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Properties of Modulated and Demodulated Systems with Implications to Feedback Limitations

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Abstract

It is well known that the poles, zeros and delay of a system play an important role in determining the associated feedback performance limitations. In this paper, we first derive an approximate transfer function for a modulated and demodulated system of a particular form. We next analyse the behaviour of the poles, zeros, and delay of this transfer function when the modulation frequency is varied. Some implications of these results are also briefly discussed.

Key words: Performance limitations, performance trade-offs, poles and zeros, modulated and demodulated systems

1 Introduction

Modulated and demodulated control systems are found in certain specific applications. An early example of a modulated control system is the 'envelope feedback for a radio frequency transmitter' discussed in [3, Sect. 19.3]. More recent examples of modulated and demodulated systems include vibratory microgyroscopes, such as those described in [5], and rotating gravity gradiometers [3]. Modulated control can also be used to damp vibrations of flexible structures [10].

The above examples provide motivation for the study of modulated control systems. The microgyro system [5], in particular, provided the original motivation for this paper. The drive control loop of this system uses an automatic gain controller (AGC) to maintain an oscillation at the resonance frequency of the device. In [9], it was shown that the AGC essentially consists of a demodulator, a nonlinear amplitude controller, and a modulator. It was observed in [5] that, with this scheme, it is possible to regulate the relatively fast oscillation despite a large time delay. An alternative explanation (to that given in [5]) for this apparent paradox is given in Sect. 5 of the current paper.

In [10], the application of modulated control to vibration damping in flexible structures is considered. The control scheme relies on the fact that the response of a flexible structure can be viewed as a sum of modulated signals, corresponding to the resonant modes of the structure. Thus, each resonant mode can be controlled by first demodulating the response, passing the low frequency signal through a (baseband) controller, and then modulating the resulting signal back up to the resonance frequency. The advantage of this approach is that it allows the wide bandwidth flexible structure to be controlled with a bank of low bandwidth baseband controllers.

Motivated by the above applications, the goal of the current paper is to gain an understanding of the properties of modulated systems and to lay a foundation for understanding the associated design trade-offs.

Thus, we consider a modulated and demodulated system of the type shown in Fig. 1. In this figure, $G(s)$ denotes the transfer function of a linear system and $d_0(t)$ represents an output disturbance. The input to $G(s)$ is $\cos \omega_0 t$ modulated (i.e., multiplied) by $u(t)$. The output is demodulated by correlating it with $\cos(\omega_0 t + \phi)$ (where ϕ is an appropriate phase shift) and passing the resulting signal through a low pass filter $F(s)$. We refer to $G(s)$ as the base system.

In this paper, we first derive a (approximate) transfer function for the system in Fig. 1. We then analyse the behaviour of the poles, zeros and delay of this transfer function as ω_0 is varied. One of the motivations for this study is the fact that the closed loop performance limitations for a linear system are, to a large extent, determined by its poles, zeros and delay. In particular, we refer to the time and frequency domain integral constraints discussed in detail in [11] and [13].

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[3] See, for example, [7], [2]. A description of the Bell rotating gradiometer can also be found at http://www.bellgeo.com under the heading 'FTG'.

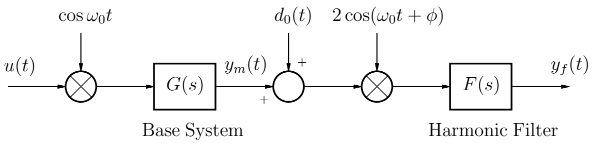

Fig. 1. Block diagram of modulated and demodulated system

The structure of the remainder of the paper is as follows: In Sect. 2, we derive the approximate transfer function. Then, in Sect. 3, we provide example time responses. Sect. 4 contains the main results of the paper. This is followed by some brief comments on the implications of these results on feedback performance trade-offs (Sect. 5). Sect. 6 concludes the paper.

1.1 Notation

In this paper, $\arg z$ denotes the argument and $\operatorname{Arg} z$ denotes the principal argument of z. Thus, $-\pi < \operatorname{Arg} z \leq \pi$. $f(x_0^+)$ is used to denote $\lim_{x \to x_0^+} f(x)$. $f(x_0^-)$ is defined similarly. Upper case is often used to denote the Laplace transform of a signal.

2 General System Description

We return to the modulated and demodulated system shown in Fig. 1. We note that ϕ is a function of ω_0 defined by $\phi(\omega_0) = \operatorname{Arg}[G(j\omega_0)]$. However, we omit the argument of ϕ when it is clear from the context.

