebra of CFT type, then the category of V-modules forms a modular tensor category; that is, a ribbon fusion category with a nondegenerate S-matrix [H2]. The Verlinde formula holds in any modular tensor category.

Lemma 2.5.6. *Let V be a rational vertex operator algebra of CFT type. Then V is simple.*

Proof. Let I be a nonzero ideal of V. Since V is rational, V is completely reducible as a V-module. Hence, there exists a V-module W such that

$$V \cong I \oplus W.$$

Because V is of CFT type, we have

$$V_0 = \mathbb{C}\mathbf{1}.$$

Therefore, the vacuum vector $\mathbf{1}$ must belong to either I or W. If $\mathbf{1} \in W$, for any $u \in V$, $u = u_{-1}\mathbf{1} \in W$. This forces $W = V$, and $I = 0$. This contradicts our assumption that

$I \neq 0$. Therefore, $\mathbf{1} \in I$ and $I = V$. V is simple. □

By this lemma, the simplicity assumption in the previous theorem can be removed.

Remark 2.5.7. Let V be a vertex operator algebra with vertex operator Y. Vertex operator Y is an intertwining operator of type $I_V\begin{pmatrix} V \\ V\,V \end{pmatrix}$. If W is a V-module, its vertex operator Y_W is an intertwining operator of type $I_V\begin{pmatrix} W \\ V\,W \end{pmatrix}$.

If V is a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type, its representation category is a modular tensor category and V is the unit object in this category. Hence, we know $V \boxtimes W \cong W$ for any V-module W.

Corollary 2.5.8. *Let V be a rational, C_2-cofinite, and self-dual vertex operator algebra of CFT type, and let M_i and M_j be two irreducible V-modules.*

$$dim\ I_V\begin{pmatrix} M_j \\ V\,M_i \end{pmatrix} = 1$$

if $M_i \cong M_j$ and

$$dim\ I_V\begin{pmatrix} M_j \\ V\,M_i \end{pmatrix} = 0$$

otherwise.

The following theorem also plays an important role in computing the fusion rules of $V_{L_2}^{S_4}$ [DL], [A]:

Theorem 2.5.9. *Let V be a vertex operator algebra, M_1, M_2 be irreducible V-modules, and M_3 be a V-module. Let U be a vertex operator subalgebra of V with the same*

Virasoro element and N_i be a U-module in M_i, $(i = 1, 2)$. We have:

$$dim\, I_V \begin{pmatrix} M_3 \\ M_1\, M_2 \end{pmatrix} \leq dim\, I_U \begin{pmatrix} M_3 \\ N_1\, N_2 \end{pmatrix}.$$

§ 2.6 Quantum Dimensions

Definition 2.6.1. Let V be a vertex operator algebra and let M be a V-module such that $Z_V(\tau)$ and $Z_M(\tau)$ exist. The quantum dimension of M over V is defined as

$$\operatorname{qdim}_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)}$$

where y is real and positive. [DJX]

From Remark 3.5 of [DJX], if $\operatorname{qdim}_V M$ exists, then

$$\operatorname{qdim}_V M = \operatorname{qdim}_V M'.$$

In the calculation of Lemma 4.2 in [DJX], if V is a simple, rational, C_2-cofinite vertex operator algebra of CFT type and the conformal weights of all irreducible V-modules are positive except that of V itself, then

$$\operatorname{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}},$$

where M^i is an irreducible V-module and $S_{i,j}$ denotes the (i, j)-entry of the S-matrix. By convention, we let $M^0 = V$.

Theorem 2.6.2. *The following result is due to* [DJX]. *Let V be a simple, rational, C_2-cofinite vertex operator algebra of CFT type. Let $M_0, M_1, \ldots, M_d$ be a complete list of equivalence classes of irreducible V-modules such that $M_0 = V$, and the conformal* *weights of all irreducible V-modules except M_0 are positive. Also, assume that $V \cong V'$ as a V-module. Then,*

$$\mathrm{qdim}_V(M_i \boxtimes M_j) = \mathrm{qdim}_V(M_i)\,\mathrm{qdim}_V M_j.$$

Definition 2.6.3. Let V be a simple vertex operator algebra. An irreducible V-module M is called a simple current if for any irreducible V-module W, the tensor product $M \boxtimes W$ exists, and $M \boxtimes W$ is an irreducible V-module.

Theorem 2.6.4. *By* [DJX]*, let V be a vertex operator algebra as in Theorem 2.6.2. Then M is a simple current if and only if* $\mathrm{qdim}_V(M) = 1$.

Example 1. [DJX] *Let $M(1)$ be the Heisenberg vertex operator algebra constructed above and H has dimension d. Then,*

$$ch_q M(1,\lambda) = \frac{q^{\frac{(\lambda,\lambda)}{2}-\frac{d}{24}}}{\Pi_{n\geq 1}(1-q^n)^d}, \lambda \in H \tag{2.6.1}$$

By calculating the limit, we see that $qdim_{M(1)}M(1,\lambda) = 1, \forall \lambda \in H$.

Example 2. *Let $V_{\mathbb{Z}\alpha}$ be the rank-one lattice vertex operator algebra with $(\alpha,\alpha) = 2$. Every irreducible $V_{\mathbb{Z}\alpha}$-module is a simple current. Hence, all irreducible $V_{\mathbb{Z}\alpha}$-modules have quantum dimension* 1.

Definition 2.6.5. The global dimension of V is defined as $\mathrm{glob}(V) = \sum_{i=0}^{d}(\mathrm{qdim}\, M_i)^2$.

Remark 2.6.6. Assume that V is a vertex operator algebra satisfying all the conditions in Theorem 2.6.2. Then the representation category of V is a modular tensor category.

Theorem 2.6.2 shows that the quantum dimension defines a character of the Grothendieck ring whose values are always positive. In a modular tensor category, there is a unique character with this property, called the Frobenius–Perron dimension. Therefore, under the assumptions of Theorem 2.6.2, the Frobenius–Perron dimension coincides with the quantum dimension.

In [DJX], Dong proposed the following open question: Let V be a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type, and let

$$\{M_1, \cdots, M_n\}$$

be a complete list of inequivalent irreducible V-modules with conformal weights λ_i. Are the modules with the lowest conformal weight among all irreducible V-modules unique?

If the answer is affirmative, then some assumptions in Theorem 2.6.2 can be removed, and similar results concerning quantum dimensions would still hold.

Theorem 2.6.7 ([DRX])**.** *For any irreducible g-twisted V-module M, we have*

$$\mathrm{qdim}_{V^G} M = |G| \,\mathrm{qdim}_V M.$$

Theorem 2.6.8. *(*[DRX]*).*

$$\mathrm{glob}(V^G) = |G|^2 \,\mathrm{glob}(V).$$

Chapter 3

Fusion Rules for $V_{L_2}^{S_4}$

§ 3.1 Irreducible Modules of $V_{L_2}^{S_4}$

Let L be a rank-one even lattice satisfying $(\alpha, \alpha) = 2k$ and V_L the corresponding lattice vertex operator algebra.

L° is the dual lattice of L:

$$L^\circ = \{\lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z}\} = \frac{1}{2k}L. \tag{3.1.1}$$

Then L° admits the coset decomposition

$$L^\circ = \bigcup_{i=-k+1}^{k} (L + \lambda_i), \quad \text{where } \lambda_i = \frac{i}{2k}\alpha.$$

Set $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$. The modules $V_{L+\lambda_i}$, for $i = -k+1, \ldots, k$, are all inequivalent irreducible V_L-modules.

Let θ be the order-two automorphism defined at the end of section 2.3. We can extend θ to a linear isomorphism from $V_{L+\lambda_i} \to V_{L-\lambda_i}$ for $i = -k+1, \cdots, k$.

$$\theta : \alpha(-n_1) \cdots \alpha(-n_l) \otimes e^{\beta+\lambda_i} \to (-1)^l \alpha(-n_1) \cdots \alpha(-n_l) \otimes e^{-\beta-\lambda_i},$$

where $n_j > 0$, $1 \le j \le l$, and $\beta \in L$. It follows that θ induces an automorphism on $M(1)$. For V_L and $V_{L+\lambda_k}$, θ is a linear isomorphism from the space to itself. Since θ has order 2, its eigenvalues must be ± 1.

We use $M(1)^+$, V_L^+, and $V_{L+\lambda_k}^+$ to denote the $+1$ eigenspaces of $M(1)$, V_L, and $V_{L+\lambda_k}$, respectively. Similarly, we use $M(1)^-$, V_L^-, and $V_{L+\lambda_k}^-$ to denote the -1 eigenspaces of $M(1)$, V_L, and $V_{L+\lambda_k}$, respectively.

Similarly, we can define an action of θ on $V_L^{T_i}$, $i = 1, 2$:

$$\theta : \alpha(-n_1) \cdots \alpha(-n_l) \otimes t \longrightarrow (-1)^l \alpha(-n_1) \cdots \alpha(-n_l) \otimes t,$$

for $n_i \in \frac{1}{2} + \mathbb{Z}_+, t \in T_{\chi_i}, i = 1, 2$. We also use $(V_L^{T_i})^{\pm}$ to denote the ± 1 eigenspace of θ on $V_L^{T_i}$, $i = 1, 2$.

$V_L^+ = V_L^{<\theta>}$ is also a vertex operator algebra.

Theorem 3.1.1 ([CM]). *Let T be a simple non-negatively graded regular vertex operator algebra with a non-singular invariant bilinear form and let G be a finite solvable group of automorphisms of T. Then the fixed-point vertex operator subalgebra T^G is simple, non-negatively graded, regular, and admits a non-singular invariant bilinear form.*

Definition 3.1.2. A vertex operator algebra V is called regular if every weak V-module is a direct sum of irreducible ordinary V-modules.

Theorem 3.1.3. [ABD] *A vertex operator algebra of CFT type is regular if and only if it is rational and C_2-cofinite.*

Definition 3.1.4. Let V be a vertex operator algebra and let M be an ordinary V-module. A bilinear form $(\cdot, \cdot)$ on M is called invariant if for any $u \in V$ and $w_1, w_2 \in M$, we have

$$(Y(u, z)w_1, w_2) = \left(w_1, Y\left(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1}\right) w_2\right).$$

In particular, we have

$$(L(n)w_1, w_2) = (w_1, L(-n)w_2), \qquad \forall n \in \mathbb{Z}.$$

Proposition 3.1.5. *Let V be a vertex operator algebra and M an ordinary V-module. There exists a nondegenerate invariant bilinear form on M if and only if M is self-dual.*

Proposition 3.1.6. *Let V be a vertex operator algebra of CFT type and $(\cdot,\cdot)$ a nonzero invariant bilinear form on V. Then it is nondegenerate if and only if V is simple.*

Proposition 3.1.7. *Any invariant bilinear form on V is symmetric.*

Combining these propositions, we obtain the following consequence of Theorem 3.1.1:

Let V be a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type, and let G be a finite solvable subgroup of $\mathrm{Aut}(V)$. Then V^G is also a rational, C_2-cofinite, and self-dual vertex operator algebra of CFT type.

Hence, $V_{L_2}^+, V_{L_2}^{A_4}, V_{L_2}^{S_4}$ are all rational and C_2-cofinite vertex operator algebras since $\mathbb{Z}_2, A_4, S_4$ are all solvable groups.

Remark 3.1.8. This theorem implies the rationality of $V_{L_2}^+$, $V_{L_2}^{A_4}$, and $V_{L_2}^{S_4}$. However, this theorem does not apply to $V_{L_2}^{A_5}$, since A_5 is not solvable. Without knowing the rationality of $V_{L_2}^{A_5}$, we cannot guarantee a complete classification of all irreducible $V_{L_2}^{A_5}$-modules via orbifold theory, nor can we compute the fusion rules using the Verlinde formula.

In [DN], all inequivalent V_L^+-modules are classified:

Theorem 3.1.9. *Any irreducible V_L^+-module is isomorphic to one of the following modules:*

$$V_L^\pm, V_{\lambda_i+L}(i = 1, 2, \cdots, k-1), V_{\lambda_k+L}^\pm, (V_L^{T_1})^\pm, (V_L^{T_2})^\pm.$$

Let L_2 be the rank-one positive definite even lattice with $(\alpha,\alpha)=2$. Then x_1,x_2,x_3 form an orthonormal basis of $(V_{L_2})_1$

$$x_1=\frac{1}{\sqrt{2}}\alpha(-1)\mathbf{1},x_2=\frac{1}{\sqrt{2}}(e^{\alpha}+e^{-\alpha}),x_3=\frac{i}{\sqrt{2}}(e^{\alpha}-e^{-\alpha}).$$

Let $\delta,\rho\in\mathrm{Aut}(V_{L_2})$, and let $\tau_i\in\mathrm{Aut}(V_{L_2}), i=1,2$ be defined as follows.

$$\delta\left(x_1,x_2,x_3\right)=(x_1,x_2,x_3)\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}.$$

$$\tau_1\left(x_1,x_2,x_3\right)=(x_1,x_2,x_3)\begin{bmatrix}1&&\\&-1&\\&&-1\end{bmatrix},$$

$$\tau_2\left(x_1,x_2,x_3\right)=(x_1,x_2,x_3)\begin{bmatrix}-1&&\\&1&\\&&-1\end{bmatrix},$$

$$\rho\left(x_1,x_2,x_3\right)=(x_1,x_2,x_3)\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

Clearly,

$$\langle\tau_1,\tau_2,\delta,\rho\rangle\cong S_4,\quad\langle\tau_1,\tau_2,\delta\rangle\cong A_4,\quad\langle\tau_1,\tau_2\rangle\cong K_4,$$

and the K_4 generated by τ_1 and τ_2 is a normal subgroup of both S_4 and A_4.

Remark 3.1.10. A direct calculation shows τ_2 is the automorphism θ we defined above.

Lemma 3.1.11. *By* [DG],[WZ], *we have* $V_{L_2}^K=V_{\mathbb{Z}\beta}^+, V_{L_2}^{A_4}=(V_{\mathbb{Z}\beta}^+)^{<\delta>}, V_{L_2}^{S_4}=(V_{\mathbb{Z}\beta}^+)^{<\delta,\rho>}=$

$(V_{L_2}^{A_4})^{\rho}$.

