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Transport Equations for the QCD Gluon Wigner Operator

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Abstract:

We define a gauge covariant Wigner Operator for SU(N) gauge fields and derive its quantum transport equation. In the semiclassical limit we eliminate a mean field and derive a gauge covariant Vlasov equation with non-Abelian modifications for gluon fluctuations. For slowly varying mean field we decompose it into N(N-1) charged gluon equations and N-1 for neutral ones, which decouple.

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Ultra-relativistic nuclear collision experiments are soon to be realized at CERN and BNL with the goal of studying quark-gluon plasmas [1,2,3]. In order to understand properties of this new phase of matter as predicted by QCD it is necessary to investigate their relation to the variety of proposed experimental signatures. In particular the time evolution of the plasma from its initial formation towards thermal and chemical equilibrium has to be carefully studied. This can be achieved by a transport theory [4,5,6] for quarks and gluons which, when suitably generalized, incorporates the strong interactions as described by the SU(3)-color gauge theory [7,8,9].

Recently in ref.[9] (hereafter referred to as I) we studied the relativistic non-Abelian transport theory for spin-1/2 quarks interacting via SU(N) gauge fields. We showed how gauge covariant quantum transport equations can be derived from Dirac's equation rigorously. Furthermore, having for example an application to the semiclassical flux tube picture of plasma formation in mind (see ref.[10] and references therein), we developed a systematic expansion procedure for the transport equations. Thus we derived previously known classical transport equations [7,8,11] from field theory and calculated quantum corrections in a transparent way.

The purpose of this paper is to complement I in an important aspect: The gauge field was treated as an operator in I, but earlier work and our previous study were aiming at a mean field description of the system. Thus for practical applications the quarks are considered to interact with each other via an external or self-consistently generated mean field (for studies of Abelian or QED plasmas see refs.[4,6,12] and references therein). However, for non-Abelian gauge theories and especially in application to the quark-gluon plasma this approximation is insufficient. Even if there were no quarks, one encounters the possibility here that gluons are spontaneously created from the classical background field. Thus there are gluons in addition to quarks which interact with that field and help to ultimately neutralize it [10,13].

Therefore, we have to introduce a proper kinetic description for the gluons, which allows us to study the fluctuating part of the field separately from its mean field behavior in a coherent state. In the following we define a *gauge covariant Wigner operator for the gluon field* and derive its quantum transport equation. We will also discuss an appropriate semiclassical limit and show how a non-Abelian Vlasov type equation arises for gluon fluctuations. In this way we complete the outline of a consistent quark-gluon plasma transport theory which was begun in I.

Previous development of transport theory in this direction was hindered by the fact that no gauge covariant gluon (or photon) distribution function was known. Since gauge covariance, however, was a strong guiding principle in our derivation of the transport theory for quarks and provided insight in how to approximate the operator equations semiclassically, we want to keep it rather than to study an object like the gluon number operator in a specified gauge. This motivates the particular definition of a gauge covariant gluon Wigner operator proposed in eq.(3) below.

Our notation follows that of I. We use the metric $g^{\mu\nu} \equiv diag(1,-1,-1,-1)$, $a \cdot b \equiv a_\mu b^\mu$, and choose units such that $\hbar = c = 1$. The gauge field potential is defined by $A_\mu \equiv A^j_\mu t_j$, with the $N^2 - 1$ Hermitian generators of SU(N) in the fundamental representation satisfying $Tr\ t_j = 0$, $Tr\ t_i t_j = \delta_{ij}/2$, and $[t_i, t_j] = i f_{ijk} t_k$. The covariant derivative $D_\mu \equiv \partial_\mu + igA_\mu$ defines the field strength tensor, $F_{\mu\nu} \equiv [D_\mu, D_\nu]/(ig)$, which obeys the field equation

$$g^{-1} \times [D_\mu, F^{\mu\nu}] = J^\nu \equiv t_j \bar{\psi}\gamma^\nu t_j \psi = \int d^4p\, t_a\, Tr\ \gamma^\mu t_a \hat{W}(x,p) \ , \tag{1}$$

where J is the quark color current operator, the trace refers to spinor and color indices, and where $\hat{W}$ is the *gauge covariant Wigner operator for spin-1/2 quarks* from I:

$$\hat{W}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\ \bar{\psi}(x) e^{\frac{1}{2}y\cdot D^\dagger_x} \otimes e^{-\frac{1}{2}y\cdot D_x}\psi(x) \ . \tag{2}$$