The following assumptions are made:

Assumptions

(1) $u(t)$ is a band-limited signal having bandwidth ω_b rad/s (by this we mean that $|U(j\omega)|$ is small for $\omega > \omega_b$).
(2) $\omega_0 > \omega_b$.
(3) $j\omega_0$ is not a pole or zero of $G(s)$ (i.e., $\phi(\omega_0)$ is well defined).
(4) $F(s)$ is a low pass filter which rolls off between ω_b and $2\omega_0 - \omega_b$.

Note that the role of $F(s)$ is to significantly reduce the demodulated output components appearing at the base frequencies shifted by $2\omega_0$ relative to the base frequencies.

For any u which stabilises the modulated system, it is readily seen that $Y_f(s)$ is given by

$$F(s)\Big[G_m(s,\omega_0)U(s) + \tfrac{1}{2}\left[e^{-j\phi}G(s+j\omega_0)U(s+2j\omega_0)\right.$$
$$\left. + e^{+j\phi}G(s-j\omega_0)U(s-2j\omega_0)\right]\Big] + D_f(s),$$

where

$$G_m(s,\omega_0) = \tfrac{1}{2}\left(e^{-j\phi}G(s+j\omega_0) + e^{j\phi}G(s-j\omega_0)\right)$$

and $D_f(s) = (e^{-j\phi}D_0(s+j\omega_0) + e^{j\phi}D_0(s-j\omega_0))F(s)$.

Assumptions 1, 2 and 4 imply that $F(j\omega)U(j\omega \pm 2j\omega_0) \approx 0$, and so we can safely approximate the output response as

$$y_f(t) \approx \mathscr{L}^{-1}\{U(s)G_m(s,\omega_0)F(s) + D_f(s)\}.$$

It follows that the modulated system has an approximate transfer function of $G_m(s,\omega_0)F(s)$ [4]. It is clear that the fidelity of this model for the modulated system will depend on the fidelity of the base system model G at the frequencies between $\omega_0 - \omega_b$ and $\omega_0 + \omega_b$ (i.e., the baseband shifted by ω_0).

The following example clarifies the relationship between y_m, y_f and G_m.

3 Example Time Responses

Consider the following base system:

$$G(s) = \frac{s-1}{(s+5)(s+10)}.$$

Suppose that this system is modulated at $\omega_0 = 7$ rad/s. Then

$$G_m(s,7) = \frac{\cos(0.15)(s^2+1.71s+66.94)(s+11.22)}{(s^2+10s+74)(s^2+20s+149)}.$$

Let $F(s) = \dfrac{89.13}{s^4+8.03s^3+32.23s^2+75.80s+89.13}$ and $U(s) = F(s)/s$. Note that $F(s)$ is a 4th order Butterworth filter with a bandwidth of approx. 3.1 rad/s.

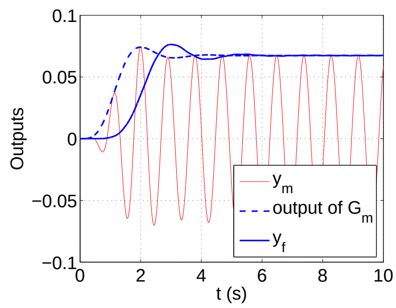

Fig. 2. Step responses

Fig. 2 contains plots of the modulated output $y_m(t)$ and the filtered output $y_f(t)$ the system in Fig. 1. The output of G_m (i.e. $\mathscr{L}^{-1}\{G_m(s,7)U(s)\}$) is also shown. It can be seen that the output of G_m is the envelope of $y_m(t)$, and that $y_f(t)$ is an approximation of the envelope filtered by $F(s)$. We note that the 'delay' observed in $y_f(t)$ relative to the output of G_m is due to the phase shift of the low pass filter.

4 Poles, Zeros, and Delays

In this section, we analyse the behaviour of the poles, zeros and delay of $G_m(s,\omega_0)$ as functions of the modulation frequency ω_0. We note that we have omitted some of

[4] We note that results which are similar or equivalent to this may be found in the literature (e.g., [8] or [4]).

the proofs and provided condensed versions of the rest. A complete set of proofs can be found in Sect. 4 of [9].

