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Explaining Infinite Series – An Exploration of Students' Images

By

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A dissertation submitted in partial satisfaction of the
requirements for the degree of

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in

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in the

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University of California, Berkeley

Committee in Charge:

Professor Alan H. Schoenfeld, chair
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Abstract

Explaining Infinite Series: An Exploration of Students' Images

by

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This study uses self-generated representations (SGR) – images produced in the act of explaining – as a means of uncovering what university calculus students understand about infinite series convergence. It makes use of student teaching episodes, in which students were asked to explain to a peer what that student might have missed had they been absent from class on the day(s) when infinite series were introduced and discussed. These student teaching episodes typically resulted in the spontaneous generation of several SGR, which provided physical referents with which both the student and an interviewer were able to interact. Students' explanations, via their SGR, included many more aspects of what they found important about that content than did the standard research technique of asking students to answer specific mathematics tasks.

This study was specifically designed to address how students construct an understanding of infinite series. It also speaks to the broader goal of examining how students use SGR as a tool for explaining concepts, rather than simply as tools for solving specific problems. The main analysis indicates that both students and their professors/textbook, when introducing the topic of infinite series, make use of the following five different image types: plots of terms, plots of partial sums, areas under curves, geometric shapes, and number lines. However, the aspects of the mathematical concepts that the students and the professors/textbooks highlight in their explanations and modes of use for those image types are different, and at times conflicting. In particular, differences emerged along three dimensions of competence – limiting processes (Tall, 1980), language, and connections.

While students using SGR generated many of the images that had been used by their professors, the limiting processes that they discussed via those images contrasted sharply. The professors and textbook chapter prioritized the limiting processes represented in particular image types to support mathematically sound conclusions. In contrast, many student explanations focused on limiting processes that did not lead to valid arguments about series convergence. There were also differences in use of language, in that students often assumed much more meaning than was intended in their professors' language choices, leading to problems with their explanations. Finally, while the experts connected their representations in meaningful ways, using other images to clarify or exemplify those that were used to define, students connected

their understanding in different ways that were not always supportive of the convergence arguments that they were trying to make.

This study expands the literature on students' understanding of infinite series topics, pointing to gaps in student understanding and ways in which students mis-applied what teachers had presented. In doing so, it suggests many avenues for improving infinite series instruction. In addition, the methods employed in this study are general, and open up ways of looking at student thinking that can be applied to many problematic areas in the curriculum. Typical studies ask students to address tasks and issues framed by a researcher. This study instead asked students to *explain* the content, thereby providing a much larger window into "what counts" from the student perspective.

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I. INTRODUCTION

Brief preview – the goals of this study. This study uses self-generated representations (SGR) – images produced in the act of explaining – as a means to uncover what university calculus students understand about the topic of infinite series convergence, when confronted with the task of *explaining* the topic. A study of what students understand about infinite series of numbers and the meaning of infinite series convergence – often a central portion of the second-semester calculus curriculum – will inform two goals. First, it will address the content-related goal of learning about how students construct an understanding of infinite series, as evidenced by their explanations and SGR. Second, it will speak to the broader goal of examining how students use their SGR as a tool for explaining, rather than simply as problem-specific diagrams to aid in solving isolated infinite series problems.

More specifically, analyzing the range of images that students produce when explaining the topic of infinite series, and then further breaking down what those images and explanations reveal about student understanding along several related dimensions, will address the following questions:

What do students' explanations via their SGR tell us about their understanding of this particular content? How do the images that they produce highlight the limiting processes (limit of terms, limit of partial sums, etc...) that students find important when explaining convergence of infinite series?

Three dimensions of student understanding, which differ from expert understandings, arise from an analysis of students' explanations and SGR. These dimensions provide lenses through which to explore student competency with the following aspects of infinite series topics:

- (a) **Limiting Processes and Objects** – To what mathematical objects do students apply limit reasoning, and how? What are the consequences of this application?
- (b) **Language** – How do students use language associated with limits in ways that are consistent or inconsistent with both the content of infinite series, and the more normative ways of communicating mathematics in this content domain?
- (c) **Connections** – In what ways do students use their SGR and language to connect the limiting processes that they find important, toward making valid mathematical conclusions about infinite series of numbers?

Exploring students' uses of SGR, more generally, may also illuminate different aspects of student explanations than asking them to work through a collection of problems would. For example, asking students to *explain* a mathematical topic provides greater latitude for students than asking them to solve problems, thus potentially demonstrating that one can gain different perspectives about what students understand than by simply posing them problems. That is, this study can also speak to the following broad theme, related to students' use of SGR:

What does the process of generating SGR tell us about how students use this practice to **explain**, rather than simply solve particular, isolated math tasks?

It is already established in the literature that students find infinite series difficult to understand (e.g. Alcock & Simpson 2004/2005; Habre, 2009, and more). The objective of this study is not simply to reaffirm this statement, nor to clarify particular aspects of this topic with which students struggle. Rather, this study seeks to identify what calculus students take away from their time spent learning this difficult topic. This study will reveal various dimensions of student proficiency with infinite series of numbers and varying levels of student proficiency within those dimensions.

Establishing need. There is a consensus that there is a need for students to master infinite series; indeed, that it may be the most important topic that students can understand from second semester calculus if they are preparing for a future in engineering, physics, and other related fields (e.g. Alcock & Simpson, 2009; Tall & Schwarzenberger 1978). Additionally, it has been documented that students are underprepared – failing to grasp the mathematical essence of the topic, unable to complete simple tasks involving infinite series, and naming it as the most difficult topic in second semester calculus (e.g. Monaghan 2001; Nardi, Biza & González-Martín 2008).

Literature suggests an unbalanced treatment of related material, as “sequences are played down, or even omitted, whilst Taylor series, geometric series, series expansions for the exponential, sine, cosine, etc are a fundamental part of sixth form work,” (Tall & Schwarzenberger 1978). Alcock and Simpson (2005) find that even when presented with a definition of convergence of an infinite series and asked to paraphrase it, directly following a unit of instruction on the topic, students differ in their descriptions of what it means for a series to converge, with a large percentage providing a mathematically incorrect description. Habre’s (2009) findings complement this, demonstrating that even multiple exposures to Taylor series do not often result in better performance on related tasks. Thus, it is not surprising that the literature documents a very wide range of ways that students think about “converge” (Monaghan 1991), in its broadest sense.

Tall and Schwarzenberger (1978) assert that series are “one of the most serious anomalies of the school syllabus” (page 11). Highlighting the idea that the notion of sequence is more fundamental than the notion of series, the authors declare that, because every series can be understood most conveniently as the limit of finite sums, “there is a strong case for banning the use of the word ‘series’ altogether!” Widely used textbooks in many college classes present series in ways that provide few opportunities for sense-making and make few explicit connections back to prior topics, instead treating series as something highly procedural. The typical approach is that series are held up against a battery of ‘convergence tests,’ which are usually applied in mechanical fashion (e.g. Stewart, 2009, section 11.7)

Although the importance and difficulty of this topic generally acknowledged, the field has not yet reached a consensus regarding what students should know about infinite series, upon exiting calculus. There do not exist clear guidelines for what it means to understand infinite series, or established instructional pathways to lead to success in this content area. In order to create well-informed instruction, with a research basis firmly grounded in and understanding of what students find difficult *and* a knowledge of how students reason with this mathematics, there must be both clear goals for what we wish students to achieve *and* a reasonable understanding of what students’ prior knowledge looks like. Thus, it will be important to identify and unpack the important characteristics of what it means to be proficient in this domain (that is, what are goals

for student performance?), and to examine students' reasoning, toward building instruction that is meaningful to students and leads them to proficiency.

Situating this study in the current mathematics education landscape. Although there is a decades-old tradition of studies underscoring the importance of studying students' understanding of limits (e.g. Cornu, 1992; Williams, 1991; Sierpiska, 1987; and more), there is little research that explores content areas, such as infinite series, that put that limit understanding into practice. Martin (2009) began to shift limit research in this direction by studying "expert/novice" understanding of Taylor and power series. Through participant responses to a variety of math tasks and survey questions, Martin demonstrated some of the differences between expert (mathematician/graduate student) and calculus student understanding. Of particular interest is his finding that experts most often appealed to dynamic partial sum and remainder arguments, while novices more frequently appealed to termwise arguments. Typical calculus instruction related to sequences and series covers sequences of numbers, then series of numbers, then power and Taylor series, in that order. Martin's finding therefore suggests that expert understanding is related to the transition period between series of numbers and power series (where partial sum and remainder are foundational), while students' more inconsistent and incorrect usage of concepts relates back to limit (limits of sequences of terms). Thus, one way to target students' understanding of this difficult domain is to study their understanding of infinite series of *numbers*, as this is one of the curricular breaking points that sets expert reasoning apart from student reasoning.

Additionally, Martin found that both experts and novices used a variety of images and modalities when reasoning with Taylor series. However, experts were fluent with a greater number of them, and were much more reliable than novices in correctly constructing and interpreting graphical images. Based on participants' usage of and fluency with the different modalities that were used when responding to Taylor series tasks, Martin concluded that the graphical understanding of Taylor series may be the most notable form of representation separating novices from experts. In discussing implications of his study, Martin also suggests that graphical imagery may be the modality that is most easily influenced through shifts in Taylor series instruction.

In Martin's study, a majority of both experts and novices appealed to images (most often graphical) when working through his tasks. This suggests that his participants found something about producing SGR to be particularly useful and/or illustrative when working in this content area, contributing something beyond what their words could express. However, exploring the various ways that experts and novices used their images differently, and the implications of their image choice on the accompanying explanations, was not the focus of his 2009 study. For these reasons, the present study gives explicit attention to the SGR people produce when explaining infinite series topics, in an attempt to unpack the specific ways in which students' image use demonstrates their varying levels of understanding.

In sum, the findings of Martin's study identified some of the features of students' and experts' problem solving with Taylor series that differ, and attempted to trace some of them back to instruction. However, the particular ways in which "experts" connect their understandings differently than "novices" is an open question. The consequences of those differences are also unexplored. Additionally, given the implication of Martin's work that many of the ways that students construct and use their understanding of Taylor series can be traced back to understanding of more basic limit ideas, it is important to start with the vast body of limit

literature and characterize the aspects of students' understanding of basic limiting processes that are consequential in the new content domain of infinite series. Thus, the present study explicitly focuses on this identified curricular breaking point - infinite series *of numbers*. It specifically makes use of students' SGR as a means of unpacking student proficiency, in ways that are richer than the literature to date (which has largely used problem sets to show where students have difficulties).

Preview of Chapters. The study proceeds in eight chapters:

I. This introduction.

II. Background – A distillation of the literature that drives such an exploration, including what is known both about students' understanding of limits, infinity, and infinite series in mathematics *and* about how students use images to explain and problem solve with limit concepts.

III. Methods and Data Collection – In brief, data for this study are in the form of student interviews, during which participants were asked to explain mathematical concepts, which often spontaneously led to their producing SGR. Guidelines for these interviews and information about the students who participated in them are included in this chapter.

IV. Content Analysis – A general analysis of the student interviews, with emphasis on their choices of what mathematical content to represent in their SGR (à la Parnafes 2009) and patterns of reasoning associated with different image types (à la Kindfield, 1993). This includes textbook and calculus instructor uses of those same image types, with the goal of identifying what comprises competence in this mathematical domain. The analysis reveals three major dimensions of competence – limiting processes (elaborated in Chapter V), language (Chapter VI), and connections (Chapter VII).

V. Limiting Processes – A characterization of the different processes to which students applied limit reasoning (e.g. sequence of terms, sequence of partial sums, etc...) and some of their ideas about the meaning of series convergence that result from focusing on the different processes. In addition to a general discussion of the varied limiting processes that students found salient, the chapter provides an in-depth look at students' use of one of Zeno's Paradoxes – an example that demonstrates that even when using the same mathematical example or image, different students come to different conclusions, depending on which limiting process(es) they emphasize.

VI. Language – a discussion of some of the ways that students' use of ambiguous limit language such as 'approaches' and 'tends to' takes specific forms when they apply it to the content domain of infinite series. Beyond discussing some of the student difficulties with limit-related language, this chapter contains two in-depth examples – on students' use of the words "converge" vs. "approach" vs. "equal" and what those imply, and a discussion of the different ways that students use the phrase "the terms go to zero."

VII. Connections – a demonstration of two contrasting cases of how students connect their understanding into a cohesive (but not necessarily correct) narrative explanation of infinite series. The juxtaposition of the two student cases is intended to show less- and more-connected explanations of infinite series of numbers that students provide. It is followed by a more general discussion of some of the ways that the larger sample of students commonly connected limiting processes in their explanations.

VIII. Implications for Research and Practice – a discussion of the ways that this study informs (a) the goal of learning about student understanding of the topic of infinite series, as evidenced by their explanations and SGR, in order to facilitate better-informed instructional design of infinite series curriculum; and (b) the broader goal of examining how students use their SGR as a tool for explaining, rather than simply as problem-specific diagrams which aid in solving isolated, particular problems about infinite series.

II. RELEVANT LITERATURE

As described in Chapter I, this study focuses on students actively creating and using images as a means for explaining a particular mathematical concept. The theoretical foundation of this research is grounded in the combination of two important perspectives that have been shown to be valuable for learning, as well as empirical evidence that these perspectives, when combined, will illuminate student understanding of infinite series of numbers.

First, a wide range of research has shown that working with “visual representations,” especially in calculus and sometimes aided by technology, enhances student learning and understanding (e.g., Larkin & Simon, 1987; Ainsworth, 1999; Borgen & Manu, 2002; Alcock & Simpson, 2004). Second, research has shown that “self-generated explanations” (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Ainsworth & Th Loizou, 2003) promote deep understanding, both in contexts in which students are prompted to self-explain, and when they do so spontaneously. Aligned with Parnafes (2009), this study combines these two perspectives by examining self-generated (visual) representations (SGR), which are used by students to *explain* the difficult concept of infinite series. Based on the findings of Martin (2009), and other related investigations in the content domain of limits, there is reason to believe that student understanding can be illuminated by these self-generated explanations that include visual images.

Outline of Chapter II -

II.A. Images in calculus

II.B. Self explanations promote deeper understanding, especially when mediated by diagrams

II.C. Self-Generated Representations - combining visual images and self-explanation

II.D. Exposing what students understand by studying their SGR

II.A. Images in Calculus

In what follows, I borrow from Latour (1987) and refer to ‘images’ as any of “graphs, tables, lists, photographs, diagrams, spreadsheets, and equations ... which are publicly and directly available, such that they can be positioned as social objects.” That is, to avoid confounding the various definitions of “visualization,” “representation,” or “inscription” (à la Presmeg, 2006), the term ‘image’ will be used to refer to an *externalization* –graphical, tabular, diagrammatic, etc... – that is used to depict a visual that is mathematically meaningful to an individual. Additionally, for this study, the images of interest will be those produced by students and professors during the course of a teaching episode. So the images here will be self-generated images, often referred to in the literature as self-generated representations (SGR).

Interest in students’ visual thinking and ability to visually represent mathematics is seen across all levels of mathematics learning. There is a particularly keen interest in studying how students understand calculus visually (e.g. Zimmermann, 1991), which is consequential when considering broader standards and curricular goals for university mathematics education. For example, in the 2004 Report from the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America, the desire for students to have fluency with visualizing calculus topics is explicitly detailed:

“Developing appropriate symbolic manipulation skills will always be a goal of mathematics instruction. But in addition graphical and numerical understanding must be

developed. Viewing topics from these analytical, graphical and numerical perspectives is necessary for true understanding. For example, in calculus courses students should experience geometric as well as algebraic viewpoints and approximate as well as exact solutions.” (page 18)

That is, the CUPM explicitly acknowledges that graphical fluency is ‘necessary’ as part of a successful calculus understanding, even going on to say, “Students should learn to visualize geometric objects, to relate graphical objects to their analytic definitions, and to see the graphical effects of varying parameters” (page 24).

Within a university mathematics setting, visual images are used by both students and their professors regularly, as a means to condense information into a more compact presentation and suggest potential approaches to proofs (e.g. Harel and Sowder, 1998), and illustrate and support results derived by algebraic means (Inglis & Mejía-Ramos, 2009). As Fischbein (1987) notes, “visualization not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.” Increasingly in undergraduate mathematics, visual images are no longer simply used as illustrations of already proven concepts or as guides to ‘get started’ with a difficult solution. Rather, they are being recognized as a key component of reasoning and problem solving (Arcavi, 2003), making mathematical arguments (Alcock & Simpson, 2004), and more. A good visual image can serve to “concretise the referent,” by making one’s image publicly available (Presmeg, 1986), thereby highlighting the “conceptual underpinnings” that may not be perceptible in formal algebraic or proof-like presentations (Arcavi, 2003).

Visual images are powerful in calculus. Yet, studying the ways that students use and adapt them is not always straightforward. For example, when Williams (1991) studied the ways that students reasoned with the concept of limit, he found that they showed “great faith” in graphing as a means of understanding limit, performing limit calculations, and justifying limit statements. His findings suggest that students believed graphing had great explanatory power, but this enthusiasm for graphing was not always consistent with students’ actual ability to produce useful graphs. According to Williams,

“It is as though problems of continuity, topological properties of the real line, and a myriad of other difficulties that they realize might arise in taking limits are magically taken care of for students in the process of drawing a graph.” (page 234)

That is, students may take an oversimplified graph as representing mathematical reality or generality, simply because they are unaware of the mathematical complexity that is condensed into a single graph.

Tall (1991) suggested similar downsides for students’ use of visual images – namely that they can be deceiving when students meet more complicated functions and geometry (e.g. a function that is continuous everywhere but differentiable nowhere). Some of this confusion may be attributed to trying to visually represent new concepts without a solid foundation in the more basic mathematical structures involved (e.g. real numbers, fractions, etc...) (Tall & Schwarzenberger, 1978). It may also be the case that students use images that they have seen in prior contexts without understanding the assumptions and implications that go along with using that particular image in that particular context. As Greeno and Hall (1997) suggest, something only becomes a representation after it has been interpreted and thus given meaning by an

individual. While some standard representations are recognized because they were created by and are most often used by particular communities with shared conventions and language, it is not guaranteed that students understand these conventions. Students may not ‘speak the same language,’ or understand the subtleties of these representations that were intended by the communities of practice in which they were created (see Roth & McGinn, 1998; Brown & Duguid, 1996; Bourdieu, 1997).

As Tall (1991) describes, one of the more significant reasons that teaching calculus is difficult for practicing mathematicians is that concepts that are regarded as intuitive to the expert are not intuitive to students. He posits, “there is no reason at all to suppose that the novice will have the same intuitions as the expert, even when considering apparently simple visual insights” (page 5). Take as an example the notion of limit. Because the formal idea of a limit is difficult for students to deal with analytically when first learning calculus, the idea is most often introduced through a series of visual images. One of these images often turns into viewing the derivative as the limit of a sequence of secant lines approaching a tangent. Students who do not understand the conceptual basis of limit more deeply will often see different things in this image than their professors do. This has strong potential to result in faulty and problematic reasoning that Oehrtman (2002) identified as students’ ‘collapsing dimension’ metaphor for limits (2002).

An additional difficulty in exploring students’ fluency and facility with visual images in calculus is that students’ understanding of a particular calculus concept is often tied to a specific graphical or algebraic representation: “For students to reason conceptually requires a translation back and forth between that representation and the problem context. It is difficult for students to see and work ... with these images” (Oehrtman, 2008, page 69). Such images, which dominate students’ reasoning, even after they have been shown to be faulty or deceptive, have been referred to as “uncontrollable images” (Aspinwall, Shaw, and Presmeg, 1997)

Finally, the role of visual images in university classrooms themselves may contribute to some student difficulties. While people often think visually and use images to begin to solve problems (mathematical or not), the move toward formal reasoning in university classes typically means that students are invited to express their ideas via either symbolic or verbal responses (e.g. Parnafes, 2009). Visual images are most often presented as illustrative of particular concepts during instruction by the lecturer (e.g. Kindfield, 1993), and not meant for expression and/or self-generation.

Research has begun to answer the following call to expand the range of ways that the undergraduate mathematics education community explores students’ ability to use images in calculus contexts:

“An understanding of just what images, both correct and incorrect, that students might construct is important if teachers are to help those students work toward connected formalizations. Continued research involving student’s work is necessary to determine just what these images might be. Knowing what incorrect images a student might have will be useful to the teacher in determining possible sources of misconceptions and should lead to better teaching.” (Borgen & Manu, 2002, page 164)

What such studies (e.g. Hacıomerğlu, Aspinwall, and Presmeg, 2010; Ferrini-Mundy & Graham, 1994) have in common is that they set out to study student use of visual images and reasoning based on images by either *asking* students to create visual images, or by providing visual images and examining student reasoning based on those provided images. While this

approach contributes to the expanding literature on how students *use* images in calculus, it does not explicitly address whether students would choose to make use of imagery. Nor does it speak to the types of images that they prefer, when producing an explanation of a concept or solution to a particular task (see also Sherin, 2000). Further work from this perspective, then, should treat as an empirical question whether and in what ways students' choices of visual imagery shape the *explanations* that they construct around them.

II.B. Self explanations promote deeper understanding, especially when mediated by diagrams

"Self-explaining" refers to the metacognitive strategy of generating explanations to oneself while learning. It has been shown to be an effective tool through which students can develop a deeper understanding of material they study. By giving mechanics students worked examples containing text and diagrams, Chi (1989) showed that students who spontaneously generated a large number of self-explanations more than doubled their post-test scores, compared with their classmates who gave fewer explanations. Such a study suggests that students can learn and understand a concept, example, or task more deeply when they verbalize an explanation, either for themselves or others, rather than simply working tasks silently or answering justification questions written into the tasks. When students are prompted to *explain why* and *explain their reasoning*, they are often required to address aspects of the concept or task that are otherwise implicit and left unaddressed (Chi, 1989).

Students often self-explain, whether prompted or not, via visual images that they have either seen previously or created themselves. Cox (1999) suggests that diagrams and other images may be a critical tool through which self-explanation becomes successful. He claims that using multiple modalities is an important component of self-explanation, so that the learner is able to fully develop an explanation and represent it from multiple angles. The use of diagrams and other visual images may encourage learners to employ multiple modalities, thereby producing more meaningful self-explanations. In some contexts, self-explanations have been shown to be particularly beneficial when they include the integration of both text and graphical means (e.g. Alevén and Koedinger (2002), with geometry instructional software). In mathematics, graphical images have significant potential benefit to aid in self-explanation, because they may help learners focus in on key components of the concept or problem at hand, thereby limiting the feedback that the image can provide to only the most relevant information required of an explanation (Cox, 1999).

The literature on self-explanation also documents some potential downsides to including visual imagery in one's self-explanation scheme. For example, Wilkin (1997) found no self-explanation effect when students were asked to generate diagrams to accompany text on two-dimensional motion. She concluded that producing the diagrams actually led to incorrect self-explanations, because students were producing familiar images that either did not fit the text, or constructing an incorrect explanation based on these diagrams.

In order to explicitly explore the particular influences of text and diagrams on the self-explanation effect, Ainsworth (2003) *asked* students to self-explain when engaging with a particular biology topic. Results showed that students given diagrams, rather than text, both performed better on post-tests and generated more self-explanations. In fact, Ainsworth found that only students who self-explained via diagrams were found to have benefited from self-explaining at all.

The benefits of self-explanation via diagrams go beyond simply producing better gains,

as measured by a post-test. Ainsworth (2003) concluded that diagrams *promote* self-explanation. This is partly attributed to the notion that diagrams can prompt causal explanations. For example, as Ainsworth showed, when prompted by “The heart is divided in half by the septum,” students who self-explain via text representations only respond with “*because it has different chambers and they need to be divided.*” However, students who self-explained with diagrams responded, “*The septum is like a wall because you need to separate the oxygenated from the deoxygenated blood.*” That is, the presence of diagrams during self-explanation encouraged actual *explanation*, and not just regurgitation of facts or procedures seen in class or a textbook. While it provides important baseline information about how students self-explain, as aided by diagrams, Ainsworth’s study neither explored the features of the diagrams that specifically promoted self-explanation nor asked students to construct the visual images themselves, when they thought they would be most useful.

Studying students’ explanations coupled with the visual images that they produce may provide a different perspective than students’ explanations as supplemented by representations provided by a textbook or professor (see Cox & Brna, 1995; Sherin, 2000) for several reasons. First, because students know a great deal about common images seen in textbooks or in lectures, it seems plausible that they may be exercising recall or attempting to reproduce professors’ explanations rather than producing their own. However, this very exposure to various images in text and lecture suggests that students may have a wide enough repertoire of available visuals in mind to draw from when constructing their own – and the ones they select are most likely to make sense to them. While asking students to use a *particular* representation or image in their explanation may promote self-explanation, it may well limit the range of things they can bring to the discussion (Sherin, 2000). At worst, it may ask them to explain using an image that does not make sense to them, eliciting a nonsensical explanation of a concept or context that they may understand very well by other means. This suggests that self-explanation that makes use of visual images may be particularly useful if the visual images are created by the students themselves (i.e. SGR), when they feel prompted to use an alternative medium to express their ideas or reason with the concept.

II.C. Self-generated Representations - Combining visual images and self-explanation

Many research programs that explicitly study SGR (e.g., Ainsworth & Loizou, 2003; Roy & Chi, 2005; Cox & Brna, 1995; Cox, 1999; Gobert & Clement, 1999) serve as examples relating self-generated explanation with visual images meant to aid said explanation. It is reasonable to assume that including visual imagery in students’ collection of self-explaining tools will further expand their interactions with new concepts, and broaden the ranges of ways that they can synthesize and connect within difficult conceptual domains. As Greeno and Hall (1997) suggest, students should “become more actively involved in learning to construct and interpret representations” by participating in discussions of their properties, advantages and limitations, and explanatory power.

Most SGR studies are based in cognitive science, pitting ‘control groups’ that are presented with existing representations against ‘experimental groups’ that are permitted to use self-constructed representations (Parnafes, 2009). Such research contributes to acknowledging the importance of SGR and its role in the changing college classroom. But in order to understand what sense students make of calculus concepts, we must also explore the particular ways that students’ SGR influence their understanding. Others have begun to address the challenge of

specifically which aspects of students' SGR are most connected with self-explanation, of either tasks or mathematical concepts (e.g. Kohl & Finkelstein, 2008; Kindfield, 1993).

For example, in the context of university biology, Kindfield (1993) focused on SGR created during reasoning about meiosis, as a means for identifying which features of this difficult biological process were prioritized in the images of both students and their professors. Participants in the study were neither prompted to produce images, nor were they provided with pre-existing images. Rather, when presented with the task of answering biology tasks, participants spontaneously drew visual images, the features of which gave insight into how novices and experts understood the process differently.

Kindfield's take on studying students' SGR, and contrasting that with biologists' SGR, contributed more than simply content-specific details about how experts and novices understand meiosis differently. It made a broader point about the interplay between knowing and representing. Kindfield demonstrated that "the development of understanding includes gains in knowledge about how to efficiently represent different aspects of meiosis knowledge and about the utility of using diagrams in particular reasoning situations" (page 21). That is, as her participants had a more sophisticated understanding of meiosis, better practices for representing what is important about this process were simultaneously developed; the understanding and ability to convey important information with one's SGR going hand-in-hand.

Parnafes (2009) conducted a similar investigation on the potential of middle school students' self-generation and elaboration of visual images, to advance their understanding of the phases of the moon. Though there are two major differences between Parnafes' and the current study – the age of participants and the fact that Parnafes explicitly asked students to draw images – her study is also very important groundwork for studies of SGR, in highlighting how students' practices of using drawings and visual representations inventively can advance understanding.

II.D. Exposing what students understand by studying their SGR

There is reason to believe that a study focusing on student understanding of infinite series of numbers will benefit greatly by examining students' SGR used during the process of explaining. First, there is empirical evidence that students are in fact able to produce visual images of what they understand of this mathematical content, correct or not. While some research asserts that students have no reliable visual imagery for infinite series (Nardi, Biza & González-Martín 2008; Martin 2009; Habre 2009), most of these studies did not set out explicitly to study students' images. Other research that provided students the opportunity to use any means they desired to explain this or related limit concepts *did* elicit infinite series images (e.g. Williams, 1991; Alcock & Simpson, 2004 & 2005; Pinto & Tall, 2002). For example, Williams (1991) found that 70% of university student participants, when asked to justify a limit statement, proceeded to draw a graph, unprompted. Alcock and Simpson (2004) also found that when presented with power series topics, half of their participants showed interest in and facility with reasoning via visual images (showing a 'tendency to visualize' based on their creation of diagrams and gestures while explaining convergence, and indication that they prefer to think of pictures or diagrams rather than algebraic representations). The other half preferred algebraic methods. Such evidence seems to suggest that students may have the ability to reason visually with infinite series, whether or not it is their inclination to do so.

Second, Martin (2009) found that both experts and novices used a variety of images (most often graphical) when reasoning with Taylor series. However, experts were fluent with a greater number of them, and were much more reliable than novices to correctly construct and

interpret graphical images. Kohl (2007) found similar results in college physics classrooms – both students and their professors used a variety of images, but the ways in which they interacted with those images greatly contrasted. These results suggest that students find something about producing SGR to be particularly useful and/or illustrative, perhaps contributing something beyond what their words could express. Based on their usage of and fluency with the different modalities that were used when responding to Taylor series tasks, Martin concluded that the graphical understanding of Taylor series may be the most notable manifestation of understanding separating novices from experts. In discussing implications of his study, he suggests that it also may be the conception that is most easily influenced through shifts in Taylor series instruction.

Finally, there is evidence suggesting that studying students' explanations and imagery associated with infinite series of numbers will address the most important breaking point that separates expert and novice understanding. Martin (2009) found that, when reasoning with tasks about Taylor series in a second semester calculus context, experts most often appealed to dynamic partial sum and remainder arguments, while novices appealed to termwise arguments more frequently. This suggests that expert understanding is related to the relationship between series of numbers and power series (where partial sum and remainder are foundational), whereas students' more inconsistent and incorrect usage of concepts relates back to limit (consideration of terms only, and limits of sequences of terms). Thus, one way to target students' understanding of this difficult domain is to study their understanding of infinite series of *numbers*, as this is one of the curricular breaking points that sets expert reasoning apart from student reasoning.

This final distinction suggests that one of the major differences in the way that students come to understand Taylor series is by relating them back to their more basic understanding of limits of sequences, while the experts tend to relate them back to series of numbers. This is essentially a difference of which *limiting process* (Tall, 1980) the different groups are prioritizing. As Tall discusses, common limiting processes that arise in calculus are either continuous (find limit, continuity, geometric limit of chord as it approaches tangent) or discrete (limit of sequence and series, decimal expansion, numerical approximation). Martin's (2009) study seems to suggest that tracing back to different limiting processes, experts and novices come to differing understandings of what it means for an infinite series to converge. Yet, further complicating this relationship is the fact that the word *converge* itself is used across multiple contexts, for multiple meanings.

For example, a student participant in Monaghan's (1991) study on student understanding of limits and limit behavior said, "The term *converges to* can mean many different things and depending upon which definition or meaning is put into practice, the answer to any question can differ," (page 22). Alcock and Simpson (2009) also find that within the context of a single figure, it is possible to ask a variety of questions all about the convergence of different aspects of what is represented (see page 28).

One possible reason that there is much confusion regarding limiting processes is that, in the context of infinite series, there are so many things to keep track of when explaining the concept or solving a problem that makes use of it. Alcock and Simpson (2009) describe how the amount of mathematical ideas that must be coordinated, and the roles that they take on during the process of problem solving (specifically within power series context) is staggering. Their "short list" of mathematical objects/processes that come into play when finding the interval of convergence for a power series, for example, comes down to the following:

- (1) A power series is a function of x
- (2) Each number x gives a series (so there are, actually, an infinite number of infinite

- series involved)
- (3) Each series adds to a number (or doesn't, but we only care if there is one, not what it is)
 - (4) We attend to the set of x values for which the series does add up to a number
 - (5) The set of x values is an interval, centered at the center of the power series, which will be communicated by another number (R , or the radius of convergence)
 - (6) The ratio test is used to find R
 - (7) The test is used by constructing a ratio of terms of the series, which depends on both x and n
 - (8) It requires a user to treat this ratio as a single expression and take a limit with n , treating as a sequence
 - (9) This requires temporarily ignoring that x is a continuous variable
 - (10) The limit of the sequence with n depends on x (another expression)
 - (11) Then, returning to viewing x as a continuous variable, one can solve for R , which is a value which yields an interval around the center which is a number
- (adapted from page 26)

As is evident from the list of mathematical objects and processes that go into reasoning with power series, coordinating which limiting processes and which values are most important may well be difficult for someone just starting out with this topic. Typical presentations of the subject may not directly address this challenge. Infinite series in the university calculus curriculum are most often treated as an algebraic enterprise; there is little exposure to visual images and/or applications of concepts in the most commonly used textbooks (Nardi, Biza & González-Martín, 2008). That is, despite empirical evidence suggesting that at least some non-trivial proportion of students choose to reason with infinite series largely by visual images, curricular materials do not tend to employ this type of presentation. Some alternative curricula, mostly technology based (e.g. Soto-Johnson, 1998; Habre, 2009; Yerushalmy & Schwartz, 1999), have attempted to infuse more visual images into units on infinite series, and have been met with mixed success.

It is curious, then, that so little visual imagery is associated with the unit on infinite series despite the fact that the base concept of limit (in calculus) has been shown to involve much imagery (e.g. Pinto & Tall, 2002; Stergiou & Patronis, 2002, Oehrtman, 2002) – especially since such imagery often prompts students to reason with limits dynamically or with motion and movement metaphors (e.g. Williams, 1991; Monaghan, 1991). Even when dynamic imagery is not prevalent, for example in Oehrtman (2002), when students were learning about limit concepts explicitly through his 'Approximation Framework,' students still demonstrated a tendency to draw and visualize behavior of graphs as they related to approximation.

As demonstrated here, visual images are important, have status in college calculus courses, and promote better self-explanation and understanding. Further, SGRs and the act of *producing* them may contribute to such gains beyond reasoning with provided visual images. In the topic area of limits, there is evidence that students have both the ability and proclivity to produce images. Limited studies have shown that this extends to students' understanding of infinite series, though it is unclear what particular aspects of their SGR are important to their understanding, and how those ideas fit into their larger scope of calculus understanding. Thus, the next step toward better understanding how students learn about and reason with infinite series is to examine the types of visual images that they create and prioritize as part of their explanations of this difficult topic.

III. DATA SYNOPSIS AND METHODS

Primary data for this study are in the form of student interviews, during which student participants were asked to explain mathematical concepts, which often spontaneously led to their producing SGR. The primary purpose for this chapter will be to describe the sample of students who participated in interviews, and the methods and protocols for said interviews. Supplementary data includes textbook and professors' images, to contrast with students' use of imagery. Thus, this chapter will also include the rationale for additional data collection, and discuss the ways in which this additional data helped provide context for students' Calculus II experiences, particularly related to the domain of infinite series.

Outline of Chapter III -

III.A. Study context and sample

III.B. Observational data – what the students' experience looks like

III.C. Methods – interviews and the data they generated

III.D. Data analysis methods

III.A. Study Context and Sample

To begin to examine students' explanations (guided by their SGR) in order to learn about their conceptions of series convergence, 37 students were interviewed, across several courses and two institutions. Figure III.1. below highlights the composition of each of the schools and their math departments.

	Large Research University (LRU)	Small Honors College (SHC)
General information:		
Total Undergraduates	26,000	2,000
	Asian American 42%	Asian American 4%
	White American 31%	White American 76%
	Hispanic America 12%	Hispanic American 4.5%
	African American 4%	African American 8%
International Students	4% (about 1,050)	2% (about 40)
Department of Mathematics:		
Faculty	51 full time, 5 with joint appointment [in physics, CS, etc...], variable number of postdoc and visiting faculty	10 full time, 2 visiting; all hold joint appointment with Mathematics and CS [joint department]
Math Majors	450	70

Figure III.1. Demographic information from study sites. Values are approximate, taken from Academic Year 2011-2012 data

These two study sites were chosen in order to include varied calculus class size and composition, while attempting to control for many other factors. In this attempt to “control” for

factors other than differing features of the calculus course, students' experience was roughly standardized along several guidelines:

- Departmental course offerings in mathematics: Both SHC and LRU offer only one course "lower" than the calculus sequence (Survey of Math, and Precalculus, respectively)
- Available majors
- Contact hours in the calculus sequence

While similar along these dimensions, SHC and LRU have important contrasts in the structure of their calculus offerings. SHC is a small (public) honors college, with calculus class size averaging about 30; LRU, a large (public) research institution, uses primarily large lectures (200 – 400 students) to teach calculus content.

For the reasons explained below, the sample (for sample size, see Figure III.2.) contains students from Calculus II, Calculus III, and Real Analysis, from each of SHC and LRU. The structure of the courses at both institutions is also included, in order to highlight similarities and differences in course structure and features at each institution. This discussion is meant to show that, while there are some differences in the course structure and staffing at SHC and LRU, students' experiences are similar in other important ways that will permit comparison and contrast in later analyses of how students orient to the topic of infinite series convergence.

	LRU	SHC
Calculus II	7	6
Calculus III	5	5
Real Analysis	6	8

Figure III.2. Sample Sizes

Calculus II: As infinite series is predominantly a "second semester" topic for schools on a semester system, second semester Calculus classes at LRU and SHC were sampled first. While there is some variation in departmental structure and style of course, all of the second semester calculus students were recruited from an instructor who was using Stewart's *Calculus* as the course textbook (either the fifth or sixth edition) (2002/2007). The instructors from whom students were recruited all assigned comparable amounts and types of homework problems from Stewart, and followed the progression that the text set forth, regarding infinite series. This meant covering Chapter 11 (in the sixth edition), with topics in the following order: general introduction, convergence tests, power series, Taylor series, remainder theorem. At LRU, infinite series composed roughly one third of the second semester syllabus, and at SHC it composed one third to one half of the syllabus, varying across two professors. Additionally, second semester calculus at both institutions began with the topic of "integration techniques," and covered "applications of integration" from Stewart (Chapter 7) before covering the "infinite series" chapter. At LRU, however, significantly more time was dedicated to differential equations (Chapters 9 and 17) than at SHC. This is perhaps explained by SHC's offering of a semester-long course dedicated entirely to ordinary differential equations – a course that LRU does not frequently offer for lower division students. However, differential equations are built into several courses at LRU, for a total of approximately 23 lecture hours during the entire lower division calculus and linear algebra sequence.

At SHC, calculus classes (with approximately 30¹ students) convene in a small classroom with whiteboards and whiteboard tables. These classes meet for 210-220 minutes per week (roughly 55 contact hours per semester), with a full-time faculty member in every class. In each of the classes, there were 1-2 undergraduate “TAs” who were responsible for holding weekly review sessions (30 contact hours per semester, optional attendance) and grading homework. At LRU, calculus classes convene as a combination of lecture and discussion section. Lectures accommodate between 225 and 500 students, in a large lecture hall, for 150 minutes per week with a full-time faculty member (about 35 contact hours per semester²). Discussion sections of approximately 25-30 students convene for an additional 150 minutes (35 contact hours per semester) and are facilitated by mathematics graduate students. Thus, while the structure and environment of the classes differ across institutions, total contact hours with full-time faculty and “assistants” are comparable, and treatment of the infinite series material was similar.

The rationale for including Calculus II students at the end of their semester in this study should be clear. As these students have just completed the course in which infinite series and convergence are covered in depth for the first time, they were interviewed at the end of the semester either just prior to or 2-3 days after their final exam. Only “passing” students (grades of A, B, or C) were chosen for interview, and their final grades represent a range of levels of “passing.” The sample only includes “passing” students in order to consider only those students who meet minimum requirements for success in the calculus course, as determined by their professors. Note, however, that this does not mean that they necessarily show a complete grasp of the material. At minimum, the students “understand” the bulk of the course material “well enough,” from the perspective of earning a minimum of a C as assessed by their given professor. The sample of Calculus II students came from two different lectures at each of LRU and SHC.

Calculus III: Third semester calculus, which at both LRU and SHC represents the final semester-long course in the calculus sequence, covers multivariable calculus topics from vectors to Stokes’ Theorem. More importantly to this study, at both institutions electing to take Calculus III signifies a more significant math component to one’s choice of major. That is, many majors require two semesters of calculus, but far fewer require additional mathematics, narrowing the potential participants’ choice of major to something more overtly related to science, mathematics, or engineering. For this reason, the goals for interviews with Calculus III students were the same as the goals for interviews with the Calculus II students; the differentiation in courses served to examine effects of self-selection into higher math courses or a “math-track.” Calculus III students were interviewed at the *beginning* of their semester, so that their prior university calculus experience was as analogous as possible to that of students in Calculus II, and that the effects of self-selection into additional mathematics were as isolated as possible. Where the Calculus II students represented a wider range of majors, the Calculus III students narrowed that pool without having the advantage (yet) of seeing additional extensions of calculus topics. At the end of the semester, students were removed from the sample if they did not complete Calculus III with a passing grade.

At SHC, Calculus III meets for approximately 210 minutes per week with a full-time faculty member (roughly 55 contact hours per semester), and has an undergraduate “TA” who hosts review sessions (up to 30 hours per semester) and grades homework. These classes have

¹ All student enrollments and contact hours are taken from Academic Year 2011-2012 reports

² While at LRU, contact “hours” are often counted as the amount of 50-minute class periods convened over the course of the semester, herein they are converted to literal 60-minute periods, for comparability with SHC.

approximately 25-30 students, and meet in a small classroom with whiteboards and whiteboard tables. At LRU these classes are again split into lecture and discussion section components. The lecture convenes with between 225 and 400 students in a large lecture hall, facilitated by a full-time faculty member (approximately 35 contact hours per semester). Discussion sections have 25-30 students and meet for an additional 35 hours per semester in a smaller classroom with a blackboard. Mathematics graduate students facilitate the discussion sections. The sampled students all attended the same lecture, at each of LRU and SHC.

Real Analysis: The first course in Real Analysis is one of few requirements that, at both institutions, distinguishes math majors' (or minors') schedules from those of their engineering or science classmates. Thus, the choice to include real analysis students in this sample represents another transition point – from students whose path is more math-general, to students whose path is very likely a major or minor in mathematics. As the actual content of real analysis, which is standard across both SHC and LRU (though they use different texts) is not related to the choice of including real analysis students in the sample, these students were again interviewed at the *beginning* of the semester in which they were enrolled. This was intended to, in some ways, ensure that these students had similar exposure to the topics of infinite series as those from the Calculus II and III samples. Infinite series is a large part of the Real Analysis curriculum, so interviewing students at the start of the semester removes some of the effects of multiple exposures to the material, and instead attempts to isolate one of the only variables distinguishing student groups as self-selection into higher mathematics. At the end of the semester, students who did not complete the course with a passing grade were removed from the sample.

At SHC, Real Analysis students have class in a small room of no more than 25 students, with a full-time faculty member, for 220 minutes per week (about 55 contact hours per semester). Undergraduate “TA’s” are sometimes employed, if attendance reaches a maximum. In the class from which SHC Analysis students were recruited, there were two undergraduate math majors who graded homework and held review sessions (a voluntarily attended 30 contact hours per semester, which most students attended). At LRU, Real Analysis classes are capped at 36 students, and facilitated by mathematics faculty or post-docs for 150 minutes per week (35 contact hours per semester) in a small classroom. LRU students were sampled from two different lectures at LRU, and one lecture at SHC.

It is important that the data sample be varied to include students from all of the groups discussed above in order to permit comparisons across different levels of student experience, and self-selection into a math-track. However, the ways in which the students’ experiences are similar or different, based on institutional differences, provide important reference points for discussing any patterns of differences that emerge across LRU and SHC students. For ease of comparison, in later chapters, the pseudonym key for identifying the different student populations is in Figure III.3., where students from the same course and institution were consistently given pseudonyms beginning with the same (randomly chosen) letter.

	LRU	SHC
Calculus II	Names beginning with “M”	Names beginning with “S”
Calculus III	Names beginning with “J”	Names beginning with “B”
Real Analysis	Names beginning with “A”	Names beginning with “T”

Figure III.3. Pseudonym codes

III.B. Observational data - what do the students' calculus experiences look like?

While it is clear that LRU and SHC did have some institutional differences in the ways that their courses were structured and staffed, both institutions followed the same outline and textbook for second semester calculus. To examine consistency in students' experience at LRU and SHC, as well as to make reasonable assumptions about what Calculus II background the Calculus III and Real Analysis students had at both institutions, a more thorough review of what happens in Calculus II classrooms at both institutions was warranted. This review consisted of both classroom observation and examination of class notes provided by the campus note-taking service, as discussed below.

At LRU, a complete catalog of the "infinite series" lectures, facilitated by two faculty members (one of whose students were recruited for interviews), provided a picture of the scope and nature of material covered in lecture. During these lectures, field notes, with attention to the lecturer's (1) language when discussing convergence, (2) images and gestures when discussing and illustrating convergence, and (3) examples when explaining the meaning of convergence or answering students' questions provided the basis for describing students' experience. The same field note/observation procedure was completed at SHC, in lectures by both faculty members whose students were recruited for the study. This classroom data was supplemented by field notes from three to five lectures on different (randomly selected) topics, throughout the semester.

Course notes, recorded by the designated note-taking service on LRU's campus for an additional three Calculus II lectures, supplemented the observation field notes. Often for Calculus I and II (among many other classes), the campus note-taking service designates "note-takers" to attend class, copy down class notes that are provided by the professors, typeset them, and distribute them for a fee to students who wish to subscribe to the service. These notes are detailed and contain both images and quotes, in addition to paraphrased portions and equations/worked examples. While they cannot be treated as an exact replica of a lecture, the images and quoted portions, in particular, were treated as representative of what was actually drawn on the blackboard and spoken in the lecture. The typeset notes for three Calculus II classes taught by different professors at LRU in 2009, 2010, and 2011, were collected and examined, along the same dimensions as discussed above. These were consistent with the field notes taken in target classrooms at both LRU and SHC.

Comparing across and within institutions was fairly simple, since (as mentioned before) all of the lecturers used the same text (Stewart's *Calculus*) in the order that the material is presented. All lectures followed the progression as outlined in figure III.4. below. The lectures were largely consistent, with many of the same examples used and images invoked at similar instances in the material.

In particular, most lecturers' first dedicated "series" images were constructed when introducing the integral test in 11.3. These images were used to show areas under a curve and to discuss error and the validity of the integral test. Some lecturers drew number-line images when demonstrating the epsilon- N definition of sequence convergence, or represented the partial sums of a series as monotonically increasing sequences, but these images were used differently according to the individual, mostly to make minor points, and were not discussed as they related to the overarching concept of infinite series. Number lines were common when discussing alternating series, and various images, mostly of continuous functions, were used when working examples using the comparison test for infinite series convergence.

Chapter 11.1	Sequences
Chapter 11.2	Series (includes definition, geometric series, telescoping series)
Chapter 11.3	Integral Test and Error
Chapter 11.4	Comparison Test and Limit Comparison Test
Chapter 11.5	Alternating Series Test and Alternating Series Estimation Theorem
Chapter 11.6	Absolute Convergence, Ratio Test, Root Test
Chapter 11.7	Strategies for Testing Series
Chapter 11.8	Power series
Chapter 11.9	Representations of Functions as Power Series
Chapter 11.10	Taylor and Maclaurin Series (including Taylor's Remainder Theorem)
Chapter 11.11	Applications of Taylor Polynomials

Figure III.4. Stewart's "Series" Chapter (from sixth edition) (2007)

A close examination of the textbook was used to put students' experience in perspective, and to produce a reasonable impression of what a typical Calculus II student would encounter at either institution in the study. While the textbook of both departments' choice, Stewart's *Calculus* (2007), contains a long treatment of infinite sequences and series, there is a notable drop-off in the manner and frequency of image use within the chapter. While graphical images of all sorts of sequences and how they do or do not converge are prevalent in Section 11.1 (Sequences). Section 11.2 ([Introduction to] Series) contains rather fewer. In fact, excluding two tables of values and a geometric proof for geometric series, which only appear in a margin, the *only* image that appears in this portion of the text is given below in Figure III.5. One could argue that more images are not necessary, as there are many examples in Section 11.1 which represent the same thing, after one has reoriented to an infinite series as an infinite sequence of partial sums. However, there is very little space dedicated to that idea in Section 11.2 (one page); hence it does not seem likely that many students would make the connection back to the images of 11.1 very easily. Independent of that, there are no images that are clearly (to a novice reader) representative of a more general case, or more general behavior that would indicate convergence of an infinite series.

