Title

COURSE IN THE THEORY AND DESIGN OF PARTICLE ACCELERATORS - LECTURE VII

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Course in the Theory and Design
of Particle Accelerators

LECTURE VII
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DETAILS OF ACCELERATING PROCESS IN THE CONSTANT FREQUENCY CYCLOTRON

In this lecture we will consider some aspects of the accelerating process in the CW (continuous wave) Cyclotron. Relations are developed by which required dee voltage may be estimated, and phase angle of the particle be predicted. Nomenclature for symbols used in this discussion is given on page 14.

Figure 1 shows the dees in plan. The path of the particle is approximately a spiral. The particle is accelerated each time it crosses the gap. Note that one cycle of accelerating voltage corresponds to one revolution of the particle. Consider a particle

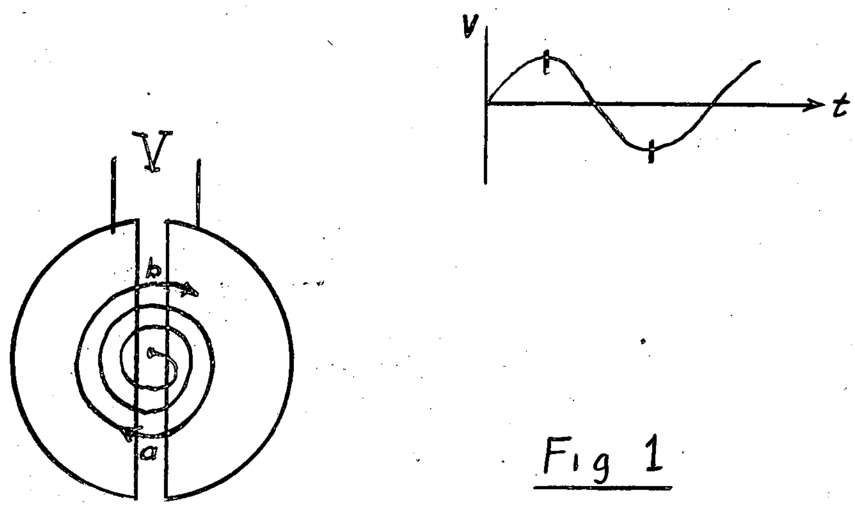


Fig 1

at "a" in phase with the voltage. It is accelerated in the direction shown. One half revolution later (at b) the voltage gradient has reversed direction but the particle is again accelerated because its direction has also reversed. Note that at any instant only one half of the gap furnishes acceleration in the correct direction.

The positions of the ion and the accelerating vector can be superimposed on the plan view of the dees for better under standing of the process. (See Figure 2).

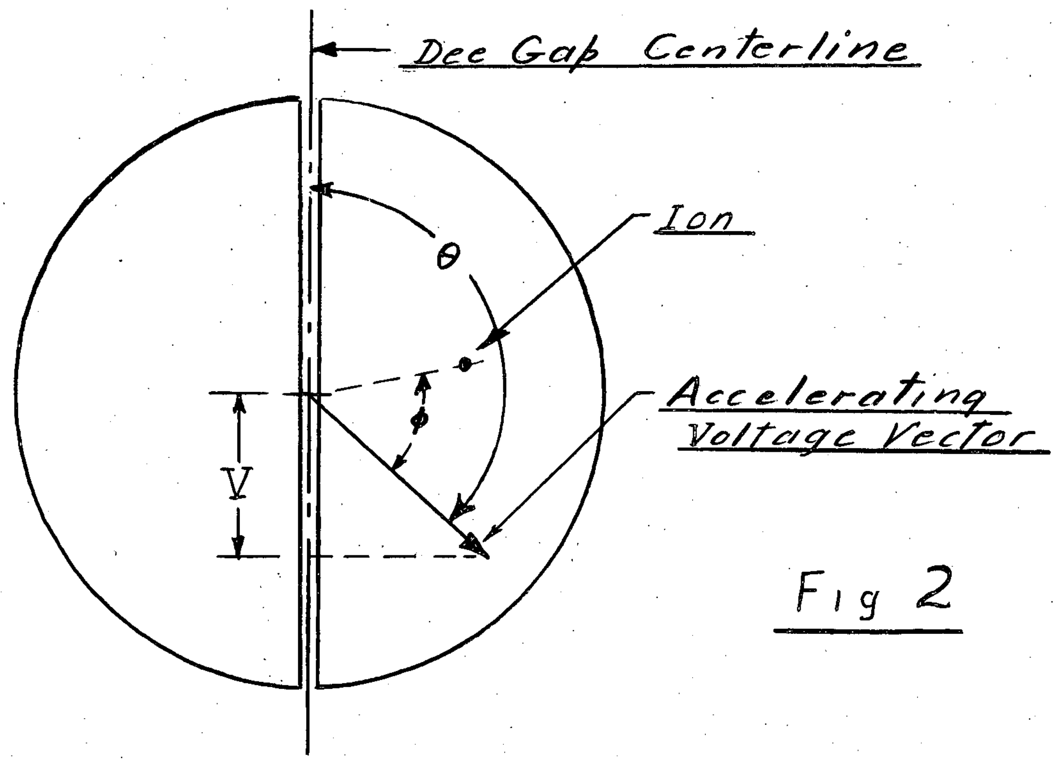
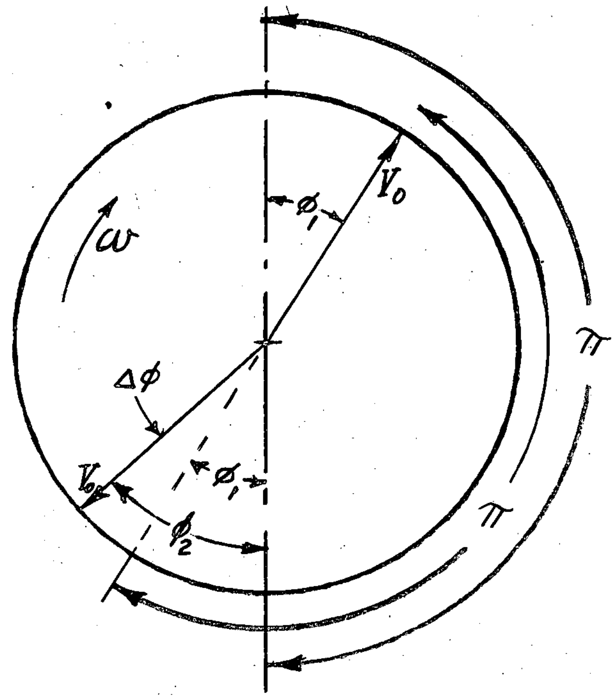