Suppose that $G(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are polynomials with real coefficients. We assume that $N(s)$ and $D(s)$ are coprime and can be written as $N(s) = \prod_{i=1}^{m}(s-z_i)$ and $D(s) = \prod_{i=1}^{n}(s-p_i)$, where $z_i,\ p_i \in \mathbf{C}$ and $\mathrm{Re}\,[z_i] \neq 0$. We also assume that $r = n - m > 0$, i.e., that $G(s)$ is strictly proper. Then

$$G_m(s,\omega_0) = \frac{1}{2}\frac{N_m(s,\omega_0)}{D_m(s,\omega_0)}, \qquad (1)$$

$$\text{where}\quad N_m(s,\omega_0) = e^{-j\phi}N(s+j\omega_0)D(s-j\omega_0) + e^{j\phi}N(s-j\omega_0)D(s+j\omega_0), \qquad (2)$$

$$\text{and}\quad D_m(s,\omega_0) = D(s+j\omega_0)D(s-j\omega_0). \qquad (3)$$

We note that $N_m(s,\omega_0)$ and $D_m(s,\omega_0)$ may have common factors for some modulation frequencies. However, we will show that this occurs only at isolated values of ω_0. Hence, the zeros of $D_m(s,\omega_0)$ will, in the sequel, be referred to as the *poles of the modulated system* (or of $G_m(s,\omega_0)$). Similarly, the zeros of $N_m(s,\omega_0)$ will be referred to as the *zeros of* $G_m(s,\omega_0)$.

4.1 Poles

An immediate consequence of (3) is the following:

Lemma 4.1 *For each $\omega_0 \in \mathbf{R}$, the zeros of $D_m(s,\omega_0)$ are given by $s = p_i \pm j\omega_0$ for $i = 1, ..., n$.*

Remark 1 We thus see that the poles of the transfer function $G_m(s,\omega_0)$ are simply shifted forms of the poles of $G(s)$. This is a straightforward connection. □

4.2 Zeros

Determining the zeros of $N_m(s,\omega_0)$ is, in general, more difficult. [5] We can, however, gain some insight into the location of the zeros by analysing the limiting behaviour of the zeros as $\omega_0 \to 0$ and as $\omega_0 \to \infty$. We first note that, for a given ω_0, $N_m(s,\omega_0)$ is a polynomial in s. Thus $N_m(s,\omega_0)$ can be written as

$$N_m(s,\omega_0) = \sum_{i=0}^{n+m} c_i(\omega_0)s^i. \qquad (4)$$

Since $N_m(a,\omega_0)$ is real $\forall\ a \in \mathbf{R}$, the coefficients $c_i(\omega_0)$ are real functions of ω_0. It is also clear that c_i is continuous at $\omega_0 = \omega_1$ if $j\omega_1$ is not a pole or zero of $G(s)$.

Suppose that $\forall\ \omega_0 \in (\omega_1,\omega_2)$, $j\omega_0$ is not a pole or zero of $G(s)$ and the degree of $N_m(s,\omega_0)$ is M. We let the zeros of $N_m(s,\omega_0)$ be denoted by $\zeta_i(\omega_0)$, $i = 1, ..., M$. Then $N_m(s,\omega_0)$ can also be expressed in the following form:

$$N_m(s,\omega_0) = c_M(\omega_0)\prod_{i=1}^{M}(s-\zeta_i(\omega_0)). \qquad (5)$$

Since the coefficients of $N_m(s,\omega_0)$ are continuous on (ω_1,ω_2), the zeros of $N_m(s,\omega_0)$ are also continuous functions of ω_0.

By comparing the coefficients we obtain the following formulae:

$$c_{n+m}(\omega_0) = 2\cos\phi(\omega_0) \qquad (6)$$

$$\text{and}\quad c_{n+m-1}(\omega_0) = -2\omega_0 r\sin\phi(\omega_0) - 2\cos\phi(\omega_0)\left[\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i\right]. \qquad (7)$$

Equations (6) and (7) imply that the degree of $N_m(s,\omega_0)$ will be $n + m - 1$ whenever $|\phi(\omega_0)| = \pi/2$ as stated in the following lemma.

Lemma 4.2 *For each $\omega_0 > 0$, the degree of $N_m(s,\omega_0)$ is*

$$m+n \quad \textit{if}\ |\phi(\omega_0)| \neq \pi/2$$
$$\textit{and}\quad m+n-1 \quad \textit{if}\ |\phi(\omega_0)| = \pi/2.$$

The following lemma describes the behaviour of ζ_i, $i = 1, ..., M$ as $\omega_0 \to \omega_1^+$. We note that the lemma is stated for the case of $\omega_0 \to \omega_1^+$ but clearly also holds for the case of $\omega_0 \to \omega_1^-$.