Note that $Tr\hat{W}(x,p) = \bar{\psi}(x)\delta^4(p - iD_x)\psi(x)$, which shows the connection to a classical phase-space distribution, with iD_x corresponding to the kinetic momentum operator. Thus, $Tr\hat{W}$ measures the Lorentz scalar density of quarks with kinetic momentum p.

Similarly we define the *gauge covariant Wigner operator for spin-1 gluons* in terms of the field operators,

$$\hat{\Gamma}_{\mu\nu}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} [e^{\frac{1}{2}y\cdot \mathcal{D}(x)} F_\mu^{\ \lambda}(x)] \otimes [e^{-\frac{1}{2}y\cdot\mathcal{D}(x)} F_{\lambda\nu}(x)] \ , \tag{3}$$

where the covariant derivative of a second-rank tensor $\mathcal{T}$ is defined by

$$\mathcal{D}(x)\mathcal{T}(x) \equiv \partial_x \mathcal{T}(x) + ig[A(x), \mathcal{T}(x)] \ . \tag{4}$$

Under local gauge transformations F, $\mathcal{D}F$, and so $\hat{\Gamma}$ transform covariantly. In eq.(3) we suppressed four color indices of $\hat{\Gamma}$ but explicitly indicated its tensor structure. $\hat{\Gamma}$ is closely related to the energy-momentum tensor of the field,

$$\hat{T}_{\mu\nu}(x) \ \equiv\ Tr\left(F_\mu^{\ \lambda} F_{\lambda\nu} + \tfrac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau}\right) \ =\ Tr\int d^4p \left(\hat{\Gamma}_{\mu\nu}(x,p) - \tfrac{1}{4} g_{\mu\nu} \hat{\Gamma}_\lambda^{\ \lambda}(x,p)\right) \ , \tag{5}$$

where the trace refers to color indices, $Tr A\otimes B \equiv A_{ab}B_{ba}$. Eq.(5) provides the connection between $\hat{\Gamma}$ and observables of the gauge field. Note that $Tr\hat{\Gamma}_{\mu\nu}(x,p) = F_\mu^{\ \lambda}(x)\delta^4(p - i\mathcal{D}(x))F^\lambda_{\ \nu}(x)$ in analogy to $Tr\hat{W}$, which measures the energy-momentum flux of gluons with kinetic momentum p.

As shown in I,

$$e^{-y\cdot\mathcal{D}(x)}\ \mathcal{T}(x) \ =\ U(x, x-y)\mathcal{T}(x-y)U(x-y, x) \ , \tag{6}$$

where a link operator U is a path ordered exponential of a line integral [14],

$$U(b,a) \equiv P\exp\left(-ig\int_a^b dz^\mu A^j_\mu(z) t_j\right) \ , \tag{7}$$

and the path of integration is the *straight line* between the end points,

$$z(s) \equiv z(b,a,s) \equiv a + (b-a)s \quad , \; 0 \leq s \leq 1 \; . \tag{8}$$

Thus we obtain from eqs.(3,6)

$$\hat{\Gamma}_{\mu\nu}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} U(x,x_2) F_\mu^{\;\lambda}(x_2) U(x_2,x) \otimes U(x,x_1) F_{\lambda\nu}(x_1) U(x_1,x) \quad , \tag{9}$$

where $x_2 \equiv x + \frac{1}{2}y$, $x_1 \equiv x - \frac{1}{2}y$. Note that $\hat{\Gamma}^\dagger_{\mu\nu} = \hat{\Gamma}_{\nu\mu}$, since the link operators are unitary. The implicit appearance of straight line paths in eq.(9) is related, as we noted in I, to the requirement that p have the physical interpretation of kinetic momentum.