The textbook of choice for both institutions aligns with the findings in Nardi, Biza, and González-Martín (2008), whose study included an examination of the surface features of the different aspects of the chapter on infinite series in several calculus texts in the UK and Canada. Nardi et al. also found very few images throughout chapters on infinite series. It can be inferred that the books in their study, much like Stewart, do not intend the images to be actively used for explaining. That is to say, they are mostly used (after a long written passage) to represent what a series is, and to provide examples of what convergent series look like. Discussions of which particular attributes of the images indicate necessary and/or sufficient conditions for convergence, or what convergence means in terms of the images, are largely absent. Also mostly absent are images that exemplify divergence of infinite series. Nardi, Biza, and González-Martín also found similar classes of images as are found in Stewart – mostly area under curve models and sequences of partial sums.

■ Figure 3 illustrates Example 6 by showing the graphs of the sequence of terms $a_n = 1/[n(n + 1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \rightarrow 0$ and $s_n \rightarrow 1$. See Exercises 62 and 63 for two geometric interpretations of Example 6.

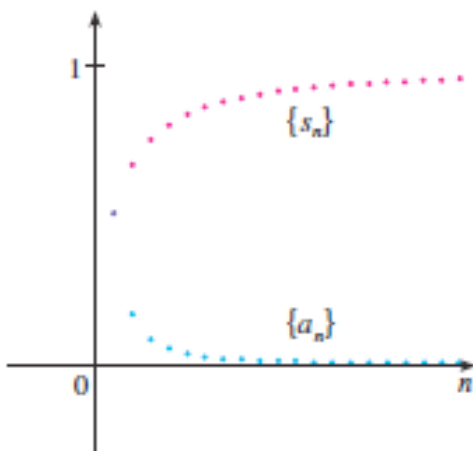


Figure III.5. Image from Chapter 11.2 of Stewart

III.C. Methods - Interviews and the data they generated

Students were recruited from the classes discussed above with the impression that they would be discussing some “Calculus II topics” with a graduate student of mathematics education. The only mathematical baseline necessary was that they had taken Calculus II – and therefore seen infinite series – so all of the students were equally suited to participate in the interviews. If some students chose to invoke higher mathematics, it was at their discretion; this was neither encouraged nor discouraged by the interviewer. At no point prior to the interview were students aware that the topic of the interview would be infinite series.

All interviews were conducted in private interview rooms with a large whiteboard and a large table, at one of the two study sites. Interviews, all of which were based on the same protocol, lasted between 45 and 100 minutes, for which students were compensated \$15. During all interviews, only the interviewer and the student participant were present.

The protocol used for the interviews was very loose at the outset, including several prompts to encourage students to elaborate on and expand their thoughts. Students were encouraged to use pen/paper or the whiteboards to communicate their thinking. Preliminary information provided to the students, with interviewer quotes *in italics*, included:

Common Pre-Interview Questions and Responses

- After verifying their consent form allows the session to be videotaped, discussion of the role of the camera: *I am videotaping our session so I don't forget what happens, and so I do not have to take notes. I want to make sure I am paying attention to what you are saying, and not too busy taking notes.* (No students declined to be videotaped)
- On the role of the interviewer's questions: *I might ask you a lot of questions. It's not because I think you're wrong. I just want to make sure that I understand what you are saying, because I can't ask you again after you leave today.*
- On the role of the whiteboard and paper: *You can use the paper or the whiteboard to write things if you think it would help you. The camera will see what you write on the whiteboard very easily, so if you have no preference I prefer you use the whiteboard. If you don't like that, feel free to use paper or nothing at all.*

The interviews were split into two parts, the first to get at what students understand, broadly, about infinite series and convergence, and the second to get at how students address problems and specific tasks around these topics. The initial prompt for all interviews is below.

Initial Interview Prompt

Imagine that I am a student in [a / your] Calculus II class, and I missed the first few lectures on infinite series. Can you help me get caught up, and tell me what I missed, so I don't fall behind?

The initial prompt establishes several things: the audience for the explanation (a student), a starting point for the material (infinite series), that the interviewer will be asking questions from the perspective of a Calculus II student, and that the expectation is to “tell me” (use of the word *tell* and not *teach*, to suggest that the objective was not to engage with a discovery-type teaching scenario, but a more “lecture-based” scenario). The initial prompt is very vague, and while some students simply launched into a teaching scenario, others had straightforward questions – see below. Standardized answers (in italics) for such questions are also found below. Most students were very comfortable to simply begin explaining the topic of infinite series³, and assumed the role of “teacher” very well, playing along that the interviewer was really a “Calculus II student.” This created remarkably few problems. Because of the structure of the interviews, these sessions will be referred to as ‘teaching episodes.’

³ It is important to note that much of what the students say and do, as reported in the coming chapters, is actually mathematically incorrect. We will not, however, refer to their understandings as ‘misconceptions’ or errors. Rather, we use these perhaps faulty and nonstandard explanations as evidence of what students *do* understand about the material. Thus, their ‘knowledge’ is essentially the set of ideas and methods that are used when explaining, solving, or reasoning with tasks, mathematically correct or not (see Schoenfeld, 2011). Recognizing that students came to this understanding, mathematically correct or not, as a result of working problems, attending lecture, reading their textbook, etc... it is important to work with what students actually say and draw as part of their explanation, as it is an authentic account of how they represent and connect the mathematical content at hand.

Common Student Questions (and responses) at Interview Launch

- Do you know about sequences yet? *Maybe? But please remind me.*
- How far should I go? *Go ahead and start, and I'll ask questions if I have them. We'll see how far we get.*
- Do you know about improper integrals? *Yes.*
- Should I use examples? *If you think it would help you make your points, then yes!*
- Should I write on the board? *You can write on the board if you like, and the camera will see it very easily. If you want to write on paper that is fine, but I might have to move the camera.*
- Can I ask you questions? *Sure, but I might not answer them. Remember, I am a Calc II student, so I will be asking you lots of questions.*

From the initial prompt, the teaching episodes were steered by the students doing the “teaching.” If they chose to highlight specific examples, they used them for the purposes they saw fit. The interviewer injected typical *Can you say more?* or *I don't quite understand* prompts to provide opportunities for the students to clarify their responses. Other standardized responses to some of the more common occurrences in teaching episodes were as follows.

Standardized responses to common interview occurrences

- If students use the word “converge” without first telling the interviewer what it means: *Wait, I don't know what converge means? Can you maybe go back and say a little more?*
- If students do not label axes or some portion of a picture: *What does [X] represent? I've never seen that particular picture before.*
- If students were excessively gesturing but not writing things down: *I see you're doing something with your hands there? Can you record that so I know what it is you're picturing? Maybe on the whiteboard?*

The interviewer was also equipped with two notecards on which some more standard examples were printed, in the event that the students could not remember anything, or could not construct a starting point. These examples, which were discussed in every lecturer's notes sampled before beginning interviews, were:

Notecard tasks for students who had difficulty getting started

- We can show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 1. Why does this make sense?
- True or false: $0.99\overline{9} = 1$.

The first was used more frequently than the second, as a way of reminding students about sigma notation, and encouraging them to talk about convergence and tell the interviewer what it means. Neither was used very often, and in both cases, these prompts only served as a reminder to students, who then began their own explanation that was only somewhat related to the contents of the notecard. A large percentage of students chose to use the first example, or a very closely related example, spontaneously.

When students indicated that they were finished with part of their explanation, several more standardized prompts may have been applied.

Standardized prompts for students who are “finished” with their explanation

- If the student has apparently completed their explanation without drawing any images, *Suppose I tell you I am more of a visual learner. Could you use a picture to help me understand?*
- If the student has apparently completed an aspect of their explanation but not exemplified it, *Can you give me an example of that? I’m not sure it makes sense.*
- If students attempt to make a claim but only provide evidence for part of it -- for example, if using the example $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$: *I think you’ve convinced me that it converges, but I’m not sure why it converges to 1. Can you maybe explain that part?*
- If the student has sufficiently explained a topic, or has nothing else to say about it, *That makes sense. Is there anything else that could happen?*

These and general *explain more* prompts were used until all students had addressed a list of unspoken questions, which were predetermined and consistent across all participants. (See below for the list.) The prompt about being a *visual learner* is particularly relevant, as the focus of the analysis is on students’ use of images/representations to explain series convergence. Therefore, if they did not produce images as part of their description, it was important to elicit them, and to understand whether and how students used these images on a more day-to-day basis, when reasoning with this material. The intention was not to direct students to draw images from the outset (by building it into the original prompt), however, because part of the exploration was to uncover how students incorporated images into their explanations when they found them to be useful/illustrative. However, during their interviews, most students created images unprompted – over 90% of them would have been categorized as ‘visualizers’ by Alcock and Simpson (2004, 2005) classification scheme for visualizers/non-visualizers responding to infinite series tasks.

To make sure that a student had at least partially addressed all aspects of this list, the interviewer kept a checklist handy, and asked additional prompts until either all aspects were addressed, or it was clear that a student was not going to address a particular aspect (or more). It is worth note that at least half of the students also discussed alternating series, though this was not explicitly on the list.

(Unspoken) List of questions to which some discussion/response was required before moving on

- What is a series?
- How can we picture a series?
- What parts of a series “matter” – what should I be attending to when thinking about series?
- What is “infinite” about infinite series?
- What does it mean for a series to converge?
- What is an example of a convergent series and why does it converge?
- What happens if a series does not converge?
- What is an example of a divergent series and why does it diverge?
- How do we know if a series converges or diverges?

Following the general explanation phase, during which the students mostly had the freedom to take the conversation in whatever direction they chose, the protocol shifted to a series of “consistency checks” to put the students’ explanations in perspective, including the following:

Consistency Checks on “Converge”

- To assess the influence of the peer-to-peer explanation scenario that was explicitly constructed at the beginning of the interview: *If I had asked you to explain this for your GSI or your professor, would you say or do anything differently?*
- To get a bottom-line, rephrasing of what convergence means, which is a response to a question that was asked of all students in the same way: *Explain the meaning of $\sum_{n=0}^{\infty} a_n = 4$ (which was written on a notecard and handed to the student). If you feel like you’re repeating yourself, that’s fine. I just want to make sure I understand.*

These were followed by a more general “problem solving” phase, during which students reasoned about tasks provided by the interviewer. For the three tasks below, which refer to specific infinite series, it was clear that most students “knew” that the harmonic series diverges (though no student in the sample could provide an adequate mathematical explanation for why), but not straightforward as to how students could leverage that information and the principles that they know about convergent and divergent series to make conclusions. While it was not an object of study to examine whether or not students answered these questions correctly, they did provide evidence of students’ competence in answering more routine questions that often appear on their homework. For example, in reasoning with the tasks that are derived from the harmonic series, students communicate many things that they know about the harmonic series in general.

Problem Solving Tasks with Specific Series of Numbers

- Make the strongest possible statement about the convergence of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$
- Make the strongest possible statement about the convergence of $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{11} + \frac{1}{12} + \frac{1}{14} + \frac{1}{15} + \dots$
- Make the strongest possible statement about the convergence of $\sum_{n=0}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

These were followed by a different sort of problem solving tasks that encouraged students to reason about more abstract scenarios, permitting them a context in which to talk about convergence and divergence more generally. These tasks, therefore, tapped into students' understanding of general behavior of series of numbers.

Problem Solving Tasks that represent General Behavior of Series

- Suppose $\sum_{n=0}^{\infty} a_n$ converges, and m is a positive integer. Does $\sum_{n=m}^{\infty} a_n$ converge, diverge, or can't you tell?
- Suppose $\sum_{n=0}^{\infty} a_n$ converges and $a_n \neq 0$. Does $\sum_{n=0}^{\infty} \frac{1}{a_n}$ converge, diverge, or can't you tell?
- Suppose $\sum_{n=0}^{\infty} a_n$ converges. Does $\sum_{n=0}^{\infty} (a_n)^2$ converge, diverge, or can't you tell?
- Suppose $\sum_{n=0}^{\infty} a_n$ converges. Does $\sum_{n=0}^{\infty} \frac{1 + a_n}{2}$ converge, diverge, or can't you tell?

The purpose of the entire set of tasks was not to test students or determine if they could get the right or wrong answer. Rather, the purpose was to hear how they reasoned, aloud, with and about convergence, and what they took as evidence of convergence.

The interviews concluded after students solved many or all of the tasks above. At the close of the interviews, all students were asked about their math backgrounds, their grades in Calculus II (if they had taken it previously), and to talk candidly about what they thought about infinite series, why they learn it, why it is useful, what is hard about it, and if they had had preconceived notions about the topic before enrolling in Calculus II.

Having described the structure of the interviews, I do want to note that examining students' use of images while *explaining* this concept brings a different dynamic to much of the existing literature on representation use, which is largely about students' use of images during problem solving. Cox (1999) describes existing literature in this domain –

“The literature on reasoning with external representations (ERs) contains two types of study. In the first kind, subjects interpret presented representations (e.g. Mayer, 1976; Newstead, 1989). Models of reasoning with external representations, such as that of Zhang (1997), have tended to focus upon tasks involving the interpretation of (and interaction with) *presented* representations. In a second type of study, subjects construct their own representations (e.g. Schwartz, 1971; Greene, 1989; Grossen & Carnine, 1990 Cox & Brna, 1995; Cox, 1996, 1997).”

Though it goes unstated, the studies to which Cox refers are all in the context of students using images as a means for reasoning with particular problems, and for problem solving. I would propose the significance of a third important way that students’ use of images is worthy of study – that is, their reasoning with images while in the act of explaining a general phenomenon or concept. Some literature starts to address this type of reasoning (e.g. Kindfield, 1993), but often characterizes the explanation of a concept as the collection of things that students do in response to a set of tasks meant to probe their understanding of it. It seems, though, that explaining a concept is cognitively different than problem solving with a particular math task, and may therefore elicit different images, or prompt students to use the same images differently. Through their explaining, it is possible to see how students use features of their images to justify why a concept is how it is, and not just how it gets utilized in the very specific context of a particular task.

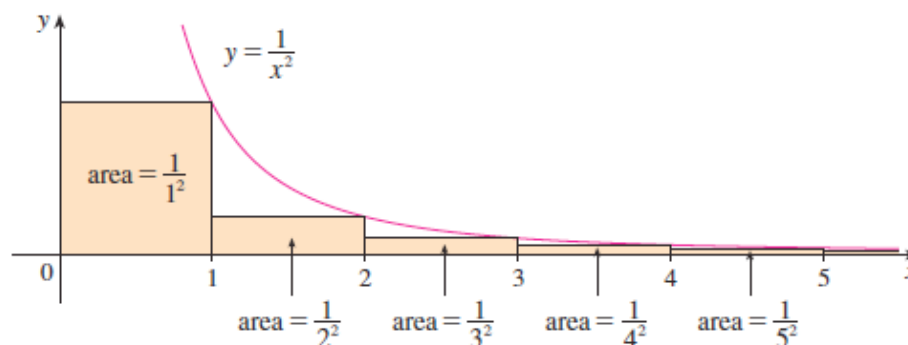


Figure III.6. Graph taken from Stewart (2007), Chapter 11.3

Take, for example, two different scenarios along which student reasoning could play out. If a student was faced with a task such as “Does $\sum 1/n^2$ converge or diverge?” she may reply by drawing the image in Figure III.6., and remarking that ‘this graph converges because the area under the curve shown is finite.’ With such a response, it is unclear if the student believes that the series converges *because* the area is finite, having carried out something like the integral test and then connecting the improper integral and infinite series, or if the appearance of decreasing terms/decreasing sizes of areas suggest to the student that the area must be finite, and therefore the associated sum must also be finite, the added areas becoming smaller and smaller. That is, with such a response to this task, the student’s understanding about the meaning of ‘convergence’ is unclear, as the response gives no indication of the way that the student is connecting the different parts of her image.

Consider now a student who responds to the prompt “Explain the meaning of the statement ‘The series $\sum 1/n^2$ converges.’” When faced with a task that requires an explanation of a more general concept, if a student draws a graph like that above, it is possible to get at the factors that the student understands to be associated with convergent series. Her response, “it converges because the areas you’re adding get smaller and smaller” is suddenly distinguishable from “it converges because we compare these areas to the improper integral, which is finite,” where in the previous example either would be a reasonable interpretation of the same response. Eliciting explanations of concepts from students, then, may uncover additional aspects and specificity of students’ knowledge that were not able to be extracted, without significant interpretation, from their responses to problem solving tasks.

III. D. Data analysis methods

Analyses of the data are specific to the purpose of the chapter at hand. When attempting to tease apart themes in students’ image use, as they correspond to their understanding, an ‘open-coding’ process to explore the themes as they arise in data was used. This process resulted in the identification of five distinct image types in the large data set, each used in various ways and iterations by students, professors, and the textbook. One fruitful exploration with these image types was to uncover which mathematical features that *could* be discussed via a particular image type were actually incorporated into a student’s explanation. Another emergent theme was in the students’ modes of use for the differing image types. The purpose of such an analysis, relevant to Chapter IV, was to explore the mathematics involved in a particular explanation, and how that mathematics is put to use accompanying explanation. Kindfield (1993) serves as a model for such an analysis. (Kindfield’s study was in the context of meiosis diagrams in biology.)

In Chapter IV, the content analysis of students’ SGR – particularly the mathematical features involved and modes of use – will speak to the ways that students use the various image types to demonstrate their understanding of what it means for infinite series to converge. Following this analysis, individual illustrative cases are chosen from the larger data set to speak to the three dimensions of competency (limiting processes, language, and connections) in Chapters V, VI, and VII respectively. A range of examples and several in-depth cases are described by providing both transcripts and photos/recreations of students’ SGR, as they become relevant.

IV. CONTENT ANALYSIS

This chapter presents the collection of image types that students used in their teaching episodes and characterizes what they did with those images. In particular, it includes a detailed discussion of what sorts of mathematical reasoning those images support, and the different ways that students used them, in addition to ways that “experts” (professors during lecture and textbook (Stewart)) used those same image types. The presentation of the affordances, drawbacks, and different groups’ modes of use for the various image types is intended to demonstrate that the students use many of the same images as the ‘experts’ when providing *explanations* of infinite series of numbers. However, the aspects of the mathematical concepts that they highlight in their explanations and modes of use for those image types are different, and at times conflicting. Documenting some of these differences will prompt further analysis of students’ understandings, along three dimensions – limiting processes, language, and connections – to help explore ways that the students’ understanding is different than that of the experts, as evidenced in the ways that they use some of the same image types, differently.

Outline of Chapter IV -

- IV.A. On the relationship of images and explanations

- IV.B. Aspects of the analysis of infinite series images

- IV.C. Different Infinite Series Image Types

- IV.C.1. Plots of Terms

- IV.C.2. Plots of Partial Sums

- IV.C.3. Areas Under Curves

- IV.C.4. Number Lines

- IV.C.5. Geometric Shapes

- IV.D. Breaking down infinite series competence

IV.A. On the relationship of images and explanations

The challenge of understanding in general how students use images for purposes of problem solving and explaining is that the types of images and ways that students interact with them are widely varied and often tied to particular prior knowledge or experience. As Sherin (2000) discusses, this already difficult task is further complicated by knowing very little about students’ general “representational capabilities,” and reasons why students gravitate toward images when reasoning or explaining. As he suggests, then, one place to look to begin to unpack this complex relationship between students and their inclinations to generate images is in contexts in which they are creating or inventing their own images. These are scenarios in which there are no limitations placed on the situation that require students to use images that are already presented to them, or particular types of images prescribed by a problem context.

That a student chooses, unprompted, to use a particular image indicates that he or she finds it conceptually useful. These images, or SGR (self-generated representations) do not have to be original or unique. Students may produce SGR that mimic or recreate other images that they have seen in lecture or their textbook, or are familiar with from other sources. In exploring the ways that students apply these images and use them to make arguments as integrated parts of their explanations, we can learn more about their understanding of particular topics. For example, what do they think the image is useful for? What do they think it tells them about the mathematical concepts at hand? What in it are they paying attention to?

Students see numerous images throughout their study of calculus, foremost from lecture and in their calculus texts. In these textbooks, there are few graphical representations, but the ones chosen by textbook authors are used with particular purposes – they are there to help clarify definitions, exemplify properties, and connect up different ideas. These images are often inserted after definitions, to clarify the text or illustrate mathematical ideas that can arise from the newly defined concept, or in the page margin as historical reference, additional justification, or demonstration of the new ideas. What is so important about these images is that they are introduced to do a specific job, for a specific reason. However, students often latch onto images from their lectures or textbooks, and use them in their own reasoning, whether or not the conclusions they are drawing are consistent with the intended use of the particular image.

In the context of infinite series, this is potentially very tricky. For example, images akin to those that students have encountered in their study of Riemann sums and improper integrals are resurrected in the section on the ‘integral test.’ These images are used to justify the particular conclusions that this convergence test allows. However, students’ tendency may be to use this type of familiar image for much broader purposes than it was intended, since it already carries some meaning for them and is recognizable. Moreover, the border between series and sequences is nuanced. Often lectures cover infinite sequences on one day and then begin infinite series on the following day, and textbooks state that ‘series convergence is convergence of sequence of partial sums’ in a compact definition (see Figure IV.1. below). Frequently there is little to no transition or additional motivation from a discussion and images of sequences to a discussion and images of series, except for them being in back-to-back sections of the same chapter.

2 DEFINITION Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. Otherwise, the series is called **divergent**.

Figure IV.1: Stewart’s definition of convergence of infinite series

In the lectures observed as part of this study, most professors did not dedicate more than five minutes’ time to discussing the connections between infinite series and sequences. More often, the exploration of infinite series was introduced as a ‘new topic.’ When references were made to the students’ study of sequences, they were as part of definitions like that above, from Stewart (2007). Note that the definition uses concepts that students identify in interviews as “new and difficult” and notation with which students frequently struggled during their teaching episodes (sigma notation) to define the new idea. So, when students saw various images through the course of calculus lecture and in their calculus textbook, they were framed mostly as new material and were not explicitly connected, except by the use of the same words, to their previously learned topics. Students did at times see different images used as part of explanations,

or to facilitate problem solving and drawing inferences. However, on homework and in class, students were not presented the opportunity to *explain* what they understood about convergence of infinite series. Instead they were asked to apply what understandings they did have to solving problems involving infinite series. By students' own admission, this often resulted in blind application of a cadre of tests or procedures, with little understanding of the phenomenon that the tests will assess, or of remembering applying some tests that didn't hold much meaning for them. For example, Mark and Tiffany indicate that they often do not challenge the meanings of such tests introduced in class.

Mark: It's not supposed to decrease but... So yea just looking at it, it's hard to tell why one converges while the other diverges. But then like ... if you ... I think it had something to do about rearranging... It was something along the lines of comparing it to another series, and...

Int: What works for you though?

Mark: In terms of?

Int: Making sense out of why one converges and why one doesn't? Because that book comparison thing might not work for you and that is fine. But what...

Mark: Honestly, **sometimes I just take it at face value.**

Tiffany: Yea I was very confused at that part too. **I just took the teacher's word for it.**

Int: Well you can try to make sense out of it though?

Tiffany: They did prove it.

Int: Ok.

Tiffany: I don't remember how.

Int: That's ok. We can try and make sense of it now.

Tiffany: So like if you did one with, (writes the harmonic series on the board)

Int: You said you think that adds up to something, or doesn't?

Tiffany: I think this is the one that does. Is this the one that does? We did this in analysis on the first day and I can't remember already.

Beverly correctly identifies a series she uses as an example as having a particular reference in second semester calculus. However, rather than talking about how to make sense of series convergence, she indicates that one must simply remember the tests and then use them to make conclusions.

Beverly: This is a p -series. And we know, and you discuss in Calc II, that $1/n^p$ has certain parameters. $p > 1$, $p = 1$, and $p < 1$ all do different things. So like **if you remember the test, you can use it.**

IV.B. Aspects of the Analysis of Infinite Series Images

While explaining and reasoning with infinite series problems in their teaching episodes, participants in this study created images of five different types: plots of terms, plots of partial sums, area under a curve, number line, and geometric shapes. This chapter details the following four aspects of *each* of the different image types, toward making claims about how individuals incorporate the different image types into an *explanation* of what it means for infinite series of numbers to converge:

(1) What mathematical features/aspects of mathematical content are included in the different image types?

Mathematical features as discussed here refer to specific aspects of mathematical content that are represented in a particular image type, whether or not the user of that image chooses to highlight those features. For example, if a particular image type contains references for both the sequence of terms of the series and the sum of the series itself, then both of those two processes could be explored with that image.

Kindfield (1993), who examined people's SGR on meiosis tasks, serves as a model for this type of analysis. She examined SGR by identifying contextually important features of meiosis SGR that are characteristic of more and less successful (expert vs. novice) reasoning. However, in this analysis, there is not a division of features that are characteristic of more or less successful reasoning. Rather, highlighting mathematical features is intended to identify what the different images can potentially elicit as part of an explanation.

(2) What are the affordances/drawbacks of the different image types?

Similarly to identifying mathematical features of each image type, affordances and drawbacks of the images themselves are identified, independent of how the user incorporates them into an explanation. This is meant to demonstrate the ways that a particular image type could facilitate productive and mathematically sound explanations, or how it could lead to misconceptions or faulty mathematical conclusions.

(3) How do individuals use features of their images to provide support for or highlight aspects of their explanations?

There were four distinct modes of use for the images, during teaching episodes and as presented in the text. These images were used for (a) defining, (b) clarifying, (c) exemplifying, and (d) drawing correspondences. When using an image for defining, students were drawing and explaining with a particular SGR with the goal of defining what is meant by series, convergence, or some other related concept. Images that were drawn for defining are those produced when students are explaining concepts for the first time, so that they use the image to introduce the concept. Images used for clarifying were those drawn and used for responding to a clarification or question that arises from their defining phase. These clarifications could arise after interaction with the interviewer, or because the student herself felt the need to clarify a point to discuss a subtlety/qualification associated with her definition. If an image was used for exemplifying, it was drawn and used when providing an example to illustrate a concept that the student had introduced previously. That is, a student could use a particular image when introducing (defining) a new concept, use another image for clarifying parts of the definition that were unclear or questionable, and perhaps yet another image for exemplifying that concept that was previously introduced – i.e. an image that was used when discussing a particular example meant to highlight the student's definition. Or, a student could use the same image for all three purposes, at various points during the teaching episode.

Finally, images were also used for drawing correspondences – that is, after the student had defined or exemplified the concept they wished to introduce, they often used an image to point out correspondences between common features across the various images already on the board and the various ideas already discussed during the teaching episode.

For this analysis, examining modes of use is presented for three groups:

(3a) Students – Most importantly, students’ teaching episodes were analyzed in order to determine their modes of use for particular image types. While not all students used the same image types in the same ways, prevailing modes of use for students’ image choices were identified. The number/percentage of students who used each image type is also documented. Since there were no level- or institution-specific patterns in which student groups most often used particular image types, the student data were collapsed into one group.

(3b) Mathematicians/professors – The observed lectures of the Calculus II classes and the lecture notes that were analyzed for instructors’ images and definitions were taken as evidence of the ways that mathematicians/professors use images in the act of explaining the concept of infinite series for the first time. Modes of use were therefore determined by the ways that the professors used the images as part of their instruction. There was substantial uniformity in the ways that professors used each image type.

(3c) Textbook – Using Stewart (2007), the textbook used by all students in the sample, modes of use for the different image types were determined by analyzing sections of the chapter on Infinite Series (Chapter 11) which pertained to sequences and series of numbers.

(4) Similarities and Differences in use of image type

For each image type, after identifying the ways that images were used, percentages of each group who used that image type, and abundant examples from the data, a discussion follows about the ways in which the different groups’ uses of that image type align or conflict.

The analysis of image types will demonstrate that determining competence with infinite series goes beyond simply identifying various ‘misconceptions’ and designing instruction to directly address those. The modes of use for the various image types across groups were very different, and at times conflicting, thereby prompting the need to examine the factors that led each group to use the images so differently. In this way, competence with infinite series is framed as a combination of the factors that differentiated the modes of use – (a) the limiting processes that are valued and prioritized, (b) the language that is used to explain and describe the concepts, and (c) the connections made among concepts. In short, this chapter documents:

- the importance of the ways that people pay attention to different **limits/limiting processes**. It shows that differentiated attention to limiting processes has great impact on the mathematical validity of students’ explanations. The selection and use of images is a very personal event (cf. Wheatley and Cobb, 1990). Different people looking at the same image will notice different things, and different people explaining the same concept will focus on different things. Different groups attend to different limiting processes with various image types, an issue pursued in Chapter V.

- the use of **language**. Language is employed during the teaching episodes in both consistent and inconsistent ways. The teaching episodes make clear that students use language differently than professors/mathematicians in ways that have an impact on understanding. Because language use is such an important factor in students’ explanations, this will be further explored in Chapter VI.

- ways in which students make **connections**. Connecting different limiting processes in particular ways, with particular language, shapes students’ understanding and explanation of

infinite series of numbers. As Wheatley and Cobb (1990) describe, “one gives meaning and structure to a spatial pattern based on his/her experiences, influenced by available conceptual structures, intentions, and the ongoing social interactions in which he/she is involved” (p. 162). That is, different individuals connect their understandings differently, and the nature of those connections, as shaped by their experiences and prior knowledge, will differ even if the base concepts are shared by multiple individuals. The remarkable differences in the ways that students connect ideas via the different image types, compared with experts’ usage of those same image types, will be investigated in Chapter VII.

IV.C. Different Infinite Series Image Types

IV.C.1. Plots of terms

Mathematical Features. A common image type in teaching episodes is a drawing of a graph representing the terms of a series, either by dots or by a smooth curve⁴. That is, a plot of the a_n , where the series is modeled by

$$\sum_{n=1}^{\infty} a_n$$

This is referred to as a “plot of terms” because the creator of the self-generated representation (SGR) is choosing to represent the terms of the series as n tends to infinity, to demonstrate the infinite behavior of the sequence of terms themselves. Examples of this image type are below, in Figure IV.2.

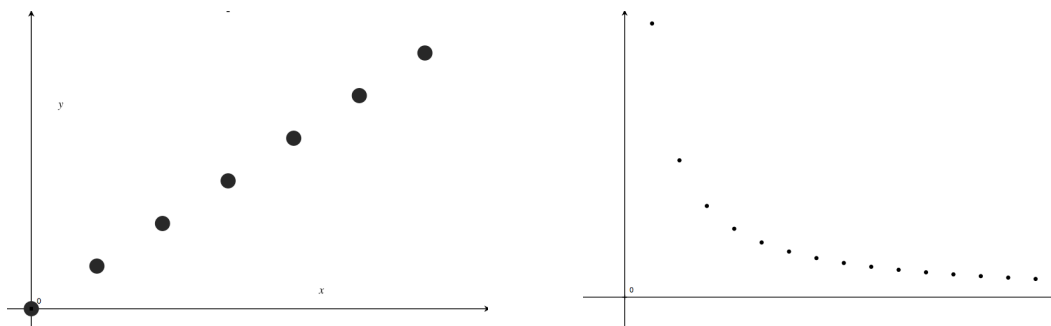


Figure IV.2. “Plots of terms” images

If used to represent the values of n and the relative sizes of the terms in the series, then the only mathematical processes or structures that can be inferred from this image type, without using it to draw correspondences to a different image type, are the different values of n , and the associated term a_n . This image type can demonstrate sequence convergence, but in no way can lead to claims about the convergence of a sum of positive terms. It can, however, potentially support claims about diverging infinite series (if the graph of the terms has a limit of anything besides zero), or claims about convergence/divergence of alternating series. While this

⁴ Previous studies (Stergiou & Patronis, 2002) have explored students’ use of discrete vs. continuous representations of sequences. They noted that when students use a smooth curve for a discretely modeled problem, they still refer to and discuss the problem attending to the discrete nature of the terms, despite the fact that their graph itself is continuous.

correspondence works in only one direction for positive terms – if the limit of the sequence is zero, one cannot remark on the sum of those terms, but if the limit is anything other than zero, the sum of those terms certainly diverges – student users of this image type often do not recognize this.

Affordances/Drawbacks. This image type does afford some productive reasoning about infinite series of numbers. For example, it can be used to justify or clarify the “divergence test” (so called by Stewart): if the sequence of terms being added does not have a limit of zero, then the sum of those terms must diverge. Thus, drawing a plot of terms is useful for making claims about the sum of those terms only when the limit of the sequence is nonzero, or it fails to exist. As another way of thinking about it, drawing a plot of terms is perhaps useful in determining if the sum of those terms has any hope of converging. If one were to draw a plot of terms and find that they limited to zero, they could conclude that their sum could *possibly* converge. However, this is the best conclusion about convergence of a series that one could make from such a graph.

One of the biggest drawbacks, then, for students using this image type, relates to the nature of conclusions that can be made in this context, when thinking about what is necessary vs. sufficient. Some research has shown (Harel & Sowder, 1998) that students at this level struggle with the notion of necessary vs. sufficient, so it is reasonable then that students faced with this type of conclusion structure may not fully appreciate that the sequence of terms tending to zero is necessary, but not sufficient, to determine that the sum of those terms is convergent.

Another potential drawback for this image type is that graphs of this type are very frequently seen in lectures about infinite sequences – often the lecture immediately preceding those on infinite series. If the topic of infinite sequences is covered quickly, or in not a lot of detail as Tall and Schwarzenberger (1978) would suggest, then the transition to infinite series may happen more quickly than students are prepared for. Consider the right side of Figure IV.2.

All one can say about the series $\sum_{n=1}^{\infty} a_n$ is that it *might* converge, assuming the terms $\{a_n\}$ converge to zero. But, what is visually salient in the figure is that the sequence $\{a_n\}$ converges. And that image, and the fact of (sequence) convergence, may be what the student remembers.

Students’ Use of Plots of Terms. “Plots of terms” images were the most frequently used image type across all students’ teaching episodes. A total of 28 people (76%) used plots of terms during their teaching episode, predominantly as a way of defining convergence of infinite series. That is, the most common mode of use for this image type for students was for drawing either smooth curves, or dots that were later connected with a curve that represented the sizes of the respective terms of the series, and using such images to define what it would mean for an infinite series to converge. For these students, remarking on the behavior of the sequence of terms was a sufficient definition. These students often drew other images at later points during their teaching episodes, but were most focused on this type when defining convergence for the first time.

For example, Becca had introduced the idea of an infinite series and provided several examples at the start of her teaching episode. After indicating that these infinite series could either converge or diverge, she went on to define what it would mean to her for a series to converge.

Becca: With series that converge, at least you know you're adding increasingly smaller increments. So much so that they're infinitesimally small that you can basically say the sum converges.

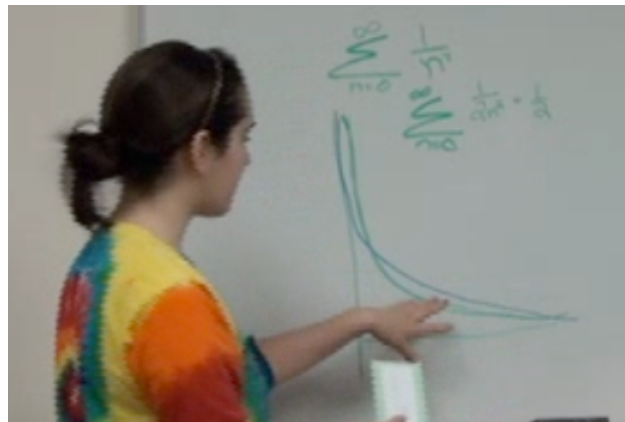


Figure IV.3. Becca's plot of terms.

Becca drew her plot of terms as she described the circumstances she believed indicated series convergence. For her, the most important aspect was that the terms were getting 'infinitesimally small,' and she chose to represent this with smooth curves for the terms of the series itself. Tim's definition was similar, and his picture just like Becca's, although he chose to create his image on the sheet of paper in front of him.

- Tim: If the numbers keep getting smaller, it's almost like you can say at some point you're not adding anything to it.
- Int: Even though we know that you're always adding another thing another thing another thing?
- Tim: Yea at some point it doesn't really matter.
- Int: So are you saying those things you're adding don't matter?
- Tim: It seems like that. That's how you can imagine that it ends. I don't mean to say you're not adding anything.
- Int: But it just seems that way?
- Tim: Yea

Additionally, Todd used the words 'converge' and 'diverge' when describing what infinite series are during his teaching episode (see Figure IV.4.). When asked what those mean by the interviewer, he drew a plot of terms and used that to define convergence.

- Int: You said something about converge and diverge. But I missed that day, I don't know what that means yet. So maybe start there?
- Todd: (draws a plot of terms) that's x and y . So if you have a series that goes this way and you're looking at a summation of this plus this plus this this this this ... if it converges then all of these added up together are equal to a value and not equal to infinity. So if you have certain values in your denominator, there's a p-series test, there's other tests you can do to determine convergence, if a function is convergent on an interval, on an infinite interval, then all of these numbers added

up together will sum up to a finite number. **As this gets infinitely smaller, you get infinitely further out. So you're adding infinitely smaller things together. so it ... the infinities almost cancel out each other.** If you [inaudible]

Int: You mean the infinite amount of terms, but the fact that they're getting infinitely small?

Todd: Yes

Int: Ok and so can you say then what that would mean one more time?

Todd: If they get small enough, fast enough, then that means that the series will converge. Because as it gets smaller you're adding up smaller and smaller numbers to the total sum. Which sort of with some hand waving makes it equal to a finite number.

Int: Small enough fast enough... how do you know what's fast enough?

Todd: Well you have a lot of tests you\\

Int: \\oh you so like different tests to tell you?

Todd: yea

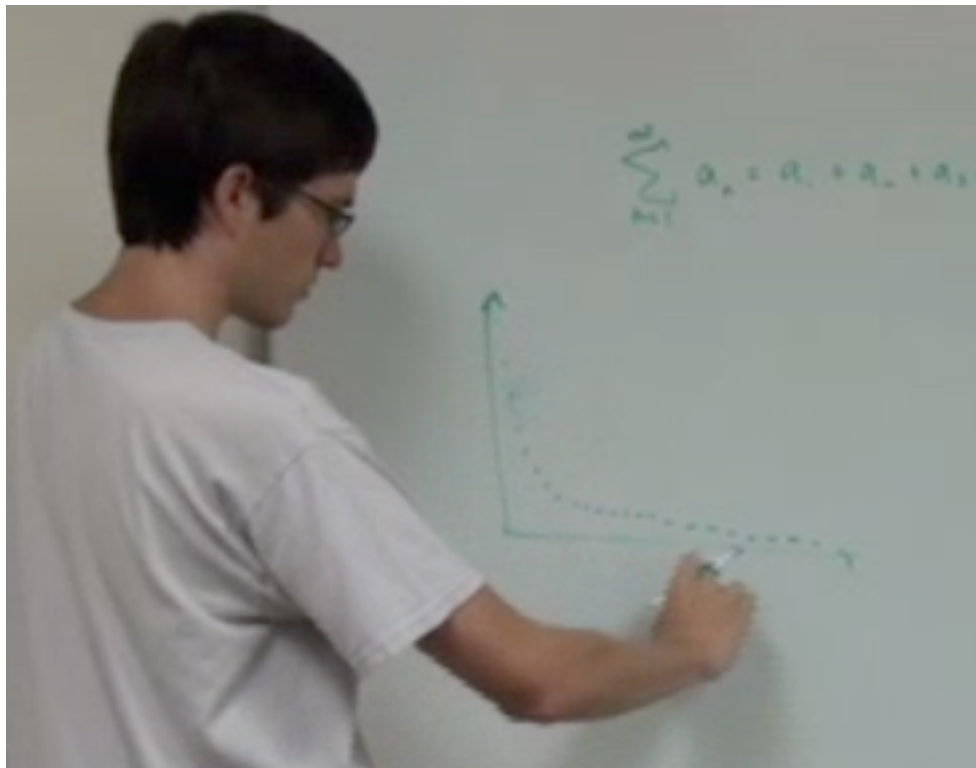


Figure IV.4. Todd's plot of terms

Todd, Tim, and Becca are typical of students who used plots of terms to define convergence. Yet other students also used this image to clarify what they meant by converge, if they had defined it with words or by writing out terms of the series. For example, it was common for students to write a sample series, such as $\sum(1/n^2)$, and declare that “convergence” meant ‘adding smaller and smaller things’ so that the sum could be finite. When probed for additional information or a more detailed explanation, students often clarified by drawing a sequence of terms graph, reiterating their previous stance while pointing at the image they had drawn.

Such a student, Brad, had defined convergence of infinite series without any image. Instead he had used ideas from computer programming and a ‘for-loop’ to describe what it meant to have an infinite series and for that series to converge or diverge. However, after providing that initial definition, Brad chose to draw a plot of terms to clarify what he meant and help the interviewer understand what was consequential in the context of series convergence.

Brad: So basically whenever you've got powers in the denominator, that's basically what you need to pay attention to. So when the power is greater than 1, that's basically when [the sum is] getting small enough so it's a finite number.

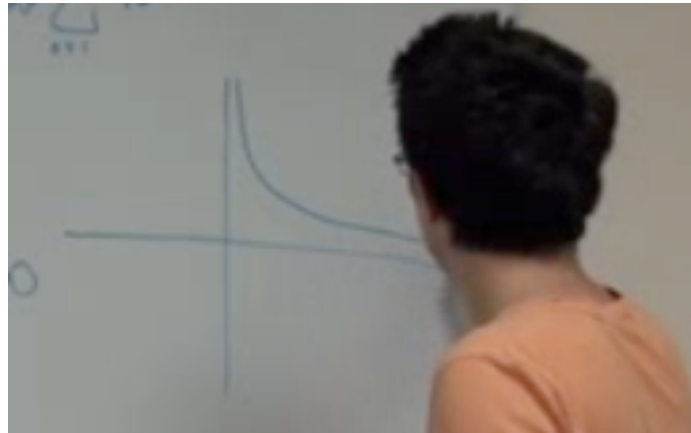


Figure IV.5. Brad's plot of terms

His initial plot of terms was intended to describe the behavior of something like $\sum 1/n^2$, which he knew to converge. Brad's idea was to use his image and that description to discuss the plausibility of an infinite series converging, as a way to clarify his computer science metaphor, which included a criterion for terminating the loop when the sizes of the terms themselves were ‘too small.’ He later added a graph representing the terms of $\sum 1/n$ to the same plot, and had some difficulty reconciling the fact that the first series converged, while the second (the harmonic series), whose terms looked similar, did not converge.

Morris also used plots of terms to clarify his definition of convergence. When defining convergence, he said:

Morris: Right. So series are really only useful, if it approaches, well, I suppose not only useful, but it's important to know if the series approaches an actual number or not. So if it approaches a number it converges, or if it goes to infinity so you just continuously keep adding numbers and it won't eventually come to some conclusion, it diverges or goes to infinity.

Int: What has to approach a number?

Morris: The series.

Int: I don't know what that means.

Morris: Actually both the sequence and the series. So first off, for a series...

Because even though you're continuously... adding an infinite number of values from a sequence, still a series has a potential to actually come to an actual value. Why, I'm not exactly sure and there's some ... I suppose there's rules to that. So the first thing is you can't add... say you have the series of 1's, so you just

continuously add 1, that would obviously go to infinity because you have an infinite number of 1's added together. Therefore the value of the sequence has to approach zero because therefore the numbers you're adding get continuously smaller and smaller, and smaller. So, therefore you know you see how ... I mean this isn't a good example, but if we had a series it would eventually level off, and let's say it converges to 2, the numbers would continuously get smaller and smaller and smaller and it would approach 2. It wouldn't actually equal 2 at any given point, imagine it as like a limit.

As this statement made use of an example, was somewhat convoluted, and included a lot of hand motions, the interviewer asked for clarification of Morris' ideas. He responded by drawing a plot of terms.

Int: So wait how is it possible that a list that is infinitely long could add up to something that is finite.

Morris: Cuz ... (long pause) I suppose based on the actual values that you are adding up, at a certain point it gets to such a small number that essentially you're adding up smaller and smaller and smaller numbers, that it just ... the amount of change is very, it won't ... it couldn't really affect it at a certain point. I suppose, actually that's right.

Additionally, though Ben shaded an area underneath the plot of terms he had drawn (see Figure IV.6.), represented by a smooth curve, he looked to the sizes of the terms themselves to define convergence. Again when clarifying his definition, he turned to a plot of terms to help make his point about the sizes of the terms decreasing and getting "considerably smaller" as being indicative of convergence of series.

Ben: So each successive area, let's take the 20th term. You'd have $n=20$, you'd have $1/20$. And that number is 0.04. (recording the values on the whiteboard). And so that's pretty small. Then you say ok let's take the 2,000,000th term.

Int: Ah that changes it quite a bit.

Ben: And 1 over 2 million is//

Int: //considerably smaller.

Ben: Many more zeroes. So you're adding such a small amount that it becomes almost negligible in the end.

Int: So that's how it could add up to something finite?

Ben: Yes. You have to accept the fact that even though it's a number you're adding, it's negligible.



Figure IV.6. Ben's plot of terms.

Professors' Use of Plots of Terms. Professors' uses of plots of terms in lecture-based explanations were limited entirely to clarifying and drawing correspondence. Professors used this for clarifying particularly in the context of introducing the "Test for Divergence" and for drawing correspondence between terms of the series and the partial sums that they generate. Plots of terms were abundant in lectures introducing sequences and sequence convergence, but *only* accompanied sequences of partial sums when used by professors in lecture.

Textbook Use of Plots of Terms. Stewart uses plots of terms only in conjunction with a graph of the partial sums; that is, to draw a correspondence between the sequence of partial sums and the terms themselves that make up those partial sums in an example (exemplifying) (see Figure IV.7.). In the first section of Chapter 11 of Stewart, which focuses entirely on sequences, graphs of sequences of terms are common. However, after the introduction of infinite series, they *only* appear on the same axes with graphs of partial sums, to show how those partial sums are generated.

■ Figure 3 illustrates Example 6 by showing the graphs of the sequence of terms $a_n = 1/[n(n + 1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \rightarrow 0$ and $s_n \rightarrow 1$. See Exercises 62 and 63 for two geometric interpretations of Example 6.

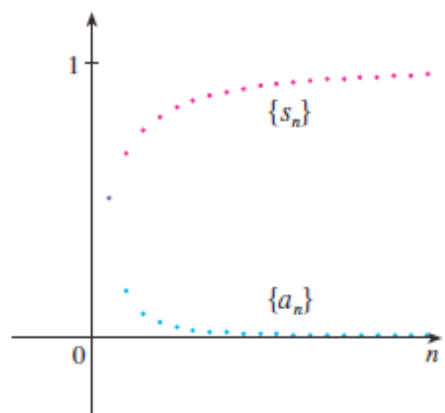


Figure IV.7. Stewart's use of a plot of terms

IV.B.2. Plots of partial sums

Mathematical Features. This image type is characterized by a graph of the sequence of partial sums, either as dots or as a smooth curve, which is continuous in the sense that it is most often used as a way of examining the infinite/eventual behavior of the sequence. That is, the plot of n against s_n , where

$$s_n = \sum_{k=1}^n a_k$$

On this image type, there is a way to reference the various values of n , as they coordinate with the values in the sequence of partial sums, which can also be broken down to show the individual sums (see Figure IV.8.). That is, while the values of the individual terms that are being added are not explicitly represented in this image type, the image itself can be broken down to show those individual terms. This image type also provides a reference for the sum of a convergent infinite series, itself, as a horizontal asymptote that the sequence of partial sums approaches. Thus, there is the potential in this single image type to refer to four different features that can be coordinated when thinking about infinite series convergence.

