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On the Modal Logic of Subset and Superset: Tense Logic over Medvedev Frames

Abstract. Viewing the language of modal logic as a language for describing directed graphs, a natural type of directed graph to study modally is one where the nodes are *sets* and the edge relation is the *subset* or *superset* relation. A well-known example from the literature on intuitionistic logic is the class of *Medvedev frames* $\langle W, R\rangle$ where W is the set of nonempty subsets of some nonempty finite set S, and xRy iff $x \supseteq y$, or more liberally, where $\langle W, R\rangle$ is isomorphic as a directed graph to $\langle \wp(S) \setminus \{\emptyset\}, \supseteq\rangle$. Prucnal [32] proved that the modal logic of Medvedev frames is not finitely axiomatizable. Here we continue the study of Medvedev frames with extended modal languages. Our results concern *definability*. We show that the class of Medvedev frames is definable by a formula in the language of *tense* logic, i.e., with a converse modality for quantifying over supersets in Medvedev frames, extended with any one of the following standard devices: nominals (for naming nodes), a difference modality (for quantifying over those y such that $x \neq y$), or a complement modality (for quantifying over those y such that $x \not\supseteq y$). It follows that either the logic of Medvedev frames in one of these tense languages is finitely axiomatizable—which would answer the open question of whether Medvedev's [31] "logic of finite problems" is decidable—or else the minimal logics in these languages extended with our defining formulas are the beginnings of infinite sequences of *frame-incomplete* logics.

Keywords: Medvedev frames, modal logic, definability, nominal tense logic, difference modality, complement modality, axiomatizability, Kripke frame incompleteness

1. Introduction

Modal logics have been found to capture properties of many important mathematical concepts—for example, the modal logic of topological closure [5], the modal logic of arithmetic provability [9], the modal logic of set-theoretic forcing [25], and more. In this paper, we are interested in modal logics for capturing properties of something very basic: the *subset* and *superset* relations between the nonempty subsets of a finite set.

By the *unimodal language*, we mean the language given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi,$$

where p belongs to a countably infinite set Prop of propositional variables, the other Boolean connectives are defined as usual, and $\Diamond\varphi := \neg\Box\neg\varphi$. We take as *models* for the unimodal language tuples $\mathcal{M} = \langle W, R, V\rangle$ based on *frames* $\langle W, R\rangle$, where $\langle W, R\rangle$ is any directed graph and $V\colon \mathsf{Prop} \to \wp(W)$. We define the truth of a formula φ at a point $w \in W$ as usual:

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$ for $p \in \mathsf{Prop}$;
- $\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$;
- $\mathcal{M}, w \vDash \varphi \wedge \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$;
- $\mathcal{M}, w \vDash \Box\varphi$ iff for all $v \in W$: if wRv, then $\mathcal{M}, v \vDash \varphi$.

A formula φ is *valid* over a frame $\langle W, R\rangle$ iff φ is true at every point in every model based on $\langle W, R\rangle$. For a class C of frames, the *unimodal logic of* C is the set of all unimodal formulas that are valid over every frame in C.

Viewing a modal language as a language for describing directed graphs in this way, a natural type of directed graph to study modally is one where the nodes in W are *sets* and the edge relation R is the *subset* or *superset* relation. We will focus on a specific class of such structures.

Definition 1.1 (Medvedev Frames). A *Medvedev frame* [31] is a frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\langle \wp(S) \setminus \{\emptyset\}, \supseteq\rangle$ for a nonempty finite set S. (A *Skvortsov frame* [37] is defined in the same way except with S a nonempty set of any cardinality.)

Equivalently, we could think in terms of $\langle \wp(S) \setminus \{S\}, \subseteq\rangle$, or we could simply define a Medvedev frame to be a frame $\langle W, R\rangle$ that is obtained from a finite Boolean lattice $\langle W, \leq\rangle$ by removing the top element. We are following the literature on intuitionistic logic (see references after Theorem 1.2 below) by thinking in terms of $\langle \wp(S) \setminus \{\emptyset\}, \supseteq\rangle$. Medvedev frames are often defined as frames $\langle \wp(S) \setminus \{\emptyset\}, \supseteq\rangle$ for a nonempty finite set S, but Definition 1.1 will be more convenient for our purposes, allowing us to speak of "defining the class of Medvedev frames" rather than always "defining the class of Medvedev frames up to isomorphism." Informally, we blur the distinction.

Where $\mathcal{M}$ is a model based on a Medvedev frame, we have:

- $\mathcal{M}, w \vDash \Box\varphi$ iff for all $v \subseteq w$: $\mathcal{M}, v \vDash \varphi$.[1]

[1] In tense logic, the box operator is usually thought of as quantifying over all points *strictly in the future* of the current point, which would suggest that we take $\mathcal{M}, w \vDash \Box_{<}\varphi$ iff for all $v \subsetneq w$: $\mathcal{M}, v \vDash \varphi$. Then our $\Box$ can be defined by $\Box\varphi := \varphi \wedge \Box_{<}\varphi$. As we will see, we do not need the expressivity of $\Box_{<}$ for our definability results in §§3-4.

Now what modal principles are valid over Medvedev frames? We will see some examples in §2, but we cannot hope for a finite axiomatization.

Theorem 1.2 (Prucnal [32]). The unimodal logic of Medvedev frames is not finitely axiomatizable.

It is an open question whether the unimodal logic of Medvedev frames is recursively axiomatizable. An affirmative answer to this question would also yield an affirmative answer to the longstanding open question of whether the *intuitionistic propositional logic* of Medvedev frames (see [12, Ch. 2]) is recursively axiomatizable; for the intuitionistic logic of Medvedev frames—Medvedev's [30, 31] "logic of finite problems"—is recursively embedded in the unimodal logic of Medvedev frames by the Gödel-McKinsey-Tarski translation of the intuitionistic language into the unimodal language. Maksimova et al. [29] showed that the intuitionistic logic of Medvedev frames is not finitely axiomatizable (cf. [14, 15]). As Shehtman [36] observes, that result, combined with the Blok-Esakia Isomorphism Theorem, gives another proof of Theorem 1.2.[2] Since the class of Medvedev frames is a recursive class of finite frames, the modal and intuitionistic logics of Medvedev frames are co-recursively enumerable; so if they are recursively axiomatizable and hence enumerable, then they are decidable.[3] We will return to this issue below.

In addition to its connection with Medvedev's logic of finite problems, the unimodal logic of Medvedev frames has recently appeared in the study of modal logics of set-theoretic forcing for certain forcing classes in [24].

The main topic of the present paper is the *modal definability* of the class of Medvedev frames in the following standard sense (see, e.g., [4]).

Definition 1.3 (Relative and Absolute Definability). For classes C and D of frames and a modal formula φ (of the unimodal language above or any of the modal languages discussed below), φ *defines* C *relative to* D iff for every frame $\langle W, R\rangle \in \mathsf{D}$, $\langle W, R\rangle \in \mathsf{C}$ iff φ is valid over $\langle W, R\rangle$; and φ *defines* C (*absolutely*) iff for every frame $\langle W, R\rangle$, $\langle W, R\rangle \in \mathsf{C}$ iff φ is valid over $\langle W, R\rangle$.

Using only the unimodal language, the class of Medvedev frames is obviously not definable, since the class is not closed under operations on frames

[2] In fact, stronger results hold: Maksimova et al. [29] showed that the intuitionistic logic of Medvedev frames (resp. Skvortsov frames) is not axiomatizable in finitely many propositional variables, and Shehtman [36] showed the same for the unimodal logic of Medvedev frames (resp. Skvortsov frames).

[3] The intuitionistic logic of Skvortsov frames is known to be recursively axiomatizable [37], but it is unknown whether it is decidable.

that preserve validity of unimodal formulas (see [8, §3.3]), e.g., taking *disjoint unions* of frames. Even relative to, e.g., the class of finite frames that are not disjoint unions of other frames, the class of Medvedev frames is not definable in the unimodal language, because it is not closed under taking *R-generated subframes* (though it is closed under taking $R \cup R^{-1}$-generated subframes). Of course, if we view Medvedev frames as structures for a *first-order* language with a relation symbol for R, then the class of Medvedev frames is not definable by a first-order formula either, given compactness.

None of this forecloses definability in an extended modal language. As a natural candidate, we first consider the *nominal tense language* [7] that extends the unimodal language with nominals and a converse modality, with which we can name sets and quantify over supersets in Medvedev frames.

