Instability of the boundary layer between a streaming plasma and a vacuum magnetic field

William Peter[a)] and Amiram Ron
Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel
Norman Rostoker
Department of Physics, University of California, Irvine, California 92717

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Analysis is carried out for the long-wavelength stability of the boundary layer formed when a plasma or neutralized ion beam is incident upon, and reflected by, an applied magnetic field. This equilibrium is similar to the Ferraro–Rosenbluth sheath which forms the basis for many models in magnetic fusion and magnetospheric physics. For wavelengths much greater than the sheath thickness a_h, the equilibrium is unstable with growth rate (in the regime $ka_h \ll m/M$) proportional to $(kg)^{1/2} \equiv (ka_h \Omega_i \Omega_e)^{1/2}$. The nonlinear evolution of the instability describes plasma propagation across the magnetic field. This instability is similar to the classical flute instability except that polarization along the boundary is due to electron, and not ion, motion.

I. INTRODUCTION

A low-beta plasma beam whose radius R is much larger than its ion gyradius $a_i (R \gg a_i)$ can penetrate an applied transverse magnetic field a distance equal to the hybrid gyroradius $a_h = (a_i a_e)^{1/2}$ before reflecting.[1-2] In this paper we shall examine the stability of the resulting boundary layer of thickness a_h which separates the streaming plasma from the vacuum magnetic field. The development of a flute-type instability on this boundary may be responsible for some experimental observations of plasma propagation across magnetic fields.[3-5]

Boundary layer equilibria separating plasma and vacuum magnetic fields are usually based on Ferraro–Rosenbluth calculations[6,7] describing particles reflected by their own self-generated fields. Recent studies[1-2] have also included an externally applied magnetic field which is of importance in cross-field injection studies in tokamaks.[8] In both schemes, the longitudinal electric field mediates the transfer of initial ion kinetic energy into electron kinetic energy. The electron velocity at the plasma–vacuum interface is then much larger than the ion velocity [Fig. 1(a)].

The cross-streaming of particles at this interface can be compared to the usual equilibrium situation of the classical Rayleigh–Taylor instability.[9] In this case, particle motion along the boundary is driven by ion and electron $\mathbf{g} \times \mathbf{B}$ drifts, where $\mathbf{g}$ is the gravitational field [Fig. 1(b)]. A comparison of Figs. 1(a) and 1(b) suggests that the present equilibrium may be subject to a flute-type instability. However, this instability would be driven by ion, and not electron, motion along the boundary. In addition, account must also be taken of the presence of a zero-order electric field in the region occupied by plasma.

In Fig. 2 we show a schematic drawing of the equilibrium.[1-2] A plasma, consisting of ions and electrons, is incident upon a transverse magnetic field $\mathbf{B}_0$ applied in the z direction, and occupying the region $x > 0$. The low-beta equations of motion are

$$v_{jx} \frac{d}{dx} \mathbf{v}_j = \frac{q_j}{m_j} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_j \times \mathbf{B}_0 \right),$$
$$\frac{dE_x}{dx} = 4\pi q (n_i - n_e), \tag{1}$$

where $\mathbf{v}_j = (v_{jx}, v_{jy}, 0)$ is the particle velocity for the jth species, and the subscript $j = (e,i)$ is the species index. The self-consistent electric field is given by $\mathbf{E} = (E_x, 0, 0)$, and the applied magnetic field by $\mathbf{B}_0 = (0, 0, B_0)$. We have denoted the charge and mass of each species (electrons, ions) by $q_j = (-q,q)$ and $m_j = (m,M)$. Finally, the quantities n_e and n_i are the electron and ion particle densities which are determined by the continuity equation

$$n_j v_{jx}(x) = 2 n_0 u_0, \tag{2}$$

where n_0 and u_0 are the plasma density and velocity at $x = 0$.