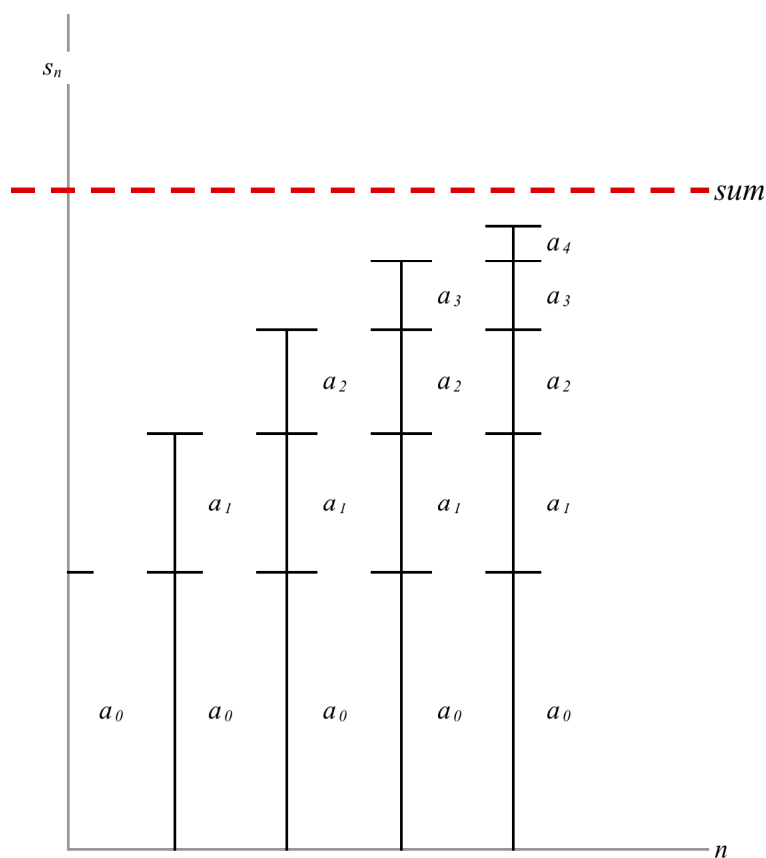


Figure IV.8. Plot of Partial Sums, broken down by term

Affordances/Drawbacks. This image type has many more affordances, compared to plots of terms, if one intends to make claims about the convergence of the sum of a particular infinite series. Because there are referents for the terms themselves, the ‘updated sum’ (the partial sums), and the value of the sum (asymptote), this image type transforms the infinite series into simply a sequence of partial sums. Because students have grappled with limits since Calculus I, and have presumably covered sequences in Calculus II prior to the unit on infinite series, this transformation turns a new and difficult concept into something connected to their previous calculus repertoire. Students have seen limits before, and are now faced with recasting the decision about whether or not infinite series converge as a decision about whether or not a limit exists. Having encountered limits and limit reasoning with graphs at multiple points prior to infinite series instruction, students can now view this image type and simply ask ‘what is the limit of this graph?’ If the graph has a horizontal asymptote, the infinite limit exists and is equal to that value, therefore the sum of the infinite series exists and is equal to that value. Thus, this image type facilitates thinking about this newer and difficult concept in ways that the students may have been thinking for some time.

This image type also facilitates students’ proclivity toward ‘process reasoning’ (Fischbein, 2001), or students’ tendency to reason with problems involving infinity as processes that must be completed. While students may use ‘process reasoning’ with plots of terms, the process there would refer to generating more terms that would later be added. While reasonable, this strategy will not help them make conclusions about *series* convergence. However, with plots

of partial sums, students can apply their process reasoning to generate ‘updated sums’ (the partial sums). This is a useful strategy if one is trying to draw a conclusion about the convergence or divergence of a particular infinite series.

Students’ Use of Plots of Partial Sums. Only eight students (22%) drew plots of partial sums SGR during their teaching episodes. While three of those students (8% of total students) did use this image for defining and clarifying, the majority of uses of this image type in students’ teaching episodes were for drawing correspondence. That is, most students who used this image type used it later in their teaching episode, as a way to illustrate the convergence of series that they had already determined converge.

Students’ use of plots of partial sums for drawing correspondences most often came after they had used the example $\sum(1/2)^n$ as part of their teaching episode, and concluded that the series converges to 1 (or 2, depending on the starting n -value). When probed about why, students like Tiffany, below (and in Figure IV.9.), drew correspondences between the way they had originally represented the example (often as groups of terms written out on the board, or as a plot of terms) and the plot of partial sums, as a way to connect up their numerical explorations with $\sum(1/2)^n$ and an image. Thus, they drew correspondences between features of their example and a graph that illustrated those features. In this context, students like Tiffany did not use the graph to define convergence, but rather to illustrate it.

- Tiffany: So I guess like for this one it would be like this (draws axes) where say this is 1 (draws asymptote). You would understand that it would go like that (draws a smooth curve approaching 1)
- Int: Wait, what's the “it” that's going like that?
- Tiffany: (laughs) So this would be the sum (labels y -axis)
- Int: So that's the sum, ok.
- Tiffany: And this would be the, I guess the “ n .”
- Int: That helps.
- Tiffany: So this would be (0,0) and this would be like (1,1/2) here.
- Int: So that's like our first guy.
- Tiffany: Yea.
- Int: So then what does that graph kind of mean? What can that graph tell us?
- Tiffany: That no matter that the higher your n value gets, the closer to 1 it gets, til it eventually reaches 1... like eventually it will hit 1 and it stabilizes at 1. And it won't keep growing.
- Int: So what are the points on there then? If that's a graph of the sum, what do the points mean I guess?
- Tiffany: It depends what you're\\
- Int: \\like where are you getting your points?
- Tiffany: By plugging them in.
- Int: Ok so like one of those things is 1/8.
- Tiffany: Yea so that would be when your $n=3$.
- Int: So that's on the graph how?
- Tiffany: It wouldn't be just 1/8. You would have to, because it's the sum it would be $1/8+1/4+1/2$ so it would be $7/8$.

Int: Ok so that somehow includes information about the $1/2$ and the $1/4$ and the $1/8$? [T: yes]. Where?

...
Tiffany: Yea how you know ... I mean I look at it like this goes up here (draws vertical line up from $n=3$) so you could extend these lines. If you wanted to know just plug 3 in without adding it would be like [the part of the vertical piece that extends past the previous sum value]

Tiffany's use of a plot of partial sums (in Figure IV.9.) is characteristic of the other students' ideas, when drawing correspondences to previous examples or definitions that they had provided. Here, she did not define anything for the first time, nor clarify a previously stated definition or example. Rather, she took an example about which she had already drawn conclusions, and used features of her plot of partial sums to identify the role of those features (terms, partial sums, etc...) in the conclusion that the series converges to 1.

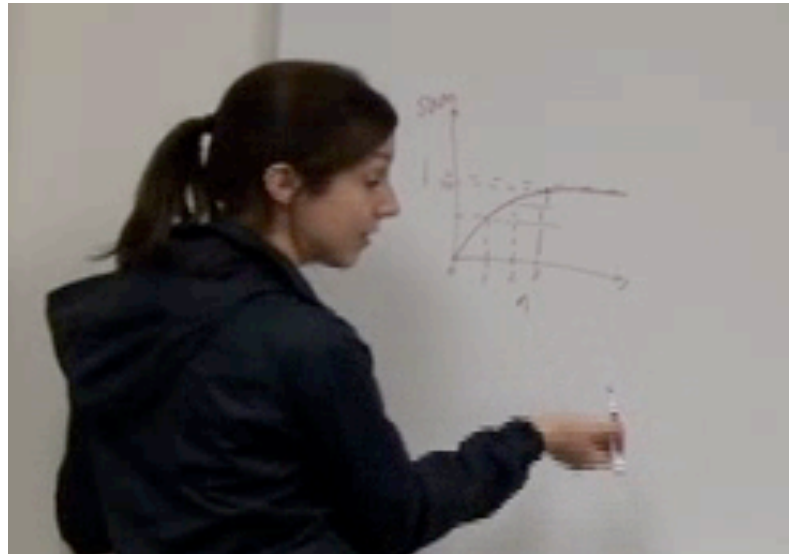


Figure IV.9. Tiffany's plot of partial sums used for drawing correspondences

Morris drew correspondences similarly to Tiffany, but for the harmonic series instead.

Morris: One example is the harmonic series, where you know, so each specific value is 1 over n so it adds all the 1 over n 's up until the point of the end. So you have the first term is 1 over 1 plus 1 over 2 , therefore the current value at $n=2$.. of n is equal to 2 is $[3/2]$. I suppose I better draw a graph. This is your n -axis, your first term is $(1,1)$. Your next term is 2 , and then $3/2$, and it continuously goes up like this.

Here, Morris had already discussed the harmonic series, showed it as an example, written out several terms, and discussed his conclusions regarding it. When he indicated 'I suppose I better draw a graph,' he used the image (in Figure IV.10.) as a way to show with a graph what happens to the sum that he had recorded on the board already, making connections between the terms and the partial sums that were already written out.

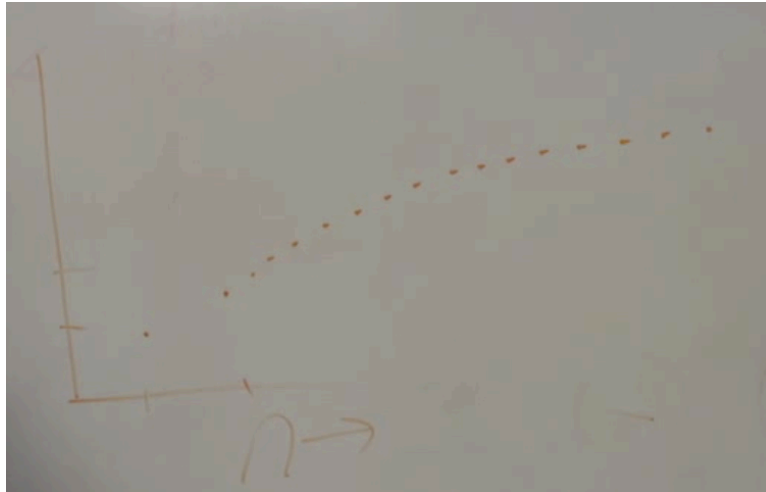


Figure IV.10. Morris' plot of partial sums for the harmonic series

The three students who used this image type for defining did so by defining series convergence with definitions similar to those used by lecturers, essentially recited while drawing a plot of partial sums. That is, the behavior of their plot captured the words that they were speaking, thereby using the image as the referent for making and clarifying the definition itself. These students, like Andrew (in Figure IV.11.), produced a plot of partial sums while stating a definition, and then interacted with or amended it when refining or clarifying their definition.

- Andrew: This is some x and y axis. (draws axes, draws dotted line at a horizontal) A converging series would go up and go to it, but a non-converging series would always increase above some line like this (pointing at asymptote), no matter which one you pick. Or will have something like, as it tends toward infinity it will be here or here, have multiple values.
- Int: I see so either multiple values or it will cross any arbitrary barrier. [A: yea] Okay so I just want to be really clear then, the curve you drew there represents what?
- A: (erases graph) It probably would be better if we draw it like steps because of each term. So like this (See Figure IV.11.)
- I: So what are those things that the steps represent?
- A: It represents each added term. Yea. I hope that makes sense.
- I: Okay, so like if I pointed at that third little step that you drew, like what does that tell me about?
- A: It's the sum of the previous ... the sum of the previous term and the new term.⁵
- I: So everything up until that term?
- A: Yeah. And that's just some arbitrary limit.
- I: What does that represent?
- A: Ummm, in the converging series it represents what it would tend to if you keep adding terms. In the [diverging] one it's just an arbitrary point that it exceeds.

⁵ We will see this student episode again in Chapter V. From a close inspection of the video, while Andrew does say "term" twice, his use is inconsistent, in that he is referring to "term" in two different ways. Video of the teaching episode makes clear that Andrew's use of "previous term" is in reference to the previous partial sum and the next term in the series.

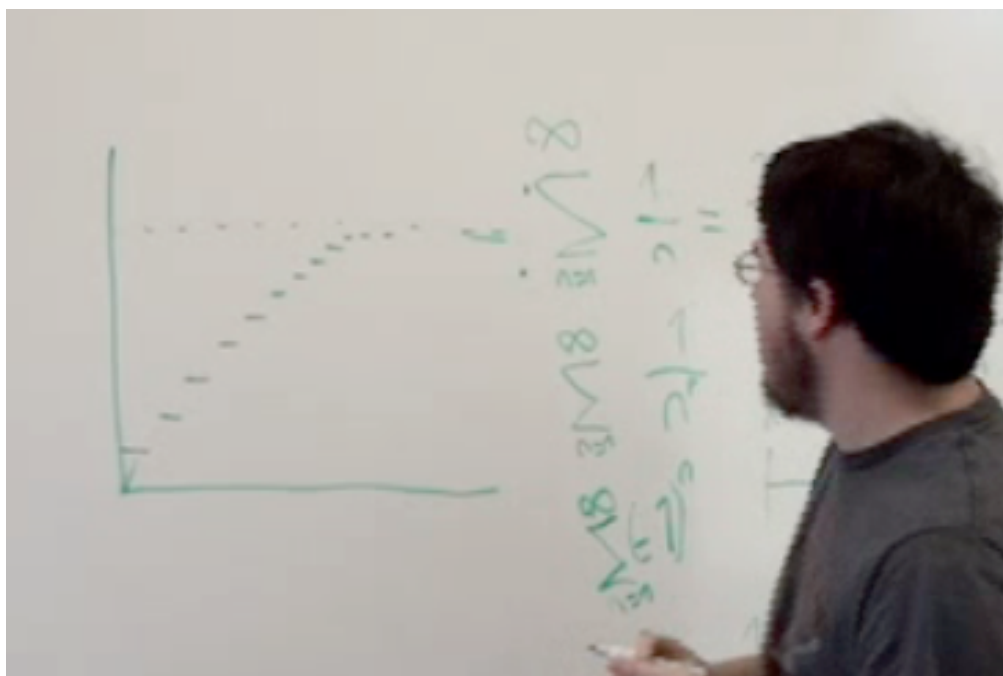


Figure IV.11. Andrew's plot of partial sums used for defining

Professors' Use of Plots of Partial Sums. During their lectures, professors used plots of partial sums either to define and clarify convergence of infinite series for the first time, or to exemplify an alternating series, after introducing the concept. When using plots of partial sums for defining, the mathematicians that were facilitating lectures often depicted these curves with dots and then sometimes connected them afterward, in order to highlight the limit of the sequence. If using this image type to exemplify alternating series, it was most often with either the alternating harmonic series or with an alternating geometric series. Not all professors defined convergence by using a plot of partial sums – however, those who did not use this image for defining convergence happened also to use no image at all when defining convergence, opting instead to use definitions and examples without images. Professors' images looked like those in the textbook (see next section), but were often not presented with a particular example. Instead, they were used to show general behavior of series.

Textbook Use of Plots of Partial Sums. Stewart's use of this image was in conjunction with a graph of the sequence of terms, and was used for exemplifying the definition of convergence, which was presented as in Figure IV.12.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s . Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

Figure IV.12. Stewart's definition for series convergence

This textbook does not use images for defining, opting instead to put text-based definitions in boxes within each chapter section. Thus, images in Stewart are mostly used for clarifying or for

exemplifying, independent of topic. The plots of partial sums used to exemplify convergence in Stewart are in Figure IV.13.

■ Figure 3 illustrates Example 6 by showing the graphs of the sequence of terms $a_n = 1/[n(n + 1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \rightarrow 0$ and $s_n \rightarrow 1$. See Exercises 62 and 63 for two geometric interpretations of Example 6.

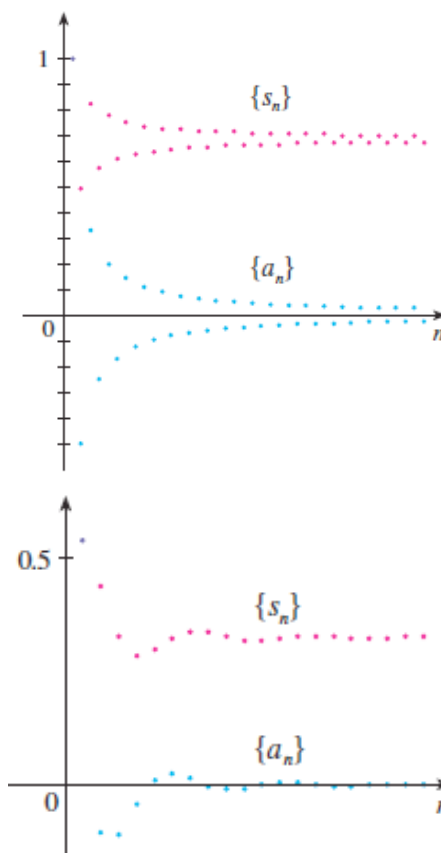
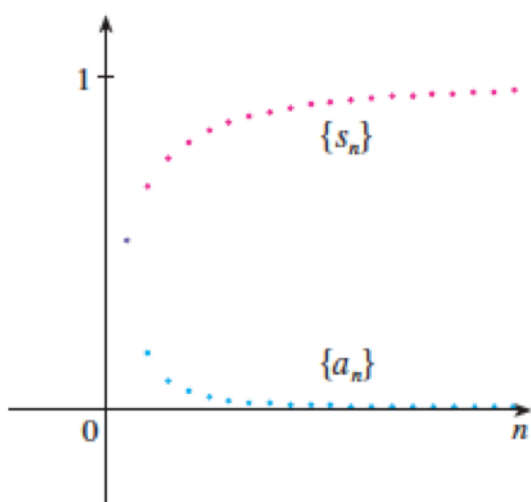


Figure IV.13. Stewart's use of plots of partial sums, for series with monotonically decreasing (positive) terms, alternating terms, and both positive and negative terms (absolute convergence) (clockwise from left)

Exploring the differences - Plots of terms vs. Plots of partial sums. Let us now explore some of the differences in the ways that students use these first two image types differently than professors/the textbook. While the calculus professors and textbook used plots of terms only for clarifying and drawing correspondences, a majority of sampled students used this image type primarily for defining during their teaching episodes (see Figure IV.14. below). This suggests that students are familiar with an important image type, as it is used frequently by 'expert' sources, but are misusing it toward making unsupported mathematical conclusions about infinite series convergence. Moreover, while the expert sources used plots of partial sums for defining and clarifying, the majority of students who used this image type only used it for drawing correspondences (though 8% of students did use it consistently with the modes of use of experts). So, not only are many students overgeneralizing plots of terms, attributing more mathematical significance to them than warranted – they are also using plots of partial sums oppositely from experts, finding the worth of that image type only for drawing correspondences to previously defined concepts.

Image type	Students	Professors	Textbooks
Plots of Terms	Define and Clarify	Clarify Draw Correspondence	Draw Correspondence
Plots of Partial sums	Draw Correspondence (Define & Clarify 8%)	Define and Clarify Exemplify	Exemplify

Figure IV.14. Different modes of use in the first two image types

That the students are using some of the same image types as the experts, but with basically opposite modes of use and intention, is alarming. It shows that students, while appreciating the significance of these images, are unclear about how these images can be used to support mathematically sound reasoning. As described earlier, some of the confusion may be attributable to student difficulties with ‘necessary’ vs. ‘sufficient’ argumentation. However, analyzing students’ modes of use for these two image types also points to one way that students’ understanding of sequence convergence may interfere with their notions of series convergence. One of the ways that we have seen the different groups use these images oppositely is that they emphasize different limiting processes when defining convergence for an infinite series – that is, students put emphasis on the limit of the sequence of *terms*, while the experts put the emphasis on the sequence of *partial sums*. Thus, the different groups are paying attention to different *limiting processes* (see Tall, 1980). This, in turn, influences which images they prioritize when explaining this concept.

Returning to the image types, three additional image types emerged in student teaching episodes. Area under curves, number lines, and geometric shapes are all image types that students used in a variety of modes (which *always* included defining) that were different than experts’ usage of these image types (which *never* included defining).

IV.B.3. Area under curve

Mathematical Features. The “area under curve” image is characterized by a graph of a smooth curve that defines rectangles between that curve and the horizontal axis, of width 1, so that the area of each rectangle represents the size of a single term in the series. This type of image lends itself to talking about the sum of the series as a value of areas of some number of rectangles. An example of this type of image is in Figure IV.15. The power in such images is to relate the area under $f(x)$ to a summation – that is, as demonstrated by Figure IV.15., if the curve is $f(x)$, we see that the area under $f(x)$ exceeds $\sum f(n+1)$. Thus, if the area is finite, the sum is bounded above.

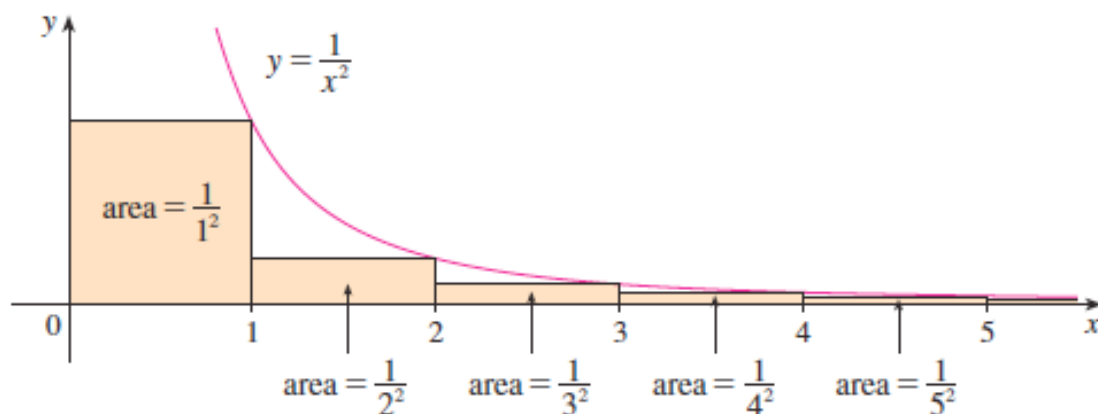


Figure IV.15. Stewart's use of an "area under curve" image

The area under curve image type has referents for the changing values of n , the terms of the series (as areas of rectangles), and the combined areas of multiple rectangles as partial sums/the sum of the series. However, there is no referent for "convergence" in this image type; it can look like a series converges if the sizes of the areas are decreasing to zero, but that fact alone is not sufficient to assert convergence. In order to conclude that a series converges, based on this image type, one would have to compare the infinite series to an improper integral, and then compute the integral (i.e. carry out the integral test). Note, however, that declaring that a series diverges based on this image type is possible.

Affordances/Drawbacks. Area under curves can be useful for students, to help connect with their earlier understanding of Riemann sums and improper integrals. Viewing the terms of the series as areas follows from students' experience approximating integrals, and is a natural connection to make, in order to accurately represent a series with a graph. However, there are also drawbacks to the use of such an image in the context of infinite series. While it is a good image to help visualize sums, thereby recasting convergence of sums as a problem of finding an area, the image itself does not give a referent for whether or not the area is finite. If students believe that the area under a positive, decreasing (to zero) graph is automatically finite, this image can be a further extension of their faulty reasoning. For students who already have difficulty understanding why sequences of terms that decrease to zero do not automatically sum to something finite, images such as area under curve graphs can serve to sway them in the wrong mathematical direction.

Additionally, this image is familiar because it is used by virtually all lecturers – but with the explicit purpose of introducing and justifying the integral test, one of many tests for convergence that students learn when studying infinite series. Unfortunately, when students see this image in lecture, as drawn by their professors, they often replicate it – but in a different context. This is a problem because, as indicated in the next section, students turn to this image type to define, while the professors do not.

Students' Use of Area Under Curves. Seventeen students in this sample (46%) used area under curves in their teaching episodes predominantly for defining convergence of infinite series. Students drew this image to describe that what it meant for an infinite series to converge was that the area of "all of the rectangles" was finite, and therefore the sum was finite. Very few

students distinguished between improper integrals and infinite series, in terms of referencing a continuous area vs. a discrete, stacked type of area.

For example, Sam began by defining series convergence in terms of improper integrals, and talked about sums of infinite series as if they were the same thing as computing improper integrals (see Figure IV.16).

Sam: Oh I guess I can show how we can compare a series to an integral.

Int: That's a good start.

Sam: So you remember Riemann Sums from Calc 1. (draws smooth curve) So we were taking either the ... (draws rectangles) So if we were taking the Riemann sum from 1 to infinity of the function $1/n^2$ we would find that we are actually taking these squares here (points at portions of rectangles that are above the curve) which are always above the line that you would be finding, which would be the area under the curve, the integral of $1/n^2$. So we can find that this function is represented by the sum of $1/n^2$. Because each of these, the width is 1 and the ... and the height is the term in the sequence. We are adding together all of these terms in the sequence.

Int: The sequence $1/n^2$?

Sam: So that is the summation of it as it goes to infinity. So I see that I add up all those areas it's the same as adding up $1/1 + 1/2^2 + 1/3^2$ and I see how they're points on that thing (referencing the curve $1/x^2$). So do you remember what defines something as being convergent or divergent?

Int: Um hm. For an improper integral?

Sam: Yea.

Int: When the area under the curve is finite?

Sam: It's ... [long pause]

Int: It's convergent?

Sam: And when it's divergent?

Int: The area under the curve is not ...?

Sam: Not defined. Yes. Ok good.



Figure IV.16. Sam's "area under curve" image used for defining

So for Sam, the discussion of convergence of improper integrals vs. infinite series is identical, because they represent the same thing. While Sam did acknowledge the error in that

the rectangles exceed the curve on his graph (using left endpoints), it did not factor into his discussion. Thus, for him, the conclusions one can draw are identical because the infinite series is another way of finding the area represented by the improper integral.

Steve also used an area under a curve to define convergence (see Figure IV.17.). In an earlier part of his teaching episode, when giving an example of an infinite series, Steve wrote $\sum 1/n^2$ on the board. Later, when he went on to say what it would mean for a series to converge, he used that example when providing a general definition. Steve's definition for what it means for something to converge came from his idea that testing infinite series for convergence was just another method for finding area under a curve.

Steve: So the infinite [sum and integral] are practically the same thing, but this is the discrete form of this, in a sense. In that it only uses integers from 1 to infinity, whereas an integral uses decimals included. So we have ... so we have um ... I don't know how to explain it.

Int: It's ok, you're doing ok. Is there anything you could write that would help explain it or would that not help?

Steve: (draws smooth curve) So this is the graph of $1/n^2$. And then uh we're taking the area under here. So this could in a sense represent the integral. Directly. And then if we look at the limit it would be if we go on...

Int: Ah all the way out there?

Steve: Yes. Til it reaches zero. So that would represent sort of the limit of how you get to there, follow it out far to zero. Then as we know an integral is a summation. So the limit of the summation would bring the area all the way out to here. But instead of doing every single number we just do 2, 3, (writing on his image) and we take the area of this, and then you add it to the area of this (pointing at rectangles) and this will be (silently computing sizes of areas and writing them on the rectangles). So in this case, the sum is almost the same thing as the integral, but the integral is more accurate measurement of the area than the actual summation.



Figure IV.17. Steve's "area under curve" image

Steve had difficulty divorcing ideas of improper integrals from infinite sums, and in his definition, defined convergence of one as being the same as the other. Maria also made heavy use of areas under curves (see Figure IV.18.) to define what convergence would mean.

Maria: This would be like term 1. Then we would know this part of the integral is, we could say is t_2 . Now we can say this entire thing is t_1 plus t_2 . So theoretically you could find t_3 , t_4 , and keep adding different terms, all the way to some infinite number as it gets smaller and smaller

Int: So, I understand everything that you're saying. So, what's the series?

Maria: So the actual series would be this $[t_1+t_2+t_3+\dots]$... So when you're talking about a series converging you're talking about like an area converging to a number. So you're going to take, go back to if the limit will equal a fixed number, like a constant...

Int: So if it equals a fixed number, then it converges?

Maria: uhhhhh sure.

Int: What's the tension in your head? You were like 'uhhhhhh sure?'

Maria: I just want to make sure I'm thinking. So can you repeat your question?

Int: The question was, what means converge? Question mark.

M: Sure. So yeah, because if you think about something that wouldn't converge, you would think of a function that looks like this (draws a function with no horizontal asymptote). And as you add each term you're gonna add more and more areas, so it's going to end up being infinity. (draws areas under the curve, larger and larger in area)

Int: So what about that picture doesn't converge? I just want to make sure I'm attending the right thing.

Maria: (pointing at a graph with infinite end behavior) So we know if you take a limit as you approach a point here, it's going to have an output. If you take the limit of the whole thing as x goes to infinity, the area under the curve is going to be infinity because you keep adding an infinite number of things that get larger and larger.

Int: Okay, so the stuff you're adding keeps getting bigger and bigger, so it's not amounting to something, anything?

Maria: Mm hm.

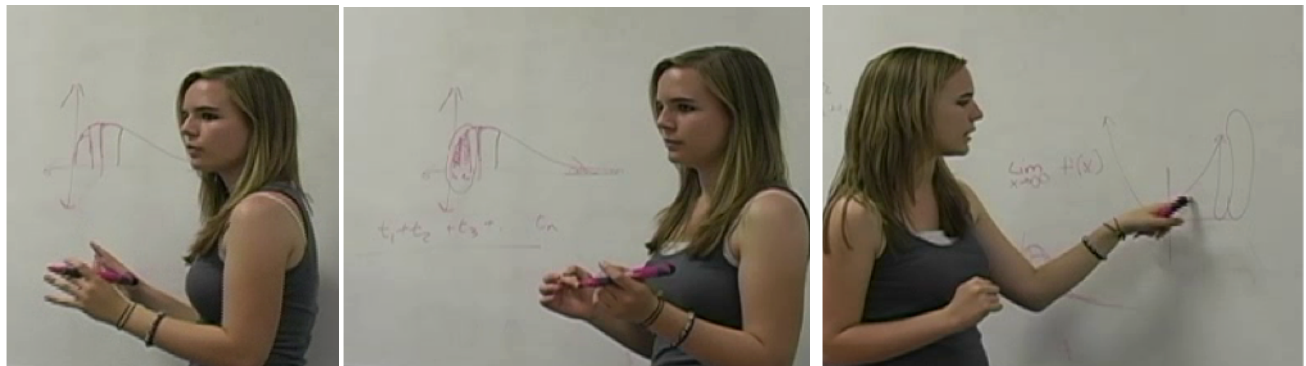


Figure IV.18. Maria's "area under curves" images

As we can see from the transcript of her teaching episode, Maria also viewed working with infinite series as a way to think about improper integrals and area under curves, and accordingly defined convergence of infinite series in terms of finding an area. She also used this part of her teaching episode to define what it would mean to *not* converge with an area under a curve, and even that discussion was rooted in finding values of areas of rectangles and adding them together.

Few students also used areas under curves to draw correspondences with other representations they had created for infinite series. For example, some students drew other graphs and then drew an area under a curve to create a correspondence between a term in a series and an area of a particular rectangle. Molly used her area under a curve to draw correspondences both to the terms of the expanded out series *and* to the terms in the sequence of partial sums. First, she showed the correspondence between terms and areas (see Figure IV.19.).

Molly: Because each of these are like a term. Their area is a term, right? And then you ... maybe?

Int: Well, convince me. I've never seen this before.

Molly: Ok. Because I vaguely remember writing it down in my notes. And there was this one part where [the professor] was like ‘This is like term $f(n_1)$, and this is like term $f(n_2)$.’ And then you like ... So with each addition of *this little area* (pointing at rectangle), this would be like *this one* (pointing at term). And *this one* [area] would be like *this one* [term].

Int: The area?

Molly: Yea. So that term represents, like I guess the area.

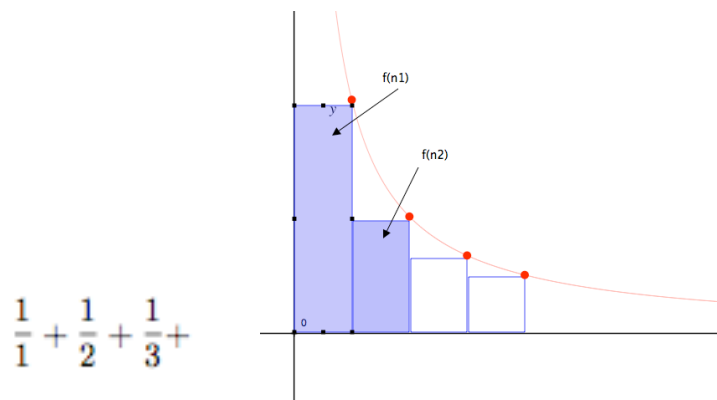


Figure IV.19. Molly’s “area under curve” image as coordinated with the harmonic series

Later in her teaching episode, Molly also drew correspondences between an area under curve and a graph of partial sums that she had constructed (see Figure IV.20.) For more detail on the ways that Molly drew correspondences between images, see Chapter VII.

Molly: Ok, I guess basically like the points [on the graph of partial sums] are just different numbers of [areas]. The first point is the first two boxes, and the second point is the first three boxes, and the areas of that.

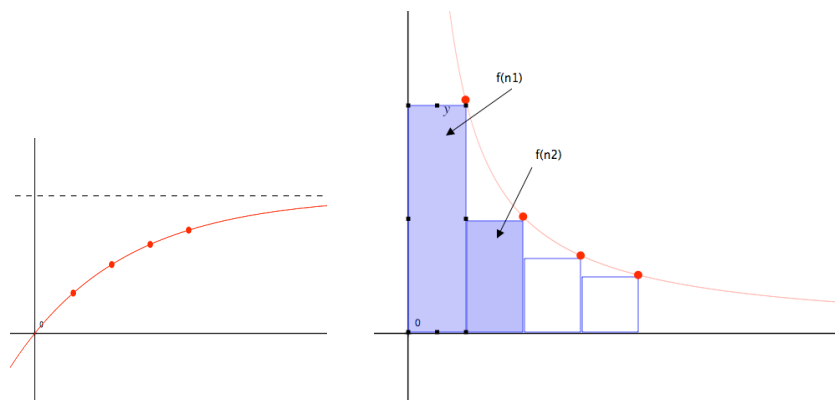


Figure IV.20. Molly's area under curve as coordinated with her plot of partial sums

Finally, several students made reference to Riemann sums, but were not able to clarify how those were related to this new context except that they knew that they needed to add areas of rectangles. Only two students referred to the integral test for convergence (both as a result of a direct question by the interviewer), even though this is the only context in which their professors and textbook used such an image. Only one student, Aiden, used this image type (Figure IV.21.) with reasoning more aligned with that of the professors (see next section), to indicate the significance of the integral test and boundedness.

Aiden: (repeating question) What is converge?

Int: I'm just wondering if there is an easier way to think about this one, you know what I mean?

Aiden: We can think about it as a function. (draws a smooth curve) Alright so let's start from zero, oh one. This is the function of $1/n^2$. The sequence goes ... from the graph we can see this one and you can draw a rectangle. With width one and height one. The area. And uhh two here and three here. So the area of this rectangle is one and this should be one quarter and one ninths. It turns out that the series or the sum of the sequence have some certain relationship, with the area under the curve of $1/x^2$. So if we can figure out whether the integral of this thing from zero to x . What is this? If we can figure out what this is, if it goes to infinity and the whole thing with area because we can see with area that finally the series is part of the area of the whole integral, so if the area is bounded then we can say the series is bounded.

Int: So that wouldn't tell you what it's bounded by, but it tells you it is bounded?

Aiden: Yeah it tells you if it is bounded.

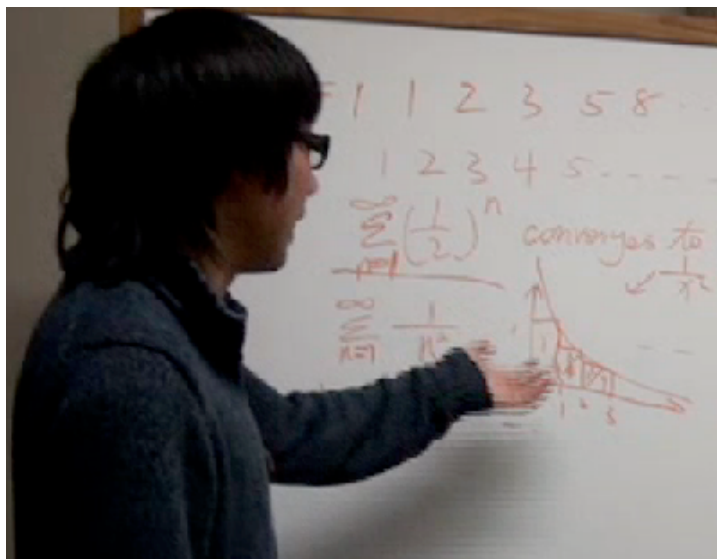


Figure IV.21. Aiden’s “area under curve” image

Professors’ Use of Area Under Curves. When used in lecture, this image was drawn to clarify the conclusions of the integral test, and sometimes to exemplify the use of the test with a specific example, as in Figure IV.22. Additionally, several different professors that were observed also used this image as a way to clarify error introduced by approximating a sum vs. an integral. The professors in this sample never drew this image type on the day that infinite series were introduced. It was only ever drawn as part of the lecture on the integral test, never when working an integral test problem, and never again for the remainder of the unit on infinite series.



Figure IV.22. An instructor’s use of an area under curve during class

For example, a professor in this sample used such images when introducing the integral test, to clarify both the relationship of the infinite series to the area under the continuous curve and the conditions that must be met when attempting to implement the integral test (see Figure IV.23.). This example, in Figure IV.23., is taken from a lecture at LRU (excerpted verbatim from the campus notetaker’s record), and demonstrates the use of areas under curves *after* the integral

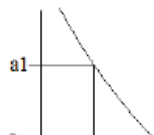
test has already been introduced, as a means for clarifying and justifying some of the important aspects of the test that was already discussed. In other words, this professor chose to use area under a curve to clarify for students the relationship between individual rectangles as determined by terms in the series, as they relate to pieces of area under the continuous curve $f(x)$. Using his image, this professor also motivates the need for the conditions that must be met in order to use the integral test to test for convergence – namely that if $\sum f(n)$ is the series in which one is interested, $f(x)$ must be both positive, continuous, and monotonically decreasing on the x -interval in question.

Another example of a professor's use of areas under a curve (in Figure IV.24.), this time for exemplifying the use of the integral test, comes from a different lecture at LRU. This professor introduced the integral test without any images, and then proceeded to use area under curves while working with the example of the harmonic series, in order to demonstrate the relationship between the infinite series and the improper integral.

To reiterate, all professors' uses of area under curves were consistent with these two examples – an image was used either when introducing the integral test for convergence as a means for clarifying the conclusions or relationships, or when working an example (most often the harmonic series) using the integral test.

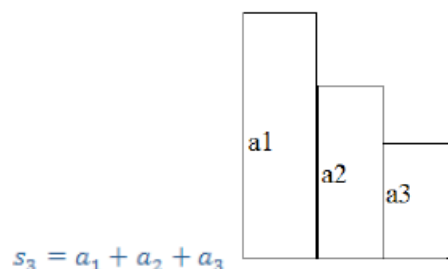
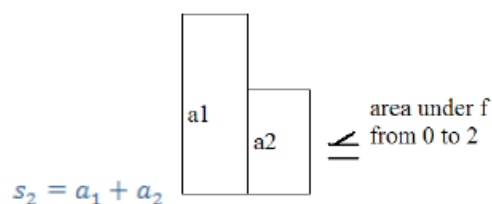
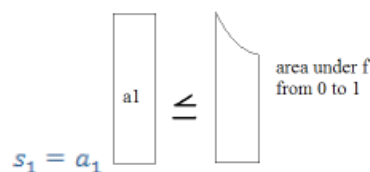
Today: Integral test

Again, there are more pictures. I'm going to start off by drawing a picture (Figure 4) and then write the general statement.



Want to show that $\sum_{n=1}^{\infty} a_n$ converges

I have some series and the numbers happen to be positive numbers. I want to show that it converges. One way to do this is to think of the area. I have to start by computing these S_n 's.



I think of my partial sums as being given by the area of these rectangles. Remember that when we're talking about a series, what we want is that these areas should converge to something. If you look at my picture, now let's get to the function, so what's that curvy thing? That curvy thing is a function. What I want is the value of the function:

$$f(1) = a_1, f(2) = a_2, \dots, f(n) = a_n$$

I also want:

*$f(x)$ decreasing function
ie $f'(x) < 0$ everywhere*

Notice that the area underneath $f(x)$ from 0 to 1 is greater than the area of a_1 , the area underneath $f(x)$ from 0 to 2 is greater than the area of $a_1 + a_2$ and so

forth. That's why I need my function to be decreasing; if my function is not decreasing, then you can see that this no longer holds true.

Look at the picture. The integral from 0 to infinity is the whole area underneath the function and the sum is less. What you have is these S_n 's, this is an increasing sequence of numbers because each time I'm adding a positive number. This is an increasing sequence and it also has an upper bound in order for it to converge. Let me say this more precisely.

Figure IV.23. A professor's use of areas under curves for clarifying the conditions and conclusions of the integral test

Compare
 $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$, or $\sum_1^{\infty} \frac{1}{n}$ with $\int_1^{\infty} \left(\frac{1}{x}\right) dx$

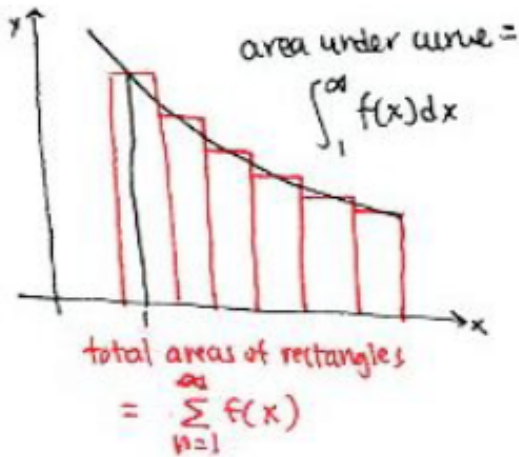


Figure IV.24. A professor’s “area under curve” image when exemplifying the integral test

Textbook Use of Area Under Curve. Stewart’s text uses areas under curves only in the section that introduces the integral test, again to clarify the written definition. This image is again only used in the textbook section on the integral test (11.3), to show the relationship between improper integrals and infinite series, as in Figure IV.25.

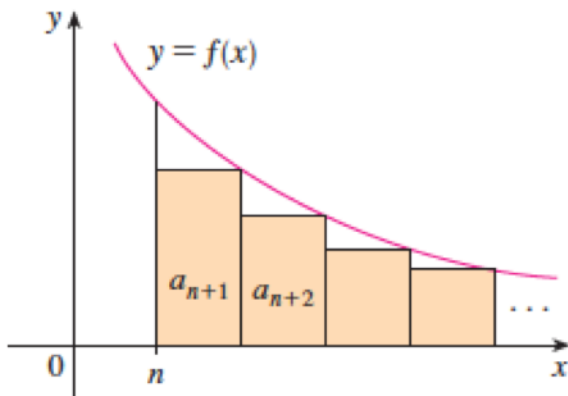


FIGURE 3

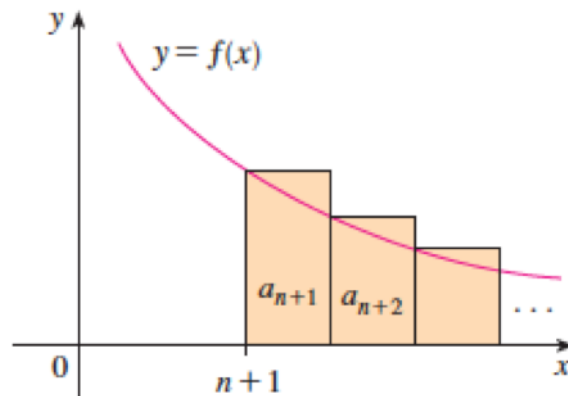


FIGURE 4

Figure IV.25. Stewart’s use of areas under curves to demonstrate error with integral test

Exploring the different uses of Area Under Curve. From examining the modes of use for the “area under curve” image type, it is once again clear that there is mismatch between the ways that students and more expert users employ it (see Figure IV.26.). While professors and mathematics texts use this image as a means for justifying and clarifying the integral test (a very specific, singular purpose), students use this image type for defining series convergence

altogether, giving this image type more and a different kind of status than the experts do. While this image cannot itself demonstrate that an infinite series converges, students' teaching episodes indicate that they pick up on the decreasing, positive terms, and the decreasing areas of subsequent rectangles, and use those features to justify their conclusions about convergent infinite series.

Image type	Students	Professors	Textbooks
Area under curve	Define (convergence) Draw Correspondence	Clarify (integral test) Exemplify (integral test)	Clarify (integral test)

Figure IV.26. Different modes of use for the “area under curve” image type

This example therefore highlights how an image type can have a very particular purpose in a lecture or textbook, but – possibly because some features are familiar or memorable to students, or because students pick up on only parts of the purposes for introducing such images – the image type can get taken up with a completely different (and broader) purpose, extending the significance of such an image beyond what is mathematically appropriate. This potentially suggests that more care should be taken when introducing such an image into a lecture on infinite series – to use mindful and purposeful *language* that has meaning to students, taking care to highlight the context in which the image type is useful, to possibly prevent some of the over-extension that students attribute to this image type. In this data, it appears that students pick up on professors' use of this image type as a means for determining convergence, and their rhetoric mimics that of the professors when using this particular image. However, what students apparently did not grasp, as evidenced by their teaching episodes, is that these conclusions were drawn by their professors and textbook in one specific instance – when using the integral test – and were not used broadly when making conclusions about series convergence.

The first three image types have shown several ways that students and experts take up various image types in different ways, leading to more or less successful mathematical conclusions about series convergence. While the final two image types are used less frequently (but are not uncommon in student teaching episodes), they also highlight instances in which students' mode of use is predominantly defining, while expert use is limited to exemplifying and clarifying.

IV.B.4. Number line

Mathematical Features. The “number line” image type is identified as any type of number line on which some representation of terms or partial sums are imposed (see Figure IV.27. for examples). This image type allows one to conceptualize the value of the sum as a distance on a number line, arrived at by concatenating the terms of the series. Number lines are most commonly associated with reasoning about alternating series, though can be generated with any infinite series explanation or problem.

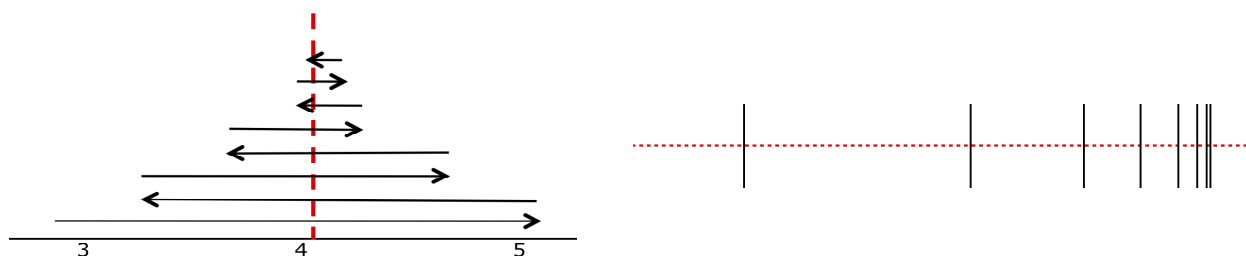


Figure IV.27. Students' number lines

Affordances/Drawbacks. Number lines, depending on how they are used, can include structures that represent the sequence of terms, the sum (if the series is convergent), and the partial sums all as distances along a unidimensional axis. As seen in Figure IV.27., these images can be used to measure terms as distances along the axis, or spaced out above the axis, where the 'updated location' along the axis, after combining several terms, represents the value of that partial sum.

Since students have presumably used number lines in classes prior to calculus, number lines may allow them to translate their skill and prior understandings of that representation into a new context. Further, conceptualizing terms and partial sums as distances may also allow students to enact their reasoning in a concrete way, moving a marker some set distance with each additional term, "acting out" this infinite process (Fischbein, 2001).

However, there are some potential drawbacks to such an image type. When using this image type for infinite series, one must be careful to move the distance determined by one term *from their previously determined location*, as dictated by the preceding terms. If a student were to plot distances corresponding to each term from the zero-point along the number line, and conclude that a series 'converges to zero' because the sequence has a limit of zero, this could be considered evidence that she lacks an appreciation for the significance of the partial sum. Also, it may be easy for individuals to lose track of n with this image type, if they are plotting distances along the number line for subsequent terms without labeling the referent to which term they are on.

Students' Use of Number Lines. About 38% of students in this sample (14 total students) employed number lines to define or draw correspondences. For students who had already drawn other image types, it was common to use number lines to make correspondences intended to highlight particular features.

For example, when reasoning with the standardized prompt about the meaning of $\sum a_n = 4$, Todd chose to draw correspondences between a general statement he had written and a number line, in order to coordinate the terms of the series and their partial sums (see Figure IV.28.).

Todd: Ok, alright so here we have (writes) so this is looking at the summation without actually putting any sort of expression on a_n . So what this is implying is that $a_0 + a_1 + a_2 +$ all the way up to ... I mean you can't write it as a value but [the infinity-th term]... so as you get further and further away you add these up the series will converge to 4.

Int: So that means that all together they are 4?

Todd: All together, this summation will approach 4 and get infini ... well finitely ... infinitely close? Yea.

Int: Infinitely close to 4?

Todd: Yes.

Int: What does that mean?

Todd: That means that if you're looking here at your at your summation, (begins to draw number line) **let's say that this is a_n , this is a_{n+1} ... or a_0, a_1, a_2 , and these respective values are the [terms] ...**

Int: Yea I see what you mean.