Lemma 4.3 *Consider the polynomial (in s) defined by (4). Let M be the degree of $N_m(s,\omega_0)$ as ω_0 approaches ω_1 from above. Suppose that $c_i(\omega_1^+)$ is finite $\forall\ i$ and let $M' \leq M$ be the degree of $N_m(s,\omega_1^+)$. Then as $\omega_0 \to \omega_1^+$, M' of the zeros of $N_m(s,\omega_0)$ tend to the zeros of $N_m(s,\omega_1^+)$. If $M - M' = 1$, then the remaining zero tends to ∞ or $-\infty$.*

Proof Outline

The first part of the result (regarding the first M' zeros) follows from Rouche's theorem. The second part follows from the fact that M' of the zeros tend to finite locations and that

$$\lim_{\omega_0\to 0^+}\left|\sum_{i=1}^{M}\zeta_i(\omega_0)\right| = \lim_{\omega_0\to 0^+}\left|\frac{c_{n+m-1}(\omega_0)}{c_{n+m}(\omega_0)}\right| = \infty. \qquad (8)$$

□

Lems. 4.2 and 4.3 imply that if $|\phi(\omega_1)| = \pi/2$, then $n + m - 1$ of the zeros are continuous at $\omega_0 = \omega_1$ and the remaining zero tends to ∞ or $-\infty$ as $\omega_0 \to \omega_1^+$ or ω_1^-. We also note that if $G(s)$ has a pole or zero of multiplicity m_1 at $j\omega_1$, then $c_i(\omega_1^+) = -c_i(\omega_1^-)$ if m_1 is odd and $c_i(\omega_1^+) = c_i(\omega_1^-)$ if m_1 is even. Thus, provided that $c_M(\omega_1^+) \neq 0$, there exist M continuous functions $\zeta_i(\omega_0)$ which satisfy (5) in the neighbourhood of ω_1.

[5] We note that the system in Fig. 1 is a periodic system. Hence, the relative degree of the system can be determined from [6, Def. 3]. However, the results in [6] on computing zeros cannot be applied to this system because it does not have a uniform relative degree.

We are now in a position to present two important results on the zero loci of the modulated system. These describe the behaviour of the zeros as $\omega_0 \to 0$ and as $\omega_0 \to \infty$, respectively.

Theorem 4.4 *(a) Let $\omega_1 > 0$ be chosen s.t. $N_m(s,\omega_0)$ has degree M on $(0,\omega_1)$. Let μ be the number of singularities (i.e., poles or zeros of $G(s)$) at the origin, and let the sets of zeros and poles of $G(s)$ be denoted by $\mathcal{Z}_G$ and $\mathcal{P}_G$, respectively. Also let*

$$\begin{aligned}\mathcal{Z}_0 &= \left\{\zeta_i(0^+) : |\zeta_i(0^+)| \neq \infty, i = 1,...,M\right\} \\ \text{and } \mathcal{Z}_1 &= \{z_0 : N_\omega(z_0) = 0\}, \end{aligned} \tag{9}$$

where $N_\omega(s) = \phi'(0^+)N(s)D(s) - N'(s)D(s) + N(s)D'(s)$. Then

$$\mathcal{Z}_0 = \begin{cases} \mathcal{Z}_G \cup \mathcal{P}_G, & \text{if } \mu \text{ is even,} \\ \mathcal{Z}_1, & \text{if } \mu \text{ is odd.} \end{cases}$$

(b) Suppose that μ is even, and α is a pole or zero (of $G(s)$) of multiplicity m_α. Let $\zeta_i(0^+) = \alpha$ for $i = 1,...,m_\alpha$ and let ζ_i^+ be defined by

$$\zeta_i^+ = \lim_{\omega_0 \to 0^+} \frac{\zeta_i(\omega_0) - \alpha}{\omega_0}, \quad i = 1,...,m_\alpha.$$

Then for each i, $\exists\ k \in \{1,...m_\alpha\}$ s.t.

$$\zeta_i^+ = \begin{cases} \tan\left(\frac{k\pi}{m_\alpha}\right), & m_\alpha \text{ odd,} \\ \tan\left(\frac{\pi}{2m_\alpha} + \frac{k\pi}{m_\alpha}\right), & m_\alpha \text{ even.} \end{cases} \tag{10}$$

Furthermore, $\zeta_i^+ = \zeta_l^+$ iff $i = l$ (i.e., the ζ_i^+'s are distinct).