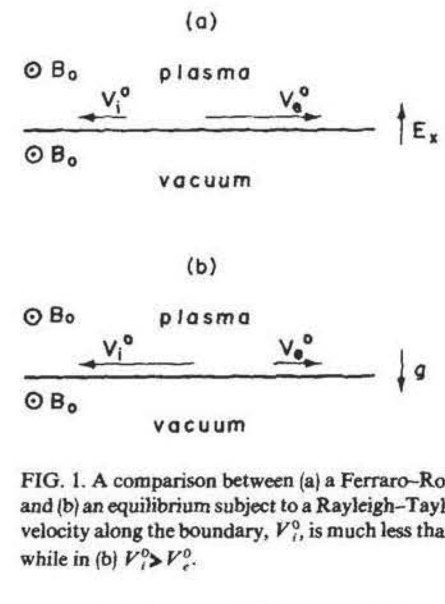

FIG. 1. A comparison between (a) a Ferraro–Rosenbluth-type equilibrium and (b) an equilibrium subject to a Rayleigh–Taylor instability. In (a) the ion velocity along the boundary, V_i^0, is much less than the electron velocity V_e^0, while in (b) $V_i^0 \gg V_e^0$.

[a)] Present address: Advanced Concepts and Plasma Physics Group, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

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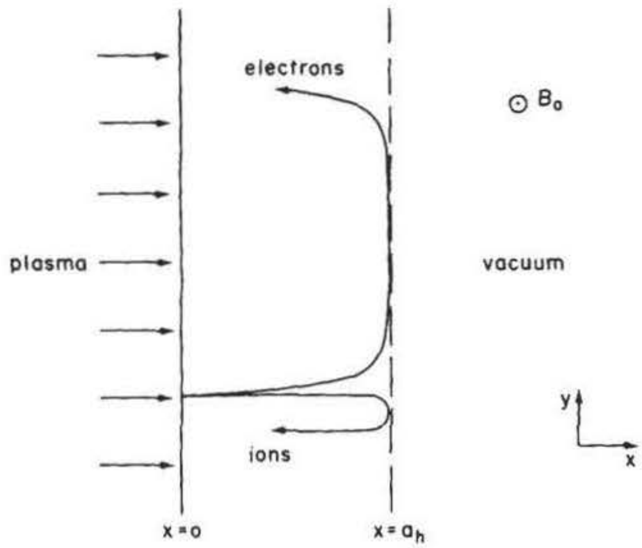

FIG. 2. The equilibrium boundary layer between an incoming plasma and an applied magnetic field. Because of magnetic forces, the plasma is turned back at a distance a_h which is the geometric mean of the electron and ion gyroradii. Strong electric forces cause the electron orbits in the y direction to be elongated, and the ions to be sharply turned around at a distance much less than the ion gyroradius.

The factor of two in Eq. (2) arises out of the fact that in the steady state there are an equal number of particles moving into and out of the magnetic field. The total plasma density at $x = 0$ is due to the presence of both incoming and reflected particles, accounting for this extra multiplicative factor.[7]

In Eqs. (1) and (2) all physical quantities were taken to be a function of x only. The fact that we can neglect the y dependence in the equations of motion is an outgrowth of our assumption that the beam radius satisfies $R \gg a_i$. For a sufficiently dense plasma, we can also assume that the plasma is quasineutral; i.e., there is local charge neutrality everywhere so that $n_i(x) = n_e(x)$. From Eq. (2) we have then

$$v_{ix} = v_{ex} \equiv u_x,$$

where u_x is defined to be either the electron or ion velocity in the x direction. The solutions to Eqs. (1) can then be written

$$u_x = u_0(1 - x^2/a_h^2)^{1/2} \tag{3a}$$

$$v_{iy} = -\Omega_i x, \tag{3b}$$

$$v_{ey} = \Omega_e x, \tag{3c}$$

where $\Omega_i = qB_0/Mc$ is the ion gyrofrequency, and $\Omega_e = qB_0/mc$ is the electron gyrofrequency. The electric field is given by

$$E_x = -(1/c)v_{ey}B_0(1 - m/M). \tag{4}$$

The equilibrium is described by the fact that the turning point of the ions and electrons is the same and equal to the geometric mean of the electric and ion gyroradii $a_h = (a_e a_i)^{1/2}$. However, the physical mechanisms for reflection of the two species are quite different. Because the scale length a_h is much less than an ion gyroradius, and much larger than an electron gyroradius, the electrons are magnetized whereas the ions are not. As the electrons are turned back by magnetic forces, a strong electric field E_x is produced to keep the plasma quasineutral. The result is that the field mediates the transfer of the initial ion kinetic energy to electron motion. The ions are thus turned back at $x = a_h$ by electric, and not magnetic, forces. The electrons in turn are pulled forward to $x = a_h$ by the electric field before reflecting. Finally, we note that the quasineutral approximation can be shown to be valid in our case provided that the plasma dielectric constant $\epsilon = 1 + \omega_{pi}^2/\Omega_i^2$ satisfies the inequality $\epsilon \gg M/m$ (Refs. 1 and 2).