Properties of link operators were studied in detail in I and were essential in our derivation of transport equations for the quark Wigner operator. The basic relation from I, which we need here also, is:

$$\begin{aligned} \delta U(b,a) &= -igA(b)\cdot db\, U(b,a) + igU(b,a)A(a)\cdot da \\ &\quad + ig\int_0^1 ds\, U(b,z(s))F_{\mu\nu}(z(s))U(z(s),a)(b-a)^\mu(da+(db-da)s)^\nu \; . \end{aligned} \tag{10}$$

Eq.(10) expresses the first-order change of a link operator due to infinitesimal shifts of the endpoints of its path of integration. It allows us to calculate derivatives of link operators with respect to one or both endpoints.

After these preparations we sketch the derivation of the quantum transport equation for $\hat{\Gamma}$ proceeding in several steps in parallel to I:

1. We want to calculate $p\cdot\tilde{D}\hat{\Gamma}_{\mu\nu}$, where $\tilde{D}$ is defined by

$$\tilde{D} A\otimes B \equiv [DA]\otimes B + A\otimes[DB] \quad , \tag{11}$$

since this is the gauge covariant generalization of $p\cdot\partial_x\hat{\Gamma}$, which would arise in the absence of interactions.

2. Representing $\hat{\Gamma}$ as in eq.(9), we carry $p\cdot\tilde{D}$ into the integrand and eventually convert $p \to -i\partial_y$ (by partial integration) and $y \to i\partial_p$, if we want to pull ∂_p out of the integral (cf. I).

3. We calculate all necessary derivatives of link operators which occur in the integrand by repeatedly applying eq.(10). These are straightforward though tedious manipulations. To help keeping track of the ordering of terms, it is useful to adopt the notation:

$$\begin{aligned} \hat{O}A\otimes B &\equiv \hat{O}_{ab}A_{bc}B_{de} \; , \; A\otimes B\hat{O} \equiv A_{ab}B_{cd}\hat{O}_{de} \; , \\ \hat{O}_R A\otimes B &\equiv A_{ab}\hat{O}_{bc}B_{de} \; , \; A\otimes B\hat{O}_L \equiv A_{ab}\hat{O}_{cd}B_{de} \; , \end{aligned} \tag{12}$$

where color indices are inserted for clarity.

4. By applying eq.(6) together with the group property of link operators, $U(a,b)U(b,c) = U(a,c)$, for b along the straight-line path between a and c, we obtain the result:

$$p \cdot \tilde{D}(x)\,\hat{\Gamma}_{\mu\nu}(x,p) = \tag{13}$$

$$+ \ \tfrac{1}{2}gp^{\sigma}\partial_p^{\tau}\int_0^1 ds\left\{[e^{(1-s)\Delta}F_{\tau\sigma} - e^{s\Delta}F_{\tau\sigma R}]\hat{\Gamma}_{\mu\nu} \ + \ \hat{\Gamma}_{\mu\nu}[e^{-s\Delta}F_{\tau\sigma} - e^{(s-1)\Delta}F_{\tau\sigma L}]\right\}$$

$$+ \ \tfrac{1}{4}ig\partial_p^{\sigma}\int_0^1 ds\left\{[(s-1)e^{(1-s)\Delta}F_{\sigma\tau} + se^{s\Delta}F_{\sigma\tau R}][\tilde{D}^{\tau}\hat{\Gamma}_{\mu\nu}]\right.$$

$$\left. + \ [\tilde{D}^{\tau}\hat{\Gamma}_{\mu\nu}][se^{-s\Delta}F_{\sigma\tau} + (s-1)e^{(s-1)\Delta}F_{\sigma\tau L}]\right\}$$

$$- \ \tfrac{1}{8}ig^2\partial_p^{\sigma}\partial_p^{\eta}\int_0^1 ds\Big(\ [(s-1)e^{(1-s)\Delta}F_{\sigma\tau} + se^{s\Delta}F_{\sigma\tau R}]$$