Proof Outline

(a) We note that if μ is even, then $\phi(0^+) = 0$ or π. The result then follows from Lem. 4.3.

If μ is odd, then $|\phi(0^+)| = \pi/2$ which implies that $c_k(0^+) = 0\ \forall\ k$. It follows from the definition of the derivative that

$$\lim_{\omega_0 \to 0^+} \frac{N_m(s,\omega_0)}{\omega_0} = \frac{\partial N_m}{\partial \omega_0}(s, 0^+) = 2je^{j\phi(0^+)}N_\omega(s).$$

Let M' be the degree of $N_\omega(s)$. Since the zeros of $N_m(s,\omega_0)/\omega_0$ and $N_m(s,\omega_0)$ are the same $\forall\ \omega_0 > 0$, Lem. 4.3 implies that $\mathcal{Z}_1 \subseteq \mathcal{Z}_0$. It can easily be shown that $M - M' \leq 1$, and hence that $\mathcal{Z}_1 = \mathcal{Z}_0$.

(b) We prove the second part of the theorem for the case of $\alpha \in \{z_i : i = 1,...,m\}$. The proof when α is a pole is similar. Suppose that $\zeta_l(0^+) = \alpha$. Since $N_m(\zeta_l(\omega_0), \omega_0) = 0$,

$$\begin{aligned}&\frac{(\zeta_l(\omega_0) - \alpha - j\omega_0)^{m_\alpha}}{(\zeta_l(\omega_0) - \alpha + j\omega_0)^{m_\alpha}} \\ &\quad = -\frac{e^{-j\phi(\omega_0)}\tilde{N}(\zeta_l(\omega_0) + j\omega_0)D(\zeta_l(\omega_0) - j\omega_0)}{e^{+j\phi(\omega_0)}\tilde{N}(\zeta_l(\omega_0) - j\omega_0)D(\zeta_l(\omega_0) + j\omega_0)},\end{aligned}$$

where $\tilde{N}(s) = \frac{N(s)}{(s-\alpha)^{m_\alpha}}$. From this equation it can be shown that

$$\left(\frac{1 + j\zeta_l^+}{1 - j\zeta_l^+}\right)^{m_\alpha} = (-1)^{m_\alpha + 1}. \tag{11}$$

Equation (11) implies that $|\frac{\zeta_l(\omega_0) - \alpha}{\omega_0}|$ does not $\to \infty$, and hence that ζ_l^+ is real. Equation (11) also implies that

$$2m_\alpha \arg\left(1 + j\zeta_l^+\right) = (m_\alpha + 1)\pi + 2k\pi \tag{12}$$

for some integer k. The result follows by solving for ζ_l^+.

We now show that the ζ_i^+'s are distinct. We first note that as $\omega_0 \to 0^+$, $\phi \to 0$ or π (μ is even), and so $\frac{\partial N_m}{\partial s}(s,\omega_0)$ has $n + m - 1$ zeros. Denote these by $\eta_i(\omega_0)$, $i = 1,...,n + m - 1$. It is clear that $\frac{\partial N_m}{\partial s}(s, 0^+)$ has exactly $m_\alpha - 1$ zeros at α. Therefore, $m_\alpha - 1$ of the η_i's $\to \alpha$. We assume without loss of generality that these are $\eta_1, ...\eta_{m_\alpha - 1}$. Let η_l^+ be defined in a similar manner to ζ_l^+. By using a similar argument to that given above, it can be shown that for each $l \in \{1,...,m_\alpha - 1\}$

$$2(m_\alpha - 1)\arg\left(1 + j\eta_l^+\right) = m_\alpha\pi + 2k\pi \tag{13}$$

for some integer k. We also note that

$$(n + m)\prod_{i=1}^{n+m-1}(s - \eta_i(\omega_0)) = \frac{\partial}{\partial s}\prod_{i=1}^{n+m}(s - \zeta_i(\omega_0)).$$

