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SMOOTHNESS OF PION AMPLITUDES AND PCAC*

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We point out the essentially obvious result that if the massless pion amplitude is the smooth finite limit of the massive pion world, the Adler self-consistency condition must be true.

There has in recent years been some interest in understanding soft pion results from the viewpoint of unitarity and analyticity. Indeed Mandelstam [1] has been able to show that most of the results of current algebra follow from the Adler self-consistency condition [2] alone, thereby emphasizing that this is the crucial result to be understood outside of the current algebra framework. Unfortunately, his attempt to derive this condition itself from the analyticity of Regge couplings at $t = 0$ did not meet with complete success.

More recently, Rosenzweig and Veneziano [3] have given a proof of the Adler condition in a world of massless pions by considering the newly

derived energy-momentum conservation sum rules for multiparticle reactions. Their proof rests on the assumption that the inclusive process $\pi\pi \rightarrow \pi\pi X$ is dominated by a single pion pole for small momentum transfer, especially when the masses of the pions tend to zero. As emphasized by Dashen and Gross [4] this need not necessarily be true, since, as $m_\pi \rightarrow 0$, multipion singularities can become equally important and so the proof of Rosenzweig and Veneziano may not, in general, be valid. So it is still an open question as to how one might derive Adler self-consistency from some acceptable set of assumptions, such as crossing, unitarity, and analyticity properties, rather than PCAC, and so place current algebra results on a rigorous footing.

In this connection the purpose of this note is to point out a somewhat trivial result, based on assuming that a smooth finite limit exists as $m_\pi \rightarrow 0$. This is the result that if the $\pi\pi$ amplitude (with all pions having mass m_π) continues to satisfy forward dispersion relations, with a finite number of subtractions, as m_π^2 is decreased to ever smaller values, then the following result must be true:

$$\lim_{m_\pi^2 \rightarrow 0} F(s = 4m_\pi^2, t = 0, u = 0) = 0. \quad (1)$$

This is, of course, just the Adler condition for massless pions.

We begin this discussion by assuming that, as $m_\pi^2 \rightarrow 0$, the $\pi\pi$ amplitude does not diverge and that the amplitude for massless pion scattering is obtained by taking the limit $m_\pi^2 \rightarrow 0$ of the massive pion amplitude, $F(s, t; m_\pi^2)$. This is the smoothness hypothesis.

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Now the amplitude $F(s, t; m_\pi^2)$ satisfies a fixed t -dispersion relation for $-28m_\pi^2 < t < 4m_\pi^2$ with a finite number of subtractions-- in fact just two. To be explicit, we choose F to be the amplitude with t -channel isospin equal to zero.* Such an amplitude is s - u symmetric and we have

$$F(s, t; m_\pi^2) = F(s_0, t; m_\pi^2) + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dy \left(\frac{1}{y-s} + \frac{1}{y+s+t-4m_\pi^2} - \frac{1}{y-s_0} - \frac{1}{y+s_0+t-4m_\pi^2} \right) \times A(y, t; m_\pi^2) \quad (2)$$

where $s = s_0$ is the subtraction point and A is the s -channel absorptive part. Now, for $m_\pi \neq 0$, such an effectively twice-subtracted dispersion relation holds rigorously for $t \in (-28m_\pi^2, 4m_\pi^2)$.† We choose $t = 0$, since this value is in this region for ever smaller m_π^2 and choose the subtraction point somewhere in the physical region, so that $s_0 \neq 0$ as $m_\pi^2 \rightarrow 0$.

* If F has $I_t = 1$, it will trivially vanish when $s = u$ from Bose statistics (in particular at the point $s = t = u = 0$). We choose $I_t = 0$, rather than $I_t = 2$, since the s -channel absorptive part is then definitely non-negative.

† Of course, these analyticity properties only rigorously hold for $m_\pi \neq 0$. However, our assumption is that $F(s, t; 0)$ is the limit of $F(s, t; m_\pi^2 \neq 0)$.