Fig 2

θ is the instantaneous angular position of the voltage vector and ϕ is the angle by which the ion position, measured backward, differs from the position of the voltage vector. The instantaneous value of the accelerating voltage (V) is as shown. At the instant the ion first crosses the gap, $\theta = \phi$ and since the acceleration takes place only while the ion is in the gap, the accelerating voltage is

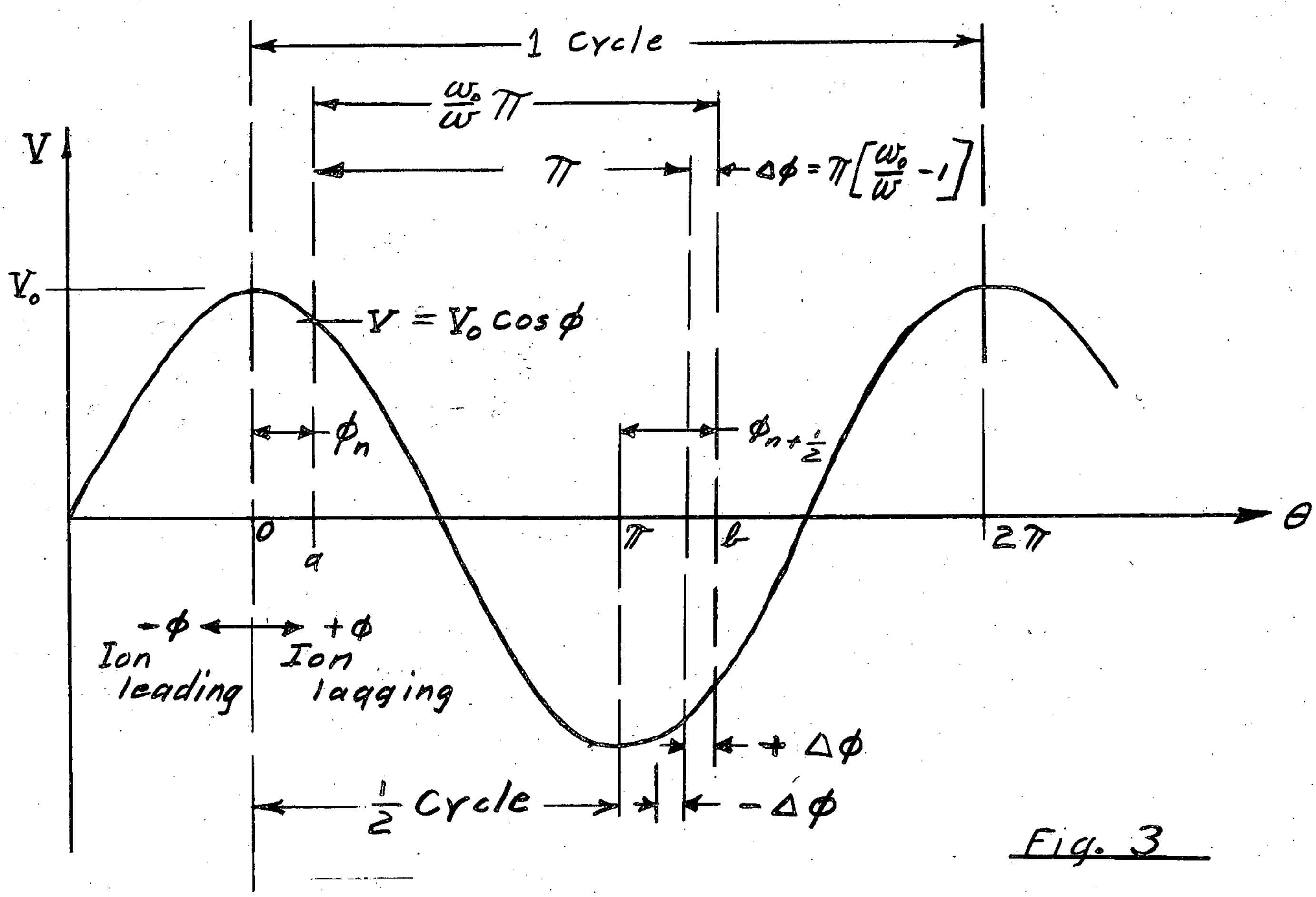
$$V = V_o \cos \theta = V_o \cos \phi \tag{1}$$

The fact that cos θ goes through both positive and negative values during one cycle (one revolution of the particle) may seem to indicate alternate accelerations and decelerations, but such is not the case. When cos θ is negative, the tangential velocity of the particle is in the negative direction and the total result is two accelerations per revolution. Note, however, that if $\phi \geq 90°$, the particle will cross the gap at a time when it is in the wrong half of the gap for acceleration and will be decelerated.



ω
V_0
ϕ_1
π
π
Δφ
V_0
ϕ_1
ϕ_2

Fig. 2a



1 Cycle
$\frac{\omega_0}{\omega}\pi$
π
$\Delta\phi = \pi\left[\frac{\omega_0}{\omega} - 1\right]$
V
V_0
$V = V_0 \cos\phi$
ϕ_n
$\phi_{n+\frac{1}{2}}$
θ
0
a
π
b
2π
−φ
+φ
Ion leading
Ion lagging
+Δφ
$\frac{1}{2}$ Cycle
−Δφ

Fig. 3

Figure 3 is another way of representing the same data shown in Figure 2a, but this picture has the advantage of showing succeeding cycles without the clutter that would result if Figure 2 were used to portray more than the events in one cycle. Figure 3 could be extended to show all the cycles (and particle revolutions) from injection to final energy; but for our purpose, one cycle will suffice.