By Remark 3.4 of [DJJJY], $SO(3)$ is the connected compact subgroup of $\mathrm{Aut}(V_{L_2})$ that contains Z_n, D_n, A_4, S_4, A_5 as discrete subgroups. It follows that $V_{L_2}^{Z_n} \cong V_{\mathbb{Z}n\alpha}$ and $V_{L_2}^{D_n} \cong V_{\mathbb{Z}n\alpha}^{+}$.

Let us denote $\beta = 2\alpha$ and $\zeta = 4\alpha$. By [DG], $V_{\mathbb{Z}\beta}^{+}$ is generated by $J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1}$, $E_\beta = e^{\beta} + e^{-\beta}$ and $V_{\mathbb{Z}\zeta}^{+}$ is generated by $J, E_\zeta = e^{\zeta} + e^{-\zeta}$ where $h = \frac{1}{\sqrt{2}}\alpha$. By [DJJJY],[DJ], all inequivalent irreducible $V_{L_2}^{A_4}$-modules are classified and their quantum dimensions are calculated. By [WZ], all inequivalent irreducible $V_{L_2}^{S_4}$-modules are classified as follows and their quantum dimensions are calculated.

Theorem 3.1.12. *There are exactly 21 irreducible modules of* $V_{L_2}^{A_4}$.

Table 3.1: Conformal weights and quantum dimensions

	$(V_{\mathbb{Z}\beta}^{+})^0$	$(V_{\mathbb{Z}\beta}^{+})^1$	$(V_{\mathbb{Z}\beta}^{+})^2$	$V_{\mathbb{Z}\beta}^{-}$	$V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$
ω	0	4	4	1	$\frac{1}{16}$	$\frac{9}{16}$
qdim	1	1	1	3	6	6

	$W^{1,T_1,1}$	$W^{1,T_1,2}$	$W^{1,T_1,3}$	$W^{2,T_1,1}$	$W^{2,T_1,2}$	$W^{2,T_1,3}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
qdim	4	4	4	4	4	4

	$W^{1,T_2,1}$	$W^{1,T_2,2}$	$W^{1,T_2,3}$	$W^{2,T_2,1}$	$W^{2,T_2,2}$	$W^{2,T_2,3}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
qdim	4	4	4	4	4	4

	$(V_{\mathbb{Z}\beta}+\frac{1}{4}\beta)^0$	$(V_{\mathbb{Z}\beta}+\frac{1}{4}\beta)^1$	$(V_{\mathbb{Z}\beta}+\frac{1}{4}\beta)^2$
ω	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{9}{4}$
$qdim$	2	2	2

Since $V_{L_2}^{A_4}=(V_{\mathbb{Z}\beta}^+)^\delta$, irreducible $V_{L_2}^{A_4}$-modules are obtained by decomposing ordinary modules, δ-twisted $V_{\mathbb{Z}\beta}^+$-modules, and δ^2-twisted $V_{\mathbb{Z}\beta}^+$-modules.

The modules $(V_{\mathbb{Z}\beta}^+)^i$ for $i=0,1,2$ are three components obtained from the decomposition of $V_{\mathbb{Z}\beta}^+$. Similarly, $(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^i$ for $i=0,1,2$ arise from the decomposition of $V_{\mathbb{Z}\beta+\frac{1}{4}\beta}$.

Moreover, $V_{\mathbb{Z}\beta}^+$ has two δ-twisted modules, W^{1,T_1} and W^{2,T_1}. Each of these decomposes into three irreducible $V_{L_2}^{A_4}$-modules:

$$W^{1,T_1,i},\quad i=1,2,3,\qquad W^{2,T_1,i},\quad i=1,2,3.$$

There are two δ^2-twisted modules, W^{1,T_2} and W^{2,T_2}. The modules $W^{1,T_2,i}$ and $W^{2,T_2,i}$ for $i=1,2,3$ are obtained in a similar way by decomposing these twisted modules.

Let

$$h=\frac{1}{3\sqrt{6}}(x_1+x_2-x_3).$$

A direct calculation shows

$$\delta=e^{2\pi i h(0)}$$

and that $e^{2\pi i h(0)}h=h$. By [DJJJY], the group $\langle \delta,\tau_i \mid i=1,2\rangle$ acts on $V_{\mathbb{Z}\frac{1}{2}\alpha}=M(1)\otimes\mathbb{C}[\frac{1}{2}\alpha\mathbb{Z}]$.

Let $\gamma=18h$. Then $V_{\mathbb{Z}\gamma}$ is the eigenspace of δ with eigenvalue 1. On the other hand, $V_{\mathbb{Z}\gamma}$ carries a vertex operator algebra structure, and

$$V_{\mathbb{Z}\gamma}\cong V_{\mathbb{Z}(3\alpha)}\cong V_{L_2}^{\langle\delta\rangle}.$$

Its irreducible modules

$$V_{\mathbb{Z}\gamma\pm\frac{1}{3}\gamma},\quad V_{\mathbb{Z}\gamma\pm\frac{1}{6}\gamma},\quad V_{\mathbb{Z}\gamma+\frac{1}{2}\gamma}$$

are also eigenspaces of δ with distinct eigenvalues. Hence, each decomposes as a direct sum of irreducible $V_{L_2}^{A_4}$-modules.

Moreover, Proposition 3.7 of [DJJJY] gives a precise identification of those irreducible $V_{L_2}^{A_4}$-modules arising from the decomposition of the δ-twisted and δ^2-twisted $V_{\mathbb{Z}\beta}^{+}$-modules.

Proposition 3.1.13. *As $V_{L_2}^{A_4}$-modules, we have the following identifications:*

$$V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma}\cong W_{\delta,1}^{0}, V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma}\cong W_{\delta,1}^{1}, V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma}\cong W_{\delta,1}^{2},$$

$$V_{\mathbb{Z}\gamma-\frac{1}{9}\gamma}\cong W_{\delta,2}^{0}, V_{\mathbb{Z}\gamma+\frac{2}{9}\gamma}\cong W_{\delta,2}^{1}, V_{\mathbb{Z}\gamma-\frac{4}{9}\gamma}\cong W_{\delta,2}^{2},$$

$$V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma}\cong W_{\delta^2,1}^{0}, V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma}\cong W_{\delta^2,1}^{1}, V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma}\cong W_{\delta^2,1}^{2},$$

$$V_{\mathbb{Z}\gamma+\frac{1}{9}\gamma}\cong W_{\delta^2,2}^{0}, V_{\mathbb{Z}\gamma-\frac{2}{9}\gamma}\cong W_{\delta^2,2}^{1}, V_{\mathbb{Z}\gamma+\frac{4}{9}\gamma}\cong W_{\delta^2,2}^{2}.$$

Here, we use a slightly different notation for some modules. In this notation, $W_{\delta,1}^{0}$ is just $W^{1,T_1,1}$ defined above, and others are similar.

Theorem 3.1.14. [WZ] *Any irreducible $V_{L_2}^{S_4}$-module is isomorphic to one of the following modules:*

$$((V_{\mathbb{Z}\beta}^{+})^{0})^{+}, ((V_{\mathbb{Z}\beta}^{+})^{0})^{-}, (V_{\mathbb{Z}\beta}^{+})^{1}, (V_{\mathbb{Z}\beta}^{-})^{+}, (V_{\mathbb{Z}\beta}^{-})^{-}, \tag{3.1.2}$$

(3.1.3) $$(V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta})^+, (V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta})^-, V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta},$$

(3.1.4) $$(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^-, (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^-,$$

(3.1.5) $$V_{\mathbb{Z}\gamma+\frac{r}{18}\gamma}, \quad \textit{for } 1 \leq r \leq 8 \textit{ and } r \neq 0 \textit{ (mod } 3),$$

(3.1.6) $$V_{\mathbb{Z}\zeta+\frac{s}{32}\zeta}, \quad \textit{for } 1 \leq s \leq 15 \textit{ and } s \neq 0 \textit{ (mod } 2),$$

(3.1.7) $$V^{T_2,+}_{\mathbb{Z}\zeta}, V^{T_2,-}_{\mathbb{Z}\zeta}$$

Here $((V^+_{\mathbb{Z}\beta})^0)^+$ is just the eigenspace of $\psi(\rho)$ with eigenvalue $+1$. It is an irreducible $V_{L_2}^{S_4}$-module that is obtained by decomposing $V_{L_2}^{A_4}$-module $(V^+_{\mathbb{Z}\beta})^0$.

Theorem 3.1.15. [WZ] *The quantum dimensions for all irreducible $V_{L_2}^{S_4}$-modules over $V_{L_2}^{S_4}$ are given by the following tables.*

Table 3.2: Quantum dimensions and notation for the irreducible modules

	$\left((V^+_{\mathbb{Z}\beta})^0\right)^+$	$\left((V^+_{\mathbb{Z}\beta})^0\right)^-$	$(V^+_{\mathbb{Z}\beta})^1$	$(V^-_{\mathbb{Z}\beta})^+$	$(V^-_{\mathbb{Z}\beta})^-$
qdim	1	1	2	3	3
Module	M^0	M^1	M^2	M^3	M^4

	$\left(V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta}\right)^+$	$\left(V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta}\right)^-$	$V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta}$
qdim	2	2	4
Module	M^5	M^6	M^7

	$(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^+$	$(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^-$	$(V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^+$	$(V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^-$
qdim	6	6	6	6
Module	M^8	M^9	M^{10}	M^{11}

	$V_{\mathbb{Z}\gamma+\frac{r}{18}\gamma}$	$V_{\mathbb{Z}\zeta+\frac{s}{32}\zeta}$	$V_{\mathbb{Z}\zeta}^{T_2,+}$	$V_{\mathbb{Z}\zeta}^{T_2,-}$
qdim	8	6	12	12
Module	$M^{12}, \ldots, M^{17}$	$M^{18}, \ldots, M^{25}$	M^{26}	M^{27}

In the last table, $r, s \in \mathbb{Z}$*,* $1 \leq r \leq 8$*,* $1 \leq s \leq 15$*,* $r \neq 0$ *(mod 3), and* $s \neq 0$ *(mod 2).*

Let $P = \rho \circ \tau_2$ and note that

$$\langle P, \tau_2 \rangle = \langle \tau_1, \tau_2, \rho \rangle \cong D_8.$$

By direct computation, P maps $\alpha(-1)$ to $\alpha(-1)$, e^{α} to ie^{α}, and $e^{-\alpha}$ to $-ie^{-\alpha}$, so P fixes e^{ζ} and $\zeta(-1)$. Therefore, $V_{L_2}^{\langle P \rangle} = V_{\mathbb{Z}\zeta}$, not just isomorphic. Furthermore, τ_2 is exactly the automorphism θ discussed above, so

$$(V_{L_2}^{\langle P \rangle})^{\langle \tau_2 \rangle} = (V_{L_2})^{\langle P, \tau_2 \rangle} = V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle \rho \rangle}.$$

By abuse of notation, we write $M^{\pm}$ for the ± 1-eigenspace of $\psi(\rho)$ on M if M is a $V_{\mathbb{Z}\beta}^+$-module.

Proposition 3.1.16. [DN] *The following tables give the scalars obtained from the construction of these modules:*

Table 3.3: Eigenvalues of ω, E, and J

	V_L^+	V_L^-	$V_{L+\frac{r}{2k}}$, $(1 \leq r \leq k-1)$	$V_{L+\frac{\alpha}{2}}^+$	$V_{L+\frac{\alpha}{2}}^-$
ω	0	1	$\frac{r^2}{4k}$	$\frac{k}{4}$	$\frac{k}{4}$
E	0	0	0	1	-1
J	0	-6	$c^4 - \frac{c^2}{2}$, $\quad c^2 = \frac{r^2}{2k}$	$\frac{k^2}{4} - \frac{k}{4}$	$\frac{k^2}{4} - \frac{k}{4}$

	$V_L^{T_1,+}$	$V_L^{T_1,-}$	$V_L^{T_2,+}$	$V_L^{T_2,-}$
ω	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{9}{16}$
E	2^{-2k+1}	$-2^{-2k+1}(4k-1)$	-2^{-2k+1}	$2^{-2k+1}(4k-1)$
J	$\frac{3}{128}$	$\frac{-45}{128}$	$\frac{3}{128}$	$\frac{-45}{128}$

Proposition 3.1.17. *As* $V_{L_2}^{S_4}$*-modules, we have the following identifications:*

(3.1.8) $$V_{\mathbb{Z}\beta}^{++} \cong V_{\mathbb{Z}\zeta}^{+} \cong ((V_{\mathbb{Z}\beta}^{+})^0)^+ \oplus (V_{\mathbb{Z}\beta}^{+})^1$$

(3.1.9) $$V_{\mathbb{Z}\beta}^{+-} \cong V_{\mathbb{Z}\zeta+\frac{1}{2}\zeta}^{+} \cong ((V_{\mathbb{Z}\beta}^{+})^0)^- \oplus (V_{\mathbb{Z}\beta}^{+})^1$$

(3.1.10) $$V_{\mathbb{Z}\beta}^{-+} \cong V_{\mathbb{Z}\zeta}^{-}$$

(3.1.11) $$V_{\mathbb{Z}\beta}^{--} \cong V_{\mathbb{Z}\zeta+\frac{1}{2}\zeta}^{-}$$

(3.1.12) $$V_{\frac{1}{8}\beta+\mathbb{Z}\beta}^{+} \cong V_{\frac{2}{32}\zeta+\mathbb{Z}\zeta}$$

(3.1.13) $$V_{\frac{3}{8}\beta+\mathbb{Z}\beta}^{+} \cong V_{\frac{6}{32}\zeta+\mathbb{Z}\zeta}$$