The essential points in deriving Eq. (1) are (i) to recall that $F(s, 0; m_\pi^2)$ and $A(y, 0; m_\pi^2)$ do not diverge as $m_\pi^2 \rightarrow 0$, by assumption, and (ii) to notice that, if $s = 0(m_\pi^2)$ and s_0 is fixed, then we will have a singularity, as $m_\pi^2 \rightarrow 0$, from the end-point of integration at $y = 4m_\pi^2$, unless $A(y, 0; m_\pi^2)$ vanishes sufficiently fast. In fact fast enough that Eq. (1) is true.

To see this more explicitly consider the near threshold integral

$$I(x; m_\pi^2) = \int_{4m_\pi^2}^{Y+4m_\pi^2} \frac{dy}{y-x} A(y, 0) \quad (3)$$

where Y is some fixed positive number, independent of m_π^2 , but close to threshold. Now

$$A(y, 0) = \sum_{\ell} (2\ell + 1) \operatorname{Im} f_{\ell}(y) \quad (4)$$

and on using partial wave unitarity we have

$$A(y, 0) = \left(\frac{y - 4m_\pi^2}{y} \right)^{\frac{1}{2}} \sum_{\ell} (2\ell + 1) \times \left[|f_{\ell}(y)|^2 + \frac{1}{4} \left(\frac{y}{y - 4m_\pi^2} \right) \left(1 - \eta_{\ell}(y)^2 \right) \right] \quad (5)$$

where η_{ℓ} is the inelasticity factor, $0 \leq \eta_{\ell} \leq 1$, and $\eta_{\ell}(y) \equiv 1$ for $4m_\pi^2 \leq y \leq 16m_\pi^2$. For $y \sim 4m_\pi^2$, the partial waves have the threshold behavior

$$\text{Re } f_\ell(y) \sim (y - 4m_\pi^2)^\ell, \quad \text{Im } f_\ell(y) \sim (y - 4m_\pi^2)^{2\ell+1/2} y^{-1/2}. \quad (6)$$

Let us consider the leading s-wave contribution to Eq. (3)

$$I_0(x; m_\pi^2) = \int_{4m_\pi^2}^{Y+4m_\pi^2} \frac{dy}{y-x} \left(\frac{y - 4m_\pi^2}{y} \right)^{\frac{1}{2}} \times \left(\rho^2 + \left(\frac{y}{y - 4m_\pi^2} \right) \tau^2 \theta(y - 16m_\pi^2) \right) \quad (7)$$

where $\rho = \text{Re } f_0(4m_\pi^2)$, $\tau^2 = \frac{1}{4}[1 - \eta_0(y = n^2 m_\pi^2)^2]$. We have included the inelasticity factor, even though $\eta_0 = 1$ for $y < 16m_\pi^2$, because, as $m_\pi^2 \rightarrow 0$, the integration range of Eq. (7) will include all multi-pion thresholds ($y = n^2 m_\pi^2$) and not just two pion threshold.

We can clearly evaluate Eq. (7) explicitly. We distinguish two different situations:

$$(a) \quad |x| > Y + m_\pi^2$$

$$I_0(x; m_\pi^2) = \rho^2 \left(1 - \left(1 - \frac{4m_\pi^2}{x} \right)^{\frac{1}{2}} \right) \ln \left(\frac{Y}{m_\pi^2} \right) + \tau^2 \left(1 - \frac{1}{\left(1 - \frac{4m_\pi^2}{x} \right)^{\frac{1}{2}}} \right) \ln \left(\frac{\alpha Y}{m_\pi^2} \right) + \left(\rho^2 \left(1 - \frac{4m_\pi^2}{x} \right)^{\frac{1}{2}} - \frac{\tau^2}{\left(1 - \frac{4m_\pi^2}{x} \right)^{\frac{1}{2}}} \right) \ln \left(\frac{x - Y}{x} \right) \quad (8)$$

where $\alpha = (7 - 4\sqrt{3})$. Since $x \neq 0$ as $m_\pi^2 \rightarrow 0$ this integral is perfectly well-behaved in this limit.