Todd: It will eventually get so much smaller then it will get almost all the way to 4. So as you're adding all of these up, it will get ... **it will approach 4.**

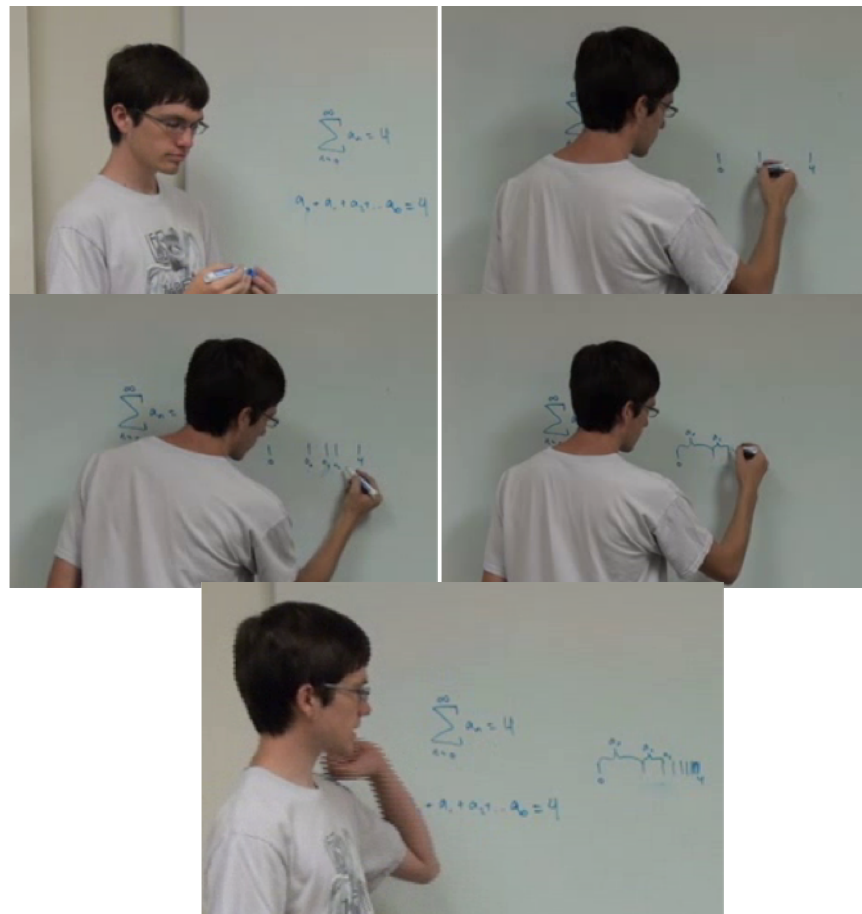


Figure IV.28. Todd's process for drawing correspondences with a number line

In this exchange, Todd had written out several terms of the infinite series $\sum a_n$. In order to demonstrate his description of what it means for that sum to converge to 4, he drew a number line starting at 0 and ending at 4, and segmented it according to the terms of the series. While he at first mistakenly segmented it by labeling points along the number line as corresponding to the individual terms, Todd later revised his work to show that the distance moved along his number line each time corresponded to the terms in his series. In making the number line, Todd literally pointed at the correspondences between the physical distance between each hash mark on the number line and the terms in his series, providing a referent for what it meant that the series

converged to 4. Todd's correspondences between the written out series and the number line made it possible for him to show what he meant by 'converge to 4' in a different manner.

Other students also used number lines to coordinate features with areas under curves and plots of partial sums – for example, stating that the sum of the series (pointing at a point on a number line) corresponds with the value of the area under the curve.

Ben is representative of those students who used number lines for *alternating* series, to demonstrate the behavior for an alternating geometric series by drawing correspondences between written out terms and partial sums as represented by locations on a number line (see Figure IV.29.). That is, these students wrote out and drew conclusions about alternating geometric series, and then demonstrated their conclusions by drawing correspondences with a number line.

- Ben: We also saw here ... it alternates with positive and negative [terms in the harmonic series] (referencing a series he wrote on the board).
Int: What would change then?
Ben: Instead of continually going forward, you would go ... (draws a number line) halfway to 1 and then halfway back to zero, and then halfway back to $1/2$, and then you just keep alternating and I don't think this one has ... or no, it does.
Int: Has what?
Ben: Like a solvable exact value.
Int: It does have a value, or does not?
Ben: I think it does.
Int: At the beginning you said you thought it didn't? What made you change your mind?
Ben: Because it oscillates. But in this case it looks like it's approaching a certain//
Int: //so your picture sort of //
Ben: //somewhere in here. Yea the picture sort of helps.

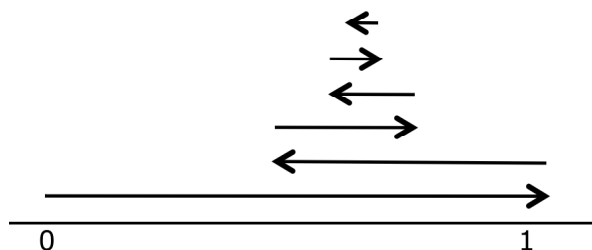


Figure IV.29. Ben's number line

For Ben, drawing the number line helped him believe that the series could potentially converge, though he had already concluded that it did. This image was a way for him to represent a conclusion he had already drawn, and link up the expanded series with a representation that he found accessible and able to represent his conclusion clearly.

When images were used for defining convergence, students were most often explaining what it would mean for *alternating* series to converge.

- Terrell: [For an alternating series to converge] what I'm saying is that like the negative of the largest a_n is there and then something smaller than it is being added to it. So absolute value [of the n^{th} partial sum] shrinks and its absolute value gets bigger and then it shrinks again. And sort of like, this [partial sum] is going to be smaller than that [partial sum]
- Int: For it to converge?
- Terrell: Well yea if this is true the a_3 is less than a_2 and ... so it will never be more negative than $-a_1-a_3$, and so $-a_1-a_3$ converges, then that's ... if I had written all that out I think I could have used that as a formal proof.

As Terrell described his definition of convergence for alternating series, he wrote “ $-a_1+a_2-a_3+a_4-\dots$ ” and made the motions of them along a number line, when discussing the distances between the subsequent terms as shrinking. Thus, he defined *convergence* for alternating series via his number line. The image itself was not used to clarify a definition already given, nor to exemplify a definition with an example. It was created in the act of defining what convergence meant to Terrell, here – a case he viewed as distinct from “regular” series convergence (presumably referring to series with only positive terms). Terrell is representative of many students who chose to discuss alternating series via number lines, and even defined the concept separately, saying,

- Terrell: You could have what's called an *alternating series*, as well. But I always think of that as a different case because it's like really obvious sometimes why alternating series can converge, and why others can't.

Use of this image type was not restricted to alternating series, however. When Aaron used the word ‘converge’ for the first time, without defining it, the interviewer asked him to detail what it meant. His response was to define it with an example (seen in Figure IV.30).

- Aaron: Ummm, usually about this stuff we would answer like, does it converge, or does it diverge.
- Int: Maybe say what that means... ?
- Aaron: If you take a path if you divide it basically. If you are traveling from point A to point B at some point you are going to pass this halfway point and another point then you pass another halfway point between the halfway point and then you are going to pass another half way point. And so on and so forth. And you have to pass this infinite number of halfway points but you will ... but as you are passing them you are eventually reaching B but as you are passing them you have to pass an infinite number of points. But if you think of this infinite sum, it's this same sort of thing. You're adding a half plus a fourth plus an eighth and a sixteenth and so on and so forth.
- Int: And you still get to B? So you're saying you could get a finite sum is because you can still get to B?
- Aaron: Yeah

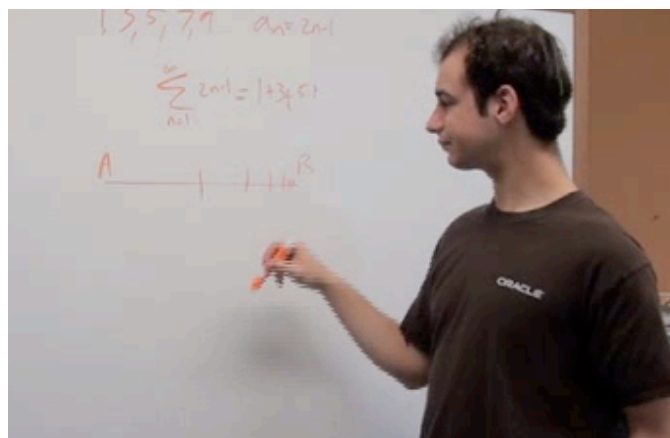


Figure IV.30. Aaron's use of a geometric argument to define convergence

Aaron focused on the geometric properties of the number line he chose to draw, rather than labeling points on it. Aaron described what he meant by “convergence” through the example of $\sum 1/2^n$. He did not use this particular series as a way of exemplifying or clarifying a previously provided definition – for him, the definition was given through that example itself.

Professors' Use of Number Lines. In lecture, professors used number lines when exemplifying alternating series and when drawing correspondences between plots of partial sums, in order to show the limit of the sequence of partial sums on a unidimensional representation rather than a two-dimensional one.

For example, one LRU professor used number line imagery first when working with the alternating harmonic series, as a means to represent the behavior of the sequence of partial sums (see Figure IV.31.). It is worth noting that this professor had not used number line imagery at all during the unit on infinite series, until introducing the example of the alternating harmonic series. This was *after* defining the “Alternating Series Test for Convergence.”

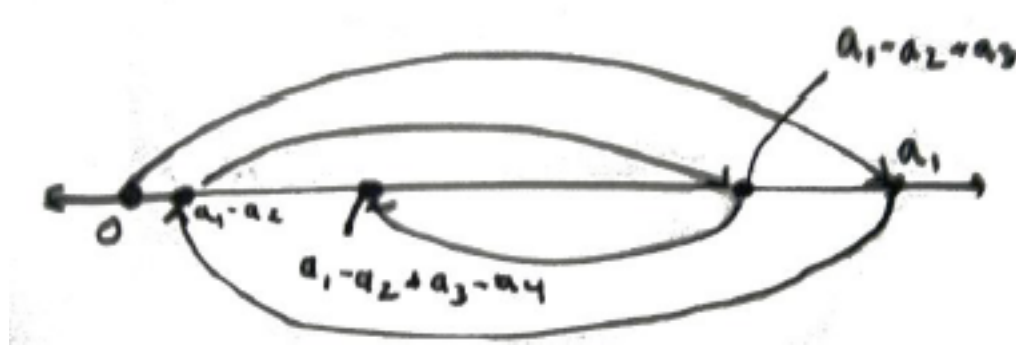


Figure IV.31. A professor's use of a number line to show terms in the sequence of partial sums

This professor then continued with the example of the alternating harmonic series, but changed his use of a number line slightly, instead using it to draw correspondences between the first image of the general sequence of partial sums and the odd and even partial sums (see Figure IV.32.). The second number line was used to show the correspondence of the terms and locations

on the first image with the number of terms in each partial sum. Thus, the number lines were coordinated, and used both to exemplify why the alternating harmonic series converges.

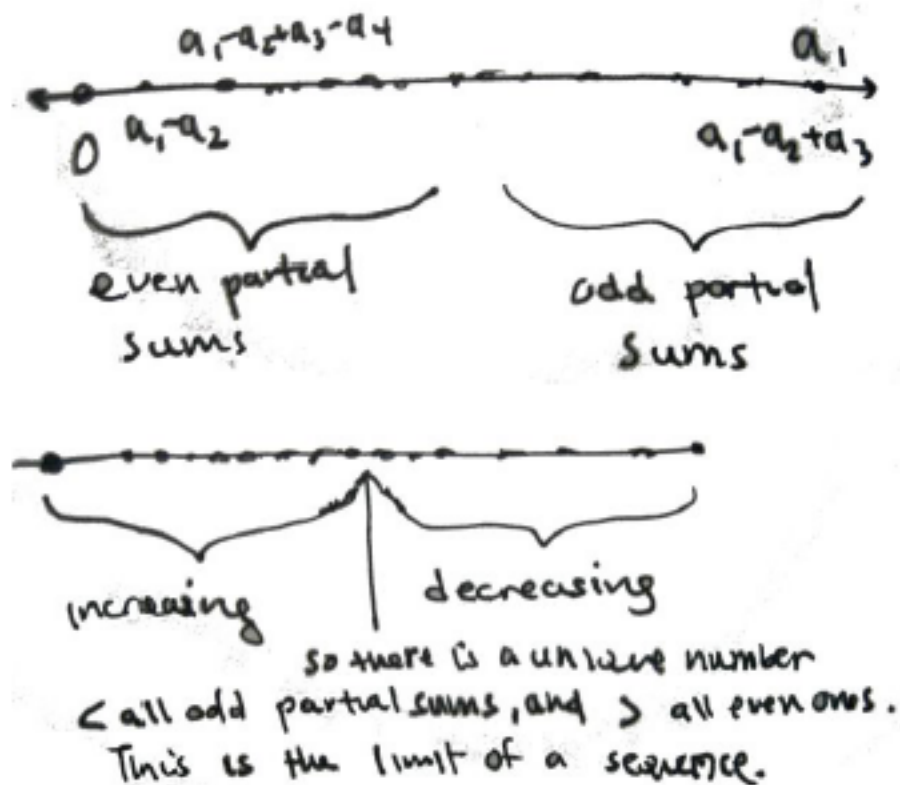


Figure IV.32. Sequences of odd and even partial sums

An SHC professor used a number line both for exemplifying and drawing correspondences when working the example of $\sum(-1)^{n-1}/n^3$, first to “show that the series bounces back and forth” and then to describe the relationships of the terms and the partial sums, as they were labeled in the expanded out series on the board (see Figure IV.33.).

That was pretty straightforward for part a. Let's think about part b now. You should always start off with a picture. The picture is that the series bounces back and forth. The length of the jump is the absolute value of the next term. As you can see, it overshoots it each time.

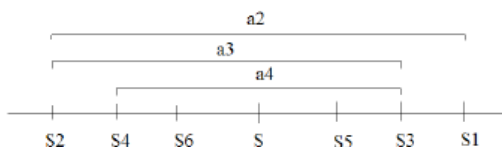


Figure 1

Figure IV.33. A professor's reproduced lecture notes when discussing the problem $\sum(-1)^{n-1}/n^3$

Though the examples each come from single lectures, the number lines were only used by professors in this sample in situations such as the two described above, either when working examples of alternating series or when students were struggling with coordinating terms and partial sums.

Textbook Use of Number Lines. Stewart (2007) uses a number line for clarifying its definition of convergence for alternating series (in Section 11.5) and for exemplifying that same concept in the same section. This image (in Figure IV.34.) was again used after the definition for convergence of alternating series, in order to provide a referent for the text-based definition of the “Alternating Series Test.”

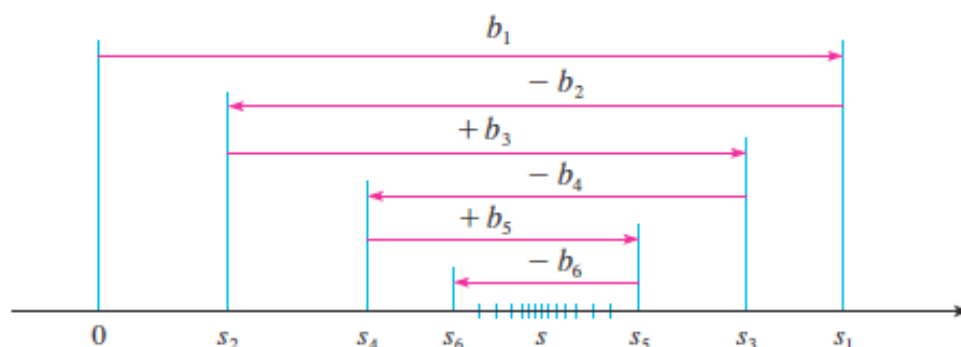


Figure IV.34. Stewart's use of a number line

IV.B.5. Geometric shapes

Mathematical Features. “Geometric shapes” images were identified as those that were not necessarily drawn on a Cartesian plane, but rather those for which areas or distances accumulated into some geometric shape whose partitions have significance to an infinite series. These often took the form of partitioning a box into areas of sizes that correspond to terms of the series, where the area of the box represented the sum of the series (see Figure IV.35.). Other versions of geometric shapes were of ‘slices of pie’ drawn as sectors of circles, where the sum represented a total amount of pie, and pieces of a stick, where the sum represented the entire length of a stick.

Geometric shapes provide a way to represent the sequence of terms (as physical quantities), as well as the sum of those terms if convergent (as an area or distance), and the partial sums as areas or distances of groups of individual terms.

Affordances/Drawbacks. These images may be more approachable to students because they present decisions about convergence of series in terms of shapes with which they are presumably familiar. Partitioning shapes into pieces that correspond to terms is somewhat straightforward to students who have a reasonable understanding of fractions. However, this image type is only readily useful when dealing with geometric series. There are (at least) two ways that a student could reason with a geometric shape in the context of infinite series. First, a student could assume the value of the final (convergent) sum and then partition the shape so that the sizes of the terms add to the value of the sum. Second, the student could partition a shape and then, generating a series from that, find the value of the sum as it relates to the total area. Among

students that used this image type, there was apparent difficulty in distinguishing the difference between these two.

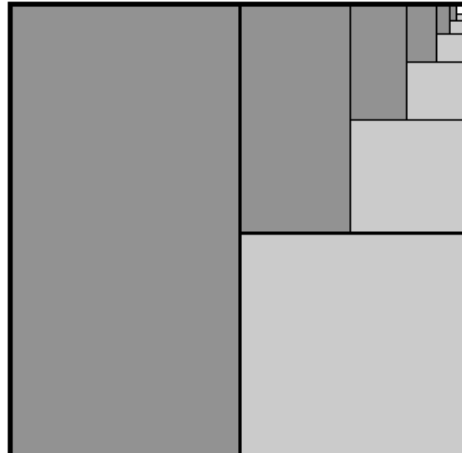


Figure IV.35. A geometric shape for the series $\sum (1/2)^n$, with alternating shading (light/dark) to show subsequent terms

Students' Use of Geometric Shapes. Ten students (27%) used geometric shapes in their teaching episodes to define and draw correspondences about infinite series convergence. Geometric shapes were most commonly drawn as some version of Figure IV.35., above, where students used this image to define what it meant for a series to converge by discussing area as the value of the sum. The students who used this image type for defining did so by discussing the example of $\sum (1/2)^n$, rather than choosing that particular geometric series to exemplify some aspect of series convergence. However, the students who used this image type for drawing correspondences did so by pointing out features of the geometric shape that corresponded to written work that they had recorded on the whiteboard. For example, in Figure IV.36., Tracy wrote out several partial sums, and drew correspondences for each of those partial sums to the area of the total box that those terms represent.

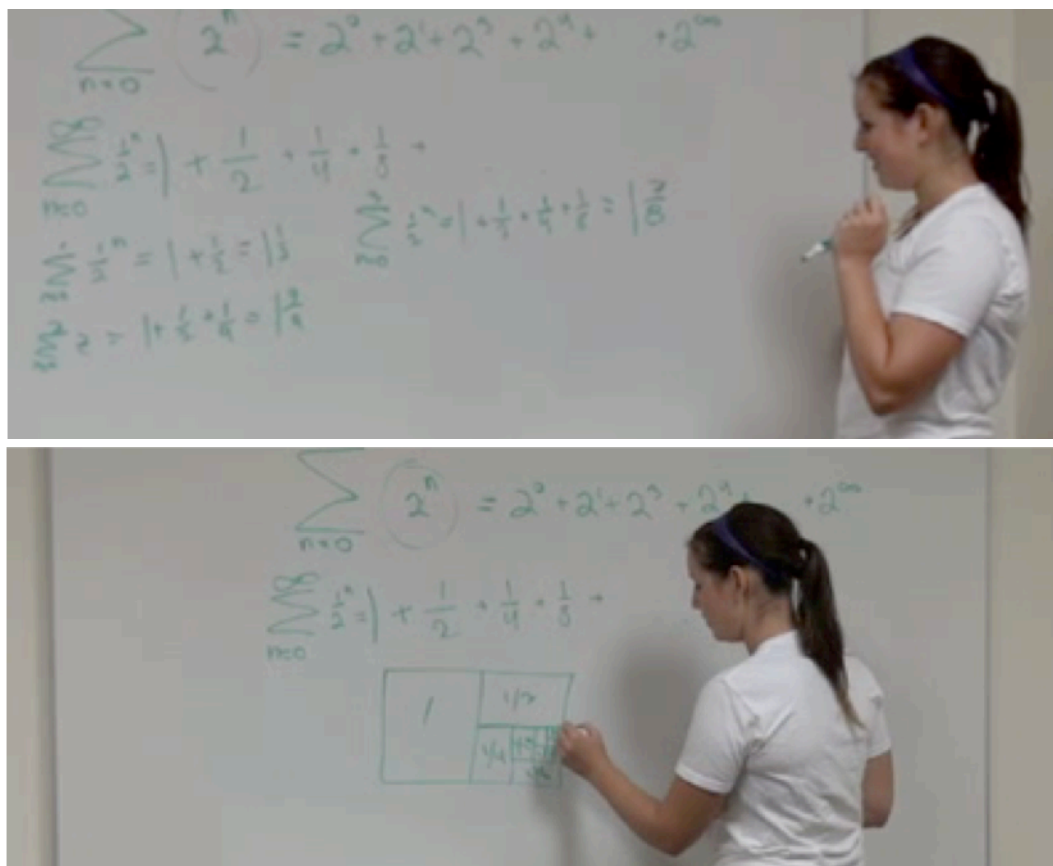


Figure IV.36. Student (Tracy) drawing correspondences between groups of terms in her written work and areas in her “geometric shapes” image

Tracy’s use of her images to draw correspondences between something like Figure IV.35. and the partial sums that she had listed on the board was typical of other students’ use of geometric shapes for drawing correspondences. Other students used geometric shapes of pies and cookies much like Tracy used Figure IV.36., to make correspondences to the terms of a series, rather than the partial sums. For example, Tina chose to use a geometric shape to describe the relative sizes of terms being added in terms of pieces of pie. After writing out several terms of the harmonic series, Tina began to draw ‘pieces of pie’ as a way to draw correspondences with the terms in her series and shapes she could use and visualize to describe them.

- Tina: So it's like you're taking pies. Here's a whole pie. Here's half of a pie. Here's a third of a pie. And a fourth of a pie. (drawing sectors of circles)
- Int: You're better at drawing them than I am.
- Tina: [laughs] And you know a fifth of a pie. The pies are getting smaller and smaller and smaller\\
- Int: \\the slices are getting smaller?
- Tina: Exactly. And so you notice that you have one whole pie and you have half. and then you have three quarters. Or a quarter. And they're getting smaller and smaller. So you're adding smaller and smaller numbers.

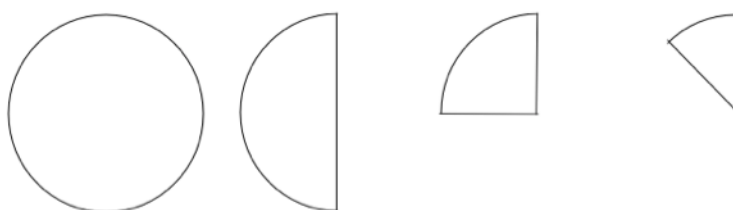
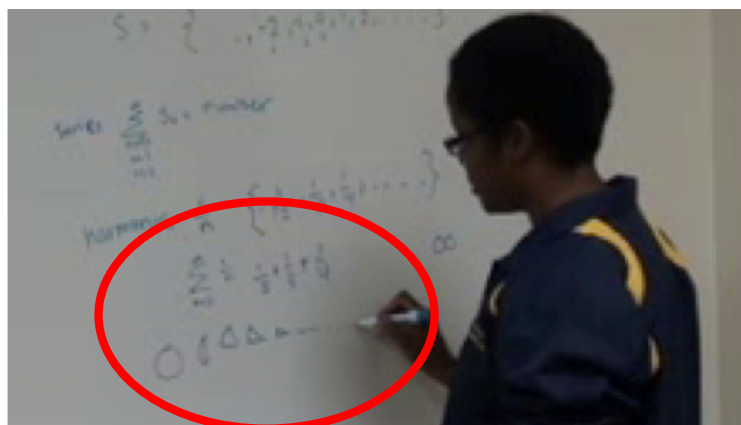


Figure IV.37. Student (Tina) use a geometric shapes when discussing convergence

Additionally, Aiden drew and used what he called the “stick theorem” to define convergence, stating:

Aiden: If the series converges, it means that you can break a stick into infinite pieces, but they still make up the whole stick,

referring to the total ‘stick’ as the value of the sum of the series – an image that he drew on the board.

Students also used geometric shapes as a way to define what it would mean for a series to converge. Though their shapes look very similar to those in the examples above, these students attempted to define the meaning of convergence by the properties of a particular geometric shape. Travis, when asked to clarify his use of the word “convergence,” requested a moment to look for an example in a textbook. He remembered and settled on $\sum (1/2)^n$, and proceeded to use a geometric shape (in Figure IV.38.) to explain for the first time what ‘converge’ would mean. While there are several problems with the way he used this example, the important feature here is that he is used the shape as a way to define how he thought it was possible that the sum of infinitely many terms could be finite.

Int: So someone in my position, a student who missed doesn’t know what [converge] means.

Travis: If I had a textbook I would show them a geometric series.

Int: Do you remember what that is, or do you want a textbook to remember what it is?

Travis: I would love a textbook. (finds an example)

...

Travis: So the first one we have [the sum from] 1 to infinity. We have $1/2 + 1/4 + 1/8$. We have all these terms. And I think that this is never going to be greater than 2. So we can look at it in a different way. Say we have a box and it has an area of 2. First one, you want to fill in half the box. Next one you want to fill in a quarter of the box with that half.

Int: So you have both of the shaded things together?

Travis: Right. Then we're going to do an eighth of a box, half a quarter. Then a sixteenth. Thirty-second. Sixty-fourth. And so on. So we see that we can keep adding them infinitely many times however it's only half of the remainder of the area. So regardless of how many times, you're only gonna get half of the area that's still left. And that will never exceed 2.

Int: So it's never gonna like add to the box?

Travis: No, but the limit as it approaches the box is gonna get so close. And you're gonna get the limit of $1/n$. And as n goes to infinity, that's going to be the area left. So do you know anything about the limit as n goes to infinity of $1/n$, does that approach?

Int: Yea sure, zero. So what's the $1/n$? I think I know what you're saying but I wanna make sure I get the whole point.

Travis: [laughs] Right, the $1/n$ then represents the remainder of the box.

Int: And you're saying it's getting filled up all the way? Cuz the limit is zero?

Travis: Right.

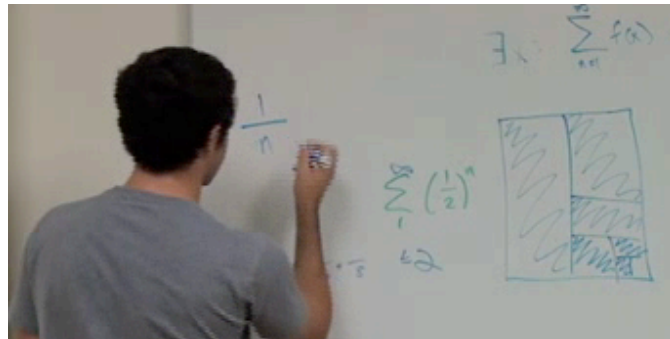


Figure IV.38. Travis' use of a geometric shape to define convergence

This is different than students who used these images to draw correspondences with some other images or work they had done, because the student did not use this shape to illustrate features he already discussed, or supplement a previous example. Rather, he used the figure to define the meaning of a concept for the first time.

Professors' Use of Geometric Shapes. In lectures, geometric images were used for exemplifying, most often with the series $\sum (1/2)^n$, though not by all professors. Three professors used an image like Figure IV.35. when demonstrating for students that $\sum (1/2)^n$ did in fact converge to 1 (or 2, depending on the starting n -value). When exemplifying geometric series with geometric images, two of these professors also drew correspondences between the terms of the particular geometric series example ($1, 1/2, 1/4$, etc...) and the limit of the partial sums. That is, they connected the relative sizes of the boxes in Figure IV.35 to the sequence of partial sums definition, where the n^{th} partial sum $(2^{n+1}-1)/2^{n+1}$ has a finite limit.

Textbook Use of Geometric Shapes. Stewart uses a geometric shape (see Figure IV.39.) in the margin of Section 11.2, the introductory section of infinite series, to clarify geometric series by providing a demonstration/proof for the sum of a geometric series. This particular geometric shape (of a triangle) used in Stewart was not used by any professor or student in the sample, nor is it emphasized in the textbook section. Rather, it is placed in the margin in order to provide additional clarification for the conclusions of the definition for geometric series convergence. It is the only geometric shape used in this chapter of the text.

■ Figure 1 provides a geometric demonstration of the result in Example 1. If the triangles are constructed as shown and s is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar} \quad \text{so} \quad s = \frac{a}{1 - r}$$

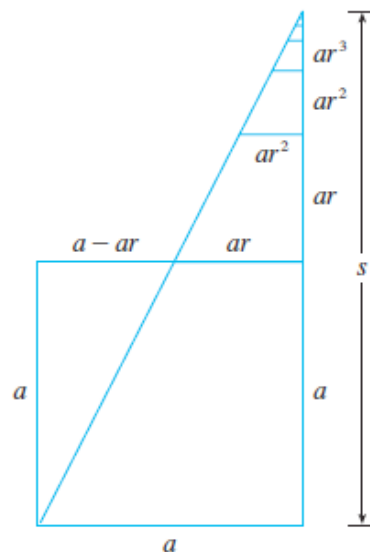


Figure IV.39. Stewart's geometric shapes figure demonstrating the sum of a geometric series

Exploring the different uses of number line and geometric shapes. Number line and geometric shapes are powerful images for clarifying definitions and exemplifying particular infinite series, perhaps through worked examples in lecture or in the textbook (most commonly for alternating and geometric series, respectively). However, we again find that students' modes of use for these image types carry more weight than simply exemplifying – students use these image types as a means for *defining* series convergence, when they are more suited for visually illustrating particular examples. For the experts, these image types were particularly well-suited as context-dependent tools useful for illustrating particular properties that are well staged by working specific examples, but not necessarily for making broad conclusions and generalizations. However, in students' use of them for defining, we find that students attend to different features of these image types than experts do, thereby leading them to use the images in ways that are not necessarily mathematically sound (see Figure IV.40).

Image type	Students	Professors	Textbooks
Number line	Define Draw Correspondence	Exemplify Draw Correspondence	Exemplify Clarify
Geometric shapes	Define Draw Correspondence	Exemplify Draw Correspondence	Clarify

Figure IV.40. Different modes of use for number lines and geometric shapes

A few other images, none of which were incorporated into an explanation by more than one user, were not classifiable as one the five prevailing image types. These images included a story/metaphor (relating an infinite series to sand on a beach), a unique/original image that was one student's attempt at combining partial sums and individual terms into a plot, and the textbook use of a tabular representation depicting partial sums (in Figure IV.41.).

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Figure IV.41. Stewart's tabular representation of partial sums

To summarize the main thread of this chapter, see Figure IV.42: Students largely operated with the different image types with very different modes of use than the experts (professors and textbooks) *when explaining*. Some of the prevailing differences relate to the ways that students and experts differently view what exactly is useful about each image type, and the language with which the different groups talk about each image type. While experts view key features of the different image types as the 'important part,' and base their explanations with that image type on those features, students often focus on entirely different mathematical features of the same image types. The differentiated attention within the image types, based on the user of those images, is therefore leading to different modes of use that are not always mathematically appropriate. These modes of use were also apparent in students' justifications of various convergence tests and specific infinite series tasks, discussed in the next section of this chapter.

Image type	Students	Professors	Textbooks
Plots of Terms	Define and Clarify	Clarify Draw Correspondence	Draw Correspondence
Plots of Partial sums	Draw Correspondence (Define & Clarify 8%)	Define and Clarify Exemplify	Exemplify
Area under curve	Define (convergence) Draw Correspondence	Clarify (integral test) Exemplify (integral test)	Clarify (integral test)
Number line	Define Draw Correspondence	Exemplify Draw Correspondence	Exemplify Clarify
Geometric	Define Draw Correspondence	Exemplify Draw Correspondence	Clarify

Figure IV.42. Combined table of modes of use for different image types, populations

IV.C. Image use for explaining vs. justifying answers to math tasks

The previous sections have detailed the different image types used by students, professors, and Stewart's (2007) textbook, and the prevailing modes of use when incorporated into *explanations*. During their teaching episodes, students were also asked to *reason* about infinite series by solving several math problems related to convergence and divergence. Recall from Chapter III that there were two types of problems during this portion of the teaching episodes – infinite series problems about $\sum a_n$, where the a_n is known, and others for which it is left ambiguous. Two different things happened during students' problem solving. First, many students were able to apply convergence tests to series problems where the a_n was known, and correctly arrive at an *answer*, despite having faulty definitions of convergence at earlier points in their teaching episodes. These students were rarely able to answer the questions for which the a_n was not known. Second, students' *justification* for the answers that they provided was often tied to the predominant imagery from their teaching episode, through which they defined converge/diverge. Through the following two examples (which are representative of the larger data set), we can see evidence of the ways that students' modes of use play out in their justification, often toward faulty conclusions about series convergence. During both examples, the students were asked to address the following question:

(*) Make the strongest possible statement about the convergence of $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \dots$

Example 1 – Steve's teaching episode.

As described earlier in the chapter, Steve's primary way of explaining infinite series convergence was to use an area under a curve. He believed that testing infinite series for convergence was just another method for finding area under a curve. Using an 'area under a curve' image, Steve expressed that convergence of improper integrals was the same as convergence of infinite series. He continued to apply this type of reasoning to his justifications for why (*) would diverge:

Steve: I want to say this diverges.

Int: Ok how come?

Steve: Adding one half and one ... (starts adding fractions)

Int: Oh so you're adding up pairs of terms?

Steve: Yea. it's ... (long pause) So we're just ... all I'm thinking of is when I see this, is that it is really similar to the natural logarithm function.

Int: Oh, ok why is that?

Steve: Because you have the function but it's just not defined at certain points.

Int: I see so there's like missing stuff?

Steve: Holes, yea. Specifically at where [the denominator is] $3 \bmod 1$. It makes me think that this function is divergent because we know that $\ln(x)$ is divergent.

Int: What about $\ln(x)$ is divergent? Can you say what you mean by that?

Steve: The area under a curve. Well, the area under the curve of $1/x$.

Int: So that's what this makes you think about?

Steve: Cuz it's very close to $1/x$, except it's taking out some of those terms.

Int: I see, ok.

Steve remembered how to find the area under the curve of $1/x$, and also that the antiderivative on the interval in question is $\ln(x)$. Because in his explanation he expressed his belief that infinite series are simply another method for computing improper integrals, asking if something of the model $1/n$ converges is the same as asking if the area under $1/x$ converges, for positive values of x (or n). So, while his answer was correct, and parts of his justification may have suggested that he was applying the integral test, in some fashion, the explanation phase of his teaching episode sheds additional light for interpreting the justification of his 'diverge' response.

Example 2 – Ben's teaching episode.

As discussed earlier in this chapter, Ben's ideas of infinite series convergence was related to the plot of the terms a_n . Namely, that as the terms themselves got 'smaller and smaller,' tending to zero and becoming "negligible," their sum would definitely converge on a value. His justification for why he thought (*) would converge utilized this same reasoning.

Ben: We're adding a smaller quantity to the previous quantity. I would think it converges to a number. But ... because we're adding them forever so [the sum] is getting bigger and bigger, it also makes me think it would diverge.

Int: Oh gosh, ok.

Ben: But (reading the task), so the wording kinda makes me think that it actually would converge. So ok. I would, like the case I would make for this would be that because we're adding by smaller numbers, it's kinda like the number line where we start at a specific point, like one half, and we're adding smaller and smaller quantities, but we're still going in the positive direction, but because those quantities are getting so much smaller, we're not going to go past a certain number.

Int: So you think there will be some number, somewhere ... that it won't go past?

Ben: Yes.

Differently than Steve, Ben did not arrive at the correct answer to (*). However, because of the additional context from his explanation phase of his teaching episode, it is clear that his

definition of convergence for infinite series is fairly well-developed when put into practice (though incorrect).

Ben and Steve are just two examples from the set of teaching episodes for which students' justification for their answers to infinite series tasks were consistent with the predominant imagery from their explanations. Ben's entire notion of convergence was based on plots of terms decreasing to zero, and his subsequent justifications all traced back to relative sizes of terms. Steve's explanation of convergence was that improper integrals and their related infinite series were essentially two ways of displaying the same information. Consistent with this explanation, his justification of why $1/2+1/3+1/5+1/6+\dots$ diverged was that the area under the curve of $1/x$ was divergent. While some students, like Steve, were able to answer (*) (and similar questions) correctly, looking more closely at their justifications, and the ways that those related to the students' definitions of convergence, demonstrate that asking for explanations rather than simply responses to math tasks bring different information about the students' understanding to bear.

IV.D. Examining infinite series competence based on limiting processes, language, and connections

The contrast between how students and experts (professor and textbook) use the different image types can often be traced to highlighting or putting the emphasis on different features of the various image types. One way of characterizing the differences is as follows - What did the experts "do right?" The professors and textbook chapter prioritized the limiting processes represented in a particular image type in appropriate ways so that they could be used to support mathematically sound conclusions. They used careful language so that the images were not overextended to contexts in which they did not support conclusions. And they connected representations in meaningful ways, using other images to clarify or exemplify those that were used to define.

Limiting processes. For example, students used plots of terms for defining convergence because they associated the limit of the terms as implying convergence, while experts used plots of partial sums, because they recognized the association of limits of sequences of partial sums for convergence. This finding is consistent with Martin (2009), who found that students made arguments about Taylor series that were rooted in basic limit reasoning, while mathematicians made Taylor series arguments based in their understanding of the importance of partial sums. Also, that students associated the decreasing sizes of the areas under curves as the marker of convergence lead them to use those images for defining, when experts knew the limitations of such images, and instead used them to clarify. This focus on particular limiting processes is captured in the ways that students vs. experts used the different image types. For example, when using areas under curves, experts attended to the connection with improper integrals, in the specific context of clarifying the integral test. In contrast, some students focused on the decreasing sizes of the terms of the series, while other students focused on the increasing size of the area under the curve as each new 'rectangle' was added to the previous rectangles. Thus, when using a particular image, lecturers paid attention to one feature (tied to a particular limiting process) while students paid attention a number of different ones. In order to disentangle the different limiting processes that students attend to, and the ways that they do so, there is a need to study with more depth the mathematical features that students value in their SGR.

Language. Professors used particular language associated with each image type, meant to draw attention to the context in which that image is useful, and the conclusions that can be reliably drawn/inferred from it. For example, professors' use of area under curves was intended to highlight aspects of convergence that were germane to discussing boundedness and the integral test. When students used the same image type, however, the context was sometimes lost, and only the surface features and broad idea of accumulation were retained. Also, when discussing alternating series professors tended to use number lines to highlight ideas about partial sums. Students often took and used those same images and language. However, in their application of those ideas, students often interpreted alternating series as being different from 'regular series,' thus requiring a new definition of convergence and a new way of demonstrating it. Given that students emulate professors and use some of the same language and images in their own explanations, but in ways that fail to capture the mathematical intentions of the expert, it is important to understand what the students believe is meant by their choices of language. Thus we should also pay attention to the language used as part of the explanations.

Connections. There were no patterns that distinguished between groups of students (by institution or level of mathematics) according to image type. However, there *were* patterns in how the different image types got connected by students during their teaching episodes. For example, certain image types were more often used by students for drawing correspondences with alternating series or geometric series. Plots of terms and plots of partial sums were often connected by experts when explaining convergence, while students connected those image types in different ways or not at all. Because (most) students used multiple SGR through the course of their explanations, it is instructive to look at the ways that they connect their multiple images and the instances in which they find the different image types to be useful over others (modes of use).

V. LIMITING PROCESSES

Students' explanations of infinite series convergence differed substantially from experts' explanations along three distinct dimensions of competence. The focus of this chapter is on exploring the first dimension of competence related to infinite series - Limiting Processes (see Tall, 1980). This chapter opens with the presentation of the variety of ways that students applied limit-type reasoning to different aspects of infinite series, as evidenced by their teaching episodes. An in-depth example follows, documenting how students' attention to different limiting processes, even while using the same language and/or types of imagery, can shape the more or less successful ways that they are able to explain infinite series. A particular example – students' use of Zeno's Paradox – demonstrates that even though many of them ground their explanations in the same basic example, their paying attention to different features of that example has profound effects on the mathematical conclusions that are drawn. And in this case, the different features of the example correspond to different limiting processes inherent in it.

Outline of Chapter V -

V.A. What is meant by limiting processes?

V.B. The collection of limiting processes

V.B.1. Vague limit notions

V.B.2. Terms in the series

V.B.3. Infinite series

V.C. An example from the broader set – Zeno's Paradox reasoning

V.A. What is meant by “limiting processes?”

'Limiting processes' here is used to refer to mathematical structures to which students apply their limit understanding and properties (see Tall, 1980). Common limiting processes that arise in calculus are continuous, such as finding limit of a function, or considering the geometric limit of chord as it approaches tangency, and discrete, such as considering sequences and series, and doing various numerical approximations.

As discussed in Chapter IV, prioritizing particular limiting processes may be more productive than others, depending on the context in which the student is working. For example, if one is attempting to explain the meaning of convergence of an infinite series of numbers, the limit of the sequence of terms is only useful if it is non-zero. Thus, an argument for convergence based on the limit of the sequence of individual terms is inadequate. Despite this, students often focus on and overextend properties of the limit of individual terms to make mathematically unwarranted claims about convergence of the resultant infinite series (e.g., asserting that if the sequence of terms $\{a_n\}$ has limit zero, then the series $\sum a_n$ must converge). Thus, an examination of what particular limiting processes that students are attending to *and* why/in which contexts can help frame findings about what features of this difficult mathematical domain students find important. Additionally, examining students' more mathematically normative and non-normative appeals to the different limiting processes can serve to identify the ways that students are connecting their mathematical knowledge, and provide points to trace back to instruction, as a means of uncovering what we as teachers do to facilitate these connections (for better or worse).

V.B. The collection of limiting processes

Nine limiting processes referenced by students during their teaching episodes were

identified using an approach aligned with grounded theory (Glaser & Strauss, 1967; Strauss 1987). These limiting processes correspond with vague limit notions, terms in the series, or reasoning about the series itself. These nine limiting processes provide a common language with which to talk about the different dimensions of competence (Summarized in Figure V.1.).

Limiting Process	Working Definition	To <i>what</i> is the student applying limit ideas?
Vague Limit		
(1) Unattached limit language	Vague discussion of “ n ” or a non-oriented limiting process that is not explicitly linked to terms, sum, etc.; vague limiting language without a referent	Nothing/unclear
(2) Limit symbol outside the sigma	Students choice to rewrite the standard sigma notation with an explicit limit statement, to help reason with the infinite nature of the mathematics	The index of the series, or stepping through the values of n in the sum
Terms		
(3) Individual terms	Attention is on the individual terms and their arising from a pattern, but there is no attention on examining their sequence or order	The model of an n^{th} term, as generated from the first several terms
(4) Terms as physical quantities	Terms are representative of physical quantities, whether areas under a curve, geometric shapes, steps in a trajectory, etc.; this is independent of whether or not the student reasons about the accumulation of terms	Physical quantities that are being described as taking on the mathematical properties of limits
(5) Sequence of terms	Attention is on the ordering of the individual terms and their progression; all claims about convergence are about “where the terms go”	The sequence of terms themselves
Sums		
(6) Sigma symbol as addition	Focus on the sigma symbol as referring to a sum, but not on applying limit properties to find a value for that sum	The (temporal) process of adding a bunch of numbers together
(7) Partial sums as collections of individual terms	The attention is on partial sums as collections of terms, whether ordered or not; while these partial sums may not be ordered into a sequence so that the limiting process can be attributed to the progression, they may be grouped in such a way that shows their relative sizes and impacts of adding more terms	Groupings of terms with increasingly more entries
(8) Partial sums as a sequence	Attention is on generating and examining the sequence of ordered partial sums	Sequence of partial sums
(9) Compilation	The sum has a value that is the result of putting together/compiling some units, where the sum is a physical quantity, and the emphasis is on accumulation of terms	The process of compiling terms or objects toward a final state represented by their sum

Figure V.1. The nine limiting processes

Reporting on percentages of students who appealed to each limiting process is not particularly useful in this context, simply because many students appealed to many different limiting processes at different times and for different reasons. Therefore, the following sections will contain many instances of students' use of each limiting process, taken as representative of the entire sample of participants.

V.B.1. Vague limit notions.

Students often referred to behavior of a sequence with vague colloquial language that indicated limit reasoning. For example, saying things like “that approaches some number” without being able to clarify what the “that” is, is identified as students' use of unattached limit language. If students used unattached limit language, which was followed by a clarification of the particular limiting process to which they were referring, this was not classified as a use of unattached limit language. This category was reserved for uses of words like “approaches” and “tends to” and “has a limit” that were unattached to any specific mathematical object.

For example, below Terrell used the phrase “it goes to infinity.” However, there was no reference for the “it”; and despite attempts for clarification, Terrell's response provided no referent for the limiting behavior.

Terrell: So we've already said it diverges if it goes to infinity.
 Int: What's the [second] ‘it’?
 Terrell: The *sum* diverges if it goes to infinity.

That is, the second “it” could refer to the limit of the terms themselves, the upper n -value on the sum, the behavior of some undisclosed graph, or something else. There are no additional contextual clues in this portion of Terrell's teaching episode that would indicate exactly what mathematics is taking on the limit properties of “going to infinity.”

It was also common for students to rewrite or reframe the infinite series about which they were talking explicitly in terms of a limit, as below

$$\lim_{m \rightarrow \infty} \sum_{n=0}^m a_n \text{ instead of } \sum_{n=0}^{\infty} a_n,$$

thereby placing the limit symbol outside sigma. Rewriting it in this way is in fact mathematically correct, and an explicit way that students *could* apply known limit properties to new content. That is, the procedural steps by which one would evaluate an infinite series are captured by evaluating the limit of the partial sums, as in the example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \lim_{m \rightarrow \infty} \sum_{n=1}^m \left(\frac{1}{2}\right)^n = \lim_{m \rightarrow \infty} \left(\frac{2^m - 1}{2^m}\right) = 1$$

However, students did not use the limit symbol outside sigma as a means for clarifying the procedure for finding a sum by computing the limit of partial sums. Rather, by rewriting series as explicit limits (with limit notation), students put the emphasis on the ‘process’ of stepping through the various values of n . That is, the students predominantly viewed the sum as being ‘in process’ (“almost going to hit 1”), rather than equaling the value of the limit itself. The

limit symbol was instead a marker to reference an infinite ‘process’ rather than a reliable procedure for finding a value of a sum (see Fischbein, 2001). It was clear that students’ ideas about limits were helping them to work with the difficult notion of having infinitely many terms, but attention to this particular limiting process did not give them any assistance in talking successfully about convergence of infinite series.

Students’ choice to rewrite the series with a limit symbol outside the summation sign was spurred by a variety of influences. For example, Brad simply rewrote his series with the limit symbol to explain the significance of the notation to a less knowledgeable peer.

- Brad: Ok so not too special, right? Just adding a couple of numbers together. Well, it's a little more interesting if, as n approaches infinity, if we look at it like this. (writes sigma with limit on the outside). Because calculus is all about limits and rules...
- Int: I have another question. I'm going to write it though. You keep writing limit?
- Brad: Which is not how we write it, I know.
- Int: No, that's ok. I'm curious about it. Like why write limit?
- Brad: Yea, I guess just because it like factually ... like it makes sense ... like it's not the traditional way of notating it, it's true. I haven't done this stuff in a while.
- Int: It seems to make sense to you, so what's the limit?
- Brad: At some point, once I was done explaining this, I would probably say yea we usually notate it as like this (writes infinity on top of the sigma) Um I guess ... probably because I was in explaining mode, I was just making it perfect. Maybe making it more obvious somehow by writing it like limit, because that's very familiar at this point, to like anybody who has taken calculus.
- Int: Yea that's true.
- Brad: So I'm saying like oh we *can* write it like this. We'll bother with the details later.