A schematic drawing of the particle orbits in our equilibrium is shown in Fig. 2. From the preceding analysis we may take the electrons to satisfy the drift approximations in our model [cf. Eq. (4) for $m/M \ll 1$]. Note also that the current in the y direction is carrier primarily by the electrons.

In this paper, we shall be analyzing the stability characteristics of the boundary between the plasma and the vacuum magnetic field at $x = a_h$. We describe a stability analysis similar to the particle-orbit approach of Rosenbluth and Longmire[9] for long-wavelength perturbations in Sec. II. In the last section, we compare our results with experiment and present the conclusions.

II. STABILITY ANALYSIS

In Fig. 3, perturbations on the boundary between the plasma and the vacuum magnetic field are pictured. We will use zero superscripts to denote zero-order quantities, so that V_j^0 is defined to be the equilibrium velocity of the jth species along the plasma–vacuum interface at $x = a_h$. From Eqs. (3) we then have $V_e^0 = \Omega_e a_h$ and $V_i^0 = -\Omega_i a_h$.

At the boundary the density $n(x)$ becomes weakly infinite.[1,7] It then drops to zero for $x > a_h$. More realistically, we shall assume that the density has a large, but finite, value N_0 at a_h.

The perturbations will be taken to have the form

$$\delta\xi = a \exp[i(ky - \omega t)]. \tag{5}$$

Because of the magnetic field, the electron boundary will drift away from the ion boundary at the velocity $V_e^0 = \Omega_e a_h$. The motion of the ions along the interface is a factor m/M smaller than V_e^0, and can be neglected in calculating the relative velocity between these two boundaries. Because of this charge separation, a surface charge density $\delta\sigma$ will appear on the boundary with a value

$$\frac{\partial}{\partial t}\delta\sigma = \delta\mathbf{J}\cdot\hat{n} + \mathbf{J}\cdot\delta\hat{n} = \delta J_x = J_y \delta n_y, \tag{6}$$

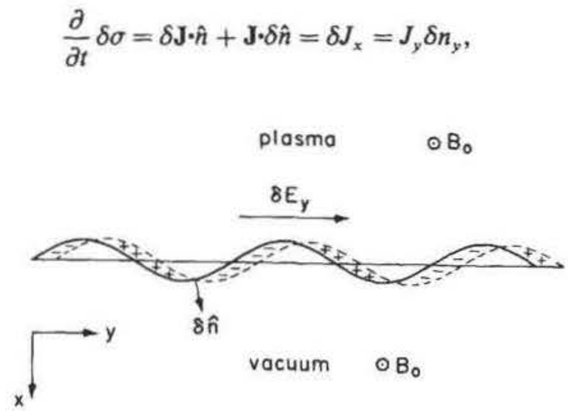

FIG. 3. Schematic drawing of perturbations on the plasma–vacuum boundary in the long-wavelength case.

where $\hat{n}(\delta\hat{n})$ is a unit vector normal to the unperturbed (perturbed) plasma–vacuum interface, and pointing away from the plasma (Fig. 3).

For infinitesimal perturbations the normal to the perturbed interface is given by

$$\delta n_y = -\frac{\partial}{\partial y}(\delta\xi) = -ik\delta\xi. \tag{7}$$

The equilibrium current density J_y at the interface $x = a_h$ is due mainly to electrons, and is given by

$$J_y = N_0 q(V_i^0 - V_e^0) \simeq -N_0 q V^0 \tag{8}$$

We are neglecting terms of order m/M with respect to unity. Equations (6)–(8) allow us to write

$$\frac{\partial}{\partial t}\delta\sigma = \delta J_x + ikN_0 q V_e^0 \delta\xi. \tag{9}$$

Note the presence of the perturbed current density $\delta J_x = N_0 q(\delta v_{ix} - \delta v_{ex})$ in Eq. (9). If it were not present, our stability analysis would be similar to the classical Rayleigh–Taylor case. As it is, this term is analogous to the inclusion of finite-gyroradius effects in this classical problem.[10] For the finite-gyroradius case, δJ_x is calculated from

$$\delta J_x = N_0 q\left(\frac{c\delta E_y(i)}{B_0} - \frac{c\delta E_y(e)}{B_0}\right),$$

where $\delta E_y(i)$ and $\delta E_y(e)$ are the perturbed polarization electric fields felt by the ions and electrons, respectively. In the magnetohydrodynamic limit when the particle gyroradii are zero, the electrons and ions feel the same electric field, and this term is zero.