(3.1.14) $$V_{\frac{1}{8}\beta+\mathbb{Z}\beta}^{-} \cong V_{\frac{14}{32}\zeta+\mathbb{Z}\zeta}$$

$$V^{-}_{\frac{3}{8}\beta+\mathbb{Z}\beta} \cong V_{\frac{10}{32}\zeta+\mathbb{Z}\zeta} \tag{3.1.15}$$

$$V^{+}_{\frac{1}{4}\beta+\mathbb{Z}\beta} \cong V_{\frac{4}{32}\zeta+\mathbb{Z}\zeta} \cong (V^{0}_{\frac{1}{4}\beta+\mathbb{Z}\beta})^{+} \oplus V^{1}_{\frac{1}{4}\beta+\mathbb{Z}\beta} \tag{3.1.16}$$

$$V^{-}_{\frac{1}{4}\beta+\mathbb{Z}\beta} \cong V_{\frac{12}{32}\zeta+\mathbb{Z}\zeta} \cong (V^{0}_{\frac{1}{4}\beta+\mathbb{Z}\beta})^{-} \oplus V^{1}_{\frac{1}{4}\beta+\mathbb{Z}\beta} \tag{3.1.17}$$

$$V^{T_1,+}_{\mathbb{Z}\zeta} \cong (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \tag{3.1.18}$$

$$V^{T_1,-}_{\mathbb{Z}\zeta} \cong (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-} \tag{3.1.19}$$

$$V^{+}_{\mathbb{Z}\beta+\frac{1}{2}\beta} \cong V^{-}_{\mathbb{Z}\beta+\frac{\beta}{2}} \cong V_{\frac{8\zeta}{32}+\mathbb{Z}\zeta} \tag{3.1.20}$$

Proof. First, we prove (3.1.12)–(3.1.15). By the definition of ρ, we have $\rho(J) = J$ and $\rho(E_\beta) = -E_\beta$. On the one hand, by [WZ], we have

$$V_{\frac{\beta}{8}+\mathbb{Z}\beta} \cong V^{+}_{\frac{\beta}{8}+\mathbb{Z}\beta} \oplus V^{-}_{\frac{\beta}{8}+\mathbb{Z}\beta}$$

as a $V_{L_2}^{S_4}$-module. On the other hand, applying Theorem 2.4.2 to decompose $V_{\frac{\beta}{8}+\mathbb{Z}\beta}$ into $V^{+}_{\mathbb{Z}\zeta}$-modules, we again obtain

$$V_{\frac{\beta}{8}+\mathbb{Z}\beta} \cong V^{+}_{\frac{\beta}{8}+\mathbb{Z}\beta} \oplus V^{-}_{\frac{\beta}{8}+\mathbb{Z}\beta}$$

as a $V^{+}_{\mathbb{Z}\zeta}$-module. Thus, $V^{+}_{\frac{\beta}{8}+\mathbb{Z}\beta}$ and $V^{-}_{\frac{\beta}{8}+\mathbb{Z}\beta}$ must be isomorphic to irreducible $V^{+}_{\mathbb{Z}\zeta}$-modules listed in Theorem 3.1.9.

Without loss of generality, we may assume that $\psi(\rho)(e^{\frac{\beta}{8}}) = e^{\frac{\beta}{8}}$. Then the conformal weight of $V^+_{\frac{\beta}{8}+\mathbb{Z}\beta}$ is $\frac{1}{16}$. By the definition of $\psi(\rho)$, we have

$$\psi(\rho)Y(E_\beta, z) = -Y(E_\beta, z)\psi(\rho).$$

Hence,

$$(E_\beta)_{(n)}e^{\frac{\beta}{8}} \in V^-_{\frac{\beta}{8}+\mathbb{Z}\beta}$$

for every n such that $(E_\beta)_{(n)}e^{\frac{\beta}{8}} \neq 0$. By direct calculation,

$$Y(E_\beta, z)e^{\frac{\beta}{8}} = zE^-(-\beta, z)e^{\frac{9\beta}{8}} + z^{-1}E^-(\beta, z)e^{-\frac{7\beta}{8}}.$$

Comparing with the quantum dimensions and conformal weights of irreducible $V^+_{\mathbb{Z}\zeta}$-modules, it follows that the conformal weight of $V^-_{\frac{\beta}{8}+\mathbb{Z}\beta}$ is $\frac{49}{16}$. We obtain (3.1.12) and (3.1.14). Applying the same argument yields (3.1.13) and (3.1.15).

Next, we prove (3.1.16) and (3.1.17). Using a similar method, we find that $V^+_{\frac{\beta}{4}+\mathbb{Z}\beta}$ has conformal weight $\frac{1}{4}$ and $V^-_{\frac{\beta}{4}+\mathbb{Z}\beta}$ has conformal weight $\frac{9}{4}$. Hence,

$$V^+_{\frac{\beta}{4}+\mathbb{Z}\beta} \cong V_{\frac{4}{32}\zeta+\mathbb{Z}\zeta}, \qquad V^-_{\frac{\beta}{4}+\mathbb{Z}\beta} \cong V_{\frac{12}{32}\zeta+\mathbb{Z}\zeta}$$

as $V^+_{\mathbb{Z}\zeta}$-modules. Therefore, $V^+_{\frac{\beta}{4}+\mathbb{Z}\beta}$ and $V^-_{\frac{\beta}{4}+\mathbb{Z}\beta}$ have the same quantum dimension. By [DJ], we have

$$V_{\frac{\beta}{4}+\mathbb{Z}\beta} = V^0_{\frac{\beta}{4}+\mathbb{Z}\beta} \oplus V^1_{\frac{\beta}{4}+\mathbb{Z}\beta} \oplus V^2_{\frac{\beta}{4}+\mathbb{Z}\beta}.$$

as $V_{L_2}^{A_4}$-module. By [WZ], we have

$$V_{\frac{\beta}{4}+\mathbb{Z}\beta} = (V^0_{\frac{\beta}{4}+\mathbb{Z}\beta})^+ \oplus (V^0_{\frac{\beta}{4}+\mathbb{Z}\beta})^- \oplus 2V^1_{\frac{\beta}{4}+\mathbb{Z}\beta}.$$

By counting the quantum dimensions, we have $V_{\frac{4}{32}\zeta+\mathbb{Z}\zeta} \cong (V^0_{\frac{1}{4}\beta+\mathbb{Z}\beta})^+ \oplus V^1_{\frac{1}{4}\beta+\mathbb{Z}\beta}$ and $V_{\frac{12}{32}\zeta+\mathbb{Z}\zeta} \cong (V^0_{\frac{1}{4}\beta+\mathbb{Z}\beta})^- \oplus V^1_{\frac{1}{4}\beta+\mathbb{Z}\beta}$

A direct calculation shows $\rho(E_\beta) = -E_\beta$ and $\rho(J) = J$. Using the scalars in the above proposition from [DN] and combining this with the action of ρ on $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$, it follows that $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}} \circ \rho \cong V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$. So, $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$ and $V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$ are in the same orbit. This implies:

$$V^+_{\mathbb{Z}\beta+\frac{\beta}{2}} \cong V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$$

as $V^+_{\mathbb{Z}\zeta}$-modules and $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$ is an irreducible $V^+_{\mathbb{Z}\zeta}$-module. Comparing the conformal weight and the quantum dimensions of $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$ with those of the remaining $V^+_{\mathbb{Z}\zeta}$-modules, we see that $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}} \cong V_{\frac{8}{32}\zeta+\mathbb{Z}\zeta}$.

By [WZ],

$$V^{T_1,+}_{\mathbb{Z}\beta} \cong V^{T_2,+}_{\mathbb{Z}\beta} \cong V^{T_1,+}_{\mathbb{Z}\zeta}, \qquad V^{T_1,-}_{\mathbb{Z}\beta} \cong V^{T_2,-}_{\mathbb{Z}\beta} \cong V^{T_1,-}_{\mathbb{Z}\zeta}$$

as $V^+_{\mathbb{Z}\zeta}$-modules. On the other hand, by [DJ],

$$V_{\mathbb{Z}\beta+\frac{\beta}{8}} \cong V^{T_1,+}_{\mathbb{Z}\beta} \cong V^{T_2,+}_{\mathbb{Z}\beta}, \qquad V_{\mathbb{Z}\beta+\frac{3\beta}{8}} \cong V^{T_1,-}_{\mathbb{Z}\beta} \cong V^{T_2,-}_{\mathbb{Z}\beta}$$

as $V^{A_4}_{L_2}$-modules. Thus, (3.1.18) and (3.1.19) follow.

Comparing with the remaining inequivalent $V^+_{\mathbb{Z}\zeta}$-modules, it is clear that

$$V^{++}_{\mathbb{Z}\beta} \cong V^+_{\mathbb{Z}\zeta}, \qquad V^{-+}_{\mathbb{Z}\beta} \cong V^-_{\mathbb{Z}\zeta}.$$

By an argument similar to that used to prove (3.1.12), we see that $V^{+-}_{\mathbb{Z}\beta}$ is generated by E_β. A direct computation shows that

$$(E_\zeta)_{(15)}(E_\beta) = E_\beta.$$

Therefore, by [DN],

$$V^{+-}_{\mathbb{Z}\beta} \cong V^+_{\mathbb{Z}\zeta+\frac{1}{2}\zeta}, \qquad V^{--}_{\mathbb{Z}\beta} \cong V^-_{\mathbb{Z}\zeta+\frac{1}{2}\zeta}$$

as $V_{\mathbb{Z}\zeta}^{+}$-modules and hence as $V_{L_2}^{S_4}$-modules. Finally, (3.1.8) and (3.1.9) follow from the quantum dimensions.

□

Corollary 3.1.18. *As $V_{L_2}^{S_4}$-modules:*

$$V_{\mathbb{Z}\zeta} \cong ((V_{\mathbb{Z}\beta}^{+})^{0})^{+} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \tag{3.1.21}$$

$$V_{\mathbb{Z}\zeta+\frac{\zeta}{4}} \cong (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \tag{3.1.22}$$

$$V_{\mathbb{Z}\zeta+\frac{\zeta}{2}} \cong ((V_{\mathbb{Z}\beta}^{+})^{0})^{-} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \tag{3.1.23}$$

$$V_{\mathbb{Z}\zeta+\frac{3\zeta}{4}} \cong (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \tag{3.1.24}$$

$$V_{\mathbb{Z}\zeta+\frac{\zeta}{8}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}) \tag{3.1.25}$$

$$V_{\mathbb{Z}\zeta+\frac{3\zeta}{8}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}) \tag{3.1.26}$$

$$V_{\mathbb{Z}\zeta+\frac{5\zeta}{8}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}) \tag{3.1.27}$$

$$V_{\mathbb{Z}\zeta+\frac{7\zeta}{8}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}) \tag{3.1.28}$$

(3.1.22) follows from Lemma 5.1 [DJ]. Applying the method in [DJJJY], let $\gamma = \sqrt{6}(x_1 + x_2 - x_3)$ and $\epsilon = 2(x_2 + x_3)$, we have the following identifications:

Proposition 3.1.19. *As* $V_{L_2}^{S_4}$*-modules:*

(3.1.29) $$V_{\mathbb{Z}\gamma}\cong((V_{\mathbb{Z}\beta}^{+})^{0})^{+}\oplus((V_{\mathbb{Z}\beta}^{+})^{0})^{-}\oplus(V_{\mathbb{Z}\beta}^{-})^{+}\oplus(V_{\mathbb{Z}\beta}^{-})^{-}$$

(3.1.30) $$V_{\mathbb{Z}\gamma+\frac{\gamma}{3}} \cong (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}$$

(3.1.31) $$V_{\mathbb{Z}\gamma+\frac{2\gamma}{3}} \cong (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}$$

(3.1.32) $$V_{\mathbb{Z}\gamma+\frac{\gamma}{6}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})$$

(3.1.33) $$V_{\mathbb{Z}\gamma+\frac{\gamma}{2}} \cong 2V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}$$

(3.1.34) $$V_{\mathbb{Z}\gamma+\frac{5\gamma}{6}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus (V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})$$

(3.1.35) $$V_{\mathbb{Z}\epsilon} \cong ((V_{\mathbb{Z}\beta}^{+})^{0})^{+} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus 2(V_{\mathbb{Z}\beta}^{-})^{-}$$

(3.1.36) $$V_{\mathbb{Z}\epsilon+\frac{\epsilon}{4}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus 2(V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})$$

(3.1.37) $$V_{\mathbb{Z}\epsilon+\frac{\epsilon}{2}} \cong ((V_{\mathbb{Z}\beta}^{+})^{0})^{-} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus 2(V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}$$

(3.1.38) $$V_{\mathbb{Z}\epsilon+\frac{3\epsilon}{4}} \cong (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+} \oplus (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-} \oplus 2(V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})$$

Proof. Applying the method in [DJJJY] and Corollary 3.1.18, we have

(3.1.39)

$$V_{L_2} \cong ((V_{\mathbb{Z}\beta}^{+})^{0})^{+} \otimes W_1^0 \oplus ((V_{\mathbb{Z}\beta}^{+})^{0})^{-} \otimes W_1^1 \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \otimes W_2^0 \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \otimes W_3^1 \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \otimes W_3^0,$$

and

$$V_{L_2+\frac{\alpha}{2}} \cong (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \otimes W_2^1 \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \otimes W_2^2 \oplus (V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}) \otimes W_4. \tag{3.1.40}$$

Here W_i^j denotes an irreducible representation of $GL(2,3)$, where i indicates the dimension of the irreducible module and j distinguishes inequivalent irreducible modules of the same dimension. Combining Corollary 3.1.18 with the eigenspace decomposition (with respect to the eigenvalues of the linear map P) in [DJJJY], we obtain that W_1^0 has eigenvalue 1, W_1^1 has eigenvalue -1, W_2^0 has eigenvalues ± 1, W_2^1 has eigenvalues $e^{\pm\frac{3\pi}{4}i}$, W_2^2 has eigenvalues $e^{\pm\frac{\pi}{4}i}$, W_3^0 has eigenvalues $-1, \pm i$, W_3^1 has eigenvalues $1, \pm i$, and W_4 has eigenvalues $e^{\pm\frac{\pi}{4}i}, e^{\pm\frac{3\pi}{4}i}$.

Therefore, calculating the eigenvalues of W_i^j with respect to ρ and δ yields (3.1.29)–(3.1.38).

