UNIVERSITY OF CALIFORNIA, SAN DIEGO

Shadows and Intersections

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2012

The dissertation of Ameerah Naz Chowdhury is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

2012

DEDICATION

To my mother.

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ACKNOWLEDGEMENTS

Over the course of my graduate career, many mathematicians have provided crucial advice and assistance. I am grateful to my advisor, Jacques Verstraëte, for his generosity, kindness, and patience. Ada Chan, Chris Godsil, and Rick Wilson deserve special thanks for instilling me with a love for mathematics and for their ongoing support. I am indebted to Benny Sudakov for allowing me to be a part of his tightly run ship at UCLA. My studies in Hungarian-style combinatorics have been enriched by László Babai's and Péter Frankl's wonderful unpublished manuscript *Linear Algebra Methods in Combinatorics* and by conversations with many members of the Rényi Institute, such as my coauthor, Balázs Patkós.

A number of funding agencies and institutes made my research visits productive and memorable. Specifically, I thank the NSF, CESRI, Fulbright, and the Rényi Institute for providing me with valuable experiences abroad. I am grateful to IPAM for hosting me during their CMA 2009 program, and especially to their associate director, Amber Puha, for her encouragement.

I am fortunate to have friends like Matthew Drescher and Tony Huynh, who fought valiantly beside me in the trenches, and who lived through it all, from cookies to cobras. Thanking Matthew with words for his exemplariness as a human being is impossible, but a backrub might suffice. Claudine Chen, Tristan McLoughlin, Carlos Pin, and Beverly Tomov always provided a friendly ear and a helping hand. Lastly and most importantly, I am grateful to my family for all of their sacrifices, big and small.

Chapter 5, in part, is a reprint of the material as it appears in “Shadows and intersections in vector spaces," 2010. Chowdhury, Ameera; Patkós, Balázs. *J. Combin. Theory Ser. A*, 117(8):1095–1106, 2010. The dissertation author was the primary investigator and author of this paper.

Chapter 6, in part, is a reprint of the material as it appears in “On a conjecture of Frankl and Füredi," 2011. Chowdhury, Ameera. *Electron. J. Combin.*, 18(1):Paper 56, 16, 2011. The dissertation author was the primary investigator and author of this paper.

Chapter 7, in part, is currently being prepared for submission for publication of the material. Chowdhury, Ameera. The dissertation author was the primary investigator and author of this material.

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A. Chowdhury, “On the Manickam-Miklós-Singhi Conjecture," in preparation.

ABSTRACT OF THE DISSERTATION

Shadows and Intersections

by

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Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

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This thesis makes contributions to extremal combinatorics, specifically extremal set theory questions and their analogs in other structures. Extremal set theory studies how large or small a family of subsets of a finite set X can be under various constraints. By replacing the set X with another finite object, one can pose similar questions about families of other structures. Remarkably, a question and its analogs essentially have the same answer, regardless of the object. Despite these similarities, not much is known about analogs because standard techniques do not always apply. Our main results establish analogs of extremal set theory results for structures such as vector spaces and subsums of a finite sum. We also study intersecting families and shadows in their classical context of sets by researching a conjecture of Frankl and Füredi.

Chapter 1

Introduction

This thesis makes contributions to the field of extremal combinatorics, specifically extremal set theory questions and their analogs for other discrete structures. Given a finite set X, the general problem in extremal set theory asks how large or small a family of subsets of X can be if it satisfies certain restrictions. Naturally, this type of question appears throughout mathematics, and so extremal set theory can be applied in areas ranging from topology [74] to theoretical computer science [40]. On the other hand, extremal set theory borrows tools from algebra and probability, and its connections to other branches of mathematics is one of its most beautiful features.

Two core concepts in extremal set theory are intersecting families and shadows. The main results for intersecting families are the Erdős-Ko-Rado and Hilton-Milner theorems, and the principal result for shadows is the Kruskal-Katona theorem. By defining suitable notions of “intersecting" and “shadow," one can find remarkable analogs of these theorems for other structures such as vector spaces and permutations. Tantalizingly, while many results about sets should generalize to different settings, not much is known about analogs because standard techniques do not always apply.

This thesis aims to further understanding of shadows and intersecting families in sets and other structures. In the latter case, the goals are to identify objects with analogs of the Erdős-Ko-Rado, Hilton-Milner, and Kruskal-Katona theorems and to find integrated approaches to their proofs. The ultimate objective is to have a unified theory that characterizes the structures for which analogs exist and that proves results simultaneously for broad classes of objects.

Since this research applies to a variety of structures, it injects new questions, applications, and techniques into many areas. Vector space analogs, for example, influence the fields of finite geometry, algebraic combinatorics, and coding theory; analogs for permutations affect the theory of group representations. Each new object with an analog touches further areas.

1.1 Outline of Thesis

We now outline the chapters of this thesis. We first present our main results in Section 1.2 and compile a list of frequently used notation in Section 1.3. In Chapter 2, we discuss combinatorial methods in extremal set theory, paying particular attention to the shifting technique. We then examine the use of algebraic methods in extremal set theory in Chapter 3. Next, we give background on vector spaces over finite fields in Chapter 4; we also highlight the difficulties in generalizing purely combinatorial techniques in extremal set theory to vector spaces, and discuss algebraic methods that have been successfully used to prove theorems about both sets and vector spaces. In Chapter 5, we demonstrate new combinatorial techniques for vector spaces by proving vector space analogs of both Lovász's version of the Kruskal-Katona theorem and Frankl's r-wise intersection theorem; our proof of the latter also yields a new proof of the Erdős-Ko-Rado theorem for vector spaces. We end Chapter 5 with remarks on three of our favorite open problems in this area. In Chapter 6, we discuss our results on a conjecture of Frankl and Füredi. Finally, in Chapter 7, we end by presenting some new results on the Manickam-Miklós-Singhi conjecture and discussing some related open problems.

1.2 Main Results and Significance

A classical question in extremal set theory is to bound the size of a family of subsets of X whose members have size k and pairwise intersect; such a family is called *intersecting*. Erdős, Ko, and Rado [47] showed that if X is large enough, then the unique intersecting family of maximum size consists of the k-element subsets containing a fixed point. If no point of X may lie in all sets, then the Hilton-Milner theorem [70] determines the largest intersecting family in this case.

Another important property of a family $\mathscr{F}$ of k-element subsets of X is its *shadow*, which consists of all $(k-1)$-element subsets of X contained in at least one member of $\mathscr{F}$. Kruskal and Katona [75, 80] determined the minimum size of the shadow, and their result implies many others, such as the Erdős-Ko-Rado theorem. In practice, however, their theorem is not used in its full generality and a weaker but more convenient version due to Lovász [84, Ex 13.31(b)] is applied.

1.2.1 Shadows and Intersections in Vector Spaces

In Chapter 5, our main results show that striking analogs of extremal set theory results exist for vector spaces. Our proofs are not straightforward generalizations of the corresponding ones for sets because standard techniques do not often apply; one example is that the complement of a subset of X is another subset while the complement of a subspace of V is not another subspace. Consequently, we develop new methods.

1.2.1.1 Results

Patkós and I [27] discovered a unified proof of Lovász's version of the Kruskal-Katona theorem that works for sets and vector spaces [77]. Our result, Theorem 5.0.12, is one of the first about shadows in vector spaces although the problem is over thirty years old [67]. Since Lovász's theorem has many corollaries, its analog provides a way to extend them to vector spaces. Applications are not straightforward, however, because combinatorial techniques do not often apply. Our Theorem 5.0.14 yields unified proofs of the Erdős-Ko-Rado and Frankl's r-wise intersecting theorems for sets and vector spaces as corollaries. Three nice features of our proofs are that they are inductive, don't involve tedious computations, and characterize the case of equality.

We also state, but do not prove our other results in this area, namely obtaining a vector space analog of the Hilton-Milner theorem, Theorem 5.4.1, and determining the chromatic number of the q-Kneser graph, Theorem 5.4.5 and Theorem 5.4.6. The latter results were motivated by the longstanding problem of coloring the Kneser graph, whose solution involved a novel use of algebraic topology [9, 83].

1.2.1.2 Significance

Vector space analogs bring new questions and techniques to finite geometry since many of its problems can be reformulated in these terms. They also provide applications for the q-analog identities studied by algebraic combinatorialists. Recently, coding theorists such as Vardy are studying vector space analogs because they imply results about projective codes [18, 49, 50]. Since codes are used in communication systems, research in this area may yield practical applications.

1.2.1.3 Future Work

Our Lovász analog, Theorem 5.0.12, establishes shadows as a viable method for proving vector space analogs such as Theorem 5.0.14, and more applications are expected. Recently, Wang [113] used Theorem 5.0.12 to prove a conjecture of Erdős, Faigle, and Kern [48]. The method of proof in our Frankl analog, Theorem 5.0.14, has also been used to prove results on r-cross intersecting families of sets [58]; Patkós and I are currently working with Frankl and Tokushige to extend these results to vector spaces.

1.2.2 On a Conjecture of Frankl and Füredi

Fisher's Inequality bounds the size of λ*-intersecting* families, which are families of subsets of X whose members pairwise intersect in exactly λ points. Fisher's Inequality proves that if $\mathscr{F}$ is λ-intersecting, then the number of points in X that are covered by a set in $\mathscr{F}$ is at least $|\mathscr{F}|$. Inspired by Fisher's Inequality, Frankl and Füredi conjectured that if $\mathscr{F}$ is a λ-intersecting family, for which no point in X lies in all the sets in $\mathscr{F}$, then the number of pairs of points in X that are covered by some set in $\mathscr{F}$ is at least $\binom{|\mathscr{F}|}{2}$.

1.2.2.1 Results

We reformulated the Frankl-Füredi conjecture in terms of shadows, and proved it in some special cases using linear programming [25]. We first show in Theorem 6.1.1 that Frankl and Füredi's conjecture holds for nontrivial λ-intersecting families that satisfy a reasonable extra condition and characterize the extremal families. If a conjecture of Hall [68] is true, Theorem 6.1.1 would verify the Frankl-Füredi conjecture for λ-intersecting families whose members have fixed size k, when k is large with respect to λ. While Hall's conjecture remains open, we applied Theorem 6.1.1 in Theorem 6.1.2 to prove the Frankl-Füredi conjecture, when $\mathscr{F}$ is additionally required to be uniform and λ is small.

1.2.2.2 Significance

The λ-intersecting family in the Frankl-Füredi conjecture is a central concept in coding theory so any new results about it may have applications in communication systems. A proof of the Frankl-Füredi conjecture would be interesting from a mathematical standpoint because it would yield a new proof of Fisher's Inequality. As we have seen, it also relates to many other conjectures in design theory such as Hall's.

1.2.2.3 Future Work

Not much is known about the Frankl-Füredi conjecture, although it is twenty years old, and my paper is the first to consider it since it was published. I pose a conjecture about 2-intersecting families in my paper, and if this conjecture is true, it would imply the Frankl-Füredi conjecture when $\lambda = 2$. I intend to continue working on both of these conjectures.

1.2.3 On the Manickam-Miklós-Singhi Conjecture

We used combinatorial and algebraic methods for the previous two questions. The probabilistic method, however, also plays a role in extremal set theory problems and their analogs. Alon, Huang, and Sudakov's recent work [1] on the Manickam-Miklós-Singhi conjecture [88, 89] uses probabilistic arguments to prove partial analogs of the Erdős-Ko-Rado and Hilton-Milner theorems for subsums of a finite sum. More precisely, they showed that if $n \geq 33k^2$, then every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative. The conjecture is thought to hold when $n \geq 4k$, and Chapter 7 presents ideas for verifying it when k is small and for improving Alon, Huang, and Sudakov's result.

1.2.3.1 Results

For $k \in \mathbb{Z}^+$, let $f(k)$ be the minimum integer $N \in \mathbb{Z}^+$ such that for all $n \geq N$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative. Manickam, Miklós, and Singhi [88, 89] proved that $f(k)$ exists and conjectured that $f(k) \leq 4k$. Alon, Huang, and Sudakov [1] recently showed that $f(k) \leq \min\{33k^2, 2k^3\}$, which substantially improves previous bounds. We prove $f(3) = 11$ and $f(4) \leq 24$, which improves the previous upper bounds of $f(3) \leq 12$ from [87, 91] and $f(4) \leq 128$ from [1]. Although Conjecture 7.1.1 for $k = 3$ was previously tackled, our result is stronger because we determine $f(3)$ exactly and we characterize the case of equality; moreover our proof is simpler and gives a nice application of the Kruskal-Katona theorem. We also show how our method could potentially yield a quadratic upper bound on $f(k)$ that improves on that of Alon, Huang, and Sudakov. We end Chapter 7 by discussing a related open problem, the vector space analog of the Manickam-Miklós-Singhi conjecture.

1.3 Notation and Terminology

Here, we collect a list of frequently used notation and terminology.

- Number Systems

  1. $\mathbb{Z}$ denotes the integers. $\mathbb{Z}^+$ denotes the positive integers.
  2. If $a, b \in \mathbb{Z}$, then $a|b$ if there exists $x \in \mathbb{Z}$ such that $ax = b$.
  3. $\mathbb{N} = \{0, 1, \ldots\}$ denotes the natural numbers.
  4. $\mathbb{Q}$ denotes the rationals.
  5. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}^+$ denotes the positive real numbers.
  6. $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$
  7. If $a, b \in \mathbb{R}$ and $a < b$, then $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.
  8. $\mathbb{F}_q$ denotes the finite field of order q.
  9. If A is a set, then $A^d = A \times \cdots \times A$ denotes the *d-fold product* of A with itself.

- Sets.

  1. $X := [n] = \{1,2,\ldots,n\}$.
  2. $\emptyset$ denotes the *empty set*.
  3. S_X denotes the family of all permutations of the set X.
  4. $|X|$ denotes the *cardinality* of X.
  5. $x \in X$ denotes an *element* of X.
  6. $X \cup Y$ denotes the *union* of X and Y.
  7. $X \cap Y$ denotes the *intersection* of X and Y.
  8. $X \setminus Y$ denotes the *set complement* of Y with respect to X.

- Families of Sets

  1. 2^X is the family of all subsets of X.
  2. $\mathscr{F} \subset 2^X$ denotes a family of sets. When using the notation $\mathscr{F} = \{F_1,\ldots,F_m\}$, the sets F_i are assumed to be distinct, unless otherwise stated.
  3. $\binom{X}{k}$ denotes the family of all k-element subsets of X. We also call $\binom{X}{k}$ the complete k-uniform hypergraph on n vertices.
  4. $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ is the binomial coefficient. Note $|\binom{X}{k}| = |\binom{n}{k}|$.
  5. $\mathscr{C}_k^m$ denotes the first m sets of $\binom{X}{k}$ in the colex order.

- Properties of Families of Sets

  1. A family $\mathscr{F} \subset 2^X$ is *k-uniform* if $\mathscr{F} \subset \binom{X}{k}$.
  2. For $\mathscr{F} \subset 2^X$ and $x \in X$, the *degree* of x is $\deg(x) := |\{F \in \mathscr{F} : x \in F\}|$, the number of elements in $\mathscr{F}$ that contain x.
  3. For $\mathscr{F} \subset 2^X$ and $S \subset X$, the *co-degree* of S is $\text{codeg}\,(S) := |\{F \in \mathscr{F} : S \subset F\}|$, the number of sets in $\mathscr{F}$ that contain S.
  4. A family $\mathscr{F} \subset 2^X$ is *r-regular* if $\deg(x) = r$ for all $x \in X$.
  5. A family $\mathscr{F} \subset 2^X$ is *trivial* if there exists $x \in X$ with $\deg(x) = |\mathscr{F}|$, and is *nontrivial* otherwise.

- Intersection Properties of Families of Sets

  1. A family $\mathscr{F} \subset 2^X$ is *intersecting* if $F \cap F' \neq \emptyset$ for any $F, F' \in \mathscr{F}$.
  2. A family $\mathscr{F} \subset 2^X$ is *r-wise intersecting* if, for all $F_1, \ldots, F_r \in \mathscr{F}$, we have $\bigcap_{i=1}^r F_i \neq \emptyset$.
  3. For $t \in \mathbb{Z}^+$, a family $\mathscr{F} \subset 2^X$ is *t-intersecting* if $|F \cap F'| \geq t$ when $F, F' \in \mathscr{F}$.
  4. For $\lambda \in \mathbb{N}$, a family $\mathscr{F} \subset 2^X$ is *λ-intersecting* if $|F_1 \cap F_2| = \lambda$ for any $\{F_1, F_2\} \in \binom{\mathscr{F}}{2}$.
  5. For a finite set $L \subset \mathbb{N}$, a family $\mathscr{F} \subset 2^X$ is *L-intersecting* if $|F_1 \cap F_2| \in L$ for any $\{F_1, F_2\} \in \binom{\mathscr{F}}{2}$.

- Operators on Families of Sets

  1. If $\mathscr{F} \subset 2^X$ then $\partial^i \mathscr{F} := \left\{ E \in \binom{X}{i} : E \subset F \in \mathscr{F} \right\}$ is the *i-shadow* of $\mathscr{F}$.
  2. If $\mathscr{F} \subset 2^X$ and $\{i, j\} \in \binom{X}{2}$, then $\tilde{S}_{ij}(\mathscr{F})$ is the *shift operator* that replaces the element j by the element i whenever possible.
  3. A family $\mathscr{F} \subset 2^X$ has an associated $|\mathscr{F}| \times |X|$ incidence matrix M.

- Graphs

  1. $G = (V, E)$ denotes an undirected *graph*.
  2. $V(G) = V$ denotes the set of *vertices* in the context of a graph.
  3. $E \subset \binom{V}{2}$ denotes the family of *edges* of a graph.
  4. $v \sim w$ denotes vertex *adjacency*.
  5. A graph $G = (V, E)$ has an associated $|V| \times |V|$ adjacency matrix A.
  6. $\chi(G)$ denotes the chromatic number of a graph.
  7. If $G = (V, E)$ is a graph, then the *degree* of a vertex $v \in G$, denoted $\deg(v)$, is the number of vertices adjacent to v; that is, $\deg(v) := |\{w : v \sim w\}|$.
  8. A graph $G = (V, E)$ is *k-regular* if all vertices $v \in V$ have degree $\deg(v) = k$.
  9. $K_{n:k}$ denotes the Kneser graph.
  10. $qK_{n:k}$ denotes the q-Kneser graph.

- Digraphs

  1. $D=(V,A)$ denotes a *digraph*
  2. $A \subset V \times V$ is a family of ordered pairs of vertices called *arcs*.
  3. If $a=(v,w)$ is an arc in the digraph $D=(V,A)$, then a is said to be directed from v to w. We call v the *tail* of a and w the *head*.
  4. A digraph $D=(V,A)$ has an associated $|V| \times |A|$ incidence matrix M.

- Vector Spaces

  1. V denotes an n-dimensional vector space over the finite field $\mathbb{F}_q$ in the context of vector spaces.
  2. $\{0\}$ denotes the *zero subspace*.
  3. $\dim V$ denotes the dimension of the vector space V over its underlying field.
  4. $\begin{bmatrix} V \\ k \end{bmatrix}$ denotes the family of all k-dimensional subspaces of V.
  5. For $a \in \mathbb{R}$, $q \in \mathbb{R}^+$, and $k \in \mathbb{Z}^+$, the symbol $\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \le i < k} \frac{q^{a-i}-1}{q^{k-i}-1}$ denotes the *q-binomial coefficient*. When $k=1$, we write $[a]_q := \begin{bmatrix} a \\ 1 \end{bmatrix}_q$.
  6. If $A \subset V$ is a subspace of V, let $A^\perp$ denote the *orthogonal complement* of A.
  7. If $A,B \subset V$ are subspaces of V, then $A \vee B$ denotes the *linear span* of A and B. That is, $A \vee B$ is the smallest subspace that contains both A and B.
  8. $GL(V)$ denotes the family of all nonsingular linear transformations of V.

- Intersection Properties of Families of Subspaces

  1. A family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is *intersecting* if $\dim(F \cap F') \neq 0$ for any $F,F' \in \mathscr{F}$.
  2. A family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is called *r-wise intersecting* if $\dim(\bigcap_{i=1}^r F_i) \neq 0$ whenever $F_1,\dots,F_r \in \mathscr{F}$.
  3. For $t \in \mathbb{Z}^+$, a family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is *t-intersecting* if $\dim(F \cap F') \geq t$ when $F,F' \in \mathscr{F}$.

Chapter 2

Combinatorial Techniques

Let X denote the set $[n] := \{1, \ldots, n\}$, which we take as our underlying set, and let 2^X denote the family of all subsets of X. The general problem in extremal set theory asks for the maximum or minimum size of a family $\mathscr{F} \subset 2^X$ of subsets of X that satisfies certain restrictions. It is common, for example, to restrict the family to be k-uniform; that is, all sets in the family have size k. We use the symbol $\binom{X}{k}$ to denote the family of all k-subsets of X.

The sections of this chapter fall into three major themes. The first topic is shadows, and we shall discuss the Kruskal-Katona theorem [75, 80]. We next discuss intersection theorems such as the Erdős-Ko-Rado theorem [47] and its generalizations. Both of these topics allow us to demonstrate the power of the shifting technique. In this chapter, we stress the importance and usefulness of the shifting technique in extremal theory because we will see in Chapter 4 that no analog of it for vector spaces exists yet despite several attempts.

The last theme of this chapter is Fisher's inequality [17, 51, 73, 85]. We give de Bruijn and Erdős' proof [34] of the nonuniform Fisher inequality in the case $\lambda = 1$. Finally, we end the chapter with a proof of a special case of a conjecture of Frankl and Füredi [55], which generalizes de Bruijn and Erdős' result and proof, and makes use of convexity arguments. Our aim in this chapter is to give an overview of some combinatorial techniques that are frequently used in extremal set theory and in this thesis.

2.1 Shadows

Shadows are a fundamental concept in extremal set theory and appear throughout this thesis. For example, we will use shadows in the proofs of the Erdős-Ko-Rado theorem and Frankl's r-wise intersection theorem [52] in Section 2.2.4 and Section 2.2.6 respectively. We will generalize shadows to vector spaces in Chapter 5, study a conjecture of Frankl and Füredi on shadows in Chapter 6, and finally use them to prove special cases of the Manickam-Miklós-Singhi conjecture [88, 89] in Chapter 7.

In Section 2.1.1, we formally define shadows and state the main result on them, the Kruskal-Katona theorem, in its full generality. While we will need the complete Kruskal-Katona theorem in Chapter 7, in practice, it suffices to use a weaker but more convenient version of the theorem due to Lovász [84, Ex 13.31(b)]. In Section 2.1.2, we explain the statement of Lovász's version of the Kruskal-Katona theorem. We will give two proofs of Lovász's version of the Kruskal-Katona theorem in Section 2.1.5 and Section 2.1.6 respectively. The first proof is due to Frankl [53] and will make use of the shifting technique. The second is a recent and elegant proof due to Keevash [77], which we will generalize to vector spaces in Chapter 5.

2.1.1 The Kruskal-Katona Theorem

The Kruskal-Katona theorem [75, 80] gives a tight lower bound on the size of the shadow of a family $\mathscr{F} \subset \binom{X}{k}$.

Definition 2.1.1. *For a family $\mathscr{F} \subset 2^X$, we define the <u>i-shadow</u> of $\mathscr{F}$, denoted $\partial^i \mathscr{F}$, to consist of those i-subsets of X contained in at least one member of $\mathscr{F}$,*

$$\partial^i \mathscr{F} := \left\{ E \in \binom{X}{i} : E \subset F \in \mathscr{F} \right\}.$$

When $\mathscr{F} \subset \binom{X}{k}$, we define the <u>shadow</u> of $\mathscr{F}$, denoted $\partial \mathscr{F}$, to be $\partial \mathscr{F} = \partial^{k-1} \mathscr{F}$.

The Kruskal-Katona theorem also describes the structure of set-systems with minimum shadow over all set-systems with the same cardinality. To characterize these families, we need to define the colex order on $\binom{X}{k}$.

Definition 2.1.2. *Given $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ in $\binom{X}{k}$ where $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for all $i \in [k-1]$, we say that $A < B$ in the colex order if $A \neq B$, and for $s = \max\{t : a_t \neq b_t\}$, we have $a_s < b_s$. For $1 \leq m \leq \binom{n}{k}$, let $\mathscr{C}_m^k$ denote the first m elements of the colex order on $\binom{X}{k}$.*

Note that the colex order is a total order on $\binom{X}{k}$, so the definition of $\mathscr{C}_m^k$ makes sense.

We now prove a lemma that demonstrates that every positive integer $m \in \mathbb{Z}^+$ has a k-binomial representation, and this lemma will allow us to compute the size of the shadow of the first m sets in the colex order on $\binom{X}{k}$.

Lemma 2.1.3. *Given positive integers $m, k \in \mathbb{Z}^+$, there exists a unique representation of m in the form*

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where $a_k > a_{k-1} > \cdots > a_t \geq t \geq 1$. (This representation of m is called the k-binomial representation of m.)

Proof. We show by induction on k that such a representation exists and is unique. If $k = 1$, then the theorem is trivially true. Now assume that there exists a unique such representation for any $m \in \mathbb{Z}^+$ and $k = l - 1$. To show that a unique such representation exists for $m \in \mathbb{Z}^+$ and $k = l$, first note that $t = k$ and $a_t = k$ is a unique such representation for $m = 1$. Suppose then that $m \geq 2$. Since for $r \geq k$, we have

$$\sum_{j=1}^{k} \binom{r-k+j}{j} = \binom{r+1}{k} - 1 < \binom{r+1}{k},$$

we must have $a_k = \max\{r : \binom{r}{k} \leq m\}$. If $m = \binom{a_k}{k}$ then set $t = k$. If $m > \binom{a_k}{k}$, then by the induction hypothesis, we have unique positive integers $a_{k-1} > \cdots > a_t \geq t \geq 1$ such that

$$m - \binom{a_k}{k} = \binom{a_{k-1}}{k} + \cdots + \binom{a_t}{t}.$$

Hence $m = \binom{a_k}{k} + \binom{a_{k-1}}{k} + \cdots + \binom{a_t}{t}$ is a unique such representation. □

We are now ready to state the Kruskal-Katona theorem.

Theorem 2.1.4 (Kruskal-Katona)**.** *Let $\mathscr{F} \subset \binom{X}{k}$ be a family of size m and suppose that $m = \sum_{j=t}^{k} \binom{a_j}{j}$ is the k-binomial representation of m. Then*

$$|\partial \mathscr{F}| \geq |\partial \mathscr{C}_m^k| = \sum_{j=t}^{k} \binom{a_j}{j-1};$$

in other words, the size of the shadow of $\mathscr{F}$ is at least the size of the shadow of the first m sets in the colex order on $\binom{X}{k}$. Moreover, letting $m' := \sum_{j=t}^{k} \binom{a_j}{j-1}$, we have $\partial \mathscr{C}_m^k = \mathscr{C}_{m'}^{k-1}$; that is, the shadow of the first m sets in the colex order on $\binom{X}{k}$ consists of the first m' sets in the colex order on $\binom{X}{k-1}$.

2.1.2 Lovász's Version of the Kruskal-Katona Theorem

While we will need the Kruskal-Katona theorem in its full generality in Chapter 7, Lovász's weaker but more convenient version [84, Ex 13.31(b)] suffices for many applications. For example, we will use Lovász's version in the proofs of the Erdős-Ko-Rado theorem and Frankl's r-wise intersection theorem in Section 2.2.4 and Section 2.2.6 respectively. We also generalize Lovász's theorem to vector spaces in Chapter 5. In this section, we explain the statement of Lovász's result.

Recall that the binomial coefficient

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!}$$

can be defined for all $n \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. If we fix k, then $\binom{n}{k}$ is a continuous function of n that is positive and increasing; hence, by the intermediate value theorem, if $r \geq 1$ is a real number, then there exists a unique real number $n_r \geq k$ such that $r = \binom{n_r}{k}$.

Theorem 2.1.5 (Lovász)**.** *Let $\mathscr{F} \subset \binom{X}{k}$ and let $y \geq k$ be the real number defined by $|\mathscr{F}| = \binom{y}{k}$. Then $|\partial \mathscr{F}| \geq \binom{y}{k-1}$. If equality holds, then $y \in \mathbb{Z}^+$ and $\mathscr{F} = \binom{Y}{k}$, where Y is a y-subset of X.*

2.1.3 Shifting

Our first proof of Lovász's version of the Kruskal-Katona theorem will make use of the shifting technique. Also known as compression, the method was introduced by

Erdős, Ko, and Rado [47] and is one of the most important tools in extremal set theory. We will use the shifting technique again in our proof of the Erdős-Ko-Rado theorem in Section 2.2.3.

Definition 2.1.6. *For $F \subset X$ and distinct $i, j \in X$ define*

$$S_{ij}(F) := \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F \\ F & \text{otherwise.} \end{cases}$$

We see that S_{ij} replaces the element j by the element i whenever possible. Note that S_{ij} is not a one-to-one map since for $E \subset X \setminus \{i, j\}$ one has

$$S_{ij}(E \cup \{i\}) = S_{ij}(E \cup \{j\}) = E \cup \{i\}.$$

We would like to associate with S_{ij} a map $\tilde{S}_{ij}$, which sends a set system to another set system of the same size. This motivates the following definition.

Definition 2.1.7. *For a family $\mathscr{F} \subset 2^X$ and distinct $i, j \in X$ define the <u>shift operator</u>*

$$\tilde{S}_{ij}(\mathscr{F}) := \{S_{ij}(F) : F \in \mathscr{F}\} \cup \{F : F, S_{ij}(F) \in \mathscr{F}\}.$$

We note two immediate properties of S_{ij} and its associated shift operator $\tilde{S}_{ij}$.

Proposition 2.1.8. *We have $|F| = |S_{ij}(F)|$ and $|\mathscr{F}| = |\tilde{S}_{ij}(\mathscr{F})|$.*

We will see in Section 2.1.1 and Section 2.2.1 that the shift operator $\tilde{S}_{ij}$ allows us to transform our original set system into a more structured one while still preserving important properties of the original.

2.1.4 Properties of the Shift Operator

Towards a proof of Lovász's version of the Kruskal-Katona theorem via shifting, we first observe that the shifting operator $\tilde{S}_{ij}$ does not increase the size of the shadow.

Lemma 2.1.9. *For $\mathscr{F} \subset \binom{X}{k}$ and distinct $i, j \in X$, we have $|\partial \mathscr{F}| \geq |\partial \tilde{S}_{ij}(\mathscr{F})|$.*

Proof. We show that $S_{ji} : \partial \tilde{S}_{ij}(\mathscr{F}) \setminus \partial \mathscr{F} \to \partial \mathscr{F} \setminus \partial \tilde{S}_{ij}(\mathscr{F})$ is an injective map. Observe that if $E \in \partial \tilde{S}_{ij}(\mathscr{F}) \setminus \partial \mathscr{F}$ then $E \subset S_{ij}(F)$ for some $F \in \mathscr{F}$ for which $S_{ij}(F) \neq F$. Hence

$j \notin E$. If $i \notin E$, then $E = S_{ij}(F) \setminus \{i\} = F \setminus \{j\}$ so $E \in \partial\mathscr{F}$, which is impossible by our assumption; hence $i \in E$. We have shown that every $E \in \partial\tilde{S}_{ij}(\mathscr{F}) \setminus \partial\mathscr{F}$ has $i \in E$ and $j \notin E$.

We claim that $S_{ji}(E) \in \partial\mathscr{F} \setminus \partial\tilde{S}_{ij}(\mathscr{F})$. Clearly, $S_{ji}(E) \in \partial\mathscr{F}$. Suppose, for a contradiction, that $(E\setminus\{i\})\cup\{j\} = S_{ji}(E) \in \partial\tilde{S}_{ij}(\mathscr{F})$. Hence, for some $x \in X$ that is not in $(E\setminus\{i\})\cup\{j\}$, we have $(E\setminus\{i\})\cup\{j,x\} \in \tilde{S}_{ij}(\mathscr{F})$. If $x = i$, then $E\cup\{j\} \in \tilde{S}_{ij}(\mathscr{F})$ and Definition 2.1.7 implies that $E\cup\{j\} \in \mathscr{F}$ so $E \in \partial\mathscr{F}$, which is a contradiction. If $x \neq i$, then since $i \notin (E\setminus\{i\})\cup\{j,x\}$ and $j \in (E\setminus\{i\})\cup\{j,x\} \in \tilde{S}_{ij}(\mathscr{F})$, we must have

$$E\cup\{x\} = S_{ij}((E\setminus\{i\})\cup\{j,x\}) \in \mathscr{F},$$

so $E \in \partial\mathscr{F}$, which is a contradiction. This proves that $S_{ji}(E) \in \partial\mathscr{F} \setminus \partial\tilde{S}_{ij}(\mathscr{F})$.

We showed in the first paragraph that every $E \in \partial\tilde{S}_{ij}(\mathscr{F}) \setminus \partial\mathscr{F}$ has $i \in E$ and $j \notin E$. Hence, the map $S_{ji} : \partial\tilde{S}_{ij}(\mathscr{F}) \setminus \mathscr{F} \to \partial\mathscr{F} \setminus \partial\tilde{S}_{ij}(\mathscr{F})$ is injective. Consequently,

$$\begin{aligned}|\partial\mathscr{F}| &= |\mathscr{F} \cap \partial\tilde{S}_{ij}(\mathscr{F})| + |\partial\mathscr{F} \setminus \partial\tilde{S}_{ij}(\mathscr{F})| \\ &\geq |\partial\tilde{S}_{ij}(\mathscr{F}) \cap \mathscr{F}| + |\partial\tilde{S}_{ij}(\mathscr{F}) \setminus \mathscr{F}| = |\partial\tilde{S}_{ij}(\mathscr{F})|,\end{aligned}$$

as desired. □

Since shifting does not increase the size of a family's shadow, the next two lemmas show how we can shift our original family to one that is more structured.

Lemma 2.1.10. *Let $\mathscr{F} \subset \binom{X}{k}$. Define $\mathscr{H}_1 = \mathscr{F}$ and $\mathscr{H}_i = \tilde{S}_{1i}(\mathscr{H}_{i-1})$ for $i \in X \setminus \{1\}$. We then have $\tilde{S}_{1i}(\mathscr{H}_n) = \mathscr{H}_n$ for all $i \in X \setminus \{1\}$.*

Proof. Suppose, for a contradiction, that there exists $i \in X\setminus\{1\}$ such that $\tilde{S}_{1i}(\mathscr{H}_n) \neq \mathscr{H}_n$. Hence, there exists $H \in \mathscr{H}_n$ such that $i \in H$, $1 \notin H$, and $S_{1i}(H) \notin \mathscr{H}_n$. As $1 \notin H$, we have $H \in \mathscr{H}_{i-1} \cap \mathscr{H}_i$. This implies $S_{1i}(H) \in \mathscr{H}_i$ and hence $S_{1i}(H) \in \mathscr{H}_n$, which is a contradiction. Consequently, $\tilde{S}_{1i}(\mathscr{H}_n) = \mathscr{H}_n$ for all $i \in X\setminus\{1\}$. □

If $\mathscr{F} \subset \binom{X}{k}$, then we can partition $\mathscr{F}$ into two subfamilies according to whether $1 \in F \in \mathscr{F}$. We show that this partition has a special property, when $\mathscr{F} \subset \binom{X}{k}$ is a family that satisfies the conclusion of Lemma 2.1.10, namely that $\tilde{S}_{1i}(\mathscr{F}) = \mathscr{F}$ for each $i \in X\setminus\{1\}$. We will exploit this property when we inductively prove Lovász's version of the Kruskal-Katona theorem.

Lemma 2.1.11. *Suppose $\mathscr{F} \subset \binom{X}{k}$ and $\tilde{S}_{1i}(\mathscr{F}) = \mathscr{F}$ for each $i \in X \setminus \{1\}$. Let*

$$\mathscr{F}_0 := \{F \in \mathscr{F} : 1 \notin F\}, \;\; \mathscr{F}_1 := \{F \setminus \{1\} : 1 \in F \in \mathscr{F}\}.$$

We then have $|\mathscr{F}_1| \geq |\partial \mathscr{F}_0|$.

Proof. If $E \in \partial \mathscr{F}_0$ then $E \cup \{i\} \in \mathscr{F}$ for some $i \in X \setminus \{1\}$. Since $\tilde{S}_{1i}(\mathscr{F}) = \mathscr{F}$, we must have $S_{1i}(E \cup \{i\}) = E \cup \{1\} \in \mathscr{F}$. Consequently, $E \in \mathscr{F}_1$. This proves $|\mathscr{F}_1| \geq |\partial \mathscr{F}_0|$. □

2.1.5 Lovász's Result via Shifting

We now present Frankl's shifting proof of Lovász's version of the Kruskal-Katona theorem [53].

Proof of Theorem 2.1.5. Let $\mathscr{F} \subset \binom{X}{k}$ and let $y \geq k$ be the real number that is defined by $|\mathscr{F}| = \binom{y}{k}$. Define $\mathscr{H}_1 = \mathscr{F}$ and $\mathscr{H}_i = \tilde{S}_{1i}(\mathscr{H}_{i-1})$ for $i \in X \setminus \{1\}$; for notational convenience, let $\mathscr{H}^* := \mathscr{H}_n$. We have $|\mathscr{H}^*| = |\mathscr{F}| = \binom{y}{k}$ and $|\partial \mathscr{F}| \geq |\partial \mathscr{H}^*|$ by Proposition 2.1.8 and Lemma 2.1.9. Hence, the theorem will be proved for $\mathscr{F}$ if we can prove it for $\mathscr{H}^*$.