For Brad, it was not exactly a matter of one option being more correct, compared to the other. Rather, it was a means of explaining new material by drawing on mathematical features with which students would already be familiar, like “lim.” Theo does something similar, but further identifies the “conventional” way of notating infinite series as ‘incorrect.’

- Theo: Well I mean I did kind of write this wrong. I wrote the full thing wrong because I wrote it in my notation because I skip steps.
- Int: That's ok, I'm curious what you think you wrote wrong.
- Theo: The proper form would really be (writes limit outside sum notation). So that there's actually a limit in there.
- Int: So you're saying that's more correct than what you had before?
- Theo: It's more correct, yes, because this is really the same thing if you understand it, but I guess if someone doesn't understand it, that would make more sense to them.

That is, while Theo’s choice to write the limit outside the sigma was motivated by its ability to help others who do not understand the mathematical content relate it to something that they know from prior instruction, he also views writing the notation without the explicit limit

statement as incorrect. His view is that the series should be written with the limit statement because it is more correct, but if someone understands the meaning of the more compact notation, it is acceptable to use it.

Slightly differently, Todd views the limit notation as a means for being able to justify that the infinite sum could *equal* a value, whereas the sum written without the limit has a narrower and less powerful interpretation.

- Todd: So it's eventually get to the point where it's going to almost hit 1.
Int: Will it hit 1?
Todd: No but you can treat it more as a limit, like it approaches 1, and therefore the summation would have a value of 1.
Int: So does that mean the summation equals 1? Or does it mean something different?
Todd: Actually I think ... I don't remember but I think you can treat it more as a limit as x goes to infinity of $\sum_{n=1}^x (1/2)^n$, so basically it's the limit.
Int: So when you do the limit, can you write equals 1?
Todd: Yes. Well the limit would equal 1. But the sum would approach 1.

For Todd, writing the limit outside the sigma was a way to make a stronger mathematical conclusion. Brad, Theo, and Todd were only three of many students who appealed to this particular limiting process during their teaching episode. This limiting process does not have particular bearing on how students reason with what infinite series *are*, but it does influence the way they interpret convergence, and it does help them get a foothold with some of the difficulties of dealing with infinity.

V.B.2. Terms in the series

Students also often applied limit reasoning to the terms of the infinite series. There were three ways that students applied limit knowledge to the terms themselves – attending to the generation of the individual terms, the terms as physical quantities for which the limit was associated with the quantity, and sequences of terms.

Individual terms. Many students' goal was to examine a list of terms to identify the pattern that generated the individual terms. Attention to individual terms, then, was identified as reasoning focused on how to identify numerical patterns and generate a closed form for the terms of a series. For example,

- Brittany: So we know that a series is a (pause)... a list of numbers? Yea list of numbers. Usually like a_1, a_2, a_3 , all the way to a_n . And then infinite usually means there's no set end. So we're trying to understand how all these numbers are related, til the end. And usually series have patterns. So we have like 1, 2, 3, ..., up to 50. That you're adding 1 every time. And then sometimes you have 2, 4, 6, ... and then you have things that are like $1/2, 1/4, 1/16$, and you keep going. And as they go on, you try to figure out, usually in infinite series there's some sort of pattern so you can figure out what the 100th term is. So in order to understand that there's a long list of numbers and you want to find this pattern, you have to write them down. So here I wrote the first three. And so with these, we can see

that ... it would be a_1+1 and then it would be a_2+1 . Like it would be the first number and then\\

Int: \\Oh so you're adding on 1?

Brittany: But then there are other ways to write it.

Int: Other ways to write the same one?

Brittany: Yea like I could ... this is a_1 , this is a_2 , this is a_3 (pointing at numbers) and so with the series you'd say well you're adding 1 every time, so it would be a_1+1 and then a_1+2 would be that one. But [inaudible] simpler. And then usually you start seeing things like fractions. And here you see that the numerator stays the same, but the denominator is being squared every time. So one way to write it is $1/n^2$. And so I guess like an infinite series is just a list of numbers and when you have usually like the first four you start seeing the pattern and in that pattern you can create overall formula to find whatever specific term you want.

As Brittany looked at more and more terms, her focus was on attributing limit ideas to the pattern of terms, to eventually be able to write a model of the individual terms, in terms of n . She was not concerned with the limit of the sequence of terms, nor with adding them up. Rather, she was most focused on coming up with a model so that for any value n , she would be able to express a name for the particular term associated with that n . Thus, she was looking at modeling the individual terms for particular values of n , as n tended to infinity (a general model for a_n). Given the emphasis on this type of activity, both in homework exercises and in lectures, it is reasonable that students' first step would be to condense a long list of terms into a simpler presentation. However, this was not always followed by a move to apply any particular convergence tests or other reasoning, in order to remark on convergence of the series.

Terms as physical quantities. As a way of describing infinite series, students often attributed limit properties to terms of a series as if they were physical quantities that could take on those properties. Often, students described terms of a series as shapes, distances, areas, and with food analogies that could embody limit properties as a result of changing their shape, length, or other physical attributes. For example, Tina likened terms in a series to slices of pie (as in Figure V.2.), which she was able to connect to real life, draw in an image, and describe amounts of food quantity as the relative sizes of the terms.

Tina: When looking at the terms of fractions, just because the number on the denominator gets bigger does not mean that the actual fraction is bigger. So it's like you're taking pies. Here's a whole pie. Here's half of a pie. Here's a third of a pie. And a fourth of a pie. And you know, a fifth of a pie⁶. The pies are getting smaller and smaller and smaller\\

Int: \\the slices are getting smaller?

Tina: Exactly. And so you notice that you have one whole pie and you have half. And then you have three quarters. Or a quarter. And they're getting smaller and smaller. So you're adding smaller and smaller numbers.

Int: So what would converge mean with this pie thing?

⁶ The figures presented in Figure V.2. are as Tina drew them.

Tina: It means that eventually the amount of pie slices that you can add becomes so small that it almost doesn't seem like you're adding anything at all. If that makes sense.

Int: That tells me about the pieces. But does that tell me about the series? It's a question; I don't know.

Tina: It tells you something about the pieces *in* the series. It's the pieces that you're adding here become so small and essentially irrelevant. Or not irrelevant, but essentially worthless to add?

Int: Why? Why would they be worthless?

Tina: Because it's like you're adding ... say 1 over you know (writing)

Int: Lots of zeroes?

Tina: Piece of a pie, like that I wouldn't be able to draw, you wouldn't be able to cut.

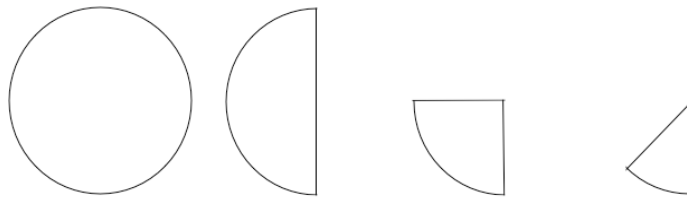


Figure V.2. Tina's geometric shapes image that was explained via terms as physical quantities

While Tina did hint at the notion that she wanted to add up the slices of pie, she attached her limit understanding to the relative sizes of the pieces, making smaller and smaller pieces of pie with each iteration. For Molly, the terms were interpreted as the sizes of walking distances that a person might take on a journey. Using reasoning akin to Zeno's Paradox, Molly described the terms of the series $\sum (1/2)^n$ as physical distances between herself and a wall (see Figure V.3.), which she was able to "halve," thereby attributing properties of limit to the sizes of the steps she was taking and/or the distances each step was moving her along her journey.

Molly: I'm thinking like ... You have some wall (draws line on board). And that wall is like some number I guess. So make it like S . You're some point here. And you're trying to go there by halving the distance. So you half it, and you're here, then you half it and you're here, and then you half it and you're here (Molly's drawing approaches her solid line as she acts out the steps)... And um the limit is that you're like approaching that number, so that number would be the limit, I guess? But like the distance ... I don't know, I guess it would be like the distance between you and here, would be the sum of the series. Because you're adding like this distance and then like this distance (imitating walking steps as coordinated with her drawing).



Figure V.3. Molly's 'geometric shapes' image that was explained via terms as physical quantities

Molly enacted this use of terms as physical quantities, but also created a referent of her journey on the whiteboard to record what she meant by "going half it and you're here." For Molly, the terms themselves were the distances traversed, and the location at which she arrived was the updated partial sum. However, of interest here is the fact that she viewed the terms as physical quantities that could take on properties of limits, such as "halving the distance each time" she steps.

Sequence of terms. Beyond applying limit ideas to the generation of terms or the relative sizes/descriptions of terms, most students applied their limit knowledge to the sequence of terms that composed the infinite series, as evidenced in Chapter IV with the wide use of plots of terms. There is merit to reasoning with limits of sequences of terms. For example, if $\sum a_n$ converges, then the limit of the a_n must be zero. However, the converse is not true. That is, if you want to know if an infinite series converges, it is necessary (but not sufficient) that the sequence of terms has this particular limiting behavior. Student difficulty with applying limit reasoning to sequences of terms, then, often results because they view this condition as sufficient for convergence. Students also often appeal to the 'process' aspect of limit again in this context, remarking on the relative rates at which sequences of terms approach zero ("getting small very fast")⁷.

For example, Todd believed that as the terms get smaller and smaller, the 'relative infinities' of the terms being 'infinitely small' even though you have 'infinitely many' of them will balance out, allowing the series to converge. That is, Todd's limit understanding was tied to the limit of the sequence of terms, and the sequence of terms getting "small enough fast enough."

- Todd: As this gets infinitely smaller, you get infinitely further out. So you're adding infinitely smaller things together. So it ... the infinities almost cancel out each other.
- Int: You mean the infinite amount of terms, but the fact that they're getting infinitely small?
- Todd: Yes.
- Int: Ok and so can you say then what that would mean one more time?
- Todd: If they get small enough, fast enough, then that means that the series will converge. Because as it gets smaller you're adding up smaller and smaller numbers to the total sum. Which sort of with some hand waving makes it equal to a finite number.
- Int: Small enough fast enough... how do you know what's fast enough?

⁷ This may be the result of a partial understanding of what was said in lecture. It was not unusual for a lecturer to distinguish between the terms of a divergent series (e.g., the harmonic series), where the terms "get small," and a convergent series such as $\sum 1/n^2$ where the terms get small "very fast."

Todd: Well you have a lot of tests.

While he could not entirely describe what this would mean, Todd's attention to the sequence of terms getting "small enough fast enough" indicates that he was interested in the limit of the terms themselves. More specifically, he was interested in the rate at which the sequence of terms itself converged.

Alex also applied limit properties to the sequence of terms, but did so for a different reason than Todd. Alex believed that if you examine the sequence of terms, there would come a time when the terms become "too small," so that they eventually are not important, resulting in essentially adding an infinite amount of zeroes. He says, of the series $\sum (1/2)^n$,

Alex: So the last term is too small. So if you think this is 0.5 and 0.25 and you get this later you get too many zeros. You can approximately equal to zero. So this [sum has] a value [which] is determined by the terms $[n=]1, 2, \text{ and } 3$.

For Alex, Todd, and many other students, the limit of the sequence of terms partially (or entirely) determined the convergence of their sum. While these students did remark on the sum as part of their explanations, the limit reasoning was applied to the sequences of terms (the sequence approaches zero, for example, as the consequential distinction). Different students applied limit properties to the sequences with varying degrees of mathematical correctness, but one of the most predominant limiting processes to which students attended was the sequence of the terms themselves.

V.B.3. Infinite series

Beyond simply applying limit principles to terms and sequences of terms, many students were also able to successfully talk about the infinite series itself. Students were also able to use limit ideas to explain convergence beyond simply attending to the terms themselves. This happened in a variety of ways, including looking at the sum as a generic addition, studying partial sums as collections of terms, orienting to the sequence of partial sums, and finally to a general notion of compilation or accumulation.

Sigma symbol as addition. Most students were familiar with the notation associated with infinite series, and were able to work with the notation and ideas it represents with varying levels of success. For some students, rather than associate limit ideas with a particular process tied to infinite series, the sigma was seen as a symbol that meant "add." This command 'to add' was a reference to many individual terms. That is, the sigma symbol itself referred to a collection of things, with no specific mechanism by which a finite sum could result, except the generic notion of "adding." For example, Brittany had an idea that addition was involved with reasoning with infinite series, but was unable to associate the mathematics involved with anything more than a general collection of summed terms.

Brittany: This is sigma. And it means sum. Which when we have numbers, sometimes we want to find out what they all add up to. So this is basically telling you that you want to sum up all the numbers. An introduction is that it's a list of numbers and usually you do the sum of them and use this notation.

For students who viewed the sigma symbol as a collection of things, the limit ideas were attributed to the notion of spending time adding individual terms. These students often had successful ways of talking about infinite series of numbers as mathematical objects that could (or could not) be represented by a single numerical value, but did not have an idea of how to apply limit ideas in order to describe the meaning of that value.

Partial sums as collections of terms. Some students had a notion of partial sums, and were able to flexibly talk about the meaning of partial sums, though they did not necessarily reframe the infinite series as a sequence of those partial sums. These students tended to focus on groupings of terms, each subsequent grouping of terms containing one additional term. Examining the groupings allowed students to study the patterns of the resultant values. With this kind of reasoning, students attributed limit ideas to resulting groupings of terms, without formally viewing them as a sequence. For example (as in Figure V.4.), Tracy viewed partial sums as collections of terms by writing out several finite series and comparing their outcomes to look for a pattern, or limit.

Tracy: Um I guess you might start ... You could take a bunch of finite series instead of the infinite one and see if the numbers started to approach something. If you start out by just saying that it starts at zero and goes to 1, then you get $1 + 1/2$. So you have 1 and $1/2$. And then you make it go to 2. That's $1 + 1/2 + 1/4$. Which is 1 and $3/4$. [The sum from 0 to 3] is $1 + 1/2 + 1/4 + 1/8$, which would be 1 and $7/8$. Yea so if you just sort of take all of these examples.

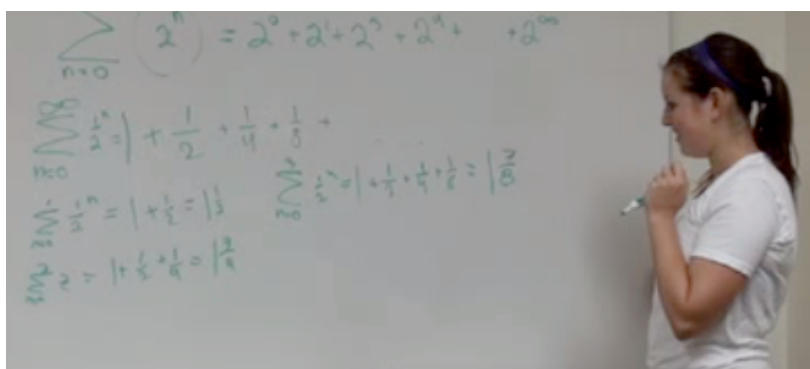


Figure V.4. Tracy's partial sums

Int: The examples being where you add one \\
Tracy: \the examples being yes that they just sort of add on to one another because they're all the same function, they're just going up to a different number of terms that you're adding. I guess if you did a lot more of them it would get closer and closer to 2 but it would never actually hit it. Because ok so for this one ... you have 1 and $7/8$ and you're $1/8$ from hitting 2. But then in the next series you know you're going to be adding something less than $1/8$. So you're not going to hit 2...
Int: Is this convincing to you that it would go to 2, this method?
Tracy: Um I think probably if I wrote out a few more.
Int: What if I tell you the next couple? (points to the calculator at the table)
Tracy: That would be good

- Int: [reads off the next several terms]
 Tracy: So the way the pattern's going, it sounds like every single series you add up, the sum is going to be 2 minus the last term. You know that for the next series that you write out, you're not going to be adding something equal to or larger than this next term.

Tracy used her limit knowledge to make conclusions about the results of partial sums, which were all displayed out on the board. From the way that she displayed them, she was able to make the conclusion that the result is always 2 minus the previous term. This may not have been obvious if Tracy had recorded her thoughts differently. Molly also attributed her limit ideas to partial sums as groups of terms, which facilitated by her choice of SGR, served as an intermediate tool that she later used (when discussing the harmonic series) to transition to thinking about a sequence of partial sums.

Figure V.5. Molly's work while reasoning with partial sums as collections of terms

- Molly: ...the boxes are showing the addition of partial sums I guess? So like you have ... This is the sum of this, like this is the value of the area. And this is this one. So if you add them together, I drew a box around it.
 Int: In terms of the picture, what is that?
 Molly: It's like this whole [area under the curve]. Because ... yea because that would be like, this would be a partial sum of the whole area. I guess. And I just drew another box because when you have series, isn't it like you add ... like if you were drawing terms of partial sums or something, it would be $1 + 1/2$ and then $1 + 1/2 + 1/3$, and then $1 + 1/2 + 1/3 + 1/4$...
 Int: So what is that thing that you're writing down now?
 Molly: That is what the boxes represent.
 Int: So you're saying each box is a different thing in your list?
 Molly: Um hm. Like this box ... Cuz I look at it like you have a whole bunch of terms 1, $1/2$, $1/3$, $1/4$, and then how I at least think about it... I think of like a box in my mind, and you like add another term in the box, and add another term to that box

In Molly's boxing scheme, as in Figure V.5., the limit ideas were applied when she added another term with each iteration, creating a new box. Thus, viewing partial sums as collections of boxed terms, Molly was able to use limit reasoning to think about the influence of the new terms that were included in each new box.

Sequence of partial sums. Beyond viewing partial sums as collections containing more terms than previous iterations, some students were able to successfully recast infinite series as

sequences of partial sums. This is aligned with the mathematically acceptable definition of series convergence:

Define $s_n = \sum_{i=1}^n a_i$. **If** $\lim_{n \rightarrow \infty} s_n = s$, **where s exists and is a real number, then** $\sum_{n=1}^{\infty} a_n = s$.

Students' attention to sequences of partial sums was most obvious when they committed to graphing those sequences on the whiteboard, although it was also possible to talk about behavior of sequences of partial sums without using a graph as a referent. In the following example, Andrew drew a sequence of partial sums on a graph (see Figure V.6.), where he viewed each added term as an additional step, and the graph as the result of that addition – a staircase.

Andrew: It probably would be better if we draw it like steps because of each term. So like this.

Int: So what are those things that the steps represent?

Andrew: It represents each added term. Yea. I hope that makes sense.

Int: Okay, so like if I pointed at that third little step that you drew, like what does that tell me about?

Andrew: It's the sum of the previous ... the sum of the previous term and the new term⁸.

Int: So everything up until that term?

Andrew: Yeah

Int: Okay, so your graph is basically the graph of the updated sum? [A: yea] And the dotted line?

Andrew: That's just some arbitrary limit.

Int: What does that represent?

Andrew: Ummm, in the converging series it represents what it would tend to if you keep adding terms. In the [divergent] one it's just an arbitrary point that it exceeds.

Andrew's graph of the sequence of partial sums also contained an asymptote that holds meaning as the value of the infinite series itself, if convergent. Though he never recorded an example of what the sequence of partial sums would look like themselves, Andrew was able to reason with the process of "updating" the sum from each addition of a new term as if it represented a limit of partial sums.

⁸ While Andrew does say "term" twice, his use is inconsistent, in that he is referring to "term" in two different ways. Video of the teaching episode makes clear that Andrew's use of "previous term" is in reference to the previous partial sum and the next term in the series.

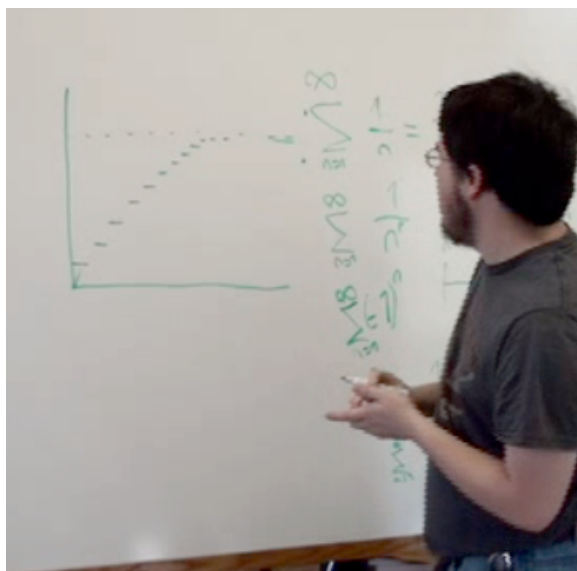


Figure V.6. Andrew's continuous curve of partial sums image explained via sequence of partial sum reasoning

As discussed above, Molly also had success in reasoning about the sequence of partial sums with her knowledge of limits. For Molly, however, the sequence (See Figure V.7.) was recorded after creating the intermediate boxed-term image, where she was viewing partial sums as collections of terms. This intermediate image led Molly to list out the “contents of each box.”

- Molly: Like this box ... Cuz I look at it like you have a whole bunch of terms 1, 1/2, 1/3, 1/4, and then how I at least think about it... I think of like a box in my mind, and you like add another term in the box, and add another term to that box.
- Int: Okay, so you're listing out the contents of each box?
- Molly: Yea.
- Int: So what does that list represent? I see where it comes from but I want to see what it is.
- Molly: Like, would it be all the different partial sums? Like this is the first partial sum, and this is the second partial sum?
- Int: It makes sense, cuz that's like the contents of each box. But why do we need that list?
- Molly: Because then I can think of it like a *sequence* ... But it's gonna be like approaching some value. (draws graph)
- Int: So what is that picture? That's weird.
- Molly: [laughing]
- Int: No no, use it. Just explain it. Cuz it looks like sort of the opposite of the first one.
- Molly: I just kinda drew it like this because you're increasing your values. So like this is gonna be a greater number than this number. And so I guess like this would be like the first point here (points to graph of partial sums with first term in written out sequence). And then this would be like the second point, and a third point. I guess I tried to give it a sequence. So like you have all those different

things. And this would be the sequence of the partial sums. And then that is approaching some value, like if it would converge it would just have some limit here.

$$1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

Figure V.7. Molly's transition from groups as partial sums to sequence of partial sums

Each of Molly and Andrew drew an image that reflects some understandings that can be extended and built upon. For Andrew, the steps in the function represent the partial sums. Thus, if the heights of those steps have a limit, the series converges. For Molly, the nested boxes represent the partial sums. Thus, she now has a sequence to evaluate, which can help her make claims about convergence of infinite series.

Compilation. Finally, some students applied their limit knowledge to the more general process of accumulation or compilation, without using explicit partial sum reasoning. This limit process is different than sums as collections because students who appeal to this type of reasoning are attending to the process of adding more and more terms in an infinite series, rather than simply the result of having put several individual terms together. For example, in response to the interviewer question “what would be a good example to help [me] illustrate some of the things that we learn about with series? Do you have some examples that would help since I never heard of it before?” Tim used a metaphor to demonstrate the result of his compilation reasoning.

- Tim: When you asked that question, I thought about a beach and sand. It seems like there's an infinite amount of sand on the beach. But it adds up to a beach. So it's like this bigger picture, but it's composed of these really tiny, almost infinite pieces. The sum of a series would be like that.
- Int: Like putting all the sand together?
- Tim: To get a beach.

Tim's compilation reasoning is more showcased in the result of his simile, but his limit ideas were based on the accumulation of sand on the beach. Similarly, Maria's compilation reasoning is evident in her attention to what happens to areas under a curve that she chose to represent a divergent infinite series. In discussing a divergent series, Maria used limit reasoning when thinking about the area under a curve growing larger and larger, unbounded.

- Maria: Sure. So yeah, because if you think about something that wouldn't converge, you would think of a function that looks like this (draws graph with no horizontal asymptote). And as you add each term you're gonna add more and more areas, so it's going to end up being infinity.
- Int: So what about that picture doesn't converge? I just want to make sure I'm attending the right thing.

Maria: (pointing at a graph with infinite end behavior) So we know if you take a limit as you approach a point here, it's going to have an output. If you take the limit of the whole thing as x goes to infinity, the area under the curve is going to be infinity because you keep adding an infinite number of things that get larger and larger. So the stuff you're adding keeps getting bigger and bigger, so it's not amounting to something, anything.

Tina's reasoning about pieces of pies as terms in an infinite series (as discussed earlier, in reference to her reasoning with terms as physical quantities) also lent itself to compilation reasoning, in terms of accumulated "pie."

Int: You're making me hungry. Can I ask another question about the pies?
Tina: Yes [laughing].
Int: So if the piece of pie that is that small, if I can barely perceive that I'm eating it because it's so small, does that mean that the amount of pie in total is finite, or no?
Tina: It means that the... it doesn't mean that the amount of pie itself is finite. It means that our perception of the amount of pie is finite. And so the perception of this number of overall pie has essentially stopped growing. So it seems like the series ... or the summation of all these pieces of pie ... has approached a number. Like I don't know what the number would be, but say it's 3 and 1/4 pieces of pie.
Int: Sure, so there could be some number.
Tina: Right.

Here, Tina displayed reasoning about accumulated or compiled pie ("overall pie has essentially stopped growing") as she talked about pieces of pie. However, her attention to compilations also highlights other aspects of limit reasoning that were part of her understanding. For example, if the overall pie has stopped growing, the partial sums are essentially the same, which means that each additional term added must be 'smaller and smaller.' Additionally, the imprecise language referencing "perception of the amount of pie" suggests difficulty with the idea of infinity and how things might 'accumulate.'

Other students, like Aaron, used physical examples of compilation to explain the meaning of divergence of an infinite series. For Aaron, numbers of pieces of paper helped explain how the series $1+3+5+\dots$ diverges, by stacking the paper and seeing that the stack will grow, unbounded.

Aaron: If you assume that this sheet of paper is the first and this is one and then we go add the next one and then we had three and five and if we keep one adding that grows infinitely tall.

Students often appealed to compilation reasoning in conjunction with terms as physical quantities. This permitted them to discuss convergence as accumulation of physical things, which amount to a finite (tangible) end result. Without compiling/accumulating physical things, it was difficult for these students to have a referent for explaining convergence.

A summary of the above nine limiting processes is below, in Figure V.8. Two ways that students' use of limiting processes can help us grapple with what students understand of infinite

series are to explore (1) the ways in which particular images seemed to facilitate reasoning based on differing limiting processes, and how the particular images suggest which limiting processes are relevant, and (2) the ways that students make connections among the limiting processes that they do find relevant. While the latter will be discussed in depth in Chapter VII, an example of the former makes up the remainder of this chapter.

Limiting Process	Working Definition	To <i>what</i> is the student applying limit ideas?
Vague Limit		
(1) Unattached limit language	Vague discussion of “ n ” or a non-oriented limiting process that is not explicitly linked to terms, sum, etc.; vague limiting language without a referent	Nothing/unclear
(2) Limit symbol outside the sigma	Students choice to rewrite the standard sigma notation with an explicit limit statement, to help reason with the infinite nature of the mathematics	The index of the series, or stepping through the values of n in the sum
Terms		
(3) Individual terms	Attention is on the individual terms and their arising from a pattern, but there is no attention on examining their sequence or order	The model of an n^{th} term, as generated from the first several terms
(4) Terms as physical quantities	Terms are representative of physical quantities, whether areas under a curve, geometric shapes, steps in a trajectory, etc.; this is independent of whether or not the student reasons about the accumulation of terms	Physical quantities that are being described as taking on the mathematical properties of limits
(5) Sequence of terms	Attention is on the ordering of the individual terms and their progression; all claims about convergence are about “where the terms go”	The sequence of terms themselves
Sums		
(6) Sigma symbol as addition	Focus on the sigma symbol as referring to a sum, but not on applying limit properties to find a value for that sum	The (temporal) process of adding a bunch of numbers together
(7) Partial sums as collections of individual terms	The attention is on partial sums as collections of terms, whether ordered or not; while these partial sums may not be ordered into a sequence so that the limiting process can be attributed to the progression, they may be grouped in such a way that shows their relative sizes and impacts of adding more terms	Groupings of terms with increasingly more entries
(8) Partial sums as a sequence	Attention is on generating and examining the sequence of ordered partial sums	Sequence of partial sums
(9) Compilation	The sum has a value that is the result of putting together/compiling some units, where the sum is a physical quantity, and the emphasis is on accumulation of terms	The process of compiling terms or objects toward a final state represented by their sum

Figure V.8. The nine limiting processes, revisited

V.C. Example – Zeno’s Paradox reasoning

One rich example that comes up both in lectures and students’ teaching episodes is one of Zeno’s paradoxes - *That which is in locomotion must arrive at the half-way stage before it arrives at the goal*. Students use this single example to lead up to a variety of different ‘punch lines,’ which are mathematically productive to varying degrees. Interestingly, the varying ‘morals of the story’ are often the consequence of focusing on different limiting processes. In what follows, we will see the multiple ways in which students (partially) make sense of that story. This discussion will show ways that many of the partial understandings discussed in the first part of this chapter arise in the context of this mathematically rich example.

Zeno’s Paradox in the literature. The literature is sometimes vague when distinguishing between the *idea* of limit and the specific, mathematical construct of limit. For example, Williams (1991) declares that using a “walk to wall” description, whereby the walker halves the distance between himself and the wall (essentially an embodied version of Zeno’s Paradox) provides a “compelling metaphor of limit.” But – a compelling metaphor of limit *of what*? That the distance one is walking is aggregating in smaller and smaller bits, but converging to a value equal to the distance between the start point and the wall? That the distance left to travel is “approaching” zero, with every step? Something else? Those with a more sophisticated understanding of limit see the continuity of this reasoning and likely do not question that this embodied example is indeed a metaphor for the *idea* of limit that instructors wish students to understand. But students, early in their learning about limit, struggle to orient themselves to which specific aspect of the metaphor represents a limit. Further, the value of the limit in question depends on which frame one is using to orient the metaphor.

One cannot blankly say that Zeno’s paradox is a good metaphor of anything in particular. Rather, the success of the metaphor depends on what one is attempting to explain or justify. Some students focus on the decreasing sizes of steps, using reasoning consistent with their understanding of the ‘sequence of terms’ limiting process. Other ways that sequence of terms reasoning could become salient are if students attend to the distances left to travel, or the notion that movement is ‘impossible’ because before taking one step, one would have to take a step half the size, etc... Alternatively, if students focus on the total distance traveled when “walking to a wall,” rather than the steps it takes to “get there,” they are focusing more on ‘partial sums.’ Yet other students attend to the ‘compilation of terms,’ recast as physical quantities (areas, distances, etc.). This often comes across as ‘filling a geometric shape,’ closing in on a finite distance traveled, reaching a destination on a number line, etc. Interestingly, of several professors that used this same example as part of their lecture and description, most were relatively vague as to the particular way in which they wanted students to orient to the metaphor. So, it may not be surprising that even as a result of direct instruction, students came away from class with unclear ideas about which parts of the metaphor were useful and in what ways. However, beyond seeing it in lecture, students generally found this paradox to be a familiar and easy-to-adapt way of explaining convergence of infinite series. Possibly for this reason, many of them were drawn to this example.

Students’ use of Zeno’s Paradox. During the 37 teaching episodes, 15 students included in their explanations some reference to Zeno’s Paradox. These students all also created some sort of SGR to represent the aspects of the walking to the wall metaphor that they were keying in on in their explanation. Student uses of Zeno’s Paradox were in reference to either (a) sequences of terms (sizes of steps), (b) partial sums (updated distance), or (c) physical quantities and

compilation (with a physical referent for the value of the sum). The three contrasting limiting processes show how particular aspects of the metaphor can play out differently for students, leading to different punch lines. While all of the different limiting processes that can be illustrated with the metaphor are powerful for thinking about limit and about infinite series, they are powerful in different ways, only some of which translate to successful mathematical conclusions. What will follow is a description of what it looked like when students focused on particular limiting processes in their use of Zeno's Paradox, with some examples of student reasoning, and what those mean about students' understanding, more generally.

Sequence of terms. Students who focused on the sequence of terms when using Zeno's Paradox were attributing limit behavior to the sequence of $1/2$, $1/4$, $1/8$, $1/16$ and so on, and discussing the limit of the sequence of either numbers, steps, distances, or areas that corresponded to the terms in that sequence. These students, to varying degrees, relied on the fact that their numbers, steps, distances, or areas were getting smaller in size, and used that as a means of arguing for the convergence of the series $\sum (1/2)^n$. In verbal descriptions alone, it was not always clear that the students were referring to the terms themselves. Therefore, students' descriptions along with their images can help clarify the ways in which they find this example illuminating.

Largely, these students made arguments that had some (limited) validity, but overextended them. These students were often not able to make well-supported conclusions that would be valid beyond this particular context. Below are several examples of sequence of terms reasoning with Zeno's Paradox.

Todd: Let's do this. So you have $1/2 + 1/4 + 1/8$ plus smaller and smaller and smaller. So this is probably approaching 1.

Int: Where did the 1 come from?

Todd: You're adding smaller and smaller terms each time. So it's eventually going to get to the point where it's going to almost hit 1 ... Well you know that because of the rules of calculus, but you can sort of visualize it as each time you add it up, you have sort of an interval that's left, and from that interval that's left you're adding half of that. So you're getting halfway close to 1 each time.

Abby: So half the area. Now we're adding a fourth of the area. And you just sorta keep doing that and keep dividing each piece into a half cuz that's what the next term is always doing. And you never quite get to the end because you are always just taking half.

Andrew: You can go half way and go half way again and go half way the remaining distance and you will ne... just by this you would never actually exceed 1.

Int: Because there is always a halfway to go?

Andrew: Yeah. So, even though you are taking an infinite number of steps you never quite get to 1. But you never go past it.

Theo: Yea like the sum from $n=1$ to infinity of $[(1/2)^n]$. Because this equals (corrects starting index to zero) ... Because $1 + 1/2 + 1/4 + 1/8 + \dots$ and that means we've got

[a number line] from zero to 2, halfway, halfway, halfway, and it just limits [to 2].

Brad: 1 wall distance. That's equal to $1/2$ plus $1/4$ and then you have to go $1/2$ of a $1/4$ again, so you go $1/8$ of the distance. And then you have to go $1/2$ of that again, so $1/16$. And what Zeno said is basically you can keep cutting numbers in half, but since there are infinitely many real numbers, you'll always gonna get something that's larger than zero. So you're always going to get some amount of distance you'll have to travel again to get to the wall.

In using the $\sum(1/2)^n$ example, students who applied limit reasoning to a sequence of terms were most often focused on the halving of the sizes of the steps or the shrinking of the “remaining distance.” This assumed they knew the final value to which the series converged, in advance. These students’ reasoning was aligned with one of the most common errors that students make when explaining infinite series – namely, stating that the sum of terms that decrease to zero is automatically finite. This type of reasoning was most connected with *labeled* number line SGR, such as in Figure V.9, where the students focused on moving half the previous distance, along the number line. It was also common that students acted out this type of reasoning, connecting terms they had written out with the sizes of the steps they were simulating.



Figure V.9. Labeled number line

Partial sum as sequence. Students who instead focused on partial sum as sequence reasoning when discussing Zeno’s Paradox examples were more likely to be able to reason in mathematically appropriate ways about why $\sum(1/2)^n = 1$ (or 2, depending on the starting n -value). These students focused not only on the steps or terms in the series, but also on an “updated location” or total distance traveled up to a certain point. That is, appeals to partial sum reasoning took the form of expressing the important features of Zeno’s Paradox in terms of “progress” made from carrying out the process, rather than solely describing the process of taking consecutively smaller steps with each iteration. Some sample student reasoning is below.

Terrell: This is interesting. You go halfway. Then you're at $1/2 + 1/4$ of the way. Then you're $1/2 + 1/4 + 1/8$ of the way. And so on. And you keep taking half away from it again and again. Without getting into a horrible proof, as you keep taking halves away, we know this is an asymptotic relationship. We know there's no “a-sub-finite-number” where this becomes 1. But we also understand that as we keep add up an infinite number of them, and after you do all of them, you'll add up to 1. You can't ever do all of them. If at any point you stop, you're not at 1.

Andrew: Because you're trying [inaudible] each half step, for example you go one half here and then at this point you're $3/4$ of the way, you're at $1/2$ there you're at $3/4$ there, which equals $1/2 + 1/4$, and you get $7/8$, that's what it is, and then $15/16$, each time you're adding a power of $1/2$.

Molly: I'm thinking like ... You have some wall. And that wall is like some number I guess. So make it like S . You're some point here. And you're trying to go there by halving the distance. So you half it, and you're here, then you half it and you're here, and then you half it and you're here (recording hashmarks on the whiteboard)... And um the limit is that you're like approaching that number, so that number would be the limit, I guess? But like the distance ... I don't know, I guess it would be like the distance between you and here, would be the sum of the series. Because you're adding like this distance and then like this distance.

Tracy: If you start out by just saying that it starts at zero and goes to 1, then you get $1+1/2$. So you have 1 and $1/2$. And then you make it go to 2. That's $1+1/2+1/4$. Which is 1 and $3/4$. [The sum from 0 to 3] is $1+1/2+1/4+1/8$, which would be 1 and $7/8$. So if you just sort of take all of these examples ...

Int: The examples being where you add one?

Tracy: The examples being yes that they just sort of add on to one another because they're all the same function, they're just going up to a different number of terms that you're adding. I guess if you did a lot more of them it would get closer and closer to 2 but it would never actually hit it. Because ok so for this one, but I guess this requires you knowing 2 as well, well no you have 1 and $7/8$ and you're $1/8$ from hitting 2. But then in the next series you know you're going to be adding something less than $1/8$. So you're not going to hit 2.

These students were more successful than those who attended to the sequence of terms limiting process, because recasting the problem in terms of partial sums provides a basis on which students can make more mathematically appropriate claims about the convergence of the infinite series. The analogy of the “updated location” for those students who were ‘acting out’ Zeno’s Paradox allowed students to coordinate both the sizes of the steps with the amount of distance traveled, and the location that that sequence of steps produced. Reasoning with this limiting process most often coincided with an *unlabeled* number line (like in Figure V.10.) or with grouping written out terms on the whiteboard (such as Tracy, in Figure V.4.).



Figure V.10. Unlabeled number line

Terms as physical quantities + Compilation. The most common type of explanation that students made was related to the ‘terms as physical quantities’ and ‘compilation’ limiting processes. Though the most common, students’ appeals to these two limiting processes were mixed in terms of success. Some students used this type of reasoning to describe the meaning of convergence from the perspective of deciding whether or not the series converged, building up to the value of the sum. Other students started their explanations with the assumption that the series summed to 1 (or 2), and worked backwards, decomposing their image or description to make it match the known final result.

To aid in distinguishing the two ways that students used terms as physical quantities and compilation reasoning when explaining Zeno’s Paradox, consider Figure V.11. – an image that

frequently appeared in students' teaching episodes. A hypothetical student might start by saying "this is one box." This student may then proceed to identify that they have 'shaded half of the box' and the 'half of half of the box' and so on. This type of reasoning starts with identifying the sequence of terms, but then proceeds toward reorganizing that information to identify that if the student has shaded the rectangles that correspond with $1/2$ and $1/4$ and $1/8$, etc. of the single box, they have shaded in total $7/8$ of the box. That is, using the image this way allows students to transition their sequence of terms reasoning to partial sum reasoning. On the other hand, a hypothetical student might start by saying "this box has an area of 1." They could then proceed to decompose the known area of 1 into parts that correspond to the terms of the series $\sum (1/2)^n$. The major difference is that in the former ('independent of final state'), the object that students are compiling/examining is one object, and they are using properties of its area to reason about infinite series. However, the latter has a predetermined area of 1 (or 2). The latter ('already known final state') essentially supposes that one knows the series to converge, and is seeking a way to break it into sectors that represent that total area.

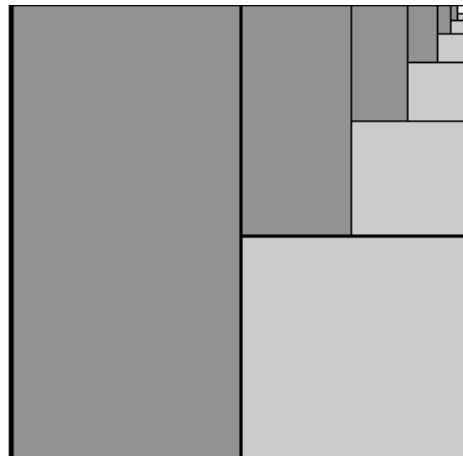


Figure V.11. Geometric shape for $\sum (1/2)^n$

Students' reasoning strategies in the following examples illustrate the different ways that focusing on the limiting processes associated with terms as physical quantities and compilation played out differently.

Independent of final state:

Aiden: So the question is to show that this series converges to one. Then how does this make sense or where does this make sense. It's kind of ... if you write out the series... then you kinda see, if you have a stick and you first take half of it then you take a quarter then an eighth and you take that finite times you can not complete the whole ... you cannot exhaust the whole thing, but this series if you repeat your action infinite times then you can take out a whole stick ... It's kinda like it's an interesting part of that because in common sense you can never complete the stick but in math, in magical math you can do this ... The stick theorem ha ha ha.

Brad: Let's see, as I understand Zeno's Paradox, the idea is that you have a finite distance between a person and a wall, and the first thing you do to get to the wall from where you are, as a person, is to go half the distance. You have to do that, then you have to go half the distance again. So you go a quarter of the total distance. And then you keep doing that. So you go from ... so you have 1 total unit length wall distance.

Already known final state:

Tracy: The way that I kind of think about it is you have this box, and this box equals 2. And you cut it in half, and this half equals 1. And you cut this in half and this part equals one half. And then this is $1/4$ and this is $1/8$. And $1/16$ and $1/32$. And you keep going and you keep cutting them in half. But you never add on to the box. But you can keep cutting these little tiny rectangles in half. Forever.

Travis: Say we have a box and it has an area of 2. First one, you want to fill in half the box. Next one you want to fill in a quarter of the box with that half.

Int: So you have both of the shaded things together?

Travis: Right. Then we're going to do an eighth of a box, half a quarter. Then a sixteenth. Thirty-second. Sixty-fourth. And so on. So we see that we can keep adding them infinitely many times however it's only half of the remainder of the area. So regardless of how many times, you're only gonna get half of the area that's still left. And that will never exceed 2.

Int: So it's never gonna like add to the box?

Travis: No, but the limit as it approaches the box is gonna get so close. And you're gonna get the limit of $1/n$. And as n goes to infinity, that's going to be the area left. So do you know anything about the limit as n goes to infinity of $1/n$, does that approaches?

Int: Yea sure, zero. So what's the $1/n$? I think I know what you're saying but I wanna make sure I get the whole point.

Travis: [laughs] Right, the $1/n$ then represents the remainder of the box.

Int: And you're saying it's getting filled up all the way? Cuz the limit is zero?

Travis: Right.

Abby: (drawing a box with areas as terms) It's like you're always just dividing by two. Once you get zero - you won't. What the point of this is, is that it converges to one. So As you're like, if we can imagine going off into infinity pretending we reach some sort of "end" (air-quotes) we can see that this gets closer and closer to the whole box.

Molly: My calculus teacher in high school for AB, when she was like trying to explain I don't know, something, I always remembered though. She like walked to, say there's like a wall there, right? And she walked to it. And she walked like halfway. And then she went another halfway. And another halfway. And another halfway. You can like never reach the wall. Cuz you'll always be

getting closer to it, but you'll never actually hit it. And so I guess it's one of those things where like, you're like adding the distance you walk. Maybe? And you're saying that that sum would be like the distance from you to the wall, but you never actually get there. I don't know if that's a good analogy, but ...

Int: I think ... so that example ... so you thought of series when you think of that example?

Molly: Ummm I think of limit.

Int: You think of *limit* when you think of that example?

Molly: Like it's approaching something, but this series would show the sum of the distance. And ... I don't know if that makes sense.

...

Int: So where is the “infinity” in this example?

Molly: That you can never reach the actual number. Like you can never actually, physically hit the wall.

...

Molly: Yea but when you find if a series ... or I mean if a sequence converges or diverges, one of the ways you do it is you take the limit of it and you like see what value it approaches. Or if it approaches a value. And so that's just like the way I've been thinking about them. With series, they're more of some kind of unit - like distance, or area, or something like that. So it's more of a compilation of values. I think that makes sense.

...

Molly: You only have a certain amount of distance you can go. Or only a certain amount of area that can be underneath the graph. If that makes sense. I guess it goes back to [the person and wall diagram]. Because you have some physical distance you can go (gestures), and that is the only distance you can go. And even though have like ... you're taking your distance or your area in certain parts, and you're thinking ok this is one part, and you're adding another little part, and another little partial sum, right? But you're only going to have a certain amount of area that you can ever achieve with that.

In viewing the terms as physical objects that can take on limit properties such as halving the size of the previous step, or stacking, or accumulating bits into a single stick, for example, students created a physical reference for the value of the sum, and drew correspondences for both the individual terms and the partial sums along the way. For students who used this limiting process after first assuming that they knew the value to which the series converged, it was most natural to connect the compilation reasoning to the sequence of terms. They had already decomposed the image with the fixed area/destination/final distance. However, for students who used compilation reasoning to lead up to conclusions about convergence, it was most natural to build in sequence of partial sum reasoning that lead to a conclusion about the value of the (convergent) sum.

The above three ways that students used Zeno's Paradox when explaining infinite series – attending to sequences of terms (sizes of steps), partial sums (updated distance), or physical quantities and compilation (with a physical referent for the value of the sum) – show the differences in how the metaphor can play out for students, based on which limiting process they prioritize in their explanations. As evidenced by the various examples, focusing on different

parts of the metaphor, which require students to apply limit reasoning differently, results in thinking that is mathematically productive to varying degrees.

In terms of student understanding, examining the different limiting processes that are elicited as a result of students' appeals to Zeno's Paradox provide examples of which aspects of infinite series of numbers students think are consequential for explaining convergence and/or divergence. Common concrete examples such as students' use of Zeno's Paradox allow us to identify the different ways that students see the various limiting processes as important, and trace where that might come from as a result of instruction. For example, the group of students who focused on taking steps and measuring the sizes of their steps represent students for whom paying attention to the sequence of terms of a series is sufficient to discuss convergence of that series. In asking students to enact metaphors such as those elicited in Zeno's Paradox-type reasoning, many of these more common reasoning patterns emerge. The identified patterns can lead to innovative curriculum, meant to address these habits that form them, which often lead to mathematically unsound conclusions.

While all of the limiting processes are important and worthy of student attention, this scheme that breaks down students' justifications based on reasoning with the different limiting processes brings about a better understanding of how the students value each one as part of an explanation of infinite series convergence. When compared to the experts (textbook and professors), student vs. expert use of particular images is differentiated mostly in the ways that the different groups focus on the limiting processes represented. For example, when using area under curves images, experts' usage for clarification of the integral test shows that they prioritize different limiting processes than students who use the same image to define. Students focus on either the decreasing sizes of the terms of the series (sequence of terms), or the size of the area under the curve as each new 'rectangle' is added to the previous rectangles (compilation) to "define" convergence. Experts, on the other hand, focus on partial sums reasoning to make a case for error associated with integral test. Thus, when using this particular image type, lecturers pay attention to one feature (tied to a particular limiting process) while students pay attention to a number of other ones.

In addition to students emphasizing different limiting processes than experts, students also attend to different limiting processes when using the same example – Zeno's Paradox. As Wheatley and Cobb (1990) discuss, different people explaining the same concept or using the same image will focus on different things. Breaking down students' use of Zeno's Paradox reasoning, it is possible to see the different ways that students support their claims about infinite series convergence, through attention to different limiting processes.

VI. LANGUAGE

As described in previous chapters, students' explanations of infinite series convergence were distinct from experts' explanations in three different aspects. The emphasis of this chapter is on exploring the second aspect of competence – Language. As identified in Chapter IV, professors characterized various image types using particular, intentional language choices. These language choices were meant to draw attention to the context in which a specific image type is useful, and the conclusions that can be drawn or inferred from it. Students used the same language in their explanations, but in different ways, sometimes understanding professors' language to mean something different than intended. In exploring this aspect of competence, we will consider language used during the teaching episodes in both consistent and inconsistent ways, and how students' use of language has an impact on their understanding.

To begin, the two most common examples from the larger data set related to students' difficulty with language are presented. First, the ways that students perceived and discussed differences between 'converge,' 'approach,' and 'equal' highlight some of the ways that students see distinctions in experts' language use that were not intended. Second, students' use of 'terms tend to zero' reasoning highlights the ways that students' SGR (self generated representations) help to disambiguate colloquialisms and imprecise language, allowing for a more reliable interpretation of their understanding of infinite series.