□

Lemma 3.1.20. *As $V_{L_2}^{S_4}$-modules:*

$$V_{\mathbb{Z}\zeta}^{T_2,+} \cong V_{\mathbb{Z}\epsilon+\frac{\epsilon}{8}} \tag{3.1.41}$$

$$V_{\mathbb{Z}\zeta}^{T_2,-} \cong V_{\mathbb{Z}\epsilon+\frac{3\epsilon}{8}} \tag{3.1.42}$$

Proof. By Theorems 2.4.1–2.4.3, we know that any irreducible $V_{L_2}^{S_4}$-module arises from a decomposition of irreducible V_{L_2}-modules, τ_1-twisted V_{L_2}-modules, δ-twisted V_{L_2}-modules, ρ-twisted V_{L_2}-modules, or P-twisted V_{L_2}-modules. By [WZ], the irreducible $V_{L_2}^{S_4}$-modules arising from ρ-twisted and P-twisted V_{L_2}-modules are M_{18}–M_{27} (using the indexing in Theorem 3.1.15).

Both V_{L_2} and $V_{\mathbb{Z}\alpha+\frac{\alpha}{2}}$ are inequivalent irreducible V_{L_2}-modules, and both are stabilized by P. By [DLM2], there are exactly two P-twisted V_{L_2}-modules with conformal weights $\frac{1}{64}$ and $\frac{9}{64}$. Thus, the stabilizer of the action is equal to the centralizer of P, namely

$$C_G(P) = \langle P \rangle \cong \mathbb{Z}_4.$$

The Schur multiplier of $\mathbb{Z}_4$ is trivial, so there are eight inequivalent irreducible $V_{L_2}^{S_4}$-modules arising from the decomposition of those two P-twisted V_{L_2} modules.

Applying Theorem 2.4.1 to $G = \langle P \rangle$, we see that these eight modules are also inequivalent irreducible $V_{L_2}^{\langle P \rangle} = V_{\mathbb{Z}\zeta}$-modules. Therefore, these eight modules correspond exactly to M_{18}–M_{25}. Consequently, M_{26} and M_{27} arise from ρ-twisted V_{L_2}-modules. By applying a similar argument, we see that M_{26} and M_{27} are also inequivalent irreducible $V_{L_2}^{\langle \rho \rangle} \cong V_{\mathbb{Z}\epsilon}$-modules. Comparing conformal weights, we obtain (3.1.41) and (3.1.42).

□

The identifications above are useful for calculating the fusion rules of modules that are contained in untwisted irreducible V_{L_2}-modules. In order to determine the fusion rules involving irreducible $V_{L_2}^{S_4}$-modules that arise from twisted V_{L_2}-modules, we need to compute the S-matrix of $V_{L_2}^{S_4}$. The main method follows Lemma 5.1 of [DJJJY]; however, for completeness, we provide a proof here.

Lemma 3.1.21. *The entries of the S-matrix that involve irreducible twisted modules of $V_{L_2}^{S_4}$ are given in the Appendix.*

Proof. We give the proof for $M_{18}, \ldots, M_{25}$; the other cases are similar. Consider irreducible $V_{\mathbb{Z}\zeta}$-modules. They are of the form

$$V_{\mathbb{Z}\zeta+\lambda_k}, \qquad \lambda_k = \frac{k\zeta}{32}, \quad k = 0, \ldots, 31.$$

By [S], we have

$$Z_{V_{\mathbb{Z}\zeta+\lambda_k}}\left(-\frac{1}{\tau}\right) = \sum_{j=0}^{31} \frac{1}{\sqrt{32}} e^{-2\pi i\langle\lambda_j,\lambda_k\rangle} Z_{V_{\mathbb{Z}\zeta+\lambda_j}}(\tau). \tag{3.1.43}$$

Thus,

$$S_{\lambda_j,\lambda_k} = \frac{1}{\sqrt{32}} e^{-2\pi i\langle\lambda_j,\lambda_k\rangle},$$

where S_{λ_j,λ_k} denotes the (λ_j, λ_k)-entry of the S-matrix of $V_{\mathbb{Z}\zeta}$.

Let $(S_{j,k})$ denote the S-matrix of $V_{L_2}^{S_4}$. For $j = 18, \ldots, 25$, we have

$$S_{j,0} = \frac{1}{\sqrt{32}} = \mathrm{qdim}_{V_{L_2}^{S_4}}(M_j) \cdot S_{0,0} = 6S_{0,0}.$$

Therefore,

$$S_{0,0} = \frac{1}{6\sqrt{32}},$$

and hence

$$S_{j,0} = \frac{\mathrm{qdim}_{V_{L_2}^{S_4}}(M_j)}{6\sqrt{32}} \qquad \text{for all } j = 0, \ldots, 27.$$

Fix $M^j \cong V_{\mathbb{Z}\zeta+\lambda_l}$ for some $l \in \{0, \ldots, 31\}$. Let M^k be an irreducible $V_{L_2}^{S_4}$-module. If M^k does not appear as a submodule of any $V_{\mathbb{Z}\zeta+\lambda_s}$, $s = 0, \ldots, 31$, then

$$S_{j,k} = 0.$$

Otherwise, suppose M^k appears in

$$V_{\mathbb{Z}\zeta+\lambda_{k_1}}, \ldots, V_{\mathbb{Z}\zeta+\lambda_{k_r}}$$

with multiplicities $s_1, \ldots, s_r$, respectively. Then

$$S_{j,k} = \sum_{i=1}^{r} s_i\, S_{\lambda_l,\lambda_{k_i}}.$$

The same method applies to compute the S-matrix entries involving $M^{12} - M^{17}$ and M^{26}, M^{27}.

□

By Proposition 3.1.17, $V^{\pm}_{\mathbb{Z}\beta+\frac{\beta}{8}}$ and $V^{\pm}_{\mathbb{Z}\beta+\frac{3\beta}{8}}$ can also be computed in this way. The next lemma calculates the S-matrix entries related to $(V^{-}_{\mathbb{Z}\beta})^{\pm}$.

Lemma 3.1.22.

$$\begin{aligned}
Z_{(V^{-}_{\mathbb{Z}\beta})^{+}}(-\frac{1}{\tau}) \;=\;& \frac{1}{2\sqrt{32}}Z_{((V^{+}_{\mathbb{Z}\beta})^{0})^{+}}(\tau) + \frac{1}{2\sqrt{32}}Z_{((V^{+}_{\mathbb{Z}\beta})^{0})^{-}}(\tau) + \frac{1}{\sqrt{32}}Z_{(V^{+}_{\mathbb{Z}\beta})^{1}}(\tau) \\
&+\frac{3}{2\sqrt{32}}Z_{(V^{-}_{\mathbb{Z}\beta})^{+}}(\tau) + \frac{3}{2\sqrt{32}}Z_{(V^{-}_{\mathbb{Z}\beta})^{-}}(\tau) + \frac{1}{\sqrt{32}}Z_{(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+}}(\tau) \\
&+\frac{1}{\sqrt{32}}Z_{(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-}}(\tau) + \frac{2}{\sqrt{32}}Z_{(V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+}}(\tau) \\
&-\frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-}}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+}}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}}(\tau) \\
&+\frac{1}{\sqrt{32}}\sum_{s=1,s\not\equiv 0 \pmod 2}^{15} Z_{V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}}(\tau) - \frac{2}{\sqrt{32}}Z_{V^{T_2,+}_{\mathbb{Z}\zeta}}(\tau) - \frac{2}{\sqrt{32}}Z_{V^{T_2,-}_{\mathbb{Z}\zeta}}(\tau).
\end{aligned} \tag{3.1.44}$$

$$\begin{aligned}
Z_{(V^{-}_{\mathbb{Z}\beta})^{-}}(-\frac{1}{\tau}) \;=\;& \frac{1}{2\sqrt{32}}Z_{((V^{+}_{\mathbb{Z}\beta})^{0})^{+}}(\tau) + \frac{1}{2\sqrt{32}}Z_{((V^{+}_{\mathbb{Z}\beta})^{0})^{-}}(\tau) + \frac{1}{\sqrt{32}}Z_{(V^{+}_{\mathbb{Z}\beta})^{1}}(\tau) \\
&+\frac{3}{2\sqrt{32}}Z_{(V^{-}_{\mathbb{Z}\beta})^{+}}(\tau) + \frac{3}{2\sqrt{32}}Z_{(V^{-}_{\mathbb{Z}\beta})^{-}}(\tau) + \frac{1}{\sqrt{32}}Z_{(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+}}(\tau) \\
&+\frac{1}{\sqrt{32}}Z_{(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-}}(\tau) + \frac{2}{\sqrt{32}}Z_{(V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}})}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+}}(\tau) \\
&-\frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-}}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+}}(\tau) - \frac{1}{\sqrt{32}}Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}}(\tau) \\
&-\frac{1}{\sqrt{32}}\sum_{s=1,s\not\equiv 0 \pmod 2}^{15} Z_{V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}}(\tau) + \frac{2}{\sqrt{32}}Z_{V^{T_2,+}_{\mathbb{Z}\zeta}}(\tau) + \frac{2}{\sqrt{32}}Z_{V^{T_2,-}_{\mathbb{Z}\zeta}}(\tau).
\end{aligned} \tag{3.1.45}$$

Proof. Let us start with the proof of (3.1.44). By Proposition 3.1.17, we have $(V_{\mathbb{Z}\beta}^-)^+ \cong V_{\mathbb{Z}\zeta}^-$ as $V_{L_2}^{S_4}$-module. So,

$$Z_{V_{\mathbb{Z}\zeta}}(-\frac{1}{\tau}) = Z_{V_{\mathbb{Z}\zeta}^+}(-\frac{1}{\tau}) + Z_{V_{\mathbb{Z}\zeta}^-}(-\frac{1}{\tau}) = \sum_{j=0}^{31} \frac{1}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta+\frac{j}{32}\zeta}}(\tau). \tag{3.1.46}$$

By Theorems 2.6.7 and 2.6.8, we can calculate the quantum dimension of $V_{\mathbb{Z}\zeta}^+$-modules. $V_{\mathbb{Z}\zeta}^{\pm}, V_{\zeta+\frac{\zeta}{2}}^{\pm}$ have quantum dimensions 1, $V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}$ have quantum dimensions 2 for ($s = 1, \ldots, 15$) and $V_{\mathbb{Z}\zeta}^{T_1,\pm}, V_{\mathbb{Z}\zeta}^{T_2,\pm}$ have quantum dimensions 4. Applying the technique from the previous lemma, we know $S_{0,0} = \frac{1}{2\sqrt{32}}$. Thus, we have

$$\begin{aligned} Z_{V_{\mathbb{Z}\zeta}^+}(-\frac{1}{\tau}) &= \frac{1}{2\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^+}(\tau) + \frac{1}{2\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^-}(\tau) + \frac{1}{2\sqrt{32}} Z_{V^+_{\mathbb{Z}\zeta+\frac{\zeta}{2}}}(\tau) + \frac{1}{2\sqrt{32}} Z_{V^-_{\mathbb{Z}\zeta+\frac{\zeta}{2}}}(\tau) \\ &\quad + \sum_{j=1}^{15} \frac{1}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta+\frac{j}{32}\zeta}}(\tau) + \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_1,+}}(\tau) + \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_1,-}}(\tau) + \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_2,+}}(\tau) \\ &\quad + \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_2,-}}(\tau). \end{aligned} \tag{3.1.47}$$

Then, we have

$$\begin{aligned} Z_{V_{\mathbb{Z}\zeta}^-}(-\frac{1}{\tau}) &= \frac{1}{2\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^+}(\tau) + \frac{1}{2\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^-}(\tau) + \frac{1}{2\sqrt{32}} Z_{V^+_{\mathbb{Z}\zeta+\frac{\zeta}{2}}}(\tau) + \frac{1}{2\sqrt{32}} Z_{V^-_{\mathbb{Z}\zeta+\frac{\zeta}{2}}}(\tau) \\ &\quad + \sum_{j=1}^{15} \frac{1}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta+\frac{j}{32}\zeta}}(\tau) - \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_1,+}}(\tau) - \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_1,-}}(\tau) - \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_2,+}}(\tau) \\ &\quad - \frac{2}{\sqrt{32}} Z_{V_{\mathbb{Z}\zeta}^{T_2,-}}(\tau). \end{aligned} \tag{3.1.48}$$

Combining this with Proposition 3.1.17, we obtain (3.1.44). The proof of (3.1.45) is similar. We can compute the quantum dimensions of $V_{\mathbb{Z}\beta}^+$ modules. $V_{\mathbb{Z}\beta}^{\pm}, V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{\pm}$ have quantum dimensions 1, and $V_{\mathbb{Z}\beta+\frac{s\beta}{8}}, V_{\mathbb{Z}\beta}^{T_1,\pm}, V_{\mathbb{Z}\beta}^{T_2,\pm}$ have quantum dimensions 2. We have

$$Z_{V_{\mathbb{Z}\beta}}(-\frac{1}{\tau}) = Z_{V_{\mathbb{Z}\beta}^+}(-\frac{1}{\tau}) + Z_{V_{\mathbb{Z}\beta}^-}(-\frac{1}{\tau}) = \sum_{i=0}^{7} \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta+\frac{i\beta}{8}}}(\tau) \tag{3.1.49}$$

and

$$\begin{aligned} Z_{V_{\mathbb{Z}\beta}^+}(-\frac{1}{\tau}) &= \frac{1}{2\sqrt{8}}(Z_{V_{\mathbb{Z}\beta}^+}(\tau) + Z_{V_{\mathbb{Z}\beta}^-}(\tau) + Z_{V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}}(\tau) + Z_{V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}}(\tau)) + \frac{1}{\sqrt{8}}\sum_{i=1}^{3} Z_{V_{\mathbb{Z}\beta+\frac{i}{8}\beta}}(\tau) \\ &+\frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_1,+}}(\tau) + \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_1,-}}(\tau) + \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_2,+}}(\tau) + \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_2,-}}(\tau). \end{aligned} \tag{3.1.50}$$