We use double induction on k and $m = |\mathscr{F}| = |\mathscr{H}^*|$. Define

$$\mathscr{H}_0^* := \{H \in \mathscr{H}^* : 1 \notin H\}, \;\; \mathscr{H}_1^* := \{H \setminus \{1\} : 1 \in H \in \mathscr{H}^*\}.$$

Suppose, for a contradiction, that $|\mathscr{H}_1^*| < \binom{y-1}{k-1}$. We then have

$$|\mathscr{H}_0^*| = |\mathscr{H}^*| - |\mathscr{H}_1^*| > \binom{y}{k} - \binom{y-1}{k-1} = \binom{y-1}{k}.$$

Hence, by the induction hypothesis, $|\partial \mathscr{H}_0^*| \geq \binom{y-1}{k-1}$. Lemma 2.1.10 and Lemma 2.1.11 then imply that $|\mathscr{H}_1^*| \geq |\partial \mathscr{H}_0^*| \geq \binom{y-1}{k-1}$, which contradicts our original assumption. Therefore, we must have $|\mathscr{H}_1^*| \geq \binom{y-1}{k-1}$, and so $|\partial \mathscr{H}_1^*| \geq \binom{y-1}{k-2}$ by the induction hypothesis. As $\partial \mathscr{H}_0^* \subset \mathscr{H}_1^*$ by Lemma 2.1.11, we obtain the desired conclusion that

$$|\partial \mathscr{F}| \geq |\partial \mathscr{H}^*| = |\mathscr{H}_1^*| + |\partial \mathscr{H}_1^*| \geq \binom{y-1}{k-1} + \binom{y-1}{k-2} = \binom{y}{k-1}. \tag{2.1.1}$$

We now characterize the case of equality in (2.1.1). If $|\partial \mathscr{F}| = \binom{y}{k-1}$, then equality holds everywhere in (2.1.1). We then have that $|\partial \mathscr{H}_1^*| = \binom{y-1}{k-1}$, and hence

$|\mathscr{H}_0^*| = \binom{y-1}{k}$. The induction hypothesis and Lemma 2.1.11 imply that $|\partial\mathscr{H}_0^*| = \binom{y-1}{k-1}$ and $\partial\mathscr{H}_0^* = \mathscr{H}_1^*$. Since $|\mathscr{H}_0^*| = \binom{y-1}{k}$ and $|\partial\mathscr{H}_0^*| = \binom{y-1}{k-1}$, the induction hypothesis implies that $\mathscr{H}_0^* = \binom{Z}{k}$ where $|Z| = y-1$. Hence $\mathscr{H}^* = \binom{Y'}{k}$, where $Y' = Z \cup \{1\}$, as $\mathscr{H}^* = \mathscr{H}_0^* \cup \{H \cup \{1\} : H \in \mathscr{H}_1^*\}$ and $\partial\mathscr{H}_0^* = \mathscr{H}_1^*$.

If $\mathscr{F} = \mathscr{H}^*$, then setting $Y = Y'$, we are done since $|Y| = |Y'| = y$. If $\mathscr{F} \neq \mathscr{H}^*$, let $i = \min\{j : \mathscr{H}_j = \mathscr{H}^*\}$. Of course $i \geq 2$, since $\mathscr{H}_1 = \mathscr{F}$ and we have assumed that $\mathscr{F} \neq \mathscr{H}^*$. When we apply the shift operator $\tilde{S}_{1i}$ to $\mathscr{H}_{i-1}$ only the element i can be deleted; we consequently have $\partial^1\mathscr{H}_{i-1} \subset Z \cup \{i\}$ because $\mathscr{H}_i = \mathscr{H}^* = \binom{Z\cup\{1\}}{k}$. Hence, $|\partial^1\mathscr{H}_{i-1}| \leq |Z \cup \{i\}| = y$. We have that $|\partial\mathscr{F}| = \binom{y}{k-1} = |\partial\mathscr{H}^*|$, so Lemma 2.1.9 implies that $|\partial\mathscr{H}_{i-1}| = \binom{y}{k-1}$. By (2.1.1), we have $|\partial^1\mathscr{H}_{i-1}| \geq y$, so $|\partial^1\mathscr{H}_{i-1}| = |Z \cup \{i\}| = y$ and $\partial^1\mathscr{H}_{i-1} = Z \cup \{i\}$. Consequently, $\mathscr{H}_{i-1} = \binom{Z\cup\{i\}}{k}$. Since $1 \notin Z \cup \{i\}$, Definition 2.1.7 implies that $\mathscr{F} = \binom{Z\cup\{i\}}{k}$ so setting $Y = Z \cup \{i\}$ finishes the proof. □

2.1.6 Keevash's Proof of Lovász's Result

As we mentioned previously, no analog of the shifting technique for vector spaces exists yet. Consequently, it is not clear whether Frankl's proof in Section 2.1.5 can be generalized to vector spaces. We will give more examples of problems that arise when trying to generalize purely combinatorial proofs about sets to vector spaces in Chapter 4. Since the techniques that tend to work for both sets and vector spaces are typically algebraic in nature, we were pleasantly surprised to find that Keevash's purely combinatorial proof of Lovász's result [77] generalizes to vector spaces. We present Keevash's proof now, and discuss its generalization in Chapter 5.

We first collect definitions that will be used in Keevash's proof of Theorem 2.1.5.

Definition 2.1.12. *For $\mathscr{F} \subset \binom{X}{k}$ and $x \in X$, define*

$$K_{k+1}^k(\mathscr{F}) := \left\{T \in \binom{X}{k+1} : \binom{T}{k} \subset \mathscr{F}\right\}$$

to be the family of $(k+1)$-subsets of X all of whose k-subsets lie in $\mathscr{F}$ and

$$K_{k+1}^k(x) := \{T \in K_{k+1}^k(\mathscr{F}) : x \in T\}$$

to be the family of $(k+1)$-subsets in $K_{k+1}^k(\mathscr{F})$ that contain x.

Definition 2.1.13. *For $\mathscr{F} \subset 2^X$ and $x \in X$, define the* <u>*degree*</u> *of x, which is denoted by* $\deg(x)$, *to be the number of elements of $\mathscr{F}$ that contain x,*

$$\deg(x) := |\{F \in \mathscr{F} : x \in F\}|.$$

Definition 2.1.14. *For $\mathscr{F} \subset \binom{X}{k}$ and $x \in X$, define the* <u>*link*</u> *of x to be the family of $(k-1)$-subsets in $X \setminus \{x\}$ whose union with x is an element of $\mathscr{F}$,*

$$L(x) := \{A \subset X \setminus \{x\} : |A| = k-1,\, A \cup \{x\} \in \mathscr{F}\} \subset \begin{bmatrix} X \\ k-1 \end{bmatrix}.$$

In Theorem 2.1.15, we establish an upper bound on $|K^k_{k+1}(\mathscr{F})|$ in terms of $|\mathscr{F}|$; we will see that Theorem 2.1.5 follows as a simple corollary.

Theorem 2.1.15 (Keevash)**.** *Let $\mathscr{F} \subset \binom{X}{k}$ and let $y \geq k$ be the real number defined by $|\mathscr{F}| = \binom{y}{k}$. Then*

$$|K^k_{k+1}(\mathscr{F})| \leq \binom{y}{k+1}.$$

Equality holds if and only if $y \in \mathbb{Z}^+$ and $\mathscr{F} = \binom{Y}{k}$ for some y-subset $Y \subset X$.

Proof. We argue by induction on k. The base case $k = 1$ is easy, so assume the statement is true for $k-1$. We first show that if $x \in X$, then $|K^k_{k+1}(x)| \leq ((y-k)/k)\deg(x)$; we will then sum this inequality over all $x \in X$ and double count to obtain the desired upper bound on $|K^k_{k+1}(\mathscr{F})|$. If $\deg(x) = 0$, then clearly $|K^k_{k+1}(x)| \leq ((y-k)/k)\deg(x)$, so we will assume that $\deg(x) \neq 0$. We will need to consider the cases where $\deg(x)$ is large and where $\deg(x)$ is small separately.

First, let's consider the case when $\deg(x) \geq \binom{y-1}{k-1}$. If $F \cup \{x\} \in K^k_{k+1}(x)$, then $F \in \mathscr{F}$ does not contain x; consequently,

$$\begin{aligned} |K^k_{k+1}(x)| &\leq |\mathscr{F}| - \deg(x) \leq \binom{y}{k} - \binom{y-1}{k-1} = \binom{y-1}{k} \\ &= \frac{y-k}{k}\binom{y-1}{k-1} \leq \frac{y-k}{k}\deg(x). \end{aligned} \tag{2.1.2}$$

We have equality in (2.1.2) if and only if $\deg(x) = \binom{y-1}{k-1}$ since $\deg(x) \neq 0$.

Now we'll consider the case where $\deg(x) \leq \binom{y-1}{k-1}$. If $F \cup \{v\} \in K^k_{k+1}(x)$, then $F \in K^{k-1}_k(L(x))$ and so $|K^k_{k+1}(x)| \leq |K^k_{k-1}(L(x))|$. We define the real number $y_x \geq k$

by $\deg(x) = \binom{y_x-1}{k-1}$. Since $L(x) \subset \left[{X \atop k-1}\right]$ and $|L(x)| = \deg(x) = \binom{y_v-1}{k-1}$, the induction hypothesis yields that

$$|K_{k+1}^k(x)| \le |K_{k-1}^k(L(x))| \le \binom{y_x-1}{k} = \frac{y_x-k}{k}\binom{y_x-1}{k-1} \le \frac{y-k}{k}\deg(x), \qquad (2.1.3)$$

where the last inequality follows because $y_x \le y$, by our assumption on $\deg(x)$. We have equality in (2.1.3) if and only if $\deg(x) = \binom{y-1}{k-1}$ by our assumption that $\deg(x) \neq 0$.

To finish the proof, we sum the inequality $|K_{k+1}^k(x)| \le ((y-k)/k)\deg(x)$ over all $x \in X$ and double-count to obtain the desired inequality on $|K_{k+1}^k(\mathscr{F})|$. We have

$$\begin{aligned}(k+1)|K_{k+1}^k(\mathscr{F})| &= \sum_{x\in X}|K_{k+1}^k(x)| \le \frac{y-k}{k}\sum_{x\in X}\deg(x) = \frac{y-k}{k}\sum_{F\in\mathscr{F}}|F| \qquad (2.1.4)\\ &= \frac{y-k}{k}(k|\mathscr{F}|) = (y-k)\binom{y}{k} = (k+1)\binom{y}{k+1}.\end{aligned}$$

We now characterize the case of equality in (2.1.4). We see that equality holds in (2.1.4) if and only if $|K_{k+1}^k(x)| = \frac{y-k}{k}\deg(x)$ for all $x \in X$. We saw previously that if $\deg(x) \neq 0$, then equality holds in (2.1.2) and (2.1.3) if and only if $\deg(x) = \binom{y-1}{k-1}$. Consequently,

$$|\partial^1\mathscr{F}|\binom{y-1}{k-1} = \sum_{x\in X}\deg(x) = \sum_{F\in\mathscr{F}}|F| = k\binom{y}{k}$$

so $|\partial^1\mathscr{F}| = y$. As $\binom{y}{k} = |\mathscr{F}| \le \binom{|\partial^1\mathscr{F}|}{k} = \binom{y}{k}$, we have $\mathscr{F} = \binom{Y}{k}$, where $Y = \partial^1\mathscr{F}$. □

We now show that Theorem 2.1.5 follows as a corollary of Theorem 2.1.15.

Keevash's Proof of Theorem 2.1.5 Let $\mathscr{F}$ be as in Theorem 2.1.5, and let $x \ge k-1$ be the real number defined by $|\partial\mathscr{F}| = \binom{x}{k-1}$. By Theorem 2.1.15,

$$\binom{y}{k} = |\mathscr{F}| \le |K_k^{k-1}(\partial\mathscr{F})| \le \binom{x}{k}$$

because $\mathscr{F} \subset K_k^{k-1}(\partial\mathscr{F})$. Hence $x \ge y$, so $|\partial\mathscr{F}| = \binom{x}{k-1} \ge \binom{y}{k-1}$. If $|\partial\mathscr{F}| = \binom{y}{k-1}$ then $x = y$. Hence, $|K_k^{k-1}(\partial\mathscr{F})| = \binom{y}{k}$ and $\mathscr{F} = K_k^{k-1}(\partial\mathscr{F})$. By Theorem 2.1.15, this implies $y \in \mathbb{Z}^+$ and $\partial\mathscr{F} = \binom{Y}{k-1}$ for some y-subset $Y \subset X$. Clearly, $\binom{Y}{k} = K_k^{k-1}(\partial\mathscr{F}) = \mathscr{F}$. □

2.2 Intersection Theorems

We introduce the Erdős-Ko-Rado theorem in Section 2.2.1 and its extension to t-intersecting families [116] in Section 2.2.8. We will give three proofs of the Erdős-Ko-Rado theorem in Section 2.2.3, Section 2.2.4, and Section 2.2.5. We also discuss generalizations of the Erdős-Ko-Rado theorem such as Frankl's r-wise intersection theorem [52] in Section 2.2.6 and the Hilton-Milner theorem [70] in Section 2.2.9. In Chapter 4 and Chapter 5, we will see vector space analogs of all these theorems.

2.2.1 The Erdős-Ko-Rado Theorem

We present three proofs of the Erdős-Ko-Rado theorem. The first is the original proof by Erdős, Ko, and Rado [47], which introduced the shifting technique. The second is Daykin's elegant proof [33], which shows that the Erdős-Ko-Rado theorem is a simple corollary of Lovász's version of the Kruskal-Katona theorem. The last is a proof by Katona [76], which uses the cyclic permutation method. In Chapter 4 and Chapter 7, we will use arguments similar to Katona's cyclic permutation method to prove special cases of the Erdős-Ko-Rado theorem for vector spaces and the Manickam-Miklós-Singhi conjecture respectively.

Definition 2.2.1. *We say a family $\mathscr{F} \subset 2^X$ is intersecting, if any two sets in $\mathscr{F}$ have nonempty intersection; that is, for all $F, F' \in \mathscr{F}$ we have $F \cap F' \neq \emptyset$.*

The Erdős-Ko-Rado problem asks for the maximum size of an intersecting family $\mathscr{F} \subset \binom{X}{k}$ of k-subsets of X. Of course, the problem is interesting only when $n \geq 2k$, as otherwise any two sets of $\binom{X}{k}$ intersect.

Theorem 2.2.2 (Erdős-Ko-Rado)**.** *Suppose $\mathscr{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$. Then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if $\mathscr{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$, excepting the case $n = 2k$.*

2.2.2 More Properties of the Shift Operator

Towards a proof of the Erdős-Ko-Rado theorem using shifting, we first observe that the shifting operator $\tilde{S}_{ij}$ preserves the intersecting property of set systems.

Proposition 2.2.3. *If $\mathscr{F} \subset 2^X$ is intersecting, then so is its shift $\tilde{S}_{ij}(\mathscr{F})$.*

Proof. Suppose, for a contradiction, that $\mathscr{F} \subset 2^X$ is intersecting but its shift $\tilde{S}_{ij}(\mathscr{F})$ is not. Let $F_1, F_2 \in \tilde{S}_{ij}(\mathscr{F})$ be disjoint sets. We cannot have both $F_1, F_2 \in \mathscr{F}$ since $\mathscr{F}$ is intersecting and this would contradict our assumption that F_1 and F_2 are disjoint. Observe that Definition 2.1.7 implies that if $F \in \tilde{S}_{ij}(\mathscr{F})$ and $F \notin \mathscr{F}$ then $i \in F$ and there exists $E \in \mathscr{F}$ such that $j \in E$, $i \notin E$, and $F = S_{ij}(E)$. Hence, we cannot have both $F_1, F_2 \notin \mathscr{F}$ as then $i \in F_1 \cap F_2$. Without loss of generality, assume $F_1 \notin \mathscr{F}$ and $F_2 \in \mathscr{F}$. Let $E_1 \in \mathscr{F}$ such that $j \in E_1$, $i \notin E_1$, and $F_1 = S_{ij}(E_1)$. We have $E_1 \cap F_2 \neq \emptyset$ as $\mathscr{F}$ is intersecting and $E_1, F_2 \in \mathscr{F}$. Since $F_1 = S_{ij}(E_1) = (E_1 \setminus \{j\}) \cup \{i\}$ and $F_1 \cap F_2 = \emptyset$, we must have $E_1 \cap F_2 = \{j\}$. Now $F_1 \notin \mathscr{F}$ implies that $i \in F_1$; consequently $i \notin F_2$ as $F_1 \cap F_2 = \emptyset$. We have shown that $i \notin F_2$ and $j \in F_2$ so $S_{ij}(F_2) \neq F_2$. Definition 2.1.7 implies that $S_{ij}(F_2) \in \mathscr{F}$ as $F_2 \in \tilde{S}_{ij}(\mathscr{F}) \cap \mathscr{F}$. Since $i \notin E_1$ and $E_1 \cap F_2 = \{j\}$, we have

$$E_1 \cap S_{ij}(F_2) = E_1 \cap ((F_2 \setminus \{j\}) \cup \{i\}) = (E_1 \cap F_2) \setminus \{j\} = \emptyset,$$

which contradicts that $\mathscr{F}$ is intersecting. Hence, if $\mathscr{F}$ is intersecting then so is $\tilde{S}_{ij}(\mathscr{F})$. □

We now prove a lemma that will enable us to inductively prove the Erdős-Ko-Rado theorem via shifting.

Lemma 2.2.4. *Suppose $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family and $n \geq 2k$. Define $\mathscr{H}_0 = \mathscr{F}$ and $\mathscr{H}_i = \tilde{S}_{in}(\mathscr{H}_{i-1})$ for $1 \leq i \leq n-1$. For all $H, H' \in \mathscr{H}_{n-1}$, we must have*

$$H \cap H' \cap [n-1] \neq \emptyset.$$

Proof. Suppose, for a contradiction, that there exist $H, H' \in \mathscr{H}_{n-1}$ such that

$$H \cap H' \cap [n-1] = \emptyset.$$

We have $\mathscr{H}_{n-1}$ is intersecting by Proposition 2.2.3 and so $H \cap H' = \{n\}$. As $n > 2k$ and $|H \cup H'| = 2k-1$, there exists $i \in [n-1]$ such that $i \notin H \cup H'$. Since $n \in H$, we have $H \in \mathscr{F}$, and since $H \in \mathscr{H}_{n-1}$, we have $H \in \mathscr{H}_{i-1} \cap \mathscr{H}_i$. Since $i \notin H$, $n \in H$, and $H \in \mathscr{H}_i = \tilde{S}_{in}(\mathscr{H}_{i-1})$, Definition 2.1.7 implies that $\hat{H} := (H \setminus \{n\}) \cup \{i\} \in \mathscr{H}_{i-1}$ and hence $\hat{H} \in \mathscr{H}_{n-1}$. As $H \cap H' = \{n\}$ and $i \notin H'$, we have $\hat{H} \cap H' = \emptyset$ which contradicts $\mathscr{H}_{n-1}$ being intersecting. This proves that $H \cap H' \cap [n-1] \neq \emptyset$ for all H, H' in $\mathscr{H}_{n-1}$. □

2.2.3 Erdős-Ko-Rado via Shifting

We now prove the Erdős-Ko-Rado theorem via the shifting technique [47]. We will see that Proposition 2.2.3 and Lemma 2.2.4 allow us to transform our original intersecting family into a more structured intersecting family of the same size.

Proof of Theorem 2.2.2. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family. We will apply induction on n and prove the theorem simultaneously for all $k \leq n/2$. When $n = 2k$, each k-set can be paired up with its complement, which is also a k-set. Since any intersecting family $\mathscr{F} \subset \binom{X}{k}$ can contain at most one k-set from each pair, we have

$$|\mathscr{F}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}. \tag{2.2.1}$$

This argument also shows that there are exponentially many ways for equality to hold in (2.2.1) as any $\mathscr{F} \subset \binom{X}{k}$ that chooses exactly one k-set from each pair is an extremal intersecting family.

For $n > 2k$, define $\mathscr{H}_0 = \mathscr{F}$ and $\mathscr{H}_i = \tilde{S}_{in}(\mathscr{H}_{i-1})$ for $1 \leq i \leq n-1$. By Proposition 2.1.8 and Proposition 2.2.3, we have $|\mathscr{H}_{n-1}| = |\mathscr{F}|$ and $\mathscr{H}_{n-1}$ is intersecting. Define families $\mathscr{G}_{k-1} \subset \binom{[n-1]}{k-1}$ and $\mathscr{G}_k \subset \binom{[n-1]}{k}$ as follows:

$$\mathscr{G}_{k-1} = \{H \cap [n-1] : H \in \mathscr{H}_{n-1}, |H \cap [n-1]| = k-1\},$$
$$\mathscr{G}_k = \{H \cap [n-1] : H \in \mathscr{H}_{n-1}, |H \cap [n-1]| = k\}.$$

Lemma 2.2.4 implies that both $\mathscr{G}_{k-1}$ and $\mathscr{G}_k$ are intersecting families. By the inductive hypothesis,

$$|\mathscr{G}_{k-1}| \leq \binom{n-2}{k-2}, \quad |\mathscr{G}_k| \leq \binom{n-2}{k-1}.$$

Observe that for fixed $G \in G_{k-1} \cup G_k$, there is exactly one k-set $H \in \mathscr{H}_{n-1}$ such that $H \cap [n-1] = G$ so by Pascal's identity

$$|\mathscr{F}| = |\mathscr{H}_{n-1}| = |\mathscr{G}_{k-1}| + |\mathscr{G}_k| \leq \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1}. \tag{2.2.2}$$

We have proved that if $\mathscr{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$.

We now show that if equality holds and $n > 2k$, then $\mathscr{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$. We apply induction on k and prove the theorem simultaneously for all $n > 2k$. Clearly, the statement is true for $k = 2$ and $n > 4$ since, in this case, any intersecting family is contained in $\mathscr{F}_1 = \binom{[3]}{2}$ or $\mathscr{F}_x = \left\{F \in \binom{X}{2} : y \in F\right\}$ for some $y \in X$. If $|\mathscr{F}| = \binom{n-1}{k-1}$ then (2.2.2) implies that $|\mathscr{G}_{k-1}| = \binom{n-2}{k-2}$ and $|\mathscr{G}_k| = \binom{n-1}{k-1}$. Since $\mathscr{G}_{k-1} \subset \binom{[n-1]}{k-1}$ is intersecting and $n-1 > 2(k-1)$, the induction hypothesis implies that $\mathscr{G}_{k-1} = \left\{G \in \binom{[n-1]}{k-1} : y \in G\right\}$ for some $y \in [n-1]$. Lemma 2.2.4 implies that if $G \in \mathscr{G}_{k-1}$ and $G' \in \mathscr{G}_k$ then $G \cap G' \neq \emptyset$ so we conclude that $\mathscr{G}_k = \left\{G \in \binom{[n-1]}{k} : y \in G\right\}$ and $\mathscr{H}_{n-1} = \left\{F \in \binom{X}{k} : y \in F\right\}$. As a result, $\mathscr{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$. □

2.2.4 Erdős-Ko-Rado via Shadows

We present Daykin's elegant proof [33] of the Erdős-Ko-Rado theorem, which shows that it is a simple corollary of Lovász's version of the Kruskal-Katona theorem. We will return to this proof in Chapter 4, as it provides a simple example of a purely combinatorial proof that does not readily generalize to vector spaces.

Proof of Theorem 2.2.2. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family. Define

$$\mathscr{G} := \{X \setminus F : F \in \mathscr{F}\} \subset \binom{X}{n-k}$$

to be the family of $(n-k)$-sets that are the complements of the sets in $\mathscr{F}$. Note that $n-k \geq k$ under our assumption that $n \geq 2k$. Since $\mathscr{F}$ is intersecting, there does not exist $F \in \mathscr{F}$ such that $F \subset G$ for some $G \in \mathscr{G}$. Hence, $\partial^k \mathscr{G}$ and $\mathscr{F}$ are disjoint subfamilies of $\binom{X}{k}$ so

$$|\partial^k \mathscr{G}| + |\mathscr{F}| \leq \binom{n}{k}. \tag{2.2.3}$$

Suppose, for a contradiction, that $|\mathscr{F}| > \binom{n-1}{k-1}$. Hence, $|\mathscr{G}| = |\mathscr{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Applying Theorem 2.1.5 repeatedly, we find

$$|\partial^{n-k-1}\mathscr{G}| > \binom{n-1}{n-k-1}, \ldots, |\partial^k \mathscr{G}| > \binom{n-1}{k}. \tag{2.2.4}$$

We arrive at a contradiction by (2.2.3) since

$$\binom{n}{k} \geq |\partial^k \mathscr{G}| + |\partial \mathscr{F}| > \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k},$$

and so we must have $|\mathscr{F}| \leq \binom{n-1}{k-1}$. We now characterize the case of equality. Suppose that $|\mathscr{F}| = \binom{n-1}{k-1}$, then $|\mathscr{G}| = \binom{n-1}{k-1} = \binom{n-1}{n-k}$. We have $|\partial\mathscr{G}| \geq \binom{n-1}{n-k-1}$ by Theorem 2.1.5. On the other hand, (2.2.3) and (2.2.4) imply that $|\partial\mathscr{G}| \leq \binom{n-1}{k-1}$. We consequently have $\mathscr{G} \subset \binom{X}{n-k}$, $|\mathscr{G}| = \binom{n-1}{n-k}$, and $|\partial\mathscr{G}| = \binom{n-1}{n-k-1}$. Theorem 2.1.5 therefore implies that $\mathscr{G} = \binom{Y}{n-k}$ for some $Y \in \binom{X}{n-1}$. Let $x \in X$ be the unique element in $X \setminus Y$; we then have $\mathscr{F} = \{F \in \binom{X}{k} : x \in F\}$ as desired. □

2.2.5 Erdős-Ko-Rado via Cyclic Permutations

Our final proof of the Erdős-Ko-Rado theorem in this chapter is via Katona's cyclic permutation method [76]. We will use similar arguments in Chapter 4 and Chapter 7 to prove special cases of the Erdős-Ko-Rado theorem for vector spaces and the Manickam-Miklós-Singhi conjectures respectively. Let S_X denote the symmetric group on X. We say $\alpha \in S_X$ is a *cyclic permutation* of X if α has exactly one cycle when written in cycle notation. Clearly, there are $(n-1)!$ cyclic permutations of X. If $F \in \binom{X}{k}$, we say that α *contains* F if the elements of F are consecutive in α. We first prove a lemma that shows that a cyclic permutation can contain at most k elements of an intersecting family $\mathscr{F} \subset \binom{X}{k}$.

Lemma 2.2.5. *If $\alpha \in S_X$ is a cyclic permutation and $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family, then α contains at most k elements of $\mathscr{F}$.*

Proof. Suppose that $F \in \mathscr{F}$ appears as consecutive elements $x_1, \ldots, x_k$ of α. Since $\mathscr{F}$ is intersecting, the only sets of k consecutive elements of α which can be sets in $\mathscr{F}$ are the $k-1$ sets beginning with the elements $x_2, \ldots, x_{k-1}$ respectively and the $k-1$ sets ending with the elements $x_1, \ldots, x_{k-1}$ respectively. Without loss of generality, we can assume that one of the $k-1$ sets beginning with $x_2, \ldots, x_k$ lies in $\mathscr{F}$; choose the last such x_j, where $2 \leq j \leq k$, for which this is true. Since $\mathscr{F}$ is intersecting, none of the sets of k consecutive elements ending with $x_1, \ldots, x_{j-1}$ can lie in $\mathscr{F}$. Hence, there are at most $j + (k-j) = k$ elements of $\mathscr{F}$ contained in α. □

Proof of Theorem 2.2.2. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family. Let

$$S := \{(\alpha, F) : \alpha \in S_X \text{ is a cyclic permutation, } F \in \mathscr{F}\text{, and } \alpha \text{ contains } F\}.$$

We will double-count the cardinality of S. On one hand, each $F \in \mathscr{F}$ is contained in $k!(n-k)!$ cyclic permutations α because there are $k!$ ways to arrange the elements of F consecutively and $(n-k)!$ ways to arrange the elements of $X \setminus F$. On the other hand, Lemma 2.2.5 implies that each of the $(n-1)!$ cyclic permutations α can contain at most k elements of $\mathscr{F}$. Putting these observations together yields

$$|\mathscr{F}|k!(n-k)! = |S| \leq (n-1)!k,$$

and so $|\mathscr{F}| \leq \binom{n-1}{k-1}$. For a characterization of equality using the method of cyclic permutations see [65, Theorem 7.8.1]. □

2.2.6 Frankl's *r*-wise Intersection Theorem

The Erdős-Ko-Rado theorem asserts that if $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family and $n \geq 2k$, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. Frankl [52] showed in his r-wise intersection theorem that we can get the same bound on $|\mathscr{F}|$ for smaller values of n if the pairwise intersecting condition is strengthened. We will prove a generalization of Frankl's result for vector spaces in Chapter 5.

Definition 2.2.6. *A family $\mathscr{F} \subset 2^X$ is called r-wise intersecting if any r sets in $\mathscr{F}$ have nonempty intersection; that is, for all $F_1, \ldots, F_r \in \mathscr{F}$ we have $\bigcap_{i=1}^{r} F_i \neq \emptyset$.*

When $r = 2$, then r-wise will be omitted since, in this case, an r-wise intersecting family is simply intersecting. We saw that the Erdős-Ko-Rado problem was only interesting in the case that $n \geq 2k$, otherwise any two elements of $\binom{X}{k}$ intersect. Similarly, the question of determining the largest r-wise intersecting family is only interesting when $n \geq (r/r-1)k$, otherwise any r elements of $\binom{X}{k}$ intersect. On the other hand, since an r-wise intersecting family $\mathscr{F} \subset \binom{X}{k}$ is intersecting, the Erdős-Ko-Rado theorem shows that $|\mathscr{F}| \leq \binom{n-1}{k-1}$ when $n \geq 2k$. Frankl proved that this same bound on $|\mathscr{F}|$ holds in the range $(r/r-1)k \leq n < 2k$.

Theorem 2.2.7 (Frankl)**.** *Suppose that $\mathscr{F} \subset \binom{X}{k}$ is r-wise intersecting and $(r-1)n \geq rk$. Then $|\mathscr{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if $\mathscr{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$, excepting the case $r = 2$ and $n = 2k$.*

Note that when $r = 2$, Frankl's r-wise intersection theorem reduces to the Erdős-Ko-Rado theorem. We will prove Frankl's result using Lovász's version of the Kruskal-Katona theorem, Theorem 2.1.5. First, we need a lemma due to Kleitman.

Lemma 2.2.8 (Kleitman)**.** *Let $k_1, \dots, k_r \in \mathbb{Z}^+$ such that $k_1 + \cdots + k_r = n$ and suppose $\mathscr{F}_i \subset \binom{X}{k_i}$ is a family of k_i-subsets of X for $i \in [r]$. If there do not exist $F_i \in \mathscr{F}_i$ with $i \in [r]$ and $F_1 \cup \cdots \cup F_r = X$, then*

$$\sum_{i \in [r]} \frac{|\mathscr{F}_i|}{\binom{n}{k_i}} \leq r - 1.$$

Moreover, equality holds if and only if for every ordered partition $G_1 \cup \cdots \cup G_r = X$ with $|G_i| = k_i$, there is exactly one $i \in [r]$ for which $G_i \notin \mathscr{F}_i$.

Proof. Consider all ordered partitions $G_1 \cup \cdots \cup G_r = X$ with $|G_i| = k_i$. Say there are T of them. Define the $T \times r$ matrix A by

$$A_{(G_1 \cup \cdots \cup G_r, \mathscr{F}_i)} = \begin{cases} 1 & \text{if } G_i \in \mathscr{F}_i \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed $F \in \binom{X}{k_i}$, one has $F \in G_i$ for a fraction $1/\binom{n}{k_i}$ of all these partitions. Hence, $G_i \in \mathscr{F}_i$ holds for a fraction $|\mathscr{F}_i|/\binom{n}{k_i}$ of these partitions. The number of nonzero entries in the matrix A is thus

$$T \sum_{i \in [r]} \frac{|\mathscr{F}_i|}{\binom{n}{k_i}}.$$

By assumption, however, there are at most $r - 1$ nonzero entries in each row of A. Hence,

$$T \sum_{i \in [r]} \frac{|\mathscr{F}_i|}{\binom{n}{k_i}} \leq (r-1)T \quad \Rightarrow \sum_{i \in [r]} \frac{|\mathscr{F}_i|}{\binom{n}{k_i}} \leq r - 1,$$

and the characterization of equality is clear. □

Now we give Frankl's proof of his r-wise intersection theorem [115].

Proof of Theorem 2.2.7 Define $\mathscr{F}^c := \{X \setminus F : F \in \mathscr{F}\} \subset \binom{X}{n-k}$ to be the family of complements of sets in $\mathscr{F}$ with respect to X. Choose $k_1, \dots, k_r \in [n-k]$ such that $\sum_{i \in [r]} k_i = n$. For $i \in [r]$, define $\mathscr{F}_i := \partial^{k_i} \mathscr{F}^c$. As $\mathscr{F} \subset \binom{X}{k}$ is r-wise intersecting, this implies that $\mathscr{F}^c \subset \binom{X}{n-k}$ does not contain r sets whose union is X. Thus $\mathscr{F}_1, \dots, \mathscr{F}_r$ satisfy the assumptions of Lemma 2.2.8.

We have assumed that $(r-1)n \geq rk$ so first suppose that $(r-1)n = rk$. In this case, $(n-k)r = n$ so $k_1 = \cdots = k_r = n-k$ and $\mathscr{F}_1 = \cdots = \mathscr{F}_r = \mathscr{F}^c$. Hence, by Lemma 2.2.8,

$$|\mathscr{F}| = |\mathscr{F}^c| \leq \frac{r-1}{r}\binom{n}{n-k} = \frac{n}{k}\binom{n}{k} = \binom{n-1}{k-1}, \tag{2.2.5}$$

which proves the bound in this case.

Now, we characterize the case of equality in (2.2.5). By Lemma 2.2.8, if equality holds, then for every ordered partition $G_1 \cup \cdots \cup G_r = X$, there is exactly one $i \in [r]$ for which $G_i \notin \mathscr{F}^c$. As a result, if equality holds in (2.2.5), then there cannot exist two disjoint sets in $\binom{X}{n-k} \setminus \mathscr{F}^c$, as then we could form an ordered partition of X with two sets missing from $\mathscr{F}^c$. Hence $\binom{X}{n-k} \setminus \mathscr{F}^c$ is an intersecting family of size

$$\left|\binom{X}{n-k} \setminus \mathscr{F}^c\right| = \left|\binom{X}{n-k}\right| - |\mathscr{F}^c| = \binom{n}{n-k} - \binom{n-1}{n-k} = \binom{n-1}{n-k-1}.$$

Since $n = r(n-k)$ and $\binom{X}{n-k} \setminus \mathscr{F}^c \subset \binom{X}{n-k}$, we see that for $r \geq 3$, the uniqueness of the extremal families in the Erdős-Ko-Rado theorem implies that there exists $x \in X$ such that $\binom{X}{n-k} \setminus \mathscr{F}^c = \{G \in \binom{X}{n-k} : x \in G\}$. Consequently, there exists $\hat{x} \in X$ such that $\mathscr{F} = \{F \in \binom{X}{k} : \hat{x} \in F\}$.

Now assume that $(r-1)n > rk$ so that $(n-k)r > n$ and some $k_i < n-k$. Suppose that $|\mathscr{F}| \geq \binom{n-1}{k-1}$ so that $|\mathscr{F}^c| \geq \binom{n-1}{n-k}$. Applying Theorem 2.1.5 repeatedly yields that $|\mathscr{F}_i| \geq \binom{n-1}{k_i}$. Since $\sum_{i \in r} k_i = n$, we thus have

$$\sum_{i \in [r]} \frac{|\mathscr{F}_i|}{\binom{n}{k_i}} \geq \sum_{i \in [r]} 1 - \frac{k_i}{n} = r-1. \tag{2.2.6}$$

Lemma 2.2.8 and (2.2.6) imply that $|\mathscr{F}_i| = \binom{n-1}{k_i}$ and so $|\mathscr{F}^c| = \binom{n-1}{n-k}$ and $|\mathscr{F}| = \binom{n-1}{k-1}$. Moreover, by Theorem 2.1.5, we must have $\mathscr{F}^c = \binom{X \setminus \{x\}}{n-k}$ for some $x \in X$ since there exists $i \in [r]$ such that $k_i < n-k$. Consequently, $\mathscr{F} = \{F \in \binom{X}{k} : x \in F\}$. We have shown that if $|\mathscr{F}| \geq \binom{n-1}{k-1}$, then $\mathscr{F} = \{F \in \binom{X}{k} : x \in F\}$. Hence, we always have $|\mathscr{F}| \leq \binom{n-1}{k-1}$, and if equality holds then $\mathscr{F} = \{F \in \binom{X}{k} : x \in F\}$. $\square$

2.2.7 Forbidding Triangles

In the Erdős-Ko-Rado theorem, Theorem 2.2.2, and Frankl's r-wise intersection theorem, Theorem 2.2.7, the extremal family has the form $\{F \in \binom{X}{k} : x \in F\}$ for some

$x \in X$. Such a family is called a *star.* What other conditions on a uniform family $\mathscr{F} \subset \binom{X}{k}$ imply that the extremal family is a star? One particularly nice example, Theorem 2.2.10, was posed by Erdős and proved by Mubayi and Verstraëte [93].

Definition 2.2.9. *A triangle $\mathscr{T} = \{A,B,C\} \subset \binom{X}{k}$ is a family consisting of three sets A,B,C such that $A\cap B, A\cap C, B\cap C$ are each nonempty but $A\cap B\cap C = \emptyset$.*

We may ask for the size of the largest family $\mathscr{F} \subset \binom{X}{k}$ not containing a triangle. We see that the question is uninteresting unless $n \geq 3k/2$, otherwise $\binom{X}{k}$ is the extremal example. If $k = 2$, then Mantel's theorem [90] asserts that $|\mathscr{F}| \leq \lfloor n^2/4 \rfloor$ and that the extremal family is a complete bipartite graph. The answer is quite different when $k \geq 3$.

Theorem 2.2.10 (Mubayi-Verstraëte)**.** *Suppose that $\mathscr{F} \subset \binom{X}{k}$ contains no triangle and that $k \geq 3$ and $n \geq 3k/2$. Equality holds if and only if $\mathscr{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$.*

Note that if $3k/2 \leq n < 2k$ and $\mathscr{F}$ contains no triangle, then $\mathscr{F}$ is 3-wise intersecting. As a result, Frankl's r-wise intersection theorem establishes Theorem 2.2.10 in this range. We will return to Theorem 2.2.10 in Section 5.5.

2.2.8 On t-intersecting Families

The Erdős-Ko-Rado theorem gives an upper bound on families $\mathscr{F} \subset \binom{X}{k}$ for which any two sets in $\mathscr{F}$ have nonempty intersection. If the intersection condition is strengthened to any two sets in $\mathscr{F}$ have intersection of size at least t, where $t \in \mathbb{Z}^+$, then Erdős, Ko, and Rado [47] showed that a stronger upper bound on $|\mathscr{F}|$ holds for sufficiently large n. We will see in Chapter 4 that an analog of this result holds for vector spaces.