Since we are discussing long-wavelength perturbations, we need not consider finite gyroradius effects at all. However, in the present case, there is a finite δJ_X due to the presence of the equilibrium electric field E_x^0. This field builds up a charge separation out of phase with the characteristic charge separation due to particle drifts in the simple model of Rosenbluth and Longmire.[9] The fact that ions and electrons do not now move together across the field can also be seen by noting that the scale length a_h of the problem is much less than an ion gyroradius but much larger than an electron gyroradius.

In the present case, the quantity δJ_x will be calculated in the following way. In the presence of a perturbation, charge separation along the boundary will result in perturbed electric fields δE_x and δE_y. The equations of motion for a particle of charge q_j and mass m_j which enters the sheath region is

$$\frac{d}{dt}\delta v_{jx} = \frac{q_j}{m_j}\delta E_x + \Omega_j \delta v_{jy}, \tag{10a}$$

$$\frac{d}{dt}\delta v_{jy} = \frac{q_j}{m_j}\delta E_y - \Omega_j \delta v_{jx}, \tag{10b}$$

where $\Omega_j = q_j B_0/m_j c$. Using Eq. (5) and eliminating δv_{jy}, we obtain

$$\delta v_{jx} = i\frac{\omega/\Omega_j}{(\omega/\Omega_j)^2 - 1}\left(\frac{c\delta E_x}{B_0}\right) - \frac{1}{(\omega/\Omega_j)^2 - 1}\left(\frac{c\delta E_y}{B_0}\right), \tag{11}$$

so that $\delta J_x = N_0 q(\delta v_{ix} - \delta v_{ex})$ can be determined.

The electric field $\delta\mathbf{E}$ is described by Poisson's equation

$$\nabla\cdot[\epsilon\nabla(\delta\varphi)] = -4\pi\delta\rho, \tag{12}$$

where $\delta\varphi$ is the perturbed electric potential, and we have introduced the plasma dielectric constant $\epsilon = 1 + \omega_{pi}^2/\Omega_i^2$. The introduction of ϵ is a way of including the polarizability of the plasma in the model equations without directly considering the polarization drift.[9] In Eq. (12), the quantity $\delta\rho$ is the perturbed charge density at the boundary $x = a_h$, and can be expressed as

$$\delta\rho = \delta\sigma[\delta(x - a_h)],$$

where $\delta(x - a_h)$ is the Dirac delta function. Equation (12) has the following exponentially decaying solutions in terms of the perturbation amplitude $\hat{\varphi}$ (where $\delta\varphi = \hat{\varphi}\exp[i(ky - \omega t)]$),

$$\varphi = \begin{cases}\hat{\varphi}_1 \exp[k(x - a_h)], & x < a_h,\\ \hat{\varphi}_2 \exp[-k(x - a_h)], & x > a_h.\end{cases}$$

To match these solutions at the interface, we can make use of the boundary conditions at $x = a_h$,

$$\frac{d\hat{\varphi}}{dx}(a_h + 0) - \epsilon\frac{d\hat{\varphi}}{dx}(a_h - 0) = -4\pi\hat{\sigma}, \tag{13a}$$

$$\hat{\varphi}(a_h + 0) = \hat{\varphi}(a_h - 0). \tag{13b}$$

From Eq. (13b), $\hat{\varphi}_1 = \hat{\varphi}_2$. Hence, (13a) yields

$$\hat{\varphi}_0 = \frac{4\pi\hat{\sigma}}{k(1+\epsilon)} \simeq \frac{4\pi}{k\epsilon}\hat{\sigma}, \tag{14}$$

where $\hat{\varphi}_0 \equiv \hat{\varphi}_1 = \hat{\varphi}_2$, and we have approximated $\epsilon \gg 1$. Due to the perturbation, the boundary will move initially with a velocity

$$\delta v_x(x = a_h) = \frac{\partial}{\partial t}\delta\xi\Big|_{x = a_h} = \frac{c\delta E_y(x = a_h)}{B_0} \tag{15}$$