Thus,

$$\begin{aligned} Z_{V_{\mathbb{Z}\beta}^-}(-\frac{1}{\tau}) &= \frac{1}{2\sqrt{8}}(Z_{V_{\mathbb{Z}\beta}^+}(\tau) + Z_{V_{\mathbb{Z}\beta}^-}(\tau) + Z_{V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}}(\tau) + Z_{V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}}(\tau)) + \frac{1}{\sqrt{8}}\sum_{i=1}^{3} Z_{V_{\mathbb{Z}\beta+\frac{i}{8}\beta}}(\tau) \\ &-\frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_1,+}}(\tau) - \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_1,-}}(\tau) - \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_2,+}}(\tau) - \frac{1}{\sqrt{8}} Z_{V_{\mathbb{Z}\beta}^{T_2,-}}(\tau). \end{aligned} \tag{3.1.51}$$

By Proposition 3.1.17, we have

$$\begin{aligned} Z_{V_{\mathbb{Z}\beta}^-}(-\frac{1}{\tau}) &= \frac{1}{2\sqrt{8}}(Z_{((V_{\mathbb{Z}\beta}^+)^0)^+}(\tau) + Z_{((V_{\mathbb{Z}\beta}^+)^0)^-}(\tau) + 2Z_{(V_{\mathbb{Z}\beta}^+)^1}(\tau)) + \frac{3}{2\sqrt{8}}(Z_{(V_{\mathbb{Z}\beta}^-)^+}(\tau) + Z_{(V_{\mathbb{Z}\beta}^-)^-}(\tau)) \\ &+\frac{1}{\sqrt{8}}(Z_{(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+}(\tau) + Z_{(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^-}(\tau) + 2Z_{V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}}(\tau)) \\ &+\frac{1}{\sqrt{8}}(Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+}(\tau) + Z_{(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^-}(\tau) + Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+}(\tau) + Z_{(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-}(\tau)) \end{aligned} \tag{3.1.52}$$

Subtracting (3.1.44), we obtain (3.1.45). □

§ 3.2 Fusion Rules for $V_{L_2}^{S_4}$

In this section, we compute the full fusion rules of $V_{L_2}^{S_4}$-modules. The indices of the modules follow from Theorem 3.1.15. Throughout this section, we use 0 to denote +

and 1 to denote $-$ when appropriate. For example, $(V_{\mathbb{Z}\beta}^{-})^0$ denotes $(V_{\mathbb{Z}\beta}^{-})^+$, and $(V_{\mathbb{Z}\beta}^{-})^1$ denotes $(V_{\mathbb{Z}\beta}^{-})^-$.

Lemma 3.2.1. *Every irreducible $V_{L_2}^{S_4}$-module is self-dual.*

Proof. By the construction of the contragredient module, we know that a module W and its contragredient module W' have the same conformal weight. On the other hand, W and W' have the same q-character, since the weight-λ subspace of W' is the dual space of the weight-λ subspace of W, and each weight space of W is finite-dimensional. Therefore, W and W' have the same quantum dimension.

By comparing the conformal weights and the quantum dimensions of each irreducible $V_{L_2}^{S_4}$-module in Theorem 3.1.15, we conclude that every irreducible $V_{L_2}^{S_4}$-module is self-dual. $\square$

Theorem 3.2.2.

(3.2.1)

$$(1).\quad (V^{j}_{\mathbb{Z}\beta+\frac{i\beta}{8}}) \boxtimes (V_{\mathbb{Z}\gamma+\frac{r}{18}\gamma}) = \bigoplus_{\substack{1\leq t\leq 8\\ t\not\equiv 0 \pmod 3}} V_{\mathbb{Z}\gamma+\frac{t\gamma}{18}},$$

$$(i = 1, 3,\ j = 0, 1,\ 1 \leq r \leq 8,\ r \not\equiv 0 \pmod 3).$$

(3.2.2)

(2).

$$M_8 \boxtimes M_{18} = M_{18} \oplus M_{19} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_8 \boxtimes M_{19} = M_{18} \oplus M_{20} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-},$$

$$M_8 \boxtimes M_{20} = M_{19} \oplus M_{21} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_8 \boxtimes M_{21} = M_{20} \oplus M_{22} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_8 \boxtimes M_{22} = M_{21} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_8 \boxtimes M_{23} = M_{22} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_8 \boxtimes M_{24} = M_{23} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_8 \boxtimes M_{25} = M_{24} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_9 \boxtimes M_{18} = M_{24} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_9 \boxtimes M_{19} = M_{23} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_9 \boxtimes M_{20} = M_{22} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_9 \boxtimes M_{21} = M_{21} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_9 \boxtimes M_{22} = M_{20} \oplus M_{22} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_9 \boxtimes M_{23} = M_{19} \oplus M_{21} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_9 \boxtimes M_{24} = M_{18} \oplus M_{20} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_9 \boxtimes M_{25} = M_{18} \oplus M_{19} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{10} \boxtimes M_{18} = M_{20} \oplus M_{21} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{10} \boxtimes M_{19} = M_{19} \oplus M_{22} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{10} \boxtimes M_{20} = M_{18} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{10} \boxtimes M_{21} = M_{18} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{10} \boxtimes M_{22} = M_{19} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{10} \boxtimes M_{23} = M_{20} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{10} \boxtimes M_{24} = M_{21} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{10} \boxtimes M_{25} = M_{22} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{11} \boxtimes M_{18} = M_{22} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{11} \boxtimes M_{19} = M_{21} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{11} \boxtimes M_{20} = M_{20} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{11} \boxtimes M_{21} = M_{19} \oplus M_{25} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{11} \boxtimes M_{22} = M_{18} \oplus M_{24} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{11} \boxtimes M_{23} = M_{18} \oplus M_{23} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$M_{11} \boxtimes M_{24} = M_{19} \oplus M_{22} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, M_{11} \boxtimes M_{25} = M_{20} \oplus M_{21} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}$$

$$(3) \quad \left(V^j_{\mathbb{Z}\beta+\frac{i\beta}{8}}\right) \boxtimes V_{\mathbb{Z}\zeta}^{T_2,k} = \bigoplus_{\substack{1\le s\le 15 \\ s\not\equiv 0 \pmod 2}} V_{\frac{s\zeta}{32}+\mathbb{Z}\zeta} \oplus V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-}, \quad (i = 1, 3, \quad j = 0, 1, \quad k = 0, 1). \tag{3.2.3}$$

(4). For the fusion rules of type $M_i \boxtimes M_j$ *with* $12 \le i \le 17, 18 \le j \le 25$.

$$M_{12} \boxtimes M_{18} = M_{12} \boxtimes M_{21} = M_{12} \boxtimes M_{22} = M_{12} \boxtimes M_{25} = M_{13} \boxtimes M_{19}$$

$$= M_{13} \boxtimes M_{20} = M_{13} \boxtimes M_{23} = M_{13} \boxtimes M_{24} = M_{14} \boxtimes M_{19}$$

$$= M_{14} \boxtimes M_{20} = M_{14} \boxtimes M_{23} = M_{14} \boxtimes M_{24} = M_{15} \boxtimes M_{18}$$

$$= M_{15} \boxtimes M_{21} = M_{15} \boxtimes M_{22} = M_{15} \boxtimes M_{25} = M_{16} \boxtimes M_{18}$$

$$= M_{16} \boxtimes M_{21} = M_{16} \boxtimes M_{22} = M_{16} \boxtimes M_{25} = M_{17} \boxtimes M_{19}$$

$$= M_{17} \boxtimes M_{21} = M_{17} \boxtimes M_{22} = M_{17} \boxtimes M_{25}$$

$$= M_{18} \oplus M_{21} \oplus M_{22} \oplus M_{25} \oplus M_{26} \oplus M_{27}. \tag{3.2.4}$$

All other products of this type are equal to: $M_{19} \oplus M_{20} \oplus M_{23} \oplus M_{24} \oplus M_{26} \oplus M_{27}$.

$$\begin{aligned}(5) \quad V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}} \boxtimes V_{\mathbb{Z}\zeta}^{T_2,j} = &\bigoplus_{\substack{1\le s\le 15\\ s\not\equiv 0 \pmod 2}} V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}\\ &\oplus 2V_{\mathbb{Z}\zeta}^{T_2,+} \oplus 2V_{\mathbb{Z}\zeta}^{T_2,-},\\ &(1 \le r \le 8, \quad r \not\equiv 0 \pmod 3, \quad j = 0, 1).\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}(6) \quad V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} \boxtimes V_{\mathbb{Z}\zeta}^{T_2,j} = &\bigoplus_{\substack{1\le r\le 8\\ r\not\equiv 0 \pmod 3}} V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}}\\ &\oplus \bigoplus_{\substack{a=1,3\\ t=0,1}} V^{t}_{\frac{a\beta}{8}+\mathbb{Z}\beta},\\ &(1 \le s \le 15, \quad s \not\equiv 0 \pmod 2)).\end{aligned} \tag{3.2.6}$$

$$(7). \quad (V_{\mathbb{Z}\beta}^{-})^{i} \boxtimes V_{\mathbb{Z}\zeta}^{T_2,j} = V_{\mathbb{Z}\zeta}^{T_2,+} \oplus 2V_{\mathbb{Z}\zeta}^{T_2,-} \text{ for } i = j, (i, j = 0, 1). \tag{3.2.7}$$

$$(8). \quad (V_{\mathbb{Z}\beta}^{-})^{i} \boxtimes V_{\mathbb{Z}\zeta}^{T_2,j} = 2V_{\mathbb{Z}\zeta}^{T_2,+} \oplus V_{\mathbb{Z}\zeta}^{T_2,-} \text{ for } i \neq j, (i, j = 0, 1). \tag{3.2.8}$$

Proof. This follows directly from Theorem 2.5.4, Lemma 3.1.21, Lemma 3.1.22. □

We use the following conventions in the next theorem. Let $i, j \in \mathbb{Z}$. We write $\overline{i+j}$

for $i+j$ mod 2 and $\overline{\overline{i+j}}$ for $i+j$ mod 3. By [DJJJY] and [WZ], we have the following identifications as $V_{L_2}^{S_4}$-modules:

$$W_{\delta,1}^{0} \cong V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \tag{3.2.9}$$

$$W_{\delta,2}^{0} \cong V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \tag{3.2.10}$$

$$W_{\delta,2}^{1} \cong V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \tag{3.2.11}$$

$$W_{\delta,1}^{1} \cong V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \tag{3.2.12}$$

$$W_{\delta,1}^{2} \cong V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \tag{3.2.13}$$

$$W_{\delta,2}^{2} \cong V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \tag{3.2.14}$$

Theorem 3.2.3.

$$(1).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes ((V_{\mathbb{Z}\beta}^{+})^{0})^{j} = ((V_{\mathbb{Z}\beta}^{+})^{0})^{\overline{i+j}}, (i,j=0,1). \tag{3.2.15}$$

$$(2).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes (V_{\mathbb{Z}\beta}^{+})^{1} = (V_{\mathbb{Z}\beta}^{+})^{1}, (i=0,1). \tag{3.2.16}$$

$$(3).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes (V_{\mathbb{Z}\beta}^{-})^{j} = (V_{\mathbb{Z}\beta}^{-})^{\overline{i+j}}, (i,j=0,1). \tag{3.2.17}$$

$$(4).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{j} = (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{\overline{i+j}}, (i,j=0,1). \tag{3.2.18}$$

(3.2.19) $$(5).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} = V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}, (i=0,1).$$

(3.2.20) $$(6).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes (V_{\mathbb{Z}\beta+\frac{j}{8}\beta})^{k} = (V_{\mathbb{Z}\beta+\frac{j}{8}\beta})^{\overline{i+k}}, (i,k=0,1, j=1,3).$$

(3.2.21) $$(7).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}}, (i=0,1).$$

(3.2.22) $$(8).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}, (i=0).$$

(3.2.23) $$(9).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{(16-s)\zeta}{32}}, (i=1).$$

(3.2.24) $$(10).\quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{i} \boxtimes V_{\mathbb{Z}\zeta}^{T_2,j} = V_{\mathbb{Z}\zeta}^{T_2,\overline{i+j}}, (i,j=0,1).$$

(3.2.25) $$(11).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes (V_{\mathbb{Z}\beta}^{+})^{1} = ((V_{\mathbb{Z}\beta}^{+})^{0})^{+} \oplus ((V_{\mathbb{Z}\beta}^{+})^{0})^{-} \oplus (V_{\mathbb{Z}\beta}^{+})^{1}.$$

(3.2.26) $$(12).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes (V_{\mathbb{Z}\beta}^{-})^{j} = (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}, (j=0,1).$$

(3.2.27) $$(13).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{i} = V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}, (i=0,1).$$

(3.2.28) $$(14).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} = (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}.$$

(3.2.29) $$(15).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes (V_{\mathbb{Z}\beta+\frac{j\beta}{8}})^{k} = (V_{\mathbb{Z}\beta+\frac{j\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{j\beta}{8}})^{-}, (j=1,3, k=0,1).$$

(3.2.30) $$(16).\quad (V_{\mathbb{Z}\beta}^{+})^{1} \boxtimes V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}.$$

$$(V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}}=V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}}\oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}. \tag{3.2.31}$$

$$(V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}}=V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}}\oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}. \tag{3.2.32}$$

$$(V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}}=V_{\mathbb{Z}\gamma+\frac{\gamma}{18}}\oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}. \tag{3.2.33}$$

$$(V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}=V_{\mathbb{Z}\gamma+\frac{\gamma}{18}}\oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}}. \tag{3.2.34}$$

$$(V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}=V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}}\oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}}. \tag{3.2.35}$$

$$(17).\quad (V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}=V_{\frac{s\zeta}{32}+\mathbb{Z}\zeta}\oplus V_{\mathbb{Z}\zeta+\frac{(16-s)\zeta}{32}}. \tag{3.2.36}$$

$$(18).\quad (V_{\mathbb{Z}\beta}^{+})^{1}\boxtimes V_{\mathbb{Z}\zeta}^{T_2,\pm}=V_{\mathbb{Z}\zeta}^{T_2,+}\oplus V_{\mathbb{Z}\zeta}^{T_2,-}. \tag{3.2.37}$$

$$\begin{aligned}(19).\quad (V_{\mathbb{Z}\beta}^{-})^{i}\boxtimes (V_{\mathbb{Z}\beta}^{-})^{j}&=((V_{\mathbb{Z}\beta}^{+})^{0})^{\overline{i+j}}\oplus (V_{\mathbb{Z}\beta}^{+})^{1}\\ &\oplus (V_{\mathbb{Z}\beta}^{-})^{+}\oplus (V_{\mathbb{Z}\beta}^{-})^{-},\qquad (i,j=0,1).\end{aligned} \tag{3.2.38}$$

$$(20).\quad (V_{\mathbb{Z}\beta}^{-})^{i}\boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{j}=(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{\overline{i+j}}\oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}. \tag{3.2.39}$$

$$(21).\quad (V_{\mathbb{Z}\beta}^{-})^{i}\boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}=(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+}\oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-}\oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}. \tag{3.2.40}$$