Definition 1.4 (Language). Fixing a countably infinite set Nom—the set of nominals—such that $\mathsf{Prop} \cap \mathsf{Nom} = \emptyset$, the *nominal tense language* is given by grammar

$$\varphi ::= p \mid i \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Box^{-1}\varphi,$$

where $p \in \mathsf{Prop}$, $i \in \mathsf{Nom}$, and $\Diamond^{-1}\varphi := \neg\Box^{-1}\neg\varphi$. The *tense language* is the fragment without nominals in formulas. The *nominal modal language* is the fragment without the converse modality $\Box^{-1}$ in formulas.

The intended semantics for this language is as follows.

Definition 1.5 (Models, Truth, and Validity). A *nominal model* is a tuple $\mathcal{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and $V\colon \mathsf{Prop} \cup \mathsf{Nom} \to \wp(W)$ is such that for all $i \in \mathsf{Nom}$, $V(i)$ is a singleton set.

The definition of truth of a nominal tense formula at a point w in a model $\mathcal{M}$ is defined in the same way as above, except for the new clauses:

- $\mathcal{M}, w \vDash i$ iff $w \in V(i)$ for $i \in \mathsf{Nom}$;
- $\mathcal{M}, w \vDash \Box^{-1}\varphi$ iff for all $v \in W$: if vRw, then $\mathcal{M}, v \vDash \varphi$.

So where $\mathcal{M}$ is based on a Medvedev frame:

- $\mathcal{M}, w \vDash \Box^{-1}\varphi$ iff for all $v \supseteq w$: $\mathcal{M}, v \vDash \varphi$.

A formula of the nominal tense language is *valid* over a frame $\langle W, R \rangle$ iff it is true at every point in every *nominal* model based on $\langle W, R \rangle$. For a class C of frames, the *nominal tense logic of* C is the set of all nominal tense formulas that are valid over every frame in C.

For the theory of modal languages with nominals, in addition to [7], see, e.g., [17, 10, 6, 11]. For general results on frame definability in nominal modal languages, see [7, 17, 10]. Note, however, that since the class of Medvedev frames is not a first-order definable class of frames, some of the Goldblatt-Thomason-style definability theorems do not apply.

Since the class of Medvedev frames is not closed under R-generated subframes, it is not definable by a formula of the *nominal modal* language, for the validity of such formulas is preserved under R-generated subframes; and since it is not closed under disjoint unions, it is not definable by a formula of the *tense* language. However, we shall see that with their powers combined, the *nominal tense* language can define the class of Medvedev frames.

A language with as much frame-defining power as the nominal tense language is the tense language with a *difference* modality (see, e.g., [34, 33, 27] and [8, §7.1]), given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Box^{-1}\varphi \mid [\neq]\varphi$$

and the following semantics:

- $\mathcal{M}, w \vDash [\neq]\varphi$ iff for all $v \in W$: if $w \neq v$, then $\mathcal{M}, v \vDash \varphi$.

For lack of a better term, we will call this the *differential tense language.* The following result follows easily from the proof in §4.1.II of [17] concerning the unimodal language extended with $[\neq]$.

Theorem 1.6 (Gargov and Goranko [17]). If a class of frames is definable in the nominal tense language, it is definable in the differential tense language.

Thus, by our result that the class of Medvedev frames is definable in the nominal tense language, it is also definable in the differential tense language.

As shown in [17], the converse of Theorem 1.6 holds if we add to the nominal tense language the universal modality, which is already definable in the differential language by $\blacksquare\varphi := \varphi \wedge [\neq]\varphi$. Otherwise the differential tense language has more frame-defining power. For example, the formula $\neg[\neq]\bot$ defines a class of frames that is not definable in the nominal tense language.

For an alternative route to defining the class of Medvedev frames, we can extend the unimodal language with not only a modality $\Box^{-1}$ for the converse of R, but also a modality $\boxminus$ for the *complement* of R [2, 26, 19, 18, 21, 20, 3].

Definition 1.7 (Language). The *complementary tense language* is given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Box^{-1}\varphi \mid \boxminus\varphi,$$

where $p \in \mathsf{Prop}$ and $\diamondminus\varphi := \neg\boxminus\neg\varphi$.

The intended semantics for this language is as follows.

DEFINITION 1.8 (Models, Truth, and Validity). Models for the complementary tense language are the same as models for the unimodal language. The truth clause for the $\boxminus$ modality is:

- $\mathcal{M}, w \vDash \boxminus\varphi$ iff for all $v \in W$: if *not* wRv, then $\mathcal{M}, v \vDash \varphi$.

So where $\mathcal{M}$ is based on a Medvedev frame:

- $\mathcal{M}, w \vDash \boxminus\varphi$ iff for all $v \not\subseteq w$: $\mathcal{M}, v \vDash \varphi$.

Validity over frames is defined as in the case of the unimodal language. For a class C of frames, the *complementary tense logic of* C is the set of all complementary tense formulas that are valid over every frame in C.

For general results on frame definability in this language, see [21, §6].

The nominal tense language and complementary tense language are incomparable in frame-defining power. For example, the nominal tense formula $i \to \Diamond\neg i$ defines the class of frames such that $\forall x\exists y(xRy \wedge x \neq y)$; yet this class is not definable by a formula of the complementary tense language [21, Lemma 6.3(i)], because the validity of such formulas is preserved under taking (the tense version of) *p-morphic images* (see §5), whereas the class is not closed under that operation. Conversely, the complementary tense formula $\diamondminus\top$ defines the class of frames such that $\forall x\exists y\,\neg xRy$; yet this class is not definable by a formula of the nominal tense language, because the validity of such formulas is preserved under taking $R \cup R^{-1}$-generated subframes [7, p. 62], whereas the class is not closed under that operation.

Given this incomparability in expressive power, it is informative to know that not only the nominal and differential tense languages but also the complementary tense language can define the class of Medvedev frames.

From our main definability results, a tantalizing disjunction follows: *either* the logic of Medvedev frames in one of the three tense languages is finitely axiomatizable—which would answer the open question of whether Medvedev's logic of finite problems is decidable, as above—*or* the minimal logics in the three languages extended with our Medvedev-defining formulas as axioms are the beginnings of infinite sequences of *frame-incomplete* logics. For if the minimal logic with our Medvedev-defining formula as an axiom—call it $\mathbf{L}_0$—is the logic of any class of frames at all, then it is the logic of the class of Medvedev frames, which is then finitely axiomatizable. Thus, if the logic of Medvedev frames is *not* finitely axiomatizable, then not only is $\mathbf{L}_0$ frame incomplete, but also there is an infinite sequence $\mathbf{L}_1, \mathbf{L}_2, \ldots$ of logics such that the following holds: $\mathbf{L}_{n+1}$ extends $\mathbf{L}_n$ with a non-theorem of $\mathbf{L}_n$ that is valid over Medvedev frames, which is possible by the hypothesis of $\mathbf{L}_n$'s incompleteness; then $\mathbf{L}_{n+1}$ still defines the class of Medvedev frames, but by its finite axiomatization it is not complete with respect to the class of Medvedev frames; so $\mathbf{L}_{n+1}$ is not complete with respect to any class of frames. By similar reasoning, if the logic of Medvedev frames is not even recursively axiomatizable, then adding any recursive set of formulas from the logic of Medvedev frames to $\mathbf{L}_0$ will result in a frame-incomplete logic.

In §§2-3, we prove our definability theorem for the nominal tense language. It is then a short step in §4 to prove the result for the complementary tense language. In §§5-6, we draw some morals about the definability and prospects for axiomatizability of the class of Medvedev frames in different languages. In our conclusion in §7, we return to the disjunction above.

2. Nominal Tense Axioms

Let AX_1 be the set of the following axioms:

$$\begin{array}{rl} \mathrm{T}_N & i \to \Diamond i \\ 4_N & \Diamond\Diamond i \to \Diamond i \\ \mathrm{Anti}_N & i \to \Box(\Diamond i \to i) \\ \mathrm{Sep}_N & \Box\Diamond\Diamond^{-1} i \to \Diamond^{-1} i \\ \mathrm{Con}_N^{-1} & \Diamond^{-1}\Diamond i \\ \mathrm{Uni}_N & (\Diamond i \wedge \Diamond j) \to \Diamond(\Diamond i \wedge \Diamond j \wedge \Box\Diamond(\Diamond^{-1} i \vee \Diamond^{-1} j)). \end{array}$$

Although these axioms will not define the class of Medvedev frames absolutely, but only relative to the class of finite frames (Theorem 3.5), we single them out as a group because they are all *pure* nominal formulas, i.e., containing no propositional variables, a point to which we will return below. The T_N and 4_N axioms are just the pure nominal, diamond versions of the standard modal axioms T ($\Box p \to p$) and 4 ($\Box p \to \Box\Box p$), respectively.