Recalling that $\delta E_y = ik\delta\varphi$, and using the definition of $\delta\xi$ in Eq. (5), we now obtain

$$i\omega\hat{a} = (ikc/B_0)\hat{\varphi}_0,$$

or from $\hat{\varphi}_0 = (4\pi/k\epsilon)\hat{\sigma}$,

$$\hat{a} = (4\pi c/\omega B_0 \epsilon)\hat{\sigma}. \tag{16}$$

This gives us a relation between the perturbation and surface charge density amplitudes. From Eq. (9) we can write

$$-i\omega\delta\sigma = \delta J_x + N_0 q\Omega_e a_h ik\delta\xi,$$

or

$$(\omega^2 + ka_h\Omega_i\Omega_e)\delta\sigma = i\omega\delta J_x, \tag{17}$$

where we have used Eq. (16). In the simple magnetohydrodynamic (MHD) case where $\delta J_x = 0$, Eq. (17) describes an instability with

$$\omega^2 = -ka_h\Omega_i\Omega_e. \tag{18}$$

We can identify the quantity $a_h\Omega_i\Omega_e$ with a gravity g so that $\omega^2 = -kg$ in analogy with the classical flute case. In Eq. (17), $\delta J_x = N_0 q(\delta v_{ix} - \delta v_{ex})$, and hence from Eq. (11),

$$\delta J_x = N_0 q\left[i\frac{\omega}{\Omega_i}\frac{c\delta E_x}{B_0} - \left(\frac{\omega}{\Omega_i}\right)^2\frac{c\delta E_y}{B_0}\right] \times\left[\left(\frac{\omega}{\Omega_i}\right)^2 - 1\right]^{-1}. \quad (19)$$

The electrons satisfy the drift approximation so that we have taken $\omega/\Omega_e \ll 1$ in Eq. (19).

Substituting for $\delta\mathbf{E}$ in terms of the electric potential $\delta\varphi$, Eq. (19) becomes

$$\delta J_x = -N_0 q(ikc/B_0)(\omega/\Omega_i)[(\omega/\Omega_i) - 1]^{-1}\delta\varphi. \quad (20)$$

If we now insert this δJ_x in Eq. (17), and express $\delta\varphi$ in terms of $\delta\sigma$ from Eq. (14), we obtain the following dispersion relation

$$\omega^3 - 2\Omega_i\omega^2 + kg\omega - kg\Omega_i = 0, \quad (21)$$

which is a cubic equation in the frequency ω. In Eq. (21), we have used the definition $g = a_h\Omega_i\Omega_e$. From the theory of cubic equations[11] we construct quantities

$$q = \tfrac{1}{3}kg - \tfrac{4}{9}\Omega_i^2, \quad r = \tfrac{1}{6}kg\Omega_i + \tfrac{8}{27}\Omega_i^3, \quad (22)$$

and determine the range of parameters for which $q^3 + r^2 > 0$ (i.e., for which the equilibrium is unstable). Substituting for q and r from Eq. (22), this condition can be written

$$\alpha[8 - (13/4)\alpha + \alpha^2] > 0, \quad (23)$$

where $\alpha \equiv kg/\Omega_i^2 = ka_h(M/m)$. The critical value of α for which the left-hand side of Eq. (23) changes sign is determined by the quadratic equation in square brackets. Since α is always positive, this inequality is always satisfied and the equilibrium is unstable to long wavelengths $ka_h \ll 1$. The growth rate of the instability is given by

$$\gamma = (\sqrt{3}/2)(s_+ - s_-),$$

where $s_\pm = [r \pm (q^3 + r^2)^{1/2}]^{1/3}$, and q and r are given by Eqs. (22). For $\alpha \ll 1$, or $ka_h \ll (m/M)$, the growth rate is proportional to $(kg)^{1/2}$. It is evident that in the long-wavelength case we are considering here, the oscillation frequencies given by Eq. (21) satisfy our approximation that $\omega/\Omega_e \ll 1$. We will discuss the interpretation of these results in the following section.

III. DISCUSSION AND CONCLUSIONS

In the previous two sections we demonstrated the instability of the equilibrium boundary layer formed when a plasma streams across an externally applied magnetic field. This equilibrium is possible provided that the beam is wide, or $R \gg a_i$, and the plasma dielectric constant satisfies $\epsilon \gg M/m$.

Experimentally, cross-field motion of small-gyroradius beams is usually characterized by the formation of thin sheets of plasma oriented along the magnetic field[3,4] or discrete plasma blobs similar to the plasmoids of Bostick.[12,14] Bostick[12] may have been the first to attribute the formation of such distinct plasma entities to the onset of a Rayleigh–Taylor instability. Other experiments by Marcovic and Scott,[3] Speck *et al.*,[4] and Wessel and Robertson[5] have reported similar flute-like behavior.