(3.2.41)

$$(22). \quad (V_{\mathbb{Z}\beta}^{-})^i \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^j = (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^{\overline{i+j}} \oplus V_{\mathbb{Z}\beta+\frac{(4-k)\beta}{8}}^{+} \oplus V_{\mathbb{Z}\beta+\frac{(4-k)\beta}{8}}^{-}, \qquad (k=1,3,\ j=0,1,\ i=0,1).$$

(3.2.42) (23). $(V_{\mathbb{Z}\beta}^{-})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}$, *for r odd.*

(3.2.43) $(V_{\mathbb{Z}\beta}^{-})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}$, *for r even.*

(3.2.44) (24)

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{-} \boxtimes V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{-} \boxtimes V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{-} \boxtimes V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^{-})^{-} \boxtimes V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$

$$(V_{\mathbb{Z}\beta}^-)^- \boxtimes V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$
$$(V_{\mathbb{Z}\beta}^-)^- \boxtimes V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$
$$(V_{\mathbb{Z}\beta}^-)^- \boxtimes V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}},$$
$$(V_{\mathbb{Z}\beta}^-)^- \boxtimes V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}}.$$

$$(25).\quad (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^j = ((V_{\mathbb{Z}\beta}^+)^0)^{\overline{i+j}} \oplus (V_{\mathbb{Z}\beta}^-)^{\overline{i+j}}, (i,j=0,1). \tag{3.2.45}$$

$$(26).\quad (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} = (V_{\mathbb{Z}\beta}^+)^1 \oplus (V_{\mathbb{Z}\beta}^-)^+ \oplus (V_{\mathbb{Z}\beta}^-)^-, (i=0,1). \tag{3.2.46}$$

$$(27).\quad (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ = V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- = V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ = V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^- = V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ = V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- = V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ = V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^- = V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}}. \tag{3.2.47}$$

$$(28).\quad (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}},$$

$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} = V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}. \tag{3.2.48}$$

(29). $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}}$,

$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}},$$
$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}},$$

$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}},$$

$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \boxtimes V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}}. \tag{3.2.49}$$

$$(30). \quad (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^i \boxtimes V^{T_2,j}_{\mathbb{Z}\zeta} = V^{T_2,+}_{\mathbb{Z}\zeta} \oplus V^{T_2,-}_{\mathbb{Z}\zeta}. \quad (i=0,1,\ j=0,1) \tag{3.2.50}$$

$$(31). \quad V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} = ((V^+_{\mathbb{Z}\beta})^0)^+ \oplus ((V^+_{\mathbb{Z}\beta})^0)^- \oplus (V^+_{\mathbb{Z}\beta})^1 \oplus 2(V^-_{\mathbb{Z}\beta})^+ \oplus 2(V^-_{\mathbb{Z}\beta})^-. \tag{3.2.51}$$

$$(32). \quad V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^j = \bigoplus_{k=1,3,s=0,1} (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^s. \tag{3.2.52}$$

(33).

$$\begin{aligned}
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{\gamma}{18}}) &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus 2V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}},\\
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}}) &= V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus 2V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}},\\
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}}) &= V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus 2V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}},\\
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}}) &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus 2V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}},\\
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}) &= 2V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}},\\
V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes (V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}) &= 2V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}},
\end{aligned} \tag{3.2.53}$$

$$(34). \quad V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{3\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{5\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{11\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{13\zeta}{32}}, \quad (s=1,7,9,15).$$

(3.2.54)

$$V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{7\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{9\zeta}{32}} \oplus V_{\mathbb{Z}\zeta+\frac{15\zeta}{32}}, \quad (s=3,5,11,13).$$

$$\begin{aligned}(35).\quad &(V_{\mathbb{Z}\beta+\frac{i\beta}{8}})^j \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^l \\ &= ((V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^{\overline{j+l}}) \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ &\quad \oplus ((V^+_{\mathbb{Z}\beta})^0)^{\overline{j+l}} \oplus (V^+_{\mathbb{Z}\beta})^1 \\ &\quad \oplus (V^-_{\mathbb{Z}\beta})^{\overline{j+l}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}} \\ &\quad \oplus V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}}, \qquad \textit{for } i=k=1,\ j,l=0,1.\end{aligned} \tag{3.2.55}$$

$$\begin{aligned}&(V_{\mathbb{Z}\beta+\frac{i\beta}{8}})^j \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^l \\ &= ((V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^{\overline{j+l+1}}) \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ &\quad \oplus ((V^+_{\mathbb{Z}\beta})^0)^{\overline{j+l}} \oplus (V^+_{\mathbb{Z}\beta})^1 \\ &\quad \oplus (V^-_{\mathbb{Z}\beta})^{\overline{j+l}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}} \\ &\quad \oplus V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}}, \qquad \textit{for } i=k=3,\ j,l=0,1.\end{aligned} \tag{3.2.56}$$

$$\begin{aligned}(V_{\mathbb{Z}\beta+\frac{i\beta}{8}})^j \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^l \;=\; & ((V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^{\overline{j+l}}) \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V^-_{\mathbb{Z}\beta})^+ \oplus (V^-_{\mathbb{Z}\beta})^- \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \\ & \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{3\beta}{8}},\ \textit{for } i\neq k,\ j,l=0,1.\end{aligned} \tag{3.2.57}$$

$$\begin{aligned}(36).\quad W^k_{\delta,1} \boxtimes W^l_{\delta,1} = {} & W^0_{\delta,1} \oplus W^1_{\delta,1} \oplus W^2_{\delta,1} \oplus W^{\overline{\overline{k+l}}}_{\delta,2} \\ & \oplus ((V^+_{\mathbb{Z}\beta})^0)^+ \oplus ((V^+_{\mathbb{Z}\beta})^0)^- \oplus (V^-_{\mathbb{Z}\beta})^+ \oplus (V^-_{\mathbb{Z}\beta})^- \\ & \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-, \\ & \textit{for } k=l, \quad k,l=0,1,2.\end{aligned} \tag{3.2.58}$$

$$W_{\delta,1}^{k} \boxtimes W_{\delta,1}^{l} = W_{\delta,1}^{0} \oplus W_{\delta,1}^{1} \oplus W_{\delta,1}^{2} \oplus W_{\delta,2}^{\overline{\overline{\overline{k+l}}}} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}$$
$$\oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-},\ \textit{for } k \neq l, k, l = 0, 1, 2.$$

(3.2.59)

$$W_{\delta,2}^{k} \boxtimes W_{\delta,2}^{l} = W_{\delta,1}^{0} \oplus W_{\delta,1}^{1} \oplus W_{\delta,1}^{2} \oplus W_{\delta,2}^{\overline{\overline{\overline{1-k-l}}}} \oplus ((V_{\mathbb{Z}\beta}^{+})^{0})^{+} \oplus ((V_{\mathbb{Z}\beta}^{+})^{0})^{-} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}$$
$$\oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-},\ \textit{for } k = l, k, l = 0, 1, 2.$$

(3.2.60)

$$\begin{aligned} W_{\delta,2}^{k} \boxtimes W_{\delta,2}^{l} = {} & W_{\delta,1}^{0} \oplus W_{\delta,1}^{1} \oplus W_{\delta,1}^{2} \oplus W_{\delta,2}^{\overline{\overline{\overline{1-k-l}}}} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \\ & \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}, \\ & \textit{for } k \neq l, k, l = 0, 1, 2. \end{aligned} \tag{3.2.61}$$

(3.2.62)

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} = {} & V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} \\ & \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}. \end{aligned} \tag{3.2.63}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} = {} & V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} \\ & \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}. \end{aligned} \tag{3.2.64}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} = {} & V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \\ & \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^{-} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^{-}. \end{aligned} \tag{3.2.65}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \\ &\oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.66}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus 2V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ &\oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.67}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \\ &\oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.68}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus 2V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ &\oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.69}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \\ &\oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.70}$$

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \boxtimes V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} &= V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}} \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^- \\ &\oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^- \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^+ \oplus (V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^-. \end{aligned} \tag{3.2.71}$$

Proof. We begin with the proof of (1)–(10). By Theorem 3.1.15, $((V^+_{\mathbb{Z}\beta})^0)^i$ is a simple current, and its tensor product with any irreducible module is again an irreducible $V_{L_2}^{S_4}$-module. By Corollary 3.1.18, we have

$$((V^+_{\mathbb{Z}\beta})^0)^+ \subseteq V_{\mathbb{Z}\zeta} \quad \text{and} \quad ((V^+_{\mathbb{Z}\beta})^0)^- \subseteq V_{\mathbb{Z}\zeta+\frac{\zeta}{2}}.$$

We first show that

$$((V_{\mathbb{Z}\beta}^+)^0)^+ \boxtimes ((V_{\mathbb{Z}\beta}^+)^0)^+ = ((V_{\mathbb{Z}\beta}^+)^0)^+.$$

By Theorem 2.5.9, some irreducible $V_{L_2}^{S_4}$-modules appearing in $V_{\mathbb{Z}\zeta}$ must also appear in $((V_{\mathbb{Z}\beta}^+)^0)^+ \boxtimes ((V_{\mathbb{Z}\beta}^+)^0)^+$. The equality then follows from a comparison of the quantum dimensions. The remaining cases in (1) are proved in a similar manner.

The same method applies to the proofs of (1)–(10) when combined with Corollary 3.1.18, Proposition 3.1.19, and Lemma 3.1.20.

*Proof of (*11*)*: By Theorem 2.5.9 and the fusion rules for $V_{L_2}^{A_4}$-modules [DJJJY], $(V_{\mathbb{Z}\beta}^+)^1$ must be contained in $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes (V_{\mathbb{Z}\beta}^+)^1$. From Lemma 3.2.1, we know that all irreducible $V_{L_2}^{S_4}$-modules are self-dual.

By Corollary 2.5.8, we know $\dim I_{((V_{\mathbb{Z}\beta}^+)^0)^+}\begin{pmatrix} M \\ ((V_{\mathbb{Z}\beta}^+)^0)^+ \ M \end{pmatrix} = 1$ for all irreducible $V_{L_2}^{S_4}$-modules M. Thus, by Theorem 2.5.2, $\dim I_{((V_{\mathbb{Z}\beta}^+)^0)^+}\begin{pmatrix} ((V_{\mathbb{Z}\beta}^+)^0)^+ \\ M \ M \end{pmatrix} = 1$. That is to say there is one and only one copy of $((V_{\mathbb{Z}\beta}^+)^0)^+$ in $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes (V_{\mathbb{Z}\beta}^+)^1$. By counting the quantum dimensions, we know $((V_{\mathbb{Z}\beta}^+)^0)^- \in (V_{\mathbb{Z}\beta}^+)^1 \boxtimes (V_{\mathbb{Z}\beta}^+)^1$ and get (11).

*Proof of (*12*)*: By the fusion rules for $V_{L_2}^{A_4}$-modules, we have

$$(V_{\mathbb{Z}\beta}^+)^1 \boxtimes_{A_4} V_{\mathbb{Z}\beta}^- = V_{\mathbb{Z}\beta}^-.$$

Let I be an intertwining operator of type $I_{V_{L_2}^{A_4}}\begin{pmatrix} V_{\mathbb{Z}\beta}^- \\ (V_{\mathbb{Z}\beta}^+)^1 \ V_{\mathbb{Z}\beta}^- \end{pmatrix}$. By [DL], the restriction of I yields a nonzero intertwining operator of type

$$I_{V_{L_2}^{S_4}}\begin{pmatrix} V_{\mathbb{Z}\beta}^- \\ (V_{\mathbb{Z}\beta}^+)^1 \ (V_{\mathbb{Z}\beta}^-)^j \end{pmatrix}, \qquad j = 0, 1.$$

Let $v \in (V_{\mathbb{Z}\beta}^-)^j$. Then

$$\langle u_n v | u \in (V_{\mathbb{Z}\beta}^+)^1, n \in \mathbb{Z}\rangle = V_{\mathbb{Z}\beta}^-,$$

where $u_n v$ denotes the coefficient of z^{-n-1} in $I(u, z)v$. This follows from the fact that $V_{\mathbb{Z}\beta}^-$ is an irreducible $V_{L_2}^{A_4}$-module. Consequently, $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes (V_{\mathbb{Z}\beta}^-)^j$ must contain both $(V_{\mathbb{Z}\beta}^-)^+$ and $(V_{\mathbb{Z}\beta}^-)^-$. The equality then follows by comparing the quantum dimensions of both sides.

Proof of (13): This follows from Theorem 2.5.9 and the fusion rules for $V_{L_2}^{A_4}$-modules.

Proof of (14): By the fusion rules for $V_{L_2}^{A_4}$-modules, we have $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes_{A_4} V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \cong V^2_{\mathbb{Z}\beta+\frac{\beta}{4}}$ and $V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes_{A_4} V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \cong (V_{\mathbb{Z}\beta})^- \oplus (V_{\mathbb{Z}\beta}^+)^1$. From the first identity, we know $V^2_{\mathbb{Z}\beta+\frac{\beta}{4}} \cong V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}$ as $V_{L_2}^{S_4}$-modules is in $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}$. From the second identity, we know $1 = \dim I_{V_{L_2}^{A_4}}\begin{pmatrix} (V_{\mathbb{Z}\beta}^+)^1 \\ V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} \; V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \end{pmatrix} \leq \dim I_{V_{L_2}^{S_4}}\begin{pmatrix} (V_{\mathbb{Z}\beta}^+)^1 \\ (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^\pm \; V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \end{pmatrix} = \dim I_{V_{L_2}^{S_4}}\begin{pmatrix} (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^\pm \\ (V_{\mathbb{Z}\beta}^+)^1 \; V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \end{pmatrix}$. This means $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^\pm$ must be in $(V_{\mathbb{Z}\beta}^+)^1 \boxtimes V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}$. Equality follows by counting the quantum dimensions.