Outline of Chapter VI -

VI.A. What is meant by language

VI.B. Example 1 – converge, approach, and equal

VI.B.1. Limit outside sigma

VI.B.2. Vague limit language

VI.B.3. Mathematics vs. common sense

VI.C. Example 2 – “terms tend to zero”

VI.C.1. Three examples of 'terms tend to zero' reasoning

VI.C.2. Mark's Teaching Episode

VI.A. What is meant by “language?”

The language associated with explaining and reasoning with limits in mathematics – converges, limits, tends to, approaches, etc. – relies on words that not only have specific mathematical meanings (which to a mathematician are entirely consistent) but also various colloquial meanings that may or may not align perfectly with the mathematical meaning. For example, simply considering the word “limit,” one could conceive of a variety of meanings, only some of which resonate with mathematical limits. Referencing a *speed limit*, for example, as a fixed speed, which one can technically exceed but may result in penalty, is not taken in the same spirit as a limit in a calculus course. Even further, colloquial phrases like “pushed to the limit” load more meaning onto that simple word that influences the way it is used, whether or not there is an intention of being mathematically correct. Many of the same types of commentary could be posed about all of the limit related words listed above, and others (see Monaghan, 1991).

The past several decades of research on student understanding of limits firmly establish that students continue to struggle with the language of *converge*, even within an isolated context of limits of continuous functions (e.g., Monaghan, 1991; Williams, 1991; Cornu, 1992). Monaghan (1991) showed that students considered “converges” to be the most foreign and

difficult phrase of “limit,” “approaches,” “tends to,” and “converges.” To the students in his study, *converges* came to represent an instance of two continuous objects coming nearer to one another, eventually touching in most cases. It also became apparent that students were not sure what *converge* meant in mathematical situations. For example, one student stated (on page 23):

“I don’t really see how numbers can converge. Converge means light from a thing coming in. It’s two separate parts. You’d have to have two sequences coming in on each other. I don’t think you can have one sequence converging.”

Other students indicated that *converges* means that “two lines are going to sort of touch each other” and “I was thinking of the word converge as coming from two sides.” This is perhaps the most confusing of the phrases to students because the everyday meaning, according to Monaghan, is so closely associated with lines converging. One final student discusses, “... the term converges to can mean many different things and depending upon which definition or meaning is put into practice, the answer to any question can differ,” highlighting that *converges to* has slightly different meanings across functions, sequences, series, etc.

Monaghan (2001) also provides a perspective on students’ ideas about infinity in which he explicitly tries to divorce his analysis from limit concepts. When faced with something like $0.1 + 0.01 + 0.001 + \dots$, mathematicians see a series that converges to one ninth; they do not find it problematic to say things like “going on forever and getting an answer.” It is not viewed as a temporal statement of taking an infinite summation, over time. Put differently, examples such as Zeno’s Paradox are not problematic to someone who does not see the mathematical complication. But an analysis such as Monaghan’s, though it tries to function independently of limit entailments, cannot help but acknowledge them and the many linguistic and imagistic problems that they create for students. In many instances of people trying to study students’ work with Zeno’s Paradoxes (e.g. Fischbein, 2001), phrasing changes that are very subtle can cause huge shifts in interpretation (meaning of words such as “reach” for example). Phrases like “approaches” and “tends to” have been shown in some studies to indicate ideas of motion (e.g. Cornu, 1992), and in other studies to indicate ideas of “closeness” (Oehrtman, 2008).

The phrases that instructors use when conveying the content of infinite series for the first time may also present additional language issues for their students. For instructors, phrases like “if this converges, it means the infinite sum is finite” are not problematic. However, for the students in their classes, these may not make sense. How can the sum to which you are referring, with the adjective of ‘infinite,’ possibly also be finite? Herein is a clash of limiting processes. For instructors, such a phrase is not problematic, because the phrase “the infinite sum is finite” invokes two separate limiting processes. The ‘infinite’ is in reference to the series itself as composed of infinitely many terms. And ‘finite’ is in reference to the result of adding those terms – the value to which the sum converges. However, difficulties such as this may only compound students’ troubles with the language of infinite series in second semester calculus.

There are a couple of ways that this difficulty with language is revealed in students’ explanations. The first way that this difficulty may be revealed is in the ways that students interpret language that mathematicians consider synonymous, and therefore use interchangeably in lectures and textbooks. For example, an instructor may use the word ‘tends to’ in one lecture and ‘approaches’ in another, in reference to the same thing. She may not even notice that she has changed her word choice, because the words carry the same meaning to her. However, students may pick up on the different words used, and attribute some difference to the choice, tied to the

day's lesson, even though none was intended. Thus, when students adapt and reuse the language, they often use it much more narrowly (in the context of limit) than experts would intend (see Monaghan, 1991). Secondly, we may identify students' difficulties with language by examining how their images help illuminate their thinking/understanding, when language alone would be ambiguous or insufficient. That is, if a student's explanation contained much jargon or inconsistently used phrases, we could look to their SGR to help disentangle some of the differences they perceive in the language.

These two ways that language difficulties can manifest themselves in students' explanations are significant because, since students can use the same limit language to refer to different limiting processes, there is a need to sort out the ways that they apply this language to the different limiting processes. Now that some of the ways in which students find particular limiting processes important have been explored, the goal of understanding student thinking shifts to studying how they are using their language to describe the ways that those processes play out. Since students' language use is so often ambiguous and idiosyncratic, examining students' use of images can help to disentangle students' meanings and the ways that they apply limit ideas to the different limiting processes.

Students' difficulties in interpreting and using mathematical language surfaced most prominently in their uses of 'converges,' 'equals,' and 'approaches' to describe convergent infinite series. Students identified particular features of different scenarios that signaled use of one word vs. another. Additionally, students' use of language related to the sequence of terms tending to zero was often ambiguous. These two examples will be explored in the remainder of this chapter.

VI.B. Converge, Approach, and Equal – on student difficulties with interpreting and using mathematical language

Mathematicians typically feel comfortable using the terms 'converge,' 'equal,' and 'approach' interchangeably (Monaghan 1991), and they do not necessarily see any features of one that is not captured by another. However, students often draw very clear distinctions between the different choices of limit words, each associated with very particular properties that the others do not necessarily share. Thus, for students, it is not trivial to decide whether $\sum (1/2)^n$ approaches 1, converges to 1, or equals 1. Even after they have decided, many different ideas about limits and what it means to have a limit in the first place further complicate students' choices of language. Williams (1991) found that an "overwhelming majority" of students in his study believed a limit unreachable – meaning that limits "approach but do not reach" or "get close but never reach." Though many of the students in this current study held this same belief, it had different implications for all of them. What follows here are three examples of the ways that students struggled with the language of converge, approach, and equal during their teaching episodes.

VI.B.1. Limit symbol outside the sigma

It was a common theme for students to reconcile their struggles with 'converge' and 'equals' by putting the limit symbol outside the sigma. Students often appeared unsure of the different ways they could conclude that a series 'converged,' and unclear if that meant that a sum equaled the value to which it converged. However, from previous calculus experience, they understood that when working with limits, they are allowed to use an equal sign to show the value of a limit. That is, in order to sort out language issues around 'converge' and 'equal,' for

some students it was as simple as transforming the sigma notation to include an explicit limit statement. This allowed them to recast all conclusions that they would make in terms of limits, thus avoiding the question of equality. The example student reasoning below highlights the ways that students adapted their language and limit ideas to make conclusions about convergence.

Abby: I think it's equals.

Int: So you think equals is okay, so what makes you not hesitate? Even though [you said] that list can't be completed you said that list is ok? .

Abby: Right. Well we are taking ... ok so this is basically if we replaced this guy and we took the limit as t approaches infinity. (Amends sigma notation with upper bound of t and then writes limit in front of it)

Int: Will it hit 1?

Todd: No but you can treat it more as a limit, like it approaches 1, and therefore the summation would have a value of 1.

Int: So does that mean the summation equals 1? Or does it mean something different?

Todd: Actually I think ... I don't remember but I think you can treat it more as a limit as x goes to infinity of $\sum_{n=1}^x (1/2)^n$, so basically it's the limit.

Int: So when you do the limit, can you write equals 1?

Todd: Yes. Well the limit would equal 1. But the sum would approach 1.

Andrew: Oh yeah hm oh yea that's actually a really illustrative example. For example, you have a [horizontal line]. Say you're here [at x] and you want to get here [at another x]. Um you can go half way and go half way again and go half way the remaining distance and you will ne-... just by this you would never actually exceed 1.

Int: Because there is always a halfway to go?

Andrew: Yeah. So, even though you are taking an infinite number of steps you never quite get to 1. But you never go past it.

Int: So how does this relate to the idea of series?

Andrew: Because basically this is like the series where you add all the half steps together. 1 over 2 to the n , which would tend to 1 based on this logic.

Int: So would you write equals 1, or no?

Andrew: You would write equals 1 because... (pause) now I have to explain [inaudible] equals 1, which ...

Int: What's causing you tension here?

Andrew: Yeah you can write that, I'm just trying to think of why. It never ... oh I know why. Because this technically equals limit (writes the sigma notation with a lim on the outside)

Int: Oh I see you're just basically replacing that infinity with a//

Andrew: //with a limit, that's just basically what it is, which does equal 1.

Int: Without the limit you were feeling uncomfortable?

Andrew: It's implied that that is the limit.

Int: So if you are writing that thing you are sorta implying that you mean//

Andrew: //the limit.

For Abby, Todd, and Andrew (and others), amending the notation was enough to overcome their hesitance with uncertain language of converge vs. equal. However, they did not actually resolve the problem. Students who appealed to the limiting process of putting the limit outside the sigma only transferred the problem so that they could talk about mathematics with which they were more comfortable. Their issues with whether an infinite series equals its convergent value or not were mediated by the use of limit, and not necessarily resolved. Todd's statement, "The limit would equal 1. But the sum would approach 1," indicates that he feels comfortable writing the equal sign when working with a limit problem. However, he was unable to commit to language, instead using the more ambiguous 'approach,' when talking about the sum without the limit symbol. Andrew viewed the problem in a similar way, and by stating "you are taking an infinite number of steps you never quite get to 1. But you never go past it," indicates that he is thinking in terms of limit as well (which he later inscribes on the whiteboard). However, without the limit symbol attached to the sigma itself, none of Abby, Todd, Andrew, nor many other students, were willing to commit to the (convergent) sum being "equal" to its value. As one student shares, "I wouldn't write [equals] unless I wrote limit. That might not be necessary, but I wouldn't do it without it."

Other students took their commitment to needing the limit symbol outside the sigma farther. Theo (below, and also described in Chapter V) viewed the notation as mathematically incorrect, unless written with a limit symbol.

- Theo: Say we have some number x that is less than 2. No matter what x is, this sum will get closer to 2 than x . So let's assume that the full sum from $n=0$ to infinity of $(1/2)^n$ is in fact less than or equal to $2-x$. And we're saying also [x is positive]. Then that would mean that you would never ever ever get past x , but if you go far enough you'll see that you get past x . And with very small values of x , it's hard to demonstrate it.
- Int: Ok and where do you see limit in that, even if you don't use the actual word limit?
- Theo: You've got an infinite thing and you can't have an infinite thing equal a finite thing without a limit. So you're limiting what you care about as it gets to 2. Because as n gets arbitrarily large, $(1/2)^n$ gets arbitrarily small. To the point where limit as n goes to infinity of $(1/2)^n$ is zero. Eventually we'll just be adding things that really don't matter anymore. As we showed with this, we're always going to get something slightly bigger than any $2-x$.
- Int: So you're saying the farther and farther you get in your sum, the x is getting smaller and smaller?
- Theo: Ok
- Int: So then my question is, is that thing actually equal to 2? Or does it just get closer and closer and closer? And let me say it differently. Is it like [$Y < \text{sum} < 2$] or [$Y < \text{sum} \leq 2$] (written on whiteboard)
- Theo: The second one.
- Int: Ok so how is that possible?
- Theo: Well I mean I did kind of write this wrong. I wrote the full thing wrong because I wrote it in my notation because I skip steps.
- Int: That's ok, I'm curious what you think you wrote wrong.

Theo: The proper form would really be [with limit notation outside sum]. So that there's actually a limit in there.

For Theo, the limit was applied in multiple ways. First, as a limit of n tending to infinity, with a limit symbol in front of the sigma. Then, as a limit of the difference of the value of the sum and 2, shrinking to zero. In clarifying the latter, Theo permitted a non-strict inequality, indicating that the sum could in fact equal 2, only if he was explicit about expressing the limit.

Further evidence that students see a difference between “equals” and “approaches” is exemplified by Tracy, who indicates that there is a significant difference between saying that the sum “approaches 2” vs. “equals 2.” To Tracy, who is representative of others, the best we can say is that the sum “approaches 2,” until the limit symbol is added. When it is, we can still say that the sum is approaching 2, but can with more confidence write the equal sign, since it is known that “limits can ‘equal.’”

Int: So that would say that it equals to 2?

Tracy: No? No cuz that still only says that it approaches 2.

Int: Here's a question, you had it but you erased it, that it equals 2. Right? But now you're sort of using the word approaches, and you erased the ‘equals 2.’ Is there a difference?

Tracy: Well not if you put ... if you put a limit in front of the series and you make this m and say that m approaches infinity, because that's just asking what it's approaching, which is 2.

Int: So would you write equals 2?

Tracy: So yes. We can write that.

Other students made less progress with resolving “equals” vs. “converges,” as they were unable to sort out their understanding of infinity and limit beyond referring to it as an ‘infinite process.’ As discussed in the next section, some students’ attempts at sorting out issues with “converges” vs. “equals” were thwarted by what Fischbein (2001) would refer to as conflicting space and time models.

VI.B.2. Vague limit language – time, motion, and space models of infinity

Another way that students worked on resolving ‘converge’ and ‘equal’ was through their use of vague limit language, which targeted their general ideas of infinity, rather than a particular sequence or series of terms. Fischbein (2001) discusses genuine difficulties inherent in reasoning with infinity and infinite processes, and the ways that these difficulties are essentially psychological. He posits that one of the most significant difficulties with which people struggle is in reconciling the space vs. time models, as they relate to infinite concepts. In thinking about infinity, we often mix aspects of space and time in ways that may not be comparable. We divide, measure, and estimate time, but can only do so by taking advantage of a space model. Space can be infinitely divided and then recomposed, but with time this proves difficult, as infinitely divided time, into non-dimensional instants, cannot be recomposed into a non-zero duration. Thus, space acts as a substitute to interpret these infinite “rest moments” that are so common in reasoning with, for example Zeno’s Paradoxes. If one decomposes the Paradoxes down into geometric series, she has thereby eliminated both time and motion; as neither motion nor time is infinitely divisible, there is no such thing as an infinity of instances that underlie a real interval of

time. Instead, if we decompose it with spatial terms, we eliminate measurement problems of re-composing the original time interval from an infinity of instants of rest.

Put differently, it is somewhat possible to consider a line segment composed of an infinity of real points, but not so with motion composed exclusively of an infinity of rest instants. Therefore, we appeal to the tacit space model of time, and use it to make predictions. It is embedded in the language of the mathematician, such that they do not feel a discrepancy in reasoning with time in spatial terms. That is, the mathematician does not “view a convergent series in a temporal light” (Monaghan 1991). Textbooks make similar assertions – as Stewart (2007) asserts, “What do we really mean when we say that the sum of the series in Example 2 is 3? Of course we can’t literally add an infinite number of terms, one by one” (page 690). However, students wrestling with difficult ideas related to infinity may not be at the point yet when they are able to make these associations.

Many students were unable to reconcile their thoughts about dividing time and infinite motion to come to a clear consensus on “converges” vs. “equals.” In fact, some students became so fixated on viewing the notion of infinity as an infinite process of continual motion and infinite dividing, temporally, that they were not able to conceive of a way that an infinite series could have a finite sum. They could not divorce the mathematical object from the (temporal) process one would have to carry out in order to determine it. Brittany and Malcolm’s teaching episodes demonstrate this difficulty. For them and others, ‘converge’ referred to the process of getting very close, but never being able to be “equal.”

Brittany: They don't - well, yea, if you go to like spend the rest of your life adding, you'll find ... I guess maybe not, you wouldn't get to it. But I think there's a way to figure out. I don't remember it. But like how ... that this means something. This is gonna eventually equal but not really equal ‘one number’ [air quotes]

Int: What do you mean equal but not really equal?

Brittany: Well because we can't add an infinite number of numbers. So we have to understand like in the limit that it's going toward one number (pause) and there's a way to figure out how it goes towards that number and why.

Int: But it won't equal that number?

Brittany: No because if you keep adding, it will just keep going.

Malcolm: We will say 0, s_1 , s_2 , s_3 , s_4 , ... and then the very small amount will reach, will get closer and closer without ever reaching it. The good comparison would be with the Greek runner. When we have here the start and here the end, and when we start we still have the one half the start of the whole distance. And eventually we will be here and eventually we will have one half like this, so $1/4$. But if we reason like that we will never reach the end. Because we will always have something that we always have to divide by two, so we would never reach one or the end. We will always ...

Int: Okay so what are you saying about that series? Are you saying that that thing equals 1, doesn't equal 1?

Malcolm: That this equals 1 because that is the limit.

Int: It equals one because that is the limit.

Malcolm: Yeah, it's not strictly the sum of all the infinite of all the terms from one to infinity, because if we do that we would actually have, we would actually spend an infinite amount of time adding up the terms, it's just that mathematically we know that at some point it will ... this equaling 1, it does not mean that it will eventually equal 1. It means that it goes closer and closer to 1 with out ever reaching it.

As Brittany indicates, the convergent infinite series and its sum are “equal but not really equal,” because she cannot conceive of a time at which the process of adding will be complete. Thus, she was unable to remove the temporal element from this process, and it prevented her from viewing ‘converge’ and ‘equal’ as equivalent language. As she indicated, “if you go to like spend the rest of your life adding, you'll find ... I guess maybe not, you wouldn't get to it.” Malcolm similarly could not conceive of the infinite series without the component that he would have to “spend an infinite amount of time adding up the terms.” Thus for him the limit equaling 1 “does not mean [the sum] will eventually equal 1.” Instead, it meant that the sum will approach and never reach 1, but only because there is always more to be added.

For Aiden and his “stick theorem,” which describes infinitely dividing a stick, and recomposing the stick from the infinity of pieces, the mathematics that permits one to talk about how an infinite series could in fact *equal* a specific value is “magical.” While he knew that the stick theorem makes sense ‘mathematically,’ he could not conceive of a process by which he could actually (literally) infinitely divide a stick and recompose it of its infinite parts. This leads Aiden to the conclusion that the only way that this is possible is by “magic.”

Aiden: It's kinda like it's an interesting part of that because in common sense you can never complete the stick but in math, in magical math you can do this.

VI.B.3. Mathematical reasoning vs. common sense

Other students were unable to reconcile their language use around ‘approach,’ ‘converge,’ and ‘equal,’ expressing their difficulties as mismatches in their intuition and what they know to be mathematically correct, rather than any limiting process in particular. For example, when considering the example of $0.\bar{9} = 1$, Tina was able to acknowledge that there is indeed a limit, and that the limit is 1. However she attributed all of this to abstract mathematical thinking that did not resonate with her own.

Tina: But I know mathematically that it is a limit that approaches 1. It gets to that point that eventually you just call it 1.

Int: So you said 'mathematically' - as if that's different than the way you're thinking about it?

Tina: Yes. In my head there's the mathematical sense and there's reality sense. Math is sometimes removed from reality. In the sense that math is sometimes removed from reality, which sometimes it is, like with infinities and other random ideas. And so mathematically I know that there's a limit as the number of 9's approaches infinity, it becomes so long in the 9-sense that this limit would be equal to 1.

Int: So you're saying with a limit it makes sense, but in \\\

Tina: \ Correct, but to just say this for me, does not automatically make me want to say that this is equal to 1. Because if there's never a 0.000...1... because of the fact that there's this repeating bar for you know the repeating 9's, you can never have a number that's gonna have this many zeroes, because the repeating 9's is infinite.

For Tim, the meaning of 'equals' had a more precise mathematical definition than 'converges,' and therefore the use of the two was not interchangeable. In his thinking, 'converge' was the more abstract concept.

Int: Wait, is [converge vs. equals] a different question, or is it the same question?
 Tim: I guess they're different questions but the result is the same. You know that this equals 1 ...
 Int: You *do* know that that equals 1?
 Tim: Yea, in other words it converges to 1 [inaudible] more precise.
 Int: So the words like 'equals' and 'converges,' do they mean the same thing?
 Tim: Um almost, yea.
 Int: Almost? What's not the same?
 Tim: Um cuz equal to me has more of a precise definition like $2=2$. Like there's nothing that can break that. Or $2+2=4$. And you can work with that. But converge would be more like it's getting to that point, and if you could ever add up all these numbers, it would equal 1. So it's getting closer to that number, but it's ... I mean it abstractly equals 1, but there's no real way to think about it equaling 1 because it's infinite.

All of these struggles with the language used to describe limits highlight the idea that students ascribe very specific meaning into particular words that mathematicians would see as synonymous and use interchangeably in lecture. Some students are able to reconcile their issues with language by recasting the problem on which they are working in terms of more familiar concepts – i.e. with “lim” outside the sigma symbol. However, even for those students who are able to reconcile their language struggles, the language is still viewed as distinct. Instruction targeted at eliciting students’ ideas about converge, approach, and equals will work toward building language bridges so that students’ experiences are better integrated and closer to the mathematical meaning intended with this language.

VI.C. “Terms tend to zero” – on the ambiguity of students’ language and how images help disambiguate what they are talking about

The second most salient way that students’ language difficulties play into their explanations concerns their discussion of ‘terms tending to zero.’ In this context, students’ images help illuminate their thinking/understanding, when their language alone is ambiguous or insufficiently precise. When students include various versions of ‘the limit of the terms implies that the sum converges’ in their explanations, one possible (reasonable) instructional reaction is to indicate that it is wrong, and counter with certain examples (most often the harmonic series) that contradict the students’ ideas. However, such an intervention may only be warranted if students were referencing the terms of the sequence being summed, and not some other sequence.

When students using a phrase such as ‘the terms go to zero’ in discussing the convergence of their sum, it is not always clear which limiting process students are referencing. In fact, by examining students’ SGR in the context of this study, it becomes clear that the students who use this argument often reference different things. However, considering their words alone, the students’ explanations may appear faulty or without merit. What will follow are three ways that students’ uses of variants of the phrase ‘the terms go to zero imply series convergence’ differ. This discussion includes an in-depth example of how one student (Mark) came to realize, by drawing in image to aid in his explanation, that his use of the phrase was faulty.

V.C.1. Three examples of ‘terms tend to zero’ reasoning

Example 1: Individual Terms get smaller and smaller. More than 75% of students argued, at some point during their teaching episode (either while defining or while providing an example), that the sequence of terms tending to zero was enough evidence that their sum would converge. While some of these students later revised their thinking, for most of them it was enough to cite the limit of the sequence of terms as adequate evidence of the convergence of the sum. While it is certainly true that the sequence of terms having a limit of zero is necessary for their sum to converge, it is of course not sufficient. Taking Adam’s thinking (below) as an example, it is unclear what limiting process he is referencing. However, his SGR and additional clarifying questions provide further insight.

- Int: Just so I’m clear, that’s an example of converging why?
Adam: Because it starts at some number and gradually goes down to zero. And like I said, for approximation you could neglect from this point on.
Int: Ah, so the thing that makes it converge is like if it gets so small that it gets to a point you can stop paying attention to it?
Adam: Yup. It gradually gets smaller.
Int: And what’s the “it?”
Adam: The numbers we’re adding up.

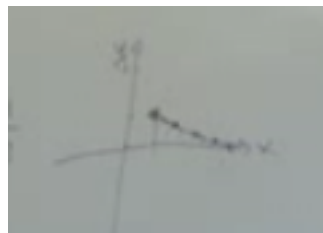


Figure VI.1. Adam’s SGR

Adam’s image (in Figure VI.1.) allowed him to clarify that he was in fact referring to the sequence of terms as having a limit of zero, as he interacted with and pointed to it during his explanation. Adam professed, however, that his drawing of that image was only to help clarify his explanation.

- Adam: In Calc II you don’t draw that many pictures. Because of the series and the rules for defining whether they’re converging or diverging.
Int: So there’s not really pictures for those rules?

A: No, well the logic is like this: if you see that the series is decreasing then eventually you'll get to a finite number. It's getting smaller.

Adam's statement "if the series is decreasing then eventually you'll get to a finite number," offers no indication of what he means by 'the series is decreasing.' Thus, even though he professed that he doesn't find images to be particularly useful, his use of the image was the only way to be sure which limiting process Adam was referencing. Other students like Abby and Aiden (below) are additionally representative of students whose 'tending to zero' reasoning focused on sequences of terms. Their explanations are more explicitly about individual terms in the sequence.

Abby: The way I think of it is each thing you are adding becomes smaller. So this guy can't possibly be, it's smaller than this guy, so it's not like you're adding this and this picture. You never see it sorta go above, sorta pass above at least the 1, right? You're never adding a half plus a half. That would be 2. You would always be adding something smaller than 2. That's how I think of it.

Aiden: If the limit of the sequence is zero, as the sequence goes, it's actually dying out. It actually becomes zero, so finally you're adding a bunch of infinite zeroes. So this turns out that this whole series, which is the sum of your sequence, is bounded by certain numbers ... like at the really end we're kinda adding useless terms because they will be going to zero or whatever.

While Abby used a geometric area model to represent her terms, her reasoning about sizes of terms tending to zero played out as smaller and smaller shapes adding on to a larger area. The focus of her reasoning was on the relative sizes of those shapes, where she concluded that because the shapes were getting smaller and smaller, the total area was finite. Aiden was also explicit about his reasoning about the sequence of terms tending to zero signaling that their sum would be finite, referring to the sequence of terms as 'dying out' and becoming 'useless' because they are so small.

Finally, Matt's image use (in Figure VI.2.) as a way of juxtaposing a 'converge' vs. 'diverge' scenario allowed him to pinpoint the features of his 'converge' graph with more specific language that would otherwise be ambiguous.

Int: Between those two examples the bouncing one and the one you have on the board now, what can you say about the sum? It looks like you drew the a_n 's?

Matt: The first one I drew would be divergent because the a_n approaches two different points. The other converges because as n goes to infinity a_n approaches 0.

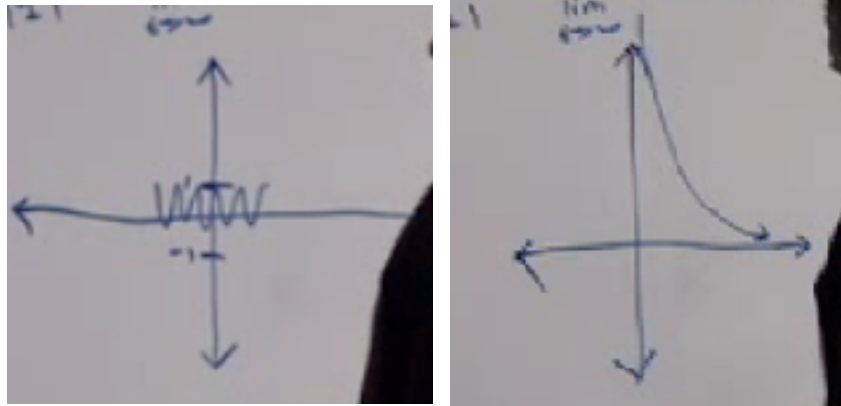


Figure VI.2. Matt's contrasting images

While his creation of the images and general descriptions were ambiguous, in interacting with his SGR, Matt was able to clarify that the sequence of terms tending to zero was sufficient for him to declare that the sum of those terms is finite.

Example 2: Partial Sums get smaller and smaller. Some students used 'terms tending to zero' reasoning with reference to the partial sums themselves, toward faulty conclusions that make little sense in context. For these students, when probed for a particular referent for their 'terms tend to zero' language, the limiting process that is invoked is the notion of partial sums. This was often the result of either being unsure of how to discuss their reasoning or lack of understanding of what the term 'partial sum' actually meant. While it was less common for students to use 'terms tend to zero' language in this way, students like Beverly and Tina characterize the ways that this reasoning played out.

Tina: (Responding to the interviewer's proposition of modeling the n^{th} partial sum for $\sum(1/2^n)$) It makes sense. But I don't like it. Because I don't know where the model came from. The '-1' is not so much that the impact gets smaller, it's that the whole numbers $(1/2^n)$ get smaller. The whole partial sum gets smaller. When you take all the added up pieces, each consecutive partial sum gets smaller and smaller and smaller

Beverly: 1 over something ... (long pause) it would start to approach zero because ...

Int: What's the "it?" I just want to make sure I understand.

Beverly: The sum of this number, of 1 over um $n+1$ would ... that's the way it's not going to approach infinity anymore [inaudible]. (long pause) So it would start at 1 and then go toward 0.

It is unclear if Tina is using the language of 'partial sum' to actually mean terms in the series, or whether she is referring to partial sums in a more mathematically correct sense, by looking at her language alone. For example, if Tina were just confusing the word 'partial sum' with 'term' then perhaps her reasoning is no different than that in Example 1. However, as Tina explained aloud, she interacted with her SGR, indicating that she truly was talking about partial sums by gesturing and writing over groups of terms at a time, each iteration with an additional term.

Beverly showed similar confusion, indicating that she understood that the first term of the series is 1, and it then ‘goes toward zero.’ Again, as with Tina, from the language alone it would be reasonable to assume that Beverly is referring to the terms of the series, as in Example 1. However, by her reference to the “sum of $1/(n+1)$ ” (an example she generated) and the way in which Beverly interacted with her SGR (a graph), it became clear that she was in fact referring to the partial sums. Without their SGR, it would be difficult to tease apart how these students’ reasoning differed from those in Example 1.

Example 3: Partial Sums as a Sequence; the differences of partial sums tend to zero. Different still was the reasoning that focused on the *differences* of partial sums. Multiple students essentially made arguments that the differences between partial sums tending to zero implied that the sum would converge, applying ‘terms tending to zero’ language to the sequence of *differences* between consecutive partial sums. Some students like Beverly went on to do this in a somewhat convoluted way, focusing both on the sequence of terms themselves and the differences between partial sums.

Beverly: Ok it's not (long pause) I guess it's not more than 1 because we're adding such small quantities, so even so early in the series like we're adding $1/16$, and then $1/32$, and the next term would be ... we're adding such small number so that we aren't really gaining any ground on the number line. So we've gained enough that we're moving a little bit. But it's kind of like exponentially, but not growth. It's not [inaudible] it's ... like an inverse. Like we're getting smaller and smaller and smaller and smaller. So we're taking less and less forward. So it's not going to be less than 1 because we're still gaining ground, we still are moving positively. But then it's not going to be more than 1 because we're not taking big enough steps. Like we're not adding big enough quantities to get us past 1 at any point. And that kind of relates back to this ratio (points at $1/2$), and then whatever exponent we take it to. So like $1/2$ to the 10, is going to be some really small number, $1 \text{ over } 2^{10}$. And that's going to be a really small number that we are still moving forward by that much, so it's still pushing us in a positive direction towards 1. Um but it is so small that we're never going to be pushed past 1 because ... because it's such a small number. We are moving in a positive direction but not by leaps and bounds

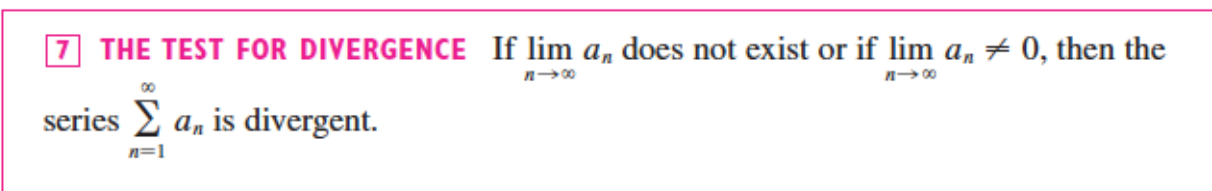
Her references to the sum being ‘pushed past 1’ and the sum moving ‘but not by leaps and bounds’ indicate that Beverly focused on both to the sizes of the terms in the sequence and also to their impact on the resulting sum. That is, Beverly’s focus at the end of this section of the teaching episode is on both the sizes of the added terms *and* the decreasing (tending to zero) impact of the addition of those consecutive terms (differences of partial sums).

Another student, Mark, similarly applied his ‘terms tend to zero’ language to the differences between partial sums. However, his application of this idea was unclear until he discussed it via his SGR. In interacting with and labeling his image, Mark both clarified that he was using ‘terms tend to zero’ language to refer to differences between partial sums, and also recognized errors in his explanation. A portion of his teaching episode, in more depth, follows.

VI.C.2. The story of Mark – non-normative ‘terms tend to zero’ reasoning.

One particular student’s use of the ‘sequence tends to zero’ line of reasoning serves to illustrate the ways that over-interpreting, based only on language, can provide a less-than-accurate sense of the student’s understanding of a concept. Mark’s use of language to explain convergence of infinite series sounds very similar to that of other students. However, when examining his language in conjunction with his SGR, the limiting processes that he emphasized become clear, and provide a referent of what Mark finds important to represent. In Mark’s discussion of the ‘sequence tending to zero,’ his SGR provided the opportunity to label axes and explain behavior with a physical referent for the language. Mark’s teaching episode demonstrates the ways that students’ images often aid in their explanations, when language alone may be insufficient.

Mark’s Teaching Episode. At the time of Mark’s teaching episode, he was days away from taking his Calculus II final exam at LRU. Upon completion of the course, Mark earned A+, with one of the highest scores awarded. From the start of the teaching episode, Mark was able to talk coherently about the differences between sequences and series, the meaning and importance of partial sums, what it meant for a series to converge, and what he knew about the ‘divergence test’ (see Figure VI.3.).



7 THE TEST FOR DIVERGENCE If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Figure VI.3. Stewart’s ‘Test for Divergence’

In Mark’s own words:

On sequences and series:

“A sequence is any, just a list of numbers. Putting it in layman’s terms, a series is just a sequence, but instead of just listing term after term, you just sum all of them together. So in terms of infinite series, you want to take the limit of the ... that series toward infinity.”

On partial sums:

“You have to determine whether or not a series converges or diverges. And basically one way to look ... this isn’t the whole picture, but in order to find out if an infinite series converges or diverges, you have to look at the sequence of partial sums. And determine whether or not that converges or diverges. And what a partial sum is, is basically you just take each term and then just add it up.”

On series convergence:

“You know what a limit is, so you know that if the limit of something as your terms tend to infinity approaches some number, which is finite, then you will have a convergent sequence. And if you’re able to determine that your sequence of partial sums is convergent, then you can determine that that series is also convergent.”

On the ‘divergence test’:

“And then so you can determine that if, by any pattern, your infinite series ... or your infinite sequence of terms doesn't tend to 0, then you're not going to have a convergent series.”

While there were still some gaps in Mark's ideas about infinite series, it was clear that he had a fairly robust understanding of these topics. However, through further interaction, it became apparent that Mark had some difficulties that were not illuminated by his choices of words, alone.

Mark provided an explanation during which he stated that the sequence tending to zero implied that the infinite series converged. As discussed above, at this point the standard instructional move would likely be to present Mark with examples such as the harmonic series, as a way of demonstrating the error in this logic. However, Mark's statement was ambiguous, and it was unclear what sequence Mark was referring to – the sequence of terms, sequence of partial sums, or something else. If either of the former, it would seem that this statement was at odds with Mark's earlier discussion. But if the latter, to what sequence was Mark referring? Thus, this prompted the request to explain further, draw, and provide clarification. The response illuminated that Mark was talking about neither the sequence of terms nor partial sums:

“Eventually the difference between the ... let's say your one millionth term and your one millionth and first term is so small that it's insignificant ... Ok let's take the bigger terms, s_{90} and s_{89} will be smaller than s_{88} and s_{89} . Which means the distance between your partial sums is getting smaller and smaller and smaller. It will eventually, as n heads toward infinity, equal zero.”

To demonstrate this, Mark drew a graph (Figure VI.4, left), and a second graph to demonstrate that “Technically your partial sums will always be increasing. In this [example]. Because you're only adding positive numbers.” (Figure VI.4, right)

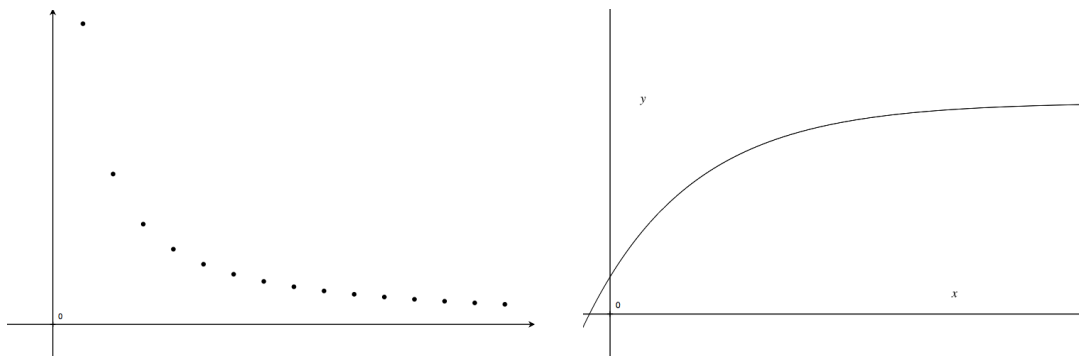


Figure VI.4. Mark's images

Mark's move to label the vertical axes (in Figure VI.4.) as ‘differences between s_n ’ makes clear that the sequence he is considering is essentially a reframed way of thinking about the contribution from one term to the next. Mark then went on to use his now clarified idea to explain why $\sum (6/5)^n$ would not converge. He reasoned that, “the difference between your consecutive partial sums is going to get greater and greater.”

With Mark's use of language alone, related to his notion that the 'sequence tending to zero implied that the infinite series converged,' one would infer that he was referencing the sequence of terms. However, in his move to label the image he drew, and further clarify with an example, it became clear that Mark was in fact looking at an entirely different sequence, with different implications. Though his words were similar to students with the belief that decreasing terms sum to a finite value, his understanding was very different, and incorporated partial sums.

While his line of reasoning has its own problems, one may argue that it would require a very different strategy for addressing than what would have been the more standard interpretation of a 'sequence tends to zero' line of reasoning. For example, Mark recognized the importance of partial sums and had a more robust idea of how a sequence of partial sums is significant in this context, beyond simply looking at the individual terms themselves. Mark himself recognized the problems with his approach at the end of his teaching episode.

- Int: Ok. This difference between partial sums thing is kinda cool. Did you sort of come up with that to help you, or where did you come up with that?
- Mark: I don't even... yeah I guess.
- Int: So you're saying ... so we'll want absolute value, of the difference between partial sums, is tending to zero? Is that what you're saying?
- Mark: Yea.
- Int: This thing? I just want to make sure I get it right. From one partial sum to the previous one, that their difference in size, tending to 0?
- Mark: Actually. (pause) Yea I kind of see a problem with that.
- Int: Oh do you? I haven't thought about it too much...
- Mark: No cuz I mean if you take the harmonic series then this... (pause)
- Int: Does that hold? What do you think?
- Mark: It's kind of hard to picture because you're taking infinite values.
- Int: Sure.
- Mark: I think this way of thinking is just another representation of your divergence test.

Mark's explanation, including the interaction with his image and the labeling of the axes, helped him and the interviewer reach consensus about the specific limiting process that he found important. It also made him realize the shortcomings of his explanation. By making use of an image, both Mark and the interviewer were able to engage in his non-normative ideas, whereas without it, the standard reactions to his 'sequence of terms' reasoning would have been the next teaching move.

As we can see both through the example of Mark, and with the more general discussion of the different ways that the 'terms go to zero' language is intended, students' use of language with limit-related content such as infinite series is widely varied in its intention, even if it sounds the same on the surface. Without Mark's labeled image, for example, it would have been incredibly difficult to tease apart the way he was using that language from the exact same phrases uttered by his peers, with entirely different meaning. That is, since students' language use with limit concepts is inconsistent and often idiosyncratic, without a visual referent, such statements are difficult to interpret, diagnose, and address with instruction. Taking language at

face value, therefore, may have lead to unfounded or mismatched instructional interventions, particularly in the case of Mark.

It became apparent in the differentiated uses of images, across experts and students, that experts used particular language meant to draw attention to the context in which that image is useful, and the conclusions that can be drawn or inferred from that image type. Students, on the other hand, used the same language in their explanations, but in different ways. It appears that students see many more entailments associated with the terms ‘converge,’ ‘approach,’ and ‘equal,’ than were intended when they heard them in lecture or a textbook. For those who resolved the difference by taking advantage of putting the limit symbol outside the sigma, the difficulties with the language were not fully resolved. For those students who used vague limit notions to talk about the differences between these three words, the terms were most often recast as temporal statements that require someone to spend infinite time carrying out an addition process. There, ‘converge’ and ‘equal’ were seen as starkly different – and students had difficulty conceiving of a way that an infinite series, framed as a process in time, could possibly ever be *equal* to a number. Knowing the various ways that students use this language differently from experts can help in both disambiguating what students really hear and infer when instructors use particular words in lecture, and what students mean when they say, for example, that converge means something different than equal. Additionally, we have seen that commonly used phrases like ‘the terms tend to zero’ can take on a variety of meanings, depending on which limiting process the student is referencing. For students who reference different limiting processes with this phrase, different teaching interventions would be warranted.

VII. CONNECTIONS

This chapter focuses on exploring the final of three aspects of competence identified in Chapter IV that distinguished student and expert understandings – Connections. There were remarkable differences in the ways that students connected their ideas via the different image types, compared with experts' usage of those same image types. Thus it is important to study the nature of those connections. Because (most) students use multiple images through the course of their explanation, it is instructive to look at the ways that they connect those images (since they often don't choose one or the other) and the instances in which they find the different image types to be useful over others (modes of use).

After discussing what is meant by 'connections' in more depth, this chapter will proceed in two parts. First, two contrasting teaching episodes from students Molly and Tina will demonstrate how the limiting processes highlighted in their SGR can facilitate meaningful connections (or not) within their understanding of infinite series. After the cases of Molly and Tina, the final analysis will detail some of the more common ways that students draw connections between limiting processes, from the larger data set. The presentation of both the cases *and* the broader patterns will also show how students' connections are made salient in their SGR.

Outline of Chapter VII -

- VII.A. What is meant by connections

- VII.B. Contrasting the Teaching Episodes of Tina and Molly

- VII.B.1. Molly teaching episode

- VII.B.2. Tina teaching episode

- VII.C. Connections in the broader data set

- VII.C.1. Connecting sequences of terms and partial sums

- VII.C.2. Connecting sequence of terms and sequence of partial sums

- VII.C.3. Connecting individual terms and compilation

- VII.C.4. Connecting terms as physical quantities and compilation

VII.A. What is meant by “connections?”

As discussed in the previous chapters, it is important to understand which limiting processes students find important when reasoning about infinite series. However, knowing which ones students draw on most frequently is insufficient to understand what sense students make of this topic. It is further necessary to uncover the ways that students coordinate and connect these limiting processes, while explaining. Studying the connections that students make allows insight into which limiting processes are prioritized and why, and which limiting processes are linked to one another in a student's attempt to produce a coherent, mathematical story that accounts for the convergence of infinite series of numbers. Understanding which resources are prioritized and connected (and which are not) helps us to understand what information students are taking from their classes, and what might need to be emphasized or corrected when things go awry.

One common example of connecting limiting processes (or not) in this larger data set is among students that connect sequences of terms to sequences of partial sums when explaining how to determine whether or not infinite series converge. That is, some students can flexibly transform sequences of terms into sequences of partial sums, thereby connecting their more general limit knowledge to a new context and finding mathematically relevant correspondences.

Other students may be able to talk about the meaning of partial sums, as well as discuss sequences of terms, but not connect the two in meaningful ways that would help them draw mathematical conclusions. Often, when students are able to connect their understandings of multiple limiting processes, it is because they are able to discuss their understanding via their SGR (self generated representations), drawing correspondences across the multiple images with which they are working.

Broadly, the literature on “multiple representations” discusses how important it is for students to grapple with multiple images when solving a problem. NCTM’s *Principles and Standards* (NCTM, 2000) stresses the importance of flexibly translating between representations, and choosing appropriate representations to solve particular problems, and the reform calculus movement (e.g. Ferrini-Mundy & Graham, 1994) places a similar emphasis on adaptability and use of multiple representations. However, a focus on students’ self-generated representations is less widespread (see Parnafes 2009), even more so when considering SGR provided for explaining a concept rather than solving a particular problem (e.g. Kindfield 1993). In this content domain, it is clear that students’ SGR can illuminate how students are connecting limiting processes in a very physical way, with less ambiguity than when focusing on language alone. Producing a physical referent with which both the interviewer and the student can interact through the course of the teaching episode offers the opportunity to see how these limiting processes that students find salient are linked, and affect one another. Thus, in terms of studying the connections that students are making within this difficult content, visual instantiations of their thinking provide the medium with which they can interact to demonstrate how the connections between those processes play out.

VII.B. Contrasting the Teaching Episodes of Tina and Molly

In order to demonstrate the richness of students’ images – for highlighting connections and showing where there are lacking connections – we turn now to the contrasting cases of students Molly and Tina, who each used their images to discuss the ways that they connected the limiting processes that they deemed important.

VII.B.1. Molly’s Teaching Episode. We first consider now the case of Molly. Molly engaged with eight separate images of infinite series of numbers over the course of her teaching episode. She generated them all spontaneously, and used each to build additional images that brought the topic of infinite series, with which she admittedly struggled, closer to the topic of sequences of numbers, about which she felt confident. In her teaching episode, Molly was both able to communicate her ideas about infinite series *and* demonstrate the growth and evolution of her ideas. She connected the many images she drew, drawing correspondences between important features, and providing visual referents for difficult concepts such as “the sum of the series.” In the following narrative, Molly’s process of constructing her eight distinct images for infinite series is played out, with particular emphasis on the connections across the images, and how she used the images to help move her explanation forward.

The context – Molly’s teaching episode. Molly, a second year engineering student at LRU, participated in a teaching episode during finals week of the semester in which she took second semester calculus (and earned a course grade of B). Rather than participate for monetary reward, Molly asked for help with a few problems related to her upcoming final exam as compensation for her time. The following is taken from a detailed narrative of the first 24

minutes of her teaching episode, during which Molly transitioned from an insecure memory of some of the features of infinite series of numbers, to the construction of several images, one built on another, and ultimately to a coherent mathematical story in which she connected all of the images. The teaching episode is broken down according to the limiting processes that Molly chose to focus on with her various images. Through following her explanations, it will become clear that Molly's images were tied to her understanding of particular limiting processes, which then served as the *connective thread* by which she built her explanation of infinite series.

While Molly did not arrive at an entirely reliable method by which she could determine series convergence, she did draw connections across the many representations that she produced, end with a reasonable idea about what it would mean for a series to converge, and find meaning in images that were previously mystifying to her. Because writing and gesture are critically important in this story, Molly's quotes are presented side-by-side with descriptions of her simultaneous drawing and/or gesturing. In these transcripts, Molly is denoted "M," the Interviewer "I." All of Molly's images were reproduced using software, rather than including photographs of her drawn images, due to poor image-capture quality.

Figure VII.1. provides an overview of the ensuing narrative. As it indicates, Molly's use of images demonstrates her emphasis on considering individual terms, terms as physical quantities, sequences of terms, partial sums as collections of terms, sequences of partial sums, and finally compilation. In the closing minutes of her teaching episode, Molly used this string of images, each of which represent her attention to a different limiting process, to tell a coherent story of what it means for a series of numbers to converge.