By letting $s = \zeta_1(\omega_0)$, dividing both sides by $\omega_0^{m_\alpha - 1}$ and taking limits we obtain

$$\begin{aligned}&\lim_{\omega_0 \to 0^+}\left[(n + m)\prod_{i=1}^{m_\alpha - 1}\frac{\Delta_{\eta i}(\omega_0)}{\omega_0}\prod_{i=m_\alpha}^{n+m-1}\Delta_{\eta i}(\omega_0)\right] \\ &\quad = \lim_{\omega_0 \to 0^+}\left[\prod_{i=2}^{m_\alpha}\frac{\Delta_{\zeta i}(\omega_0)}{\omega_0}\prod_{i=m_\alpha+1}^{n+m}\Delta_{\zeta i}(\omega_0)\right],\end{aligned}$$

where $\Delta_{\eta i}(\omega_0) = \zeta_1(\omega_0) - \eta_i(\omega_0)$ and $\Delta_{\zeta i}(\omega_0) = \zeta_1(\omega_0) - \zeta_i(\omega_0)$. Equations (12) and (13) imply that, for $i = 1,...,m_\alpha - 1$, $\lim_{\omega_0 \to 0^+}\frac{\zeta_1(\omega_0) - \eta_i(\omega_0)}{\omega_0} \neq 0$. Since $\frac{\partial N_m}{\partial s}(s, 0^+)$ has only $m_\alpha - 1$ zeros at α, $\zeta_1(0^+) - \eta_i(0^+) \neq 0$ for $i = m_\alpha,...,n + m - 1$. For $i = m_\alpha + 1,...,n + m$, $\zeta_1(0^+) - \zeta_i(0^+)$ is also nonzero and finite. It follows that $\lim_{\omega_0 \to 0^+}\frac{\zeta_1(\omega_0) - \zeta_i(\omega_0)}{\omega_0} \neq 0$ for $i = 2,...,m_\alpha$ and hence that the ζ_i^+'s are distinct. □

Remark 2 Thm. 4.4(b) implies that m_α (μ even) or $m_\alpha - 1$ (μ odd) of the ζ_i^+'s are real and non-zero. It follows that the angle of departure of each of these loci is 0 or π. If μ is odd then there is exactly one value of k s.t. $\zeta_k(0^+) = \alpha$ and $\zeta_k^+ = 0$. If α is real then ζ_k also has an angle of departure of 0 or π because complex zeros must occur in conjugate pairs. □

Next we consider the case $\omega_0 \to \infty$:

Theorem 4.5 *(a) Let $\eta_i(\omega_0) = \zeta_i(\omega_0)/\omega_0$ for $\omega_0 > 0$. As $\omega_0 \to \infty$, $2m$ of the zeros of $N_m(s,\omega_0)$ tend to $z_i + j\omega_0$ and $z_i - j\omega_0$, $i = 1, ..., m$.*
(b) If r is even, then the remaining zeros satisfy the following condition:

$$\lim_{\omega_0\to\infty} \eta_i(\omega_0) = -\tan\left(\frac{\pi}{2r} + \frac{k\pi}{r}\right), \quad k = 0, ..., r-1.$$

If r is odd, then $r-1$ of the remaining zeros satisfy the following condition:

$$\lim_{\omega_0\to\infty} \eta_i(\omega_0) = -\tan\left(\frac{\pi}{2} + \frac{k\pi}{r}\right), \quad k = 1, ..., r-1,$$

and the final η_i tends to ∞ or $-\infty$.

Proof Outline

(a) We observe that for all $\varepsilon > 0$ and all $\theta \in [0, 2\pi)$,

$$\left|\frac{G\left(z_i \pm 2j\omega_0 + \varepsilon e^{j\theta}\right)}{G\left(z_i + \varepsilon e^{j\theta}\right)}\right| \to 0 \text{ as } \omega_0 \to \infty \qquad (14)$$

because $G(s)$ is proper. The result can be proved by combining this observation with Rouche's theorem.

(b) To establish the second part of the theorem we observe that for $\omega_0 > 0$, $\eta_i(\omega_0)$ is a zero of

$$\begin{aligned} P(s,\varepsilon) :=& e^{-j\phi(\omega_0)} \prod_{i=1}^{m}(s + j - \varepsilon z_i) \prod_{i=1}^{n}(s - j - \varepsilon p_i) \\ &+ e^{j\phi(\omega_0)} \prod_{i=1}^{m}(s - j - \varepsilon z_i) \prod_{i=1}^{n}(s + j - \varepsilon p_i), \end{aligned}$$

where $\varepsilon = 1/\omega_0$. Let $M' + 2m$ be the degree of $P(s, 0^+)$. Since

$$P(s,0^+) = (s^2+1)^m \left[e^{-j\phi(\infty)}(s-j)^r + e^{j\phi(\infty)}(s+j)^r\right]$$