$$\times \int_0^1 d\tilde{s}\left\{[e^{(1-\tilde{s})\Delta}F_{\eta}^{\ \tau} - e^{\tilde{s}\Delta}F_{\eta R}^{\ \tau}]\hat{\Gamma}_{\mu\nu} \ + \ \hat{\Gamma}_{\mu\nu}[e^{-\tilde{s}\Delta}F_{\eta}^{\ \tau} - e^{(\tilde{s}-1)\Delta}F_{\eta L}^{\ \tau}]\right\}$$

$$+ \int_0^1 d\tilde{s}\left\{[e^{(1-\tilde{s})\Delta}F_{\eta}^{\ \tau} - e^{\tilde{s}\Delta}F_{\eta R}^{\ \tau}]\hat{\Gamma}_{\mu\nu} \ + \ \hat{\Gamma}_{\mu\nu}[e^{-\tilde{s}\Delta}F_{\eta}^{\ \tau} - e^{(\tilde{s}-1)\Delta}F_{\eta L}^{\ \tau}]\right\}$$

$$\times \ [se^{-s\Delta}F_{\sigma\tau} + (s-1)e^{(s-1)\Delta}F_{\sigma\tau L}] \ \Big)$$

$$- \ \tfrac{1}{2}i\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\{[e^{\frac{1}{2}y\cdot D}D^2F_{\mu}^{\ \lambda}]\otimes[e^{-\frac{1}{2}y\cdot D}F_{\lambda\nu}] \ - \ [e^{\frac{1}{2}y\cdot D}F_{\mu}^{\ \lambda}]\otimes[e^{-\frac{1}{2}y\cdot D}D^2F_{\lambda\nu}]\right\} \ ,$$

where $\Delta \equiv \frac{1}{2}i\partial_p \cdot D$, $D \equiv D(x)$, $F \equiv F(x)$, $\hat{\Gamma} \equiv \hat{\Gamma}(x,p)$, and where D always acts within brackets [...].

5. The final step to convert eq.(13) into a dynamical equation for $\hat{\Gamma}$ consists in evaluating D^2F according to the quadratic Yang-Mills equation, which follows from eq.(1):

$$g^{-1}D^2F_{\mu\nu} \ = \ D_{\mu}J_{\nu} - D_{\nu}J_{\mu} \ - \ 2i\,[F_{\mu\sigma}, F_{\ \nu}^{\sigma}] \ . \tag{14}$$

Then, by the definition of the Wigner operator, eq.(3), the D^2F-terms in eq.(13) can be expressed as

$$g^{-1}\times\int ...D^2F..._{\mu\nu} \ = \tag{15}$$

$$-\tfrac{1}{2}i\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\{[e^{\frac{1}{2}y\cdot D}(D_{\mu}J^{\lambda} - D^{\lambda}J_{\mu} - 2iTr'\int d^4\tilde{p}\{\hat{\Gamma}_{\mu}^{\ \lambda} - \hat{\Gamma}^{\lambda}_{\ \mu}\})]\otimes[e^{-\frac{1}{2}y\cdot D}F_{\lambda\nu}]\right.$$

$$\left. - \ [e^{\frac{1}{2}y\cdot D}F_{\mu}^{\ \lambda}]\otimes[e^{-\frac{1}{2}y\cdot D}(D_{\lambda}J_{\nu} - D_{\nu}J_{\lambda} - 2iTr'\int d^4\tilde{p}\{\hat{\Gamma}_{\lambda\nu} - \hat{\Gamma}_{\nu\lambda}\})]\right\},$$

with $(Tr'A \otimes B)_{ac} \equiv A_{ab}B_{bc}$ so that $(Tr'\hat{\Gamma})_{ac} = \hat{\Gamma}_{abbc}$. Eq.(13) together with eq.(15) constitute the *gauge covariant quantum transport equation for the QCD gluon Wigner operator* as defined in eq.(3).