Dropping the restriction to pure formulas, let AX_2 be the set of the following axioms:

$$\begin{array}{rl} \mathrm{Grz} & \Box(\Box(p \to \Box p) \to p) \to p \\ \mathrm{Grz}^{-1} & \Box^{-1}(\Box^{-1}(p \to \Box^{-1} p) \to p) \to p \\ \mathrm{Sep}_N & \Box\Diamond\Diamond^{-1} i \to \Diamond^{-1} i \\ \mathrm{Con}_N^{-1} & \Diamond^{-1}\Diamond i \\ \mathrm{Uni} & (\Diamond(p_1 \wedge \Box q) \wedge \Diamond(p_2 \wedge \Box q)) \to \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q). \end{array}$$

One can think of the Grz axiom in AX_2 as replacing the T_N, 4_N, and Anti_N axioms in AX_1, based on the semantic Fact 2.2 below. Syntactically, any *normal modal logic* that contains the Grz axiom also contains the T axiom and—much less obviously—the 4 axiom [1].[4]

REMARK 2.1. Let us briefly note some syntactic facts about the Uni axiom. First, consider the generalizations of Uni for arbitrary $n \geq 2$:

$$(\bigwedge_{1\leq i\leq n} \Diamond(p_i \wedge \Box q)) \to \Diamond(\bigwedge_{1\leq i\leq n} \Diamond p_i \wedge \Box\Diamond q).$$

One can show that for each $n \geq 2$, the formula above belongs to the smallest normal unimodal logic containing Uni and the 4 axiom. Second, the same can be shown for the formula

$$(\Diamond\Box p_1 \wedge \Diamond\Box p_2 \wedge \Diamond\Box p_3 \wedge \neg\Diamond((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3))) \to \Diamond(\Diamond\Box p_1 \wedge \Diamond\Box p_2 \wedge \neg\Diamond\Box p_3)$$

and its generalizations for arbitrary $n \geq 3$, which Hamkins et al. [24] note are valid over Medvedev frames. We prove these facts in the Appendix. In addition, using results of [13], we show that there are still continuum-many normal unimodal logics between, on the one hand, the logic axiomatized by Grz and Uni, and on the other, the unimodal logic of Medvedev frames.

The axioms T_N, 4_N, Anti_N, and $\mathrm{Grz}/\mathrm{Grz}^{-1}$ are standard axioms, whose associated classes of frames are plain to see—except perhaps in the case of Grz, the *Grzegorczyk axiom*, one of the famous examples of a modal formula that defines a non-first-order class of frames (see, e.g., [12, p. 83]).

FACT 2.2 (Frames for the Standard Axioms). For any frame $\langle W, R\rangle$:

1. $\langle W, R\rangle$ validates T_N iff R is reflexive;
2. $\langle W, R\rangle$ validates 4_N iff R is transitive;
3. $\langle W, R\rangle$ validates Anti_N iff R is antisymmetric;
4. $\langle W, R\rangle$ validates Grz (resp. Grz^{-1}) iff R (resp. R^{-1}) is a *Noetherian* partial order, i.e., a reflexive, transitive, and antisymmetric relation that contains no infinite ascending chain of distinct elements.

[4] As usual, a set of unimodal formulas is a *normal modal logic* iff it contains all tautologies of classical propositional logic, contains the formula $\Box(p \to q) \to (\Box p \to \Box q)$, and is closed under modus ponens and the operations of prefixing $\Box$ to any formula and of uniformly substituting formulas for propositional variables in any formula.

The other axioms are not standard, but their associated classes of frames are also easy to work out. The term ‘Sep_N’ suggests *separative*; ‘Con_N^{-1}’ suggests *convergence* in the R^{-1} direction; and ‘Uni’ suggests *union*, for reasons that will become clear. The notion of a separative poset comes from the literature on set-theoretic forcing (see, e.g., [28, p. 4]).

Fact 2.3 (Frames for the New Axioms). For any frame $\langle W, R\rangle$:

1. $\langle W, R\rangle$ validates Sep_N iff R is separative, i.e.,
$$\forall x \forall y(\forall y'(yRy' \to \exists y''(y'Ry'' \wedge xRy'')) \to xRy); \qquad (\mathsf{Sep})$$
2. $\langle W, R\rangle$ validates Con_N^{-1} iff every two points have a common predecessor, i.e.,
$$\forall y_1 \forall y_2 \exists x(xRy_1 \wedge xRy_2); \qquad (\mathsf{Con}^{-1})$$
3. $\langle W, R\rangle$ validates Uni_N iff it validates Uni iff R satisfies
$$\forall x \forall y_1 \forall y_2((xRy_1 \wedge xRy_2) \to$$
$$\exists u(xRu \wedge uRy_1 \wedge uRy_2 \wedge \forall v(uRv \to \exists w(vRw \wedge (y_1Rw \vee y_2Rw))))). \quad (\mathsf{Uni})$$

Proof. Part 1, part 2, and the right-to-left direction of part 3 are straightforward. For the left-to-right direction of part 3 in the case of Uni, suppose $\langle W, R\rangle$ does not satisfy (Uni), so
$$\exists x \exists y_1 \exists y_2((xRy_1 \wedge xRy_2) \wedge$$
$$\forall u((xRu \wedge uRy_1 \wedge uRy_2) \to \exists v(uRv \wedge \forall w(vRw \to \neg(y_1Rw \vee y_2Rw))))). \quad \text{(I)}$$
Define a model $\mathcal{M} = \langle W, R, V\rangle$ such that $V(p_1) = \{y_1\}$, $V(p_2) = \{y_2\}$, and $V(q) = \{z \in W \mid y_1Rz \text{ or } y_2Rz\}$. Thus, from xRy_1 and xRy_2, we have $\mathcal{M}, x \vDash \Diamond(p_1 \wedge \Box q) \wedge \Diamond(p_2 \wedge \Box q)$. Now suppose for reductio that $\mathcal{M}, x \vDash \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q)$, so there is a u with xRu and $\mathcal{M}, u \vDash \Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q$. Since $\mathcal{M}, u \vDash \Diamond p_1 \wedge \Diamond p_2$, we have uRy_1 and uRy_2, so there is a v as in (I). Since for all w with vRw, we have neither y_1Rw nor y_2Rw, it follows that $\mathcal{M}, v \nvDash \Diamond q$, which with uRv implies $\mathcal{M}, u \nvDash \Box\Diamond q$, which contradicts the description of u. Thus, $\mathcal{M}, x \nvDash \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q)$, so $\mathcal{M}, x$ falsifies Uni. ∎

Remark 2.4. We can now see semantically or syntactically that the Uni axiom is a modal version of the superintuitionistic Kreisel-Putnam axiom $\boldsymbol{kp} := (\neg a \to (b \vee c)) \to ((\neg a \to b) \vee (\neg a \to c))$, which belongs to Medvedev's logic of finite problems [12, p. 54]. Semantically, the condition (Uni) is necessary and sufficient for a poset $\langle W, R\rangle$ to validate $\boldsymbol{kp}$ according to intuitionistic Kripke semantics [12, Exercise 2.10] (thanks to an anonymous referee for pointing out this correspondence). Syntactically, if we take the Gödel translation [12, §3.9] of $\boldsymbol{kp}$, contrapose the main conditional, drive negations in, and rearrange some subformulas, we obtain the formula $\boldsymbol{kpm} := \Box((\Diamond(\Diamond\neg b \wedge \Box\Diamond\neg a) \wedge \Diamond(\Diamond\neg c \wedge \Box\Diamond\neg a)) \to \Diamond(\Diamond\neg b \wedge \Diamond\neg c \wedge \Box\Diamond\neg a))$; and it is easy to see that if a normal modal logic contains the 4 and T axioms, then it contains $\boldsymbol{kpm}$ iff it contains the simpler Uni axiom.

All Skvortsov frames satisfy the properties corresponding to the axioms of AX_1, and all Medvedev frames satisfy those corresponding to the axioms of AX_2. (Sep) reflects extensionality and the nonemptiness of our sets: contrapositively, if $x \not\supseteq y$, then there is an $s \in y \setminus x$, so there is a $y' = \{s\}$ such that $y \supseteq y'$ but for all y'' with $y' \supseteq y''$ (i.e., $y' = y''$ since y' is a singleton and $y'' \neq \emptyset$), we have $x \not\supseteq y''$. For (Con^{-1}), take x to be the union of the sets y_1 and y_2. For (Uni), take u to be the union of the sets y_1 and y_2.

Fact 2.5 (Soundness). The formulas of AX_1 are valid over all Skvortsov frames. The formulas of AX_2 are valid over all Medvedev frames.