and $e^{-j\phi(\infty)}/e^{j\phi(\infty)} = (-1)^r$, it can be deduced that $M' = r$ if r is even and $M' = r-1$ if r is odd. It follows from Lem. 4.3 that as $\omega_0 \to \infty$ (or as $\varepsilon \to 0$), $2m$ of the η_i's $\to \pm j$ (these correspond to the $2m$ ζ_i's in part (a)) and M' of the η_i's tend to the (finite) solutions of

$$\left(\frac{s+j}{s-j}\right)^r = -(-1)^r.$$

□

For almost all $\omega_0 > 0$, the zeros and poles of $G_m(s,\omega_0)$ will be the same as the zeros of $N_m(s,\omega_0)$ and $D_m(s,\omega_0)$, respectively. However, at isolated values of ω_0 we may have 'pole-zero' cancellations as stated in the following lemma.

Lemma 4.6 *For each $\omega_1 > 0$, $N_m(s,\omega_1)$ and $D_m(s,\omega_1)$ have a common zero iff $\exists\, k, l \in \{1, ..., n\}$ s.t.*

$$p_l = p_k + 2j\omega_1. \qquad (15)$$

Let m_i denote the multiplicity of p_i for $i = 1, ..., n$. If $\omega_1 > 0$, and condition (15) is satisfied, then $N_m(s,\omega_1)$ has at least $\min\{m_k, m_l\}$ zeros at $p_k + j\omega_1 = p_l - j\omega_1$.

Remark 3 We have thus seen that the zeros of the transfer function $G_m(s,\omega_0)$ are, in general, not simply related to the zeros of $G(s)$. However, Thm. 4.5 shows that for large ω_0 (relative to the location of the poles of $G(s)$), the zeros of $G_m(s,\omega_0)$ approach the shifted forms of the zeros of $G(s)$ together with some extra zeros which converge to specific asymptotes. □

Remark 4 The situation described in Remark 3, and formalised by Thm. 4.5, is reminiscent of the zeros of unmodulated sampled data systems having zero order hold input. We recall that, when expressed in the equivalent delta domain [12], the zeros of these systems tend, as the sampling rate is increased, to the zeros of the underlying continuous time system, together with some extra zeros (sometimes called the sampling zeros) which converge to specific locations ([1], [12]). □

We illustrate the above results with an example.

Example 1

Consider the following base system:

$$G(s) = \frac{(s-0.5)(s+1)}{(s+0.5)(s+2)^2}.$$

Notice that this system has zeros at 0.5 and -1. For this example, we have $\phi(0) = -\pi$ and $|\phi(\omega_0)| = \pi/2$ has one solution at $\omega_0 = \omega_x \approx 0.4885$. $G(s)$ has no poles or zeros at the origin and hence μ is even. It follows that the loci of the zeros of $G_m(s,\omega_0)$ start at the poles and zeros of $G(s)$ and as $\omega_0 \to \infty$, two of the loci tend to $0.5 \pm j\omega_0$ and two tend to $-1 \pm j\omega_0$. The relative degree is one ($r = 1$), and so the remaining zero tends to $-\infty$.

The loci (in the complex plane) of the five zeros of $N_m(s,\omega_0)$ are shown in Fig. 3(a). The arrows indicate the direction of increasing ω_0. The real and imaginary parts of the zeros are plotted against ω_0 in Figs. 3(b) and 3(c). The zeros are labelled ζ_i, $i = 1, ..., 5$. As predicted by Thm. 4.5 (see also Remark 3), ζ_3, ζ_4 converge to $0.5 \pm j\omega_0$ and ζ_1, ζ_2 converge to $-1 \pm j\omega_0$. Note that ζ_4 and ζ_5 both begin at -2. From Figs. 3(b) and 3(c) it can be seen that $\zeta_5 \to -\infty$ along the real axis. ζ_4 is discontinuous and tends to $-\infty$ and ∞ as $\omega_0 \to \omega_x$ from below and above, respectively. Hence, in Fig. 3(a), ζ_4 is in two parts and the part on the left overlaps ζ_5.