We remark that besides the proper transport equation for $\hat{\Gamma}$ one expects a second one describing a generalized mass-shell constraint (cf. I). It can be derived by considering $p^2\hat{\Gamma}$ and following similar steps as 1. - 5. above. However, we do not consider it here.

As was shown in I, the *semiclassical limit of transport equations* is described by the lowest-order terms of an expansion in powers of $\triangle \sim \hbar\partial_p\mathcal{D}$ and by replacing the Wigner operator by an ensemble average in the end. However, for gluon fields there is an important intermediate step. We want to separate $\hat{\Gamma}$ into the interesting *incoherent fluctuation part* $\hat{G}$ plus $\bar{\Gamma}$ corresponding to the external or self-consistent mean field $\bar{F}$ representing the *coherent* part of the field,

$$\hat{G}_{\mu\nu} \equiv \hat{\Gamma}_{\mu\nu} - \bar{\Gamma}_{\mu\nu}\ , \ \ a_\mu \equiv A_\mu - \langle A_\mu \rangle \equiv A_\mu - \bar{A}_\mu\ . \tag{16}$$

Here a_μ denotes the quantum fluctuations around the classical potential $\bar{A}_\mu$ and $\bar{\Gamma}_{\mu\nu}$ is defined by eq.(3) with $F \to \bar{F}$ (barred quantities henceforth refer to $\bar{A}$). It is important to realize that $\langle\hat{\Gamma}\rangle \neq \bar{\Gamma}$ and therefore $G \equiv \langle\hat{G}\rangle \neq 0$. The classical mean field is given by (cf. eq.(1))

$$[\bar{D}_\mu, \bar{F}^{\mu\nu}] = g(\langle J^\nu\rangle + \langle j^\nu\rangle)\ , \tag{17}$$

where $\langle j^\nu\rangle$ denotes the average color current caused by the field fluctuations as determined below.

To derive the analog of Vlasov's equation for gluons, i.e. an equation for $G \equiv \langle\hat{G}\rangle$, we first observe that formally $\bar{\Gamma}$ also obeys eq.(13) involving the corresponding barred quantities. Next, we keep only the zeroth-order terms from an expansion in powers of $\triangle$ of both the equations for $\hat{\Gamma}$ and $\bar{\Gamma}$. We insert eqs.(16) into the resulting equation for $\hat{\Gamma}$ and keep only the zeroth-order terms in a_μ; this is equivalent to *linearizing in the fluctuations* a_μ *and neglecting correlations* when an ensemble average is taken with $\langle a_\mu\rangle = 0$. Thus for example $\langle A\hat{G}\rangle \approx \bar{A}G$. Subtracting the mean field equation for $\bar{\Gamma}$ from the equation for $\hat{\Gamma}$ we obtain the approximate gauge covariant transport equation for gluon fluctuations:

$$\begin{aligned}
p\cdot\bar{\bar{\mathcal{D}}}\, G_{\mu\nu} \;=\; & \tfrac{1}{2} g p^\sigma \partial_p^\tau \left\{ \left[\bar{F}_{\tau\sigma}, G_{\mu\nu}\right]_R + \left[G_{\mu\nu}, \bar{F}_{\tau\sigma}\right]_L \right\} \\
& + \tfrac{1}{8} i g \partial_p^\sigma \left\{ \left[\bar{F}^\tau_{\ \sigma}, \bar{\bar{\mathcal{D}}}_\tau G_{\mu\nu}\right]_R - \left[\bar{\bar{\mathcal{D}}}_\tau G_{\mu\nu}, \bar{F}^\tau_{\ \sigma}\right]_L \right\} \\
+ \tfrac{1}{16} i g^2 \partial_p^\sigma \partial_p^\eta & \left\{ \left[\, \bar{F}_{\tau\sigma}\, ,\, \left[\bar{F}^\tau_{\ \eta}, G_{\mu\nu}\right]_R + \left[G_{\mu\nu}, \bar{F}^\tau_{\ \eta}\right]_L \,\right]_R \right. \\
& \left. - \left[\, \left[\bar{F}^\tau_{\ \eta}, G_{\mu\nu}\right]_R + \left[G_{\mu\nu}, \bar{F}^\tau_{\ \eta}\right]_L\, ,\, \bar{F}_{\tau\sigma} \,\right]_L \right\}\ ,
\end{aligned} \tag{18}$$