For completeness, we can start with the nominal tense logic of all frames, which can be axiomatized in various ways (see, e.g., [7, 17, 10]) and then add the axioms of AX_1 or AX_2 (closing under rules). For AX_1, the result will be a logic that is complete with respect to the class of frames defined by the conjunction of the axioms in AX_1. This follows from the fact that AX_1 contains only *pure* nominal formulas. As in the case of Sahlqvist axioms [8, §3], so too in the case of pure axioms, there is an automatic completeness theorem.[5] To apply this theorem, which is usually stated for an arbitrary nominal multimodal language, note that we may freely switch between interpreting the nominal tense language in frames $\langle W, R\rangle$ and interpreting it in birelational frames $\langle W, R_1, R_2\rangle$ where $\Box$ is the box modality for R_1, $\Box^{-1}$ is the box modality for R_2, and $R_2 = R_1^{-1}$ (cf. the notion of a *polymodal base* in [20, 21]). Within the class of birelational frames, the class of frames with $R_2 = R_1^{-1}$ is defined by the pure axioms $i \to \Box\Diamond^{-1}i$ and $i \to \Box^{-1}\Diamond i$. Thus, the following result for arbitrary nominal multimodal logics gives us automatic completeness results for nominal tense logics as well.

Theorem 2.6 (Gargov and Goranko [17]). If a class C of multirelational frames is defined by a set Σ of pure nominal multimodal formulas, then the nominal multimodal logic of C is obtained by adding to the minimal nominal multimodal logic the formulas of Σ as axioms (and closing under rules).

[5] It is also worth noting that in the case of nominal *tense* logic, for every Sahlqvist axiom there is a pure axiom that defines the same class of frames [23].

In the case of AX_2, which has non-pure axioms, the question of completeness does not have such an easy answer. We will return to it in §6.

3. Definability of Medvedev Frames

In this section, we show that AX_1 defines the class of Medvedev frames relative to the class of finite frames (Theorem 3.5), and AX_2 defines the class of Medvedev frames absolutely (Theorem 3.6). Our helpers in this task are the *endpoints* in Medvedev frames, representing singleton sets.

DEFINITION 3.1 (Endpoints and McKinsey Frames). Given a frame $\langle W, R\rangle$, an *endpoint* is a $w \in W$ such that for all $v \in W$, if wRv, then $w = v$.[6] Let $\mathsf{end}\langle W, R\rangle$ be the set of all endpoints in $\langle W, R\rangle$, and for $w \in W$, let $\mathsf{end}(w)$ be the set of all endpoints $v \in W$ such that wRv.

A *McKinsey frame* [12, p. 82] is a frame $\langle W, R\rangle$ such that for every $w \in W$, $\mathsf{end}(w) \neq \emptyset$.[7]

Note that any finite poset and any Noetherian poset is a McKinsey frame. The reason we call attention to McKinsey frames is that it is worthwhile to clearly separate which parts of the argument to follow depend on *finiteness* and which parts depend on the weaker McKinsey property that holds not only of Medvedev frames but also of Skvortsov frames.

The argument involves three lemmas, the first of which is easy to check.

LEMMA 3.2 (McKinsey and Separativity). For any frame $\langle W, R\rangle$:

1. if $\langle W, R\rangle$ is a transitive frame, then for all $w, v \in W$, wRv implies $\mathsf{end}(w) \supseteq \mathsf{end}(v)$;
2. if $\langle W, R\rangle$ is a separative and transitive McKinsey frame, then for all $w, v \in W$, $\mathsf{end}(w) \supseteq \mathsf{end}(v)$ implies wRv;
3. if $\langle W, R\rangle$ is such that for all $w, v \in W$, $\mathsf{end}(w) \supseteq \mathsf{end}(v)$ implies wRv, then $\langle W, R\rangle$ is separative.

The second lemma is an alternative characterization of Medvedev frames.

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[6] Note that this is not the same as the notion of an atom/coatom in a poset, since an atom/coatom is required to be a non-minimum/maximum element. Nonetheless, what we call 'McKinsey frames' are sometimes called 'atomic frames'.

[7] As is well known, the class of transitive McKinsey frames is definable by the conjunction of the 4 axiom $\Box p \to \Box\Box p$ and the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$. Blackburn [7, p. 64f] shows that no purely nominal formula defines the class of transitive McKinsey frames. His proof also shows that no such formula defines the Noetherian posets.

LEMMA 3.3 (Characterization of Medvedev Frames). For any frame $\langle W, R\rangle$, the following are equivalent:

1. $\langle W, R\rangle$ is a finite separative poset such that
$$\forall y_1, y_2 \in W\, \exists u \in W \colon \mathsf{end}(u) = \mathsf{end}(y_1) \cup \mathsf{end}(y_2); \qquad (\cup)$$
2. $\langle W, R\rangle$ is finite and isomorphic as a frame to $\langle \wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\}, \supseteq\rangle$;
3. $\langle W, R\rangle$ is a Medvedev frame.

PROOF. From 1 to 2, the isomorphism $f \colon W \to \wp(\mathsf{end}\langle W, R\rangle)\setminus\{\emptyset\}$ is defined by $f(w) = \mathsf{end}(w)$. Since $\langle W, R\rangle$ is a finite poset, it is McKinsey, so for all $w \in W$, $\mathsf{end}(w) \neq \emptyset$, which means that f is indeed a function from W to $\wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\}$. To see that f is injective: if $\mathsf{end}(w) = \mathsf{end}(v)$, then since $\langle W, R\rangle$ is separative, transitive, and McKinsey, we have wRv and vRw by Lemma 3.2.2, so $w = v$ since $\langle W, R\rangle$ is antisymmetric. To see that f is surjective: if $Y \in \wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\}$, so Y is a nonempty set of endpoints, then since R is reflexive, for all $y \in Y$, $\mathsf{end}(y) = \{y\}$. Hence
$$Y = \bigcup_{y\in Y} \mathsf{end}(y).$$
Then since W is finite, by $(\cup)$ there is a $u \in W$ such that $\mathsf{end}(u) = Y$, so $f(u) = Y$. Finally, since $\langle W, R\rangle$ is separative, transitive, and McKinsey, by Lemma 3.2.1-2 we have wRv iff $f(w) \supseteq f(v)$. Thus, f is an isomorphism.

From 2 to 3, since $W \neq \emptyset$, if $\langle W, R\rangle$ is isomorphic to $\langle \wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\}, \supseteq\rangle$, then $\wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\} \neq \emptyset$, so $\mathsf{end}\langle W, R\rangle \neq \emptyset$. Then $\langle W, R\rangle$ is a Medvedev frame as in Definition 1.1 with $S = \mathsf{end}\langle W, R\rangle$.

From 3 to 1, observe that 1 holds whenever $\langle W, R\rangle = \langle \wp(S)\setminus\{\emptyset\}, \supseteq\rangle$ for a nonempty finite S, and 1 is preserved under directed graph isomorphism. ■

The third lemma relates the property (Uni) from Fact 2.3 with the property $(\cup)$ from Lemma 3.3. Recall that (Uni) is $\forall x \forall y_1 \forall y_2((xRy_1 \wedge xRy_2) \to \exists u(xRu \wedge uRy_1 \wedge uRy_2 \wedge \forall v(uRv \to \exists w(vRw \wedge (y_1Rw \vee y_2Rw)))))$.

LEMMA 3.4 (Relating (Uni) and $(\cup)$). For any frame $\langle W, R\rangle$:

1. if $\langle W, R\rangle$ is a separative and transitive McKinsey frame satisfying $(\cup)$, then $\langle W, R\rangle$ satisfies (Uni);
2. if $\langle W, R\rangle$ is a transitive frame satisfying (Con^{-1}) and (Uni), then $\langle W, R\rangle$ satisfies $(\cup)$.

Proof. For part 1, suppose $\langle W, R\rangle$ is a separative and transitive McKinsey frame satisfying ($\cup$). To show (Uni), suppose xRy_1 and xRy_2. Then by ($\cup$), there is a u with (i) $\mathsf{end}(u) = \mathsf{end}(y_1) \cup \mathsf{end}(y_2)$. Since xRy_1 and xRy_2, we have $\mathsf{end}(x) \supseteq \mathsf{end}(y_1)$ and $\mathsf{end}(x) \supseteq \mathsf{end}(y_2)$ by Lemma 3.2.1, so $\mathsf{end}(x) \supseteq \mathsf{end}(u)$ and hence xRu by Lemma 3.2.2. Since $\mathsf{end}(u) \supseteq \mathsf{end}(y_1)$ and $\mathsf{end}(u) \supseteq \mathsf{end}(y_2)$, we also have uRy_1 and uRy_2 by Lemma 3.2.2. Consider any v such that uRv. Since $\langle W, R\rangle$ is a McKinsey frame, there is a $w \in \mathsf{end}(v)$. By Lemma 3.2.1, uRv implies $\mathsf{end}(u) \supseteq \mathsf{end}(v)$, so $\mathsf{end}(y_1) \cup \mathsf{end}(y_2) \supseteq \mathsf{end}(v)$ by (i). Then since $w \in \mathsf{end}(v)$, we have either $w \in \mathsf{end}(y_1)$, in which case y_1Rw, or $w \in \mathsf{end}(y_2)$, in which case y_2Rw. Hence $\langle W, R\rangle$ satisfies (Uni).