Definition 2.2.11. *For a positive integer $t \in \mathbb{Z}^+$, a family $\mathscr{F} \subset 2^X$ is t-intersecting if any two sets in $\mathscr{F}$ have intersection of size at least t; that is, $|F \cap F'| \geq t$ for all $F, F' \in \mathscr{F}$.*

Using algebraic methods similar to those in Section 3.3.2, Wilson [116] proved the following generalization of the Erdős-Ko-Rado theorem. Note that when $t = 1$, Wilson's result reduces to the Erdős-Ko-Rado theorem.

Theorem 2.2.12 (Wilson)**.** *Suppose* $\mathscr{F} \subset \binom{X}{k}$ *is* t*-intersecting and* $n \geq (t+1)(k-t+1)$*. Then* $|\mathscr{F}| \leq \binom{n-t}{k-t}$*. Equality holds if and only if* $\mathscr{F} = \{F \in \binom{X}{k} : S \subset T\}$ *for some* t*-subset* $S \in \binom{X}{t}$*, excepting the case* $n = (t+1)(k-t+1)$*.*

2.2.9 The Hilton-Milner Theorem

Finally, we state the Hilton-Milner theorem [70], which is the last generalization of the Erdős-Ko-Rado theorem we will discuss. Like the other results, we will see in Chapter 5 that Hilton and Milner's theorem can also be generalized to vector spaces. Frankl and Füredi [54] gave an elegant proof of the Hilton-Milner theorem using the shifting technique.

Theorem 2.2.13 (Hilton-Milner)**.** *Let* $\mathscr{F} \subset \binom{X}{k}$ *be an intersecting family with* $k \geq 2$*,* $n \geq 2k+1$*, and such that there does not exist* $x \in X$ *such that* $\mathscr{F} \subset \{F \in \binom{X}{k} : x \in F\}$*. We then have*

$$|\mathscr{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality holds if and only if

(i) $\mathscr{F} = \{F\} \cup \{G \in \binom{X}{k} : x \in G,\ F \cap G \neq \emptyset\}$ *for some* k*-subset* F *and* $x \in X \setminus F$*.*

(ii) $\mathscr{F} = \{F \in \binom{X}{3} : |F \cap S| \geq 2\}$ *for some 3-subset* S *if* $k = 3$*.*

2.3 Fisher's Inequality

In Section 2.2.1, we were concerned with an upper bound for intersecting families, that is those families $\mathscr{F} \subset \binom{X}{k}$ for which $|F_1 \cap F_2| \geq 1$ for every $F_1, F_2 \in \mathscr{F}$. In this section, we will drop the uniformity condition, but insist that distinct elements F_1, F_2 of our family $\mathscr{F} \subset 2^X$ satisfy $|F_1 \cap F_2| = \lambda$, where $\lambda \in \mathbb{N}$. Such families are called λ-intersecting.

In Section 2.3.1, we present the de Bruijn-Erdős theorem [34], which bounds the size of maximum 1-intersecting families and characterizes the extremal examples. We then generalize the de-Bruijn Erdős theorem in Section 2.3.3 by stating Fisher's Inequality [17, 51, 73, 85], which handles the same question for general λ. We will also state a conjecture of Frankl and Füredi [55] that generalizes Fisher's Inequality, and then present a proof of their conjecture in the special case that $\lambda = 1$. Frankl and Füredi's proof is similar to that of de Bruijn and Erdős and will make use of convexity, an important tool in extremal combinatorics.

2.3.1 The de Bruijn-Erdős Theorem

We will state the de Bruijn-Erdős theorem [34], which bounds the size of maximum 1-intersecting families and characterizes the extremal examples.

Definition 2.3.1. *Given $\lambda \in \mathbb{N}$, a family $\mathscr{F} \subset 2^X$ is* <u>*λ-intersecting*</u> *if, for any distinct $F_1, F_2 \in \mathscr{F}$, we have $|F_1 \cap F_2| = \lambda$.*

Definition 2.3.2. *A family $\mathscr{F} \subset 2^X$ is* <u>*k-uniform*</u> *if $\mathscr{F} \subset \binom{X}{k}$.*

Definition 2.3.3. *A family $\mathscr{F} \subset 2^X$ is* <u>*r-regular*</u> *if $\deg(x) = r$ for all $x \in X$.*

We now give some examples of 1-intersecting families.

Definition 2.3.4. *We say $\mathscr{F}$ is* <u>*trivial*</u> *if there exists $x \in X$ with $\deg(x) = |\mathscr{F}|$, and is* <u>*nontrivial*</u> *otherwise.*

Definition 2.3.5. *A family $\mathscr{F} \subset 2^X$ is called a* <u>*near-pencil*</u> *if*

$$\mathscr{F} = \{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\}.$$

Definition 2.3.6. *A family $\mathscr{F} \subset \binom{X}{k}$ is called a* <u>*projective plane*</u> *if $\mathscr{F}$ is 1-intersecting, uniform, and $|\mathscr{F}| = n$.*

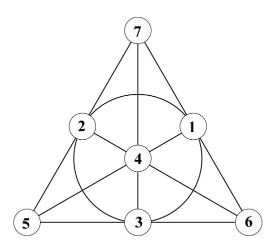

Figure 2.1: The Fano plane is an example of a projective plane.

Theorem 2.3.7 (de Bruijn-Erdős)**.** *Suppose $\mathscr{F} \subset 2^X$ is a 1-intersecting family of size m. We have $m \leq n$. Moreover, if equality holds, then exactly one of the following is true:*

(i) *there is $F \in \mathscr{F}$ such that $|F| = 1$,*

(ii) *$\mathscr{F}$ is a near-pencil,*

(iii) *$\mathscr{F}$ is a projective plane.*

2.3.2 Proof of the de Bruijn-Erdős Theorem

In this section, we will prove the de Bruijn-Erdős theorem, Theorem 2.3.7. First, we state and prove a lemma to which the theorem reduces.

Lemma 2.3.8. *Let $\mathscr{F} \subset 2^X$ be a family of size m. Suppose $\deg(x) < m$ and $|F| < n$ for every $x \in X$ and $F \in \mathscr{F}$. Assume further that if $x \notin F$, then $\deg(x) \leq |F|$. We then have $m \leq n$.*

Proof. Suppose, for a contradiction, that $m > n$. Given any pair (x,F) for which $x \notin F$, we then have

$$\frac{\deg(x)}{m - \deg(x)} < \frac{|F|}{n - |F|}.$$

Summing this inequality over all pairs (x,F) for which $x \notin F$ yields a contradiction,

$$\begin{aligned} \sum_{F \in \mathscr{F}} |F| = \sum_{x \in X} \deg(x) = \sum_{x \in X} (m - \deg(x)) \frac{\deg(x)}{m - \deg(x)} < \sum_{(x,F), x \notin F} \frac{\deg(x)}{m - \deg(x)} \\ < \sum_{(x,F), x \notin F} \frac{|F|}{n - |F|} = \sum_{F \in \mathscr{F}} (n - |F|) \frac{|F|}{n - |F|} = \sum_{F \in \mathscr{F}} |F|. \end{aligned} \tag{2.3.1}$$

Hence, $m \leq n$. Also observe that if $m = n$, then (2.3.1) yields that $\deg(x) = |F|$ if $x \notin F$. □

Now we prove the de Bruijn-Erdős theorem [34].

Proof of Theorem 2.3.7. If there exists $F \in \mathscr{F}$ such that $|F| = 1$, then all the other sets contain F and are disjoint otherwise. It follows that $m \leq (n-1) + 1 = n$.

Hence, for each $x \in X$, we can assume $\deg(x) < m$; otherwise we can add $\{x\}$ to $\mathscr{F}$ if it is not there already, and since $|\{x\}| = 1$, we are done by the previous paragraph.

We can also assume for each $F \in \mathscr{F}$ that $|F| < n$, otherwise $m = n = 1$. We claim that if $x \notin F$, then $\deg(x) \leq |F|$. This is because every $E \in \mathscr{F}(x) := \{E \in \mathscr{F} : x \in E\}$ intersects F in precisely one element and no two distinct $E \in \mathscr{F}(x)$ intersect F in the same element as $\mathscr{F}$ is 1-intersecting. Hence, we are in the situation of Lemma 2.3.8, so $m \leq n$.

We now characterize the nontrivial extremal families. We first prove that every pair of points is contained in some $F \in \mathscr{F}$. Suppose, for a contradiction, that there exist distinct $x, y \in X$ such that there is no $F \in \mathscr{F}$ for which $x, y \in F$. Since $m = n$, we must have $\deg(y) \neq 0$, so choose $F' \in \mathscr{F}$ such that $y \in F'$. We have $x \notin F'$ by assumption, so Lemma 2.3.8 implies that $\deg(x) = |F'|$. As $\mathscr{F}$ is 1-intersecting, every $E \in \mathscr{F}(x)$ intersects F' in precisely one element and no two distinct $E \in \mathscr{F}(x)$ intersect F' in the same element. Consequently, $\deg(x) = |F'|$ implies that some $E \in \mathscr{F}(x)$ must contain y, which contradicts our initial assumption about x and y. This proves that if $\mathscr{F}$ is a nontrivial 1-intersecting family with $m = n$, then every pair of points is contained in some $F \in \mathscr{F}$.

We distinguish two cases according to whether there exist distinct $F_1, F_2 \in \mathscr{F}$ such that $X = F_1 \cup F_2$. First, assume there exist distinct $F_1, F_2 \in \mathscr{F}$ such that $X = F_1 \cup F_2$. Suppose, for a contradiction, that $|F_1|, |F_2| \geq 3$. Let $x = F_1 \cap F_2$ and let y_1, y_2 and z_1, z_2 be two other points on F_1 and F_2 respectively. By the result of the previous paragraph, there exist $F_3, F_4 \in \mathscr{F}$ such that $y_1, z_1 \in F_3$ and $y_2, z_2 \in F_4$. Since $\mathscr{F}$ is 1-intersecting and $X = F_1 \cup F_2$, we see $F_3 = \{y_1, z_1\}$ and $F_4 = \{y_2, z_2\}$. We obtain a contradiction since $F_3 \cap F_4 = \emptyset$. Hence, we cannot have both $|F_1|, |F_2| \geq 3$. Without loss of generality, assume $|F_1| < 3$; we must have $|F_1| = 2$ and $|F_2| = n - 1$ since $\mathscr{F}$ is nontrivial and $X = F_1 \cup F_2$. As every pair of points belongs to some $F \in \mathscr{F}$, we see that $\mathscr{F}$ must be a near-pencil.

Now suppose there does not exist $F_1, F_2 \in \mathscr{F}$ such that $X = F_1 \cup F_2$. Fix some $F \in \mathscr{F}$ and suppose $|F| = k$. For any other $F' \in \mathscr{F}$, we can choose $x \in X$ such that $x \notin F \cup F'$. By Lemma 2.3.8, we have $|F'| = \deg(x) = |F| = k$ so $\mathscr{F}$ is k-uniform. Hence $\mathscr{F}$ is a projective plane. $\square$

2.3.3 The Frankl-Füredi Conjecture

The de Bruijn-Erdős theorem proved that if $\mathscr{F} \subset 2^X$ is a 1-intersecting family of size m, then $m \leq n$. The well-known Fisher's Inequality establishes the same conclusion more generally for λ-intersecting families.

Theorem 2.3.9 (Fisher's Inequality)**.** *If $\lambda \in \mathbb{Z}^+$ and $\mathscr{F} \subset 2^X$ is a λ-intersecting family of size m, then $m \leq n$.*

Fisher [51] proved Theorem 2.3.9 under the additional constraint that $\mathscr{F}$ is regular (and hence uniform). Bose [17] proved Theorem 2.3.9 when $\mathscr{F}$ is required to be uniform. Majindar [85] first proved Theorem 2.3.9 as stated, and his proof was later rediscovered by Isbell [73]. We will defer a proof of Theorem 2.3.9 until Chapter 3, as its proof was one of the first to demonstrate the power of linear algebra.

Another way to restate Fisher's Inequality is that if $\mathscr{F} \subset 2^X$ is a λ-intersecting family of size m then $|\partial^1 \mathscr{F}| \geq m$. Inspired by Fisher's Inequality, Frankl and Füredi [55] conjectured a similar inequality for $|\partial^2 \mathscr{F}|$ and verified it when $\lambda = 1$.

Conjecture 2.3.10 (Frankl-Füredi)**.** *If $\lambda \in \mathbb{Z}^+$ and $\mathscr{F} \subset 2^X$ is a nontrivial λ-intersecting family of size m, then*

$$|\partial^2 \mathscr{F}| \geq \binom{m}{2}.$$

Theorem 2.3.11 (Frankl-Füredi)**.** *If $\mathscr{F} \subset 2^X$ is a nontrivial 1-intersecting family of size m then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$.*

2.3.4 Convexity

We will give more background on Conjecture 2.3.10 in Chapter 6. In this section, we will concern ourselves with Frankl and Füredi's proof of Theorem 2.3.11; their argument is similar to de Bruijn and Erdős's proof of Theorem 2.3.7 and uses convexity. We now define the concepts of convexity and Schur convexity and state a theorem that will be used in the proof of Frankl and Füredi's Conjecture 2.3.10 for $\lambda = 1$.

Definition 2.3.12. *We say a function $f : \mathbb{R} \to \mathbb{R}$ is <u>convex</u> if for all $x_1, x_2 \in \mathbb{R}$ and any $t \in [0,1]$ we have*

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2).$$

Definition 2.3.13. *For* $\alpha := (\alpha_1 \dots, \alpha_d) \in \mathbb{R}^d$*, we let* $\alpha^* := (\alpha_1^*, \dots, \alpha_d^*)$ *denote the vector with the same components as* α *but sorted in decreasing order. Given* $\alpha, \beta \in \mathbb{R}^d$*, we say* α *majorizes* β *and we write* $\alpha \succeq \beta$ *if for all* $k \in [d]$*, we have*

$$\sum_{i=1}^{k} \alpha_i^* \geq \sum_{i=1}^{k} \beta_i^*, \quad \sum_{i=1}^{d} \alpha_i^* = \sum_{i=1}^{d} \beta_i^*.$$

Definition 2.3.14. *We say* $F : \mathbb{R}^d \to \mathbb{R}$ *is Schur convex if* $F(\alpha) \geq F(\beta)$ *for all* $\alpha, \beta \in \mathbb{R}^d$ *for which* $\alpha \succeq \beta$*.*

Theorem 2.3.15. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a convex function. We then have that the function* $F : \mathbb{R}^d \to \mathbb{R}$ *defined by the sum*

$$F(x_1, \dots, x_d) = \sum_{k=1}^{d} f(x_k)$$

is Schur convex.

For a proof of Theorem 2.3.15 see [105, Chapter 13]. We now present Frankl and Füredi's proof [55] of Conjecture 2.3.10 when $\lambda = 1$.

Proof of Theorem 2.3.11 Let $\mathscr{F} \subset 2^X$ be a nontrivial 1-intersecting family of size m. For $x_i \in X$ and $F_j \in \mathscr{F}$, let $d_i := \deg(x_i)$ and $e_j := |F_j|$. Order $F_1, \dots, F_m$ so that $e_1 \geq \dots \geq e_m$. Similarly order $x_1, \dots, x_n$ so that $d_1 \geq \dots \geq d_n$. We will show that

$$e_i \geq d_i. \tag{2.3.2}$$

Since $\mathscr{F}$ is 1-intersecting, we have that if $x \notin F$, then $\deg(x) \leq |F|$; this is because every $E \in \mathscr{F}(x) := \{E \in \mathscr{F} : x \in E\}$ intersects F in precisely one element and no two distinct $E \in \mathscr{F}(x)$ intersect F in the same element. Let $F_j \in \mathscr{F}$ be any set that does not contain $\{x_1, \dots, x_i\}$. Hence, for some $k \in [i]$ we have $x_k \notin F$ so $e_j \geq d_k \geq d_i$. Consequently, (2.3.2) holds if we have i sets $F \in \mathscr{F}$ that do not contain $\{x_1, \dots, x_i\}$. For $i = 1$, we certainly have one set $F \in \mathscr{F}$ that does not contain x_1 since $\mathscr{F}$ is nontrivial. For $i > 1$, suppose there are only at most $i - 1$ sets $F \in \mathscr{F}$ that do not contain $\{x_1, \dots, x_i\}$. Since $i \geq 2$ and $\mathscr{F}$ is 1-intersecting, there can be at most one set in $\mathscr{F}$ that contains $\{x_1, \dots, x_i\}$ so $m \leq i$. As there are i sets and $d_1 \geq \dots \geq d_i$, we have $d_i \leq 1$ so $e_i \geq 2 > 1 \geq d_i$. We see that in all cases (2.3.2) holds.

Let $\delta := (d_1, \dots, d_n)$, $\varepsilon := (e_1, \dots, e_m, 0, \dots, 0) \in \mathbb{R}^n$. We have $\varepsilon \succeq \delta$ since $e_i \geq d_i$ and $\sum_{i=1}^m e_i = \sum_{i=1}^n d_i$. The function $f(x) = \binom{x}{2}$ is convex so by Theorem 2.3.15, the function $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x_1, \dots, x_n) := \sum_{i=1}^n \binom{x_i}{2}$$

is Schur convex. Since $\mathscr{F}$ is 1-intersecting, we have

$$F(\delta) = F(d_1, \dots, d_n) = \sum_{i=1}^n \binom{d_i}{2} = |\{(x, F, F') : F, F' \in \mathscr{F}, F \cap F' = \{x\}\}| = \binom{m}{2},$$

$$F(\varepsilon) = F(e_1, \dots, e_m, 0, \dots, 0) = \sum_{i=1}^m \binom{e_i}{2} = |\{(\{x,y\}, F) : F \in \mathscr{F}, x, y \in F\}| = |\partial^2 \mathscr{F}|.$$

We showed $\varepsilon \succeq \delta$ and so

$$|\partial^2 \mathscr{F}| = F(\varepsilon) > F(\delta) = \binom{m}{2},$$

as desired. □

Chapter 3

Algebraic Techniques

In this chapter, we describe algebraic methods that are frequently used in extremal set theory and in this thesis. First, we make good on our promise in Chapter 2 and prove Fisher's Inequality, Theorem 2.3.9. We also discuss the major open problem of characterizing the extremal families in Fisher's Inequality, which is known as the λ-design conjecture [99, 117]. Next, we discuss a generalization of Fisher's Inequality to L-intersecting families [59], whose proof illustrates the polynomial method. This result and its variants have powerful combinatorial and geometric consequences; they have been used, for example, to disprove Borsuk's conjecture in topology [74]. We then prove the Erdős-Ko-Rado theorem via the eigenvalue method [66, 116]; variants of this method have yielded Erdős-Ko-Rado analogs in many different structures. Finally, we discuss linear programming and use this technique to prove Baranyai's theorem [10] on decompositions of the complete hypergraph into perfect matchings. We will use Baranyai's theorem in Chapter 5 and Chapter 7.

3.1 Fisher's Inequality via Linear Algebra

We prove Fisher's Inequality, Theorem 2.3.9, via linear algebra. Given a family of sets $\mathscr{F} \subset 2^X$, a natural way to represent it is via its incidence matrix. Majindar's [85] and Isbell's [73] ingenious proof of Fisher's Inequality analyzes the rank of the incidence matrix.

Definition 3.1.1. *If $\mathscr{F} \subset 2^X$ is a family of sets with size $|\mathscr{F}| = m$, then its associated* <u>*incidence matrix*</u> *M is defined to be the $m \times n$ matrix with*

$$M_{ij} := \begin{cases} 1 & \text{if} \quad j \in F_i \\ 0 & \text{if} \quad j \notin F_i. \end{cases}$$

Proof of Theorem 2.3.9. Let $\mathscr{F} \subset 2^X$ be a λ-intersecting family. First, we consider the case where some $F \in \mathscr{F}$ has exactly λ elements. Since $\mathscr{F}$ is λ-intersecting, all the other sets in $\mathscr{F}$ contain F and are disjoint otherwise. Hence $m \leq n+1-\lambda \leq n$ as $\lambda \in \mathbb{Z}^+$.

We may thus assume that the numbers $\gamma_i := |F_i| - \lambda$ are all positive for $1 \leq i \leq m$. Let M be the incidence matrix of $\mathscr{F}$, and define the matrix $A := MM^T$. Observe that $A_{ij} = |F_i \cap F_j|$; since $\mathscr{F}$ is λ-intersecting, we have $A = \lambda J + C$, where J is the $m \times m$ all ones matrix and C is the diagonal matrix with entries $C_{ii} := \gamma_i$. Note that λJ is a positive semidefinite matrix and that C is a positive definite matrix since $\gamma_i > 0$. Consequently, A is a positive definite matrix so $m = \operatorname{rank} A \leq \operatorname{rank} M \leq n$. □

3.1.1 The Extremal Families in Fisher's Inequality

Characterizing the extremal families $\mathscr{F} \subset 2^X$ in Fisher's Inequality is a major open problem and is known as the *λ-design conjecture*. In the de Bruijn-Erdős theorem, Theorem 2.3.7, the crucial step in characterizing the extremal families is proving that every pair of points in a nontrivial extremal family $\mathscr{F}$ is contained in some $F \in \mathscr{F}$; this is the content of Frankl and Füredi's generalization, Theorem 2.3.11. While Babai [3] has shown that Frankl and Füredi's Conjecture 2.3.10 is true for all $\lambda \geq 1$ in the case that $m = n$, this does not lead to a characterization of the extremal families in Fisher's Inequality unfortunately. We will partition the extremal families in Fisher's Inequality according to whether they are uniform.

Definition 3.1.2. *For $\lambda \in \mathbb{Z}^+$, a λ-intersecting family $\mathscr{F} \subset \binom{X}{k}$ is a* <u>*symmetric design*</u> *if it is uniform and has cardinality $|\mathscr{F}| = n$.*

Definition 3.1.3. *For $\lambda \in \mathbb{Z}^+$, a λ-intersecting family $\mathscr{F} \subset 2^X$ is a* <u>*λ-design*</u> *if $\mathscr{F}$ is not uniform and $|\mathscr{F}| = n$.*

3.1.2 Symmetric Designs

We now give Ryser's [98] elegant algebraic proof which shows that a symmetric design $\mathscr{F} \subset \binom{X}{k}$ is k-regular and that any pair of points $\{x,y\} \in \binom{X}{2}$ is contained in exactly λ sets in $\mathscr{F}$. Ryser's proof provides another application of incidence matrices.

Theorem 3.1.4 (Ryser)**.** *If $\mathscr{F} \subset \binom{X}{k}$ is a symmetric design, then $\mathscr{F}$ is k-regular and any pair of points $\{x,y\} \in \binom{X}{2}$ is contained in exactly λ sets in $\mathscr{F}$.*

Proof. Let M be the $n \times n$ incidence matrix of $\mathscr{F}$ and observe that

$$M^T M_{ij} := \begin{cases} \deg(x_i) & \text{if} \quad i = j \\ \operatorname{codeg}(x_i, x_j) & \text{if} \quad i \neq j, \end{cases} \tag{3.1.1}$$

where $\operatorname{codeg}(x_i,x_j) := |\{F \in \mathscr{F} : \{x_i,x_j\} \in F\}|$ denotes the number of sets in $\mathscr{F}$ that contain $\{x_i,x_j\}$. We will show that $M^T M = \lambda J + (k-\lambda)I$; hence (3.1.1) implies that $\mathscr{F}$ is k-regular and that any pair of points $\{x,y\} \in \binom{X}{2}$ is contained in exactly λ sets in $\mathscr{F}$. Note that $MM^T_{ij} = |F_i \cap F_j|$ so $MM^T = \lambda J + (k-\lambda)I$ since $\mathscr{F} \subset \binom{X}{k}$ is λ-intersecting. As in the proof of Theorem 2.3.9, we conclude that MM^T is positive definite so M^{-1} exists. We thus have that

$$M^T = M^{-1}(\lambda J + (k-\lambda)I) \;\Rightarrow\; M^T M = \lambda M^{-1} J M + (k-\lambda)I. \tag{3.1.2}$$

Since $\mathscr{F}$ is k-uniform, we have $MJ = kJ$, which implies that $M^{-1}J = k^{-1}J$. Also note that $JM_{ij} = \deg(x_j)$. Hence, by (3.1.2), we have

$$M^T M_{ij} := \begin{cases} \lambda k^{-1} \deg(x_i) + (k-\lambda) & \text{if} \quad i = j \\ \lambda k^{-1} \deg(x_j) & \text{if} \quad i \neq j. \end{cases} \tag{3.1.3}$$

Since $M^T M$ is symmetric, (3.1.3) implies that $\deg(x_1) = \deg(x_j)$ for all $2 \leq j \leq n$. We also have $nk = \sum_{x \in X} \deg(x) = n \deg(x_1)$ since the sum of the rows of M equals the sum of the columns of M; hence $\mathscr{F}$ is k-regular. By (3.1.1), we see

$$\operatorname{codeg}(x_i,x_j) = M^T M_{ij} = \lambda k^{-1} \deg(x_j) = \lambda.$$

Hence, any pair of points $\{x,y\} \in \binom{X}{2}$ is contained in exactly λ sets in $\mathscr{F}$. □

Recall that projective planes are symmetric designs for $\lambda = 1$. By Theorem 3.1.4, every pair of points in a projective plane $\mathscr{F}$ is contained in exactly one set so

$$\binom{n}{2} = n\binom{k}{2} \;\;\Rightarrow\;\; n = (k-1)^2 + (k-1) + 1.$$

This observation motivates the definition of the order of a projective plane.

Definition 3.1.5. *If $\mathscr{F} \subset \binom{X}{k}$ is a projective plane, then its* <u>order</u> *is $k-1$.*

The projective plane of order 1 is the triangle $\binom{[3]}{2}$ and the projective plane of order 2 is the Fano plane, Figure 2.1. We now show that if $q \in \mathbb{Z}^+$ is a prime power, then a projective plane of order q exists.

Lemma 3.1.6. *If $q \in \mathbb{Z}^+$ is a prime power, then a projective plane of order q exists.*

Proof. Since q is a prime power, there exists a finite field $\mathbb{F}_q$ of order q. Let $V = \mathbb{F}_q^3$ be a three-dimensional vector space over $\mathbb{F}_q$. Let $\left[{V \atop i}\right] := \{S \subset V : \dim(S) = i\}$ be the family of i-dimensional subspaces of V for $i \in [3]$. Let $X = \left[{V \atop 1}\right]$ and let $\mathscr{F} := \left[{V \atop 2}\right]$. We have

$$|X| = \frac{q^3-1}{q-1} = q^2+q+1 = \frac{(q^3-1)(q^3-q)}{(q^2-1)(q^2-q)} = |\mathscr{F}|.$$

If $F \in \mathscr{F}$ then $|F| = (q^2-1)/(q-1) = q+1$. We have $\dim(F_1 \cap F_2) = 1$ for distinct $F_1, F_2 \in \mathscr{F}$ since $\dim V = 3$ and $\dim F_1 = \dim F_2 = 2$. Consequently, $\mathscr{F} \subset \binom{X}{q+1}$ is a 1-intersecting family with $|\mathscr{F}| = |X|$, so $\mathscr{F}$ is a projective plane of order q. $\square$

By Lemma 3.1.6, there is a projective plane of order $q \in \{2,3,4,5,7,8,9\}$. Is there a projective plane of order 6 or of order 10? The celebrated Bruck-Ryser-Chowla theorem [23, 28] answers the first question in the negative, and is the definitive tool in proving the nonexistence of symmetric designs. Note that the existence of a projective plane of order 10 is not ruled out by the Bruck-Ryser-Chowla theorem, but has been ruled out by a massive computer search [81].

Theorem 3.1.7 (Bruck-Ryser-Chowla)**.** *Let $\mathscr{F} \subset \binom{X}{k}$ be a symmetric design.*

(i) If $|\mathscr{F}|$ is even, then $k-\lambda$ is a square.

(ii) If $|\mathscr{F}|$ is odd, then the equation $z^2 = (k-\lambda)x^2 + (-1)^{(|\mathscr{F}|-1)/2}\lambda y^2$ has a solution in integers x, y, z, not all zero.

Lemma 3.1.6 proves there are infinitely many symmetric designs for $\lambda = 1$. We do not know if the same is true for $\lambda > 1$, and a folklore conjecture asserts that the answer is no.

Conjecture 3.1.8 (Folklore)**.** *There are finitely many symmetric designs for fixed* $\lambda > 1$.

3.1.3 The λ-design Conjecture

Ryser [99] and Woodall [117] have shown that the points in a λ-design have only two degrees, say r_1 and r_2.

Theorem 3.1.9 (Ryser-Woodall)**.** *If* $\mathscr{F} \subset 2^X$ *is a* λ*-design, then* $\deg(x) \in \{r_1, r_2\}$ *and* $r_1 + r_2 = n + 1$.

Woodall [117] has shown that for each $\lambda > 1$, there are only finitely many λ designs, which is interesting because the corresponding problem for symmetric designs, Conjecture 3.1.8, is not yet solved.

Theorem 3.1.10 (Woodall)**.** *If* $\lambda > 1$*, there are finitely many* λ*-designs.*

The only class of nontrivial λ-designs known are the point-complemented block designs.

Definition 3.1.11. *Let* $\mathscr{F} \subset \binom{X}{k}$ *be a symmetric design. A* λ*-intersecting family* $\mathscr{G} \subset 2^X$ *is a* <u>*point-complemented block design*</u> *if there exists* $F' \in \mathscr{F}$ *such that*

$$\mathscr{G} := \{F'\} \cup \{F \triangle F' : F' \neq F \in \mathscr{F}\}.$$

In the de Bruijn-Erdős theorem, Theorem 2.3.7, the 1-designs of type (ii) are point complements of the symmetric designs $\binom{X}{n-1}$. Ryser [99] has shown that the unique 2-design $\hat{\mathscr{F}}$ is the point complement of the Fano plane,

$$\hat{\mathscr{F}} := \{\{1,2,4\},\{1,4,6,7\},\{1,2,5,7\},\{1,2,3,6\},\{1,3,4,5\},\{2,3,4,7\},\{2,4,5,6\}\}.$$

The λ-design conjecture [99, 117] asserts that all nontrivial λ-designs are of this type.

Conjecture 3.1.12 (Ryser-Woodall)**.** *All nontrivial* λ*-designs are point-complemented block designs.*

The de Bruijn-Erdős theorem, Theorem 2.3.7, proves Conjecture 3.1.12 in the case $\lambda = 1$. Conjecture 3.1.12 has been verified for all $\lambda \leq 34$ [19, 20, 21, 79, 99, 101, 108, 114], and is also true for a few infinite families of λ [102, 103].

3.2 L-intersecting Families

We now discuss a generalization of Fisher's Inequality to L-intersecting families, whose proof illustrates the polynomial method. This result and its many variants have powerful combinatorial and geometric consequences; they have been used, for example, in the counterexample to Borsuk's conjecture from topology [74].

Definition 3.2.1. *Given a finite set $L \subset \mathbb{N}$ of nonnegative integers, we say a family $\mathscr{F} \subset 2^X$ is L-intersecting if for all distinct $F_1, F_2 \in \mathscr{F}$, we have $|F_1 \cap F_2| \in L$.*

A celebrated theorem of Frankl and Wilson [59] bounds the size of L-intersecting families as a function of $|L|$.

Theorem 3.2.2 (Frankl-Wilson)**.** *Suppose $L \subset \mathbb{N}$ has size $|L| = s$. If $\mathscr{F} \subset 2^X$ is an L-intersecting family, then*

$$|\mathscr{F}| \leq \sum_{i=0}^{s} \binom{n}{i}. \tag{3.2.1}$$

This result is best possible in terms of the parameters n and s, as demonstrated by taking $L = \{0, \ldots, s-1\}$ and $\mathscr{F}$ to be the family of all subsets of X of size at most s. However, it is possible to get sharper bounds by specifying L. For example, if $\mathscr{F} \subset 2^X$ is λ-intersecting, where $\lambda \in \mathbb{Z}^+$, then Fisher's Inequality yields that $|\mathscr{F}| \leq n$, whereas the *statement* of Theorem 3.2.2 gives the weaker bound $|\mathscr{F}| \leq n+1$ since $|L| = |\{\lambda\}| = 1$. We shall see, however, that the *proof* of Theorem 3.2.2 can be modified in this case to yield that $|\mathscr{F}| \leq n$. In general, though, it is an open problem to even determine the order of magnitude of the largest L-intersecting family for a specific set L.

3.2.1 Polynomial Spaces

Given a field $\mathbb{F}$, the *polynomial ring* $\mathbb{F}[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$ is the set of all finite sums of *monomial terms*, which are elements of the form

$$a x_1^{d_1} \cdots x_n^{d_n}, \tag{3.2.2}$$

where $a \in \mathbb{F}$, and $d_i \in \mathbb{N}$. A monomial is called *monic* if $a = 1$. In (3.2.2), the exponent d_i is called the *degree in* x_i of the term and the sum $d = d_1 + \cdots + d_n$ is called the *degree* of the term. A *polynomial* is a finite sum of nonzero monomial terms, and its degree is the largest degree of any of its monomial terms. A polynomial is *homogeneous of degree* k if each of its monomial terms has degree k. The monomial (3.2.2) is called *multilinear* if each $d_i \in \{0,1\}$, and a *multilinear polynomial* is a finite sum of nonzero multilinear monomials. Basic properties of the polynomial ring $\mathbb{F}[x_1,\ldots,x_n]$ are covered in standard abstract algebra texts such as [41] and will be assumed.

We will now show that certain subsets of $\mathbb{F}[x_1,\ldots,x_n]$ form finite-dimensional vector spaces over $\mathbb{F}$ and compute their dimension.

Lemma 3.2.3. *The set T of multilinear homogeneous polynomials of degree k with coefficients in $\mathbb{F}$ forms a vector space over $\mathbb{F}$ with dimension* $\dim T = \binom{n}{k}$.

Proof. We can more concretely write T as

$$T = \{f \in \mathbb{F}[x_1,\ldots,x_n] : f \text{ is multilinear and homogeneous of degree } k\} \cup \{0\}.$$

Clearly, T forms a vector space over $\mathbb{F}$ and has basis $\{x_{i_1} \ldots x_{i_k} : i_1 < \cdots < i_k\}$. Hence, $\dim T = \binom{n}{k}$. □

A similar argument yields that the space W of multilinear polynomials of degree at most s and with coefficients in $\mathbb{F}$ has dimension $\dim W = \sum_{i=0}^{s} \binom{n}{i}$.

Lemma 3.2.4. *The set $W := \{f \in \mathbb{F}[x_1,\ldots,x_n] : f$ is multilinear of degree at most $s\}$ forms a vector space over $\mathbb{F}$ with dimension*

$$\dim W = \sum_{i=0}^{s} \binom{n}{i}.$$

Given a finite set Ω, it is easy to see that the set $\mathbb{F}^{\Omega} := \{f : \Omega \to \mathbb{F}\}$ forms a vector space over $\mathbb{F}$ called the *function space*. The Triangular Criterion, Lemma 3.2.5, gives a sufficient condition for showing that a set of functions in $\mathbb{F}^{\Omega}$ is linearly independent.

Lemma 3.2.5. *For $i \in [m]$, let $f_i : \Omega \to \mathbb{F}$ be functions and $a_i \in \Omega$ be elements such that*

$$f_i(a_j) \begin{cases} \neq 0 & \text{if} \quad i = j \\ = 0 & \text{if} \quad j < i. \end{cases} \tag{3.2.3}$$

Then $f_1, \ldots, f_m$ are linearly independent members of the function space $\mathbb{F}^{\Omega}$.

Proof. Suppose, for a contradiction, that there exist constants $c_i \in \mathbb{F}$ not all zero such that $\sum_{i=1}^{m} c_i f_i = 0$. Let i' be the smallest i such that $c_i \neq 0$. By (3.2.3),

$$0 = \sum_{i=1}^{m} c_i f_i(a_{i'}) = c_{i'} f_{i'}(a_{i'}),$$

which implies that $c_{i'} = 0$, a contradiction. Hence, $f_1, \ldots, f_m$ are linearly independent members of $\mathbb{F}^{\Omega}$. □

The last lemma we need before commencing with the proof of Theorem 3.2.2 shows that for any polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most s, there exists a unique multilinear polynomial $\hat{f} \in \mathbb{F}[x_1, \ldots, x_n]$, which has the same values as f on the set $\{0,1\}^n$.

Lemma 3.2.6. *For any polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most s, there exists a unique multilinear polynomial $\hat{f} \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most s such that $f(x) = \hat{f}(x)$ for all $x \in \{0,1\}^n$.*

Proof. Expand f and use the identity $x_i^2 = x_i$, which is valid over $\{0,1\}^n$. □

3.2.2 Proof of the Frankl-Wilson Theorem

We present Babai's [4] elegant proof of Theorem 3.2.2. We first define the notion of the characteristic vector of a set.

Definition 3.2.7. *The characteristic vector of a set $F \subset X$, denoted $v_F \in \{0,1\}^n$, is defined by*

$$(v_F)_i := \begin{cases} 1 & \text{if} \quad i \in F \\ 0 & \text{if} \quad i \notin F. \end{cases}$$

Proof of Theorem 3.2.2 Let $\mathscr{F} = \{F_1, \ldots, F_m\}$, where $|F_1| \leq \cdots \leq |F_m|$, and suppose $L = \{l_1, \ldots, l_s\}$. With each set $F_i \in \mathscr{F}$, associate its characteristic vector $v_i \in \mathbb{R}^n$. Note that $v_i \cdot v_j = |F_i \cap F_j|$. For $i \in [m]$, define the polynomial $f_i \in \mathbb{R}[x_1, \ldots, x_n]$ by

$$f_i(x) := \prod_{k: l_k < |F_i|} (v_i \cdot x - l_k). \tag{3.2.4}$$

Each polynomial f_i has degree at most s because $|L| = s$. Since $\mathscr{F}$ is L-intersecting and $|F_1| \leq \cdots \leq |F_m|$, we have

$$f_i(v_j) = \prod_{k:l_k<|F_i|} (v_i \cdot v_j - l_k) = \prod_{k:l_k<|F_i|} (|F_i \cap F_j| - l_k)$$
$$\begin{cases} \neq 0 & \text{if} \quad i = j \\ = 0 & \text{if} \quad j < i. \end{cases} \tag{3.2.5}$$

We now use Lemma 3.2.6 to replace each f_i by the unique multilinear polynomial $\hat{f}_i$ of degree at most s such that $f_i(x) = \hat{f}_i(x)$ for all $x \in \{0,1\}^n$. Since $v_i \in \{0,1\}^n$, the polynomials $\hat{f}_1, \ldots, \hat{f}_m$ and the elements $v_1, \ldots, v_m \in \{0,1\}^n$ satisfy (3.2.5); that is they satisfy the Triangular Criterion, Lemma 3.2.5. Hence, $\hat{f}_1, \ldots, \hat{f}_m$ are linearly independent members of W, the space of multilinear polynomials of degree at most s and with coefficients in $\mathbb{R}$. By Lemma 3.2.4, we conclude that

$$|\mathscr{F}| = m \leq \sum_{i=0}^{s} \binom{n}{i}. \qquad \square$$

If $\lambda \in \mathbb{Z}^+$ and $\mathscr{F} \subset 2^X$ is λ-intersecting, then Theorem 3.2.2 gives $m \leq n+1$, which is weaker than Fisher's Inequality. However, the proof can easily be modified to give the correct bound. If $\lambda \in \mathbb{Z}^+$, then $\emptyset \notin \mathscr{F}$. Hence, in (3.2.4), $f_i(x) = v_i \cdot x - \lambda$ is a multilinear polynomial of degree 1. By Lemma 3.2.3, we conclude that $m \leq n$.