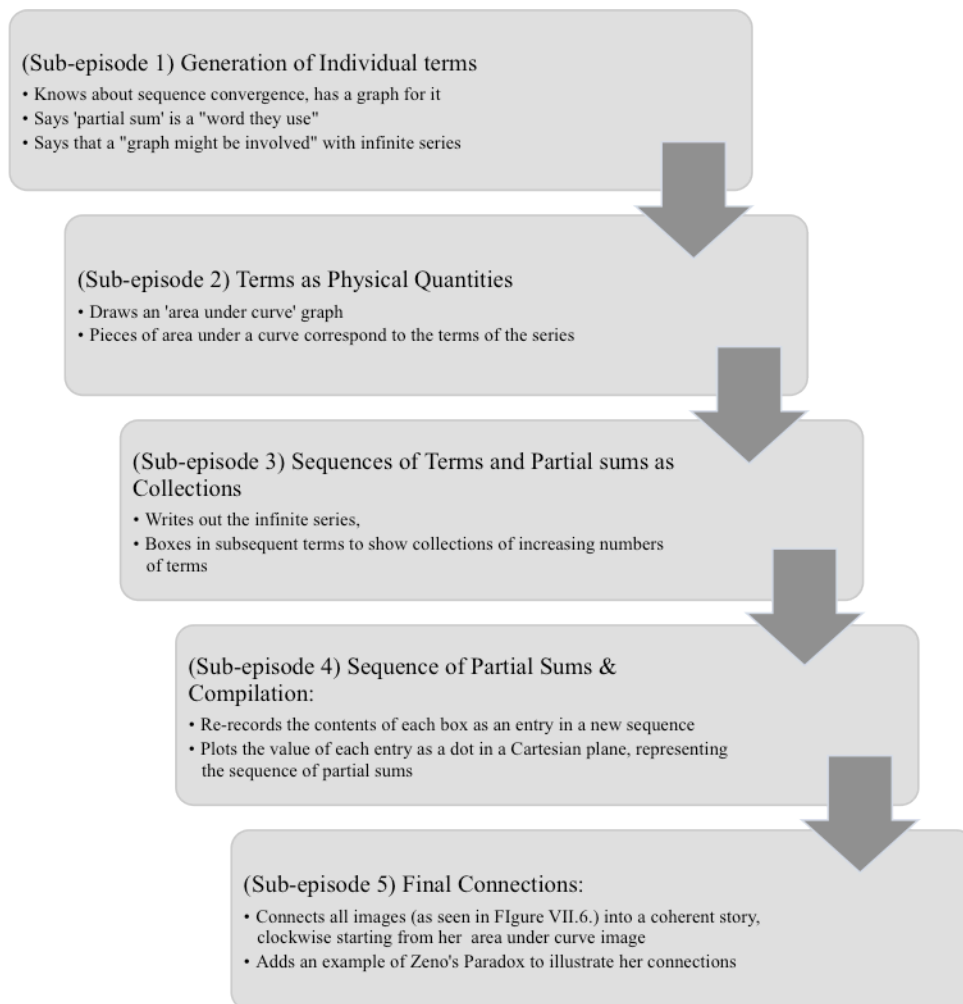


Figure VII.1. Overview of Molly's teaching episode

Sub-episode 1: Generation of Individual Terms. Molly's teaching episode began with the standard prompt to 'explain infinite series as if to a peer who had missed class.' Her initial response included a confident mention of sequences, and much apprehension around the topic of series. She viewed sequences as lists of numbers, "separated by commas," and recorded the example of **1,2,3,4,5** on the whiteboard, but had more difficulty with series, stating: "What is a series? A series is just like... I don't know, not adding those but... (long pause) Like, I don't know." One thing that she *did* remember, however, was that the term 'partial sum' is somehow affiliated with the topic. It became clear in the early stages of her teaching episode that Molly knew neither what this term meant, nor how it related to the topic of infinite series. For her, 'partial sum' was simply "the word *they* use," and nothing more.

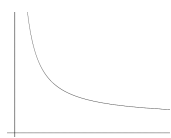
Very early, though, Molly's inclination to produce some sort of visual representation of her thinking was evident. She spent a good deal of time staring at the white board, trying to conceptualize what sort of image might be instructive, since she knew "a graph is involved with both of them." Finding no success, Molly returned to her thinking about sequences, because she "[understood] sequences better than series." There was reason to believe her, at this point, because she produced an acceptable (informal) definition of what it would mean for a sequence to converge.

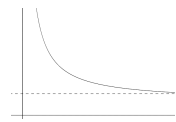
<i>Transcript</i>	<i>Gesture/Drawing</i>
	(previously written on board): 1,2,3,4,5
M: Yea, and I know like with a sequence ... I think I understand sequences better than series. Because with a sequence, you have numbers like that ...	Points at 1,2,3,4,5
M: if they converge or diverge or whatever, they approach some number.	
M: The individual numbers themselves	Points at each of 1,2,3,4,5 individually
M: will go to, like 5 or something like that.	Points far to the right of the 1,2,3,4,5 list on the board
M: Or if you have 1, and 1 and 1/2, and 1 and 3/4 or something like that, and then they'd approach 2 .	Gestures air quotes when saying “approach 2.”

At this point, it was clear that Molly had some operational way of thinking about sequence convergence, which for her was linked to ideas of limit. That is, she viewed sequence convergence as a specific limiting process by which the values that are listed are said to “approach” a fixed value. While the use of “approach” could be taken in a variety of ways, her reference to the sequence 1, $1\frac{1}{2}$, $1\frac{3}{4}$, provided a concrete referent to indicate that her view of “approach” included monotonically getting “closer” to some fixed value, but not “reaching” that value.

At that time, it was also clear that Molly considered sequences and series to be distinct mathematical objects. While she was not quite sure how to explain what is meant by infinite series (of numbers), she had confidence in referring to sequences, and remarked that each *could be* represented graphically, though these graphs would be distinct. She also had ready examples for sequence convergence, and was able to articulate what this meant to her, but had no analogy for infinite series.

After she provided the above description, the interviewer challenged Molly to produce some sort of picture for her description of a sequence converging, since she suggested earlier that producing a graph was possible.

M: So like that I guess.	Draws smooth curve: 
--------------------------	--

M: If there is some number	Draws horizontal asymptote below curve: 
M: that the values go to.	Traces hand across asymptote from right to left.

In producing such a graph, Molly demonstrated first that she viewed the value to which a sequence converges as a fixed number, and additionally could identify this number with the asymptote on the graph. To Molly, the asymptote represented more than just a feature of a graph that a continuous curve “can’t touch.” She recognized that the asymptote stands for a fixed value that the terms of the sequence, here depicted by her continuous curve, will “approach.” Molly’s use of a continuous curve to represent a sequence, when a set of isolated points would be more mathematically appropriate, was not unexpected (see Stergiou & Patronis, 2002). While incorrect, at this stage Molly’s image was appropriate for the point she was trying to express about the meaning of the asymptote on her graph.

Drawing her version of what it would look like for a sequence to converge prompted Molly to consider what the “graph of a series” might look like. However, this consideration was accompanied by very different body language, confidence, and willingness to record her thoughts on the whiteboard. While putting the cap on the whiteboard marker, playing with it in her hands, facing away from the interviewer, scratching her head, playing with her ponytail, and becoming much quieter, Molly considered:

“And then I think with a series... it might be like the area? It might have something to do with area now that I'm thinking about it. Like I don't know. Cuz [another student] had those things, and I remember seeing them lecture. With the graphs like underneath and above [the curve].”

Her hesitance here, and admission that while she remembered seeing something related to areas above and below a curve, she was not sure what it means or how it connects, are consistent with her apprehensive body language. She even asserted, about her experience in class:

“I was like, this doesn't make any sense. Just like the whole thing, it was very confusing. I think I was just overwhelmed. And [the professor] confused me even more.”

The interviewer, in an attempt to keep her talking, suggested that what her friend had in his notes “was just some stuff that was in the book. So whatever works for you...” Molly thus abandoned the idea of producing a graph, temporarily, and decided to focus on something she *did* remember – notation for infinite series. Shifting the flow of the interview allowed her to regain some control over the topics discussed, and make headway in addressing the original prompt of explaining infinite series to someone who had ‘missed class.’

M: Let's see. The notation is like that,	Writes: $\sum_{n=1}^{\infty}$
M: And you have 1 over n or something. So that's the harmonic series. That is the series I always think of for some reason, because it's 1 over n so it's really easy to remember.	Writes: $\sum_{n=1}^{\infty} \frac{1}{n}$
I: What does that mean, that series?	
M: It means like if you write out the terms of the series, like 1 over 1 plus 1 over 2 plus 1 over 3 ... I: So there's adding?	Writes while talking: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$
M: Yea. So it is like a sum	Runs hand along the bottom of the written out sum
M: of all these different points. Maybe, and ... I don't know.	Points to individual terms in the written out sum

This move also allowed Molly to express some knowledge that she had of infinite series that she was not sure how to represent graphically. While she expressed that she often thinks with graphs in math class, “I normally think of [series] with the notation, because the graph, when I get to series, really confuses me.” This provided another opportunity for Molly to contrast her understandings of sequences and series as distinct, because she follows:

- I: So you think of a graph when you think of sequences, and it makes sense?
M: Yea.
I: Why does it make sense?
M: Because you are just looking at the x values or the y values and you're not thinking about it.

This also provided an opportunity for the interviewer to ask Molly to be very specific about what particular graphs or drawings from class were confusing.

Molly's response is entirely focused on an 'area under curve' image (see Figure VII.2.). Molly recalled seeing such a graph in the text, in her friend's notes, and again in lecture, when it

was used to demonstrate the conclusions of the integral test. Molly's class was one of those observed by the interviewer to examine professors' uses of images and graphs, and this particular graph was only used by her professor in one lesson, to discuss the conclusions of the integral test.

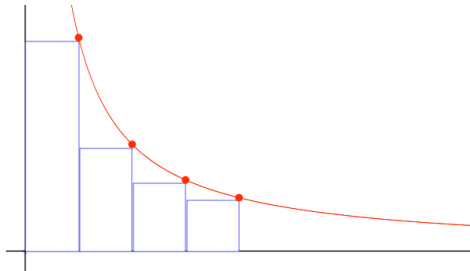


Figure VII.2. Molly's 'area under curve' image

Molly's rationale for the professor's use of the graph was not related to the integral test, nor any particular convergence test. Rather, she connected such a graph back to first semester calculus, as a method for approximating area under a curve:

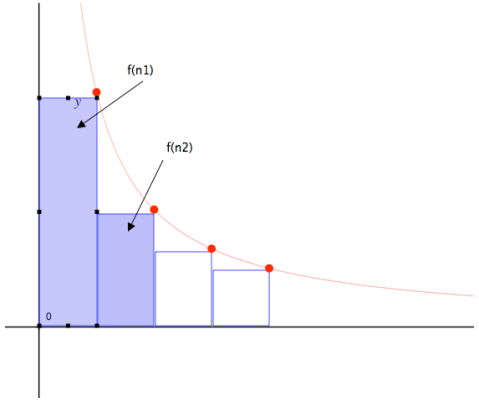
"... he's trying to show that ... maybe he's showing that like the **area under the graph**, with the right end point, you know what I mean? And then you have midpoint and trapezoidal rules. All those ways to have the area. Maybe he's trying to show the area under the curve or something? Like it adds up to a certain value? Like **approaches** a value?"

While a reasonable, informal way of describing convergence of improper integrals, Molly's description did not relate to infinite series in the way she earlier described them. In the first five minutes of the teaching episode, Molly remarked on all of the following:

- Ideas about sequence convergence and an asymptote representing the value to which a sequence converges,
- Ideas about convergence and divergence of improper integrals,
- Ideas about the notation of infinite series and what it means to translate "sigma notation" into an expanded version of a sum,
- The vocabulary of "partial sums,"
- Right-, left-endpoint, trapezoidal, and midpoint rules for approximating definite integral, and
- A graph resembling right-endpoint integral approximation that was produced during lecture and reproduced by Molly's peers.

In an attempt to refocus the student on infinite series, the interviewer asked the question "But how would that relate to series?" With no other discussion or prompting, Molly's entire demeanor then shifted dramatically as she exclaimed, "Be – ohhhh haha ha!" Something shifted for Molly in this literal "aha!" moment. What follows is the second distinct segment of Molly's teaching episode, during which she used the "area graph" to create and connect several other images that she constructed along the way.

Sub-episode 2: Terms as Physical Quantities. Upon exclaiming “ohhhh ha ha ha!,” Molly smiled, laughed, stood up straighter, and started talking faster. She also began recording additional images and notation on the whiteboard at a much faster rate than previously. Her first revelation came as she began to attribute meaning to the features of the area graph that she recalled from lecture.

<p>M: Because each of these are like a term. Their area is a term, right? And then you ... maybe? (not directed at interviewer – this “maybe?” is introspective)</p>	<p>Points at the first individual area, then at the first number in:</p> $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$
<p>I: Well, convince me. [Like] I've never seen this before.</p> <p>M: Ok. Because I vaguely remember writing it down in my notes. And there was this one part where [the professor] was like ‘This is like term $f(n_1)$, and this is like term $f(n_2)$.’ And then you like ...</p>	<p>Draws the following, writing the labels as she says them aloud</p> 
<p>M: So with each addition of <i>this little area</i>, this would be like <i>this one</i>.</p>	<p>Shades in first area with marker. Points at first area as she says “this little area”, and then circles the $1/1$ term as she says “this one.”</p>
<p>M: And <i>this one</i> would be like <i>this one</i>.</p>	<p>Points to second area as she says the first “this one,” and circles the $(1/2)$ term, as she says the second “this one.”</p>
<p>I: The area?</p> <p>M: Yea. So that term represents, like I guess the area. I don't know if it is right endpoint or left endpoint or whatever.</p>	<p>Motions as if drawing lines from area to term, showing correspondence.</p>

At this point during the teaching episode, the whiteboard was arranged as in Figure VII.3.

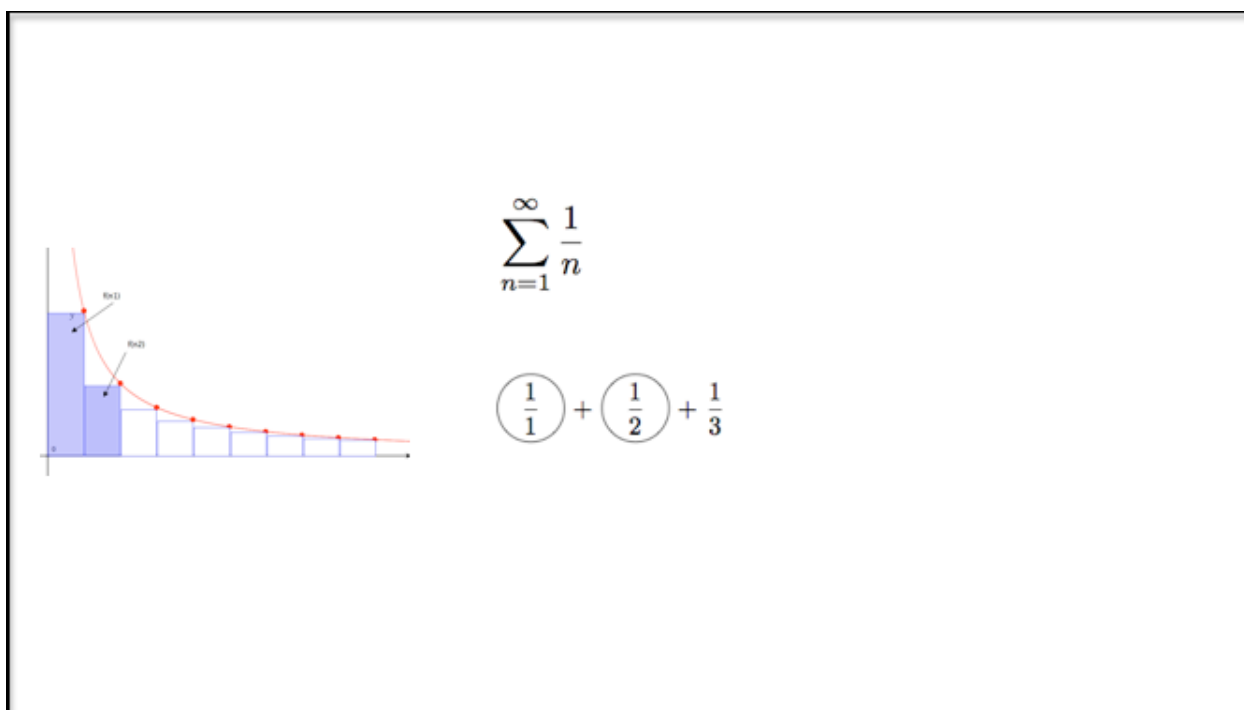


Figure VII.3. Molly's whiteboard, early in her teaching episode

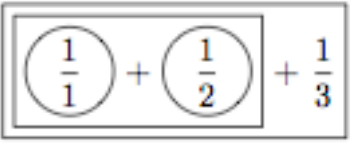
In this exchange, Molly connected her representation of the terms of the series to the graph that was previously confusing and had little meaning. Her correspondence of the numerical terms as values of areas in the area graph show that she started to connect not only the areas to the written out terms, but also the values of those terms as derived from the heights of the rectangles, drawn along the curve ($f(n_1), f(n_2)$, etc...). While at this point, Molly had only begun to connect the images term-wise, her next move was to connect them in such a way that clearly related to her notion that series have something to do with summing up terms.

M: And then the thing with series I guess, when it converges or diverges, is when you *add this*, right? Add that together, that would be like 1 and 1/2.

Draws a box around first two terms as she says "add this"

$$\left(\frac{1}{1} + \frac{1}{2} \right) + \frac{1}{3}$$

Then writes "1+1/2" below it, as she says it aloud.

M: And <i>this</i> would be like something else. The harmonic series diverges, but...	Draws another box as she says “this” 
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When asked for clarification about the circles and boxes, Molly continued:

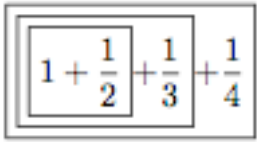
M: So the circles are this area.	Points from circled terms to the areas under curve that they represent, as she identifies each
M: And the boxes are showing	Points at the boxed terms
M: the addition of partial sums I guess?	Air quotes for “addition of partial sums”
M: So like you have ... <i>This</i> is the sum of <i>this</i> .	Points at first circled term and first area, as she says each “this”
M: Like this is the value of the area.	Motions over, as if coloring in the first area
M: And <i>this</i> is <i>this</i> one.	Points at second circled term and second area, as she says each “this”
M: So if you <i>add them together</i> , I drew a box around it.	Traces boxed terms as she says “add them together”
I: In terms of the picture, what is that [boxed terms]? M: It's like this whole area right here.	Traces with pen the total area enclosed by the first two rectangles
M: Because ... yea because that would be like, this would be a partial sum	Shading the total area of first two rectangles
M: of the whole area.	Points to the written out sum of numbers, then back and forth between drawings

In this short exchange, Molly did a lot of connecting back and forth, between drawings of the areas under the curve and her amended series expansion, which she enhanced to include circled terms and boxed sets of terms. The use of circling and boxing terms and sets of terms show how she connected the two drawings, shifting her attention to two distinct mathematical

objects – the collection of circled individual terms, and the sets of boxed terms, which for her meant the difference between the sequence and the series. We see evidence of Molly beginning to build into her scheme for infinite series the idea of partial sums, though informally – a concept that she, at the outset, could only identify as related vocabulary. The circling and boxing scheme turned out to be a stepping-stone for Molly, which supported her next move of more carefully transforming her example into a sequence of partial sums. She then applied her more robust knowledge of sequences and sequence convergence.

Sub-episode 3: Sequence of Terms *and* Partial Sums as Collections of Terms.

Molly's move to generate a sequence from the expanded series she had written before unfolded as follows:

<p>M: Like if you were drawing terms of partial sums or something, it would be $1 + \frac{1}{2}$ and then $1 + \frac{1}{2} + \frac{1}{3}$ and then $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$...</p>	<p>Writes on the board as she says aloud:</p> $1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
<p>I: What is that thing that you're writing down now? M: That is what the boxes represent.</p>	<p>Points back at the boxed sets of terms, vaguely.</p>
<p>I: So you're saying each box is a different thing in your list? M: Um hm. Like this [first] box ... Cuz I look at it like you have a whole bunch of terms 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.</p>	<p>Rewrites the sum of terms above her current work on the board:</p> $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$
<p>M: And then how I at least think about it... I don't know if this is right or not ... But I think of <i>like a box</i> in my mind, and you like <i>add another term</i> in the box, and <i>add another term</i> to that box.</p>	<p>Boxes in first two terms, then first three, then all four, each time she says "add another term"</p> 
<p>I: Oh, so you're listing out the contents of each box? M: Yea I: So is that list ... I see where it comes from but I don't see what it is?</p>	

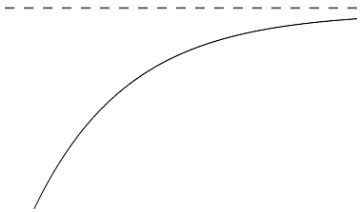
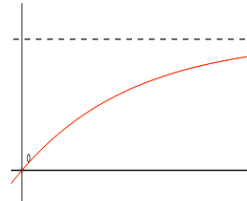
M: Like, would it be all the different partial sums? (Introspectively – not looking at the interviewer for approval – facing the whiteboard). Like <i>this</i> is the first partial sum, and <i>this</i> is the second partial sum?	Underlines each entry in the list as she says it: $1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
I: Hm it makes sense, cuz that's like the contents of each box. But why have that list? M: Because then I can think of it like a sequence?!	Smiles

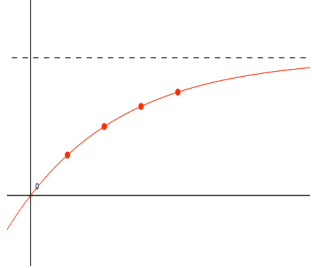
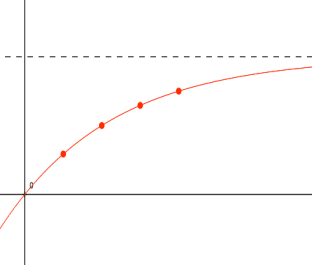
Although she ended this segment with a slight questioning tone (to which the interviewer did not respond with affirmation), Molly used her circling and boxing scheme to give rise to the notion of partial sums. She recorded them as a sequence, with which she showed both confidence and skill earlier in the teaching episode. Thus, she ‘invented’ the notion of ‘sequence of partial sums’ while creating correspondences between terms from the familiar notation and areas in the graph she recognized from lecture. While Molly used the term ‘partial sum’ at earlier points in the teaching episode, this exchange marks the first time that she used it appropriately, or tried to give it meaning.

This point in the interview was the first moment that Molly even slightly appealed to authority. It is possibly because the questions were somewhat like those that a teacher would use to probe students’ understanding, or possibly because Molly knew that the interviewer is not *really* a second semester calculus student. However, it is unclear if that was even what she was doing – she faced the whiteboard the entire time, and looked inquisitively at what she had written. The interviewer’s intention was to ask Molly why such a list was necessary, if we already have the boxes. Thus, Molly’s move to “think of it like a sequence” is taken here as her way of expressing the need to transform her intermediate, boxing image into something more familiar, that has properties with which she was familiar and could already work. While her work in this sub-episode is recorded in a more symbolic/numeric way, with superimposed features like circles and boxes, Molly’s next move was to relate these to yet another graphical representation, distinct from the areas under the curve.

Sub-episode 4: Sequences of Partial Sums and Compilation. Through the transition period when Molly moved on to consider the limiting processes of sequences of partial sums and compilation, her SGR begin to look different. Rather than areas under curves and grouped up terms representing partial sums, Molly drew a plot of partial sums, as a means of plotting the values in the sequence of partial sums, much like in the initial stages of her teaching episode, during which she ‘graphed’ an example sequence.

M: Then it's like <i>this</i> is some area, and <i>this</i> is some area, and <i>this</i> is some area,	Each time she mentions an area, she points to the individual terms in her list of partial sums
M: but it's gonna be like	Draws a smooth curve, increasing at a decreasing rate (see below, in next row)

<p>M: <i>approaching some value.</i></p>	<p>Draws dotted line [asymptote] as she says “approaching some value”</p> 
<p>I: So what is that picture? That's weird. M: [laughing] It's a graph of “ln”! Haha I don't know! I: No no, Just explain it. Cuz it looks like sort of the opposite of the first one, right?</p>	<p>Draws a vertical axis onto her picture, then erases it. Then redraws:</p> 
<p>M: I just kinda drew it like this because you're increasing your values.</p>	<p>Pointing at the sequence of terms she's written, as partial sums, from left to right.</p>
<p>M: So like this is gonna be a greater number than this number.</p>	<p>Comparing the first and second terms in her sequence by pointing at them.</p>
<p>M: And so I guess like this would be like the <i>first point</i> here.</p>	<p>Circles first term in sequence of partial sums, then draws a point on the curve, below it (see cell below)</p>

<p>M: And then <i>this</i> would be like the second point, and a <i>third point</i>.</p>	<p>Points at the next terms in her sequence, and then draws points on the curve to correspond with them, one at a time:</p> 
<p>M: I guess I tried to give it a <i>sequence</i>? So like you have all those different things.</p>	<p>Points at individual terms in sequence of partial sums</p> $1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
<p>M: And this would be the sequence <i>of the partial sums</i>.</p>	<p>Points at newest graph while saying “of the partial sums”</p> 
<p>M: And then that is approaching <i>some value</i>,</p>	<p>Points at the asymptote while saying “some value.”</p>
<p>M: like if it would converge it would just have some limit here.</p>	<p>Makes air quotes for “converge.” Points to asymptote when says “limit here.”</p>
<p>M: If that makes sense.</p>	<p>Turns to interviewer, while leaving marker pointed at graph, signaling the end of her explanation.</p>

Thus, in this newest move, Molly created another image to coordinate her ideas about how the terms in her sequence of partial sums could be represented visually, providing a referent (the asymptote) for what it would mean for this sequence to converge. While she had forgotten that her example was of the harmonic series, which she “knows diverges,” (as will be evident in the next phase of her teaching episode), Molly accomplished a lot in a relatively short time. At that stage, her collection of images were all linked, and limiting processes of each are consistently represented in multiple ways:

Terms of the series as (1) areas of rectangles, (2) numbers in a series expansion, and (3) circled to indicate that the collection of circles constitutes the collection of terms, and

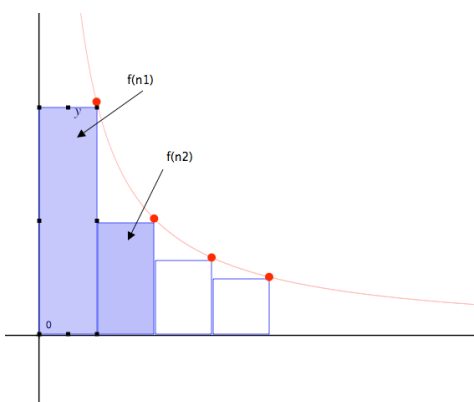
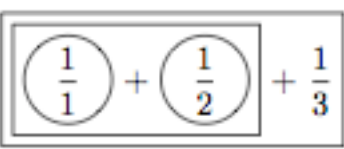
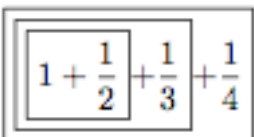
Terms in a sequence of partial sums, as (1) boxed collections of terms, (2) numerically as a sequence, and (3) graphically as points on a curve representing ‘the series.’

At this point, the interviewer was confident that Molly had connected her various representations satisfactorily, but wished for more detail around what it might mean or look like for an infinite series to converge. Molly responded to this prompt by connecting her collection of images in another way:

I: You've just told me what a series is. And I think you've showed me a couple different ways to graph it. Right? Is that right? But you said ‘converge’ and I don't know what it means yet for a series to converge.	
M: A series to converge means the area, I guess, under the graph, or however. Some kind of area.	Points vaguely to her first area graph , as a whole
M: I just think of it like ... the sum, so like all those terms added together,	Points at the first boxed in sum
M: if you add another term to [the box] like <i>this</i> ,	Points at individual terms in sequence of partial sums
M: <i>this</i> will approach some number.	Motions across the length of the sequence of partial sums as she says “some number”
M: So you have like the sum ...	Air quotes for “sum”
M: Well like this graph. This is points of your different partial sums, right?	Traces smooth curve, point by point
M: And if they approach some number, I don't know, some number <i>N</i> ,	Points at asymptote and labels it “ <i>N</i> ”
M: then you will ... it converges because it is	Retraces the curve as it travels along

approaching that value.	the asymptote
<p>I: So what does that value mean?</p> <p>M: That value would be like ... I don't know. Like the answer to what it is, like the sum of the series.</p>	Throws up hands as if to mean "the answer!"

By the end of this phase, Molly had constructed six separate images that represent infinite series, to her, each with their own affordances and potential to connect to her other images to tell the story of what it might mean or look like when a series converges. These are found in Figure VII.4., in the order of constructing them.

(1)		Area under curves
(2)	$\sum_{n=1}^{\infty} \frac{1}{n} \quad \frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$	Sigma notation and series expansion
(3)		Collections of circled and boxed terms
(4)		Boxed partial sums

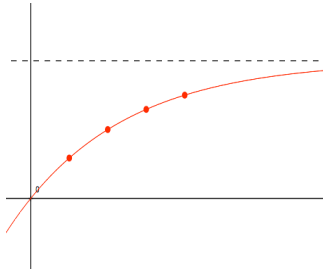
(5)	$1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$	Sequence of partial sums
(6)		Plot of partial sums

Figure VII.4. Molly's collection of images

These six images were arranged as in Figure VII.5. Molly made clear how she coordinated the mathematical processes in her various images (what is meant by terms? What is meant by partial sums? How are each represented, and distinct?). She also provided a physical referent for what it would mean for a series to converge (the horizontal asymptote on the graph of partial sums). At this point in the teaching episode (not described here), Molly proceeded to discuss her understanding of Zeno's Paradox in depth, using a walk-to-the-wall metaphor. During her use of this example, she created another image and used an example of sigma notation to illustrate her thinking. Following that segment of the teaching episode, Molly reiterated all of the connections that she had drawn throughout her narrative.

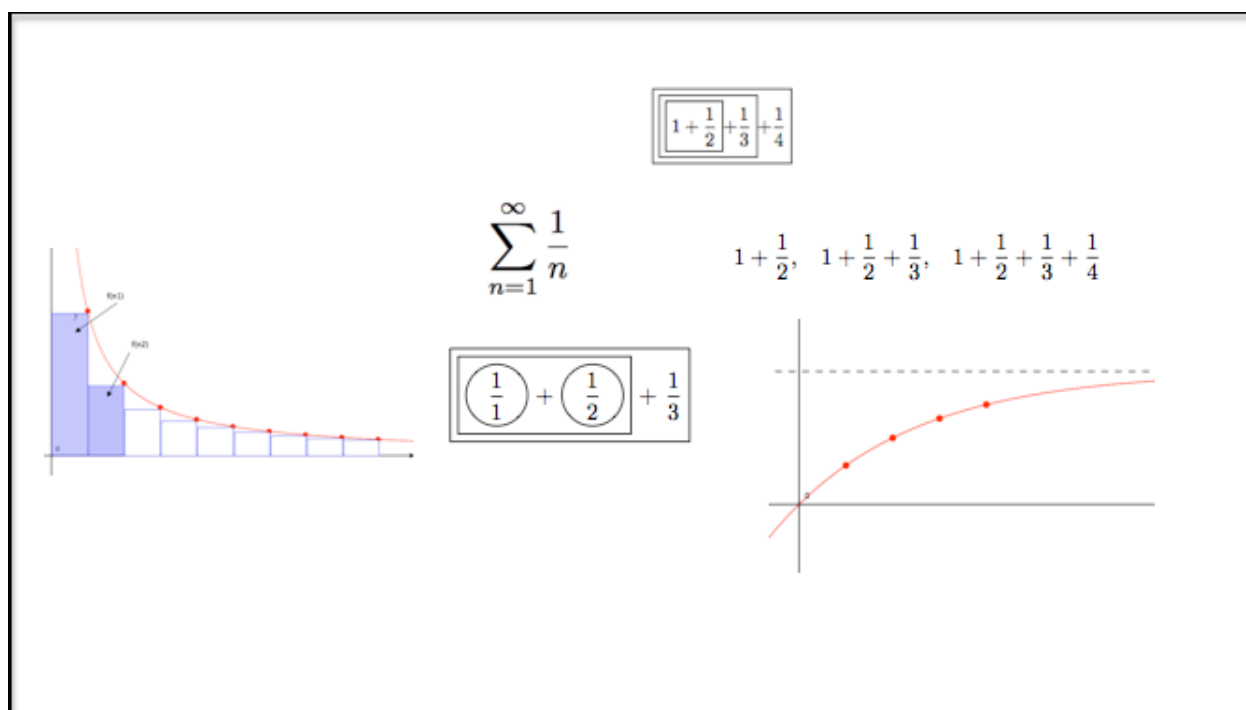


Figure VII.5. Molly's whiteboard, mid-teaching episode

Sub-episode 5: A final connecting phase. In one final attempt to encourage Molly to express the meaning of infinite series convergence, the interviewer responded, “So then say one more time what it means for a *series* to converge. Like why is it different than a sequence? Or is it different than a sequence? It may or may not be.” Molly’s final re-expression of the work that she had done to this point again connected all of the images that were still on the board, as well as the walk-to-the-wall metaphor, into a coherent narrative of what it means for a series of numbers to converge.

First, she described the relation that, if a series converges, it can be represented by some fixed value, which may have some units attached to it. Knowing that the expanded out sum of terms, if convergent, could be equivalently represented by a single value is not something that most student participants were able to grasp and/or express in their teaching episodes.

<p>M: I think a sequence is just like points. So it is just like a normal limit. I think it's just really easy to think about it like that. Cuz a lot of times if you ... I mean I'm not very good at taking limits//</p> <p>I: //That's ok, I'm not going to ask you to compute anything.</p>	
<p>M: Yea but when you find if a series ... or I mean if a sequence converges or diverges, one of the ways you do it is you take the limit of it and you like see what value it approaches. Or if it approaches a value. And so that's just like the way I've been thinking about them.</p>	<p>Hands waving back and forth in the air around “take the limit of it” and “see what value it approaches”</p>
<p>M: With series, they're more of some kind of unit. Like <i>distance</i>, or <i>area</i>, or something like that</p>	<p>Bounces body up and down as she says “distance” and “area”</p>
<p>M: So it's more of a compilation of values.</p> <p>I: Ok I think I understand what you’re saying.</p> <p>M: So there’s a number that’s that compilation of values. I think that makes sense.</p>	<p>Side to side hand motions around “compilation” – as if ‘putting together’</p>

With the layout of the whiteboard as below, in Figure VII.6., Molly acknowledged that the harmonic series does not converge (though she cannot explain why), but wished to proceed to talk through her story one final time. With agreement that the discussion could proceed as if the example she had chosen did in fact converge, she used her collection of images to tell a well-connected mathematical story, highlighting the connections between images.

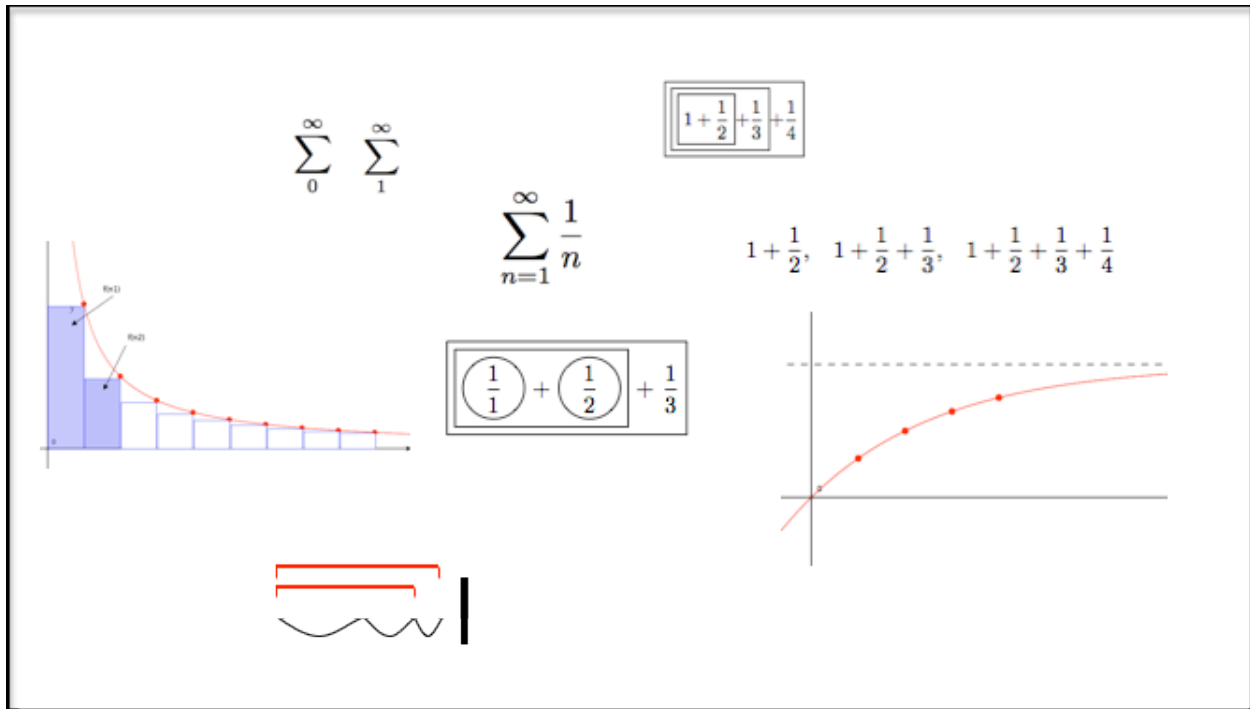


Figure VII.6. Molly's whiteboard, at the end of her teaching episode

M: Ok, I guess basically like the points [on the smooth curve] are just different numbers of [areas].	Points first at the plot of partial sums , then at individual areas, from left to right, in the area graph
M: The first point is the first two boxes, and	Points at plot of partial sums , then puts first two fingers over first two areas below area graph
M: the second point is the first three boxes, and the areas of that.	Points at plot of partial sums , then puts first three fingers over first three areas below area graph
M: Because the areas are getting really really small in these boxes ...	Pointing at areas as they get smaller on the right side of area graph
M: Like the total ... the value on the end of that term.	Points at sequence of partial sums
M: Like say you have like 1 over something ... well something really small.	Runs over and writes something on to the end of the sequence [illegible], then runs back to area picture, pointing at a small area at the end
M: Then that would be less of a difference. I don't want to say it will be like a <i>negligible</i> term, but it would be pretty pretty darn small.	Air quotes for 'negligible'

<p>M: And so I guess it would be like ... I don't want to say ... I know that these are ways to approximate the area underneath that curve, right?</p> <p>I: Sure, I think that was a [Calc I] topic. Right endpoints or something? I think so.</p> <p>M: Something like that. But I don't really think of it like integrals. I don't know. I don't know how to explain it if I would think of it like an integral.</p>	Crinkles up face, looking at the collection of images she's drawn
<p>I: Think of what like an integral?</p> <p>M: Like, trying to add in the factor that this is an approximation as an integral. Because I feel like that just kind of confuses me.</p>	
<p>M: I just like thinking if it (pause) as a sum of little boxes. Does that answer your question?</p>	Pointing at individual areas , left to right
<p>I: I think so. You said the first two [areas] - the area is like the first point [on the partial sum graph]?</p> <p>M: Um hm, yeah. Because that [curve] is like ...</p>	
<p>M: These [areas]</p>	Points at areas under curve
<p>M: represent each number.</p>	Points at circled values that she boxed in, in the series expansion
<p>M: And if you're just adding them like the partial sums, like with the boxes here,</p>	Points to the boxed sets of numbers
<p>M: and making those [into] points,</p>	Points at the sequence of partial sums she's written out
<p>M: then you would just add up those [areas].</p>	Returns to area graph , points at individual areas under curve

Thus, in her final pass through the collection of images drawn on the whiteboard, Molly pointed again to all of the relationships between images, from right to left, and highlighted the related features of each one; she coordinated terms of the series, terms in the sequence of partial sums, areas of rectangles, points on the graph of partial sums, and the total area and asymptote on partial sum graph as the value to which the sum would be converging. Spanning a total of eight spontaneously generated images, Molly's explanation not only communicated to the interviewer her understanding of both what infinite series are and what it means for them to converge. It also made apparent that her understanding was evolving as she actively connected the various images, and translated features of one to another. Molly created intermediate images like the boxed and circled terms in order to enable herself to represent series as sequences, with

which she was clearly more comfortable from the outset. That is, Molly began the interview with (admittedly) very little idea of what it might mean for an infinite series to converge, specifying only that she knew it was “something about adding,” and that “partial sums is a word *they* use,” and recognizing an area under curve graph as familiar from her textbook and friend’s notes. But, by the end of only 24 minutes of explaining, with her SGR drawn to make sense of and explain the ideas behind infinite series convergence, Molly came to a much more robust understanding of the meaning of the concept. While it was not without flaws, such as the misconception that shrinking areas guaranteed convergence of area under the curve, it was internally consistent and connected, and she was able to create references from image to image.

VII.B.2. Tina’s Teaching Episode.

Tina’s SGR also highlight the various limiting processes discussed during her teaching episode. However, the connections she was able to make did not permit her to construct a coherent mathematical story about the convergence of infinite series. While Molly was able to communicate about how her mathematical ideas were connected, and to represent the various limiting processes as they related to one another, Tina struggled to represent her understandings, demonstrating less sophisticated connections between limiting processes.

The context – Tina’s teaching episode. Tina, a third-year Real Analysis student and sciences major at SHC, participated in a teaching episode in Fall 2011. After showing some aversion and hesitation to the topic of infinite series at first (“a series is just a collection of numbers that is ... divided? Added?”), Tina eventually settled on the idea that a series and sequence are distinct, but neither are so bad, as they simply refer to collections of numbers:

“...the difference between series and sequence is that sequence is just a collection of numbers, but series is the summation of the collection of numbers ... You’re adding the sequence to create the series. Does that make sense? You have the sequence of numbers that you’re adding together. And those numbers that you’re adding together collectively give you a series.”

Tina drew several images to explain what she understood about series convergence. Tina first used a number line image to communicate the idea that smaller and smaller progressions on such a number line meant that eventually a series *could* converge. In her words:

“Whenever you’re counting up infinite series it’s not always going to give you an infinity. Because summation of numbers could eventually approach ... 10 ... if you have small enough numbers that you’re adding ... each number [that you’re adding] is getting smaller and smaller and smaller so ... it’s either going to converge, or approach a number, or it’s going to diverge, or go away from a number. As a summation.”

Tina was not able to use her number line image to convey the meaning of a *diverging* infinite series. Moreover, her notions of *convergence* demonstrated that she viewed series through the coordination of **sequences of terms** and the idea of **compilation**, with no referent for what the *sum* or final state might be.

Tina then represented a **sequence of terms**, now as **physical quantities**, in a different way – as a geometric progression of smaller and smaller ‘pieces of pie,’ whereby she drew a full

circle, a semicircle, and then smaller and smaller pieces of circles (see earlier chapters). While these, for Tina, represented terms in some sequence, her notion of **compilation** lead her to explain convergence, through this image, as a total amount of pie:

“Here’s a whole pie. Here’s half a pie. Here’s a third of a pie. And a fourth of a pie ... and you know, a fifth. The pies are getting smaller and smaller and smaller ... so you’re adding smaller and smaller numbers ... It means that eventually the amount of pie slices that you can add becomes so small it almost doesn’t seem like you’re adding anything at all.”

Tina was slightly more confident with this image, and she made some claims about the meaning of convergence in this context, stating, “The perception of this number of overall pie has essentially stopped growing.” That is, by representing the **terms as physical quantities**, Tina concluded, “So it seems like the series... or the summation of all these pieces of pie ... has approached a number. Like, I don’t know what the number would be, but say it’s 3 and 1/4 pieces of pie.” The final state of *convergence* was therefore represented by a value, in this case a physical referent of some amount of pie.

Tina’s attention to **sequence of terms** and **compilation** were slightly vague in these two early images. Further, as these were both versions of images that she had seen in lecture, it was unclear whether Tina was choosing to attempt to recreate them and justify them with some ad hoc description, or whether she was attempting to use them to convey something that had specific meaning to her. However, Tina’s next move was to create an entirely different image, in response to the standard interview prompt:

$$\text{Explain the meaning of } \sum_{n=0}^{\infty} a_n = 4$$

Tina’s choice to draw a new image was unprompted by the interviewer, and entirely self-guided, with few interviewer interjections. Her response to this prompt is detailed below (summarized in Figure VII.7.), followed by a close analysis of how her creation of and work with her SGR highlights how she connects the different limiting processes to come to *her* understanding of infinite series convergence.

(Sub-episode 1) Generation of Individual terms

- Writes out several terms of the series, with subscripts
- Includes a "last term" with a subscript of infinity
- Says that most of the terms themselves become very small



(Sub-episode 2) Compilation and Value of the Sum

- Draws a graph with the 'sequence of terms' on the vertical, and the 'value of the sum' on the horizontal
- Includes no reference to partial sums, nor the advancing values of n
- Connects the " a_0 " term on the vertical with the "4" on the horizontal, to indicate convergence of the infinite series

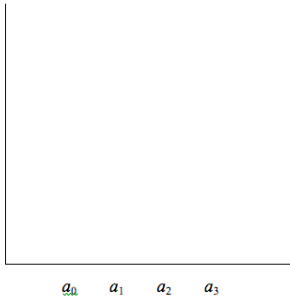
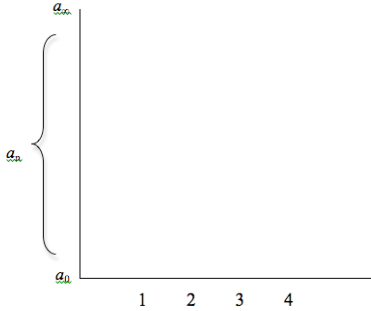
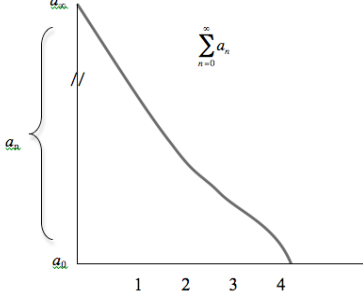
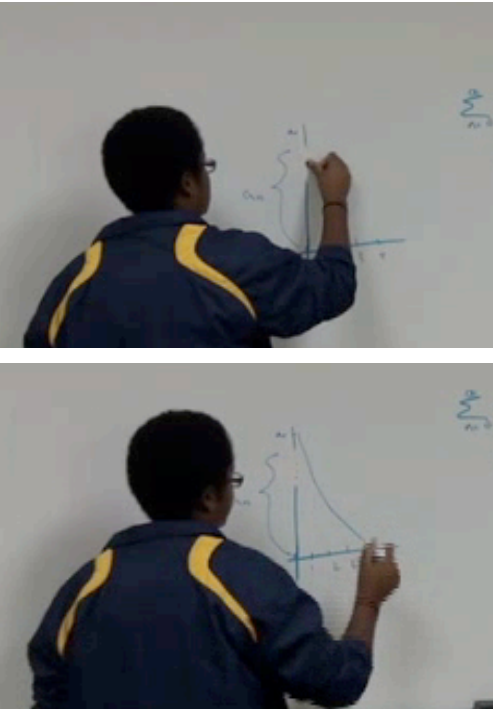
Figure VII.7. Overview of Tina's teaching episode

Sub-episode 1: Generation of Individual Terms. Tina began the interaction by attempting to represent a_n as a list. Recognizing that she did not know the function $f(n)$ which would produce a specific model for a_n , she proceeded to expand it out with subscripts, as

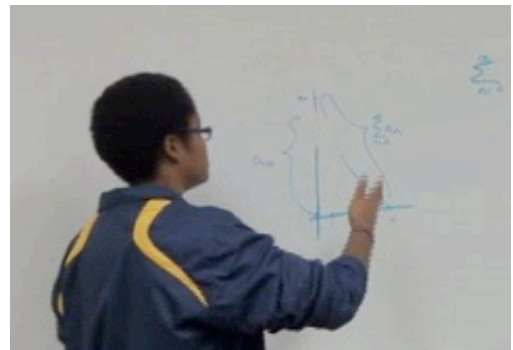
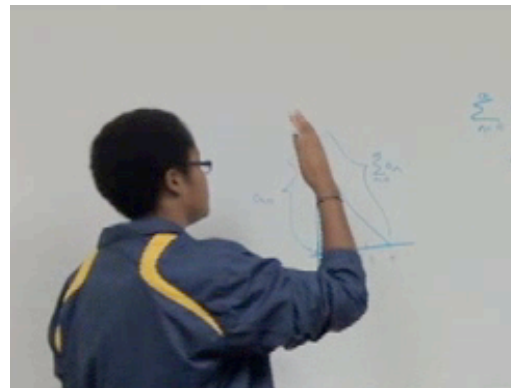
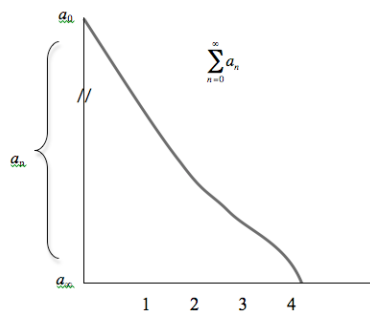
$$a_1 + a_2 + a_3 + a_4 + \dots + a_\infty.$$

Her choice to include an a_∞ term, thereby indicating that there is a 'last term,' lead Tina to conclude that converge means "taking the bunch of pieces, bunch of parts of this series, the list of numbers, and adding all of these pieces up, they're all going to equal 4." On the terms of the series, Tina remarked, "You know in this case they become, I'm assuming anyways that they become so small that they are equal to 4 ... the pieces become so ... most of the time they get really small."