$$(b) \quad x = 0(m_\pi^2) < 4m_\pi^2. \quad \text{In particular, let } x = 4m_\pi^2/\beta, \quad \beta > 1$$

$$I_0(x; m_\pi^2) = \rho^2 \ln \left(\frac{Y}{m_\pi^2} \right) + \tau^2 \ln \left(\frac{\alpha Y}{m_\pi^2} \right) - \frac{2\tau^2}{(\beta - 1)^{\frac{1}{2}}} \times \tan^{-1} \left(\frac{\sqrt{3}}{2(\beta - 1)^{\frac{1}{2}}} \right) + 2 \left(\frac{\tau^2}{(\beta - 1)^{\frac{1}{2}}} - \rho^2 (\beta - 1)^{\frac{1}{2}} \right) \tan^{-1} \left[\frac{1}{(\beta - 1)^{\frac{1}{2}} \left(1 + \frac{4m_\pi^2}{Y} \right)^{\frac{1}{2}}} \right]. \quad (9)$$

We see that as $m_\pi^2 \rightarrow 0$ we have a logarithmic divergence unless ρ^2 and τ^2 vanish suitably (note: ρ^2 and τ^2 must vanish separately since each is non-negative definite). Situation (a) corresponds to integrating the near-threshold part of the subtraction terms in Eq. (2), when $x = s_0$ or $4m_\pi^2 - s_0$. This integral is not singular as $m_\pi^2 \rightarrow 0$. If we choose $s = 0(m_\pi^2)$ in Eq. (2), situation (b) occurs and we have the $\ln(1/m_\pi^2)$ singularity of Eq. (9). It is easy to check that the near-threshold contributions of all other partial waves, $\text{Re } f_\ell(y)$, $\ell \geq 1$, are well-behaved in the massless pion limit. However, we do obtain additional singularities from the inelasticity factor for each ℓ , i.e., we have terms like

$$\frac{1}{4} \sum_{\ell \geq 1} (2\ell + 1) [1 - \eta_\ell(n^2 m_\pi^2)^2] \ln \left(\frac{\alpha Y}{m_\pi^2} \right). \quad (10)$$

It is straightforward to see that no term, proportional to $\ln(1/m_\pi^2)$, could arise from the rest of the integration region of Eq. (2) to cancel the contribution of Eqs. (9) and (10), since $A(y, 0; m_\pi^2)$ is, by assumption, divergenceless. Since the limit as $m_\pi^2 \rightarrow 0$ of $\left(F(s = 0(m_\pi^2), t = 0; m_\pi^2) - F(s_0, 0; m_\pi^2)\right)$ is smooth, by hypothesis, we must conclude that

$$\lim_{m_\pi^2 \rightarrow 0} \eta_\ell(s = n^2 m_\pi^2) = 1 \quad (11)$$

for all $\ell \geq 0$, and

$$\lim_{m_\pi^2 \rightarrow 0} \operatorname{Re} f_0(4m_\pi^2) = 0. \quad (12)$$

This constraint on the s-wave amplitude is, in fact, just the Adler condition for the elastic scattering of massless pions:

$$\lim_{m_\pi^2 \rightarrow 0} F(s = 4m_\pi^2, t = 0; m_\pi^2) = 0,$$

since higher partial waves vanish kinematically at threshold.*

* In terms of partial waves, with definite s-channel isospin, condition (12) is

$$\lim \frac{1}{3} \left(\operatorname{Re} f_0^0(4m_\pi^2) \right)^2 + \frac{5}{3} \left(\operatorname{Re} f_0^2(4m_\pi^2) \right)^2 = 0.$$

Since each term must vanish separately Eq. (1) is, in fact,

$$\lim_{m_\pi^2 \rightarrow 0} F^I(4m_\pi^2, 0; m_\pi^2) = 0 \quad \text{for both } I_s = 0, 2.$$

Eq. (11) means all inelastic pion processes, $\pi\pi \rightarrow n\pi$, with even $n > 2$, vanish when each pion has zero four-momentum, whilst Eq. (12) implies the vanishing of elastic $\pi\pi$ scattering in the same zero momentum limit.

We have assumed that the massless pion amplitude is the finite limit of the massive pion world reached by smooth continuation. We see that this assumption ensures that the Adler self-consistency condition is true for massless pion amplitudes.

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