Figure 2a and 3 show the case where the particle is lagging the voltage by the phase angle ϕ at the instant shown at "a." Further--the circular frequency (ω) of the ion is assumed less than the circular frequency of the voltage (ω_o); so that in one half revolution the phase angle increases by an amount ($\Delta\phi$). Algebraic sign conventions are as shown, namely:

Ion lagging voltage	ϕ is (+)
Ion leading voltage	ϕ is (-)
Phase angle increasing	($\Delta\phi$) is (+)
Phase angle decreasing	($\Delta\phi$) is (-)

Now in one half revolution the angular displacements of both the ion and the voltage vector is π radians. However, in the time interval for one half revolution of the ion, the voltage vector turns through ($\pi + \Delta\phi$) radians.

$$\omega_o t = \pi + \Delta\phi$$

$$\omega t = \pi$$

combining

$$\pi \left(\frac{\omega_o}{\omega}\right) = \pi + \Delta\phi$$

or

$$\Delta\phi = \pi \left[\frac{\omega_o}{\omega} - 1\right]$$

$$\Delta\phi = -\pi \left[\frac{\omega - \omega_o}{\omega}\right] \qquad (2)$$

The energy gain for a particle accelerated through V volts is:

$$E = eV$$

and the energy increase per one half revolution in the cyclotron is:

$$\Delta E = eV = eV_o \cos\phi \qquad (3)$$

With equations (2) and (3) we can form the ratio of the energy increase per unit change in phase angle $\Delta E/\Delta\phi$.

$$\frac{\Delta E}{\Delta\phi} = \frac{eV_o \cos\phi}{-\pi\left[\frac{\omega - \omega_o}{\omega}\right]} \qquad (4)$$

Although the ΔEs occur stepwise, there are sufficient steps that we can change from finite changes to differentials without spoiling the argument. (In the 60 " cyclotron, the ions get approximately 160 accelerations)

$$\frac{dE}{d\phi} = -\frac{eV_o \cos\phi}{\pi\left[\frac{\omega - \omega_o}{\omega}\right]}$$

or $$\int_{\phi_1}^{\phi_2} \cos \phi \, d\phi = - \frac{\pi}{eV_o} \int_{E_1}^{E_2} \left[\frac{\omega - \omega_o}{\omega} \right] dE \qquad (5)$$

integrating $$\sin \phi_2 - \sin \phi_1 = - \frac{\pi}{eV_o} \int_{E_1}^{E_2} \left[\frac{\omega - \omega_o}{\omega} \right] dE \qquad (6)$$

Now recall that in a magnetic field

$$R = \frac{mvc}{Be}$$

and

$$m = \frac{E}{c^2}$$

and

$$\omega = \frac{v}{R}$$

combining $$\omega = \frac{Bec}{E} \qquad \text{Larmour frequency} \qquad (7)$$

And we define B_o as the field corresponding to oscillator frequency and rest energy of the particle.

$$\omega_o = \frac{B_o ec}{E_o} \qquad (8)$$

then $$\frac{\omega - \omega_o}{\omega} = \frac{\frac{Bec}{E} - \frac{B_o ec}{E_o}}{\frac{Bec}{E}} = \left[\frac{B}{E} - \frac{B_o}{E_o} \right] \frac{E}{B}$$

$$\frac{\omega - \omega_o}{\omega} = 1 - \frac{EB_o}{E_o B} \qquad (9)$$

since

$$E = E_k + E_o \qquad (10)$$

$$\frac{\omega - \omega_o}{\omega} = 1 - \frac{(E_k + E_o)B_o}{E_o B}$$

$$\frac{\omega - \omega_o}{\omega} = \frac{E_o B - E_k B_o - E_o B_o}{E_o B}$$

$$\Delta B = B - B_o \quad \text{or} \quad B = B_o + \Delta B \qquad (11)$$

$$\frac{\omega - \omega_o}{\omega} = \frac{E_o(B_o + \Delta B) - E_k B_o - E_o B_o}{E_o B}$$

$$= \frac{E_o \Delta B - E_k B_o}{E_o B}$$

$$\frac{\omega - \omega_o}{\omega} = \frac{\Delta B}{B} - \frac{E_k B_o}{E_o B} \qquad (12)$$

Now in actual machines B differs from B_o by only one or two percent. Therefore:

$$\frac{B}{B_o} \approx 1$$

and

$$\frac{\omega - \omega_o}{\omega} = \left[\frac{\Delta B}{B} - \frac{E_k}{E_o}\frac{B_o}{B}\right] \frac{B}{B_o}$$

equation 12 becomes

$$\frac{\omega - \omega_o}{\omega} = \frac{\Delta B}{B_o} - \frac{E_k}{E_o} \qquad (13)$$

equation 13 can be derived simpler as follows:

$$\omega = \frac{Bec}{E}$$

$$d\omega = ec\, d\left(\frac{B}{E}\right) = ec\left[\frac{EdB - BdE}{E^2}\right]$$

$$\omega - \omega_o = \Delta\omega = \frac{ec}{E^2}\left[E\Delta B - B\Delta E\right]$$

$$\frac{\omega - \omega_o}{\omega} = \frac{\Delta\omega}{\omega} = E^2\left[E\Delta B - B\Delta E\right]\frac{E}{B}$$

$$\frac{\omega - \omega_o}{\omega} = \frac{\Delta\omega}{\omega} = \frac{\Delta B}{B} - \frac{\Delta E}{E}$$

$$\frac{\omega - \omega_o}{\omega} \approx \frac{\Delta\omega}{\omega} = \frac{\Delta B}{B} - \frac{\Delta E}{E}$$

$$E \approx E_o$$

$$B \approx B_o$$

$$\frac{\omega - \omega_o}{\omega} = \frac{\Delta B}{B_o} - \frac{\Delta E}{E_o} \qquad (14)$$

combining equations 13 and 6

$$\sin\phi_2 - \sin\phi_1 = -\frac{\pi}{eV_o}\int_{E_1}^{E_2}\left[\frac{\Delta B}{B_o} - \frac{E_k}{E_o}\right] dE$$

or eliminating the (-) sin

$$\sin\phi_2 - \sin\phi_1 = \frac{\pi}{eV_o}\int_{E_1}^{E_2}\left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o}\right] dE \qquad (15)$$

This is the desired relation which predicts the change in phase angle of a particle with e units of charge and rest energy = E_o being accelerated in a CW cyclotron through the energy range $E_2 - E_1$. The integral is conveniently evaluated graphically as follows:

We assume that the field value and slope have been determined from other considerations (see figure 4 and Lecture V, page 2), that the output energy of the beam has been established and that the ions enter the dees with zero kinetic energy.