3.3 Eigenvalues

We give Godsil's and Newman's [66] proof of the Erdős-Ko-Rado theorem via eigenvalues of the Kneser graph, which is closely related to that of Wilson [116]. Variants of this method have yielded Erdős-Ko-Rado analogs for vector spaces [60] and recently, in combination with Fourier analysis, for permutations [46, 64] and graphs [45].

3.3.1 Independent Sets

We begin with basic graph theory terminology, and then prove the ratio bound on independent sets [35].

Definition 3.3.1. *A graph is an ordered pair $G = (V,E)$ comprising a set V of vertices together with a set $E \subset \binom{V}{2}$ of edges.*

Definition 3.3.2. *Let $G = (V,E)$ be a graph. Two vertices $v,w \in V$ are adjacent, denoted $v \sim w$, if $\{v,w\} \in E$.*

Definition 3.3.3. *Let $G = (V,E)$ be a graph. A subset $S \subset V(G)$ is independent if no two vertices in S are adjacent.*

Definition 3.3.4. *If $G = (V,E)$ is a graph, then its associated adjacency matrix A is defined to be the $|V| \times |V|$ matrix with*

$$A_{v,w} := \begin{cases} 1 & \text{if} \quad v \sim w \\ 0 & \text{if} \quad v \not\sim w. \end{cases}$$

Definition 3.3.5. *If $G = (V,E)$ is a graph, then the degree of a vertex $v \in G$, denoted $\deg(v)$, is the number of vertices adjacent to v; that is, $\deg(v) := |\{w : v \sim w\}|$.*

Definition 3.3.6. *A graph $G = (V,E)$ is k-regular if all vertices $v \in V$ have degree $\deg(v) = k$.*

The ratio bound relates the size of the largest independent set in a regular graph to the least eigenvalue of its adjacency matrix. The original proof by Delsarte uses linear programming techniques, but we will follow Godsil's approach [65, Lemma 9.6.2].

Theorem 3.3.7 (Delsarte)**.** *Let $G = (V,E)$ be a k-regular graph with v vertices and suppose the adjacency matrix A of G has least eigenvalue τ. Let $S \subset V$ be an independent set in G with characteristic vector z. Then*

$$|S| \leq \frac{v}{1 + \frac{k}{-\tau}};$$

if equality holds then

$$(A - \tau I)\left(z - \frac{|S|}{v}\mathbf{1}\right) = 0.$$

Proof. Let M be the matrix given by

$$M := A - \tau I - \frac{k-\tau}{v} J.$$

We claim the matrix M is positive semidefinite. It suffices to show that the eigenvalues of M are nonnegative. First observe that, since G is k-regular, the all ones vector, $\mathbf{1}$, is an eigenvector of M corresponding to the zero eigenvalue. Let $\lambda \neq 0$ be an eigenvalue of M distinct from zero. Since M is symmetric, any eigenvector $f_\lambda \in \mathbb{R}^v$ corresponding to λ must be orthogonal to $\mathbf{1}$. Hence, $Jf_\lambda = 0$ so f_λ is an eigenvector of $A - \tau I$. As τ is the least eigenvalue of A, we see that $A - \tau I$ has nonnegative eigenvalues so $\lambda \geq 0$. This proves that M is positive semidefinite.

Let $S \subset V$ be an independent set in G with characteristic vector z. As M is positive semidefinite, we see

$$0 \leq z^T Mz = z^T Az - \tau z^T z - \frac{k-\tau}{v} z^T Jz = z^T Az - \tau|S| - \frac{k-\tau}{v}|S|^2.$$

Since S is independent, we have $z^T Az = 0$, and hence

$$0 \leq -\tau|S| - \frac{k-\tau}{v}|S|^2 \;\; \Rightarrow \;\; |S| \leq \frac{v}{1+\frac{k}{-\tau}}. \tag{3.3.1}$$

This yields the bound of the theorem.

If equality holds in (3.3.1), then $z^T Mz = 0$. Since M is positive semidefinite, this implies $Mz = 0$, and accordingly

$$(A - \tau I)z = \frac{k-\tau}{v} Jz = \frac{k-\tau}{v}|S|\mathbf{1}.$$

We also have $(A - \tau I)\mathbf{1} = (k-\tau)\mathbf{1}$, which yields the second claim. □

3.3.2 Erdős-Ko-Rado via Eigenvalues

Godsil's and Newman's proof [66] of the Erdős-Ko-Rado theorem applies the ratio bound, Theorem 3.3.7, to the Kneser graph.

Definition 3.3.8. *The Kneser graph, denoted $K_{n:k}$, has vertex set $V = \binom{X}{k}$ and edge set $E = \{\{A,B\} : A,B \in \binom{X}{k},\, A \cap B = \emptyset\}$.*

Observe that independent sets in the Kneser graph $K_{n:k}$ are in bijective correspondence with intersecting families $\mathscr{F} \subset \binom{X}{k}$. Consequently, we can apply the ratio bound, Theorem 3.3.7, to the Kneser graph to prove the Erdős-Ko-Rado theorem. Fortunately, the eigenvalues and their corresponding multiplicities have been computed for the Kneser graph; a derivation can be found in [65, Section 9.4].

Theorem 3.3.9. *Let $n \geq 2k$ and let A denote the adjacency matrix of the Kneser graph $K_{n:k}$. The eigenvalues λ_i of A and their corresponding multiplicities m_i are*

$$\lambda_i = (-1)^i \binom{n-k-i}{k-i}, \quad m_i = \binom{n}{i} - \binom{n}{i-1}, \quad i = 0, \ldots, k.$$

We now give Godsil's and Newman's proof [66] of the Erdős-Ko-Rado theorem; we will prove only the bound though it is possible to use these methods to characterize the case of equality.

Proof of Theorem 2.2.2. Suppose $n \geq 2k$. Let $\mathscr{F} \subset \binom{X}{k}$ be a maximum intersecting family, and let $S \subset V$ be the independent set in the Kneser graph $K_{n:k}$ corresponding to $\mathscr{F}$. By Theorem 3.3.9, the least eigenvalue of the Kneser graph $K_{n:k}$ is

$$\tau = \lambda_1 = -\binom{n-k-1}{k-1}.$$

Observe that the Kneser graph $K_{n:k}$ is $\binom{n-k}{k}$-regular since given any k-set, there are $\binom{n-k}{k}$ k-sets disjoint from it. Hence, the ratio bound, Theorem 3.3.7, yields

$$|\mathscr{F}| = |S| \leq \frac{\binom{n}{k}}{1 + \frac{\binom{n-k}{k}}{\binom{n-k-1}{k-1}}} = \binom{n-1}{k-1},$$

which establishes the bound in Theorem 2.2.2. Using the second claim of Theorem 3.3.7 and the multiplicity $m_1 = n-1$ of the least eigenvalue λ_1, it is possible to characterize the case of equality when $n > 2k$; see [94, Section 5.4] for details. □

3.4 Linear Programming

We now discuss linear programming and some of its applications to extremal set theory. In Section 3.4.1, we state and prove the weak versions of the Duality and Complementary Slackness theorems; these results are used in Chapter 6 and provide motivation for the strong versions of these theorems [61, 112], which we will state but not prove. In Section 3.4.3, we state the Hoffman-Kruskal theorem [71] on totally unimodular matrices, and use it to prove the integrality theorem on flows [32] in Section 3.4.4. Finally, we use the flow result in Section 3.4.5 to prove Baranyai's theorem [10] on decompositions of the complete hypergraph into perfect matchings. We will need Baranyai's theorem in Chapter 5 and Chapter 7.

3.4.1 Duality

We will prove the weak versions of the Duality and Complementary Slackness theorems for the equality form, which we will use in Chapter 6. The strong versions of these theorems are much more difficult to prove, so we will only state them as they will not be needed for any of the author's results in this thesis.

Consider the problem

$$\begin{aligned} &\text{Minimize} && \sum_{j=1}^{n} c_j x_j && (3.4.1)\\ &\text{subject to:} && \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in [m] \\ & && x_j \geq 0 \quad j \in [n], \end{aligned}$$

where all coefficients and variables take real values. Let us fix some terminology.

Definition 3.4.1. *In* (3.4.1)*, the function to be optimized is called the* *objective function.*

Definition 3.4.2. *In* (3.4.1)*, the inequalities and equations to be satisfied are called the* *constraints.*

Definition 3.4.3. *A* *feasible solution* $x \in \mathbb{R}^n$ *is a point that satisfies all the constraints.*

Definition 3.4.4. *The* *feasible region* *is the set of all feasible solutions.*

Definition 3.4.5. *An* *optimal solution* *of the minimization problem* (3.4.1) *is a feasible solution* $(\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n$ *such that* $\sum_{j=1}^{n} c_j \hat{x}_j \leq \sum_{j=1}^{n} c_j x_j$ *for any feasible solution* $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

We seek a simple way to determine whether a feasible point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of the minimization problem (3.4.1) is optimal. This suggests the idea of finding lower bounds for the minimum. Observe that if we take a real multiple y_i of each equality $a_{ij} x_j = b_i$ in (3.4.1) and add these equalities so that the resulting equality $\sum_{j=1}^{n} d_j x_j = d_0$ satisfies $d_j \leq c_j$, then d_0 is a lower bound for $\sum_{j=1}^{n} c_j x_j$. The best lower bound obtainable in this way is given by

$$\begin{aligned} &\text{Maximize} && \sum_{i=1}^{m} b_i y_i && (3.4.2)\\ &\text{subject to:} && \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad j \in [n]. \end{aligned}$$

We call (3.4.2), the *dual* of (3.4.1) and refer to (3.4.1) as the *primal problem.* It is a consequence of the way we constructed the dual that every feasible solution $y \in \mathbb{R}^m$ of the dual (3.4.2) gives a lower bound for the objective value of (3.4.1). This is the content of the weak duality theorem, which we now formally state and prove.

Theorem 3.4.6 (Weak Duality)**.** *If $x \in \mathbb{R}^n$ is a feasible solution of the primal problem* (3.4.1) *and $y \in \mathbb{R}^m$ is a feasible solution of the dual problem* (3.4.2)*, then*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i.$$

Proof. The constraints in (3.4.1) and (3.4.2) yield that

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) = \sum_{i=1}^{m} b_i y_i, \tag{3.4.3}$$

which proves the theorem. □

The weak duality theorem has several useful consequences.

Corollary 3.4.7. *If $\hat{x} \in \mathbb{R}^n$ is a feasible solution of the primal problem* (3.4.1)*, $\hat{y} \in \mathbb{R}^m$ is a feasible solution of the dual problem* (3.4.2)*, and $\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i$, then $\hat{x}$ and $\hat{y}$ are optimal solutions of the primal and dual problems respectively.*

Proof. Since $\sum_{j=1}^{n} c_j x_j$ cannot be smaller than $\sum_{i=1}^{m} b_i \hat{y}_i$ for any feasible $x \in \mathbb{R}^n$, and since $\hat{x}$ achieves this bound, $\hat{x}$ is optimal. The same argument can be used for $\hat{y}$. □

By rewriting (3.4.3), we obtain the weak complementary slackness theorem.

Theorem 3.4.8 (Weak Complementary Slackness)**.** *If $\hat{x} \in \mathbb{R}^n$ is a feasible solution of the primal problem* (3.4.1)*, $\hat{y} \in \mathbb{R}^m$ is a feasible solution of the dual problem* (3.4.2)*, and $\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i$, then for each $j \in [n]$,*

$$\sum_{i=1}^{m} a_{ij} \hat{y}_i = c_j \;\; or \;\; \hat{x}_j = 0. \tag{3.4.4}$$

Proof. If $\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i$, then equality holds everywhere in (3.4.3). In particular, the first equality is equivalent to

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_j - a_{ij} \hat{y}_i \right) \hat{x}_j = 0. \tag{3.4.5}$$

Since all terms in this sum are nonnegative, we must have (3.4.4) holds. □

The weak duality theorem, Theorem 3.4.6, gives a sufficient condition for a feasible point $\hat{x} \in \mathbb{R}^n$ of (3.4.1) to be optimal, namely the existence of $\hat{y} \in \mathbb{R}^m$ satisfying (3.4.3) with equality. It is much less obvious that this condition is also necessary, which is the content of the strong duality theorem.

Theorem 3.4.9 (Strong Duality)**.** *If the primal problem* (3.4.1) *has an optimal solution* $\hat{x} \in \mathbb{R}^n$*, then the dual problem* (3.4.2) *has an optimal solution* $\hat{y} \in \mathbb{R}^m$ *and* $\hat{x}, \hat{y}$ *satisfy the relation* $\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i$.

The strong duality theorem implies the strong complementary slackness theorem.

Theorem 3.4.10 (Strong Complementary Slackness)**.** *Feasible solutions* $\hat{x} \in \mathbb{R}^n$ *and* $\hat{y} \in \mathbb{R}^m$ *of the primal and dual problems respectively are optimal if and only if* $\hat{x}_j = 0$ *or* $\sum_{i=1}^{m} a_{ij} \hat{y}_i = c_j$ *for* $j \in [n]$.

All results in this section are due to von Neumann [112] and Gale, Kuhn, and Tucker [61]. The proofs of Theorem 3.4.9 and Theorem 3.4.10 can be found in any good linear programming textbook such as [30, Section 2.5].

3.4.2 The Fundamental Theorem of Linear Programming

We stated the duality and complementary slackness theorems in the equality form because these are the results we need in Chapter 6. More generally, a linear program has the form

$$\begin{aligned} \text{Maximize} \quad & c^T x \\ \text{subject to:} \quad & Ax \leq b, \end{aligned} \tag{3.4.6}$$

where $c, x \in \mathbb{R}^n$, A is an $n \times m$ matrix with real entries, and $b \in \mathbb{R}^m$. Note that (3.4.1) can be put into the form in (3.4.6) in the following way,

$$\begin{aligned} \text{Maximize} \quad & -c^T x \\ \text{subject to:} \quad & \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}, \end{aligned}$$

where $c, x \in \mathbb{R}^n$, A is an $n \times m$ matrix with real entries $A_{ij} = a_{ij}$, I is the $m \times m$ identity matrix, and $b \in \mathbb{R}^m$.

The Fundamental Theorem of Linear Programming asserts that any linear program is exactly one of three types, and is proved in [30, Theorem 2.12]. We again fix some terminology.

Definition 3.4.11. *The linear program* (3.4.6) *is* <u>*infeasible*</u> *if there does not exist* $x \in \mathbb{R}^n$ *such that* $Ax \leq b$*.*

Definition 3.4.12. *The linear program* (3.4.6) *is* <u>*unbounded*</u> *if for each* $r \in \mathbb{R}$*, there exists feasible* $x \in \mathbb{R}^n$ *such that* $c^T x \geq r$*.*

Theorem 3.4.13 (Fundamental Theorem of Linear Programming)**.** *The linear program* (3.4.6) *is infeasible, unbounded, or has an optimal solution.*

3.4.3 Integer Programming

In practical applications, we would often like to know if a linear program has an optimal solution $\hat{x} \in \mathbb{Z}^n$ all of whose coordinates are integers. We describe a method to find such an optimal solution. We begin with some terminology.

Definition 3.4.14. *A* <u>*polyhedron*</u> *is the feasible region of the linear program* (3.4.6).

Definition 3.4.15. *A region* $C \subset \mathbb{R}^n$ *is* <u>*convex*</u> *if whenever* $x, y \in C$ *and* $\lambda \in [0,1]$*, we also have* $\lambda x + (1-\lambda)y \in C$*. In other words,* C *is convex if for any two points* $x, y \in C$*, the line segment joining* x *and* y *is also contained in* C*.*

Definition 3.4.16. *An* <u>*extreme point*</u> *of a convex set* $C \subset \mathbb{R}^n$ *is a point* $p \in C$ *such that there do not exist distinct points* $q, r \in C$ *and* $\lambda \in (0,1)$ *such that* $p = \lambda q + (1-\lambda)r$*. In other words, an extreme point of* C *is a point of* C *that is not in the interior of any line segment contained in* C*.*

Definition 3.4.17. *A set* $L \subset \mathbb{R}^n$ *is a* <u>*line*</u> *if there exist distinct* $x_1, x_2 \in \mathbb{R}^n$ *such that* $L = \{\lambda x_1 + (1-\lambda)x_2 : \lambda \in \mathbb{R}\}$*.*

Polyhedrons are convex, and the following result illustrates why their extreme points are important in linear programming; a proof is in [30, Theorem 2.27].

Theorem 3.4.18. *If a linear program* (3.4.6) *has an optimal solution and its feasible region contains no line, then it has an optimal solution that is an extreme point of its feasible region.*

Combined with Theorem 3.4.18, the Hoffman-Kruskal theorem [71] yields a way of finding an integral optimal solution.

Definition 3.4.19. *A matrix M with real entries M_{ij} is totally unimodular if every square submatrix N of M has determinant* 0, 1, *or* -1.

Theorem 3.4.20 (Hoffman-Kruskal)**.** *If A is a totally unimodular $n \times m$ matrix and $b \in \mathbb{Z}^m$, then every extreme point of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ determined by* (3.4.6) *has integral coordinates.*

Suppose we know that the $n \times m$ matrix A is totally unimodular, $b \in \mathbb{Z}^m$, and that the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ determined by (3.4.6) has an optimal solution and does not contain a line. The Hoffman-Kruskal theorem asserts that all extreme points of the polyhedron P have integral coordinates, and Theorem 3.4.18 proves that one of those integral extreme points must be an optimal solution to the linear program (3.4.6).

3.4.4 Flows

We define the notion of a transportation network, and prove the integrality theorem on flows [32] using Hoffman's and Kruskal's Theorem 3.4.20.

Definition 3.4.21. *A digraph is an ordered pair $D = (V, A)$ comprising a set V of vertices together with a set $A \subset V \times V$ of ordered pairs of vertices called arcs.*

Definition 3.4.22. *If $a = (v, w)$ is an arc in the digraph $D = (V, A)$, then a is said to be directed from v to w. We call v the tail of a and w the head.*

Definition 3.4.23. *A transportation network is a finite digraph $D = (V, A)$ together with two distinguished vertices called the source s and the sink t, and a capacity function $k : A \to \mathbb{R}_{\geq 0}$ which associates a nonnegative real number $k(a)$ to each arc $a \in A$. The source s must be the tail of all arcs which contain it and the sink t must be the head of all arcs which contain it. We further assume that A does not contain any arcs of the form $a = (v, v)$ for a vertex $v \in V$.*

Definition 3.4.24. *Given a transportation network $D = (V,A)$, a* <u>*flow*</u> *in D is a function $f : A \to \mathbb{R}_{\geq 0}$ which assigns to each arc $a \in A$ a nonnegative real number $f(a)$ such that*

(i) $0 \leq f(a) \leq k(a)$ *for all arcs $a \in A$ and*

(ii) *for each vertex $v \in V \setminus \{s,t\}$ we have*

$$\sum_{a \in A: v \text{ is head of } a} f(a) = \sum_{a \in A: v \text{ is tail of } a} f(a).$$

Definition 3.4.25. *If $D = (V,A)$ is a transportation network and $f : A \to \mathbb{R}_{\geq 0}$ is a flow on D then the* <u>*strength*</u> *of the flow, denoted $|f|$, is defined to be the sum of the values of f on the arcs leaving s,*

$$|f| := \sum_{a \in A: s \in a} f(a).$$

We will be interested in determining the maximum strength of a flow on a transportation network, and we will show how to formulate this problem as a linear program. In the following diagram, we illustrate an example of a transportation network with a maximum flow f of strength $|f| = 5$.

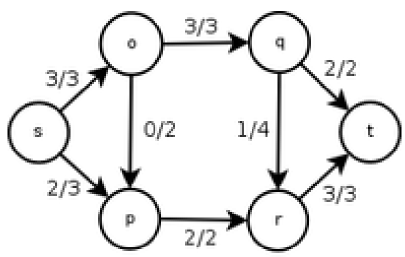

Figure 3.1: A transportation network with flow and capacity denoted f/c.

To formulate the maximum flow problem as a linear program, we need to define the concept of an incidence matrix of a digraph.

Definition 3.4.26. *If $D = (V,A)$ is a digraph, then its* <u>*incidence matrix*</u> *M is defined to be the $|V| \times |A|$ matrix with*

$$M_{v,a} := \begin{cases} -1 & \text{if} \quad v \text{ is tail of } a \\ 1 & \text{if} \quad v \text{ is head of } a \\ 0 & \text{if} \quad v \notin a. \end{cases}$$

We show that the incidence matrix of a digraph is totally unimodular.

Lemma 3.4.27. *The incidence matrix M of a digraph D is totally unimodular.*

Proof. We must show that every square submatrix N of M satisfies $\det(N) \in \{-1,0,1\}$. We prove this by induction on the order l of N. Certainly, the statement is true for $l = 1$ as the entries of M lie in $\{-1,0,1\}$. Suppose $l \geq 2$ and all square submatrices of order less than l have determinant 0 or ± 1. Observe that every column of M has exactly one $+1$ entry and one -1 entry. Now, if N has a zero column, then $\det(N) = 0$. If N has a column with exactly one ± 1 entry, then we can expand its determinant on that column, so $\det(N) = \pm\det(N')$, where N' has order $l-1$. By the induction hypothesis, we consequently have $\det(N) = \pm\det(N') \in \{-1,0,1\}$. Finally, if every column of N has exactly one $+1$ entry and one -1 entry, then its rows sum to zero, so $\det(N) = 0$. This proves that M is totally unimodular. □

Finally, we formulate the maximum flow problem as a linear program.

Lemma 3.4.28 (Maximum Flow Problem)**.** *Let $D = (V,A)$ be a transportation network. Let M denote the incidence matrix of D and let $c^T \in \mathbb{R}^{|A|}$ denote the row in M corresponding to s. Deleting the rows in M corresponding to the source s and the sink t yields a $(|V|-2) \times |A|$ matrix, which we denote $\overline{M}$. Let $k \in \mathbb{R}^{|A|}$ denote the capacity vector with entries $k_a = k(a)$. Finally, let I_w denote the $w \times w$ identity matrix and let $\vec{0}_w \in \mathbb{R}^w$ denote the all zero vector. The maximum strength of a flow on D is the optimal value of the following linear program,*

$$\begin{aligned} &\textit{Maximize} && -c^T x \\ &\textit{subject to:} && \begin{bmatrix} \overline{M} \\ -\overline{M} \\ I_{|A|} \\ -I_{|A|} \end{bmatrix} x \leq \begin{bmatrix} \vec{0}_{|V|-2} \\ \vec{0}_{|V|-2} \\ k \\ \vec{0}_{|A|} \end{bmatrix}. \end{aligned} \tag{3.4.7}$$

We now state and prove the integrality theorem on flows [32].

Theorem 3.4.29 (Dantzig)**.** *If $D = (V,A)$ is a transportation network with an integral capacity vector $k \in \mathbb{Z}^{|A|}$, then there is a maximum strength flow f on D such that for each $a \in A$, the value $f(a) \in \mathbb{Z}$ is integral.*

Proof. By Lemma 3.4.27, the incidence matrix M of D is totally unimodular. Hence, $\overline{M}$, the matrix formed by removing the rows corresponding to the source s and the sink t, is totally unimodular. A similar argument to the one in Lemma 3.4.27 proves that if a matrix L is totally unimodular, then so are the matrices L^T, $[L|-L]$, $[L|I]$, and $[L|-I]$. Hence, the matrix on the left hand side of (3.4.7) is totally unimodular. As $k \in \mathbb{Z}^{|A|}$ is integral, the vector on the right hand side of (3.4.7) is integral. The Hoffman-Kruskal theorem, Theorem 3.4.20, yields that the extreme points of the polyhedron defined by (3.4.7) all have integral coordinates.

Observe that the linear program in (3.4.7) is not infeasible or unbounded because $\vec{0}_{|A|}$ is a feasible solution and the value of the objective function is bounded by $-c^T k$. Hence, (3.4.7) has an optimal solution by the Fundamental Theorem of Linear Programming, Theorem 3.4.13. Moreover, observe that the polyhedron determined by (3.4.7) does not contain a line since for each $a \in A$, we have $0 \leq x_a \leq k(a)$. Hence, by Theorem 3.4.18, one of the integral extreme points must be an optimal solution of the linear program (3.4.7). Consequently, there exists a maximum strength flow f on D such that for each $a \in A$, the value $f(a) \in \mathbb{Z}$ is integral by Lemma 3.4.28. □

For more information on flows, see [110, Chapter 7]. We will use the integrality theorem on flows, Theorem 3.4.29, to prove Baranyai's theorem.

3.4.5 Baranyai's Theorem

We prove Baranyai's theorem [10] on decompositions of the complete hypergraph into perfect matchings. This result will be needed in Chapter 5 and Chapter 7.

Definition 3.4.30. *A partition (respectively m-partition) of a set X is a family (respectively multiset) $\mathscr{P}$ of subsets of X that satisfies properties (i)-(iii) (respectively (ii)-(iv)):*

(i) $\emptyset \notin \mathscr{P}$,

(ii) $P \cap P' = \emptyset$ *for all distinct* $P, P' \in \mathscr{P}$,

(iii) $\bigcup_{P \in \mathscr{P}} P = X$,

(iv) $|\mathscr{P}| = m$.

Definition 3.4.31. *The complete k-uniform hypergraph on n vertices is the family* $\binom{X}{k}$.

Definition 3.4.32. *If* $k|n$*, then a perfect matching of the complete hypergraph* $\binom{X}{k}$ *is a family* $\mathscr{P} \subset \binom{X}{k}$ *that is an* (n/k)*-partition of* X*.*

Baranyai's theorem asserts that if $k|n$, then the complete hypergraph $\binom{X}{k}$ can be partitioned into perfect matchings. It is straightforward to verify that Baranyai's theorem is true for $k = 2$, but the case $k = 3$ is much more difficult [95]. We now formally state Baranyai's theorem and give a proof due to A.E. Brouwer and A. Schrijver [22]. All known proofs of Baranyai's theorem use some form or consequence of Theorem 3.4.29.

Theorem 3.4.33 (Baranyai)**.** *If* $k|n$*, then the complete hypergraph* $\binom{X}{k}$ *can be partitioned into* $\binom{n-1}{k-1}$ *perfect matchings.*

Proof. We prove a seemingly stronger statement. Let $m := n/k$ and let $M := \binom{n-1}{k-1}$. We assert that for any nonnegative integer l with $l \leq n$, there exists a family $\mathscr{A}_1, \ldots, \mathscr{A}_M$ of m-partitions of $[l]$ such that each subset $S \subset [l]$ appears exactly $\binom{n-l}{k-|S|}$ times among the m partitions $\mathscr{A}_i$. Observe that the case $l = n$ proves the theorem as then

$$\binom{n-l}{k-|S|} = \binom{0}{k-|S|} = \begin{cases} 1 & \text{if } k = |S| \\ 0 & \text{otherwise.} \end{cases}$$

We proceed by induction on l. The base case $l = 0$ is trivially true since each $\mathscr{A}_i$ consists of m copies of the empty set. Assume that, for some $l < n$, a family of m-partitions $\mathscr{A}_1, \ldots, \mathscr{A}_M$ with the desired properties exist. We form a transportation network with source s, sink t, vertices labeled $\mathscr{A}_i$ for $i \in [M]$, and vertices labeled S for each subset $S \subset [l]$. There is an arc with capacity 1 from the source s to each vertex labeled $\mathscr{A}_i$. If $S \in \mathscr{A}_i$, then there is an arc with capacity 1 from the vertex labeled $\mathscr{A}_i$ to the vertex labeled S; if $S = \emptyset$ we put j arcs with capacity 1 from the vertex labeled $\mathscr{A}_i$ to the vertex labeled $\emptyset$ if the empty set occurs j times in $\mathscr{A}_i$. Finally, there is an arc from each vertex labeled S to the sink t with capacity $\binom{n-l-1}{k-|S|-1}$.

We exhibit a flow in this network. Assign a flow value of 1 to the arcs leaving the source s. For an arc from a vertex labeled $\mathscr{A}_i$ to a vertex labeled S, assign the flow value $(k-|S|)/(n-l)$. For an arc from a vertex labeled S to the sink t, assign the flow value $\binom{n-l-1}{k-|S|-1}$. Clearly property (i) of a flow is satisfied so we verify property (ii). The sum of the flow values on arcs leaving a vertex labeled $\mathscr{A}_i$ is

$$\sum_{S\in\mathscr{A}_i} \frac{k-|S|}{n-l} = \frac{1}{n-l}\left(mk - \sum_{S\in\mathscr{A}_i}|S|\right) = \frac{1}{n-l}(mk-l) = 1.$$

The sum of the flow values on the arcs entering a vertex labeled S is

$$\sum_{i:S\in\mathscr{A}_i} \frac{k-|S|}{n-l} = \frac{k-|S|}{n-l}\binom{n-l}{k-|S|} = \binom{n-l-1}{k-|S|-1}.$$

For each arc a leaving the sink s, the flow value equals the capacity $f(a) = k(a)$, so this is a maximum flow with strength M. The same property holds for arcs entering into the sink t. Hence, for any maximum flow on this network, the flow value equals the capacity on any arc leaving the source s or entering the sink t.

Since all arcs have integral capacities, Theorem 3.4.29 yields an *integral*-valued maximum flow $\hat{f}$. As $\hat{f}$ is maximum, for any arc a from s to a vertex labeled $\mathscr{A}_i$, we have $f(a) = k(a) = 1$. As $\hat{f}$ is integral and all arcs leaving the vertex labeled $\mathscr{A}_i$ have capacity 1, we see that $\hat{f}$ sends one unit of flow from the vertex labeled $\mathscr{A}_i$ to exactly one vertex labeled S_i, where $S_i \in \mathscr{A}_i$. All other arcs leaving the vertex labeled $\mathscr{A}_i$ will have zero flow. As $\hat{f}$ is maximum, for any arc a leaving a vertex labeled S to the sink t, we have $f(a) = k(a) = \binom{n-l-1}{k-|S|-1}$. Consequently, we see that for each set S, the number of i such that $S_i = S$ is $\binom{n-l-1}{k-|S|-1}$.

We obtain a family of m-partitions $\mathscr{A}_1', \ldots, \mathscr{A}_M'$ of the set $[l+1]$ by letting $\mathscr{A}_i'$ be obtained from $\mathscr{A}_i$ by replacing the distinguished set S_i by $S_i \cup \{l+1\}$ for $i \in [M]$. We claim that $T \subset [l+1]$ appears exactly $\binom{n-(l+1)}{k-|T|}$ times among $\mathscr{A}_1', \ldots, \mathscr{A}_M'$. This is clear if $T = S \cup \{l+1\}$ as the number of times S is chosen to be S_i is $\binom{n-l-1}{k-|S|-1} = \binom{n-(l+1)}{k-|T|}$. Otherwise, $T \subset [l]$. Since T appears $\binom{n-l}{k-|T|}$ times among $\mathscr{A}_1, \ldots, \mathscr{A}_M$ and the number of times $l+1$ is added to T is $\binom{n-l-1}{k-|T|-1}$, we see that T appears

$$\binom{n-l}{k-|T|} - \binom{n-l-1}{k-|T|-1} = \binom{n-(l+1)}{k-|T|}$$

times among $\mathscr{A}_1', \ldots, \mathscr{A}_M'$. This completes the induction step. □

Chapter 4

Projective Prerequisites

We saw in Chapter 2 and Chapter 3 that intersecting families and shadows are two core concepts in extremal set theory. We also proved and discussed fundamental results about these concepts, such as the Erdős-Ko-Rado and Kruskal-Katona theorems. By defining suitable notions of “intersecting" and “shadow," one can find remarkable analogs of these theorems for other structures such as vector spaces. A tantalizing feature is that, although results from extremal set theory are often expected to be true for vector spaces, not much is known about analogs because standard techniques do not always apply.

We describe what vector space analogs are in Section 4.1.1 and why this area of research is significant. To explain the analogies between sets and vector spaces, we then introduce a generalization of the binomial coefficients called the q-binomial coefficients in Section 4.2. The q-binomial coefficients satisfy identities which generalize familiar ones such as Pascal’s rule; we explore these in Section 4.3 because we will need them for the analog of Lovász’s result, Theorem 2.1.5, in Chapter 5. We prove the Erdős-Ko-Rado theorem for vector spaces and introduce the closely-related q-Kneser graphs in Section 4.4. In Chapter 5, we will extend the Erdős-Ko-Rado theorem for vector spaces by proving an analog of Frankl’s r-wise intersection theorem, Theorem 2.2.7. To do this, we will need the concept of spreads, which are an analog of perfect matchings, so we discuss these in Section 4.5. Finally we end in Section 4.6 by highlighting the difficulties of adapting purely combinatorial techniques to vector spaces and dicuss algebraic methods that have worked in both the set and vector space settings.

4.1 Vector Space Analogs

In this section, we describe what vector spaces are and illustrate questions in this area by using the analog of the Erdős-Ko-Rado theorem as an example. We also discuss the significance of this area of study, and give an overview of which methods from Chapter 2 and Chapter 3 carry over to the vector space setting.

4.1.1 What are Vector Space Analogs?

In extremal set theory, our underlying set is the n-element set X. In vector space analogs, the set X is replaced by an n-dimensional vector space V over a finite field $\mathbb{F}_q$. The general question in extremal set theory concerns the maximum or minimum size of a family of subsets of X, which is usually k-uniform. In vector space analogs, the questions will usually concern families of k-dimensional subspaces of V.

To visualize the situation, consider the Fano plane in Figure 2.1. Here V is a three-dimensional vector space over the finite field $\mathbb{F}_2$. There are seven one-dimensional subspaces represented by points and seven two-dimensional subspaces represented by lines. In the picture, a point lies on a line if the one-dimensional subspace corresponding to the point lies in the two-dimensional subspace corresponding to the line.

We saw in Chapter 2 that the Erdős-Ko-Rado theorem asserts that if X is large enough, then the unique intersecting family $\mathscr{F} \subset \binom{X}{k}$ of maximum size consists of the k-element subsets containing a fixed point. Let us try to formulate a vector space analog of the Erdős-Ko-Rado theorem. On a first try, we might ask what is the maximum size of a family $\mathscr{F}$ of k-dimensional subspaces of V such that any two members of $\mathscr{F}$ have nonempty intersection. However, this doesn't quite make sense since the zero subspace is always contained in the intersection of any two subspaces. We instead stipulate that the *dimension* of the intersection of any two members of $\mathscr{F}$ is nonzero. Now, the vector space analog of Erdős-Ko-Rado asks for the maximum size of a family $\mathscr{F}$ of k-dimensional subspaces of V such that the intersection of any two members of $\mathscr{F}$ has nonzero dimension. The answer is strikingly similar to the original: if the dimension of V is large enough, then the unique intersecting family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ of maximum size consists of all k-dimensional subspaces containing a fixed one-dimensional subspace.

4.1.2 Significance

Remarkably, questions and results about intersecting families and shadows are valid not only for sets but for a wide range of mathematical objects such as vector spaces, permutations, and graphs. The ultimate objective of this research is to have a unified theory that characterizes the structures for which analogs exist and that proves results simultaneously for broad classes of objects. As we will see in the sections that follow, vector space analogs force us to analyze which methods from Chapter 2 and Chapter 3 work in a more general setting.

Vector space analogs bring new questions and techniques to finite geometry since many of its problems can be reformulated in these terms. They also provide applications for the q-analog identities studied by algebraic combinatorialists. Recently, coding theorists such as Vardy are studying vector space analogs because they imply results about projective codes [18, 49, 50]. Since codes are used in communication systems, research in this area may yield practical applications.

4.1.3 Methods

In Chapter 2 and Chapter 3, we gave four proofs of the Erdős-Ko-Rado theorem. Which of these approaches carries over to the vector space setting? In Section 4.6, we will discuss the difficulties that arise when we try to mimic the shifting proof or Daykin's proof from Section 2.2.3 and Section 2.2.4 respectively. We will see that we can mimic Katona's proof from Section 2.2.5 to yield the Erdős-Ko-Rado theorem for vector spaces in the special case that $k|n$; this argument will also be important for the analog of Frankl's r-wise intersection theorem in Chapter 5 and for the Manickam-Miklós-Singhi conjecture in Chapter 7. Finally, we will see in Section 4.4 that the eigenvalue proof from Section 3.3.2 fully carries over to the vector space setting. In general, algebraic techniques have been more successful than purely combinatorial ones for vector space analogs. We will discuss some other algebraic successes in Section 4.6. We remark that, in Chapter 5, we will give another proof of the Erdős-Ko-Rado theorem for vector spaces that is surprising because it is purely combinatorial, but does not require any tedious computations.

4.2 The q-binomial coefficients

Usually, q will denote the order of a finite field, but in this section, we will allow q to be a positive real. We define the q-binomial coefficient, a generalization of the binomial coefficient. When q is the order of the finite field $\mathbb{F}_q$, the q-binomial coefficients play the same role in the enumeration of subspaces of V that the binomial coefficients play in the enumeration of subsets of X.