For part 2, suppose $\langle W, R\rangle$ is a transitive frame satisfying (Con^{-1}). Then to prove the implication from (Uni) to ($\cup$), suppose that $\langle W, R\rangle$ does not satisfy ($\cup$), so there are $y_1, y_2 \in W$ such that (ii) for all $u \in W$, $\mathsf{end}(u) \neq \mathsf{end}(y_1) \cup \mathsf{end}(y_2)$. By ($\mathsf{Con}^{-1}$), there is an x such that xRy_1 and xRy_2. Now suppose for reductio that $\langle W, R\rangle$ satisfies (Uni). Then since xRy_1 and xRy_2, there is a u as in (Uni). Thus, uRy_1 and uRy_2, which with Lemma 3.2.1 implies $\mathsf{end}(u) \supseteq \mathsf{end}(y_1) \cup \mathsf{end}(y_2)$, which with (ii) implies $\mathsf{end}(u) \not\subseteq \mathsf{end}(y_1)\cup\mathsf{end}(y_2)$. Take a $v \in \mathsf{end}(u)\setminus(\mathsf{end}(y_1)\cup\mathsf{end}(y_2))$, so uRv but it is not the case that y_1Rv or y_2Rv. Then since v is an endpoint, we have that for all w with vRw, it is not the case that y_1Rw or y_2Rw. But this contradicts the fact that u is as in (Uni). Thus, $\langle W, R\rangle$ does not satisfy (Uni). ■

We are now ready to put everything together for AX_1.

Theorem 3.5 (Relative Definability of Medvedev Frames). For any frame $\langle W, R\rangle$, the following are equivalent:

1. $\langle W, R\rangle$ is a finite frame validating AX_1;
2. $\langle W, R\rangle$ is a Medvedev frame.

Proof. From 2 to 1, as noted with Fact 2.5, the formulas of AX_1 are valid over all Skvortsov frames and hence all Medvedev frames.

From 1 to 2, if $\langle W, R\rangle$ validates AX_1, then by Facts 2.2-2.3, $\langle W, R\rangle$ is a *separative poset* satisfying (Con^{-1}) and (Uni), which by Lemma 3.4.2 implies that it is a separative poset satisfying ($\cup$), which by Lemma 3.3 and the finiteness of W implies that $\langle W, R\rangle$ is a Medvedev frame. ■

It is worth noting that the nominal tense language can define any single finite frame up to isomorphism [7, §2] [17, §4.2] (cf. [21, p. 96] on the complementary tense language). In the case of "the" Medvedev frame $\langle W, R\rangle$ with $|W| = n$, once we have a set of axioms like AX_1 that defines the class of

Medvedev frames relative to the class of finite frames, one can add two simple axioms constraining numbers of successors in order to define the frame.

For the axiom set AX_2, we have the following stronger result.

Theorem 3.6 (Absolute Definability of Medvedev Frames). For any frame $\langle W, R\rangle$, the following are equivalent:

1. $\langle W, R\rangle$ validates AX_2;
2. $\langle W, R\rangle$ is a Medvedev frame.

Proof. Assume $\langle W, R\rangle$ validates AX_2. Then $\langle W, R\rangle$ validates AX_1, so if we can show that $\langle W, R\rangle$ is finite, it is Medvedev frame by Theorem 3.5. Since $\langle W, R\rangle$ validates Sep_N and Grz, by Facts 2.2-2.3 it is a separative McKinsey poset. Thus, as in the proof of Lemma 3.3, the $f\colon W \to \wp(\mathsf{end}\langle W, R\rangle) \setminus \{\emptyset\}$ defined by $f(w) = \mathsf{end}(w)$ is injective. So if we suppose for reductio that W is infinite, then $\mathsf{end}\langle W, R\rangle$ is also infinite. Since $\langle W, R\rangle$ validates Con_N^{-1} and Uni, it satisfies (Con^{-1}) and (Uni) by Fact 2.3, so it satisfies $(\cup)$ by Lemma 3.4.2. Now we will define an infinite sequence $x_1, x_2, \dots$ of distinct points from W such that $\mathsf{end}(x_i) \subseteq \mathsf{end}(x_{i+1})$ and $\mathsf{end}(x_{i+1})$ is finite. Take x_1 to be any element of $\mathsf{end}\langle W, R\rangle$. Having picked x_n, since $\mathsf{end}(x_n)$ is finite and $\mathsf{end}\langle W, R\rangle$ infinite, we can pick a $y \in \mathsf{end}\langle W, R\rangle \setminus \mathsf{end}(x_n)$. Then by $(\cup)$, there is an x_{n+1} such that $\mathsf{end}(x_{n+1}) = \mathsf{end}(x_n) \cup \mathsf{end}(y)$. Since $\mathsf{end}(y) = \{y\}$ and $y \notin \mathsf{end}(x_n)$, it follows that x_{n+1} is distinct from x_n. Since $\mathsf{end}(x_i) \subseteq \mathsf{end}(x_{i+1})$ and $\langle W, R\rangle$ is a separative McKinsey poset, it follows by Lemma 3.2.2 that $x_i R^{-1} x_{i+1}$. Thus, we have an infinite R^{-1}-chain of distinct points, which contradicts the assumption that Grz^{-1} is valid on $\langle W, R\rangle$, by Fact 2.2.4. Hence $\langle W, R\rangle$ is finite, which completes the proof. ∎

By Theorems 3.6 and 1.6, the class of Medvedev frames is also definable in the differential tense language. In particular, the axioms

$$\Box\Diamond\Diamond^{-1}(p \wedge [\neq]\neg p) \to \Diamond^{-1}(p \wedge [\neq]\neg p) \text{ and } (p \vee \neg[\neq]\neg p) \to \Diamond^{-1}\Diamond p$$

correspond to (Sep) and (Con^{-1}), respectively.

4. Complementary Tense Axioms

It is evident from §3 that it is now sufficient for proving that some other extended tense language can define the class of Medvedev frames that we show it can define the properties (Sep) and (Con^{-1}). For the complementary tense language discussed in §1, this is done with the following axioms.

Fact 4.1 (Complementary Axioms). For any frame $\langle W, R\rangle$:

1. $(\diamondminus p \wedge \boxminus q) \to \diamondminus(p \wedge \Diamond\Box q)$ is valid over $\langle W, R\rangle$ iff R satisfies (Sep);[8]
2. $(\Diamond p \vee \diamondminus p) \to \Diamond^{-1}\Diamond p$ is valid over $\langle W, R\rangle$ iff R satisfies (Con^{-1}).[9]

Proof. For part 1, from right to left, suppose $\langle W, R\rangle$ satisfies (Sep) and for some model $\mathcal{M} = \langle W, R, V\rangle$ and $x \in W$, $\mathcal{M}, x \vDash \diamondminus p \wedge \boxminus q$, so there is a y such that $\mathcal{M}, y \vDash p$ and *not* xRy, and (i) for all z such that *not* xRz, $\mathcal{M}, z \vDash q$. Since $\langle W, R\rangle$ satisfies (Sep), it follows from *not* xRy that there is a y' with yRy' such that for all y'' with $y'Ry''$, *not* xRy'', so $\mathcal{M}, y'' \vDash q$ by (i). Thus, $\mathcal{M}, y \vDash p \wedge \Diamond\Box q$, which with *not* xRy implies $\mathcal{M}, x \vDash \diamondminus(p \wedge \Diamond\Box q)$. From left to right, suppose $\langle W, R\rangle$ does not satisfy (Sep), so there are x, y such that *not* xRy, but (ii) for all y' with yRy' there is a y'' with $y'Ry''$ and xRy''. Define a model $\mathcal{M} = \langle W, R, V\rangle$ such that $V(p) = \{y\}$ and $V(q) = \{z \in W \mid \textit{not } xRz\}$. Then clearly $\mathcal{M}, x \vDash \diamondminus p \wedge \boxminus q$ but $\mathcal{M}, x \nvDash \diamondminus(p \wedge \Diamond\Box q)$ given (ii).