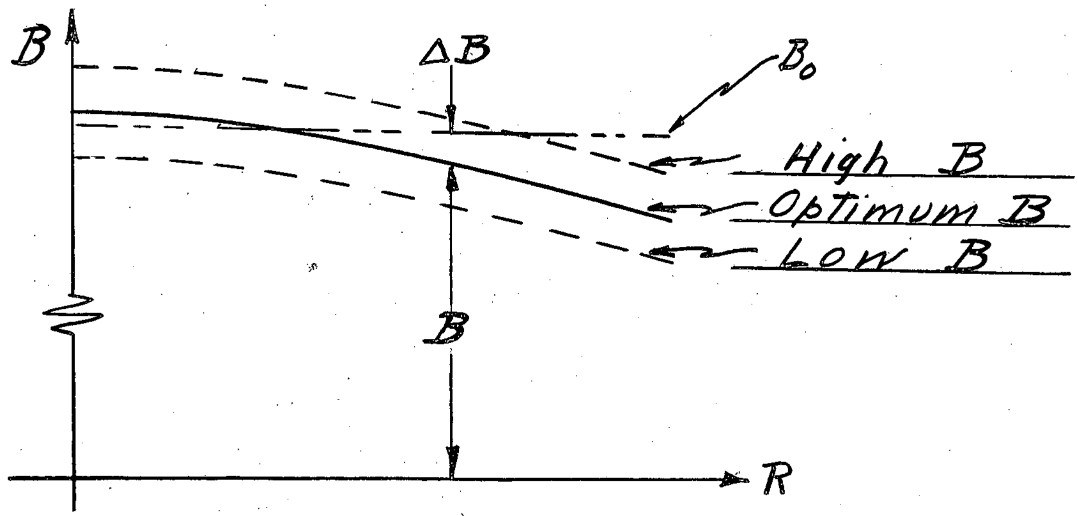


Fig. 4

NOTE: Field curve can be displaced vertically by varying magnet current.

From Figure 4 and by use of the equation for the non-relativistic case,

$$E_k = \frac{B^2e^2R^2}{2E_o} \qquad \text{Ref. (Lecture V, page 5)}$$

we can tabulate values of $\frac{\Delta B}{B_o}$ as a function of E_k

R	Fig. 4		B_o	$\frac{\Delta B}{B_o}$	$E_K = \frac{B^2e^2R^2}{2E_o}$	E_o	$\frac{E_K}{E_o}$
	ΔB	B					
			"			"	
			"			"	
			"			"	
			"			"	

Fig 5

Plotting $\frac{\Delta B}{B_o}$ and $\frac{E_k}{E_o}$ vs. E_k give curves in Figure 6.

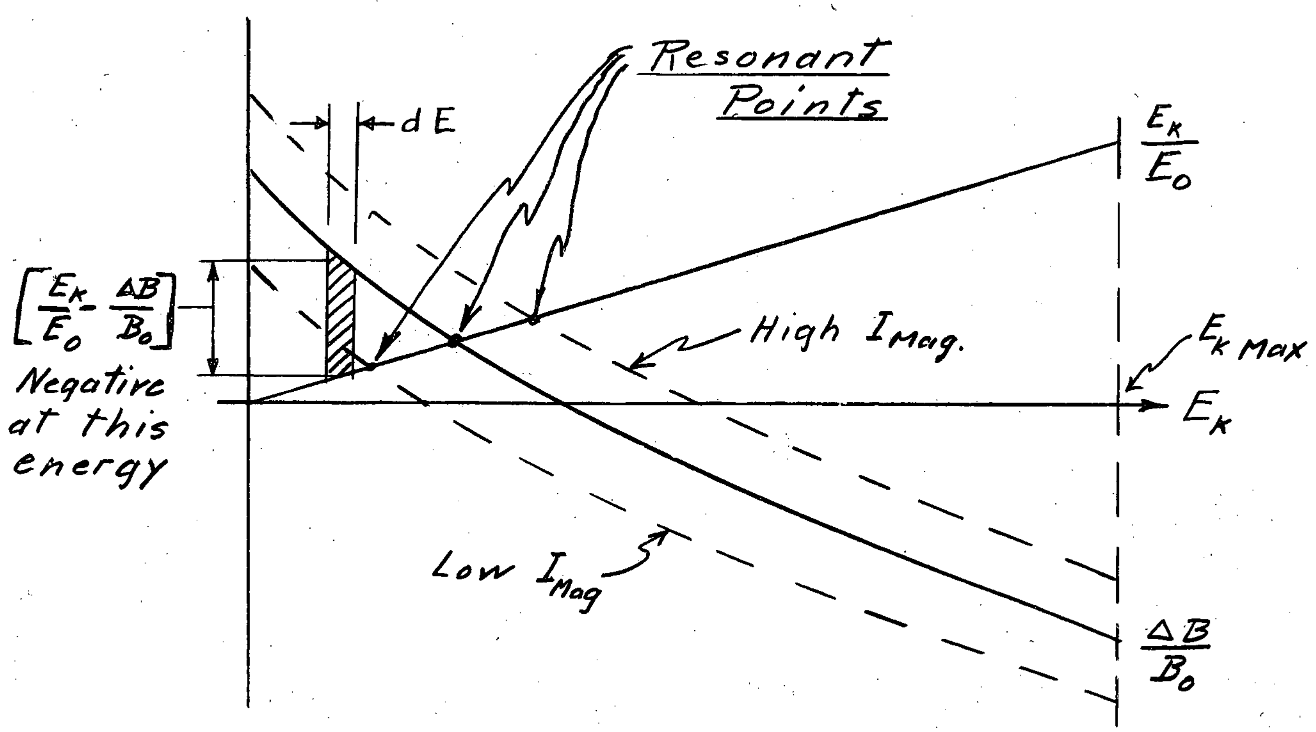


Fig 6

Summing up the differential areas between the curves gives the integral in equation (15)

$$\sin \phi_2 - \sin \phi_1 = \frac{\pi}{eV_o} \int_o^{E_k \max} \left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o}\right] dE_k \qquad (15)$$

Now assume $\phi_1 = o$
then

$$\sin \phi_2 = \frac{\pi}{eV_o} \int_o^{E_k \max} \left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o}\right] dE_k \qquad (16)$$

and the curves in Figure 7 can be drawn, (for one value of V_o).