Proof of (15): This follows from Theorem 2.5.9, Proposition 3.1.17, Corollary 3.1.18.

Proof of (16): We prove (3.2.30); the proofs of the remaining cases are similar. As a $V_{L_2}^{S_4}$-module, we have $V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \cong W^0_{\delta,1} \cong W^0_{\delta^2,1}$ by [WZ],[DJJJY]. By Theorem 2.5.9 and the fusion rules for $V_{L_2}^{A_4}$-modules, we see that $V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}}$, $V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}} \subseteq (V_{\mathbb{Z}\beta}^+)^1 \boxtimes V_{\mathbb{Z}\gamma+\frac{\gamma}{18}}$. The equality then follows from a comparison of the quantum dimensions.

Proof of (17): By Corollary 3.1.18, $(V_{\mathbb{Z}\beta}^+)^1$ is contained in both $V_{\mathbb{Z}\zeta}$ and $V_{\mathbb{Z}\zeta+\frac{\zeta}{2}}$. Using the fusion rules of $V_{\mathbb{Z}\zeta}$-modules, together with Theorem 2.5.9, quantum dimension arguments, and the fact that $V_{\mathbb{Z}\zeta+\frac{(16+s)\zeta}{32}} \cong V_{\mathbb{Z}\zeta+\frac{(16-s)\zeta}{32}}$ as $V_{L_2}^{S_4}$-modules, we obtain (17).

Proof of (18): This follows from ϵ-identification in Proposition 3.1.19 and the technique used in the proof of (17).

*Proof of (*19*)*: By the fusion rules for $V_{L_2}^{A_4}$-modules, we have

$$V_{\mathbb{Z}\beta}^{-} \boxtimes_{A_4} V_{\mathbb{Z}\beta}^{-} = (V_{\mathbb{Z}\beta}^{+})^0 \oplus (V_{\mathbb{Z}\beta}^{+})^1 \oplus (V_{\mathbb{Z}\beta}^{+})^2 \oplus 2V_{\mathbb{Z}\beta}^{-}.$$

By Theorem 2.5.9, the fusion product $(V_{\mathbb{Z}\beta}^{-})^i \boxtimes (V_{\mathbb{Z}\beta}^{-})^j$ must contain at least one irreducible module from $(V_{\mathbb{Z}\beta}^{+})^0$ and at least one copy of $(V_{\mathbb{Z}\beta}^{+})^1$. Since all irreducible $V_{L_2}^{S_4}$-modules are self-dual, we have

$$\left((V_{\mathbb{Z}\beta}^{+})^0\right)^{\overline{i+j}} \subseteq (V_{\mathbb{Z}\beta}^{-})^i \boxtimes (V_{\mathbb{Z}\beta}^{-})^j.$$

Moreover, by Proposition 3.1.17,

$$V_{\mathbb{Z}\beta}^{-+} \cong V_{\mathbb{Z}\zeta}^{-}, \qquad V_{\mathbb{Z}\beta}^{--} \cong V_{\mathbb{Z}\zeta+\frac{\zeta}{2}}^{-}$$

as $V_{L_2}^{S_4}$-modules. From [ADL], we have

$$V_{\mathbb{Z}\zeta}^{-} \boxtimes_{V_{\mathbb{Z}\zeta}^{+}} V_{\mathbb{Z}\zeta+\frac{\zeta}{4}} = V_{\mathbb{Z}\zeta+\frac{\zeta}{4}}. \tag{3.2.72}$$

By Corollary 3.1.18, $(V_{\mathbb{Z}\beta}^{-})^{+}$ is contained in $V_{\mathbb{Z}\zeta+\frac{\zeta}{4}}$. Applying the same technique as in the proof of (12), we obtain

$$(V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-} \subseteq (V_{\mathbb{Z}\beta}^{-})^i \boxtimes (V_{\mathbb{Z}\beta}^{-})^j.$$

Equality follows from comparing the quantum dimensions of both sides.

*Proof of (*20*)*: By the fusion rules for $V_{L_2}^{A_4}$-modules, the fusion product contains one copy of $V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}$ and at least one irreducible module from $V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0}$. By the fusion rules for

$V_{\mathbb{Z}\zeta}^{+}$-modules, there exists an intertwining operator I of type

$$I_{V_{\mathbb{Z}\zeta}^{+}}\begin{pmatrix} V_{\mathbb{Z}\zeta+\frac{3\zeta}{8}} \\ V_{\mathbb{Z}\zeta}^{-} \quad V_{\mathbb{Z}\zeta+\frac{3\zeta}{8}} \end{pmatrix}.$$

Restricting I, we obtain a nonzero intertwining operator of type

$$I_{V_{L_2}^{S_4}}\begin{pmatrix} V_{\mathbb{Z}\zeta+\frac{3\zeta}{8}} \\ (V_{\mathbb{Z}\beta}^{-})^{+} \quad (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} \end{pmatrix}.$$

Therefore,

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} = (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}. \tag{3.2.73}$$

Replacing $V_{\mathbb{Z}\zeta+\frac{3\zeta}{8}}$ by $V_{\mathbb{Z}\zeta+\frac{\zeta}{8}}$, we similarly obtain

$$(V_{\mathbb{Z}\beta}^{-})^{+} \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} = (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1}. \tag{3.2.74}$$

On the other hand, consider

$$\big((V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}\big) \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{j}.$$

By the fusion rules for $V_{L_2}^{A_4}$-modules, there exists an intertwining operator I of type

$$I_{A_4}\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0} \\ V_{\mathbb{Z}\beta}^{-} \quad V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0} \end{pmatrix}.$$

Then

$$I\big((V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}, z\big)\big((V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{j}\big) = (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+} \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-}.$$

This shows that $\big((V_{\mathbb{Z}\beta}^{-})^{+} \oplus (V_{\mathbb{Z}\beta}^{-})^{-}\big) \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{j}$ contains both $(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{+}$ and $(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0})^{-}$.

Consequently, $(V_{\mathbb{Z}\beta}^-)^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^j$ and $(V_{\mathbb{Z}\beta}^-)^- \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^j$ contain different irreducible $V_{L_2}^{S_4}$-modules in $V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0$. This proves (20).

Proof of (21): By the fusion rules for $V_{L_2}^{A_4}$-modules and the techniques used above, we know that $(V_{\mathbb{Z}\beta}^-)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$ must contain $(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^+ \oplus (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^- \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$. Comparing the quantum dimensions of both sides, we see that there are four remaining dimensions to be accounted for.

By the fusion rules for $V_{L_2}^{A_4}$-modules, we have

$$(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1 \supseteq (V_{\mathbb{Z}\beta}^+)^1 \oplus (V_{\mathbb{Z}\beta}^-)^+ \oplus (V_{\mathbb{Z}\beta}^-)^-.$$

This implies

$$(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1 = (V_{\mathbb{Z}\beta}^+)^1 \oplus (V_{\mathbb{Z}\beta}^-)^+ \oplus (V_{\mathbb{Z}\beta}^-)^-.$$

Since all $V_{L_2}^{S_4}$-modules are self-dual, it follows that there is exactly one copy of $(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^i$, for $i = 0, 1$, appearing in $(V_{\mathbb{Z}\beta}^-)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$. Combining this with the previous calculations, we conclude that $((V_{\mathbb{Z}\beta}^+)^0)^i$, $(V_{\mathbb{Z}\beta}^+)^1$, and $(V_{\mathbb{Z}\beta}^-)^i$ do not appear in $(V_{\mathbb{Z}\beta}^-)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$. Therefore, the only choice is $(V_{\mathbb{Z}\beta}^-)^i \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$ contains two copies of $V_{\mathbb{Z}\beta+\frac{\beta}{4}}^1$.

Proof of (22): We prove $(V_{\mathbb{Z}\beta}^-)^+ \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{8}}^+$ only. The other cases are similar. By the fusion rules of $V_{L_2}^{A_4}$, we know it contains at least one copy of an irreducible module from $V_{\mathbb{Z}\beta+\frac{\beta}{8}}$ and two copies of irreducible modules from $V_{\mathbb{Z}\beta+\frac{3\beta}{8}}$. By Proposition 3.1.17, Corollary 3.1.18, we have $(V_{\mathbb{Z}\beta}^-)^+ \subseteq V_{\mathbb{Z}\zeta}, V_{\mathbb{Z}\zeta+\frac{\zeta}{4}}, V_{\mathbb{Z}\zeta+\frac{3\zeta}{4}}$ and $V_{\mathbb{Z}\beta+\frac{\beta}{8}}^+ \cong V_{\frac{\zeta}{16}+\mathbb{Z}\zeta}$. Thus (22) follows from the fusion rules for $V_{\mathbb{Z}\zeta}$-modules and Theorem 2.5.9.

*Proof of (*23*)*: This follows directly from Theorem 2.5.9 and the fusion rules of $V_{L_2}^{A_4}$.

*Proof of (*24*)*: This proof is similar to (17).

*Proof of (*25*)*: By the fusion rules for $V_{L_2}^{A_4}$-modules, $(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^i \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0)^j$ must contain at least one module from $V_{\mathbb{Z}\beta}^-$ and $(V_{\mathbb{Z}\beta}^+)^0$. By self-duality, it is clear that $((V_{\mathbb{Z}\beta}^+)^0)^+$ is in the tensor product if $i = j$ and $((V_{\mathbb{Z}\beta}^{+})^{0})^{-}$ is in the tensor product if $i \neq j$. By (20), it is clear that $(V_{\mathbb{Z}\beta}^{-})^{+} \subseteq (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{i} \boxtimes (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{j}$ if $i = j$ and $(V_{\mathbb{Z}\beta}^{-})^{-} \subseteq (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{i} \boxtimes (V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{j}$ if $i \neq j$.

*Proof of (*26*)*: This has already been proved in the proof of (21).

*Proof of (*27*)*: This proof is similar to (17).

*Proof of (*28*)*: This follows from the fusion rules of $V_{L_2}^{A_4}$ and Theorem 2.5.9.

*Proof of (*29*)*: This proof is similar to (17).

*Proof of (*30*)*: This proof is similar to (17).

*Proof of (*31*)*: By the fusion rules for $V_{L_2}^{A_4}$-modules and the techniques used above, the fusion product contains

$$((V_{\mathbb{Z}\beta}^{+})^{0})^{+}, \quad ((V_{\mathbb{Z}\beta}^{+})^{0})^{-}, \quad (V_{\mathbb{Z}\beta}^{-})^{+}, \quad (V_{\mathbb{Z}\beta}^{-})^{-}.$$

By (3.2.28), the tensor product contains exactly one copy of $(V_{\mathbb{Z}\beta}^{+})^{1}$. By (3.2.40), there are two copies of $(V_{\mathbb{Z}\beta}^{-})^{i}$ for $i = 0, 1$. The equality then follows by comparing the quantum dimensions of both sides.

*Proof of (*32*)*: This follows from the fusion rules for $V_{L_2}^{A_4}$-modules and the technique used in the (12).

Proof of (33) will be given later.

*Proof of (*34*)*: This proof is similar to (17).

Proof of (35): The quantum dimension of $(V_{\mathbb{Z}\beta+\frac{i\beta}{8}})^{j} \boxtimes (V_{\mathbb{Z}\beta+\frac{k\beta}{8}})^{l}$ is equal to 36.

For $i = k$, the fusion rules for $V_{L_2}^{A_4}$-modules give

$$V_{\mathbb{Z}\beta+\frac{i\beta}{8}} \boxtimes_{A_4} V_{\mathbb{Z}\beta+\frac{i\beta}{8}} = V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^{2}_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta}^{+})^{0} \oplus (V_{\mathbb{Z}\beta}^{+})^{1} \oplus (V_{\mathbb{Z}\beta}^{+})^{2}$$
$$\oplus V_{\mathbb{Z}\beta}^{-} \oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus 2V_{\mathbb{Z}\beta+\frac{3\beta}{8}},$$

and

$$V_{\mathbb{Z}\beta+\frac{\beta}{8}} \boxtimes_{A_4} V_{\mathbb{Z}\beta+\frac{3\beta}{8}} = V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^2_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus 2V^-_{\mathbb{Z}\beta}$$
$$\oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus 2V_{\mathbb{Z}\beta+\frac{3\beta}{8}}.$$

The restriction of the quantum dimensions implies that there is exactly one irreducible $V_{L_2}^{S_4}$-module arising from each of the following $V_{L_2}^{A_4}$-modules if $i = k$:

$$V^0_{\mathbb{Z}\beta+\frac{\beta}{4}},\ V^1_{\mathbb{Z}\beta+\frac{\beta}{4}},\ (V^+_{\mathbb{Z}\beta})^0,\ (V^+_{\mathbb{Z}\beta})^1,\ V^-_{\mathbb{Z}\beta},$$

and two irreducible $V_{L_2}^{S_4}$-modules arising from each of $V_{\mathbb{Z}\beta+\frac{\beta}{8}}$ and $V_{\mathbb{Z}\beta+\frac{3\beta}{8}}$.

To determine whether $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+$ or $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^-$ appears in the tensor product, we use our calculation in (27). We only compute $V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \boxtimes V^+_{\mathbb{Z}\beta+\frac{\beta}{8}}$. Since

$$(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ = V^+_{\mathbb{Z}\beta+\frac{3\beta}{8}} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{8}},$$

by self-duality we conclude that $(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+ \boxtimes (V_{\mathbb{Z}\beta+\frac{\beta}{8}})^+$ contains $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+$. The other cases are similar.

By (22), we also see that the tensor product contains $(V^-_{\mathbb{Z}\beta})^{\overline{j+l}}$ if $i = k$, and $(V^-_{\mathbb{Z}\beta})^+ \oplus (V^-_{\mathbb{Z}\beta})^-$ otherwise.

By the previous arguments, it is clear that

$$((V^+_{\mathbb{Z}\beta})^0)^{\overline{j+l}}, \quad V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}, \quad (V^+_{\mathbb{Z}\beta})^1$$

all appear in the tensor product.