with the commutators defined by (cf. eqs.(12))

$$\left[\bar{F},G\right]_R \equiv (\bar{F}-\bar{F}_R)G \ , \quad \left[G,\bar{F}\right]_L \equiv G(\bar{F}-\bar{F}_L) \ . \tag{19}$$

Eq.(18) describes a gluon plasma in the so-called collisionless regime. Collision terms arise from considering correlations [5] such as $\langle JF\rangle$ and $\langle\hat{\Gamma}F\rangle$. No source terms from eq.(15) appear in eq.(18) for the average gluon fluctuations G since we require the average gluon current $\langle j\rangle$ to obey

$$\bar{D}_\mu(x)\langle j_\nu(x)\rangle + iTr'\int d^4p\ \{G_{\mu\nu}(x,p)-G_{\nu\mu}(x,p)\} = 0 \ , \tag{20}$$

with $(Tr'G)_{ac} = G_{abbc}$. Then the terms from eq.(15) only contribute to the $\bar{\Gamma}$-equation. We consider a particular case of eq.(20) below. Eqs.(17,18,20) together form a consistent set of gauge covariant equations for the study of gluon fluctuations under the influence of a classical color field. Note that we never have to solve the $\bar{\Gamma}$-equation, since eq.(17) determines the classical field. Of course, the current $\langle J\rangle$ in eq.(17) has to be determined in the quark sector of transport theory developed in I. Corrections to the above equations in powers of $\triangle \sim \hbar\partial_p D$ can be calculated systematically from eqs.(13,15).

Finally, we illustrate the simplifications which occur under the assumption that the *classical field $\bar{F}$ and potential $\bar{A}$ both can be diagonalized in color space.* Therefore, parallel to the discussion for quarks in I, we now choose a gauge which rotates $\bar{F}$ into the Abelian subalgebra of SU(N), which is always possible e.g. for fields of the form $\bar{F}_{\mu\nu}(x) = f_{\mu\nu}(x)\cdot n^i t_i$ and particularly for covariant constant fields [10]. Then $\bar{A}$ becomes diagonal in the same (global) gauge. We keep this *Abelian Dominance Approximation* [15] also for fields which vary slowly in color space.

In this case it is convenient to work in the Cartan-Weyl basis of SU(N) [10,15] which consists of $N-1$ Abelian generators h_j and $N(N-1)$ non-Abelian generators e_{ij} $(i,j=1,\cdots,N;\ i\neq j)$ satisfying

$$[h_i,h_j]=0 \ , \quad [h_i,e_{jk}] = (\vec{\eta}_{jk})_i e_{jk} \ , \quad [e_{ij},e_{jk}] = \tfrac{1}{\sqrt{2}}e_{ik}\ for\ i\neq j\neq k \ , \tag{21}$$

where the h_j are defined by $h_j \equiv (2j(j+1))^{-\frac{1}{2}}diag(1,\cdots,1,-j,0,\cdots,0)$, with $-j$ appearing in the $j+1$ column, and where

$$\vec{\eta}_{ij} \equiv \vec{\epsilon}_i - \vec{\epsilon}_j \ , \quad \vec{\epsilon}_i \equiv (\vec{h})_{ii} = ((h_1)_{ii},\cdots,(h_{N-1})_{ii}) \ , \tag{22}$$

are the adjoint representation weight vectors $\vec{\eta}$ of SU(N) as expressed in terms of the elementary weight vectors $\vec{\epsilon}$. Then, in the Abelian Dominance Approximation, we can write

$$\bar{F}_{\mu\nu}(x) \approx S(x)\vec{F}_{\mu\nu}(x)\cdot\vec{h}S^{-1}(x) \ , \quad \bar{A}(x) \approx S(x)\vec{A}(x)\cdot\vec{h}S^{-1}(x) \ , \tag{23}$$

where $S(x)$ is the diagonalizing gauge transformation.