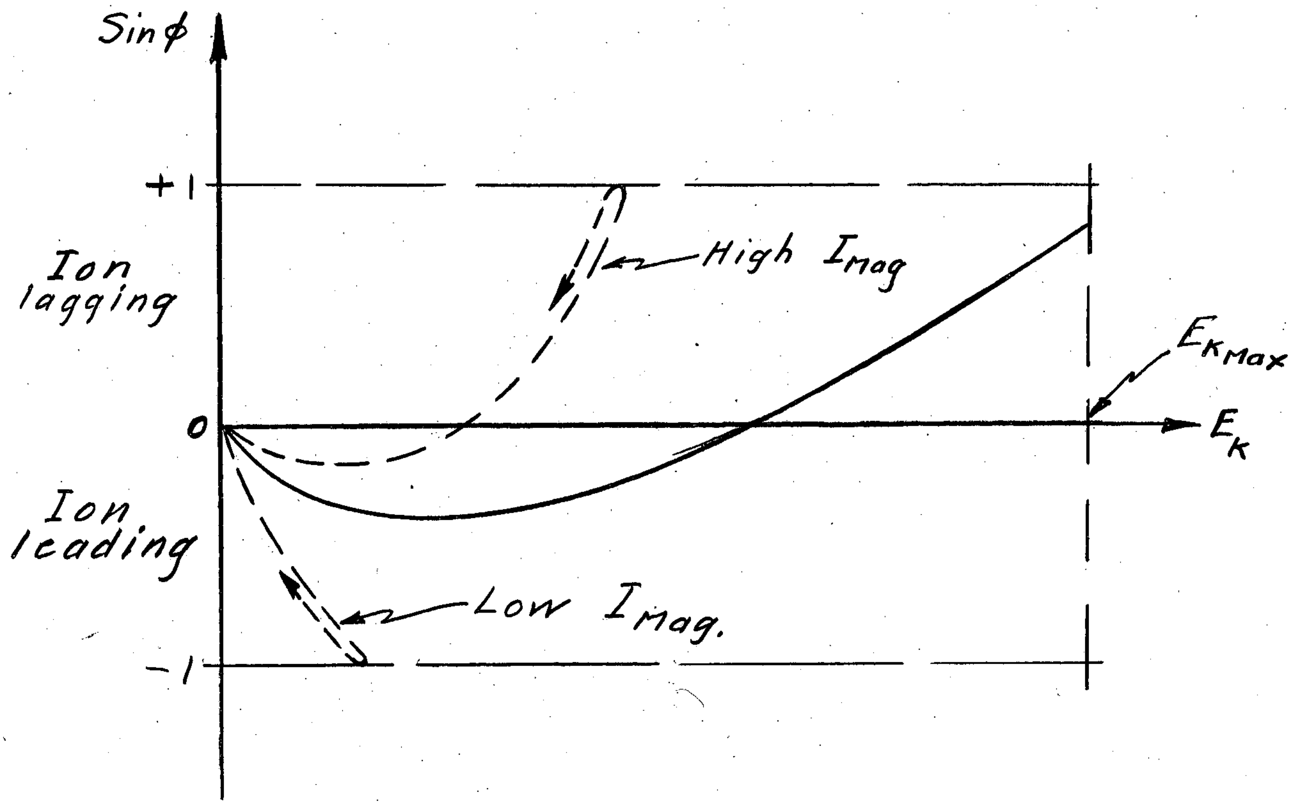


Fig 7

NOTE: Dotted curves are not displaced as shown but retrace on themselves.

Since sin ϕ can range only between -1 and +1, the curves cannot cross these lines. Once the ion lags or leads the voltage by more than 90°, it is decelerated by the dee voltage and spirals back towards the source, hence, the dotted curves are shown doubling back on themselves. Equation 3 shows that ΔE becomes negative (ion is decelerated) if ϕ exceeds $\pm \pi/2$. Then sin ϕ = ±1, corresponding to ϕ = $\pm \pi/2$, are limits beyond which the ion will decelerate and hence represent limiting energies.

Figure 7 indicates that for certain values of V_o the magnetic field can have a range of values without losing beam because of too much change of phase angle ϕ.

Now if the voltage is varied and I mag. held constant, the curves in Figure 8 result.