Definition 4.2.1. *If $a \in \mathbb{R}$, $q \in \mathbb{R}^+$, and $k \in \mathbb{Z}^+$, define the* <u>*Gaussian binomial coefficient*</u> *by*

$$\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \le i < k} \frac{q^{a-i}-1}{q^{k-i}-1}.$$

When $k = 1$, we write $[a]_q := \begin{bmatrix} a \\ 1 \end{bmatrix}_q$.

Definition 4.2.2. *For $a \in \mathbb{N}$ and $q \in \mathbb{R}^+$, define the* <u>*q-torial function*</u> *by $[0]! = 1$ and*

$$[a]_q! := \prod_{j=1}^{a} [j]_q \text{ for } a \in \mathbb{Z}^+.$$

Observe that $\binom{a}{k} = \lim_{q \to 1} \begin{bmatrix} a \\ k \end{bmatrix}_q$ and that, when $q = 1$ and $a \in \mathbb{Z}^+$, we have $[a]_1 = a$ and $[a]_1! = a!$. When $a \in \mathbb{Z}^+$, the Gaussian binomial coefficient takes the familiar form

$$\begin{bmatrix} a \\ k \end{bmatrix}_q = \frac{[a]_q!}{[k]_q![a-k]_q!}.$$

In this chapter, V always denotes an n-dimensional vector space over the finite field $\mathbb{F}_q$. We now introduce the notation $\begin{bmatrix} V \\ k \end{bmatrix}_q$, which is analogous to $\binom{X}{k}$.

Definition 4.2.3. *If $q \in \mathbb{Z}^+$ is the order of the finite field $\mathbb{F}_q$ and $k \in \mathbb{Z}^+$ is a positive integer, let $\begin{bmatrix} V \\ k \end{bmatrix}_q := \{S \subset V : \dim(S) = k\}$ denote the family of all k-dimensional subspaces of V.*

We know that the size of $\binom{X}{k}$ equals the binomial coefficient $\binom{n}{k}$; a simple counting argument shows that we similarly have that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Lemma 4.2.4. *We have $\left| \begin{bmatrix} V \\ k \end{bmatrix} \right|_q = \begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. There are $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ ways to choose k-tuple independent vectors from V. Since a given k-space has $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ distinct ordered bases, the number of k-dimensional subspaces of V is

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \begin{bmatrix} n \\ k \end{bmatrix}_q . \qquad \square$$

4.3 The q-Pascal Rule

The q-binomial coefficients satisfy identities which generalize familiar ones such as Pascal's rule. We discuss the q-Pascal rule, which we will need for our proof of the analog of Lovász's result, Theorem 2.1.5, in Chapter 5. The familiar Pascal's identity asserts that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ for } k \in [n-1]. \tag{4.3.1}$$

Often, naively changing binomial coefficients in an identity to q-binomial coefficients yields a q-identity: for example if $a \in \mathbb{Z}^+$, then

$$\binom{a}{k} = \binom{a}{a-k}, \ \begin{bmatrix} a \\ k \end{bmatrix}_q = \begin{bmatrix} a \\ a-k \end{bmatrix}_q \text{ for } k \in [a]. \tag{4.3.2}$$

In the case of Pascal's identity, however, changing binomial coefficients to q-binomial coefficients does not give a q-Pascal identity. When $q \neq 1$, we have

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1+q \neq 2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q .$$

Interestingly, there are *two* q-Pascal identities as we will show in Lemma 4.3.1. We will first give an arithmetic proof that holds when $a \in \mathbb{R}$, $q \in \mathbb{R}^+$, and $k \in \mathbb{Z}^+$. Our second proof is more conceptual, but only holds when $a \in \mathbb{Z}^+$ and q is the order of a finite field.

Lemma 4.3.1 (The q-Pascal Identities)**.** *If $a \in \mathbb{R}$, $q \in \mathbb{R}^+$, and $k \in \mathbb{Z}^+$, then*

$$\begin{bmatrix} a \\ k \end{bmatrix}_q = \begin{bmatrix} a-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} a-1 \\ k \end{bmatrix}_q = q^{a-k} \begin{bmatrix} a-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} a-1 \\ k \end{bmatrix}_q .$$

Proof. We have

$$\begin{aligned}
\begin{bmatrix} a \\ k \end{bmatrix}_q &= \left(\frac{q^a-1}{q-1}\right)\left(\frac{(q^{a-1}-1)\cdots(q^{a-k+1}-1)}{(q^2-1)\cdots(q^k-1)}\right) \\
&= \left(\frac{q^k-1}{q-1} + q^k\,\frac{q^{a-k}-1}{q-1}\right)\left(\frac{(q^{a-1}-1)\cdots(q^{a-k+1}-1)}{(q^2-1)\cdots(q^k-1)}\right) \\
&= \frac{(q^{a-1}-1)\cdots(q^{a-k+1}-1)}{(q-1)\cdots(q^{k-1}-1)} + q^k\,\frac{(q^{a-1}-1)\cdots(q^{a-k+1}-1)}{(q^2-1)\cdots(q^k-1)} \\
&= \begin{bmatrix} a-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} a-1 \\ k \end{bmatrix}_q
\end{aligned}$$

The second equality is proved in a similar manner. □

From now on, q will be restricted to be the order of the finite field $\mathbb{F}_q$, and we will drop the subscript q. Now we give a second proof of Lemma 4.3.1 using Lemma 4.2.4, but note that $a \in \mathbb{Z}^+$ must be a positive integer.

Proof. Since $a \in \mathbb{Z}^+$, let $n = a$ and let H be an $(a-1)$-dimensional subspace of V. For the second identity, we partition the k-dimensional subspaces of V into k-dimensional subspaces that are contained in H and k-dimensional subspaces that are not contained in H. By Lemma 4.2.4, there are $\begin{bmatrix} a-1 \\ k \end{bmatrix}$ k-dimensional spaces that are contained in H. Note that if a k-dimensional subspace of V does not lie in H, then it must intersect H in a $(k-1)$-dimensional space. By Lemma 4.2.4, there are $\begin{bmatrix} a-1 \\ k-1 \end{bmatrix}$ $(k-1)$-dimensional subspaces in H, each of which is contained in

$$\frac{q^a - q^{k-1}}{q^k - q^{k-1}} - \frac{q^{a-1} - q^{k-1}}{q^k - q^{k-1}} = \begin{bmatrix} a-k+1 \\ 1 \end{bmatrix} - \begin{bmatrix} a-k \\ 1 \end{bmatrix} = q^{a-k}$$

k-dimensional subspaces of V that are not contained in H. Hence,

$$\begin{aligned}
\begin{bmatrix} a \\ k \end{bmatrix} = \left|\begin{bmatrix} V \\ k \end{bmatrix}\right| &= \left|\left\{S \in \begin{bmatrix} V \\ k \end{bmatrix} : S \not\subset H\right\}\right| + \left|\left\{S \in \begin{bmatrix} V \\ k \end{bmatrix} : S \subset H\right\}\right| \\
&= q^{a-k}\begin{bmatrix} a-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} a-1 \\ k \end{bmatrix},
\end{aligned}$$

which is the second identity. Since $a \in \mathbb{Z}^+$, we can use the symmetry property of the binomial coefficients (4.3.2) to yield the first identity. □

4.4 The q-Kneser graphs

We formally state and prove the Erdős-Ko-Rado theorem for vector spaces, and introduce the closely-related q-Kneser graphs.

Definition 4.4.1. *A family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is intersecting if the intersection of any two members of $\mathscr{F}$ has nonzero dimension; that is, for all $F, F' \in \mathscr{F}$ we have* $\dim(F \cap F') \neq 0$.

We discussed the analog of the Erdős-Ko-Rado question in Section 4.1.1 and formally state it now. Hsieh [72] first proved the bound and characterized equality when $n > 2k$. His proof does not work for all relevant values of n and q, and involves many computations. Later, Frankl and Wilson [60] proved the bound and characterized equality for $n > 2k$, essentially by computing the eigenvalues of a generalized q-Kneser graph. More recently, Godsil and Newman [66, 94] used Frankl and Wilson's methods to characterize equality in the case $n = 2k$.

Theorem 4.4.2 (Hsieh, Frankl-Wilson, Godsil-Newman)**.** *Suppose $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is intersecting and $n \geq 2k$. Then $|\mathscr{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Equality holds if and only if*

(i) $\mathscr{F} = \left\{ F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F \right\}$ *for some one-dimensional subspace $v \subset V$ or*

(ii) $n = 2k$ *and* $\mathscr{F} = \begin{bmatrix} H \\ k \end{bmatrix}$ *where H is a $(2k-1)$-dimensional subspace of V.*

Recall that in the case of sets, there were exponentially many ways to obtain an extremal family in the case $n = 2k$. Since vector spaces have additional structure, the characterization of equality is stronger and there are only two extremal families when $n = 2k$. We will see further examples of results which are stronger for vector spaces than for sets in Section 4.6.3.

As in the Erdős-Ko-Rado proof from Section 3.3.2, our proof of Theorem 4.4.2 hinges on computing the eigenvalues of the q-Kneser graph, which we define now.

Definition 4.4.3. *The q-Kneser graph, denoted $qK_{n:k}$, has vertex set $\begin{bmatrix} V \\ k \end{bmatrix}$ and edge set* $E = \{\{A, B\} : A, B \in \begin{bmatrix} V \\ k \end{bmatrix}, A \cap B = \{0\}\}$.

Many parameters of the q-Kneser graphs are given by expressions that involve q-binomial coefficients and which reduce to those of the Kneser graph when we set

$q = 1$. The next two lemmas give examples of such parameters. Recall that, as part of the proof from Section 3.3.2, we showed that the Kneser graph $K_{n:k}$ has $\binom{n}{k}$ vertices and is $\binom{n-k}{k}$-regular. We now prove that the q-Kneser graph $qK_{n:k}$ has $\begin{bmatrix} n \\ k \end{bmatrix}$ vertices and is $q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}$-regular. Observe that, when $q = 1$, the expressions for the number of vertices and the degree of the q-Kneser graph reduce to that of the Kneser graph.

Lemma 4.4.4. *The q-Kneser graph $qK_{n:k}$ has $\begin{bmatrix} n \\ k \end{bmatrix}$ vertices and is regular with degree $q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}$.*

Proof. Lemma 4.2.4 proves that the q-Kneser graph $qK_{n:k}$ has $\begin{bmatrix} n \\ k \end{bmatrix}$ vertices. Now we show that the q-Kneser graph is regular and we determine its degree. Let α be a vertex of $qK_{v:k}$; it is a k-dimensional subspace and contains q^k elements of $\mathbb{F}_q^n$. There are $(q^n - q^k)(q^n - q^{k+1})\cdots(q^n - q^{2k-1})$ ways to choose k-tuple independent vectors in $\mathbb{F}_q^n$ that are not in α. Since a given k-space has $(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$ distinct ordered bases, there are

$$\frac{(q^n - q^k)(q^n - q^{k+1})\cdots(q^n - q^{2k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})} = q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}$$

k-spaces in $\mathbb{F}_q^n$ whose intersection with α is the zero subspace. Hence, the q-Kneser graph $qK_{n:k}$ is regular with degree $q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}$. □

Recall that in Theorem 3.3.9, we computed the eigenvalues of the Kneser graph's adjacency matrix and their multiplicities. Delsarte computed the corresponding quantities for the q-Kneser graph [36]. Again, when $q = 1$, the expressions for the eigenvalues and multiplicities reduce to that of the Kneser graph. Here, we interpret $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ -1 \end{bmatrix} = 0$.

Theorem 4.4.5 (Delsarte)**.** *Let $n \geq 2k$ and let A be the adjacency matrix of the q-Kneser graph $qK_{n:k}$. The eigenvalues λ_i of A and their corresponding multiplicities m_i are*

$$\lambda_i = (-1)^i q^{k(k-i)}\begin{bmatrix} n-k-i \\ k-i \end{bmatrix}, \quad m_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}, \quad i = 0, \ldots, k. \qquad \square$$

Now we prove Theorem 4.4.2 using the eigenvalue method of Section 3.3.2. As in the set case, the independent sets in the q-Kneser graph are in bijective correspondence with intersecting families $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$.

Proof of Theorem 4.4.2. Suppose $n \geq 2k$. Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum intersecting family. We know $\mathscr{F}$ is an independent set in the q-Kneser graph $qK_{n:k}$. By Theorem 4.4.5, the least eigenvalue of the Kneser graph $qK_{n:k}$ is

$$\tau = \lambda_1 = -q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}.$$

In Lemma 4.4.4, we showed that the q-Kneser graph $qK_{n:k}$ is $q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}$-regular so the ratio bound, Theorem 3.3.7, yields

$$|\mathscr{F}| \leq \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{1 + \frac{q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}}{q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}}} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

which establishes the bound in Theorem 4.4.2. Using the second claim of Theorem 3.3.7 and the multiplicity $m_1 = [n] - 1$ of the least eigenvalue λ_1, it is possible to characterize the case of equality when $n \geq 2k$; see [94, Section 5.5] for details. □

4.5 Spreads

Spreads are the vector space analogs of perfect matchings. We will need them in Section 4.6, where we give a proof of the Erdős-Ko-Rado theorem for vector spaces, Theorem 4.4.2, in the case that $k|n$. We also use spreads in Chapter 5 to prove an analog of Frankl's r-wise intersection theorem, Theorem 2.2.7.

Definition 4.5.1. *A family $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ of t-dimensional subspaces of V is called a t-spread if every one-dimensional subspace of V is contained in exactly one t-dimensional subspace in $\mathscr{S}$.*

Definition 4.5.2. *If $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ is a spread and the elements in $\mathscr{S}$ that lie in a subspace U form a t-spread of U, then we say that $\mathscr{S}$ induces a t-spread on U.*

Definition 4.5.3. *A t-spread $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ is called geometric if $\mathscr{S}$ induces a t-spread on each $2t$-dimensional subspace generated by a pair of elements in $\mathscr{S}$.*

Baer and Segre [6, 100] proved necessary and sufficient conditions for the existence of geometric t-spreads.

Lemma 4.5.4 (Baer, Segre)**.** *A geometric t-spread $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ exists if and only if $t|n$.*

Proof. If $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ is a spread, then the size of $\mathscr{S}$ is

$$|\mathscr{S}| = \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}}{\begin{bmatrix} t \\ 1 \end{bmatrix}} = \frac{q^n - 1}{q^t - 1}.$$

Since $q, |\mathscr{S}| \in \mathbb{Z}^+$ are positive integers, we must have $t|n$.

Now suppose that $n = tl$ where $l \in \mathbb{Z}^+$. We have that V is an n-dimensional vector space over the field $\mathbb{F}_q$, but we can also view V as an l-dimensional space over the field $\mathbb{F}_{q^t}$. Let $\mathscr{S} \subset \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^t}$ be the family of 1-dimensional subspaces of V as a vector space over $\mathbb{F}_{q^t}$. We have that

$$|\mathscr{S}| = \begin{bmatrix} l \\ 1 \end{bmatrix}_{q^t} = q^{t(l-1)} + q^{t(l-2)} + \cdots + q^t + 1 = \frac{q^{tl} - 1}{q^l - 1} = \frac{q^n - 1}{q^t - 1}.$$

If we think of V as an n-dimensional vector space over $\mathbb{F}_q$ again, then the members of $\mathscr{S}$ are now t-dimensional spaces over $\mathbb{F}_q$ and form a geometric t-spread of V. □

We now prove two properties of geometric spreads that we will need for our work in Section 4.6 and Chapter 5. The first proves that a geometric t-spread $\mathscr{S} \subset \begin{bmatrix} V \\ t \end{bmatrix}$ does not just induce t-spreads on the $2t$-dimensional spaces generated by a pair of elements in $\mathscr{S}$, but also on any subspace generated by elements of $\mathscr{S}$.

Lemma 4.5.5. *If $\mathscr{S}$ is a geometric t-spread of V, then $\mathscr{S}$ induces a geometric t-spread on any subspace of V that is generated by elements of $\mathscr{S}$.*

Proof. Let $\{X_1, \ldots, X_m\} \subset \mathscr{S}$, where $m \leq n/t$, and suppose $\dim(X_1 \vee \cdots \vee X_m) = mt$. We will show that $\mathscr{S}$ induces a t-spread on $X_1 \vee \cdots \vee X_m$. We proceed by induction on m. If $m \in \{1, 2\}$, the statement is true by definition of geometric. Suppose the statement is true for $m-1$ and let $\mathscr{B}$ be the spread that $\mathscr{S}$ induces on $X_1 \vee \cdots \vee X_{m-1}$.

Define $\mathscr{G} := \{X_m \vee A : A \in \mathscr{B}\} \subset \begin{bmatrix} V \\ 2t \end{bmatrix}$. We claim that every one-dimensional subspace of $X_1 \vee \cdots \vee X_m$ lies in some $G \in \mathscr{G}$. Observe that distinct $G_1, G_2 \in \mathscr{G}$ satisfy $G_1 \cap G_2 = X_m$ and that for any $G \in \mathscr{G}$, the number of one-dimensional subspaces in

$G \setminus X_m$ is $\begin{bmatrix} 2t \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}$. Hence, the number of one-dimensional subspaces in some $G \in \mathscr{G}$ is

$$\begin{aligned}\left|\left\{W \in \begin{bmatrix} V \\ 1 \end{bmatrix} : W \in G \in \mathscr{G}\right\}\right| &= |\mathscr{G}| \left(\begin{bmatrix} 2t \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}\right) + \left|\left\{W \in \begin{bmatrix} V \\ 1 \end{bmatrix} : W \in X_m\right\}\right| \\ &= \left(q^t \begin{bmatrix} m-1 \\ 1 \end{bmatrix} + 1\right)\left(\begin{bmatrix} 2t \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}\right) + \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} mt \\ 1 \end{bmatrix} \\ &= \left|\left\{W \in \begin{bmatrix} V \\ 1 \end{bmatrix} : W \in X_1 \vee \cdots \vee X_m\right\}\right|,\end{aligned}$$

which proves that every one-dimensional subspace of $X_1 \vee \cdots \vee X_m$ lies in some $G \in \mathscr{G}$.

Consequently, if $X \in \mathscr{S}$ and $X \cap (X_1 \vee \cdots \vee X_m) \neq \{0\}$, then $X \cap G \neq \{0\}$ for some $G \in \mathscr{G}$. Since $\mathscr{S}$ is geometric, this implies $X \subset G \subset X_1 \vee \cdots \vee X_m$. Hence, $\mathscr{S}$ induces a t-spread on $X_1 \vee \cdots \vee X_m$. □

The second lemma proves that an invertible linear transformation of V maps a geometric t-spread to another geometric t-spread.

Definition 4.5.6. *The set of all invertible linear transformations of V is denoted*

$$GL(V) := \{\pi : V \to V : \pi \text{ is an invertible linear transformation}\}.$$

Lemma 4.5.7. *If $\mathscr{S}$ is a geometric t-spread of V, then for any invertible linear transformation $\pi \in GL(V)$, the family $\pi(\mathscr{S}) := \{\pi(S) : S \in \mathscr{S}\}$ is also a geometric t-spread of V.*

Proof. First we check that $\pi(\mathscr{S})$ is a spread. For $S \in \mathscr{S}$, we have $\pi(S)$ is a t-dimensional subspace because π is an isomorphism of V. If $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, then $\pi^{-1}(v) \in S$ for some $S \in \mathscr{S}$ as $\mathscr{S}$ is a spread. Hence, $v = \pi(\pi^{-1}(v)) \in \pi(S)$. Also, we cannot have v in $\pi(S) \cap \pi(T)$ for distinct $S, T \in \mathscr{S}$ as otherwise $\pi^{-1}(v) \in S \cap T$, which contradicts $\mathscr{S}$ being a spread. We have shown that every one-dimensional subspace of V lies in exactly one t-dimensional subspace of $\pi(\mathscr{S})$ and so $\pi(\mathscr{S})$ is a t-spread. Given $S, T \in \mathscr{S}$, if $\mathscr{S}'$ is the spread that $\mathscr{S}$ induces on $S \vee T$, then it is easy to check that $\pi(\mathscr{S}')$ is the spread that $\pi(S)$ induces on $\pi(S) \vee \pi(T)$. □

4.6 Which Tools Work?

On a first glance, it might seem as if theorems in extremal set theory should easily extend to the vector space setting by changing identities involving binomial coefficients to their corresponding q-analogs. On closer inspection, however, we will see in Section 4.6.1 and Section 4.6.2 that adapting combinatorial techniques to vector spaces is often not straightforward. Generally, algebraic methods have been more successful than combinatorial ones for vector space analogs, as we saw in Section 4.4. We will discuss further results that have been proved using algebraic techniques in Section 4.6.3.

4.6.1 Shifting and Shadows

Adapting combinatorial techniques to vector spaces is often tricky. We cite two flawed attempts to extend the shifting technique to vector spaces. We discuss the difficulties that surface when trying to adapt Daykin's proof in Section 2.2.4 to vector spaces. We also show that no analog of the colex order exists for shadows in vector spaces.

4.6.1.1 Shifting

There have been two published attempts [31, 39] to extend the shifting proof from Section 2.2.3 to vector spaces, and both of these attempts are acknowledged to be flawed by their respective authors.

4.6.1.2 Daykin's Proof

For a simple example of the kinds of issues that occur when attempting to extend combinatorial proofs to vector spaces, consider Daykin's proof of the Erdős-Ko-Rado theorem in Section 2.2.4. The definition of a family's shadow extends naturally to vector spaces: if $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is a family of k-dimensional subspaces of V, then its shadow consists of the $(k-1)$-dimensional subspaces of V contained in at least one member of $\mathscr{F}$. In Chapter 5, we will prove an analog of Lovász's result, Theorem 2.1.5, for vector spaces. We cannot, however, mimic its application in Daykin's proof.

Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be intersecting, and suppose we try to mimic Daykin's proof to show that $|\mathscr{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. It is not clear what the analog of $\mathscr{G}$ should be as the set complement $V \setminus F$ of the subspace $F \in \mathscr{F}$ with respect to V is not itself a subspace. If we try to define $\mathscr{G} = \{F^{\perp} : F \in \mathscr{F}\} \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$ to be the family of orthogonal complements of $\mathscr{F}$, then it is possible that $F \subset G$ for some $G \in \mathscr{G}$. More importantly, the use of the Pascal identity in Daykin's proof does not work with the q-Pascal identity.

4.6.1.3 No Analog of the Colex Order

A crucial difference between sets and vector spaces is that no analog of the colex order exists for shadows in vector spaces. Recall that we defined the colex order in Definition 2.1.2. We say that a set system $\mathscr{H} \subset \binom{X}{k}$ is a solution to the shadow minimization problem with parameters k and $|\mathscr{H}|$ if the shadow of $\mathscr{H}$ is minimum over all set-systems $\mathscr{F} \subset \binom{X}{k}$ with the same cardinality,

$$|\partial \mathscr{H}| = \min_{|\mathscr{F}|=|\mathscr{H}|} |\partial \mathscr{F}|.$$

We are especially interested in the case where there are nested solutions $\{\mathscr{H}_m\}$, i.e., such that $|\mathscr{H}_m| = m$, $\mathscr{H}_m$ is a solution to the shadow minimization problem with parameters k and m, and $\mathscr{H}_{m-1} \subset \mathscr{H}_m$ for $1 \leq m \leq \binom{n}{k}$.

The Kruskal Katona theorem, Theorem 2.1.4, shows that the colex order on $\binom{X}{k}$ satisfies the following properties:

1. For $1 \leq m \leq \binom{n}{k}$, we have $\{\mathscr{C}_m^k\}$ is a family of nested solutions to the shadow minimization problem with parameters k and m.

2. The shadow of $\mathscr{C}_m^k$ is an initial segment of the colex order on $\binom{X}{k-1}$.

No analog of the colex order exists for vector spaces. For example, Bezrukov and Blokhuis [12] showed that there is no total order of the subspaces in $\mathbb{F}_2^n$ satisfying properties (i) and (ii) when $n \geq 4$. Even more remarkably, Harper [69] and Ure [109] showed that nested solutions to the shadow minimization problem for vector spaces do not always exist.

Theorem 4.6.1 (Harper, Ure)**.** *Let $V = \mathbb{F}_8^3$. If $\mathscr{F} \subset \begin{bmatrix} V \\ 2 \end{bmatrix}$ has size $|\mathscr{F}| = 24$ and minimum shadow over all families with the same cardinality, then there does not exist $\mathscr{F}' \subset \mathscr{F}$ such that $|\mathscr{F}'| = 22$ and $\mathscr{F}'$ has minimum shadow over all families with the same cardinality.*

4.6.2 Katona's Cyclic Permutation Method

Like the combinatorial methods discussed previously, Katona's cyclic permutation technique in Section 2.2.5 does not readily adapt to give a full proof of the Erdős-Ko-Rado theorem for vector spaces, Theorem 4.4.2. A similar argument, however, does yield the theorem in the special case that $k|n$. The proof uses spreads and is due to Greene and Kleitman [67]. This argument will be important for the analog of Frankl's r-wise intersection theorem in Chapter 5 and for the Manickam-Miklós-Singhi conjecture in Chapter 7. We first prove two lemmas; the first computes the size of $GL(V)$. Note that, when $q=1$, the expression for the size of $GL(V)$ reduces to that of the size of S_X, the group of permutations on X.

Lemma 4.6.2. *We have* $|GL(V)| = q^{n(n-1)/2}(q-1)^n[n]!$

Proof. Since V is an n-dimensional vector space over $\mathbb{F}_q$, there are

$$(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1}) = q^{n(n-1)/2}(q-1)^n[n]!$$

ways to choose a basis of V. The number of invertible linear transformations of V equals the number of distinct bases of V so $|GL(V)| = q^{n(n-1)/2}(q-1)^n[n]!$. □

The second lemma shows that if $A,B \in \begin{bmatrix} V \\ k \end{bmatrix}$ are k-dimensional subspaces of V, then there are $q^{n(n-1)/2}(q-1)^n[k]![n-k]!$ invertible linear transformations $\pi \in GL(V)$ such that $\pi(A) = B$. Observe that, when $q=1$, this expression reduces to the number of permutations of X that send a given k-subset to another.

Lemma 4.6.3. *If* $A,B \in \begin{bmatrix} V \\ k \end{bmatrix}$ *are* k*-dimensional subspaces of* V*, then the number of invertible linear transformations such that* $\pi(A) = B$ *is* $q^{n(n-1)/2}(q-1)^n[k]![n-k]!$

Proof. Let $v_1,\dots,v_k$ be a basis of A. Extend this basis to a basis $v_1,\dots,v_k,v_{k+1},\dots,v_n$ of V. Since $\pi(A) = B$, there are q^k-1 choices for $\pi(v_1)$. For $i \in [k]\setminus\{1\}$, we must pick $\pi(v_i)$ such that $\pi(v_i) \in B$ but is not a linear combination of $\pi(v_1),\dots,\pi(v_{i-1})$. Hence, there are q^k-q^{i-1} choices for $\pi(v_i)$ when $i \in [k]$. We must pick $\pi(v_{k+1}) \in V\setminus B$ so there are q^n-q^k choices for $\pi(v_{k+1})$. For each $j \in [n]\setminus[k+1]$, we must choose $\pi(v_j)$ in $V\setminus B$ so that $\pi(v_j)$ is not a linear combination of $\pi(v_{k+1}),\dots,\pi(v_{j-1})$. Consequently, there are $q^n - q^{j-1}$ choices for $\pi(v_j)$ when $j \in [n] \setminus [k]$. We see that the number of invertible linear transformations such that $\pi(A) = B$ is

$$(q^k - 1)\cdots(q^k - q^{k-1})(q^n - q^k)\cdots(q^n - q^{n-1}) = q^{n(n-1)/2}(q-1)^n[k]![n-k]! \qquad \square$$

Now, we give Greene and Kleitman's simple proof [67] of the Erdős-Ko-Rado theorem for vector spaces, Theorem 4.4.2, in the case that $k|n$.

Proof. Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be an intersecting family and assume that $k|n$. Let $\mathscr{S} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a spread of V and let $\pi \in GL(V)$ be an invertible linear transformation. The proof of Lemma 4.5.7 shows that $\pi(\mathscr{S})$ is also a spread. Since $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is intersecting, for any $\pi \in GL(V)$, we have $|\mathscr{F} \cap \pi(\mathscr{S})| \leq 1$. Consequently, by Lemma 4.6.2, we have

$$\sum_{\pi \in GL(V)} |\mathscr{F} \cap \pi(\mathscr{S})| \leq q^{n(n-1)/2}(q-1)^n[n]!$$

Now given $S \in \mathscr{S}$ and $F \in \mathscr{F}$, there are $q^{n(n-1)/2}(q-1)^n[k]![n-k]!$ invertible linear transformations $\pi \in GL(V)$ such that $\pi(S) = F$ by Lemma 4.6.3. Hence,

$$\begin{aligned}\left(\frac{q^n-1}{q^k-1}\right)|\mathscr{F}|q^{n(n-1)/2}(q-1)^n[k]![n-k]! &= |\mathscr{S}||\mathscr{F}||\{\pi \in GL(V) : \pi(S) = F\}| \\ &= \sum_{\pi \in GL(V)} |\mathscr{F} \cap \pi(\mathscr{S})| \\ &\leq q^{n(n-1)/2}(q-1)^n[n]!\end{aligned}$$

We consequently have that

$$|\mathscr{F}| \leq \left(\frac{q^k-1}{q^n-1}\right)\frac{[n]!}{[k]![n-k]!} = \frac{[k]}{[n]}\frac{[n]!}{[k]![n-k]!} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \qquad \square$$

4.6.3 Algebraic Successes

In this section, we mention some other important intersection theorems for vector spaces. Common to all these results is that their proofs use algebraic techniques.

First, we discuss Frankl and Graham's analog of the modular Frankl-Wilson theorem from Section 3.2. As in the Erdős-Ko-Rado theorem for vector spaces, this result provides another example of a theorem that is stronger for vector spaces than for sets. Frankl and Graham's proof makes use of the algebraic machinery of higher incidence matrices, and many variants of Fisher's Inequality, Theorem 2.3.9, were originally proved in this way as well.

Vector space analogs do not only provide applications of algebraic methods, but also inspire new ones. Lovász, for example, introduced his wedge product technique specifically to prove the vector space analog of Bollobás's Two Families Theorem. We discuss this result in Section 4.6.3.2.

Finally, in Section 4.6.3.3, we state some old and new results concerning the analog of the t-intersection theorem in Section 2.2.8 to vector spaces.

4.6.3.1 Modular Frankl-Wilson Theorem

Recall that the Frankl-Wilson theorem, Theorem 3.2.2, asserts that if $\mathscr{F} \subset 2^X$ is an L-intersecting family, where $|L| = s$, then $|\mathscr{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$. If $\mathscr{F} \subset \binom{X}{k}$ is a *uniform* L-intersecting family, then Ray-Chaudhuri and Wilson [97] proved that the upper bound on $|\mathscr{F}|$ can be strengthened.

Theorem 4.6.4 (Ray-Chaudhuri – Wilson)**.** *Suppose $L \subset \mathbb{N}$ has size $|L| = s$. If $\mathscr{F} \subset \binom{X}{k}$ is a uniform L-intersecting family, then $|\mathscr{F}| \leq \binom{n}{s}$.*

Frankl and Wilson [59] showed that the conclusion of Theorem 4.6.4 holds under the considerably weaker condition that the intersection sizes in L belong to at most s residue classes modulo a prime p.

Theorem 4.6.5 (Frankl-Wilson)**.** *Suppose $p \in \mathbb{Z}^+$ is a prime number and that $L \subset \mathbb{N}$ has size $|L| = s \leq p-1$. Also suppose that $k \in \mathbb{Z}^+$ satisfies $k \notin L \pmod{p}$. If $\mathscr{F} \subset \binom{X}{k}$ is a uniform family such that $|F \cap F'| \in L \pmod{p}$ for all distinct $F, F' \in \mathscr{F}$, then $|\mathscr{F}| \leq \binom{n}{s}$.*

The restriction that p be prime in Theorem 4.6.5 is crucial. If the prime number p is replaced by $p = 6$, for example, or even $q = p^2$, where $p \geq 7$, then Theorem 4.6.5 will be false [5, Section 5.9]!

Remarkably, Frankl and Graham [56] showed that an analog of Theorem 4.6.5 holds for vector spaces but *without* the restriction that p be prime. As in the Erdős-Ko-Rado theorem for vector spaces, Theorem 4.4.2, we obtain a stronger analog for vector spaces because of the additional structure they have.

Theorem 4.6.6 (Frankl-Graham)**.** *Suppose $b \in \mathbb{Z}^+$ is a positive integer and that $L \subset \mathbb{N}$ has size $|L| = s \leq b-1$. Also suppose that $k \in \mathbb{Z}^+$ satisfies $k \notin L$ $(mod\ b)$. If $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is a uniform family such that* $\dim(F \cap F') \in L$ $(mod\ b)$ *for all distinct $F, F' \in \mathscr{F}$, then*

$$|\mathscr{F}| \leq \begin{bmatrix} n \\ s \end{bmatrix},$$

except possibly for $q = 2$, $b = 6$, $s \in \{3, 4\}$.

4.6.3.2 Bollobás's Two Families Theorem

Bollobás's Two Families Theorem [16] is an important combinatorial result with many consequences [2, 44, 96]. An elegant proof of the theorem using the polynomial method is given in [5, Section 5.1].

Theorem 4.6.7 (Bollobas)**.** *Suppose that $A_1, \ldots, A_m \in \binom{X}{r}$ are r-element sets and that $B_1, \ldots, B_m \in \binom{X}{s}$ are s-element sets such that*

(i) A_i and B_i are disjoint for $i \in [m]$,

(ii) A_i and B_j have nonempty intersection whenever $i \neq j$.

Then $m \leq \binom{r+s}{r}$.

We have seen in Section 4.6.1 that adapting combinatorial techniques to vector spaces can be challenging. In Section 4.4 and Section 4.6.3.1, we also saw examples of algebraic proofs of extremal set theory results that generalize nicely to their vector space analogs. In the case of Bollobás's Two Families Theorem, neither the known combinatorial or algebraic proofs seemed to adapt to the case of vector spaces. As a result, Lovász [82] introduced the method of wedge products specifically to tackle the analog of Theorem 4.6.7.

Theorem 4.6.8 (Lovász)**.** *Let $U_1, \ldots, U_m \in \begin{bmatrix} V \\ r \end{bmatrix}$ be r-dimensional subspaces of V and let $W_1, \ldots, W_m \in \begin{bmatrix} V \\ s \end{bmatrix}$ be s-dimensional subspaces of V such that*

(i) $U_i \cap W_i = \{0\}$ for $i \in [m]$,

(ii) $U_i \cap W_j \neq \{0\}$ whenever $i \neq j$.

Then $m \leq \binom{r+s}{r}$.

4.6.3.3 On t-intersecting Families

In this section, we state some old and new results on t-intersecting families of vector spaces.

Definition 4.6.9. *For a positive integer $t \in \mathbb{Z}^+$, a family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ of k-dimensional spaces is called* t-intersecting *if any two elements in $\mathscr{F}$ have intersection of dimension at least t; that is, for all $F, F' \in \mathscr{F}$ we have $\dim(F \cap F') \geq t$.*

Wilson's proof [116] of the t-intersection theorem, Theorem 2.2.12, generalizes nicely to vector spaces [60].

Theorem 4.6.10 (Frankl-Wilson)**.** *If $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is t-intersecting and $n \geq 2k - t$, then*

$$|\mathscr{F}| \leq \max \left\{ \begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix} \right\}. \tag{4.6.1}$$

We also have that

(i) if $n > 2k$, then equality holds in (4.6.1) *if and only if $\mathscr{F} = \left\{ F \in \begin{bmatrix} V \\ k \end{bmatrix} : S \subset F \right\}$ for some t-dimensional subspace $S \in \begin{bmatrix} V \\ t \end{bmatrix}$;*

(ii) if $2k - t \leq n < 2k$, then equality holds in (4.6.1) *if and only if $\mathscr{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$ for some $(2k-t)$-dimensional subspace $Y \in \begin{bmatrix} V \\ 2k-t \end{bmatrix}$.*

Note that when $n = 2k$, we have $\begin{bmatrix} n-t \\ k-t \end{bmatrix} = \begin{bmatrix} 2k-t \\ k \end{bmatrix}$. Hence, the two non-isomorphic families in Theorem 4.6.10 have the same cardinality and they are conjectured to be the only extremal families when $n = 2k$. This has only been proved, however, for $t = 1$ by Godsil and Newman [66, 94].

Very recently, Tokushige [106] used Wilson's methods along with new results by Ellis, Friedgut, and Pilpel [46] to generalize Theorem 4.6.10 to cross-intersecting families. Another recent result is Frankl and Tokushige's proof [57] of an analog of Katona's t-intersection theorem using the method of higher incidence matrices discussed in Section 4.6.3.1.

Chapter 5

Shadows and Intersections in Vector Spaces

We [27] prove a vector space analog of Lovász's version of the Kruskal-Katona theorem, Theorem 2.1.5. We apply this result to extend Frankl's theorem on r-wise intersecting families, Theorem 2.2.7, to vector spaces. In particular, we obtain a short new proof of the Erdős-Ko-Rado theorem for vector spaces, Theorem 4.4.2. In Section 5.4, we cite our other results in this area, namely obtaining a vector space analog of the Hilton-Milner theorem [14, 24] and determining the chromatic number of the q-Kneser graph [14, 26]. Finally, we end the chapter in Section 5.5 with some open problems.

Before we can give precise statements of our results, we first formally define the concepts of shadow and r-wise intersection for vector spaces. We see that they are natural analogs of their set counterparts.

Definition 5.0.11. *For a family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$, we define the shadow of $\mathscr{F}$, denoted $\partial\mathscr{F}$, to consist of those $(k-1)$-dimensional subspaces of V contained in at least one member of $\mathscr{F}$,*

$$\partial\mathscr{F} := \left\{ E \in \begin{bmatrix} V \\ k-1 \end{bmatrix} : E \subset F \in \mathscr{F} \right\}.$$

As in Section 4.6.1.3, we can ask for families $\mathscr{H} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ whose shadow is minimum over all families $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ with the same cardinality,

$$|\partial\mathscr{H}| = \min_{|\mathscr{F}|=|\mathscr{H}|} |\partial\mathscr{F}|.$$

In general, this is a difficult question because of the reasons discussed in Section 4.6.1.3. Surprisingly, we can still prove an analog of Lovász's version of the Kruskal-Katona theorem. Recall that, in Lovász's result, given a family $\mathscr{F} \subset \binom{X}{k}$, we could find a real number $y \geq k$ such that $|\mathscr{F}| = \binom{y}{k}$. The same is true for q-binomial coefficients. If k and q are fixed, then $\begin{bmatrix} a \\ k \end{bmatrix}$ is a continuous function of a which is positive and strictly increasing when $a \geq k$; hence, by the intermediate value theorem, if $r \geq 1$ is a real number, then there exists a unique real number $a_r \geq k$ such that $r = \begin{bmatrix} a_r \\ k \end{bmatrix}$. Consequently, we can formulate an analog of Lovász's version of the Kruskal-Katona theorem, and this is our first result.