Since we derived gauge covariant transport equations so far, we implement the Ansatz of eq.(23) by the replacements $\bar{F} \to \vec{F} \cdot \vec{h}$, $\bar{A} \to \vec{A} \cdot \vec{h}$. The transformed G can then be expanded as

$$G \equiv G^{ijkl} e_{ij} \otimes e_{kl} + G^{i;jk} h_i \otimes e_{jk} + G^{ij;k} e_{ij} \otimes h_k + G^{ij} h_i \otimes h_j \ , \tag{24}$$

with $G \equiv G_{\mu\nu}(x,p)$ etc. Calculating the commutators in eq.(18), using eqs.(19,21,24), we obtain for example:

$$\left[\vec{F} \cdot \vec{h}, G\right]_R = \vec{F} \cdot \vec{\eta}_{ij} \left(G^{ijkl} e_{ij} \otimes e_{kl} + G^{ij;k} e_{ij} \otimes h_k\right) \equiv \vec{F} \cdot \vec{\eta} \bullet G \ . \tag{25}$$

Thus $\vec{\eta}\bullet$ projects out parts of G which contain a non-Abelian generator e_{ij} on the left side of $\otimes$, cf. eq.(24). For a precise definition of $\vec{\eta}\bullet$ and similarly $\bullet\vec{\eta}$ it is useful to consider the covariant derivatives $\bar{D}$, cf. eq.(4), and $\bar{\bar{D}}$, cf. eq.(11), which here simply become

$$\bar{D} = \partial_x + ig\vec{A} \cdot [\vec{h}, ...] \equiv \partial_x + ig\vec{A} \cdot \vec{\eta}\bullet \ ,$$

$$\bar{\bar{D}} = \partial_x + ig\vec{A} \cdot [\vec{h}, ...]_R - ig[..., \vec{h}]_L \cdot \vec{A} \equiv \partial_x + ig\vec{A} \cdot (\vec{\eta} \bullet - \bullet \vec{\eta}) \ . \tag{26}$$

From these results we conclude that if G were diagonal, $G = G^{ij} h_i \otimes h_j$, then eq.(18) would reduce to $p \cdot \partial_x G(x,p) = 0$, i.e. the transport equation for *non-interacting gluons.*

In our approximation the $N-1$ Abelian mean field components obey ($\langle j \rangle \equiv \vec{j} \cdot \vec{h}$),

$$\partial_\mu \vec{F}^{\mu\nu} = g(\vec{J}^\nu + \vec{j}^\nu) \ . \tag{27}$$

Next, we determine $\langle j \rangle$ in terms of the gluon fluctuations G from eq.(20). The particular solution with $\langle j \rangle = 0$ for $G = 0$ is

$$\langle j_\nu(x) \rangle = -Tr' \int d^4p \int d^4q \, e^{iq\cdot x} \frac{q^\mu}{q^2} \{G_{\mu\nu}(q,p) - G_{\nu\mu}(q,p)\} \ , \tag{28}$$

where $G(q,p) \equiv (2\pi)^{-4} \int d^4x e^{-iq\cdot x} G(x,p)$. Since for consistency we assumed $\langle j \rangle$ to be diagonal in our particular gauge this also imposes a restriction on G by eq.(28). Calculating $Tr'G$ is equivalent to replacing the $\otimes$-product in eq.(24) by an ordinary matrix product. Therefore, G must have the form $G = G^{ijji} e_{ij} \otimes e_{ji} + G^{ij} h_i \otimes h_j$. To see this, note that e_{ij} can be represented in terms of N orthonormal unit vectors $\hat{e}_i$ as $e_{ij} = \hat{e}_i \hat{e}_j^\dagger / \sqrt{2}$. To further specify G, we remark that from eqs.(3,5) one expects that the matrix $Tr' \int d^4p G_{\mu\nu}$ for each pair $\mu\nu$ should contain $N^2 - 1$ independent contributions. From rewriting semiclassical Yang-Mills equations in terms of charged and neutral vector fields [10,16], we also know that the effective gluon current matrix j should contain contributions from $N(N-1)/2$ pairs of charged fields only. We choose the following simplest Ansatz for G which accomodates these restrictions for the remaining degrees of freedom:

$$G_{\mu\nu}(x,p) \equiv G^{ij}_{\mu\nu}(x,p) e_{ij} \otimes e_{ji} + G^i_{\mu\nu}(x,p) h_i \otimes h_i \equiv G^{\vec{\eta}}_{\mu\nu}(x,p) + G^h_{\mu\nu}(x,p) \ , \tag{29}$$

with $G^{ij}_{\mu\nu} \equiv G^{ji}_{\nu\mu}$ *and* $G^{i}_{\mu\nu} \equiv G^{i}_{\nu\mu}$. Eq.(29) implies by the definition of $\vec{\eta}\bullet$ in eq.(26) and with eqs.(19,21,22) that

$$\vec{\eta} \bullet G \;=\; \left[\vec{h}, G\right]_R \;=\; \vec{\eta}_{ij} G^{ij} e_{ij} \otimes e_{ji} \;=\; \left[G, \vec{h}\right]_L \;=\; G \bullet \vec{\eta} \;. \tag{30}$$

Eqs.(28,29) yield the average gluon current,

$$\langle j_\nu(x)\rangle \;=\; -\int d^4p \int d^4q \; e^{iq\cdot x} \, \frac{q^\mu}{q^2} \, diag(\dots\,,\; \tfrac{1}{2} \sum_{j,j\neq i} \{G^{ij}_{\mu\nu}(q,p) - G^{ij}_{\nu\mu}(q,p)\}\;,\;\dots)\;, \tag{31}$$

which is a diagonal traceless $N \times N$ matrix. Furthermore, it is easy to check that the gluon energy-momentum tensor defined by eq.(5) with $\hat{\Gamma}$ replaced by G is symmetric by virtue of eq.(29).

Inserting eq.(29) into the transport eq.(18), we decompose it into several equations by picking out the G^{ij}, G^{i}-components by multiplying with e_{ij}, h_i respectively and taking traces. Thus we find that the $N-1$ diagonal components of G defined in eq.(29) obey a free transport equation,

$$p \cdot \partial_x \, G^i(x,p) \;=\; 0 \;. \tag{32}$$

Since G^h does not contribute to $\langle j\rangle$, eq.(31), and therefore does not influence the mean field $\vec{F}$ by eq.(27) either, the ***neutral gluons*** described by its components G^i completely decouple from the dynamics of the system in our approximation based on eqs.(18,23,29). Repeatedly using eq.(30) we derive from eq.(18) simple Vlasov type equations for the $N(N-1)$ ***charged gluons*** described by the components G^{ij}:

$$\left(p \cdot \partial_x \;-\; g p^\sigma \partial_p^\tau \vec{F}_{\tau\sigma} \cdot \vec{\eta}_{ij}\right) \, G^{ij}_{\mu\nu}(x,p) \;=\; 0 \;. \tag{33}$$

Notice the appearance of the effective coupling $g\vec{\eta}$ for charged gluons. For quarks we found the analogous effective coupling in I to be $g\vec{\epsilon}$.

In conclusion we point out that eqs.(27,29,31-33) together form a consistent set of equations determining the dynamics of gluons together with a classical mean color field in the collisionless regime. It was obtained here in the Abelian Dominance Approximation from the quantum transport equation which we derived for the gauge covariant gluon Wigner operator defined in eq.(3). Further development of quark-gluon transport theory will require the calculation of collision and source terms, which generally appear on the r.h.s. of both eqs.(32,33), by methods outlined in ref.[5]. The exact quantum transport equations derived here and in I could provide a basic starting point for such future developments.

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