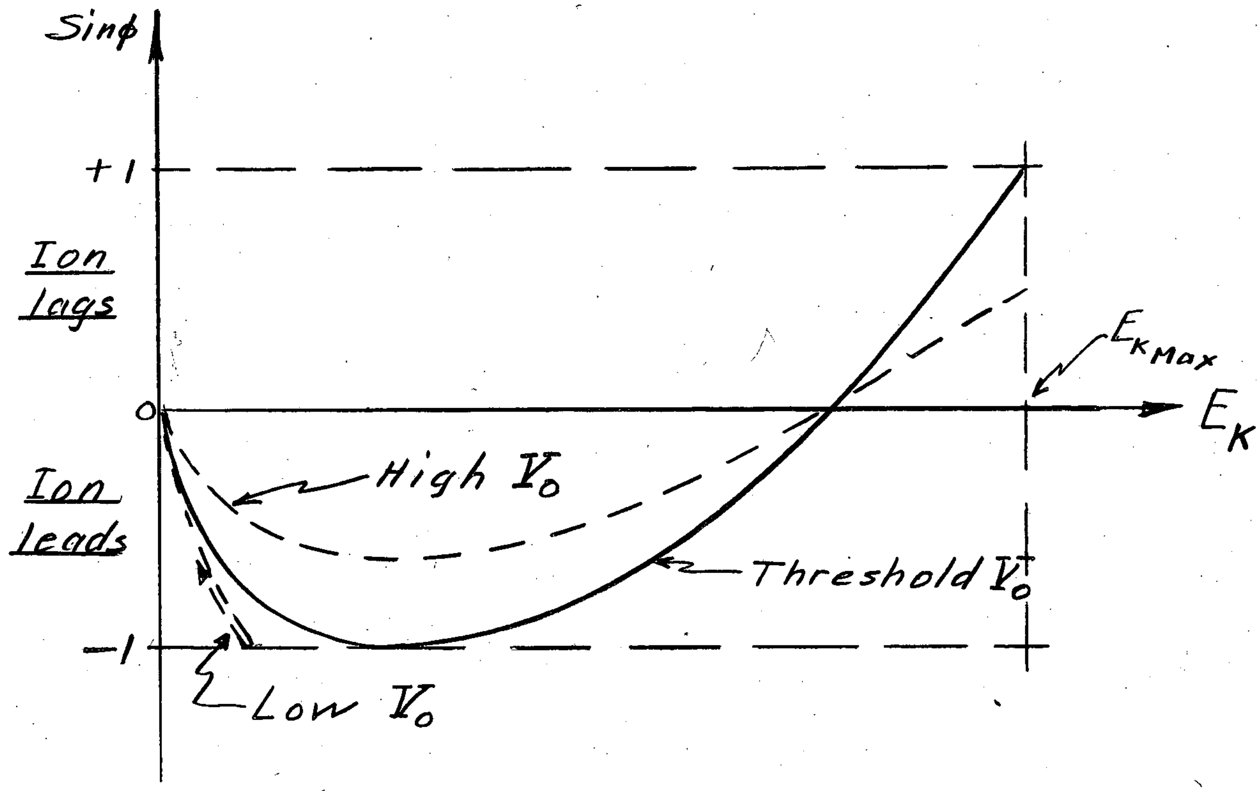


Fig 8

Note that there is a minimum voltage that will keep $-90 < \phi < 90°$. This is called the threshold voltage. Also, note that the curves are flatter with increased V_o.

Using a voltage above the threshold value, we can solve equation 15 to find the permissable range of ϕ_1 (the phase angle at beginning of acceleration process). Rewriting equation 15,

$$\sin \phi_2 = \frac{\pi}{eV_o} \int_0^{E_k} \left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o} \right] dE_k + \sin \phi_1 .$$

it is seen that the resulting curve is identical to those previously shown but displaced vertically by the distance $\sin \phi_1$.

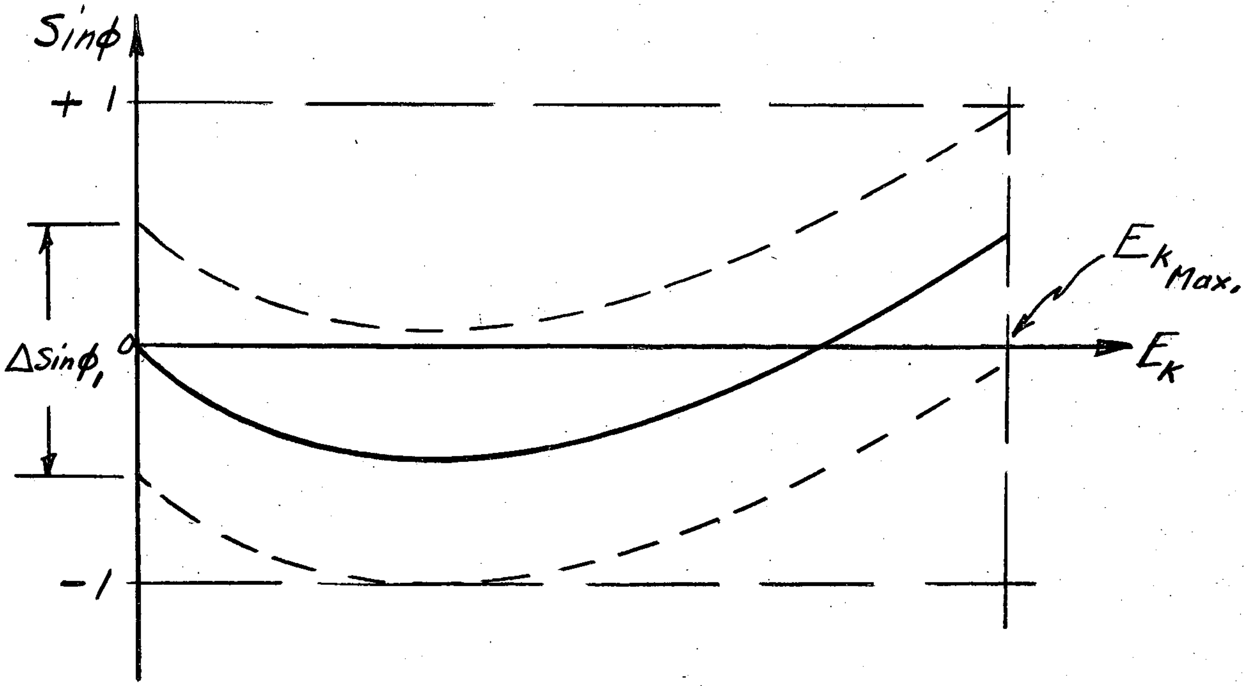


Fig 9

Now if we take the solid curve in Figure 9 which has been drawn for a given set of conditions and find the vertical displacement that will keep $-90 < \phi < 90°$, we have the permissable variation in ϕ_1. This range is called the acceptance angle; and, the greater this angle, the greater will be the possible beam current.