Theorem 5.0.12 (Chowdhury-Patkós). *Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ and let $y \geq k$ be the real number defined by $|\mathscr{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$. Then*

$$|\partial\mathscr{F}| \geq \begin{bmatrix} y \\ k-1 \end{bmatrix}.$$

If equality holds, then $y \in \mathbb{Z}^+$ and $\mathscr{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$, where Y is a y-dimensional subspace of V.

As in Section 2.2.6, we can apply our Lovász analog, Theorem 5.0.12, to yield an analog of Frankl's r-wise intersection theorem, Theorem 2.2.7. Frankl's proof from Section 2.2.6, however, does not generalize to vector spaces for the reasons discussed in Section 4.6.1. Hence, we will need new proof techniques.

Definition 5.0.13. *A family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ r-wise intersecting if the intersection of any r members of $\mathscr{F}$ has nonzero dimension; that is,* $\dim(\bigcap_{i=1}^{r} F_i) \neq 0$ *for all $F_1, \ldots, F_r \in \mathscr{F}$.*

Theorem 5.0.14 (Chowdhury-Patkós). *Suppose that $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is r-wise intersecting and that $(r-1)n \geq rk$. Then*

$$|\mathscr{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Equality holds if and only if $\mathscr{F} = \left\{ F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F \right\}$ for some one-dimensional subspace $v \subset V$, unless $r = 2$ and $n = 2k$.

Theorem 5.0.12 establish shadows as a viable method for proving vector space analogs such as Theorem 5.0.14, and more applications are expected. Recently, Wang [113] used Theorem 5.0.12 to prove a conjecture of Erdős, Faigle, and Kern [48]. The method of proof in Theorem 5.0.14 has also been used to prove results on r-cross intersecting families of sets [58]; Patkós and I are currently working with Frankl and Tokushige to extend these results to vector spaces.

5.1 Shadows and an Analog of Lovász's Theorem

Recall that, in Section 2.1.6, we gave Keevash's [77] recent and elegant proof of Lovász's version of the Kruskal-Katona theorem. In this section, we adapt his argument to prove Theorem 5.0.12. We first generalize all the definitions in Section 2.1.6 to their vector space analogs.

Definition 5.1.1. *For $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ and $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, define*

$$K_{k+1}^{k}(\mathscr{F}) := \left\{ T \in \begin{bmatrix} V \\ k+1 \end{bmatrix} : \begin{bmatrix} T \\ k \end{bmatrix} \subset \mathscr{F} \right\}$$

to be the family of $(k+1)$-dimensional spaces of V all of whose k-dimensional subspaces lie in $\mathscr{F}$ and

$$K_{k+1}^{k}(v) := \{T \in K_{k+1}^{k}(\mathscr{F}) : v \subset T\}$$

to be the family of $(k+1)$-dimensional spaces in $K_{k+1}^{k}(\mathscr{F})$ that contain v.

Definition 5.1.2. *For $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, define the <u>degree</u> of v, which is denoted by* $\deg(v)$, *to be the number of elements of $\mathscr{F}$ that contain v,*

$$\deg(v) := |\{F \in \mathscr{F} : v \subset F\}|.$$

Definition 5.1.3. *If $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \subset V$ is an $(n-1)$-dimensional subspace that does not contain v, then define the <u>link of v in U</u> to be the family of $(k-1)$-dimensional spaces in U whose linear span with v is an element of $\mathscr{F}$,*

$$L_U(v) := \left\{ A \in \begin{bmatrix} U \\ k-1 \end{bmatrix} : A \vee v \in \mathscr{F} \right\} \subset \begin{bmatrix} V \\ k-1 \end{bmatrix}.$$

As in Section 2.1.6, we first establish an upper bound on $|K_{k+1}^{k}(\mathscr{F})|$ in terms of $|\mathscr{F}|$; we will see that Theorem 5.0.12 follows as a simple corollary.

Theorem 5.1.4 (Chowdhury-Patkós). *Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ and let $y \geq k$ be the real number defined by $|\mathscr{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$. Then*

$$|K_{k+1}^{k}(\mathscr{F})| \leq \begin{bmatrix} y \\ k+1 \end{bmatrix}.$$

Equality holds if and only if $y \in \mathbb{Z}^{+}$ and $\mathscr{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$ for some y-dimensional subspace $Y \subset V$.

Proof. We argue by induction on k. The base case $k = 1$ is easy: Suppose $\mathscr{F} \subset \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $|\mathscr{F}| = [y]$. Since there are $q+1$ one-dimensional spaces in a two-dimensional space, $|K_2^1(v)| \leq (1/q)([y]-1)$ if $v \in \mathscr{F}$ and $|K_2^1(v)| = 0$ otherwise. Observe that

$$(q+1)|K_2^1(\mathscr{F})| = \sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} |K_2^1(v)| \leq \frac{[y]([y]-1)}{q}, \tag{5.1.1}$$

which implies that $|K_2^1(\mathscr{F})| \leq \begin{bmatrix} y \\ 2 \end{bmatrix}$.

Now assume the statement is true for $k-1$. We first show that if $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, then $|K_{k+1}^{k}(v)| \leq ([y-k]/[k])\deg(v)$; we will then sum this inequality over all $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and double count to obtain the desired upper bound on $|K_{k+1}^{k}(\mathscr{F})|$. If $\deg(v) = 0$, then clearly $|K_{k+1}^{k}(v)| \leq ([y-k]/[k])\deg(v)$, so we will assume that $\deg(v) \neq 0$. We will need to consider the cases where $\deg(v)$ is large and where $\deg(v)$ is small separately.

First, let's consider the case when $\deg(v) \geq \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$. If $T \in K_{k+1}^{k}(v)$, then observe that the q^k k-dimensional subspaces in T that do not contain v are elements of $\mathscr{F}$ that do not contain v. By counting pairs (S,T) where $v \not\subset S \in \begin{bmatrix} V \\ k \end{bmatrix}$ and $S \subset T \in K_{k+1}^{k}(v)$, the previous observation yields

$$q^k |K_{k+1}^{k}(v)| = \left| \left\{ S \in \begin{bmatrix} V \\ k \end{bmatrix} : v \not\subset S \subset T \in K_{k+1}^{k}(v) \right\} \right| \leq |\mathscr{F}| - d(v). \tag{5.1.2}$$

By rearranging (5.1.2) and applying the q-Pascal identity, Lemma 4.3.1, we obtain

$$\begin{aligned} |K_{k+1}^{k}(x)| &\leq \frac{|\mathscr{F}| - \deg(x)}{q^k} \leq \frac{\begin{bmatrix} y \\ k \end{bmatrix} - \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}}{q^k} = \frac{q^k \begin{bmatrix} y-1 \\ k \end{bmatrix}}{q^k} \\ &= \frac{[y-k]}{[k]} \begin{bmatrix} y-1 \\ k-1 \end{bmatrix} \leq \frac{[y-k]}{[k]} \deg(v). \end{aligned} \tag{5.1.3}$$

We have equality in (5.1.3) if and only if $\deg(v) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$ since $\deg(v) \neq 0$.

Now suppose $\deg(v) \leq \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$. Let $U \subset V$ be an $(n-1)$-dimensional subspace that does not contain v, and observe that $|L_U(v)| = \deg(v)$. Moreover, if T_1, T_2 are distinct elements of $K_{k+1}^{k}(v)$ then $T_1 \cap U$ and $T_2 \cap U$ are distinct k-dimensional spaces in

$K_k^{k-1}(L_U(v))$. Consequently, $|K_{k+1}^k(v)| \le |K_k^{k-1}(L_U(v))|$. Define the real number $y_v \ge k$ by $\deg(v) = \left[{y_v - 1 \atop k-1}\right]$. As $L_U(v) \subset \left[{V \atop k-1}\right]$ and $|L_U(v)| = \deg(v) = \left[{y_v-1 \atop k-1}\right]$, the induction hypothesis yields that

$$|K_{k+1}^k(v)| \le |K_{k-1}^k(L(x))| \le \begin{bmatrix} y_v - 1 \\ k \end{bmatrix} = \frac{[y_v - k]}{[k]}\begin{bmatrix} y_v - 1 \\ k-1 \end{bmatrix} \le \frac{[y-k]}{[k]}\deg(v), \qquad (5.1.4)$$

where the last inequality follows because $y_v \le y$, by our assumption on $\deg(v)$. We have equality in (5.1.4) if and only if $\deg(v) = \left[{y-1 \atop k-1}\right]$ by our assumption that $\deg(v) \ne 0$.

To finish the proof, we sum the inequality $|K_{k+1}^k(v)| \le ([y-k]/[k])\deg(v)$ over all $v \in \left[{V \atop 1}\right]$ and double-count to obtain the desired inequality on $|K_{k+1}^k(\mathscr{F})|$. We have

$$[k+1]|K_{k+1}^k(\mathscr{F})| = \sum_{v \in \left[{V \atop 1}\right]} |K_{k+1}^k(v)| \le \frac{[y-k]}{[k]} \sum_{v \in \left[{V \atop 1}\right]} d(v) = \frac{[y-k]}{[k]}[k]|\mathscr{F}| \qquad (5.1.5)$$

$$= [y-k]\begin{bmatrix} y \\ k \end{bmatrix} = [k+1]\begin{bmatrix} y \\ k+1 \end{bmatrix}.$$

Therefore, $|K_{k+1}^k(\mathscr{F})| \le \left[{y \atop k+1}\right]$, and equality holds only when all one-dimensional subspaces v with nonzero degree satisfy $d(v) = \left[{y-1 \atop k-1}\right]$.

We now characterize the case of equality. Again the proof proceeds by induction on k. The base case $k = 1$ is easy: Suppose $\mathscr{F} \subset \left[{V \atop 1}\right]$, $|\mathscr{F}| = [y]$, and $|K_2^1(\mathscr{F})| = \left[{y \atop 2}\right]$. Then (5.1.1) implies that $|K_2^1(v)| = (1/q)([y]-1)$ for all $v \in \mathscr{F}$. Hence, if v, w are distinct elements of $\mathscr{F}$, then every one-dimensional space in the two-dimensional space spanned by v and w lies in $\mathscr{F}$. It is easy to see by induction that if A is a subspace of dimension $1 \le d < \lceil y \rceil$ such that $\left[{A \atop 1}\right] \subset \mathscr{F}$, then there exists a subspace B of dimension $d+1$ that contains A and for which $\left[{B \atop 1}\right] \subset \mathscr{F}$. In particular, this proves that $y \in \mathbb{Z}^+$ and $\mathscr{F} = \left[{Y \atop 1}\right]$ for some y-dimensional subspace Y.

Now suppose $\mathscr{F} \subset \left[{V \atop k}\right]$, $|\mathscr{F}| = \left[{y \atop k}\right]$, and $|K_{k+1}^k(\mathscr{F})| = \left[{y \atop k+1}\right]$. Choose $v \in \left[{V \atop 1}\right]$ for which $d(v) \ne 0$. Since $|K_{k+1}^k(\mathscr{F})| = \left[{y \atop k+1}\right]$, we have $d(v) = \left[{y-1 \atop k-1}\right]$ and $|K_{k+1}^k(v)| = \left[{y-1 \atop k}\right]$. If $U \in \left[{V \atop n-1}\right]$ does not contain v, then $|L_U(v)| = d(v) = \left[{y-1 \atop k-1}\right]$. Hence,

$$\begin{bmatrix} y-1 \\ k \end{bmatrix} = |K_{k+1}^k(v)| \le |K_k^{k-1}(L_U(v))| \le \begin{bmatrix} y-1 \\ k \end{bmatrix},$$

which implies that $|K_k^{k-1}(L_U(v))| = \left[{y-1 \atop k}\right]$. Since $L_U(v) \subset \left[{V \atop k-1}\right]$, by the induction hypothesis, $L_U(v) = \left[{W \atop k-1}\right]$ for some $(y-1)$-dimensional space W, which implies $y \in \mathbb{Z}^+$.

Moreover, for every k-dimensional subspace A in $K_k^{k-1}(L_U(v)) = \begin{bmatrix} W \\ k \end{bmatrix}$, we have that $A \vee v$ is an element of $K_{k+1}^k(v)$. Hence all k-dimensional subspaces in $Y := W \vee v$ lie in $\mathscr{F}$. Since $|\mathscr{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$ and $\dim(Y) = y$, we must have $\mathscr{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$. □

5.2 Analog of Frankl's *r*-wise Intersection Theorem

In this section, we prove the bound in Theorem 5.0.14 and characterize equality when $(r-1)n > rk$. The proof proceeds by induction on $(r-1)n - rk \in \mathbb{N}$. For the base case $(r-1)n - rk = 0$, we generalize Greene and Kleitman's argument in Section 4.6.2.

Definition 5.2.1. *A family $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is <u>r-wise co-intersecting</u> if any r elements of $\mathscr{F}$ are contained in a common $(n-1)$-dimensional space.*

Suppose $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$ and let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be an r-wise intersecting family. Endow V with the usual inner product, and consider the family

$$\mathscr{F}^\perp := \{F^\perp : F \in \mathscr{F}\} \subset \begin{bmatrix} V \\ n-k \end{bmatrix}.$$

Let $\mathscr{B}$ be a geometric $(n-k)$-spread of V. We want to determine the maximum number of elements of $\mathscr{B}$ that lie in $\mathscr{F}^\perp$. Since $\mathscr{F}$ is r-wise intersecting, we have that $\mathscr{F}^\perp$ is r-wise co-intersecting. If $r = 2$ and $n = 2k$, the family $\mathscr{F}^\perp$ is both intersecting and co-intersecting; hence only one element of the spread $\mathscr{B}$ can lie in $\mathscr{F}^\perp$ in this case. Lemma 5.2.2 determines the maximum number of elements of $\mathscr{B}$ that lie in $\mathscr{F}^\perp$ whenever $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$.

Lemma 5.2.2. *Let $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$. Suppose that $\mathscr{B}$ is a geometric $(n-k)$-spread of V. If $\mathscr{B}' \subset \mathscr{B}$ is a r-wise co-intersecting subfamily, then*

$$|\mathscr{B}'| \le \frac{q^{(r-1)(n-k)} - 1}{q^{n-k} - 1}.$$

If equality holds, $\mathscr{B}'$ is a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space.

Proof. Let $B_1, \dots, B_m$ be a maximum subfamily of $\mathscr{B}'$ with $\dim\left(\bigvee_{i=1}^m B_i\right) = m(n-k)$. Hence, if $B \in \mathscr{B}'$ then $B \cap \bigvee_{i=1}^m B_i \neq \{0\}$. Since $\mathscr{B}$ is geometric, $\mathscr{B}$ induces a spread on

$\bigvee_{i=1}^{m} B_i$ by Lemma 4.5.5. As $B \cap \bigvee_{i=1}^{m} B_i \neq \{0\}$ for every B in $\mathscr{B}'$, all elements in $\mathscr{B}'$ lie in $\bigvee_{i=1}^{m} B_i$. Since $\mathscr{B}'$ is r-wise co-intersecting, we must have $m \leq r-1$. Therefore,

$$|\mathscr{B}'| \leq \frac{q^{(r-1)(n-k)}-1}{q^{n-k}-1},$$

which is the number of elements in a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space. Also, if equality holds, $\mathscr{B}'$ is a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space. □

Now we prove the base case of Theorem 5.0.14; the case $r = 2$ of Lemma 5.2.3 is the result of Greene and Kleitman that was presented in Section 4.6.2.

Lemma 5.2.3. *Suppose $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$. If $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is r-wise intersecting, then $|\mathscr{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.*

Proof. Let $\mathscr{B}$ be a geometric $(n-k)$-spread of V and let $\pi \in GL(V)$ be an isomorphism. By Lemma 4.5.7, the spread $\pi(\mathscr{B})$ is also geometric. Consider the family $\mathscr{F}^{\perp} \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$. Since $\mathscr{F}$ is r-wise intersecting, $\mathscr{F}^{\perp}$ is r-wise co-intersecting. By Lemma 5.2.2,

$$|\mathscr{F}^{\perp} \cap \pi(\mathscr{B})| \leq \frac{q^{(r-1)(n-k)}-1}{q^{n-k}-1} = \frac{q^{k}-1}{q^{n-k}-1} \tag{5.2.1}$$

because $\mathscr{F}^{\perp} \cap \pi(\mathscr{B})$ is a r-wise co-intersecting subfamily of $\pi(\mathscr{B})$ and because we have $k = (r-1)(n-k)$ when r, n, k satisfy $(r-1)n - rk = 0$.

By Lemma 4.6.2, we have $|GL(V)| = q^{n(n-1)/2}(q-1)^n[n]!$, so

$$\sum_{\pi \in GL(V)} |\mathscr{F}^{\perp} \cap \pi(\mathscr{B})| \leq \frac{q^{k}-1}{q^{n-k}-1} \cdot q^{n(n-1)/2}(q-1)^n[n]!.$$

Now, given $F^{\perp} \in \mathscr{F}^{\perp}$ and $B \in \mathscr{B}$ there are $q^{n(n-1)/2}(q-1)^n[n-k]![k]!$ isomorphisms $\pi \in GL(V)$ such that $\pi(B) = F^{\perp}$ by Lemma 4.6.3. Consequently,

$$\begin{aligned} & \left(\frac{q^{n}-1}{q^{n-k}-1}\right) |\mathscr{F}^{\perp}|\, q^{n(n-1)/2}(q-1)^n[n-k]![k]! \\ = \;& |\mathscr{B}|\, |\mathscr{F}^{\perp}|\, |\{\pi \in GL(V) : \pi(B) = F^{\perp}\}| \\ = \;& \sum_{\pi \in GL(V)} |\mathscr{F}^{\perp} \cap \pi(\mathscr{B})| \\ \leq \;& \frac{q^{k}-1}{q^{n-k}-1} \cdot q^{n(n-1)/2}(q-1)^n[n]!. \end{aligned}$$

Since $|\mathscr{F}| = |\mathscr{F}^{\perp}|$, we have

$$|\mathscr{F}| \leq \left(\frac{q^{n(n-1)/2}(q-1)^n[n]!}{q^{n(n-1)/2}(q-1)^n[n-k]![k]!}\right)\left(\frac{q^{n-k}-1}{q^n-1}\right)\left(\frac{q^k-1}{q^{n-k}-1}\right) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \qquad \square$$

Now we prove the bound in Theorem 5.0.14 and characterize equality when $(r-1)n > rk$.

Proof of Theorem 5.0.14. The proof proceeds by induction on $(r-1)n - rk \in \mathbb{N}$. The base case $(r-1)n - rk = 0$ was proved in Lemma 5.2.3. Suppose Theorem 5.0.14 holds when r, n, k satisfy $(r-1)n - rk = p$ for $p \geq 0$. We will prove Theorem 5.0.14 holds when r, n, k satisfy $(r-1)n - rk = p+1$. Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family. Now the family $\mathscr{P} := \{P \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset P\}$, where $v \subset V$ is some one-dimensional subspace, is r-wise intersecting so $|\mathscr{F}| \geq |\mathscr{P}| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Let W be an $(n+1)$-dimensional space over $\mathbb{F}_q$ that contains V. Define the family

$$\mathscr{A} := \left\{A \in \begin{bmatrix} W \\ k+1 \end{bmatrix} : \exists\, F \in \mathscr{F} \text{ with } F \subset A\right\}$$

to be the family of all $(k+1)$-dimensional spaces in W that contain some $F \in \mathscr{F}$. We will partition $\mathscr{A}$ into the following subfamilies:

$$\mathscr{A}_1 := \{A \in \mathscr{A} : A \not\subset V\}, \quad \mathscr{A}_2 := \mathscr{A} \setminus \mathscr{A}_1.$$

First let us compute the size of $\mathscr{A}_1$. Observe that if $A \in \begin{bmatrix} W \\ k+1 \end{bmatrix}$ and A does not lie in V, then A intersects V in exactly a k-dimensional space. Therefore, A cannot contain two distinct k-dimensional spaces in $\mathscr{F}$. Note that any $F \in \mathscr{F}$ can be extended to a $(k+1)$-dimensional space in $\mathscr{A}_1$ in q^{n-k} ways. Therefore, $|\mathscr{A}_1| = q^{n-k}|\mathscr{F}| \geq q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

Now we will compute the size of $\mathscr{A}_2$. By duality, we have $F \subset A \in \mathscr{A}_2$ for some $F \in \mathscr{F}$ if and only if $F^{\perp} \supset A^{\perp} \in \begin{bmatrix} V \\ n-k-1 \end{bmatrix}$. Therefore, $|\mathscr{A}_2| = \left|\partial \mathscr{F}^{\perp}\right|$. Since

$$|\mathscr{F}^{\perp}| = |\mathscr{F}| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}, \tag{5.2.2}$$

Theorem 5.0.12 yields

$$|\mathscr{A}_2| = \left|\partial \mathscr{F}^{\perp}\right| \geq \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{5.2.3}$$

As $\mathscr{A} = \mathscr{A}_1 \dot{\cup} \mathscr{A}_2$, we have by Lemma 4.3.1 that

$$|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_2| \geq q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{5.2.4}$$

Since $\mathscr{F}$ is r-wise intersecting, $\mathscr{A}$ is an r-wise intersecting family of $(k+1)$-dimensional spaces in W. Observe that $r, n+1, k+1$ satisfy

$$(r-1)(n+1) - r(k+1) = (r-1)n - rk - 1 = (p+1) - 1 = p.$$

By the induction hypothesis $|\mathscr{A}| \leq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$, which implies equality everywhere in (5.2.2), (5.2.3), and (5.2.4). Hence, $q^{n-k}|\mathscr{F}| = |\mathscr{A}_1| = q^{n-k}\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ and $|\mathscr{F}| = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$. Moreover, $|\mathscr{F}^\perp| = \left[\begin{smallmatrix} n-1 \\ n-k \end{smallmatrix}\right]$ and $\left|\partial \mathscr{F}^\perp\right| = |\mathscr{A}_2| = \left[\begin{smallmatrix} n-1 \\ n-k-1 \end{smallmatrix}\right]$. Therefore $\mathscr{F}^\perp$ satisfies equality in Theorem 5.0.12, which implies that $\mathscr{F}^\perp = \left[\begin{smallmatrix} Y \\ n-k \end{smallmatrix}\right]$ for some $(n-1)$-dimensional subspace $Y \subset V$. By duality, $\mathscr{F} = \{F \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right] : v \subset F\}$ for some one-dimensional subspace $v \subset V$. □

5.3 Characterizing Equality in the Base Case

We characterize equality in Theorem 5.0.14 when $(r-1)n - rk = 0$. Recall from Section 4.4 that Godsil and Newman characterized equality in the Erdős-Ko-Rado theorem for vector spaces using the methods of [60]. In particular, they showed

Theorem 5.3.1 (Godsil and Newman)**.** *If $n = 2k$ and $\mathscr{F} \subset \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ is an intersecting family of maximum size, then $\mathscr{F} = \{F \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right] : v \subset F\}$ for some one-dimensional subspace $v \subset V$ or $\mathscr{F} = \left[\begin{smallmatrix} U \\ k \end{smallmatrix}\right]$ for some $(2k-1)$-dimensional subspace $U \subset V$.*

We use their result to characterize equality in Theorem 5.0.14 when $r \geq 3$ and $(r-1)n - rk = 0$. The proof proceeds by induction on r; the base case $r = 2$ and $n = 2k$ is Theorem 5.3.1. Let $\mathscr{F} \subset \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ be a maximum size r-wise intersecting family. In this section, it will be more natural to state results in terms of $\mathscr{F}^\perp \subset \left[\begin{smallmatrix} V \\ n-k \end{smallmatrix}\right]$ so we make the following simple observation.

Lemma 5.3.2. *We have $\mathscr{F} \subset \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ is a maximum size r-wise intersecting family if and only if $\mathscr{F}^\perp \subset \left[\begin{smallmatrix} V \\ n-k \end{smallmatrix}\right]$ is a maximum size r-wise co-intersecting family.* □

Lemma 5.3.5 allows us to use induction. We first state two simple corollaries of Lemma 5.2.3 that will be used in the proof of Lemma 5.3.5. Since r,n,k satisfy $(r-1)n-rk=0$, note that V is $r(n-k)$-dimensional .

Corollary 5.3.3. *Suppose r,n,k satisfy $(r-1)n-rk=0$ and that $\mathscr{F}\subset\begin{bmatrix}V\\k\end{bmatrix}$ is an r-wise intersecting family. If there is a geometric $(n-k)$-spread $\mathscr{B}$ of V such that equality holds in* (5.2.1) *for all $\pi\in GL(V)$, then $\mathscr{F}$ has maximum size.* □

Corollary 5.3.4. *Suppose r,n,k satisfy $(r-1)n-rk=0$. If $\mathscr{F}\subset\begin{bmatrix}V\\k\end{bmatrix}$ is a maximum size r-wise intersecting family, then equality holds in* (5.2.1) *for every $\pi\in GL(V)$ and every geometric $(n-k)$-spread $\mathscr{B}$ of V.* □

Lemma 5.3.5. *Let $\mathscr{F}\subset\begin{bmatrix}V\\k\end{bmatrix}$ be a maximum size r-wise intersecting family. Fix $F^\perp$ in $\mathscr{F}^\perp$ and let $U\subset V$ be an $(r-1)(n-k)$-dimensional space that intersects $F^\perp$ trivially; that is $F^\perp\cap U=\{0\}$. Then*

$$\mathscr{F}^\perp|_U := \{E\in\mathscr{F}^\perp : E\subset U\}$$

is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix}U\\n-k\end{bmatrix}$.

Proof. Let $\mathscr{S}$ be a geometric $(n-k)$-spread of V. Choose $S_1,\dots,S_r$ in $\mathscr{S}$ such that $\bigvee_{i=1}^r S_i=V$. Since $F^\perp\cap U=\{0\}$, there exists an isomorphism $\rho\in GL(V)$ such that $\rho(S_1)=F^\perp$ and $\rho\left(\bigvee_{i=2}^r S_i\right)=U$. The $(n-k)$-spread $\mathscr{B}:=\rho(\mathscr{S})$ is geometric by Lemma 4.5.7, and $F^\perp\in\mathscr{B}$. Moreover $U=\bigvee_{i=2}^r\rho(S_i)$ so, by Lemma 4.5.5, we have that $\mathscr{B}$ induces a geometric $(n-k)$-spread $\mathscr{B}'$ on U.

Observe that $\mathscr{F}^\perp|_U$ is $(r-1)$-wise co-intersecting since $F^\perp\cap U=\{0\}$. To prove that $\mathscr{F}^\perp|_U\subset\begin{bmatrix}U\\n-k\end{bmatrix}$ is a maximum size $(r-1)$-wise co-intersecting family, we will apply Lemma 5.3.2 and Corollary 5.3.3. That is, we will show that if $\alpha\in GL(U)$ then equality holds in (5.2.1):

$$\left|\mathscr{F}^\perp|_U\cap\alpha(\mathscr{B}')\right| = \frac{q^{(r-2)(n-k)}-1}{q^{n-k}-1}.$$

Let $\pi\in GL(V)$ be an invertible linear transformation such that $\pi(F^\perp)=F^\perp$, $\pi(U)=U$, and $\pi|_U=\alpha$. Since $\mathscr{F}^\perp$ is a maximum size r-wise co-intersecting family, $\mathscr{F}^\perp\cap\pi(\mathscr{B})$ is an $(n-k)$-spread of an $(r-1)(n-k)$-dimensional space W_π by

Lemma 5.2.2 and Corollary 5.3.4. We have that $F^\perp$ is contained in W_π and intersects U trivially so $\dim(W_\pi \cap U) = (r-2)(n-k)$.

The spread $\pi(\mathscr{B})$ induces the spread $\mathscr{F}^\perp \cap \pi(\mathscr{B})$ on W_π and induces the spread $\alpha(\mathscr{B}')$ on U. Consider the elements of $\alpha(\mathscr{B}')$ that intersect $W_\pi \cap U$ nontrivially; as these elements are in $\pi(\mathscr{B})$ and intersect W_π, they must lie in W_π and hence in $W_\pi \cap U$. Hence, the elements of $\alpha(\mathscr{B}')$ that intersect $W_\pi \cap U$ nontrivially form a spread of $W_\pi \cap U$. Moreover, these elements lie in $\mathscr{F}^\perp \cap \pi(\mathscr{B})$ so

$$\mathscr{F}^\perp|_U \cap \alpha(\mathscr{B}') = (\mathscr{F}^\perp \cap \pi(\mathscr{B})) \cap \alpha(\mathscr{B}')$$

is the spread $\pi(\mathscr{B})$ induces on $W_\pi \cap U$. Since $W_\pi \cap U$ is $(r-2)(n-k)$-dimensional, $\left|\mathscr{F}^\perp|_U \cap \alpha(\mathscr{B}')\right|$ satisfies (5.2.1) with equality. By Lemma 5.3.2 and Corollary 5.3.3, $\mathscr{F}^\perp|_U$ is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$. □

<u>Characterizing Equality in Theorem 5.0.14 when $(r-1)n - rk = 0$ and $r \geq 3$</u>: We now characterize equality in Theorem 5.0.14 when $(r-1)n - rk = 0$ and $r \geq 3$. The proof proceeds by induction on r; the base case $r = 2$ and $n = 2k$ is Theorem 5.3.1.

Let $r \geq 3$ and suppose the statement is proved for any $2 \leq r' < r$. Let $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family and observe that $\mathscr{F}^\perp \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$ is a maximum size r-wise co-intersecting family. Our objective is to show that $\mathscr{F}^\perp = \begin{bmatrix} H \\ n-k \end{bmatrix}$, where H is a $(n-1)$-dimensional space of V. By duality, this implies that $\mathscr{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$, which is the desired conclusion.

Fix some $F^\perp \in \mathscr{F}^\perp$. By Lemma 5.3.5, if U is a $(r-1)(n-k)$-dimensional subspace that intersects $F^\perp$ trivially, then $\mathscr{F}^\perp|_U$ is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$. When $r = 3$, then $\dim U = 2(n-k)$ and $\mathscr{F}^\perp|_U$ is a maximum size intersecting and co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$; hence by Theorem 5.3.1

1. $\mathscr{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$ or
2. $\mathscr{F}^\perp|_U = \begin{bmatrix} U' \\ n-k \end{bmatrix}$ for some $(2(n-k)-1)$-dimensional subspace $U' \subset U$.

If $r > 3$ then, by the induction hypothesis and duality, $\mathscr{F}^\perp|_U = \begin{bmatrix} U' \\ n-k \end{bmatrix}$, where $U' \subset U$ is some $((r-1)(n-k)-1)$-dimensional subspace.

Our first task is to eliminate the possibility that $\mathscr{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$ in the case $r = 3$. We now show that if

$\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then every element of $\mathscr{F}^\perp$ must intersect $F^\perp \vee u$ nontrivially.

Claim 5.3.6. *If $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then for all $G \in \mathscr{F}^\perp$ we have $G \cap (F^\perp \vee u) \neq \{0\}$.*

Proof. Suppose, for a contradiction, that there exists $G \in \mathscr{F}^\perp$ such that G intersects $F^\perp \vee u$ trivially. We have $\dim((F^\perp \vee G) \cap U) = n-k$ because $F^\perp$ intersects both G and U trivially. Since u does not lie in $F^\perp \vee G$ and $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$, we can find $E' \in \mathscr{F}^\perp|_U$ that intersects $F^\perp \vee G$ trivially. Hence $F^\perp \vee G \vee E' = V$, which contradicts the fact that $\mathscr{F}^\perp$ is 3-wise co-intersecting. □

We now show that if $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then any $(n-k)$-dimensional space that meets $F^\perp$ trivially but meets $F^\perp \vee u$ nontrivially must lie in $\mathscr{F}^\perp$.

Claim 5.3.7. *Suppose $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$. If $G \in \left[{V \atop n-k}\right]$, $G \cap F^\perp = \{0\}$, and $G \cap (F^\perp \vee u) \neq \{0\}$, then $G \in \mathscr{F}^\perp$.*

Proof. There exists a geometric $(n-k)$-spread $\mathscr{B}$ of V that contains both G and $F^\perp$ because $G \cap F^\perp = \{0\}$. As $\mathscr{B}$ is a spread, all subspaces in $(\mathscr{F}^\perp \cap \mathscr{B}) \setminus \{F^\perp\}$ meet $F^\perp \vee u$ in a one-dimensional subspace that does not lie in $F^\perp$ by Claim 5.3.6. Lemma 5.2.2 and Corollary 5.3.4 imply that $\mathscr{F}^\perp \cap \mathscr{B}$ is a spread of a $2(n-k)$-dimensional space so $|(\mathscr{F}^\perp \cap \mathscr{B}) \setminus \{F^\perp\}| = q^{n-k}$. There are q^{n-k} one-dimensional subspaces in $F^\perp \vee u$ that do not lie in $F^\perp$. Hence, each one-dimensional subspace in $(F^\perp \vee u) \setminus F^\perp$ meets a unique subspace in $(\mathscr{F}^\perp \cap \mathscr{B}) \setminus \{F^\perp\}$. Since G meets $F^\perp \vee u$ in a one-dimensional subspace that does not lie in $F^\perp$ and $G \in \mathscr{B}$, we must have $G \in \mathscr{F}^\perp \cap \mathscr{B} \subset \mathscr{F}^\perp$. □

We now eliminate the possibility that $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$. We will construct three $(n-k)$-dimensional subspaces that together span V, and intersect $F^\perp \vee u$ in a one-dimensional subspace not lying in $F^\perp$. By Claim 5.3.7, these three spaces lie in $\mathscr{F}^\perp$, which contradicts $\mathscr{F}^\perp$ being 3-wise co-intersecting. To build these three subspaces, we first choose three one-dimensional subspaces v_1^1, v_2^1, v_3^1 in $(F^\perp \vee u) \setminus F^\perp$ such that $v_3^1 \not\subset v_1^1 \vee v_2^1$. These one-dimensional subspaces exist because $\dim(F^\perp \vee u) = (n-k)+1 \geq 3$ so, after picking v_1^1 and v_2^1, any one-dimensional subspace of $F^\perp \vee u$ not in $F^\perp \cup (v_1^1 \vee v_2^1)$ will do. Since the number of one-dimensional subspaces in $(F^\perp \vee u) \setminus (F^\perp \cup (v_1^1 \vee v_2^1))$ is $q^{n-k} - q > 0$, we can indeed choose v_3^1.

We construct a family of one-dimensional subspaces

$$\{v_i^j : i \in \{1,2,3\}, j \in \{1,\ldots,n-k\}\}$$

such that, for each $i \in \{1,2,3\}$, the subspace $V_i = \bigvee_{j=1}^{n-k} v_i^j$ intersects $F^\perp \vee u$ in the one-dimensional subspace $v_i^1 \not\subset F^\perp$ and $\bigvee_{i=1}^{3} V_i = V$. The subspaces V_1, V_2, V_3 are the desired three $(n-k)$-dimensional subspaces. We pick the one-dimensional subspaces one after the other; we have to show that at each step there is a possible one-dimensional subspace to pick. When picking the last one-dimensional subspace v_3^{n-k} we must choose a one-dimensional subspace from V that is not in $V_1 \vee V_2 \vee \bigvee_{j=1}^{n-k-1} v_3^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_3^j$. By inclusion-exclusion, there are $q^{3(n-k)-1} - q^{2(n-k)-2} > 0$ one-dimensional subspaces in V that do not lie in $V_1 \vee V_2 \vee \bigvee_{j=1}^{n-k-1} v_3^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_3^j$; thus it is indeed possible to construct the desired three $(n-k)$-dimensional subspaces. Therefore, we have eliminated the possibility that $\mathscr{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$ in the case $r = 3$.

We may now assume that $r \geq 3$ and that if U is a $(r-1)(n-k)$-dimensional space that intersects $F^\perp$ trivially then $\mathscr{F}^\perp|_U = \left[{U' \atop n-k}\right]$ for some $((r-1)(n-k)-1)$-dimensional subspace $U' \subset U$. Our ultimate goal is to prove that $\mathscr{F}^\perp = \left[{F^\perp \vee U' \atop n-k}\right]$. Naturally, we first show that if U_1, U_2 are two $(r-1)(n-k)$-dimensional subspaces that intersect $F^\perp$ trivially, then $F^\perp \vee U_1' = F^\perp \vee U_2'$.

Claim 5.3.8. *Let U_1, U_2 be two $(r-1)(n-k)$-dimensional subspaces of V that intersect $F^\perp$ trivially. Let U_1', U_2' be the $((r-1)(n-k)-1)$-dimensional subspaces of U_1 and U_2 such that $\mathscr{F}^\perp|_{U_1} = \left[{U_1' \atop n-k}\right]$ and $\mathscr{F}^\perp|_{U_2} = \left[{U_2' \atop n-k}\right]$. Then $F^\perp \vee U_1' = F^\perp \vee U_2'$.*

Proof. Suppose, for a contradiction, that $F^\perp \vee U_1' \neq F^\perp \vee U_2'$. We choose subspaces $W_1, \ldots, W_{r-2}$ in $\left[{U_1' \atop n-k}\right]$ such that $\dim\left(\bigvee_{i=1}^{r-2} W_i\right) = (r-2)(n-k)$ and W_1 is not contained in $F^\perp \vee U_2'$.

The subspace $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ is $(r-1)(n-k)$-dimensional because U_1 intersects $F^\perp$ trivially. The subspace U_2' is $((r-1)(n-k)-1)$-dimensional and intersects $F^\perp$

trivially so

$$(r-2)(n-k)-1 \le \dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) \le (r-2)(n-k).$$

Suppose that $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k)$ for a contradiction. By definition of W_1, we can choose a one-dimensional subspace $w \subset W_1$ that does not lie in $F^\perp \vee U_2'$. The subspace $F^\perp \vee w$ is $(n-k+1)$-dimensional. The subspace $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ is $(r-1)(n-k)$-dimensional and contains $F^\perp \vee w$. Note that $F^\perp \vee w$ must intersect U_2' nontrivially if $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k)$. This is a contradiction because w does not lie in $F^\perp \vee U_2'$ by construction. We therefore conclude that $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k)-1$.