It remains to determine the summands related to $V_{\mathbb{Z}\beta+\frac{\beta}{8}}$ and $V_{\mathbb{Z}\beta+\frac{3\beta}{8}}$. By [ADL], there exists an intertwining operator $I \in I_{V^+_{\mathbb{Z}\zeta}}\begin{pmatrix} & V^{T_1,+}_{\mathbb{Z}\zeta} & \\ V_{\mathbb{Z}\zeta+\frac{\zeta}{16}} & & V^{T_1,+}_{\mathbb{Z}\zeta}\end{pmatrix}$. Using the technique from the (12), we obtain that $V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \boxtimes V^+_{\mathbb{Z}\beta+\frac{\beta}{8}}$ contains $V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}}$, and $V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \boxtimes V^-_{\mathbb{Z}\beta+\frac{\beta}{8}}$ also contains $V^+_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{8}}$. The remaining cases are similar.

Combining all these results, we obtain (35).

Proof of (36): (3.2.58-3.2.62) follows directly from the S-matrix and previous calculations. We prove (3.2.63-3.2.71). The summands consisting of modules of the form $V_{\mathbb{Z}\gamma+\frac{i\gamma}{18}}$ are determined by the S-matrix. By the fusion rules for $V_{L_2}^{A_4}$-modules, we know the tensor product must contain at least one copy of $V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}, V^{\pm}_{\mathbb{Z}\beta+\frac{\beta}{8}}$, and $V^{\pm}_{\mathbb{Z}\beta+\frac{3\beta}{8}}$. After counting the quantum dimensions, a quantum dimension of 4 remains. From the fusion rules for $V_{L_2}^{A_4}$-modules, the tensor product contains $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+, (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^-$ unless $(k=0, l=2), (k=l=1)$, or $(k=2, l=0)$. For these three cases, we know $((V^+_{\mathbb{Z}\beta})^0)^{\pm}, (V^+_{\mathbb{Z}\beta})^1, (V^-_{\mathbb{Z}\beta})^{\pm}, (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^{\pm}$ are not in the tensor product from those fusion rules that are already calculated. Thus, we have no other choice.

Proof of (33): By Proposition 3.1.19, we know $V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}}$ must contain $V_{\mathbb{Z}\gamma+\frac{2\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{4\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{8\gamma}{18}}$ if r is odd and $V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \boxtimes V_{\mathbb{Z}\gamma+\frac{r\gamma}{18}}$ must contain $V_{\mathbb{Z}\gamma+\frac{\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{5\gamma}{18}} \oplus V_{\mathbb{Z}\gamma+\frac{7\gamma}{18}}$ if r is even. By (36) and self-duality, we obtain (33). Equality follows from counting the quantum dimensions.

$\square$

For $1 \leqslant i, j \leqslant 15$ with $i, j \neq 0 \pmod 2$, we divide the tensor product $V_{\mathbb{Z}\zeta+\frac{i\zeta}{32}} \boxtimes V_{\mathbb{Z}\zeta+\frac{j\zeta}{32}}$ into three parts.

We call all irreducible modules belonging to M_{12}–M_{17} *Part A*, and all irreducible modules belonging to M_8–M_{11} *Part B*. All remaining irreducible modules are said to belong to *Part C*. The full fusion rules are obtained by adding *Part A*, *Part B*, and *Part C*.

This decomposition will be convenient for describing the fusion rules for $V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}} \boxtimes$

$V_{\mathbb{Z}\zeta+\frac{s\zeta}{32}}$. For brevity, we will use (i, j) to denote the fusion product $M_i \boxtimes M_j$.

Theorem 3.2.4. *Part A is as follows:*

$$\begin{aligned}
&(18,18) = (18,21) = (18,22) = (18,25)\\
&= (19,19) = (19,20) = (19,23) = (19,24)\\
&= (20,20) = (20,23) = (20,24)\\
&= (21,21) = (21,22) = (21,25)\\
&= (22,22) = (22,25) = (23,23) = (23,24)\\
&= (24,24) = (25,25)\\
&= M_{12} \oplus M_{15} \oplus M_{16}. \qquad (3.2.75)
\end{aligned}$$

For other pairs, part A is $M_{13} \oplus M_{14} \oplus M_{17}$.

Part B:

$$\begin{aligned}
&(18,18) = (18,19) = (19,20) = (20,21)\\
&= (21,22) = (22,23) = (23,24) = (24,25)\\
&= (25,25) = M_8.
\end{aligned}$$

$$\begin{aligned}
&(18,24) = (18,25) = (19,23) = (19,25)\\
&= (20,22) = (20,24) = (21,21) = (21,23)\\
&= (22,22) = M_9.
\end{aligned}$$

$$\begin{aligned}
&(18,20) = (18,21) = (19,19) = (19,22)\\
&= (20,23) = (21,24) = (22,25) = (23,25)\\
&= (24,24) = M_{10}.
\end{aligned}$$

$$(18,22)=(18,23)=(19,21)=(19,24)$$
$$=(20,25)=(20,20)=(21,25)=(22,24)$$
$$=(23,23)=M_{11}. \tag{3.2.76}$$

Part C:

$$(18,18)=(19,19)=(20,20)=(21,21)=(22,22)=(23,23)=(24,24)=(25,25)$$
$$=((V_{\mathbb{Z}\beta}^{+})^{0})^{+}\oplus(V_{\mathbb{Z}\beta}^{+})^{1}\oplus(V_{\mathbb{Z}\beta}^{-})^{+}.$$
$$(18,19)=(18,20)=(19,21)=(20,22)=(21,23)=(22,24)=(23,25)=(24,25)$$
$$=(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{+}\oplus V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}.$$
$$(18,21)=(18,22)=(19,20)=(19,23)=(20,24)=(21,25)=(22,25)=(23,24)$$
$$=(V_{\mathbb{Z}\beta}^{-})^{+}\oplus(V_{\mathbb{Z}\beta}^{-})^{-}.$$
$$(18,23)=(18,24)=(19,22)=(19,25)=(20,21)=(20,25)=(21,24)=(22,23)$$
$$=(V^{0}_{\mathbb{Z}\beta+\frac{\beta}{4}})^{-}\oplus V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}.$$
$$(18,25)=(19,24)=(20,23)=(21,22)=((V_{\mathbb{Z}\beta}^{+})^{0})^{-}\oplus(V_{\mathbb{Z}\beta}^{+})^{1}\oplus(V_{\mathbb{Z}\beta}^{-})^{-}.$$

(3.2.77)

Proof. Parts A and B are obtained directly from the S-matrix using the Verlinde formula. Part C is derived from the self-duality and calculations in Theorem 3.2.3. □

Theorem 3.2.5.

$$(1)\quad V^{1}_{\mathbb{Z}\beta+\frac{\beta}{4}}\boxtimes V_{\mathbb{Z}\zeta}^{T_2,i}=2V_{\mathbb{Z}\zeta}^{T_2,+}\oplus 2V_{\mathbb{Z}\zeta}^{T_2,-},\ (i=0,1). \tag{3.2.78}$$

$$\begin{aligned}
(2)\quad & V_{\mathbb{Z}\zeta}^{T_2,i}\boxtimes V_{\mathbb{Z}\zeta}^{T_2,j}\\
&=\left(\bigoplus_{\substack{1\leqslant k\leqslant 8\\ k\not\equiv 0 \pmod 3}} 2V_{\mathbb{Z}\gamma+\frac{k\gamma}{18}}\right)\oplus\left(\bigoplus_{\substack{s=1,3\\ t=0,1}} V^t_{\mathbb{Z}\beta+\frac{s\beta}{8}}\right)\\
&\quad\oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+\oplus (V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^-\oplus ((V^+_{\mathbb{Z}\beta})^0)^{\overline{i+j}}\\
&\quad\oplus (V^+_{\mathbb{Z}\beta})^1\oplus 2V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}\oplus 2(V^-_{\mathbb{Z}\beta})^{\overline{i+j+1}}\oplus (V^-_{\mathbb{Z}\beta})^{\overline{i+j}},
\end{aligned}$$

$$(i,j=0,1). \tag{3.2.79}$$

Proof. We first prove (2). $V_{\mathbb{Z}\zeta}^{T_2,i}\boxtimes V_{\mathbb{Z}\zeta}^{T_2,j}$ contains

$$\left(\bigoplus_{k=1,k\neq 0 \bmod 3}^{8} 2V_{\mathbb{Z}\gamma+\frac{k\gamma}{18}}\right)\oplus\left(\bigoplus_{s=1,3,t=0,1} V^t_{\mathbb{Z}\beta+\frac{s\beta}{8}}\right)$$

by the S-matrix calculation. By Theorems 3.2.2 and 3.2.3, $V_{\mathbb{Z}\zeta}^{T_2,i}\boxtimes V_{\mathbb{Z}\zeta}^{T_2,j}$ contains exactly one copy of $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^+$, $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})^-$, $((V^+_{\mathbb{Z}\beta})^0)^{\overline{i+j}}$, $(V^+_{\mathbb{Z}\beta})^1$, $2(V^-_{\mathbb{Z}\beta})^{\overline{i+j+1}}$, $(V^-_{\mathbb{Z}\beta})^{\overline{i+j}}$. By counting the quantum dimensions, a quantum dimension of 8 remains. So, the only remaining summand is $2V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}$. This proves (2).

(1) follows from (2). □

Appendix

For convenience, in the following diagram, we will use W_j to denote $\frac{4}{3}(e^{\frac{j\pi i}{9}}+e^{-\frac{j\pi i}{9}})$ for $j=1,2,4,5,7,8$ and T_j to denote $e^{\frac{j\pi i}{8}}+e^{\frac{-j\pi i}{8}}$ for $j=1,3,5,7$ and P_j to denote $e^{\frac{j\pi i}{16}}+e^{\frac{-j\pi i}{16}}$ for $j=1,3,5,7,9,11,13,15$. The following is the part of the S-matrix for irreducible $V_{L_2}^{S_4}$-modules that we need:

Table 4: The S-matrix

$\sqrt{32}S_{i,j}$	0	8	9	10	11	12	13	14	15	16	17
0	$\frac{1}{6}$	1	1	1	1	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
1	$\frac{1}{6}$	1	1	1	1	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
2	$\frac{1}{3}$	2	2	2	2	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$
3	$\frac{1}{2}$	-1	-1	-1	-1	0	0	0	0	0	0
4	$\frac{1}{2}$	-1	-1	-1	-1	0	0	0	0	0	0
5	$\frac{1}{3}$	0	0	0	0	$\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$
6	$\frac{1}{3}$	0	0	0	0	$\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$
7	$\frac{2}{3}$	0	0	0	0	$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$
8	1	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	0	0	0	0	0	0
9	1	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	0	0	0	0	0	0
10	1	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0	0	0	0	0
11	1	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0	0	0	0	0
12	$\frac{4}{3}$	0	0	0	0	W_1	W_2	W_4	W_5	W_7	W_8
13	$\frac{4}{3}$	0	0	0	0	W_2	W_4	W_8	W_8	W_4	W_2
14	$\frac{4}{3}$	0	0	0	0	W_4	W_8	W_2	W_2	W_8	W_4
15	$\frac{4}{3}$	0	0	0	0	W_5	W_8	W_2	W_7	W_1	W_4

$\sqrt{32}S_{i,j}$	0	8	9	10	11	12	13	14	15	16	17
16	$\frac{4}{3}$	0	0	0	0	W_7	W_4	W_8	W_1	W_5	W_2
17	$\frac{4}{3}$	0	0	0	0	W_8	W_2	W_4	W_4	W_2	W_8
18	1	T_1	T_7	T_3	T_5	0	0	0	0	0	0
19	1	T_3	T_5	T_7	T_1	0	0	0	0	0	0
20	1	T_5	T_3	T_1	T_7	0	0	0	0	0	0
21	1	T_7	T_1	T_5	T_3	0	0	0	0	0	0
22	1	T_7	T_1	T_5	T_3	0	0	0	0	0	0
23	1	T_5	T_3	T_1	T_7	0	0	0	0	0	0
24	1	T_3	T_5	T_7	T_1	0	0	0	0	0	0
25	1	T_1	T_7	T_3	T_5	0	0	0	0	0	0
26	2	0	0	0	0	0	0	0	0	0	0
27	2	0	0	0	0	0	0	0	0	0	0

$\sqrt{32}S_{i,j}$	18	19	20	21	22	23	24	25	26	27
0	1	1	1	1	1	1	1	1	2	2
1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2
2	0	0	0	0	0	0	0	0	0	0
3	1	1	1	1	1	1	1	1	-2	-2
4	-1	-1	-1	-1	-1	-1	-1	-1	2	2
5	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0
6	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0
7	0	0	0	0	0	0	0	0	0	0
8	T_1	T_3	T_5	T_7	T_7	T_5	T_3	T_1	0	0
9	T_7	T_5	T_3	T_1	T_1	T_3	T_5	T_7	0	0
10	T_3	T_7	T_1	T_5	T_5	T_1	T_7	T_3	0	0
11	T_5	T_1	T_7	T_3	T_3	T_7	T_1	T_5	0	0
12	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0
18	P_1	P_3	P_5	P_7	P_9	P_{11}	P_{13}	P_{15}	0	0

$\sqrt{32}S_{i,j}$	18	19	20	21	22	23	24	25	26	27
19	P_3	P_9	P_{15}	P_{11}	P_5	P_1	P_7	P_{13}	0	0
20	P_5	P_{15}	P_7	P_3	P_{13}	P_9	P_1	P_{11}	0	0
21	P_7	P_{11}	P_3	P_{15}	P_1	P_{13}	P_5	P_9	0	0
22	P_9	P_5	P_{13}	P_1	P_{15}	P_3	P_{11}	P_7	0	0
23	P_{11}	P_1	P_9	P_{13}	P_3	P_7	P_{15}	P_5	0	0
24	P_{13}	P_7	P_1	P_5	P_{11}	P_{15}	P_9	P_3	0	0
25	P_{15}	P_{13}	P_{11}	P_9	P_7	P_5	P_3	P_1	0	0
26	0	0	0	0	0	0	0	0	$2\sqrt{2}$	$-2\sqrt{2}$
27	0	0	0	0	0	0	0	0	$-2\sqrt{2}$	$2\sqrt{2}$

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