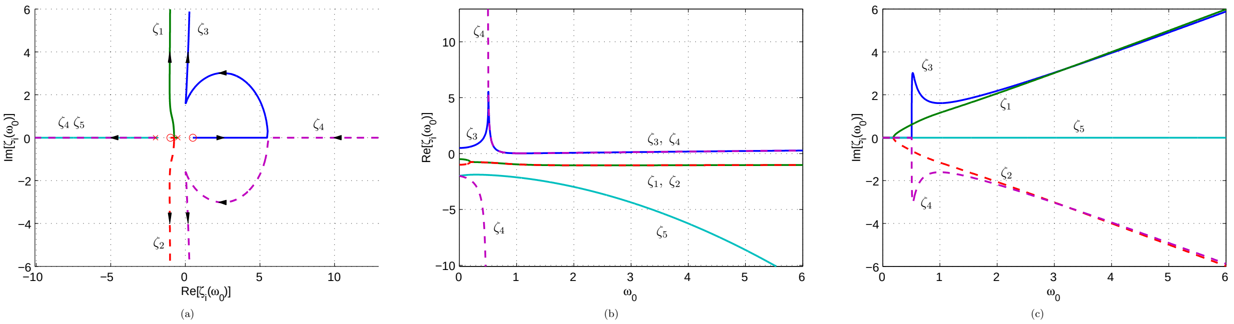

Fig. 3. Example 1 - (a) Loci of the zeros (b) real part of the loci (c) imaginary part of the loci

We also observe that the poles and zeros of $G(s)$ are all real, and so the angles of departure of the loci are all 0 or π (i.e., they depart along the real axis). It is also possible to verify that ζ_i, $i = 1, ..., 5$ have initial slopes of 0 , 0 , 0, $\tan(\pi/4) = 1$ and $\tan(3\pi/4) = -1$, respectively (Thm. 4.4). □

4.3 Delays

We next consider the impact of delays in the base system. The following lemma states that if a linear system is modulated and demodulated, then the delay is preserved.

Lemma 4.7 *Suppose that* $\tilde{G}(s) = e^{-s\tau}G(s), \quad \tau > 0.$ *Then* $\tilde{G}_m(s, \omega_0) = e^{-s\tau}G_m(s, \omega_0)$.

Proof

The proof follows from the definition of $\tilde{G}_m$ and the observation that $\tilde{\phi}(\omega_0) = \phi(\omega_0) - j\omega_0\tau$. □

4.4 Summary

In this section, we have shown that the poles of $G_m(s, \omega_0)$ are given by $p_i \pm j\omega_0$. The behaviour of the loci of the zeros is more complex. It was found that the loci are continuous (on $\mathbf{R}^+$) except at points where $|\phi(\omega_0)|$ crosses (or touches) $\pi/2$. At these points, one of the zeros 'vanishes' and the rest are continuous. As $\omega_0 \to 0$ the zeros tend to the poles and zeros of $G(s)$ (when μ is even) or the zeros of $N_\omega(s)$ (when μ odd). For large ω_0 (relative to the location of the poles and zeros of $G(s)$), $2m$ of the zeros tend to $z_i + j\omega_0$ and $z_i - j\omega_0$ and the remaining zeros tend to ∞ or $-\infty$. Finally, it was shown that the delay of a system is invariant with respect to modulation and demodulation.

5 Implications on Feedback Performance Limitations.

Once the poles, zeros and delay of $G_m(s, \omega_0)$ are known the feedback performance limitations for the modulated system can be found by applying linear system results [9]. The following observations are particularly interesting:

(1) Since the delay is preserved by modulation, the delay limits the closed loop bandwidth of the response at y_f, not the response at y_m. In particular, the speed of the oscillation at $y_m(t)$ (or the modulation frequency) is not limited by the delay. This explains the paradox referred to in the introduction, that in the microgyro system it is possible to regulate the relatively fast oscillation ($\approx$ 4.5 kHz) despite the large delay.

(2) Theorem 4.4 states that if μ is even, then the zeros of $G_m(s, \omega_0)$ tend to the poles and zeros of $G(s)$ as $\omega_0 \to 0$. It follows that if $G(s)$ has an unstable pole, then $G(s, \omega_0)$ will have an approximate pole-zero cancellation in the open right half plane (ORHP) when ω_0 is small.[6] This implies that large peaks in the closed loop sensitivity functions will be unavoidable as the right hand sides (RHS) of the Poisson Integrals for $S_{\omega_0}(s)$ and $T_{\omega_0}(s)$ will be large [13, Thms. 3.3.1 and 3.3.2]. We note that an approximate pole-zero cancellation also occurs when ω_0 is close to the resonant frequency of a conjugate pair of poles of $G(s)$ (Lem. 4.6).

6 Conclusion

In this paper, the poles, zeros and delays of modulated and demodulated systems have been analysed. It has been shown that the poles of G_m are the poles of G shifted by $\pm j\omega_0$ and that the delay is preserved. Several results on the continuity and asymptotic behaviour of the zero loci have also been given. Some implications of these results were also briefly discussed.

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