Since U_2' is $((r-1)(n-k)-1)$-dimensional, this implies that there exists a subspace Z in $\begin{bmatrix} U_2' \\ n-k \end{bmatrix}$ that intersects $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ trivially. Now $F^\perp, W_1, \ldots, W_{r-2}, Z$ lie in $\mathscr{F}^\perp$ since $\mathscr{F}^\perp|_{U_1} = \begin{bmatrix} U_1' \\ n-k \end{bmatrix}$ and $\mathscr{F}^\perp|_{U_2} = \begin{bmatrix} U_2' \\ n-k \end{bmatrix}$. By construction, $F^\perp \vee \bigvee_{i=1}^{r-2} W_i \vee Z = V$, which contradicts $\mathscr{F}^\perp$ being r-wise co-intersecting. This proves $F^\perp \vee U_1' = F^\perp \vee U_2'$. □

Now we show that any $(n-k)$-dimensional subspace in $F^\perp \vee U'$ that intersects $F^\perp$ trivially must lie in $\mathscr{F}^\perp$.

Claim 5.3.9. *If $G \in \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$ and $G \cap F^\perp = \{0\}$, then $G \in \mathscr{F}^\perp$.*

Proof. Since $G \cap F^\perp = \{0\}$, there exists a $(r-1)(n-k)$-dimensional subspace $U(G)$ that contains G and intersects $F^\perp$ trivially. Let $U(G)'$ be the $((r-1)(n-k)-1)$-dimensional subspace of $U(G)$ such that $\mathscr{F}^\perp|_{U(G)} = \begin{bmatrix} U(G)' \\ n-k \end{bmatrix}$. By Claim 5.3.8,

$$G \subset (F^\perp \vee U') \cap U(G) = (F^\perp \vee U(G)') \cap U(G) = U(G)'.$$

Hence $G \in \begin{bmatrix} U(G)' \\ n-k \end{bmatrix} \subset \mathscr{F}^\perp$. □

Now we are ready to prove $\mathscr{F}^\perp = \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. Suppose, for a contradiction, that there exists a subspace $H \in \mathscr{F}^\perp$ that is not in $\begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. We will construct $r-1$ subspaces in $\begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$ that each intersect $F^\perp$ trivially and that together with H span V. By Claim 5.3.9, these $r-1$ subspaces lie in $\mathscr{F}^\perp$ which contradicts $\mathscr{F}^\perp$ being r-wise co-intersecting.

To build these $r-1$ subspaces, we construct a family of one-dimensional subspaces

$$\{v_i^j : i \in \{1,\dots,r-1\}, j \in \{1,\dots,n-k\}\}$$

such that for each $i \in \{1,\dots,r-1\}$, the subspace $G_i = \bigvee_{j=1}^{n-k} v_i^j$ lies in $F^\perp \vee U'$, intersects $F^\perp$ trivially, and $\bigvee_{i=1}^{r-1} G_i \vee H = V$. The subspaces $G_1,\dots,G_{r-1}$ are the desired $r-1$ subspaces. We pick the one-dimensional subspaces one after the other; we have to show that at each step there is a possible one-dimensional subspace to pick. When picking the last one-dimensional subspace v_{r-1}^{n-k} we must pick a one-dimensional subspace from $F^\perp \vee U'$ that is not in $H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$. Since H is not contained in $F^\perp \vee U'$, we have

$$\dim\left(\left(F^\perp \vee U'\right) \cap \left(H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j\right)\right) = r(n-k)-2.$$

Hence, there are at least

$$q^{r(n-k)-2} - (q^{2(n-k)-2} + q^{2(n-k)-3} + \cdots + 1) > 0$$

one-dimensional subspaces of $F^\perp \vee U'$ that do not lie in $H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$; thus it is indeed possible to construct the desired $r-1$ subspaces. This proves that $\mathscr{F}^\perp \subseteq \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$, and since $|\mathscr{F}^\perp| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ we have $\mathscr{F}^\perp = \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. The subspace $F^\perp \vee U'$ is $(n-1)$-dimensional; by duality, $\mathscr{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$, which is the desired conclusion. $\square$

5.4 Other Results of the Author

In this section, we discuss the author's other results in this area, namely obtaining an analog of the Hilton-Milner theorem and determining the chromatic number of the q-Kneser graphs. The results will be stated but not proved.

5.4.1 Analog of the Hilton-Milner Theorem

Recall that Hilton's and Milner's Theorem 2.2.13 is an extension of the Erdős-Ko-Rado theorem, Theorem 2.2.2, that gives the size of the largest nontrivial intersecting family $\mathscr{F} \subset \binom{X}{k}$. The Hilton-Milner theorem was first proved by Hilton and Milner [70] using lengthy combinatorial arguments; later Frankl and Füredi produced an elegant proof using the shifting technique. Due to the issues discussed in Section 4.6.1, none of the known proofs of the Hilton-Milner theorem readily generalize to vector spaces.

We will see that the similarity between the Hilton-Milner theorem and its vector space analog, Theorem 5.4.1, is remarkable. We stress, however, that the naive analog of the extremal family (i) in Theorem 2.2.13 is *not* maximally intersecting: If $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ is a one-dimensional subspace and $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ is a k-dimensional subspace that does not contain E, then the family

$$\{U\} \cup \left\{ W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \subset W,\ \dim(W \cap U) \geq 1 \right\} \tag{5.4.1}$$

is not maximally intersecting, as we can add all subspaces in $\begin{bmatrix} E \vee U \\ k \end{bmatrix}$ that are not in (5.4.1). We will say that $\mathscr{F}$ is an *HM-type family* if

$$\mathscr{F} = \left\{ W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \subset W,\ \dim(W \cap U) \geq 1 \right\} \cup \begin{bmatrix} E \vee U \\ k \end{bmatrix}$$

for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ with $E \not\subset U$.

Theorem 5.4.1 (Blokhuis-Brouwer-Chowdhury-Frankl-Mussche-Patkós-Szőnyi)**.** *For $k \geq 3$, suppose $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$. If $\mathscr{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is an intersecting family and there does not exist $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ such that $\mathscr{F} \subset \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$, then*

$$|\mathscr{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k.$$

Equality holds if and only if

(i) $\mathscr{F}$ is an HM-type family,

(ii) $\mathscr{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(S \cap F) \geq 2\}$ for some $S \in \begin{bmatrix} V \\ 2 \end{bmatrix}$ if $k = 3$.

Variants of Theorem 5.4.1 are used to establish our next result on coloring the q-Kneser graphs.

5.4.2 Coloring the q-Kneser Graph

We discuss the question of coloring the q-Kneser graphs. First, we define what is meant by a graph coloring.

Definition 5.4.2. *Given a graph $G = (V,E)$ a proper coloring of G with l colors is a map from V to a set of colors with cardinality l such that no two adjacent vertices receive the same color.*

If V is finite, then clearly G can be properly colored with $|V|$ colors. We will be interested in the least number of colors required to properly color G.

Definition 5.4.3. *Given a graph $G = (V,E)$, the chromatic number of G, denoted $\chi(G)$, is the least integer $l \in \mathbb{Z}^+$ such that G has a proper coloring with l colors.*

Observe that a set of vertices that receives the same color in a proper coloring must be an independent set. Hence, the chromatic number of a graph is the minimum number of independent sets needed to partition its vertex set V.

Determining the chromatic number of a graph can be very difficult. In 1955, Kneser [78] conjectured that, when $n \geq 2k$, the chromatic number of the Kneser graph $K_{n:k}$ is $\chi(qK_{n:k}) = n-2k+2$. The problem remained open for twenty three years until Lovász [83] and Bárány [9] found proofs that surprisingly use algebraic topology.

Theorem 5.4.4 (Bárány, Lovász)**.** *If $n \geq 2k$, the chromatic number of the Kneser graph $K_{n:k}$ is $\chi(K_{n:k}) = n-2k+2$.*

We can easily see that $n-2k+2$ colors suffice to properly color the Kneser graph $K_{n:k}$. If α is a k-subset of $K_{n:k}$ and its largest element is greater than $2k$, define this element to be the color of α. Thus, the k-subsets not contained in $\{1,\dots,2k\}$ can be colored with $n-2k$ colors. The k-subsets not already colored induce a copy of $K_{2k:k}$, which is bipartite, so the remaining k-subsets can be colored with two colors. This proves that $\chi(K_{n:k}) \leq n-2k+2$.

Kneser's conjecture motivates the question of coloring the q-Kneser graphs. Not surprisingly, the known proofs of Theorem 5.4.4 do not generalize to the vector space setting. We have seen in Section 4.4 that many parameters of the q-Kneser graph reduce to the corresponding parameters of the Kneser graph by setting $q = 1$. Interestingly, we will see that the chromatic number is *not* one of these parameters. The author, Godsil, and Royle [26] determined the chromatic number of the q-Kneser graphs $qK_{n:2}$ when $n \geq 4$, and characterized the minimum colorings.

Theorem 5.4.5 (Chowdhury-Godsil-Royle)**.** *The chromatic number of the q-Kneser graph* $qK_{4:2}$ *is* $\chi(qK_{4:2}) = q^2 + q$*. If* $n > 4$*, then the chromatic number of the q-Kneser graph* $qK_{n:2}$ *is* $\chi(qK_{n:2}) = [n-1]$.

We see that the q-Kneser graph $qK_{5:2}$ has chromatic number $\chi(qK_{5:2}) = [4]$ by Theorem 5.4.5. The Kneser graph, $K_{5:2}$, also known as the Petersen graph, has chromatic number $\chi(K_{5:2}) = 3$, however. The relationship between the chromatic numbers of the Kneser graph and the q-Kneser graph is thus more complex than setting $q = 1$.

Using variants of Theorem 5.4.1, we can determine the chromatic number of the q-Kneser graphs $qK_{n:k}$ for $k \geq 3$, and characterize their minimum colorings.

Theorem 5.4.6 (Blokhuis-Brouwer-Chowdhury-Frankl-Mussche-Patkós-Szőnyi)**.** *Suppose that* $k \geq 3$*, and that either* $q \geq 3$ *and* $n \geq 2k+1$*, or* $q = 2$ *and* $n \geq 2k+2$*. Then the chromatic number of the q-Kneser graph* $qK_{n:k}$ *is* $\chi(qK_{n:k}) = [n-k+1]$.

Some of the remaining cases have been settled; see [15].

5.5 Open Problems

As extremal set theory questions usually have natural vector space analogs, many open problems in this area remain. We discuss three of our favorites here: the analog of Mubayi's and Verstraëte's Theorem 2.2.10, the analog of Baranyai's Theorem 3.4.33, and the question of determining the largest clique in the q-Kneser graph. The theorems in Chapter 4 and Chapter 5 seem to suggest that the obvious analogs of extremal set theory results should be true in vector spaces, even if their proofs don't readily generalize. The last question we discuss, that of determining the largest clique in the q-Kneser graph, shows that this is not always true!

5.5.1 Forbidding Triangles

Recall that Theorem 2.2.10 asserted that if $\mathscr{F} \subset \binom{X}{k}$ contains no triangles and $n \geq 3k/2$, then $|\mathscr{F}| \leq \binom{n-1}{k-1}$ with equality if and only if there exists an $x \in X$ for which $\mathscr{F} = \{F \in \mathscr{F} : x \in F\}$. We saw that if $3k/2 \leq n < 2k$, then a family $\mathscr{F} \subset \binom{X}{k}$ without triangles is necessarily 3-wise intersecting, and so the problem is solved by Frankl's r-wise intersection theorem, Theorem 2.2.7, in this case.

We can naturally ask for the largest family $\mathscr{F} \subset \left[{V \atop k}\right]$ of k-dimensional subspaces of V that does not contain a triangle.

Definition 5.5.1. *A* <u>*triangle*</u> $\mathscr{T} = \{A,B,C\} \subset \left[{V \atop k}\right]$ *is a family consisting of three sets* A,B,C *such that* $A \cap B, A \cap C, B \cap C \neq \{0\}$ *but* $A \cap B \cap C = \{0\}$.

Question 5.5.2. *What is the maximum size of a family* $\mathscr{F} \subset \left[{V \atop k}\right]$ *that does not contain a triangle? What are the extremal families?*

As in Section 2.2.7, this question is uninteresting unless $n \geq 3k/2$. Also, if $3k/2 \leq n < 2k$, then a family $\mathscr{F} \subset \left[{V \atop k}\right]$ that contains no triangle must be 3-wise intersecting. Hence, the author's and Patkós's analog of Frankl's theorem, Theorem 5.0.14, proves that, in this case, $|\mathscr{F}| \leq \left[{n-1 \atop k-1}\right]$ with equality if and only if $\mathscr{F} = \{F \in \left[{V \atop k}\right] : v \subset F\}$ for some $v \in \left[{V \atop 1}\right]$. The proof of Theorem 2.2.10 does not generalize to vector spaces because of the issues discussed in Section 4.6.1.2.

5.5.2 Baranyai's Theorem

Recall that Baranyai's Theorem 3.4.33, proved that if $k|n$, then the complete hypergraph $\binom{X}{k}$ can be partitioned into $\binom{n-1}{k-1}$ perfect matchings. Is the same true for vector spaces? We saw that the vector space analogs of perfect matchings are spreads. As the number of k-dimensional spaces in $\left[{V \atop k}\right]$ and in a spread are $\left[{n \atop k}\right]$ and $[n]/[k]$ respectively, the number of spreads needed to partition $\left[{V \atop k}\right]$ is $\left[{n-1 \atop k-1}\right]$.

Question 5.5.3. *If* $k|n$, *can* $\left[{V \atop k}\right]$ *be partitioned into* $\left[{n-1 \atop k-1}\right]$ *spreads?*

This question is surprisingly difficult, even in the case $k = 2$. We remark that a partition of $\left[{V \atop k}\right]$ into k-dimensional subspaces is called a *parallelism* by finite geometers.

Beutelspacher [11] has constructed parallelisms when $k = 2$ and $n = 2^l$ is a power of two. Baker [7, 8] answered Question 5.5.3 affirmatively when $k = 2$ and V is a vector space over $\mathbb{F}_2$. No constructions of parallelisms for $k > 2$ were known until very recently [107].

5.5.3 Cliques in the q-Kneser Graph

We saw in Section 3.3.1 that an independent set in a graph is a set of vertices S such that no two vertices in S are adjacent. In this section, we will be concerned with cliques, which are the exact opposites of independent sets.

Definition 5.5.4. *Given a graph $G = (V,E)$, a clique is a set of vertices C such that any two distinct vertices in C are adjacent.*

In Section 3.3.2 and Section 4.4, we were interested in determining the largest independent sets of the Kneser and q-Kneser graphs respectively. A clique in the Kneser graph $K_{n:k}$ corresponds to a family of k-subsets of X that are pairwise disjoint. Hence, the largest clique in the Kneser graph $K_{n:k}$ has size $\lfloor n/k \rfloor$. The corresponding quantity for the q-Kneser graph is not as easily determined.

Question 5.5.5. *What is the size and structure of the largest clique in the q-Kneser graph $qK_{n:k}$?*

A clique in the q-Kneser graph $qK_{n:k}$ is a family of k-dimensional spaces of V that pairwise intersect in the zero subspace. Clearly, when $k|n$, a spread is the largest clique in the q-Kneser graph $qK_{n:k}$ and has size $[n]/[k]$. Hence, for $k = 2$, all that remains to solve Question 5.5.5 is the case of odd n.

Theorem 5.5.6 (Beutelspacher)**.** *If n is odd, the largest clique in the q-Kneser graph $qK_{n:2}$ has size*

$$\frac{q^n - q}{q^2 - 1} - (q-1).$$

Eisfeld and Storme [42] conjectured that if $n = ck + r$, where $1 \leq r \leq k-1$, then the largest clique in the q-Kneser graph $qK_{n:k}$ has size

$$\frac{q^n - q^r}{q^k - 1} - (q^r - 1). \tag{5.5.1}$$

Note that Eisfeld and Storme's conjecture (5.5.1) reduces to Theorem 5.5.6 in the case $k = 2$. Very recently, El-Zanati et al. [43] disproved Eisfeld's and Storme's conjecture. They showed that when $q = 2$, $n = 3c+2$, and $n \geq 8$, the largest clique in the q-Kneser graph $qK_{n:3}$ has size

$$\frac{2^n - 2^2}{2^3 - 1} - 2 > \frac{2^n - 2^2}{2^3 - 1} - (2^2 - 1),$$

which is larger than the conjectured maximum in (5.5.1).

Chapter 5, in part, is a reprint of the material as it appears in “Shadows and intersections in vector spaces," 2010. Chowdhury, Ameera; Patkós, Balázs. *J. Combin. Theory Ser. A*, 117(8):1095–1106, 2010. The dissertation author was the primary investigator and author of this paper.

Chapter 6

On a Conjecture of Frankl and Füredi

Recall that Fisher's Inequality, Theorem 2.3.9, states that a λ-intersecting family $\mathscr{F} \subset 2^X$ of size m has a 1-shadow of size at least m. Frankl and Füredi [55] conjectured a similar inequality, Conjecture 2.3.10, for the 2-shadows of nontrivial λ-intersecting families. Their conjecture asserts that if $\mathscr{F} \subset 2^X$ is a nontrivial λ-intersecting family of size m, then the number of pairs $\{x,y\} \in \binom{X}{2}$ that are contained in some $F \in \mathscr{F}$ is at least $\binom{m}{2}$. Conjecture 2.3.10 generalizes Fisher's Inequality and easily implies it since $\binom{|\partial^1 \mathscr{F}|}{2} \geq |\partial^2 \mathscr{F}| \geq \binom{m}{2}$ proves $|\partial^1 \mathscr{F}| \geq m$.

One important difference between Fisher's Inequality and the Frankl-Füredi conjecture is that the latter has a nontriviality restriction. Unfortunately, this condition is necessary, and sunflowers are an example of a trivial family for which the Frankl-Füredi conjecture is not valid.

Definition 6.0.7. *A family $\mathscr{F} \subset 2^X$ is a* <u>*sunflower*</u> *if* $\deg(x) \in \{1, |\mathscr{F}|\}$ *for all* $x \in X$.

If $\mathscr{F} \subset \binom{X}{k}$ is a λ-intersecting sunflower of size m, then $|\partial^2 \mathscr{F}| \leq m\binom{k}{2} < \binom{m}{2}$ when $m > k(k-1)+1$. We note that Conjecture 2.3.10 is equivalent to the seemingly stronger statement that if $\mathscr{F} \subset 2^X$ is a λ-intersecting family of size m that is not a sunflower, then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$. Fisher's inequality and its variants are usually proved by the linear independence arguments discussed in Chapter 3. One difficulty in proving Conjecture 2.3.10 in this way is understanding how to interpret the nontriviality restriction in a linear algebra setting.

6.1 Results

Frankl and Füredi [55] verified Conjecture 2.3.10 when $\lambda = 1$. The author's work [25] appears to be the first to consider Conjecture 2.3.10 since [55] was published over twenty years ago. Several special cases of Conjecture 2.3.10 had already been proved, however, before [55] was published. For example, Ryser [99], Woodall [117], and Babai [3] showed Conjecture 2.3.10 is true when $m = n$. Majindar [86] proved Conjecture 2.3.10 for regular λ-intersecting families.

The author's main results verify the Frankl-Füredi conjecture in some special cases. We first show that their conjecture holds for nontrivial λ-intersecting families that satisfy a reasonable extra condition and characterize the extremal families. We then apply this result to verify the Frankl-Füredi conjecture when $\mathscr{F}$ is additionally required to be uniform and λ is small. More precisely, we prove the following theorems.

Theorem 6.1.1 (Chowdhury). *Let $\mathscr{F} \subset 2^X$ be a λ-intersecting family of size m. If $\mathscr{F}$ satisfies*

$$\sum_{F \in \mathscr{F}} \binom{|F|}{2} \geq \sum_{x \in X} \binom{\deg(x)}{2} = \lambda \binom{m}{2}, \tag{6.1.1}$$

then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$. Moreover, if $\lambda \geq 2$ and $\mathscr{F} \subset \binom{X}{k}$ is also k-uniform, then we have $|\partial^2 \mathscr{F}| = \binom{m}{2}$ if and only if $\mathscr{F}$ is a symmetric design.

Note that if $\mathscr{F} \subset \binom{X}{k}$ is a λ-intersecting family of size m, then (6.1.1) is equivalent to

$$m \leq \frac{k(k-1)}{\lambda} + 1. \tag{6.1.2}$$

Theorem 6.1.2 (Chowdhury). *Let $\mathscr{F} \subset \binom{X}{k}$ be a nontrivial λ-intersecting family of size m.*

(i) If $\lambda = 2$, then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$ and equality holds if and only if $\mathscr{F}$ is a symmetric design.

(ii) If $\lambda = 3$ and $k \notin \{8, 11\}$, then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$ and equality holds if and only if $\mathscr{F}$ is a symmetric design.

6.2 Old and New Conjectures

In light of (6.1.2), it is interesting to note that Stanton and Mullin [104] once conjectured that if $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial λ-intersecting family of size m, then (6.1.2) holds. Had this conjecture been true, Theorem 6.1.1 would have implied that Conjecture 2.3.10 is true for uniform families as well as characterized the case of equality. Unfortunately, Hall [68] proved that Stanton and Mullin's conjecture is true only for $\lambda \in \{1,2\}$ and produced counterexamples for every $\lambda \geq 3$.

Since (6.1.1) and (6.1.2) are equivalent for uniform families, Hall's proof of Stanton and Mullin's conjecture for $\lambda = 2$ shows that (6.1.1) is true for uniform, nontrivial, 2-intersecting families. Combined with Theorem 6.1.1, Hall's result proves Theorem 6.1.2 (i). If (6.1.1) were true for every nontrivial 2-intersecting family, then Theorem 6.1.1 would imply that Conjecture 2.3.10 is true for $\lambda = 2$. We exhibit one nontrivial 2-intersecting family that does not satisfy (6.1.1), but feel that this may be the only counterexample. Recall the unique 2-design $\hat{\mathscr{F}}$ from Section 3.1.3, which was discovered by Ryser [99]. It is easily seen that $\sum_{F \in \hat{\mathscr{F}}} \binom{|F|}{2} = 39$ while $2\binom{m}{2} = 42$. We conjecture that $\hat{\mathscr{F}}$ is the only nontrivial 2-intersecting family for which (6.1.1) does not hold.

Conjecture 6.2.1. *If $\mathscr{F} \subset 2^X$ is a nontrivial 2-intersecting family of size m that is not the unique 2-design, then* (6.1.1) *holds.*

Frankl and Füredi [55] showed (6.1.1) holds for all nontrivial 1-intersecting families, and we gave their argument in Section 2.3.4.

Theorem 6.1.1 implies that a uniform counterexample to Conjecture 2.3.10 is also a counterexample to Stanton and Mullin's conjecture. It is not difficult to see that Hall's counterexamples to Stanton and Mullin's conjecture do not give counterexamples to Conjecture 2.3.10; for definitions see [68]. Hence, we can view Conjecture 2.3.10 as a weakening of Stanton and Mullin's conjecture.

A further weakening of Stanton and Mullin's conjecture is Conjecture 6.2.2, which is due to Hall [68]. Together with Theorem 6.1.1, we see that Conjecture 6.2.2 would imply that Conjecture 2.3.10 is true if $\mathscr{F}$ is additionally required to be k-uniform and k is sufficiently large. Deza [38] showed that $k_1 = 2$; Hall [68] showed that $k_2 = 3$; our proof of Theorem 6.1.2 shows that $k_3 \leq 12$.

Conjecture 6.2.2 (Hall, 1977)**.** *For each $\lambda \in \mathbb{Z}^+$, there exists a $k_\lambda \in \mathbb{Z}^+$ such that if $k \geq k_\lambda$ and $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial λ-intersecting family of size m, then* (6.1.2) *holds.*

It is natural to wonder if the obvious analog of Conjecture 2.3.10 for higher shadows holds. By considering λ-blowups of projective planes of order q when q is large enough, we have infinitely many nontrivial λ-intersecting families $\mathscr{F}$ satisfying $|\partial^i \mathscr{F}| < \binom{m}{i}$ for each $i \geq 3$ and each $\lambda \in \mathbb{Z}^+$.

6.3 Proof of Theorem 6.1.1

We use the linear programming techniques discussed in Section 3.4 to prove Theorem 6.1.1. We will use Theorem 6.1.1 to prove Theorem 6.1.2 in Section 6.4. Though we used codegrees in the proof of Theorem 3.1.4, we formally define them now.

Definition 6.3.1. *For a subset $S \subset X$ and a family $\mathscr{F} \subset 2^X$, we define the co-degree of S, denoted* $\operatorname{codeg}(S)$*, to be the number of sets in $\mathscr{F}$ that contain S,*

$$\operatorname{codeg}(S) := |\{F \in \mathscr{F} : S \subset F\}|.$$

Proof of Theorem 6.1.1. When $\lambda = 1$, the proof of Theorem 6.1.1 is trivial because $|\partial^2 \mathscr{F}|$ equals the left hand side of (6.1.1). We therefore assume that $\lambda \geq 2$. Let $\mathscr{F} \subset 2^X$ be a λ-intersecting family of size m. Let a_i denote the number of pairs $\{x,y\} \in \binom{X}{2}$ with $\operatorname{codeg}(\{x,y\}) = i$, and observe that the following identities hold

$$\sum_{i \geq 1} i a_i = \sum_{F \in \mathscr{F}} \binom{|F|}{2}, \quad \sum_{i \geq 1} \binom{i}{2} a_i = \binom{\lambda}{2} \binom{m}{2}.$$

The first counts pairs $(\{x,y\}, F)$ where $\{x,y\} \in \binom{X}{2}$, $F \in \mathscr{F}$, and $\{x,y\} \subset F$. The second counts pairs $(\{x,y\}, \{F_1, F_2\})$ where $\{x,y\} \in \binom{X}{2}$, $\{F_1, F_2\} \subset \mathscr{F}$, and $\{x,y\} \subset F_1 \cap F_2$. Consequently, $(a_1, \ldots, a_m)$ is a feasible solution to the following linear program with objective value $|\partial^2 \mathscr{F}|$:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m} z_i && (6.3.1)\\ \text{subject to:} \quad & \sum_{i\geq 1} i z_i = \sum_{F\in\mathscr{F}} \binom{|F|}{2} \\ & \sum_{i\geq 1} \binom{i}{2} z_i = \binom{\lambda}{2}\binom{m}{2} \\ & z_i \geq 0, \quad i\in\{1,\ldots,m\}. \end{aligned}$$

The dual of this linear program is:

$$\begin{aligned} \text{Maximize} \quad & \binom{\lambda}{2}\binom{m}{2} x + \left(\sum_{F\in\mathscr{F}} \binom{|F|}{2}\right) y && (6.3.2)\\ \text{subject to:} \quad & \binom{i}{2} x + iy \leq 1, \quad i\in\{1,\ldots,m\}. \end{aligned}$$

The feasible region of the dual linear program (6.3.2) has extreme points given by

$$\left(-\frac{1}{\binom{j+1}{2}}, \frac{2}{j+1}\right), \quad j\in\{1,\ldots,m-1\}. \tag{6.3.3}$$

If $\mathscr{F}$ satisfies (6.1.1), then setting $j=\lambda-1$ in (6.3.3) and applying weak duality, Theorem 3.4.6, yields

$$|\partial^2 \mathscr{F}| \geq \binom{\lambda}{2}\binom{m}{2}\left(-\frac{1}{\binom{\lambda}{2}}\right) + \left(\sum_{F\in\mathscr{F}} \binom{|F|}{2}\right)\left(\frac{2}{\lambda}\right) \geq \binom{m}{2}, \tag{6.3.4}$$

as desired. Finally, note that equality in (6.1.1) follows from counting pairs $(x,\{F_1,F_2\})$ such that $\{F_1,F_2\}\subset\mathscr{F}$ and $x\in F_1\cap F_2$.

We now assume that $\lambda\geq 2$ and that $\mathscr{F}\subset\binom{X}{k}$ is also k-uniform. We prove that $|\partial^2\mathscr{F}| = \binom{m}{2}$ if and only if $\mathscr{F}$ is a symmetric design. Ryser [99], Woodall [117], and Babai [3] showed that if $\mathscr{F}\subset 2^X$ is a λ-intersecting family of size $m=|\partial^1\mathscr{F}|=n$, then $|\partial^2\mathscr{F}|=\binom{m}{2}$. Conversely, suppose $|\partial^2\mathscr{F}|=\binom{m}{2}$ and let a_i denote the number of pairs $\{x,y\}\in\binom{X}{2}$ with $\operatorname{codeg}(\{x,y\})=i$. We will show that $\mathscr{F}$ is k-regular, which immediately implies that $\mathscr{F}$ is a symmetric design. By (6.3.4), we see that equality holds in (6.1.2), $(a_1,\ldots,a_m)$ is an optimal solution to the primal linear program (6.3.1), and (6.3.3) with $j = \lambda - 1$ is an optimal solution to the dual linear program (6.3.2). By weak complementary slackness, Theorem 3.4.8, this implies

$$a_i = 0 \quad \text{or} \quad \binom{i}{2}\left(\frac{-1}{\binom{\lambda}{2}}\right) + i\left(\frac{2}{\lambda}\right) = 1.$$

Hence $a_i = 0$ for $i \notin \{\lambda - 1, \lambda\}$ for $i \in \{1, \dots m\}$. The constraints in the primal linear program (6.3.1) imply $a_{\lambda-1} + a_\lambda = \binom{m}{2}$ and $\binom{\lambda-1}{2}a_{\lambda-1} + \binom{\lambda}{2}a_\lambda = \binom{\lambda}{2}\binom{m}{2}$, so $a_{\lambda-1} = 0$ and $a_\lambda = \binom{m}{2}$.

Let $x \in X$ and count pairs (y, F) such that $\{x, y\} \subset F$. Since $a_\lambda = \binom{m}{2}$, we have

$$\lambda|\partial^1 L(x)| = (k-1)\deg(x) \text{ so } |\partial^1 L(x)| = \frac{k-1}{\lambda}\deg(x). \tag{6.3.5}$$

We will give a lower bound on $|\partial^1 L(x)|$ in terms of $\deg(x)$ that will allow us to prove that $\mathscr{F}$ is k-regular. For $x \in X$, let $b_{x,i}$ denote the number of vertices $y \in X$ such that $\operatorname{codeg}(\{x, y\}) = i$ and observe that the following identities hold

$$\sum_{i \geq 1} i b_{x,i} = (k-1)\deg(x), \quad \sum_{i \geq 1}\binom{i}{2} b_{x,i} = (\lambda - 1)\binom{\deg(x)}{2}.$$

The first follows from counting pairs (y, F) where $\{x, y\} \in \binom{X}{2}$, $F \in \mathscr{F}$, and $\{x, y\} \subset F$. The second follows from counting pairs $(y, \{F_1, F_2\})$ where $\{x, y\} \in \binom{X}{2}$, $\{F_1, F_2\} \subset \mathscr{F}$, and $\{x, y\} \subset F_1 \cap F_2$. Consequently, $(b_{x,1}, \dots, b_{x,m})$ is a feasible solution to the following linear program with objective value $|\partial^1 L(x)|$:

$$\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{m} w_i \\
&\text{subject to:} && \sum_{i \geq 1} i w_i = (k-1)\deg(x) \\
& && \sum_{i \geq 1}\binom{i}{2} w_i = (\lambda - 1)\binom{\deg(x)}{2} \\
& && w_i \geq 0, \quad i \in \{1, \dots, m\}.
\end{aligned}$$

The dual of this linear program is:

$$\begin{aligned}
&\text{Maximize} && (\lambda - 1)\binom{\deg(x)}{2} y + (k-1)\deg(x) z \\
&\text{subject to:} && \binom{i}{2} y + i z \leq 1, \quad i \in \{1, \dots, m\}.
\end{aligned}$$

Since (6.3.3) with $j = \lambda - 1$ is a feasible solution, using (6.3.5) yields

$$\frac{k-1}{\lambda}\deg(x) = |\partial^1 L(x)| \geq \deg(x)\left(\frac{2(k-1)}{\lambda} - (\lambda-1)\frac{\deg(x)-1}{2}\binom{\lambda}{2}^{-1}\right).$$

Hence, $\deg(x) \geq k$ for each $x \in X$. On the other hand, let $F \in \mathscr{F}$ and count pairs (x, F') such that $F \neq F' \in \mathscr{F}$ and $x \in F \cap F'$. Since equality holds in (6.1.2), we have

$$k^2 \leq \sum_{x \in F} \deg(x) = \lambda(m-1) + k = k^2$$

so $\deg(x) = k$ for all $x \in X$. Hence $\mathscr{F}$ is k-regular and is thus a symmetric design. □

6.4 Proof of Theorem 6.1.2

In light of (6.1.1) and (6.1.2), we are interested in upper bounds on the sizes of nontrivial λ-intersecting families $\mathscr{F}$ that depend only on the sizes of the sets in $\mathscr{F}$. One of the first results of this kind is Deza's theorem [38], which bounds the size of λ-intersecting families that are not sunflowers. In the case when $\mathscr{F} \subset \binom{X}{k}$ is k-uniform, the upper bound on m in (6.4.1) is bigger than the upper bound on m in (6.1.2) by a factor of roughly λ.

Theorem 6.4.1 (Deza, 1974). *Let $\mathscr{F} \subset 2^X$ be a λ-intersecting family of size m that is not a sunflower. Define $K := \max_{F \in \mathscr{F}} |F|$. Then*

$$m \leq \max\{\lambda(\lambda+1)+1, (K-\lambda)((K-\lambda)+1)+1\}. \tag{6.4.1}$$

Since nontriviality is a stronger restriction on $\mathscr{F}$ than not being a sunflower, it is plausible that (6.4.1) could be improved for nontrivial $\mathscr{F}$. Frankl and Füredi [55] did exactly this when they showed that (6.1.1) holds for all nontrivial 1-intersecting families. We mentioned in the introduction that Stanton and Mullin [104] conjectured that (6.4.1) could be improved to (6.1.2) if $\mathscr{F}$ is nontrivial and k-uniform; Theorem 6.4.1 verifies Stanton and Mullin's conjecture for $\lambda = 1$ and Hall proved Stanton and Mullin's conjecture when $\lambda = 2$.

Theorem 6.4.2 (Hall, 1977)**.** *If $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial 2-intersecting family of size m, then*

$$m \leq \binom{k}{2} + 1.$$

We adapt Hall's proof of Theorem 6.4.2 to prove Theorem 6.1.2. (For the reader's convenience, we first reproduce Hall's proof of Theorem 6.4.2.) In our proof, we will use the fact that if $\mathscr{F} \subset \binom{X}{k}$ is a λ-intersecting family, then $\deg(x)$ does not lie in a certain interval. Deza [37] showed that if $\mathscr{F} \subset \binom{X}{k}$ is a λ-intersecting family of size m then, for all $x \in X$,

$$\deg(x)(m+1-\deg(x)) \leq \max\{\lambda, k-\lambda\}(m+1). \tag{6.4.2}$$

McCarthy and Vanstone [92] adapted an argument of Connor [29], and improved this bound; they gave the following restriction on $\deg(x)$.

Theorem 6.4.3 (McCarthy-Vanstone, 1979)**.** *Let $\mathscr{F} \subset \binom{X}{k}$ be a λ-intersecting family of size m.*

(i) If $x \in X$ then,

$$\deg(x)((k-\lambda)+\lambda(m-\deg(x))) \leq (k-\lambda)((k-\lambda)+\lambda m). \tag{6.4.3}$$

(ii) Let $\{x,y\} \subset \binom{X}{2}$ and define

(a) $a_{11} = (k-\lambda)((k-\lambda)+\lambda m) - \deg(x)((k-\lambda)+\lambda(m-\deg(x)))$,

(b) $a_{12} = a_{21} = \lambda \deg(x)\deg(y) - ((k-\lambda)+\lambda m)\operatorname{codeg}(\{x,y\})$,

(c) $a_{22} = (k-\lambda)((k-\lambda)+\lambda m) - \deg(y)((k-\lambda)+\lambda(m-\deg(y)))$.

The following determinant is non-negative:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \geq 0. \tag{6.4.4}$$

We now reproduce Hall's proof of Theorem 6.4.2. Note that Hall had originally used (6.4.2) in his proof, but we will use (6.4.3) instead since it makes the argument cleaner.

Hall's Proof of Theorem 6.4.2 Suppose, for a contradiction, that there exists a nontrivial 2-intersecting $\mathscr{F} \subset \binom{X}{k}$ of size $m > \binom{k}{2} + 1$. Write

$$m = \binom{k}{2} + 1 + \varepsilon, \;\; \varepsilon \in \mathbb{Z}^+. \tag{6.4.5}$$

Note that the left hand side of (6.4.3) is quadratic in $\deg(x)$ with roots $\deg(x) = 0$ and $\deg(x) = m - 1 + k/2$. If there exists an $x \in X$ with $k \le \deg(x) \le (m - 1 + k/2) - k$, then (6.4.3) is true for $\deg(x) = k$; together with (6.4.5), this implies that $\varepsilon \le 0$, which is impossible. Hence, for all $x \in X$, either

$$\deg(x) \le k - 1 \;\text{ or }\; \deg(x) \ge m - \left\lceil \frac{k}{2} \right\rceil. \tag{6.4.6}$$

We say a vertex $x \in X$ with $\deg(x) \le k - 1$ is *light* and is *heavy* if $\deg(x) \ge m - \lceil k/2 \rceil$.