Part 2 is also straightforward. ■

Let us call the formulas in Fact 4.1 'Sep_C' and 'Con_C^{-1}', respectively. By the same reasoning as in the proofs of Theorems 3.5 and 3.6, but now using Fact 4.1 in place of Fact 2.3, we have the following.

Theorem 4.2 (Absolute Definability of Medvedev Frames). For any frame $\langle W, R\rangle$, the following are equivalent:

1. $\langle W, R\rangle$ validates Grz, Grz^{-1}, Uni, Sep_C, and Con_C^{-1};
2. $\langle W, R\rangle$ is a Medvedev frame.

5. On Separativity

In several ways, the property (Sep) is more difficult to handle than the property (Con^{-1}). For example, (Con^{-1}) can be captured in the tense language extended with only the *universal* modality $\blacksquare$ with the truth clause:

- $\mathcal{M}, w \vDash \blacksquare\varphi$ iff for all $v \in W$: $\mathcal{M}, v \vDash \varphi$.

As usual, define the existential modality by $\blacklozenge\varphi := \neg\blacksquare\neg\varphi$.

Fact 5.1 (Definability with the Universal Modality). For any frame $\langle W, R\rangle$, $\blacklozenge p \to \Diamond^{-1}\Diamond p$ is valid over $\langle W, R\rangle$ iff R satisfies (Con^{-1}).

[8] With a modality $\boxminus^{-1}$ for the complement of the *converse* of R instead of the complement of R itself, the defining formula can be simplified to $\diamondminus^{-1} r \to \Diamond\Box\diamondminus^{-1} r$.

[9] For definability relative to the class of reflexive frames, the defining formula can be simplified to $\diamondminus p \to \Diamond^{-1}\Diamond p$.

However, (Sep) eludes the power of the universal modality. To see this, recall that a *p-morphism* from a frame $\langle W, R\rangle$ to a frame $\langle W', R'\rangle$ is a function $f\colon W \to W'$ such that for all $x, y \in W$ and $y' \in W'$:

- if xRy, then $f(x)R'f(y)$ (homomorphism condition);
- if $f(x)R'y'$, then $\exists y \in W\colon xRy$ and $f(y) = y'$ (back condition).

Let us say that a *tense morphism* from $\langle W, R\rangle$ to $\langle W', R'\rangle$ is a function $f\colon W \to W'$ that is both a p-morphism from $\langle W, R\rangle$ to $\langle W', R'\rangle$ and a p-morphism from $\langle W, R^{-1}\rangle$ to $\langle W', R'^{-1}\rangle$. As usual, $\langle W', R'\rangle$ is a *p-morphic image* of $\langle W, R\rangle$ iff there is a *surjective* p-morphism from $\langle W, R\rangle$ to $\langle W', R'\rangle$. Similarly, let us say that $\langle W', R'\rangle$ is a *tense-morphic image* of $\langle W, R\rangle$ iff there is a surjective tense morphism from $\langle W, R\rangle$ to $\langle W', R'\rangle$.

Fact 5.2 (Separative Frames). The class of separative frames as in (Sep) is closed under disjoint unions and R-generated subframes. However, it is not closed under p-morphic images or even tense-morphic images.

Proof. Closure under taking disjoint unions is obvious and closure under R-generated subframes is easy to check. For the failure of closure under tense-morphic images, simply take the reflexive closure of the "forking" frame $\langle\{x_1, y_1, y_2\}, \{\langle x_1, y_1\rangle, \langle x_1, y_2\rangle\}\rangle$, which is a separative frame, the reflexive closure of the non-forking frame $\langle\{x, y\}, \{\langle x, y\rangle\}\rangle$, which is not separative, and the surjective tense morphism $\{\langle x_1, x\rangle, \langle y_1, y\rangle, \langle y_2, y\rangle\}$. ■

Since the validity of formulas in the tense language with the universal modality is preserved under taking tense-morphic images of frames (see [22] for the analogous result for the unimodal language extended with the universal modality), Fact 5.2 implies the following.

Fact 5.3 (Undefinability of Separativity). The class of separative frames is not definable in the tense language with the universal modality.

This is why we went beyond adding the universal modality and added the complement modality $\boxminus$, with which the universal modality can be expressed by $\Box\varphi \wedge \boxminus\varphi$. Unfortunately, adding the complement modality $\boxminus$ to capture separativity makes *completeness* proofs trickier, as explained in §6.

6. Considerations on Completeness

The cost of adding modalities like $\boxminus$ is well-known [20, 21], but worth repeating in our story here. It becomes clear when we think of our languages as interpreted in models with different relations for the modalities. Suppose we have a trimodal language with operators $\Box_1$, $\Box_2$, and $\Box_3$ interpreted in trirelational models $\langle W, R_1, R_2, R_3, V\rangle$ using the usual truth clauses:

- $\mathcal{M}, w \vDash \Box_i\varphi$ iff for all $v \in W$: if wR_iv, then $\mathcal{M}, v \vDash \varphi$.

Suppose we want to construct for a logic a canonical model in which R_2 is the converse of R_1, while R_3 is the complement of R_1. (Here we assume familiarity with the theory of canonical models, as in [8, §4].) As is well known, the converse part is no problem. The class of trirelational frames in which $R_2 = R_1^{-1}$ is defined by the conjunction of the axioms $p \to \Box_1\Diamond_2 p$ and $p \to \Box_2\Diamond_1 p$. Since these are Sahlqvist formulas, they are canonical, so the canonical frames for logics with these axioms will be such that $R_2 = R_1^{-1}$.

By contrast, the complement part presents difficulties. Since the validity of trimodal formulas is preserved under taking disjoint unions of trirelational frames, we cannot define the class of frames in which R_3 is the complement of R_1. We can define the class of frames in which $R_1 \cup R_3$ is an equivalence relation, using the **S5** axioms for the modality $\Box_{1,3}$ defined by $\Box_{1,3}\varphi := \Box_1\varphi \wedge \Box_3\varphi$. Since those axioms are Sahlqvist, the canonical frames for logics with those axioms will be such that $R_1 \cup R_3$ is an equivalence relation; then taking the submodel of the canonical model generated by a point w will give us a model based on a frame in which $R_1 \cup R_3$ is the universal relation, R_1, R_2, and R_3 retain properties preserved under generated subframes—such as $R_2 = R_1^{-1}$, (Sep), (Uni), and (Con^{-1})—and w satisfies the same formulas as before.[10] However, delicate work is required to not only make $R_1 \cup R_3$ the universal relation, but also make $R_1 \cap R_3 = \emptyset$, as required for R_3 to be the complement of R_1, *all the while preserving desired properties of the relations.* A "copying method" was developed by Vakarelov [38, 19] and generalized by Goranko [20] in order to achieve this purpose while preserving many standard first-order properties of the relations. But the situation becomes more complicated when we add further requirements on the relations, such as (Sep), (Uni), and (Con^{-1}), not to mention second-order requirements on R_1 and R_2 such as those imposed by the axioms Grz and Grz^{-1}.

In light of these considerations, the nominal tense language may seem to offer a simpler path. When the minimal nominal tense or modal logic is formulated with a Gabbay-style rule, one can construct canonical models in which every maximally consistent set contains a nominal [17, §5.2], which

[10] To be clear: the property (Con^{-1}) of frames $\langle W, R\rangle$ is not preserved under R-generated subframes, though it is preserved under $R \cup R^{-1}$-generated subframes. In the trirelational setting, we are thinking of (Con^{-1}) as $\forall y \forall y' \exists x(yR_2x \wedge y'R_2x)$. This property is indeed preserved by generated (i.e., $R_1 \cup R_2 \cup R_3$-generated) subframes of trirelational frames.

allows pure axioms such as those in AX_1 to enforce first-order properties on the canonical frame. (Without the Gabbay-style rule, more work is required, such as bulldozing the canonical model [7, §4].) However, since the axiom set AX_2 is only valid over *finite* frames, matters are not so simple. One may think of AX_2 as essentially AX_1 plus the axioms Grz and Grz^{-1}. As noted in §2, by themselves the axioms of AX_1 are no problem: by the general completeness result for logics with pure axioms (Theorem 2.6), the minimal nominal tense logic extended with AX_1 is complete with respect to its class of frames. By themselves the axioms Grz and Grz^{-1} in AX_2 are also not a problem: by a general completeness result of ten Cate [10, Thm. 8.2.13] and known facts about these axioms, the minimal nominal tense logic extended with Grz and Grz^{-1} is also complete with respect to its class of frames. The question is if one can repeat these success stories for the whole of AX_2.