In the cyclotron low energy ions near the ion source, moving in the field between the dees, are pulled into phase with the dee voltage. Therefore, although the phase acceptance may be wide, there are no ions to accept at angles much different than zero. Therefore, $\phi_1 = o$ is ordinarily assumed in calculating the threshold dee voltage.

It is apparent from Figure 8 that high accelerating voltages are desirable since they result in flatter curves and, therefore, allow higher energies to be reached; however, insulation and power supply problems put practical limits to the voltage. In the 60 inch cyclotron, the dee voltage is 80 to 90 KV.

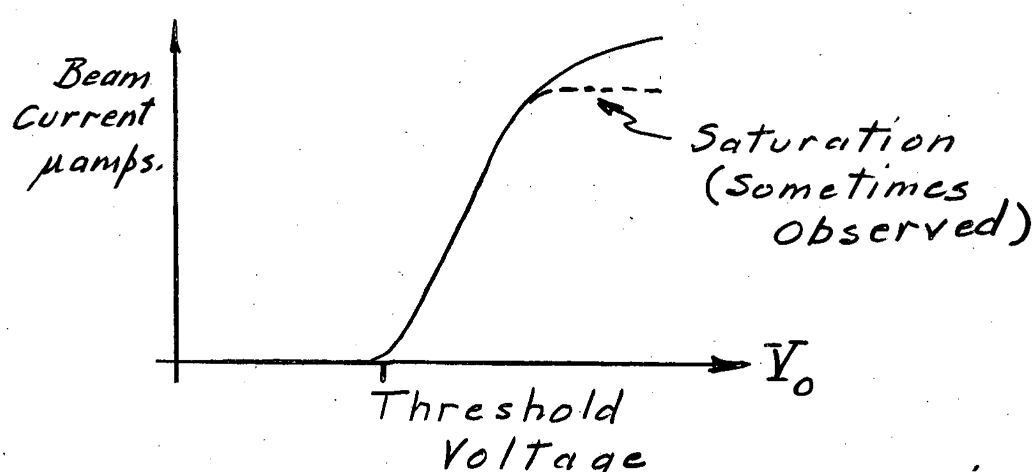


Fig 10

In Figure 10, the threshold voltage is located on the curve of beam current vs. accelerating voltage. The increase in i above saturation at high voltage is probably due to an increase in emission at the source because of the high dee voltage.

The foregoing theory has never been checked in detail because of the practical difficulties encountered in measuring ϕ; however, it can be used to predict the threshold dee voltage; and, it is undoubtedly correct in the description of the principles involved.

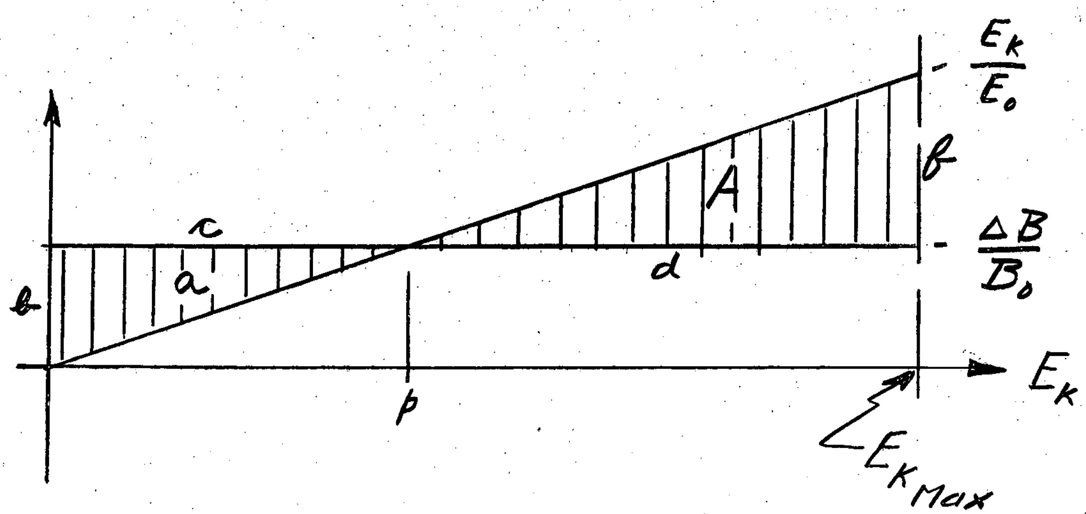
Problem: Find threshold voltage needed to accelerate particles in a CW cyclotron to final energy, assuming no magnetic field slope.

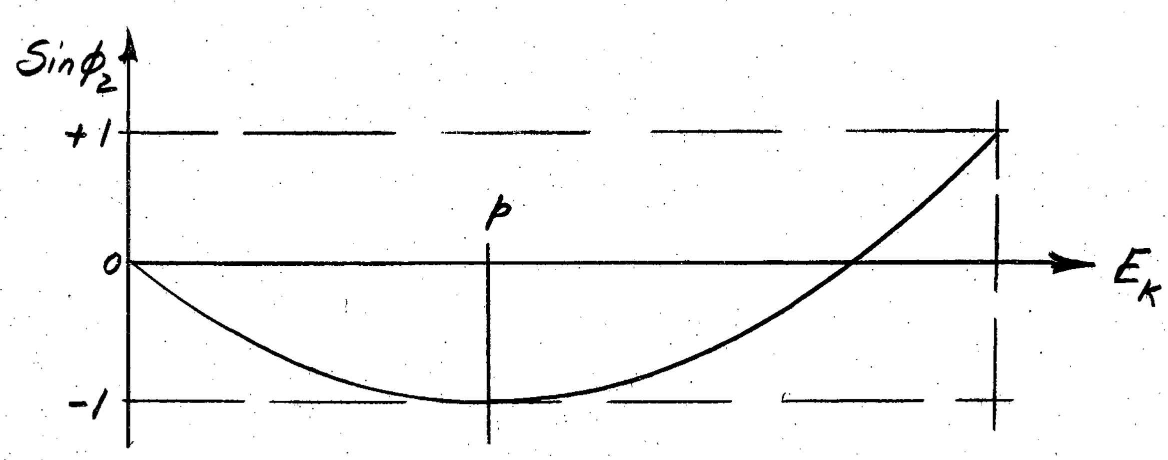
$$\sin \phi_2 = \frac{\pi}{eV_o} \int_o^{E_k} \left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o} \right] dE_k$$

$$\left. \begin{array}{l} \Delta B = B - B_o = \text{constant} \\ \frac{\Delta B}{B_o} = \text{constant} \end{array} \right\}$$

If we knew these values we could integrate directly.