A small-gyroradius beam ($R \gg a_i$) is usually produced in the laboratory by experiments utilizing plasma guns.[3–5] In most cases the plasma is observed to cross magnetic field lines, though the measurements of electric polarization and plasma transport are not very reproducible. Although the Rayleigh–Taylor instability has previously been considered to explain cross-field propagation, an explicit demonstration has not previously been published. In the sense that the instability discussed in this paper is driven by cross-streaming motion along the plasma–vacuum interface, it is similar to the classical flute instability. We believe it to be the driving mechanism for plasma transport in $R \gg a_i$ cross-field injection experiments.

Other plasma gun experiments, like those of Baker and Hammel[13] and Wessel and Robertson,[5] have reported cross-field propagation by means of an $\mathbf{E}\times\mathbf{B}$ drift. The values for the plasma beta ($\beta = 8\pi n_0 M v_0^2/B_0^2$) in these experiments were somewhat less than those in the preceeding cases.[3–5] In the Baker and Hammel experiment, $\mathbf{E}\times\mathbf{B}$ drifts were detected for $\beta \simeq 0.2$, while in the Wessel and Robertson experiment these drifts were detected only in the range of $\beta \simeq 0.01$. (For $\beta \simeq 0.1$, this latter experiment observed oscillating electric fields on a floating potential probe which suggested flute-like propagation across the magnetic field in accordance with the theory presented here.)

From previous theories[14,15] it is not clear how a small-gyroradius plasma beam ($R \gg a_i$) can $\mathbf{E}\times\mathbf{B}$ drift through a magnetic field. Such a mechanism has been demonstrated theoretically[14] for a large-gyroradius plasma beam ($R \lesssim a_i$), but for the case $R \gg a_i$ the role of charge polarization on the beam boundary should not be an important effect. Looking at it another way, $\mathbf{E}\times\mathbf{B}$ drift motion can only occur if the maximum potential difference developed across the beam does not exceed the initial energy of the ions, $qE_y(2R) > \frac{1}{2}Mu_0^2$. If the plasma drifts through the field with its initial forward velocity $u_0 = v_d = cE_y/B_0$, this inequality becomes[15]

$$R < \tfrac{1}{4}a_i.$$

We can interpret this as a condition of the beam radius R for a plasma to $\mathbf{E}\times\mathbf{B}$ drift through a magnetic field. For a

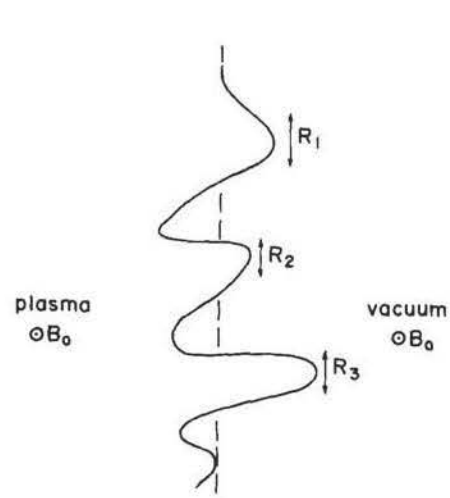

FIG. 4. Nonlinear evolution of a flute-type instability occurring on the plasma–vacuum interface. Each "beamlet" has a radius $R_n < \frac{1}{4}a_i$, and can propagate through the field by means of an $\mathbf{E}\times\mathbf{B}$ plasma drift.

beam with a radius larger than this critical value, Lindberg[15] has suggested that the beam would split up into smaller "beamlets," each of radius $R_n < \frac{1}{4}a_i$. In this case, each of these small-radius beams could then drift across the field (Fig. 4). Such a phenomenon may be due to the formation of an instability on the beam–vacuum interface, whose nonlinear evolution causes beam disintegration into smaller beamlets. In such a case, experiments may detect $\mathbf{E}\times\mathbf{B}$ drift motion of one of these individual beamlets instead of flute-type behavior of the beam as a whole. This may account for the observation of $\mathbf{E}\times\mathbf{B}$-type plasma motion in these experiments.

To summarize, we have shown that the Ferraro–Rosenbluth sheath is unstable. This probably accounts for the fact that a sheath with the predicted properties has not been observed.

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