7. Conclusion

There is a theme running through Theorems 3.6 and 4.2 and other examples of the impressive expressive power of the tense language extended with nominals or the complement modality—for example, that these languages can define the order structure of the natural numbers up to isomorphism (see [7, p. 61] and [21, p. 98]). The theme is that the nominals or complement modality are used only to express certain first-order properties of the relation R that are inexpressible in the tense language, while the second-order heavy lifting is done by formulas in the basic tense language, such as the Grzegorczyk or Löb axioms. Of course, there are many simple first-order properties of directed graphs that the tense languages with nominals or the complement modality cannot express, so success is not a foregone conclusion. But in a number of important cases, these languages can express the first-order properties we need. For Theorems 3.6 and 4.2, the first-order properties we need are just that R is separative and that any two points have a common R-predecessor. The basic tense language does the rest.

Our results raise a number of related questions. We have focused on the issue of definability with extended modal languages of the class of Medvedev frames. What about the class of Skvortsov frames? And what if we consider frames that do not contain all nonempty subsets of a set S, but only those subsets of cardinality greater than some fixed $\kappa \leq |S|$, as in [35]?

Let us finally return to the "tantalizing disjunction" at the end of §1. Logicians of the Sofia school, the Amsterdam school, and others have developed a set of techniques for proving the frame completeness (or incompleteness) of modal logics with nominals [7, 17, 10], the difference modality [33], or the complement modality [19, 18, 20]. In addition, techniques for proving completeness and filtration for the logic **Grz** are well known [9],[11] and even the nominal extension of **Grz** has been studied [6]. Could some of these techniques be used to prove the frame completeness of a logic extending the minimal nominal, differential, or complementary tense logics with our Medvedev-defining axioms, perhaps adding some recursive set of Medvedev-valid axioms? And what about the logic of Medvedev frames in the basic tense language without extra devices—is it finitely or at least recursively axiomatizable? We leave these questions as open problems for future research.

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Appendix

In this appendix, we prove the claims made in Remark 2.1. As usual, **K4** is the smallest normal unimodal logic containing the 4 axiom, $\Box p \to \Box\Box p$. Let **K4Uni** be the smallest normal unimodal logic containing 4 and

$$(\Diamond(p_1 \wedge \Box q) \wedge \Diamond(p_2 \wedge \Box q)) \to \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q). \tag{Uni}$$

We claim that for each $n \geq 2$, the following is a theorem of **K4Uni**:

$$(\bigwedge_{1\leq i\leq n} \Diamond(p_i \wedge \Box q)) \to \Diamond(\bigwedge_{1\leq i\leq n} \Diamond p_i \wedge \Box\Diamond q). \tag{II}$$

For induction, assuming we can derive (II), we will show that we can derive

$$(\bigwedge_{1\leq i\leq n+1} \Diamond(p_i \wedge \Box q)) \to \Diamond(\bigwedge_{1\leq i\leq n+1} \Diamond p_i \wedge \Box\Diamond q). \tag{III}$$

Let $\varphi := \bigwedge_{1\leq i\leq n} \Diamond p_i$, $\psi := \Diamond q$, and $\psi' := \psi \vee q$. Then (II) becomes

$$(\bigwedge_{1\leq i\leq n} \Diamond(p_i \wedge \Box q_i)) \to \Diamond(\varphi \wedge \Box\psi). \tag{IV}$$

[11] Moreover, as an anonymous referee pointed out, we can see that **GrzUni**, the smallest normal unimodal logic containing both the Grz and Uni axioms, is characterized by a class of finite frames as follows. First, by Remark 2.4, **GrzUni** is equal to the least normal extension of **Grz** with the Gödel translation of the superintuitionistic Kreisel-Putnam axiom. This implies that **GrzUni** is the *greatest modal companion* of the Kreisel-Putnam logic [12, Corollary 9.64]. Second, the Kreisel-Putnam logic is characterized by a class of finite frames [16], and this property transfers from any superintuitionistic logic to its greatest modal companion [12, p. 328]. (Incidentally, it was also claimed in [16] that Medvedev's logic is decidable, but the argument contains a mistake explained in [37].)

As a substitution instance of (Uni), we have

$$(\Diamond(\varphi \wedge \Box\psi') \wedge \Diamond(p_{n+1} \wedge \Box\psi')) \to \Diamond(\Diamond\varphi \wedge \Diamond p_{n+1} \wedge \Box\Diamond\psi'). \tag{V}$$

The antecedent of (V) is clearly derivable from that of (III) using (IV). Thus, to show that (III) is derivable, it suffices to establish the derivability of

$$\Diamond(\Diamond\varphi \wedge \Diamond p_{n+1} \wedge \Box\Diamond\psi') \to \Diamond(\bigwedge_{1\leq i\leq n+1} \Diamond p_i \wedge \Box\Diamond q). \tag{VI}$$

Unpacking φ and ψ', (VI) works out to

$$\Diamond(\Diamond(\bigwedge_{1\leq i\leq n} \Diamond p_i) \wedge \Diamond p_{n+1} \wedge \Box\Diamond(\Diamond q \vee q)) \to \Diamond(\bigwedge_{1\leq i\leq n+1} \Diamond p_i \wedge \Box\Diamond q),$$

which is easily seen to be a theorem of **K4**.

For the second part of Remark 2.1, we claim that for each $m \geq 3$, the following formula is a theorem of **K4Uni**:

$$\big((\bigwedge_{1\leq i\leq m} \Diamond\Box r_i) \wedge \neg\Diamond(\bigvee_{1\leq i\leq m-1} (r_i \wedge r_m)\big) \to \Diamond(\bigwedge_{1\leq i\leq m-1} \Diamond\Box r_i \wedge \neg\Diamond\Box r_m). \tag{VII}$$

First, observe that the following is a theorem of **K4**:

$$\big((\bigwedge_{1\leq i\leq m} \Diamond\Box r_i) \wedge \neg\Diamond(\bigvee_{1\leq i\leq m-1} (r_i \wedge r_m)\big) \to \big(\bigwedge_{1\leq i\leq m-1} \Diamond(\Box r_i \wedge \Box\neg r_m)\big). \tag{VIII}$$

Next, observe that as an instance of (II) for $n = m-1$, substituting $\Box r_i$ for p_i and $\neg r_m$ for q, we have:

$$\big(\bigwedge_{1\leq i\leq m-1} \Diamond(\Box r_i \wedge \Box\neg r_m)\big) \to \Diamond(\bigwedge_{1\leq i\leq m-1} \Diamond\Box r_i \wedge \Box\Diamond\neg r_m). \tag{IX}$$

Putting together (VIII), (IX), and $\Box\Diamond\neg r_m \to \neg\Diamond\Box r_m$, we can derive (VII). From here it is not difficult to show that for $1 < k \leq m$, we can also derive

$$\big((\bigwedge_{1\leq i\leq m} \Diamond\Box r_i) \wedge \neg\Diamond(\bigvee_{1\leq i,j\leq m,\, i\neq j} (r_i \wedge r_j)\big) \to \Diamond(\bigwedge_{1\leq i\leq k-1} \Diamond\Box r_i \wedge \bigwedge_{k\leq j\leq m} \neg\Diamond\Box r_j).$$

Finally, let **GrzUni** be the smallest normal unimodal logic containing Grz and Uni, and let **Medv** be the unimodal logic of Medvedev frames.

PROPOSITION 7.1. There are continuum-many normal unimodal logics between **GrzUni** and **Medv**.

The following proof combines observations of Matthew Harrison-Trainor and James Walsh (personal communication) together with the well-known results of Fine [13]. We will assume that the reader has Fine's paper at hand, so we will not repeat his definitions.

PROOF. Consider the frames $\mathfrak{F}_0$, $\mathfrak{F}_1$, etc. in [13]. Let us show that for each $\mathfrak{F}_n$ and Medvedev frame $\mathfrak{G}$, there is no p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}_n$. Suppose for reductio that there is such a p-morphism f. Let us treat $\mathfrak{G}$ as $\langle \wp(S) \setminus \{\emptyset\}, \supseteq \rangle$, and let R_n be the edge relation in $\mathfrak{F}_n$. Since f is onto, for the endpoints 1 and 2 in $\mathfrak{F}_n$ there are $x, y \in \mathfrak{G}$ such that $f(x) = 1$ and $f(y) = 2$. Then we can pick $\{a\} \subseteq x$ and $\{b\} \subseteq y$ in $\mathfrak{G}$, and we must have $f(\{a\}) = 1$ and $f(\{b\}) = 2$ by the homomorphism condition on f. In addition, $\{a, b\}$ is in $\mathfrak{G}$, and $\{a, b\} \supseteq \{a\}$ and $\{a, b\} \supseteq \{b\}$, so $f(\{a, b\})R_n 1$ and $f(\{a, b\})R_n 2$. It follows by the definition of $\mathfrak{F}_n$ that either $f(\{a, b\}) = 0$ or $f(\{a, b\}) \geq 5$. In either case, $f(\{a, b\})R_n 3$ or $f(\{a, b\})R_n 4$. Where $k \in \{3, 4\}$, suppose $f(\{a, b\})R_n k$. Then by the back condition on f, there is a $z \in \mathfrak{G}$ such that $\{a, b\} \supseteq z$ and $f(z) = k$. But the only candidates for z are $\{a, b\}$, $\{a\}$, and $\{b\}$, and f does not map any of these sets to k, so we have a contradiction.