Sub-episode 2: Compilation and Value of the Sum. In an attempt to demonstrate how she coordinated the **sequence of terms** a_n with the notion that she was putting all terms together to obtain "4," Tina did the following:

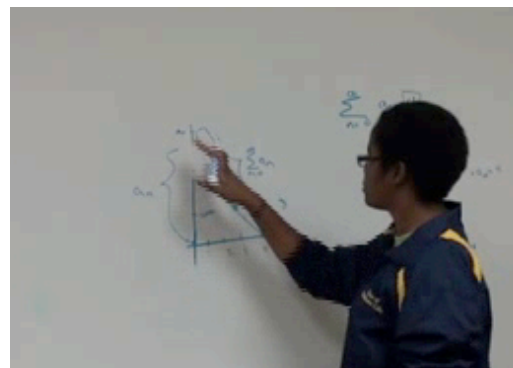
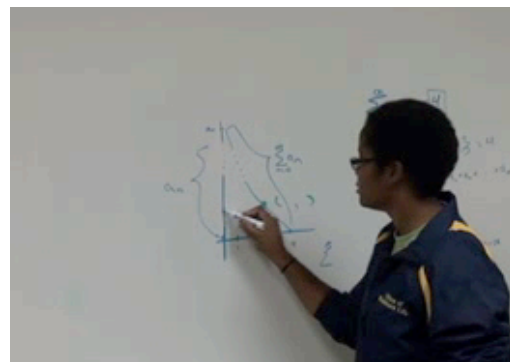
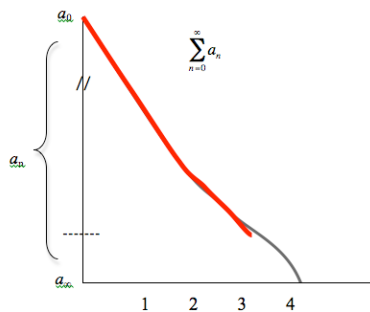
Transcript	Image	Gesture/Drawing
<p>T: Like in some sort of graphing. Ok I'm gonna go back to when I said here is a_0, a_1, a_2, dot dot dot ... wait this needs to be on the other side.</p>		<p>Lists the terms themselves on the horizontal axis</p>
<p>T: And here are all your parts of a_n. And you can start out where a_0 is down here, and then a_∞ is up here, and then it approaches 4 ...</p>		<p>Lists the individual terms on the vertical axis, and the integers (through 4) on the horizontal axis</p>
<p>T: Cuz the way to draw this graphically, you would want ... there's a break in the graph I can't draw. It would start and go down.</p>		

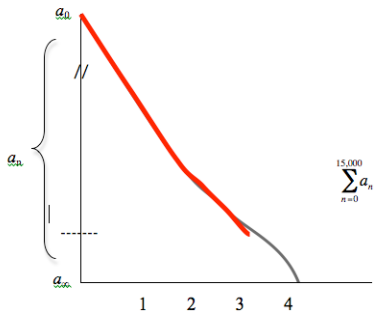
T: And then this curve is essentially the summation of a_n from zero to infinity. So like that term itself, the curve if you wanted to draw it like that, is the actual summation.



I: Like this point ... what is the meaning of it?

T: This right here would be the summation from ... I'm just gonna pick an arbitrary ... let's say this is hmmm ... a_{15000} .



<p>T: This curve right here – the red curve- would be the summation of a_n from 15,000 to ... no that's 0 to 15,000. So this would no longer be an infinite series. So then this coordinate itself would be $a_{15,000}$ to 3.</p>		
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In her resultant image, seen in Figure VII.8., Tina demonstrated how she connected the **sequence of terms**, as represented on the vertical axis of her graph, with the values of the sum, represented on the horizontal axis. Further, the curve that she drew was meant to represent the **compilation** of these values as the sequence of terms continues, over time. While originally orienting the sequence of terms from first to last, vertically, Tina’s move to reverse that seems to result from her desire to “end” the series on the horizontal at “4.”

What is also apparent is that Tina did not incorporate other limiting processes into her scheme for explaining convergence – there was no focus on anything more than generic compilation, no reference to partial sums in any way, and of the biggest consequence, no mechanism for how an infinite series would converge. Rather, her entire scheme for uniting the limiting processes of **sequence of terms** and **compilation** hinged on the notion that somehow, when adding “eventually small terms,” the terms would become sufficiently small that they would sum to a finite quantity. For Tina, the location on the horizontal axis was the only referent for the value of the series itself, and the process of arriving there only included a vague notion of decreasing terms.

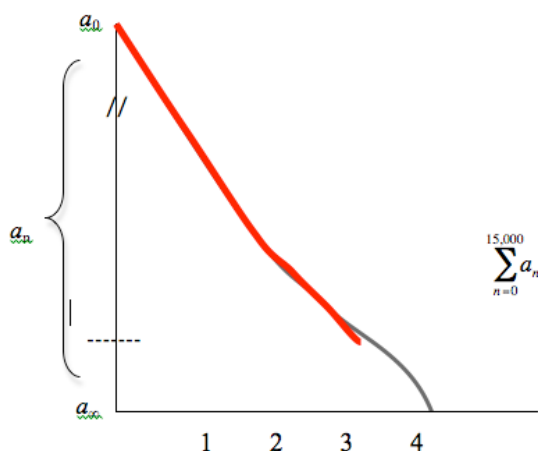


Figure VII.8. Tina’s infinite series image

Tina's choice to label the vertical axis as corresponding to the sequence of terms themselves, combined with her decision to horizontally represent values that the sum 'takes on' en route to its 'final state,' showed that she was attending both to the limiting processes representing the sequence of terms and the ideas of compilation. However, in attempting to connect the terms of the series with the value of the convergent series, Tina's thinking (as evidenced by her original image) neglected to connect to intermediate concepts, such as sequence of partial sums. These intermediate concepts would have given her some means for discussing convergence. Further, as evidenced by her image, Tina had not connected the two limiting processes to which she is attending in an appropriate way, so that their coordination on the image itself was even sensible. When Tina interacted with the image, it became clear that she was treating the vertical axis as the 'compilation' of *values through a_n* , and not as the values themselves (which was how she labeled the image). However, divorced from her explanation, this would not be easily interpretable.

VII.C. Connections in the broader data set

The contrasting cases of Molly and Tina highlight the ways that different students' SGR can illuminate not simply which limiting processes are prioritized, but also the ways in which students are connecting them to explain and draw conclusions. It is now instructive to look at more general patterns in the broader set of data, beyond exemplar students such as Molly and Tina. As discussed in Chapter V, the broad set of data was analyzed according to the limiting processes that were foregrounded in students' explanations and reasoning. Using this analysis as a starting point, the most common associations of limiting processes were identified, in order to pinpoint broader patterns in the connections that students were making. The results of this analysis follow, with exemplar students found in previous chapters.

VII.C.1. Connecting individual terms and compilation

Often, students' first connection was to link the individual terms in a series with the general idea of compilation. This indicates that students *were* able to correctly recognize a few things: how to expand a series written with sigma notation, how to incorporate the increasing values of n into the model a_n provided in the sigma notation, that looking at the sequence of terms is an important first step in determining series convergence, and that the sigma notation indicates that the terms should be added. However, for many students, this was the only connection that they were able to make. That is, for these students who did not go on to make more connections between the limiting processes they associated with infinite series, there was no way to apply limit properties that were related to the sum itself. Their only description of how a sum could converge, as a result of drawing this connection, was that they 'add up to something,' citing the general notion of compilation of 'things.' As a result, when these students declared the value of any convergent sum, these values were either guesses (based on the relative size of the terms) or remembered results from lecture or other examples. There was also no mechanism for distinguishing features of convergent and divergent series, except by making a judgment call based on the relative sizes of the terms being added. Imagery associated with connecting these limiting processes was vague, and students appeared less committed to it than others, who made more sophisticated connections.

VII.C.2. Connecting terms as physical quantities and compilation

Beyond simply connecting terms to general ideas of compilation, some students also treated those terms as if they were physical quantities, and included this ‘terms as physical quantities’ reasoning into their connection. That is, for students who only connected terms to general ideas of compilation, there was often no way of distinguishing between convergent or divergent series except by judging relative sizes of terms. With the added component of also connecting terms as physical quantities, these students often had more reliable ways of making their judgment call about series convergence, by relating term size to something (a shape, food, item, etc...) that they could visualize and reason with. While this was not always correct, it was at least more systematic than less connected reasoning.

For example, Tina’s “pie” reasoning (see Figure VII.9.), through which she drew out pieces of pie as a way to imagine term size, allowed her to examine the sizes of terms through a medium with which she was more familiar. In this way, she not only connected terms to the general idea of them compiling, but also based her argument about whether they would compile into a finite amount or not on the size of perceived, accumulated pie. In Tina’s case, where she used her pie argument with the example of the harmonic series, her conclusion was faulty. However, it was clear that the added connection with terms as physical quantities gave her something on which to base a convergence argument, even though it was incorrect.

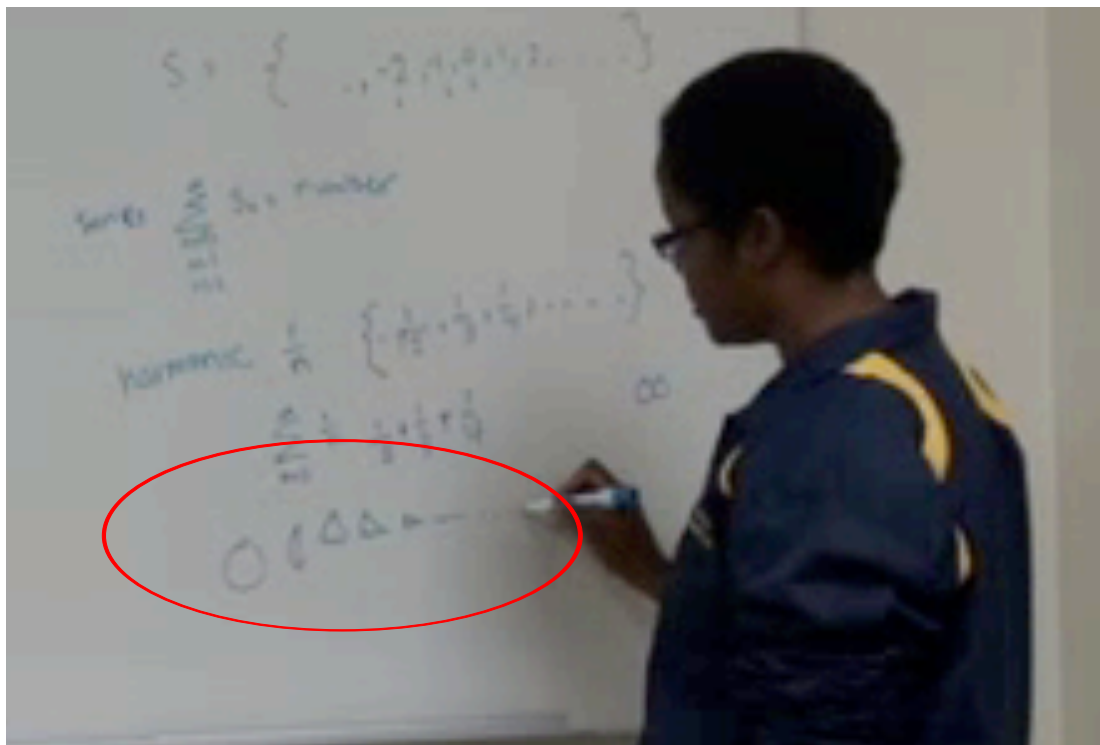


Figure VII.9. Tina’s terms as physical quantities

VII.C.3. Connecting sequence of terms and grouped partial sums

A very common connection that students frequently exploited during their teaching episodes is that of using sequences of terms to illustrate partial sums, by grouping and listing as groups. These students often expanded an example infinite series (often the harmonic series) and proceeded to circle or box groups of terms (like Molly), or write out first several partial sums

separately (not as a sequence). As with Tracy (in Figure VII.10.), this strategy most often accompanied a ‘Zeno’s Paradox’ or ‘walk to the wall’ example, where the results of the first few partial sums were easy to compute. By making the connection between the limiting processes of sequence of terms and groups of partial sums, students were able to explain the behavior of the series itself approaching a value. However, these students did not frame their results of the grouped partial sums as a sequence, of which they could take a limit.

Connecting the sequence of terms and grouped partial sums is a first productive step in uncovering a mechanism by which an infinite series of numbers could actually converge. Rather than applying limit principles to the sequence of terms alone, connecting those ideas to sums of groups of them, each with an additional term, potentially shows that students recognize the relationship between limit of terms and limit of partial sums, in that they look to the results of the grouped terms, relative to the previous result(s).

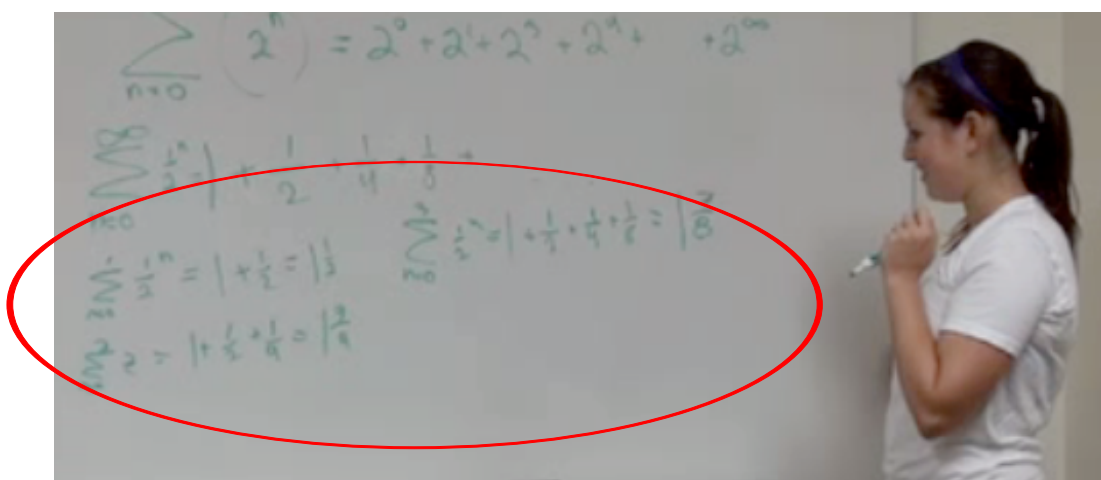


Figure VII.10. Tracy’s grouped partial sums

VII.C.4. Connecting sequence of terms and sequence of partial sums

Some students were able to connect the sequence of terms to groups of partial sums by turning them into *sequences* of partial sums, as in the mathematical definition of series convergence. These students were among the only ones who were able to provide some description of a mechanism by which a series could converge, beyond simply examining the sequence of terms and deciding if they ‘get small enough, fast enough.’ For students who connected these two limiting processes, the conclusions have the potential to be slightly more systematic than those that result from connecting sequence of terms and grouped partial sums. For example, even though connecting terms and grouped partial sums is a good first step toward applying limit principles to a partial sums argument, there is still a layer of ‘small enough, fast enough’ reasoning that requires determining some criteria, and applying them individually to the results of each grouped partial sum, toward making a convergence decision. Connecting sequences of terms and sequences of partial sums, on the other hand, poses limit as the mechanism by which convergence is decided – a non-subjective measure that can be consistently computed. Interestingly, though, of all of the students who connected sequences of terms with sequences of partial sums, the predominant way of reasoning with this connection was via a graph like Andrew’s, in Figure VII.11. That is, students did not produce a means for actually modeling the sequence of partial sums, and finding the limit of that, nor explain how this connection played out in the variety of convergence tests that were brought up in the explanation.

Rather, students produced images of what a convergent sequence of partial sums (most often for monotonic sequences with only positive terms) would look like on a graph.

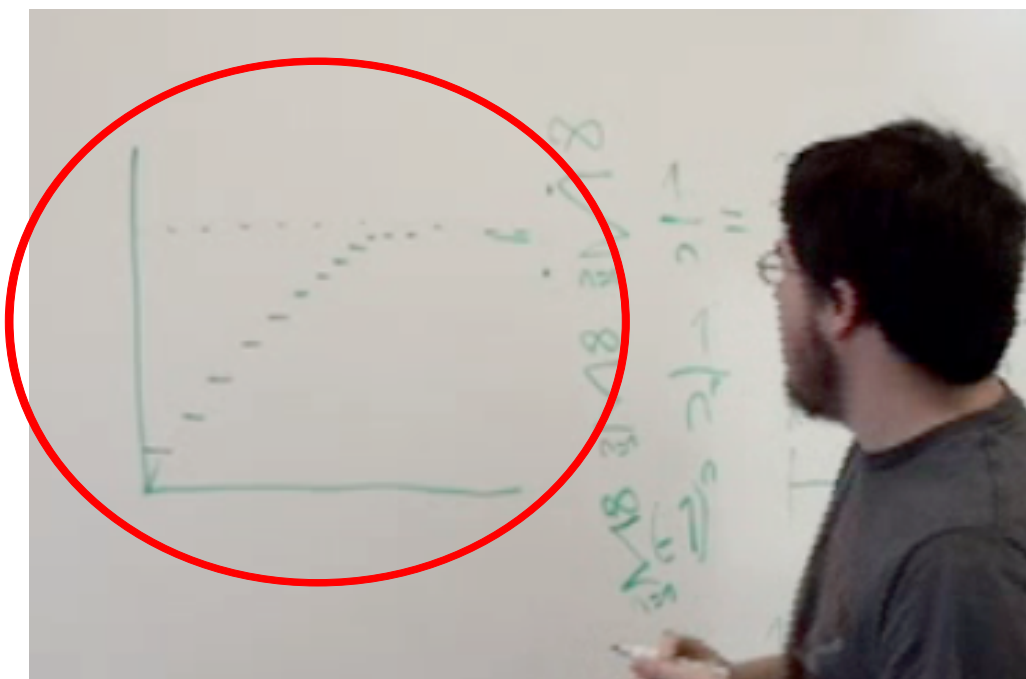


Figure VII.11. Andrew's plot of partial sums

As shown here, different individuals connect their understandings differently, and the nature of those connections, as shaped by their experiences and prior knowledge, will differ even if the base concepts are shared by multiple individuals. Even though there were no patterns according to which groups of students used which image types, there *were* patterns in how the different image types got connected by students during their teaching episodes (such as C1-C4, above). Because (most) students use multiple images through the course of their explanation, looking at the ways that they connect their multiple images and the instances in which they find the different image types to be useful over others (modes of use) helps to paint a clearer picture of what students understand about this particular mathematical content.

The varying levels of connections that were made between limiting processes during the course of the teaching episodes, as well as the examples of how those were made via Tina's and Molly's teaching episodes, are meant to showcase the range of ways that students not only refer to different limiting processes, but connect them in unique ways. Identifying the most common patterns of connections can provide instructional starting points, for instruction based on the patterns that naturally make sense to students. Looking at the fine detail of how students more and less successfully connect limiting processes provides further insight into constructing lesson sequences that take advantage of the subtleties that students pick up on when extending limit principles to contexts in which they are applied.

VIII. IMPLICATIONS FOR RESEARCH AND PRACTICE

Summary.

This study was designed to provide insight on the following two questions:

*What do students' explanations (via SGR) tell us about their understanding of infinite series?
How do the images that they produce highlight the limiting processes (limit of terms, limit of partial sums, etc...) that students find important when explaining convergence of infinite series?*
and
What does the process of generating SGR tell us about how students use this practice to explain, rather than simply solve particular, isolated math tasks?

Evidence suggests that studying students' explanations associated with infinite series of numbers will address the most important breaking point that separates expert and novice understanding (Martin, 2009). Martin also found that both experts and novices used a variety of (most often graphical) images when reasoning with Taylor series. Thus, this study focused on student understanding of infinite series, honing in on student understanding by the close examination of students' imagery associated with the topic. Because Martin and others (e.g. Kohl, 2007; Kindfield, 1993) found that both students and their professors used a variety of images, but interacted with those images in very different ways, comparisons were constructed in the present study to identify some of the ways that students and their professors used imagery similarly and differently while explaining aspects of the topic. The results here align with Kohl and others to suggest that students find something about producing SGR to be particularly useful and/or illustrative, when perhaps words would not convey the same information.

Students' teaching episodes, during which they were asked to explain to a peer what they might have missed had they been absent from class on the day(s) when infinite series were introduced and discussed, most often resulted in the spontaneous generation of several SGR – visual images used in the act of explaining. These SGR were then studied and compared with experts' SGR (including textbook and professor images) in order to explore what sense students make of infinite series topics, and how their understanding compares with a more normative understanding.

Outline of chapter -

- VIII.A. Findings from Chapter IV
- VIII.B. Findings from Chapter V
- VIII.C. Findings from Chapter VI
- VIII.D. Findings from Chapter VII
- VIII.E. Instructional implications and directions for future research

VIII.A. Findings from Chapter IV

Chapter IV demonstrated the five major image types that students, professors, and their Calculus II textbooks use when explaining infinite series convergence. Perhaps the most significant finding was that students use the same image types as their professors and textbooks, but often use them for purposes that over-extend the power of the particular image. In Chapter IV, students' and experts' (professors and textbooks) modes of use for their infinite series images

were often contrasted, with students using images that would be appropriate in some contexts inappropriately in others.

The five image types – plots of terms, plots of partial sums, area under curve, geometric shape, and number lines – were used for a number of purposes, predominantly including defining, clarifying, exemplifying, and drawing correspondences. While students used *all* image types (to varying degrees) for defining convergence for infinite series of numbers, the most frequently used was plots of terms and the least frequently used was plots of partial sums (for examples, see Figure VIII.1.). This contrasted sharply with professors, who only used the single image type of plots of partial sums when defining infinite series convergence. This finding is significant, first, because while some students did recognize and discuss partial sums as an important component of understanding infinite series, only 8% of this student sample used this image type for defining. This may indicate that some students understand that partial sums are important, but do not recognize their significance in the mathematical definition of convergence of series. And second, this finding is important because we see that students are overgeneralizing results based on graphs of individual terms, attributing more mathematical significance to them than is warranted.

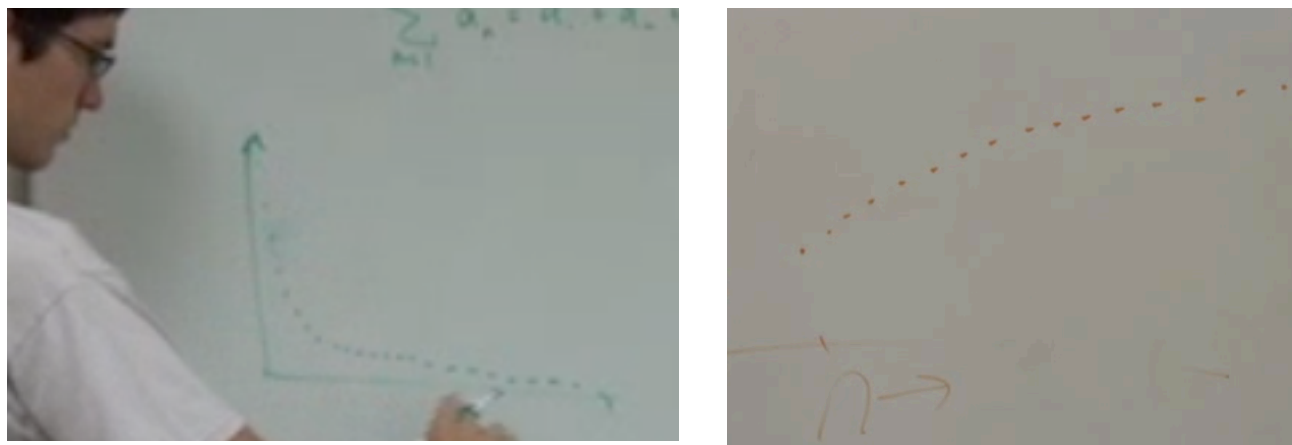


Figure VIII.1. Students' plot of terms and plot of partial sum images

Differences emerged around area under curves images (for an example, see VIII.2.) as well. Students used such images for both defining convergence and drawing correspondences between areas and individual terms written in an expanded out infinite series, while experts used them exclusively for clarifying and exemplifying the integral test for convergence. That is, experts used this image type in a very specific context, to explain and demonstrate one test for convergence. Forty-six percent of students in this sample (who may have seen similar images in Calculus I when studying Riemann sums and areas under curves) introduced such images into their teaching episode for the purpose of defining what it would mean for a series to converge. These explanations most often contained conclusions such as 'because the areas are getting smaller and smaller, at some point it does not matter' – essentially an extension of students' proclivities to lean on plots-of-terms reasoning to support their ideas about series convergence.



Figure VIII.2. A student's 'area under curve' image

Finally, students used number lines and geometric shapes for defining (among other purposes), while experts used them mostly to illustrate (exemplify or clarify) specific worked examples.

After the explanation phase of the teaching episode, students were also asked to reason with a few sample problems about series $\sum a_n$, to see how they connected their explanation to their reasoning strategies. Many students were able to apply various convergence tests to series problems where the a_n was known, often correctly arriving at a *conclusion*, despite having faulty definitions of convergence at earlier points in their teaching episodes. However, these students were rarely able to answer questions for which the a_n was unknown. In either case, students' *justification* for the answers that they provided was often tied to the predominant imagery from their teaching episode, through which they defined converge/diverge. That is, while students were often able to apply convergence tests on actual examples, their justification for why such a test would lead to a conclusion was often subject to the same errors associated with their predominant image choices. Note that a research methodology that focused on students' being able to correctly answer questions regarding particular series' convergence would be completely insensitive to the fact that the students were obtaining some right answers for the wrong reasons.

In this study, students were neither asked to use a given image, nor initially prompted to draw an image when explaining or reasoning. Rather, their move to spontaneously produce SGR instead of using other modes of explanation indicates that they have available imagery associated with this topic. It further demonstrates that the students think those images are important enough to use as a basis for an explanation of series convergence. Whether their default would be to reproduce an image previously seen, or create an original image, the important aspect is that they are producing such an image because they find it salient in the moment. Other research (e.g. Nardi, Biza & González-Martín 2008; Habre 2009) that has found that students have no ready images for infinite series has perhaps come to such a conclusion because the studies looked for evidence of students' images in their responses to specific tasks. As students in this study admitted, drawing was not their first response during the reasoning portion of the teaching episode. However, when in the mode of explaining a topic, rather than reasoning through a specific task, the students leaned much more heavily on their SGR to give insight into their understanding. And it was only because of the in-depth explanation phases of these teaching episodes, which were rife with SGR, that students' reasoning patterns were able to be related back to their problem-solving justifications from the reasoning portion of the episodes.

That the students are using some of the same image types as the experts, but with basically opposite modes of use and intention, is alarming. It shows that students, while

appreciating the significance of these images, are unclear about how these images can be used to support mathematically sound reasoning. As described in Chapter IV, some of the confusion may be attributable to student difficulties with ‘necessary’ vs. ‘sufficient’ argumentation. However, analyzing students’ modes of use for these two image types also points to one way that students’ understanding of sequence convergence may interfere with their notions of series convergence. One of the ways that we have seen the different groups use these images oppositely is that they emphasize different limiting processes when defining convergence for an infinite series – that is, students put emphasis on the limit of the sequence of *terms*, while the experts put the emphasis on the sequence of *partial sums*. Thus, the different groups are paying attention to different *limiting processes* (see Tall, 1980). This, in turn, influences which images they prioritize when explaining this concept.

One way of characterizing the differences found in Chapter IV is as follows: What did the experts “do right?” The professors and textbook prioritized the limiting processes represented in a particular image type in appropriate ways, so that those processes could be used to support mathematically sound conclusions. They used careful language so that the images were not overextended to contexts in which they did not support conclusions. And they connected representations in meaningful ways, using other images to clarify or exemplify those that were used to define. In brief, student explanations stood in clear contrast to the care and precision in experts’ use of limiting processes.

Further exploration in subsequent chapters examined how students used limit reasoning to assign properties of limits to the different mathematical objects associated with infinite series, the language that they used when explaining this topic, and the connections between concepts that they drew throughout their teaching episodes.

VIII.B. Findings from Chapter V

The main objective in the analysis in Chapter V was to demonstrate that some of the differences in students’ images of choice could be traced back to the particular limiting processes (see Tall, 1980) that they find important to emphasize when explaining infinite series. The rationale behind uncovering the limiting processes that students find important is simple: different instructional interventions would be warranted, based on what aspects of infinite series students find to be most important and consequential in determining series convergence. For example, students who found the sequence of terms to be the most important aspect to examine when determining whether or not their sum converged to a value may profit from different instructional strategies than students who were focused on partial sums.

Nine different limiting processes were identified in students’ teaching episodes (see Figure V.1.). These limiting processes - roughly corresponding to vague notions about limit, limits relating to terms of a series, and limits relating to the behavior of the series itself - were apparent in students’ explanations, and played out in students’ images.

Vague limit notions - picking up on the changing value of n , recognizing that limit could be explicitly related to the process of finding a sum (by writing the sum with a finite upper bound and then applying the limit to that bound), and using ‘limit words’ like “approach” - gave students some ideas about the behavior related to infinite series. But beyond this, students who only had vague notions of how limit related to infinite series did not make much progress in defining convergence. Students were able to make a little more headway when applying limit properties to terms of a series. Looking at limits of terms is helpful when modeling individual terms or thinking about them as physical quantities, but one can only make convergence

(actually, divergence) arguments when the limit of the terms is non-zero. Thus, students who applied limit processes to sequences of terms had the potential to make *some* claims about the convergence of their sum. However, this regularly used limiting process often resulted in overextending the principles of convergence of a series. Student reasoning about sums took the form of reasoning about groups of terms informally, or re-forming groups of terms into sequences of partial sums. Other students had more imprecise notions of compilation that, while more robust than simply using vague limit language, did not contain a mechanism by which a series could actually be shown to equal a value.

The example of Zeno's Paradox, a common thread in many student teaching episodes, is used in Chapter V to demonstrate that a single example affords the potential for discussions of a variety of different limiting processes. Students who focused on different limiting processes picked up on different parts of the example – size of steps decreasing, distance to the 'finish line' decreasing, distance traveled increasing by half a step size each time – thereby leading to differential success when explaining convergence.

VIII.C. Findings from Chapter VI

It became apparent, in the differentiated uses of images, that experts used particular language meant to draw attention to the context in which that image is useful, and the conclusions that can be drawn or inferred from that image type. Students used the same language in their explanations, but in different ways. Exploring this dimension of competence provided insight into the various ways that students inferred broader meanings to the words converge, approach, and equal than experts intended. Second, the analysis demonstrated the ways that students were able to use their images to make clearer which limiting processes they were actually attending to when using particular language.

It appears that students see many more entailments associated with the terms 'converge,' 'approach,' and 'equal,' than were intended when they heard them in lecture or a textbook. For those students who resolved the difference between "converge" and "equals" by taking advantage of putting the limit symbol outside the sigma, the difficulties with the language were not fully resolved. For those who used vague limit notions to talk about the differences between these three words, the terms were most often recast as temporal statements that require someone to spend infinite time carrying out an addition process. There, "converge" and "equal" were seen as starkly different – and students had difficulty conceiving of a way that an infinite series, framed as a process in time, could possibly ever be *equal* to a number. Knowing the various ways that students use this language differently from experts can help in both disambiguating what students really hear and infer when we use particular words in lecture, and what students mean when they say, for example, that 'converge' means something different than 'equal.'

Additionally, Chapter VI demonstrated that commonly used phrases like 'the terms tend to zero' take on a variety of meanings, depending on which limiting process a student is referencing. For students who reference different limiting processes with this phrase, different teaching interventions may be warranted.

VIII.D. Findings from Chapter VII

Knowing which limiting processes students draw on most frequently is insufficient to understand what sense students make of this topic. So, in Chapter VII, the analysis focused on patterns of coordination and common connections drawn between limiting processes that students emphasized while explaining. Studying the connections that students made gave insight

into which limiting processes are prioritized and why, and which limiting processes are linked to one another in an attempt to produce a coherent, mathematical story that accounts for the convergence of infinite series of numbers.

Contrasting the teaching episodes of Tina and Molly demonstrated the different ways that connections were made between the various limiting processes and limit ideas during students' teaching episodes. Analysis of teaching episodes illuminate not only the different ways that students connect their ideas, but also some of the structural differences in creating an original image vs. connecting multiple, familiar images.

VIII.E. Instructional implications and directions for future research

One clear avenue for future research is to rethink instruction, in order to explicitly implement several core ideas found in this study and examine their individualized effects on students' changing conceptions of infinite series and convergence. An entire instructional unit could be devised to start with students' understandings of limit, and proceed to develop more thoroughly their ideas of sequences, before delving into infinite series topics. Or, several of the key findings of this study could be used to make well-informed changes to current instructional practices. Since imagery was such a critical component of student understandings, given that such a large percentage of students spontaneously produced SGR, future instruction on this topic could rely heavily on visual images. Some of these core ideas that could be incorporated into infinite series instruction are as follows:

- (1) **Start with sequences.** Students showed a lot of facility with sequences, and were able to make claims about convergence and limits of sequences with both graphical and algebraic arguments. Often, when using plot of terms reasoning, students were able to correctly make claims and construct examples for various things that could happen with the terms of a series, even if their conclusions about the sums of those terms were off the mark. Thus, instruction that starts with more than a brief introduction to sequences – which Tall and Schwarzenberger (1978) posit is the norm, and is consistent with the observations that were part of this study – could be better leveraged into the design of a unit on infinite series.
- (2) **Purposeful imagery.** Part of instruction with topics that make heavy use of images must be a focus on explaining not only what the images show and how they are used, but also *why* they are particularly useful for certain contexts. Designing instruction centered on constructing images that are useful for particular contexts and claims, discussing the benefits and strategies for reasoning with imagery, and then using images while developing and proving convergence tests has multiple benefits. Because students are so prone to this type of reasoning in the context of infinite series, and as graphing has been identified as the modality easiest to influence with careful instruction (Martin, 2009), such instruction could make significant improvements in how students use images to reason *productively* with infinite series. But also, an instructional unit based around visualizations in calculus (lecture) could help students build representational skills that will extend beyond this singular context. That is, carefully designed instruction that takes advantage of students' proclivities to create images when working with infinite series could also speak to larger issues of best practices for fostering representational competency, more generally.
- (3) **Caution with certain images.** Many students used area under curves imagery for the purpose of defining convergence. As we saw in Chapter IV, this differs from instructors' use of this image type, which was largely for clarifying the integral test. When examining

students' use of this image, it became apparent (as in Chapter V) that this was a compelling image type for them for defining because the limiting process of sequence of terms was prioritized in their explanations, and held a lot of bearing. If spending a little more time discussing what features of particular images, like area under curve, are able to tell us about convergence, and what they cannot (i.e. the affordances and drawbacks of the image, in that context), it may bring students' and experts' use of imagery into better alignment. This requires that explicit attention be paid to the common image types that are associated with the most common errors in student explanations – namely those related to sequence of terms reasoning.

- (4) **Caution with certain examples.** In addition to spending more time discussing the affordances and drawbacks of specific images that are most aligned with students' more common errors, instruction must also take care to present examples with adequate justification (what are they examples *of*) and proper timing. For example, most instructors defaulted to the harmonic series as an example of why terms tending to zero do not guarantee convergence. As a first example, and one that is not yet synced with students' ideas of how a series could possibly converge, many students identified this as a 'special case' or 'something they should memorize' but could not quite justify. Carefully timed instructional examples that make general points, as well as those that illustrate specific behaviors, should be framed as such.
- (5) **Consistent use of language.** Students already struggle with limit-related language (e.g. Williams, 1991). This struggle is compounded when students have to apply their limit understandings to a new context like infinite series. While mathematicians would use terms like 'converge' and 'equal' interchangeably (see Monaghan, 1991), the very act of doing so in lecture, without discussing the synonymy of those words, results in students' interpretations of subtle differences that are not compatible. Care is also warranted in the language of labeling and connecting images and concepts during infinite series lectures. This includes being verbally explicit about the connections intended when going from reasoning about terms to reasoning about their sum, and by showing those connections not just in mathematical definitions and with limit statements, but also with consistent use of labeling on pictures.
- (6) **Geometric arguments.** Often in infinite series instruction, geometric series are among the first examples shown. This is reasonable, considering that they are among the easiest class of examples to use when first practicing generating terms, relating infinite series to approachable contexts (like walking to the wall in a version of Zeno's Paradox), and using geometric imagery and geometric area to demonstrate convergence visually. However, perhaps given that it is among the first class of examples seen in infinite series lecture, students often inferred *incorrect* geometric arguments, during their teaching episodes, on series where those arguments are difficult (at best) to make. Students often referred to any number of series as geometric, even when they were not, and proceeded to try to explain them in the same way as a geometric series – with geometric images and attempts to find a common ratio. Thus, in redesigned instruction, in which explicit attention is paid to discussing the reasons that particular images are best suited for certain contexts, an in-depth discussion about recognizing and working with geometric series would also be beneficial.

Beyond the specific take-aways that could be considered in revamped infinite series instruction, this study has demonstrated a different way to encourage students to reveal what they

know – correct or incorrect. Soliciting students’ *explanations*, particularly in contexts in which their SGR help to elucidate their understandings where language alone could be confusing, provides pathways for both proactive and responsive instruction. This instruction, based on students’ SGR in the context of an explanation, is proactive in the sense that it helps to identify patterns of connections that students make. In so doing, it will reveal where students are more or less likely to stray from mathematically sound reasoning, making it is easier to anticipate problems and design instruction to address them. Further, it is responsive in the sense that the more an instructor knows about and tailors instruction to the knowledge that students bring to her class, the more the instructor can position herself to mold and change it. That is, the more an instructor knows about the connections that her students are likely to make, and the ways that they are using prior knowledge in the new contexts, the easier it is to form instruction that starts with what students bring to bear to the context. Such instruction only builds from there, helping students to form appropriate connections.

The emphasis of this work on SGR goes beyond examining students’ SGR in the context of series. It also demonstrates that, even in broader content areas, the act of producing and explaining with SGR may be a productive window into student thinking that provides insight beyond what could be elicited with problems that may have a narrower focus. This type of exploration, then, is potentially applicable to a wider range of calculus topics, and beyond. Eliciting students’ SGR and working with them as an instructional starting point is one way to create artifacts with which to mold and shape students’ ideas, as well as put on the table the variety of ways that students see how mathematical ideas connect and work together.

Finally, SPOT diagrams (Structure Perceived Over Time) (Yoon, 2012), can also aid in identifying some of the ways that students connect their ideas about infinite series similarly and differently from their professors, and how the connections appear different when connecting vs. creating representations. However, in the context of the teaching episodes elicited as part of this study, they can also (perhaps more importantly) help to identify whether some learning or perceptual shifts occurred *during* the process of explaining in the interview setting. That is, when students were invited to participate in teaching episodes, they were not aware that they would be asked to *explain* mathematical topics. Perhaps some of them had never been prompted to do so, or had never participated in such an activity prior to their interview. However, following the Latin proverb ‘Docendo discimus,’ (“we learn by teaching”), there is a time-tested belief that the act of explaining a topic may lead to more robust and connected understanding. So, it stands to reason that some ‘learning’ or realizations may have occurred during the process of participating in the teaching episodes, themselves. This was not of concern to the present study, because whether students were explaining previous understandings of infinite series, or whether they were constructing them on the spot, the focus was on the content of those and the connections made throughout the explanation. However, with the existing data, it may be possible to (aided by SPOT diagrams) identify some conditions or sequences of conversation that lead to “aha! moments” (Yoon, 2012), with the future goal of exploring ways that carefully designed, explanation-focused teaching episodes can complement regular instruction.

Overall, this study contributes more than simple insight into students’ understanding of infinite series topics. It also opens up ways of looking at student thinking that can be applied to many problematic areas in the curriculum. While typical studies ask students to address tasks and issues framed by a researcher, this study instead asked students to *explain* the content, thereby providing a much larger window into “what counts” from the student perspective. This in turn revealed much about students’ understanding that instructors should heed, when planning and

carrying out instruction in this difficult domain. This awareness promises to be important for future research, toward exploring a range of student understandings in this same way. It is also promising for future instruction, where, in a wide range of topics, attention to students' explanations and SGR can shape more productive instruction, starting with what makes sense to the students, themselves.

REFERENCES

- Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education*, 33, 131-152.
- Ainsworth, S. & Loizou, A. (2003). The effects of self-explaining when learning with text or diagrams. *Cognitive Science*, 27, 669-681.
- Alcock, L. & Simpson, A. (2004). Convergence of Sequences and Series: Interactions Between Visual Reasoning and the Learner's Beliefs about their own Role. *Educational Studies in Mathematics*, 57, 1-32.
- Alcock, L. & Simpson, A. (2005). Convergence of Sequences and Series 2: Interactions Between Non-visual Reasoning and the Learner's Beliefs about their own Role. *Educational Studies in Mathematics*, 58, 77-100.
- Alcock, L. & Simpson, A. (2009) Ideas from mathematics education: An introduction for mathematicians.
- Aleven, V. A., & Koedinger, K. R. (2002). An effective metacognitive strategy: Learning by doing and explaining with a computer-based Cognitive Tutor. *Cognitive Science*, 26(2), 147-179.
- Apsinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33(3), 301-317.
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215-241.
- Borgen, K. & Manu, S. (2002). What do students really understand? *Journal of Mathematical Behavior*, 21, 151-165.
- Bourdieu, P. (1997). *Meditations pascaliennes* [Pascalian meditations]. Paris: Seuil.
- Brown, J. S., & Duguid, P. (1996). Keeping it simple. In T. Winograd (Ed.), *Bringing design to software* (pp. 129-145). New York: ACM Press.
- Chi, M., Bassok, M., Lewis, M., Reimann, P., & Glaser, R. (1989). Self-Explanations: How students study and use examples in learning to solve problems. *Cognitive Science*, 13(2), 145-189.
- Cornu, B. (1992). Limits. In D. Tall (Ed.) *Advanced mathematical thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic.
- Cox, R. (1999). Representation construction, externalized cognition and individual differences. *Learning and Instruction*, 9, 343-363.

Cox, R. & Brna, P. (1995). Supporting the use of external representations in problem solving: The need for flexible learning environments. *Journal of Artificial Intelligence in Education*, 6(2), 239-302.

The Committee on the Undergraduate Program in Mathematics. (2004). Undergraduate programs and courses in the mathematical sciences: CUPM curriculum guide 2004. *Mathematical Association of America*.

Ferrini-Mundy, Joan, and Karen Graham. "Research in calculus learning: Understanding of limits, derivatives, and integrals." *MAA NOTES* (1994): 31-46.

Fischbein, H. (1987). *Intuition in science and mathematics: An educational approach* (Vol. 5). Springer.

Fischbein, H. (2001). Tacit models and infinity. *Educational Studies in Mathematics*, 48(2/3), 309-329.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Gobert, J. & Clement, J. (1999). Effects of student-generated diagrams versus student-generated summaries on conceptual understanding of causal and dynamic knowledge in plate tectonics. *Journal of Research in Science Teaching*, 36(1), 39-53.

Greeno, J. & Hall, R. (1997). Practicing representation: Learning with and about representational forms. *The Phi Delta Kappan*, 78(5), 361-367.

Habre, S. (2009). Multiple representations and the understanding of Taylor polynomials. *PRIMUS*, 19(5), 417-432.

Haciomerglu, E., Aspinwall, L., & Presmeg, N. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning*, 12(2), 152-176.

Harel, G. & Sowder, L. (1998). Students' proof schemes. *Research on Collegiate Mathematics Education, Vol. III*. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), AMS, 234-283.

Inglis, M. & Mejía-Ramos, J. P. (2009). On the persuasiveness of visual arguments in mathematics. *Foundations of Science*, 14, 97-110.

Kindfield, A. (1993). Biology diagrams: Tools to think with. *The Journal of the Learning Sciences*, 3(1), 1-36.

Kohl, P. (2007). Toward an understanding of how students use representations in physics problem solving. Doctoral Dissertation, University of Colorado.

- Kohl, P. & Finkelstein, N. (2008). Patterns of multiple representation use by experts and novices during physics problem solving. *Physical Review Special Topics: Physics Education Research*, 4, 010111:1-13.
- Larkin, J. & Simon, H. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science*, 11, 65-99.
- Latour, B. (1987). Visualization and cognition: Thinking with eyes and hands. *Knowledge and Society*, 6, 1-40.
- Martin, J. (2009). Expert conceptualizations of the convergence of Taylor series: Yesterday, today, and tomorrow. Doctoral Dissertation, University of Oklahoma.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking Mathematically*. London: Addison Wesley
- Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics*, 11(3), 20-24.
- Monaghan, J. (2001). Young people's ideas of infinity. *Educational Studies in Mathematics*, 48 (2&3), 239-257.
- Nardi, Biza, & González-Martín. (2008). Introducing the concept of infinite sum: Preliminary analyses of curriculum content. In Joubert, M. (Ed.) *Proceedings of the British Society for Research into Learning Mathematics*, 28(3), 84-89.
- Oehrtman, M. (2002). Collapsing dimensions, physics limitation, and other student metaphors for limit concepts: an instrumentalist investigation into calculus students' spontaneous reasoning. Doctoral Dissertation, University of Texas at Austin.
- Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In Carlson, M. & Rasmussen, C. (Eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, (pp. 65-80). MAA Notes #73.
- Parnafes, O. (2009). Self-generated representations (SGR) for promoting conceptual understanding. *Proceedings of the Chais conference on instructional technologies research: Learning in the technological era*. Y. Eshet-Alkalai, A. Caspi, S. Eden, N. Geri, Y. Yair (Eds.), Raanana: The Open University of Israel, 147-152.
- Pinto, M. & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics*, 22(1), 2-10.
- Presmeg, N. (1986). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17, 297-311.
- Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. *Handbook of research on the psychology of mathematics education*, 205-235.

Roth, W. & McGinn, M. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of Educational Research*, 68(1), 35-59.

Roy, M. & Chi, M. (2005). Self-explanation in a multi-media context. In R. Mayer (Ed.), *Cambridge Handbook of Multimedia Learning* (pp. 271-286). Cambridge Press.

Schoenfeld, A. H. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. Taylor & Francis US.

Sherin, B. (2000). How students invent representations of motion: A genetic account. *Journal of Mathematical Behavior*, 19, 399-441.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, 18(4), 371-397.

Soto-Johnson, H. (1998). The Impact of Technology on Infinite Series, *International Journal of Computer Algebra in Mathematics Education*, 5(2), p. 95-109.

Stergiou, V. & Patronis, T. (2002). Exploring a modern path to infinitesimals: mathematics students' understanding of the concept of rate of convergence. *International Journal of Mathematical Education in Science and Technology*, 33(5), 651-659.

Stewart, J. (2007). *Calculus, Sixth Edition*. Thomson Brooks/Cole.

Tall, D. (1980). Mathematical intuition, with special reference to limiting processes. *Proceedings from the Fourth International Conference for the Psychology of Mathematics Education*, 170-176.

Tall, D. (1991). Intuition and rigor: The role of visualization in the calculus. In Zimmermann & Cunningham (Eds.) *Visualization in Mathematics, MAA Notes #19*, 105-119.

Tall, D. & Schwarzenberger, R. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, 82, 44-49.

Wheatley, G. H., & Cobb, P. (1990). Analysis of young children's spatial constructions. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: International Perspectives* (pp. 167-173). Hillsdale, NJ: Lawrence Erlbaum Associates.

Wilkin, B. (1997). Learning from explanations: Diagrams can 'inhibit' the self-explanation effect. *AAAI Technical Report FS-97-03*, 136-143.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematical Education*, 22(3), 219-236.

Yerushalmy, M., Schwartz, J.L. (1999) A procedural approach to exploration in calculus. *International Journal of Mathematical Education in Science and Technology*, 30 (6), 903-914.

Yoon, C. (2012). Mapping mathematical leaps of insight. Proceedings from the 12th Annual International Congress on Mathematical Education.

Zimmermann, W. (1991). Visual thinking in calculus. In W. Zimmermann and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 127-137). Mathematical Association of America, Washington DC.