Graphing $\frac{E_k}{E_o}$ and $\frac{\Delta B}{B_o}$ vs. E_k and drawing the curve of $\sin \phi_2$ corresponding to threshold voltage.





It is seen that the area of triangle A must be twice the area of triangle a in order for the sin ϕ_2 to follow the curve.

$$A = 2a$$
$$1/2\ fd = 2\frac{bc}{2} = bc$$
$$fd = 2\ bc$$

from similar triangles

$$\frac{f}{d} = \frac{b}{c}$$
$$f = \frac{bd}{c}$$
$$d\ \frac{bd}{c} = 2bc$$
$$d^2 = 2\ c^2$$
$$d = \sqrt{2}\ c$$
$$f = \sqrt{2}\ b$$
$$f + b = \frac{E_{k\ max}}{E_o} = b(1 + \sqrt{2})$$
$$c + d = E_{k\ max} = c(1 + \sqrt{2})$$
$$b = \frac{E_{k\ max}}{E_o\ (1 + \sqrt{2})}$$
$$c = \frac{E_{k\ max}}{1 + \sqrt{2}}$$
$$a = -\frac{1}{2}\ \frac{E_{k\ max}}{E_o(1 + \sqrt{2})} \times \frac{E_{k\ max}}{(1 + \sqrt{2})}$$
$$a = -\frac{1}{2E_o}\ \frac{E_{k\ max}^2}{(1 + \sqrt{2})^2} = \int_0^{E_p} \left[\frac{E_k}{E_o} - \frac{\Delta B}{B_o}\right] dE_k$$

The negative sign comes from $(\frac{E_k}{E_o} - \frac{\Delta B}{B_o})$.

At (p)

$$\sin \phi_2 = -1$$

Therefore

$$-1 = \frac{\pi}{eV_o}\ a = -\frac{\pi}{2eV_oE_o} \times \frac{E_{k\ max}^2}{(1 + \sqrt{2})^2}$$

or

$$V_o = \frac{.27\ E_{k\ max}^2}{e\ E_o} \qquad (17)$$

This formula gives a rock bottom value for the accelerating voltage. In actual machines, the required V_o is some 3 or 4 times the value indicated in equation 17.

Equation 17 also indicates the difficulty encountered in designing CW cyclotrons for high energies. Increasing dee voltage as the square of the output energy puts a practical limit of 20 MEV on $E_{k\ max}$.

For a particular case, consider deuterons being accelerated to 20 MEV in the 60 inch cyclotron.

$E_{k\ max} = 20 \times 10^6$ electron (practical volts)

$E_o = 187 \times 10^6$ electron (practical volts)

$e = 1$

$$V_o = \frac{.27(20 \times 10^6)^2}{1(1874 \times 10^6)}$$

$V_o = 58{,}000$ practical volts

Actual voltage required by this machine is approximately 175 KV, approximately three times the rock bottom value above.

N O M E N C L A T U R E

B = Magnetic field intensity

B_o = Magnetic field intensity corresponding to ω_o and E_o

V = Instantaneous accelerating voltage

V_o = Peak accelerating voltage = two times dee voltage

f = frequency of dee voltage

ω_o = angular frequency of dee voltage = $2\pi F$

ω = angular velocity of particle.

ϕ = phase angle between time of peak voltage and time at which particle crosses dee gap (positive if ion is lagging)

n = number of turns made by particle during acceleration

t = time

$t_o = \frac{1}{\omega_o} = \frac{1}{2\pi f}$ = period of one cycle of dee voltage.

$\theta = \omega_o t$ = angular displacement of V_o vector.

$\theta_i = \omega t$ = angular displacement of ion

$\Delta\omega = \omega - \omega_o$

$\Delta B = B - B_o$

I_{mag} = magnet current

Other symbols as defined in Lecture 5.

12. The Bevatron consists of four quadrants of 50 ft. centerline radius connected by 20 ft. straight sections. Calculate the rotation frequency for:

Protons at 16,000 gauss field (maximum energy)
Protons at 10 MEV (injection)

Deuterons at 16,000 gauss
Deuterons at 5 MEV

13. The 184 inch cyclotron has a field of 15,000 gauss at a radius of 80 inches. Calculate the energy and velocity of protons, deuterons and alpha particles in this field by exact formulas and non-relativistic approximate formulas.

Determine at what value of Kinetic energy/rest energy the non-relativistic formulas for β and BR are in error by one and ten percent.

14. A constant frequency cyclotron is designed as follows:

B_o (at Center) = 22,000 gauss
R_{max} (maximum useful radius) = 30 inches
Magnetic field shape as follows:
- 0 to 5 inches constant
- 5 to 25 inches one percent decrease linear with radius
- 25 to 30 inches one percent linear with radius

Two dees are used
Deuterons are to be accelerated

Find the threshold dee voltage assuming the ions to start in phase with the dee voltage.

15. A Betatron is to be designed for 200 MEV electrons with a six foot diameter orbit.

1. What maximum field strength is required
2. What total change in flux through the orbit is required (in Maxwells = gauss cm^2)
3. If the maximum allowable flux in the central core is 10,000 gauss, what cross section of iron is required if the core flux density starts at zero at injection. Is this a practical design?
4. What cross section of iron is required if the flux density varies from -10,000 to +10,000 gauss during acceleration?
5. If injection is at 100 Kilovolts, what is the magnetic field at injection?

16. In the linear accelerator in building 10, calculate the drift tube number of the first and last drift tubes (the difference being the number of drift tubes in the machine). The pertinent specifications are:

Wave length 1.48 meters
Injection energy 4 MEV protons
Final energy 32 MEV protons

Resonant cavity length 40 ft.
Energy gradient uniform along length.

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