Since there is no p-morphism from a Medvedev frame onto $\mathfrak{F}_n$, and the class of Medvedev frames is closed under taking point-generated subframes, it follows by Lemma 1 of [13, §2] that the *frame-formula* $A_{\mathfrak{F}_n}$ is unsatisfiable over any Medvedev frame. Thus, $\neg A_{\mathfrak{F}_n} \in \mathbf{Medv}$ for each $n \in \mathbb{N}$. For $X \subseteq \mathbb{N}$, let **GrzUniX** be the least normal unimodal extension of **GrzUni** containing $\neg A_{\mathfrak{F}_n}$ for each $n \in X$, so $\mathbf{GrzUniX} \subseteq \mathbf{Medv}$.

Finally, it is easy to see that each $\mathfrak{F}_n$ validates **GrzUni**, by virtue of being a finite poset that satisfies the first-order correspondent for Uni given in Fact 2.3.3. Thus, by the same argument as in Theorem 1 of [13], for any distinct $X, Y \subseteq \mathbb{N}$, we have $\mathbf{GrzUniX} \neq \mathbf{GrzUniY}$. ∎

References

[1] VAN BENTHEM, J. F. A. K., and W. J. BLOK, 'Transitivity follows from Dummett's axiom', *Theoria*, 44 (1978), 2, 117–118.

[2] VAN BENTHEM, JOHAN, 'Minimal Deontic Logics (Abstract)', *Bulletin of the Section of Logic*, 8 (1979), 1, 36–41.

[3] VAN BENTHEM, JOHAN, 'The Range of Modal Logic', *Journal of Applied Non-Classical Logics*, 9 (1999), 2, 407–442.

[4] VAN BENTHEM, JOHAN, 'Correspondence Theory', in D. M. Gabbay, and F. Guenthner, (eds.), *Handbook of Philosophical Logic*, vol. 3, 2 edn., Springer, Dordrecht, 2001, pp. 325–408.

[5] van Benthem, Johan, and Guram Bezhanishvili, 'Modal Logics of Space', in Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem, (eds.), *Handbook of Spatial Logics*, Springer, 2007, pp. 217–298.

[6] Bezhanishvili, Nick, and Balder ten Cate, 'Transfer Results for Hybrid Logic. Part I: The Case Without Satisfaction Operators', *Journal of Logic and Computation*, 16 (2006), 2, 177–197.

[7] Blackburn, Patrick, 'Nominal Tense Logic', *Notre Dame Journal of Formal Logic*, 34 (1993), 1, 56–83.

[8] Blackburn, Patrick, Maarten de Rijke, and Yde Venema, *Modal Logic*, Cambridge University Press, New York, 2001.

[9] Boolos, George, *The Logic of Provability*, Cambridge University Press, New York, 1995.

[10] ten Cate, Balder, *Model theory for extended modal languages*, Ph.D. thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS-2005-01.

[11] ten Cate, Balder, and Tadeusz Litak, 'The Importance of Being Discrete', Tech. Rep. PP-2007-39, Institute for Logic, Language and Computation, University of Amsterdam, 2007.

[12] Chagrov, Alexander, and Michael Zakharyaschev, *Modal Logic*, Clarendon Press, Oxford, 1997.

[13] Fine, Kit, 'An ascending chain of S4 logics', *Theoria*, 40 (1974), 2, 110–116.

[14] Fontaine, Gaëlle, *Axiomatization of ML and Cheq*, Master's thesis, University of Amsterdam, 2006. ILLC Master of Logic Series MoL-2006-08.

[15] Fontaine, Gaëlle, 'ML is not finitely axiomatizable over Cheq', in Guido Governatori, Ian Hodkinson, and Yde Venema, (eds.), *Advances in Modal Logic*, vol. 6, College Publications, London, 2006, pp. 139–146.

[16] Gabbay, Dov M., 'The Decidability of the Kreisel-Putnam System', *The Journal of Symbolic Logic*, 35 (1970), 3, 431–437.

[17] Gargov, George, and Valentin Goranko, 'Modal Logic with Names', *Journal of Philosophical Logic*, 22 (1993), 6, 607–636.

[18] Gargov, George, and Solomon Passy, 'A Note on Boolean Modal Logic', in P. P. Petkov, (ed.), *Mathematical Logic*, Plenum Press, New York, 1990, pp. 299–309.

[19] Gargov, George, Solomon Passy, and Tinko Tinchev, 'Modal Environment for Boolean Speculations', in D. Scordev, (ed.), *Mathematical Logic and Its Applications*, Plenum Press, New York, 1987, pp. 253–263.

[20] Goranko, Valentin, 'Completeness and Incompleteness in the Bimodal Base $\mathcal{L}(R, -R)$', in P. P. Petkov, (ed.), *Mathematical Logic*, Plenum Press, New York, 1990, pp. 311–326.

[21] Goranko, Valentin, 'Modal Definability in Enriched Languages', *Notre Dame Journal of Formal Logic*, 31 (1990), 1, 81–105.

[22] Goranko, Valentin, and Solomon Passy, 'Using the Universal Modality: Gains and Questions', *Journal of Logic and Computation*, 2 (1992), 1, 5–30.

[23] Goranko, Valentin, and Dimiter Vakarelov, 'Sahlqvist Formulas in Hybrid Polyadic Modal Logics', *Journal of Logic and Computation*, 11 (2001), 5, 737–754.

[24] Hamkins, Joel David, George Leibman, and Benedikt Löwe, 'Structural connections between a forcing class and its modal logic', *Israel Journal of Mathematics*,

207 (2015), 2, 617–651.

[25] Hamkins, Joel David, and Benedikt Löwe, 'The Modal Logic of Forcing', *Transactions of the American Mathematical Society*, 360 (2008), 4, 1793–1817.

[26] Humberstone, Lloyd, 'Inaccessible Worlds', *Notre Dame Journal of Formal Logic*, 24 (1983), 3, 346–352.

[27] Humberstone, Lloyd, 'Modal Logic for Other-World Agnostics: Neutrality and Halldén Completeness', *Journal of Philosophical Logic*, 36 (2007), 1, 1–32.

[28] Jech, Thomas, *Multiple Forcing*, Cambridge University Press, Cambridge, 1986.

[29] Maksimova, L. L., V. B. Shehtman, and D. P. Skvortsov, 'The impossibility of a finite axiomatization of Medvedev's logic of finitary problems', *Soviet Mathematics Doklady*, 20 (1979), 394–398.

[30] Medvedev, Yu. T., 'Finite problems', *Soviet Mathematics Doklady*, 3 (1962), 227–230.

[31] Medvedev, Yu. T., 'Interpretation of logical formulas by means of finite problems', *Soviet Mathematics Doklady*, 7 (1966), 857–860.

[32] Prucnal, Tadeusz, 'On Two Problems of Harvey Friedman', *Studia Logica*, 38 (1979), 3, 247–262.

[33] de Rijke, Maarten, 'The Modal Logic of Inequality', *The Journal of Symbolic Logic*, 57 (1992), 2, 566–584.

[34] Segerberg, Krister, 'A note on the logic of elsewhere', *Theoria*, 46 (1980), 2-3, 183–187.

[35] Shehtman, V. B., and D. P. Skvortsov, 'Logics of Some Kripke Frames Connected with Medvedev Notion of Informational Types', *Studia Logica*, 45 (1986), 1, 101–118.

[36] Shehtman, Valentin, 'Modal Counterparts of Medvedev Logic of Finite Problems Are Not Finitely Axiomatizable', *Studia Logica*, 49 (1990), 3, 365–385.

[37] Skvortsov, D. P., 'Logic of infinite problems and Kripke models on atomic semilattices of sets', *Soviet Mathematics Doklady*, 20 (1979), 360–363.

[38] Vakarelov, Dimiter, 'Modal Characterization of the Classes of Finite and Infinite Quasi-Ordered Sets', in P. P. Petkov, (ed.), *Mathematical Logic*, Plenum Press, New York, 1990, pp. 373–387.

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