By (6.4.5), for any $F \in \mathscr{F}$, we have

$$\sum_{x \in F} \deg(x) = 2(m-1) + k = k^2 + 2\varepsilon. \tag{6.4.7}$$

Since the average degree of a vertex in $F \in \mathscr{F}$ is greater than k, every set $F \in \mathscr{F}$ contains a heavy vertex. As $\mathscr{F}$ is nontrivial, there are at least two heavy vertices x_1, x_2. Define

$$s := |\{F \in \mathscr{F} : \{x_1, x_2\} \subset \mathscr{F}\}|, \qquad t := |\{F \in \mathscr{F} : x_1 \in F, x_2 \notin F\}|,$$
$$u := |\{F \in \mathscr{F} : x_1 \notin F, x_2 \in F\}|, \qquad v := |\{F \in \mathscr{F} : x_1, x_2 \notin F\}|.$$

We have $s \le k - 1$ because $\lambda = 2$ and $\mathscr{F}$ is nontrivial. Since $u + v$ and $t + v$ count the number of sets $F \in \mathscr{F}$ not on x_1, x_2 respectively, (6.4.6) yields $t + v, u + v \le \lceil k/2 \rceil$. Consequently (6.4.5) implies,

$$\binom{k}{2} + 1 + \varepsilon = m = s + t + u + v \le s + (t + v) + (u + v) \le (k - 1) + 2\left\lceil \frac{k}{2} \right\rceil \le 2k.$$

As $\varepsilon \in \mathbb{Z}^+$, we have a contradiction for $k \ge 5$. For $k = 4$, Theorem 6.4.1 yields $m \le 7$, so we have a contradiction in this case too. We have shown that if $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial 2-intersecting family of size m then $m \le \binom{k}{2} + 1$. □

For larger λ, if we knew that a nontrivial λ-intersecting $\mathscr{F} \subset \binom{X}{k}$ that does not satisfy (6.1.2) has at least λ heavy vertices, then Hall's argument would yield a proof of Conjecture 6.2.2. Unfortunately, Hall's averaging argument only shows that any nontrivial λ-intersecting $\mathscr{F} \subset \binom{X}{k}$ that does not satisfy (6.1.2) has at least two heavy vertices. In the proof of Theorem 6.1.2, we expend a lot of effort to eliminate the possibility that there are exactly two heavy vertices when $\lambda = 3$; the key difficulty is getting a good bound on the number of sets $F \in \mathscr{F}$ that contain both the heavy vertices.

Proof of Theorem 6.1.2. We observe that Theorem 6.4.2 together with Theorem 6.1.1 yields Theorem 6.1.2 (i).

For the rest of the proof, we assume that $\lambda = 3$. We will show that if $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial 3-intersecting family, where $k \notin \{8, 11\}$, then (6.1.2) holds. Theorem 6.1.1 then implies that $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$ and that equality holds if and only if $\mathscr{F}$ is a symmetric design. First suppose $k < 6$. It is not difficult to see that if $\mathscr{F}$ is a nontrivial k-uniform 3-intersecting family of size m, where $k \in \{4, 5\}$, then $m \leq 5$; for proofs of these results in a more general setting see [62], [63], and [111]. Hence, (6.1.2) holds when $k < 6$.

Suppose, for a contradiction, that $k \geq 12$ and that $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial 3-intersecting family of size m for which (6.1.2) does not hold. Write

$$m = \frac{k(k-1)}{3} + 1 + \varepsilon, \ \ \varepsilon > 0. \tag{6.4.8}$$

Note that the left hand side of (6.4.3) is quadratic in $\deg(x)$ with roots $\deg(x) = 0$ and $\deg(x) = m - 1 + k/3$. If there exists an $x \in X$ with $k \leq \deg(x) \leq (m - 1 + k/3) - k$, then (6.4.3) is true for $\deg(x) = k$; together with (6.4.8), this implies that $\varepsilon \leq 0$, which is impossible. Hence, for all $x \in X$, either

$$\deg(x) \leq k - 1 \ \text{ or } \ \deg(x) \geq m - \left\lceil \frac{2k}{3} \right\rceil \geq m - \frac{2k+2}{3}. \tag{6.4.9}$$

Following Hall [68], we say a vertex $x \in X$ is *light* if $\deg(x) \leq k - 1$ and is *heavy* if $\deg(x) \geq m - \lceil 2k/3 \rceil$.

By (6.4.8), for any $F \in \mathscr{F}$, we have

$$\sum_{x \in F} \deg(x) = 3(m-1) + k = k^2 + 3\varepsilon. \tag{6.4.10}$$

Since the average degree of a vertex in $F \in \mathscr{F}$ is greater than k, every set $F \in \mathscr{F}$ contains a heavy vertex. As $\mathscr{F}$ is nontrivial, there are at least two heavy vertices. We consider two cases, according to whether there are exactly two or greater than two heavy vertices.

Case 1: There are exactly two heavy vertices. Let x_1, x_2 be the heavy vertices. Since $\mathscr{F}$ is nontrivial, there exists a set $F_1 \in \mathscr{F}$ which contains x_1 but not x_2, and there exists a set $F_2 \in \mathscr{F}$ which contains x_2 but not x_1. Let $F_1 \cap F_2 := \{y_1, y_2, y_3\}$. Define

$$s := |\{F \in \mathscr{F} : \{x_1, x_2\} \subset \mathscr{F}\}|, \qquad t := |\{F \in \mathscr{F} : x_1 \in F, x_2 \notin F\}|,$$
$$u := |\{F \in \mathscr{F} : x_1 \notin F, x_2 \in F\}|,$$

and observe that $m = s + t + u$ since every $F \in \mathscr{F}$ contains a heavy vertex. By (6.4.9), we have $t, u \leq \lceil 2k/3 \rceil \leq (2k+2)/3$.

We now show how to obtain an upper bound on s in terms of k. Observe that any $F \in \mathscr{F}$ that contains $\{x_1, x_2\}$ intersects $F_1 \setminus \{x_1\}$ in a subset of size two. Consequently,

$$2s = \sum_{\{x_1,x_2\} \subset F} |F \cap F_1 \setminus \{x_1\}| = \sum_{w \in F_1 \setminus \{x_1\}} \operatorname{codeg}(x_1, x_2, w). \tag{6.4.11}$$

We claim that if $w \in X \setminus \{x_1, x_2\}$ and there exists an $F \in \mathscr{F}$ such that

$$\{x_1, x_2\} \not\subset F, \;\; w \notin F, \tag{6.4.12}$$

then $\operatorname{codeg}(\{x_1, x_2, w\}) \leq (k-1)/2$. Suppose $F', \hat{F} \in \mathscr{F}$ are distinct sets in $\mathscr{F}$ that both contain $\{x_1, x_2, w\}$. Since $\lambda = 3$, the intersections of F' and $\hat{F}$ with $F \setminus \{x_1, x_2\}$ must be disjoint subsets of size two. Hence, $\operatorname{codeg}(\{x_1, x_2, w\}) \leq (k-1)/2$. Observe that if $w \in F_1 \setminus \{x_1, y_1, y_2, y_3\}$, then F_2 is a set in $\mathscr{F}$ that satisfies (6.4.12). We will consider two subcases according to whether for each $y_i \in F_1 \cap F_2$, there exists an $F \in \mathscr{F}$ that satisfies (6.4.12) for $w = y_i$.

Subcase 1: For each $y_i \in F_1 \cap F_2$, there exists an $F \in \mathscr{F}$ that satisfies (6.4.12) for $w = y_i$.

Applying (6.4.9) and (6.4.11) yields

$$\frac{k(k-1)}{3} + 1 + \varepsilon = m = s + t + u \leq \frac{(k-1)^2}{4} + 2\left\lceil \frac{2k}{3} \right\rceil \leq \frac{3k^2 + 10k + 19}{12}. \tag{6.4.13}$$

This implies that $k^2 - 14k - 7 + 12\varepsilon \leq 0$, which is a contradiction for $k \geq 15$ since $\varepsilon \geq 1/3$. For the remaining values of k, we refer the reader to the appendix.

Subcase 2: There exists a $y_i \in F_1 \cap F_2$ for which no $F \in \mathscr{F}$ satisfies (6.4.12) for $w = y_i$.

Note that if y_i is in every $F \in \mathscr{F}$ that does not contain $\{x_1, x_2\}$ then, by (6.4.9), $\text{codeg}(\{x_1, x_2, y_i\}) \leq k - 1 - (t+u)$. Suppose that every $F \in \mathscr{F}$, not containing $\{x_1, x_2\}$, contains j of the elements $\{y_1, y_2, y_3\}$ where $j \in \{1, 2, 3\}$. Applying (6.4.11) yields

$$
\begin{aligned}
\frac{k(k-1)}{3} + 1 + \varepsilon = m = s + t + u & \qquad (6.4.14)\\
&\leq \frac{1}{2}\left(\sum_{w \in F_1 \setminus \{x_1\}} \text{codeg}(x_1, x_2, w)\right) + t + u \\
&\leq \frac{1}{2}\left(j(k-1-(t+u)) + \frac{(k-1-j)(k-1)}{2}\right) + t + u \\
&= \frac{j}{2}(k-1) + \frac{(k-1-j)(k-1)}{4} + (2-j)\frac{t+u}{2} \\
&\leq \frac{j}{2}(k-1) + \frac{(k-1-j)(k-1)}{4} + (2-j)\frac{2k+2}{3} \\
&= -\left(\frac{k}{6} + \frac{7}{6}\right) j + \frac{(k-1-j)(k-1)}{4} + \frac{4k+4}{3} \\
&\leq \frac{3k^2 + 5k + 8}{12},
\end{aligned}
$$

since the penultimate expression in (6.4.14) is maximized when $j = 1$. This implies that $(k-1)(k-8) + 12(\varepsilon - 1/3) \leq 0$, which is a contradiction for $k \geq 9$.

If $k = 8$ then $\varepsilon = 1/3$. Observe that $\text{codeg}(x_1, x_2, w) \leq 3$ if w is not one of the j special vertices in $\{y_1, y_2, y_3\}$; in the bound for s in (6.4.14), we use the weaker bound $\text{codeg}(x_1, x_2, w) \leq 7/2$ for vertices w that are not one of the j special vertices in $\{y_1, y_2, y_3\}$. If we replace the weaker bound on $\text{codeg}(x_1, x_2, w)$ by the tighter bound, then we get a contradiction for $k = 8$ as well. Finally, if $k \in \{6, 7\}$, then $\varepsilon \in \mathbb{Z}^+$ so we also get a contradiction in this case.

Case 2: There are greater than two heavy vertices. Let x_1, x_2, x_3 be three heavy vertices. Define

$$
\begin{aligned}
s &:= |\{F \in \mathscr{F} : \{x_1, x_2, x_3\} \subset \mathscr{F}\}|, & t &:= |\{F \in \mathscr{F} : x_1 \in F, x_2, x_3 \notin F\}|, \\
u &:= |\{F \in \mathscr{F} : x_2 \in F, x_1, x_3 \notin F\}|, & v &:= |\{F \in \mathscr{F} : x_3 \in F, x_1, x_2 \notin F\}|, \\
w &:= |\{F \in \mathscr{F} : x_1, x_2 \in F, x_3 \notin F\}|, & x &:= |\{F \in \mathscr{F} : x_1, x_3 \in F, x_2 \notin F\}|, \\
y &:= |\{F \in \mathscr{F} : x_2, x_3 \in F, x_1 \notin F\}|, & z &:= |\{F \in \mathscr{F} : x_1, x_2, x_3 \notin F\}|.
\end{aligned}
$$

By counting the number of sets not containing x_1, x_2, or x_3 respectively we have

$$u+v+y+z,\ t+v+x+z,\ t+u+w+z \leq \left\lceil \frac{2k}{3} \right\rceil \leq \frac{2k+2}{3}, \tag{6.4.15}$$

by (6.4.9). As $\lambda = 3$ and $\mathscr{F}$ is nontrivial, we have $s \leq k-2$. Therefore (6.4.15) implies,

$$\begin{aligned} \frac{k(k-1)}{3}+1+\varepsilon = m &= s+t+u+v+w+x+y+z \\ &\leq s+(u+v+y+z)+(t+v+x+z)+(t+u+w+z) \\ &\leq (k-2)+3\left\lceil \frac{2k}{3} \right\rceil \leq (k-2)+(2k+2) = 3k. \end{aligned} \tag{6.4.16}$$

This implies $k^2-10k+3+3\varepsilon \leq 0$, so we have a contradiction for $k \geq 10$ since $\varepsilon \geq 1/3$. For the remaining values of k, we refer the reader to the appendix.

We have shown that if $\mathscr{F} \subset \binom{X}{k}$ is a nontrivial 3-intersecting family of size m and $k \notin \{8,11\}$, then $\mathscr{F}$ satisfies (6.1.2). By Theorem 6.1.1, this implies that if $\mathscr{F}$ satisfies the hypotheses of Theorem 6.1.2 (ii), then $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$ and equality holds if and only if $\mathscr{F}$ is a symmetric design. □

6.5 Appendix

Here, we collect some computations that are needed to verify Theorem 6.1.2 for small values of k. We regret that we could not prove Theorem 6.1.2 for all values of k. The missing cases are $k = 11, m = 40$ in Case 1, Subcase 1 and $k = 8, m = 20$ in Case 2.

Case Analysis for Case 1, Subcase 1:

If $k = 14$, then $\operatorname{codeg}(x_1,x_2,w) \leq 6$ for $w \in F_1 \setminus \{x_1\}$ so (6.4.11) yields $s \leq 39$. Using this value for s in (6.4.13) yields $\varepsilon < 0$, which contradicts (6.4.8).

If $k = 13$, then the penultimate inequality in (6.4.13) yields that $m = 54$, $s = 36$, and $t = u = 9$. Using these values in (6.4.4) yields a contradiction.

If $k = 12$, then $\operatorname{codeg}(x_1,x_2,w) \leq 5$ for $w \in F_1 \setminus \{x_1\}$ so (6.4.11) yields $s \leq 27$. Using this value for s in (6.4.13) yields $\varepsilon < 0$, which contradicts (6.4.8).

If $k = 11$, then (6.4.13) yields $m \in \{38, 39, 40, 41\}$. If we add the constraint $z_s = 1$ to (6.3.1), then the dual linear program becomes

$$\text{Maximize} \qquad 3\binom{m}{2}w + \binom{k}{2}mx + y \tag{6.5.1}$$
$$\text{subject to:} \qquad \binom{i}{2}w + ix \leq 1, \quad i \in \{1, \ldots, m\} \setminus \{s\}$$
$$\binom{s}{2}w + sx + y \leq 1.$$

and any feasible solution to (6.5.1) is a lower bound on $|\partial^2 \mathscr{F}|$. If $m = 38$, then (6.4.13) implies $s \geq 22$ because $t, u \leq 8$. Observe that $\binom{p}{2}\left(-\frac{1}{3}\right) + p\left(\frac{2}{3}\right) \leq \binom{q}{2}\left(-\frac{1}{3}\right) + q\left(\frac{2}{3}\right)$ if $p > q \geq 2$. We have $\binom{22}{2}\left(-\frac{1}{3}\right) + 22\left(\frac{2}{3}\right) + 63\frac{1}{3} = 1$, so the previous inequality implies that $(-1/3, 2/3, 63\frac{1}{3})$ is always a feasible solution to (6.5.1) for $m = 38$ and $k = 11$. Hence, $|\partial^2 \mathscr{F}| > \binom{m}{2}$, which contradicts our initial assumption. A similar argument eliminates the case $m = 39$. If $m = 41$, then $s = 25$ and $t = u = 8$ so $\deg(x_1) = 33$; this contradicts Theorem 6.4.3.

If $k = 10$, then the penultimate inequality in (6.4.13) yields $m = 32$ and $s = 18$. Since $\binom{18}{2}\left(-\frac{1}{3}\right) + 18\left(\frac{2}{3}\right) + 40 = 1$, we have $(-1/3, 2/3, 40)$ is a feasible solution to (6.5.1) for $m = 32$ and $k = 10$. Consequently, $|\partial^2 \mathscr{F}| > \binom{m}{2}$, which contradicts our initial assumption.

If $k = 9$, then (6.4.13) yields $\varepsilon \in \{1, 2, 3\}$ and $s \leq 16$. We consider the cases $\varepsilon = 1$ and $\varepsilon \in \{2, 3\}$ separately.

$\underline{\varepsilon = 1}$: By (6.4.9), we have $s \geq 14$ and $t, u \leq 6$. Note that $\binom{15}{2}(-\frac{1}{3}) + 15(\frac{2}{3}) + 26 = 1$ so $(-\frac{1}{3}, \frac{2}{3}, 26)$ is a feasible solution to (6.5.1) for $m = 26$, $k = 9$, and $s \geq 15$. If $s = 14$, then $t = u = 6$ so $\deg(x_1) = \deg(x_2) = 20$. Observe that (6.4.9) and (6.4.10) imply

$$84 = \sum_{x \in F_1} \deg(x) = \deg(x_1) + \sum_{x_1 \neq x \in F_1} \deg(x) \leq 20 + 8(8) = 84;$$

hence, if w is a light vertex in a set in $\mathscr{F}$ that doesn't contain both x_1, x_2, then $\deg(w) = 8$. Now suppose that z is a light vertex that is only contained in sets that contain both x_1, x_2; that is $\deg(z) = \operatorname{codeg}(x_1, x_2, z)$. Since F_1 satisfies (6.4.12) for $w = z$, we see $\deg(z) = \operatorname{codeg}(x_1, x_2, z) \leq 4$. Now let $F' \in \mathscr{F}$ be a set that contains both x_1, x_2 and observe that

$$84 = \sum_{x \in F'} \deg(x) = \deg(x_1) + \deg(x_2) + \sum_{z \in F' \setminus \{x_1, x_2\}} \deg(z) = 20 + 20 + \sum_{z \in F' \setminus \{x_1, x_2\}} \deg(z).$$

For $z \in F' \setminus \{x_1, x_2\}$, we have $\deg(z) \leq 4$ or $\deg(z) = 8$ so either F' contains four vertices of degree eight and three vertices of degree four or five vertices of degree eight and two vertices whose degrees sum to four. Hence for $x \in X$, we have $\deg(x) \in \{1,2,3,4,8,20\}$. Let n_i denote the number of vertices of degree i. Also define

$$m_1 := |\{F \in \mathscr{F} : \{x_1, x_2\} \subset \mathscr{F}, \exists\, w, z \in F \text{ with } \deg(w) = 1, \deg(z) = 3\}|,$$
$$m_2 := |\{F \in \mathscr{F} : \{x_1, x_2\} \subset \mathscr{F}, \exists\, w, z \in F \text{ with } \deg(w) = \deg(z) = 2\}|,$$
$$m_3 := |\{F \in \mathscr{F} : \{x_1, x_2\} \subset \mathscr{F}, \exists\, w \in F \text{ with } \deg(w) = 4\}|,$$

and observe that $m_1 + m_2 + m_3 = s = 14$. Note that $n_{20} = 2$, $3n_3 = n_1 = m_1$, $n_2 = m_2$, and $4n_4 = 3m_3$. In particular, m_3 is even so $n_1 + n_2 = m_1 + m_2$ is also even. Observe that

$$\begin{aligned} 234 = 9 \cdot 26 = km = \sum_{x \in X} \deg(x) &= 20n_{20} + 8n_8 + 4n_4 + 3n_3 + 2n_2 + n_1 \\ &= 20n_{20} + 8n_8 + 4n_4 + n_1 + 2n_2 + n_1 = 20n_{20} + 8n_8 + 4n_4 + 2(n_1 + n_2). \quad (6.5.2) \end{aligned}$$

Since $n_1 + n_2$ is even, (6.5.2) implies that $4|234$, which is a contradiction.

$\underline{\varepsilon \in \{2,3\}}$: Without loss of generality, we will assume $\deg(x_1) \leq \deg(x_2)$ or equivalently that $u \geq t$. Observe that (6.4.9) and (6.4.10) imply

$$\begin{aligned} \deg(x_1) = \sum_{x \in F_1} \deg(x) - \sum_{z \in F_1 \setminus \{x_1\}} \deg(z) &\geq \sum_{x \in F_1} \deg(x) - (k-1)^2 \\ &= k^2 + 3\varepsilon - (k-1)^2 = 2k - 1 + 3\varepsilon. \quad (6.5.3) \end{aligned}$$

If $\varepsilon \in \{2,3\}$, then $s \leq 16$ implies $u = 6$. When $\varepsilon = 2$, we have $\deg(x_1) = 21$, which contradicts (6.5.3). When $\varepsilon = 3$, we have $\deg(x_1) = 22$, which again contradicts (6.5.3).

The case $k \in \{6,7,8\}$ can be eliminated with an argument similar to the one for $k = 9, \varepsilon \in \{2,3\}$; we omit the details.

Case Analysis for Case 2:

If $k = 9$, then we arrive at a contradiction by using the third to last expression in (6.4.16).

If $k=8$, then $m \in \{20,21,22,23,24\}$ by (6.4.16). For $m \in \{21,22,23,24,25\}$, any heavy vertex x satisfies $\deg(x) \geq m-5$ by (6.4.3). Hence, the upper bound in (6.4.15) is improved and implies that $\deg(x_1) = s+t+w+x \leq 16$. This gives a contradiction for $m \in \{22,23,24\}$. For $m=21$, we have $s=6$, $w=x=y=5$, and $t=u=v=z=0$. Hence $\operatorname{codeg}(\{x_i,x_j\}) = 11$ for $\{i,j\} \in \{1,2,3\}$. Adding the constraint $z_{11} \geq 3$ to (6.3.1) yields that $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$, which contradicts our initial assumption; we omit the details since the computation is similar to that in (6.5.1).

If $k=7$, then $m \in \{16,17,18,19,20\}$ by (6.4.16) and (6.4.3) shows that any heavy vertex x satisfies $\deg(x) \geq m-4$. Hence, the upper bound in (6.4.15) is improved and implies that $\deg(x_1) = s+t+w+x \leq 13$. This gives a contradiction for $m \in \{18,19,20\}$.

If $k=7$ and $m=17$, then we conclude that $s=5$, $w=x=y=4$, and that $t=u=v=z=0$. Note that if $x \in X$ is heavy then $\deg(x) \geq 13$ by (6.4.9). If there is a fourth heavy vertex x_4, it can be in at most one of the five sets on x_1,x_2,x_3; moreover since heavy vertices have degree at least thirteen, x_4 is in each of the four sets on $\{x_1,x_2\}$, $\{x_1,x_3\}$, and $\{x_2,x_3\}$. As $\lambda=3$, this argument shows that there are at most four heavy vertices. If there are exactly four heavy vertices x_1,x_2,x_3,x_4, then $\operatorname{codeg}(\{x_i,x_j\}) = 9$ for $\{i,j\} \subset \{1,2,3,4\}$. Adding the constraint $z_9 \geq 6$ to (6.3.1) yields that $|\partial^2 \mathscr{F}| \geq \binom{m}{2}$, which contradicts our initial assumption; we omit the details since the computation is similar to the one in (6.5.1). Consequently, there are exactly three heavy vertices and $\deg(x_1) = \deg(x_2) = \deg(x_3) = 13$. Moreover, any set in $\mathscr{F}$ contains either exactly two or exactly three heavy vertices. Let F' be a set that contains exactly two heavy vertices. Equations (6.4.9) and (6.4.10) yield that F' contains four vertices of degree six and one of degree five. Hence, if w is a light vertex and w is contained in a set of $\mathscr{F}$ with two heavy vertices, then $\deg(w) \in \{5,6\}$; otherwise $\deg(w) = \operatorname{codeg}(x_1,x_2,x_3,w) = 1$. Now let $\hat{F}$ be a set in $\mathscr{F}$ that contains exactly three heavy vertices. Since $\deg(w) \in \{1,5,6\}$ for $w \in X \setminus \{x_1,x_2,x_3\}$, (6.4.10) yields that $\hat{F}$ contains three vertices of degree five and one vertex of degree one. As $s=5$, there are fifteen vertices of degree five and five of degree one. Let n_i denote the number of vertices of degree i. We have

$$119 = km = \sum_{x \in X} \deg(x) = 13n_{13} + 6n_6 + 5n_5 + n_1 = 39 + 6n_6 + 75 + 5,$$

which implies that $n_6 = 0$, a contradiction.

If $k=7$ and $m=16$ then (6.4.16) implies that $s\in\{4,5\}$ and $t+u+v+2z\le 1$. We conclude that $z=0$ and at most one of t,u,v equals one. We first show that the situation where exactly one of t,u,v is one is impossible. Without loss of generality, assume for a contradiction that $t=1$ and $u=v=0$. If $s=4$, then (6.4.15) implies that

$$16=m=s+t+u+v+w+x+y+z\le 4+1+0+0+0+3+3+4=15,$$

which is a contradiction. If $s=5$, then we can conclude via a similar argument that $w=x=3$ and $y=4$. Consider the unique $\hat{F}\in\mathscr{F}$ with $x_1\in F$ and $x_2,x_3\notin F$. Since light vertices have degree at most six, $\hat{F}$ must contain another heavy vertex x_4 by (6.4.10). Now $\deg(x_4)\ge 12$ and since x_4 can only be in one of the five sets on x_1,x_2,x_3, we have that x_4 is in each of the remaining 11 sets. As a result, for $i,j\in\{1,2,3,4\}$, we have $\operatorname{codeg}(\{x_i,x_j\})=8$ if $\{i,j\}\ne\{1,3\}$ and we have $\operatorname{codeg}(\{x_1,x_3\})=9$. Adding the constraints $z_9\ge 1$ and $z_8\ge 5$ to (6.3.1) yields that $|\partial^2\mathscr{F}|\ge\binom{m}{2}$, which contradicts our initial assumption; we omit the details since the computation is similar to the one in (6.5.1). Hence, we can assume $t=u=v=0$. If $s=5$, then two of w,x,y equal four and the other equals three. Hence, two of the pairs $\{x_1,x_2\},\{x_1,x_3\}.\{x_2,x_3\}$ have codegree nine and one has codegree eight. Adding the constraints $z_9\ge 2$ and $z_8\ge 1$ to (6.3.1) yields that $|\partial^2\mathscr{F}|\ge\binom{m}{2}$, which contradicts our initial assumption; we omit the details since the computation is similar to the one in (6.5.1). If $s=4$, then $w=x=y=4$ so $\deg(x_1)=\deg(x_2)=\deg(x_3)=12$ and $\operatorname{codeg}(\{x_i,x_j\})=8$ for $\{i,j\}\subset\{1,2,3\}$. If we add the constraint $z_8\ge 3$ to (6.3.1), then the corresponding dual linear program is

$$\begin{aligned}
&\text{Maximize} && 360w+336x+y && (6.5.4)\\
&\text{subject to:} && \binom{i}{2}w+ix\le 1,\quad i\in[16]\setminus\{8\}\\
& && \binom{8}{2}w+8x+y\le 1.\\
& && y\ge 0.
\end{aligned}$$

It follows that $(-1/3,2/3,5)$ is a feasible solution to (6.5.4) so $|\partial^2\mathscr{F}|\ge 119$. Since $\binom{16}{2}=120$, we obtain a contradiction unless $(-1/3,2/3,5)$ is an optimal solution to (6.5.4). If $(-1/3,2/3,5)$ is an optimal solution to (6.5.4), then by complementary slackness, $\operatorname{codeg}(\{x,y\})\in\{0,2,3,8\}$ for $\{x,y\}\in\binom{X}{2}$; moreover $z_8=3$. Observe that

$t=u=v=z=0$ implies that if $w \in X \setminus \{x_1,x_2,x_3\}$, then

$$\deg(w) = \sum_{\{i,j\}\subset\{1,2,3\}} \operatorname{codeg}(\{x_i,x_j,w\}) - 2\operatorname{codeg}(\{x_1,x_2,x_3,w\}) \le 9,$$

so (6.4.9) shows that x_1,x_2,x_3 are the only heavy vertices. Moreover, every $F \in \mathscr{F}$ contains either exactly two or exactly three heavy vertices as $t=u=v=z=0$. If $F' \in \mathscr{F}$ is a set with exactly two heavy vertices, then (6.4.9) and (6.4.10) yield that either F' contains three vertices of degree six and two vertices of degree five or four vertices of degree six and one of degree four. Now, every light vertex w must be contained in a set with exactly two heavy vertices; otherwise $\deg(w) = \operatorname{codeg}(x_1,x_2,x_3,w) = 1$, which contradicts the fact that $\operatorname{codeg}(\{x,y\}) \in \{0,2,3,8\}$. Hence, $\deg(w) \in \{4,5,6\}$ for $w \in X \setminus \{x_1,x_2,x_3\}$. As a result, if $\hat{F}$ is a set with three heavy vertices, then (6.4.10) yields that $\hat{F}$ contains four vertices of degree four. We conclude that $\deg(w) \in \{4,6\}$ for $w \in X \setminus \{x_1,x_2,x_3\}$. Since $s=4$, there are sixteen vertices of degree four. Let n_i denote the number of vertices of degree i; we have

$$112 = km = \sum_{x\in X} \deg(x) = 12n_{12} + 6n_6 + 4n_4 = 36 + 6n_6 + 64$$

so $n_6 = 2$, which is impossible.

If $k=6$, then $\varepsilon \in \{1,2\}$ by Theorem 6.4.1. Let n_i denote the number of vertices of degree i. If $\varepsilon = 2$, then $n_1 = 2$, $n_4 = 9$, and $n_{10} = 4$ by a result of Vanstone [111]. Using (6.4.10), we see that $\mathscr{F}$ is uniquely determined and must be the family

$$\begin{aligned}&\{\{1,2,3,5,6,7\},\{1,2,3,8,9,10\},\{1,2,3,11,12,13\},\{1,2,4,5,8,11\},\\ &\{1,2,4,6,9,12\},\{1,2,4,7,10,13\},\{1,3,4,5,10,12\},\{1,3,4,6,8,13\},\\ &\{1,3,4,7,9,11\},\{2,3,4,5,9,13\},\{2,3,4,6,10,11\},\{2,3,4,7,8,12\},\\ &\{1,2,3,4,14,15\}\}. \end{aligned} \tag{6.5.5}$$

Hence, $|\partial^2\mathscr{F}| = 87 > \binom{13}{2}$. If $k=6$ and $\varepsilon = 1$, then either $n_1 = 2$, $n_3 = 3$, $n_4 = 6$, $n_9 = 3$, and $n_{10} = 1$ or $n_4 = 9$ and $n_9 = 4$ by a result of Vanstone [111]. Again using (6.4.10), we can conclude that $\mathscr{F}$ is uniquely determined in both cases. In the first case, $\mathscr{F}$ must be

$$\{\{1,2,4,5,6,11\},\{1,2,4,7,8,12\},\{1,2,4,9,10,13\},\{1,3,4,7,9,11\},$$
$$\{1,3,4,5,10,12\},\{1,3,4,6,8,13\},\{2,3,4,8,10,11\},\{2,3,4,6,9,12\},$$
$$\{2,3,4,5,7,13\},\{1,2,3,5,8,9\},\{1,2,3,6,7,10\},\{1,2,3,4,14,15\}\}; \qquad (6.5.6)$$

hence, $|\partial^2\mathscr{F}| = 84 > \binom{12}{2}$. In the latter case, $\mathscr{F}$ is the complement of a projective plane of order 3 with respect to a line; hence $|\partial^2\mathscr{F}| = 78 > \binom{12}{2}$.

Chapter 6, in part, is a reprint of the material as it appears in “On a conjecture of Frankl and Füredi," 2011. Chowdhury, Ameera. *Electron. J. Combin.*, 18(1):Paper 56, 16, 2011. The dissertation author was the primary investigator and author of this paper.

Chapter 7

On the Manickam-Miklós-Singhi Conjecture

For $k \in \mathbb{Z}^+$, let $f(k)$ be the minimum integer N such that for all $n \geq N$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative. In 1988, Manickam, Miklós, and Singhi proved that $f(k)$ exists and conjectured that $f(k) \leq 4k$. In this chapter, we prove $f(3) = 11$ and $f(4) \leq 24$, which improves previous upper bounds in these cases. With more patience, our arguments could yield improved upper bounds on $f(k)$ for larger k. Moreover, we show how our method could potentially yield a quadratic upper bound on $f(k)$. We end this chapter by discussing a vector space analog of the Manickam-Miklós-Singhi conjecture.

7.1 Nonnegative Sums

Manickam, Miklós, and Singhi conjectured in [88] and [89] that

Conjecture 7.1.1. *For any integers n, k with $n \geq 4k$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative.*

This conjecture is similar to the Erdős-Ko-Rado theorem, Theorem 2.2.2, not only in the appearance of the binomial coefficient $\binom{n-1}{k-1}$, but also because of the following tight example: $x_1 = n-1, x_2 = \cdots = x_n = -1$. In this example, a k-subset of

$x_1, \ldots, x_n$ has nonnegative sum exactly when it contains x_1, and hence there are exactly $\binom{n-1}{k-1}$ k-element subsets with nonnegative sum.

As in the Erdős-Ko-Rado theorem, n must be large enough with respect to k, otherwise there exist n real numbers $x_1, \ldots, x_n \in \mathbb{R}$ with nonnegative sum and fewer than $\binom{n-1}{k-1}$ k-element subsets with nonnegative sum. Here are two counterexamples when $n < 4k$. For $n = 2k + r$, where $1 \le r \le 3k/5$, let $x_1 = \cdots = x_{2k+r-2} = 2$ and $x_{2k+r-1} = x_{2k+r} = -(2k + r - 2)$. Note that a k-subset is nonnegative exactly when it does not contain x_{2k+r-1} or x_{2k+r} and that $\binom{2k+r-2}{k} < \binom{2k+r-1}{k-1}$ when $1 \le r \le 3k/5$. For $n = 3k + r$, where $1 \le r \le k/7$, a similar counterexample sets $x_1 = \cdots = x_{3k+r-3} = 3$ and $x_{3k+r-2} = \cdots = x_{3k+r} = -(3k + r - 3)$. Again, note that a k-subset is nonnegative exactly when it does not contain x_{3k-1}, x_{3k}, or x_{3k+1} and that $\binom{3k+r-3}{k} < \binom{3k+r-1}{k-1}$ when $1 \le r \le k/7$. These counterexamples do not generalize to larger n because $t^{t-1} \not\lesssim (t-1)^t$ for $t \ge 3$.

Although the $n \ge 4k$ requirement is probably not sharp, one reason the conjecture is written with this bound is because Baranyai's theorem, Theorem 3.4.33, or the Greene-Kleitman type argument in Section 4.6.2 verifies Conjecture 7.1.1 when $k|n$; see Lemma 7.4.1.

Very recently, Alon, Huang, and Sudakov [1] verified Conjecture 7.1.1 when $n \ge \min\{33k^2, 2k^3\}$, which substantially improves previous results. They also obtained a Hilton-Milner analog. See their paper for references and historical remarks.

7.2 Notation

We begin with some definitions and notation that are special to this chapter.

Definition 7.2.1. *Given $x_1 \ge \cdots \ge x_n \in \mathbb{R}$, a subset $S \subset \{x_1, \ldots, x_n\}$ is nonnegative if $\sum_{x \in S} x \ge 0$ and is negative otherwise.*

Definition 7.2.2. *Given $x_1 \ge \cdots \ge x_n \in \mathbb{R}$, we define*

$$\mathscr{F}_k := \{S \subset \{x_1, \ldots, x_n\} : |S| = k, S \text{ is nonnegative}\}$$

to be the set of nonnegative k-subsets of $x_1, \ldots, x_n$.

Definition 7.2.3. *We say a pair* $(n,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *is* <u>*good*</u> *if whenever we are given* $x_1 \geq \cdots \geq x_n \in \mathbb{R}$ *with nonnegative sum, we have* $|\mathscr{F}_k| \geq \binom{n-1}{k-1}$.

Definition 7.2.4. *Let* <u>$f(k)$</u> *be the minimum integer* N *such that for all* $n \geq N$, *the pair* (n,k) *is good.*

Definition 7.2.5. *A family* $\mathscr{F} \subset \binom{X}{k}$ *is a* <u>*star*</u> *if there exists* $x \in X$ *such that*

$$\mathscr{F} = \left\{ F \in \binom{X}{k} : x \in F \right\}.$$

Definition 7.2.6. *We say a pair* $(n,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *is* <u>*strongly good*</u> *if it is good and* $|\mathscr{F}_k| = \binom{n-1}{k-1}$ *if and only if* $\mathscr{F}_k$ *is the star on* x_1.

7.3 Results

Alon, Huang, and Sudakov [1] proved that $f(k) \leq \min\{33k^2, 2k^3\}$ and Manickam, Miklós, and Singhi conjectured that $f(k) \leq 4k$. Our main result shows that the veracity of Conjecture 7.1.1 boils down to proving its veracity in $k-1$ base cases.

Theorem 7.3.1 (Chowdhury)**.** *If* (n,k) *is a good pair, then* $(n+k,k)$ *is a good pair. Moreover, if* (n,k) *is strongly good and* $n \geq 2k-1$, *then* $(n+k,k)$ *is strongly good.*

Hence, we can verify Conjecture 7.1.1 for small k.

Theorem 7.3.2 (Chowdhury)**.** *We have* $f(3) = 11$ *and, more precisely, that the pair* $(n,3)$ *is strongly good when* $n \geq 11$. *We have* $f(4) \leq 24$ *and, more precisely, that the pair* $(n,4)$ *is strongly good if* $n = 22$ *or* $n \geq 24$.

The previous best upper bounds for $f(3)$ and $f(4)$ were $f(3) \leq 12$ from [87, 91] and $f(4) \leq 128$ from [1]. Although Conjecture 7.1.1 for $k = 3$ was previously tackled, our result is stronger because we determine $f(3)$ exactly and we characterize the case of equality; moreover our proof is simpler and provides a nice application of the Kruskal-Katona theorem. With more patience, our arguments could yield improved upper bounds on $f(k)$ for larger k. In Section 7.4, we also show how our method might yield a quadratic upper bound on $f(k)$ that improves on the current best upper bound [1] of $f(k) \leq \min\{33k^2, 2k^3\}$.

7.4 Reductions to Base Cases

This section contains a proof of Theorem 7.3.1. We first show that Baranyai's theorem, Theorem 3.4.33, or the Greene-Kleitman type argument in Section 4.6.2 yields a lemma due to Bier and Manickam [13] that proves Conjecture 7.1.1 in the case that $k|n$.

Lemma 7.4.1 (Bier-Manickam)**.** *For $k \in \mathbb{Z}^+$, the pair (ck,k) is good for any $c \in \mathbb{Z}^+$ and is strongly good if $c \neq 2$.*

Proof. We first show that Baranyai's theorem implies that the pair (ck,k) is good for any $c \in \mathbb{Z}^+$. Let $n = ck$ and suppose that $x_1 \geq \cdots \geq x_{ck} \in \mathbb{R}$ has nonnegative sum. We will prove that if $\mathscr{G} \subset \binom{X}{k}$ is a perfect matching, then there is a nonnegative k-set $G \in \mathscr{G}$. Since the total sum of $x_1,\ldots,x_{ck}$ is nonnegative and the k-subsets in $\mathscr{G}$ are disjoint,

$$0 \leq \sum_{i=1}^{ck} x_i = \sum_{G\in\mathscr{G}} \left(\sum_{i\in G} x_i \right).$$

Hence, at least one k-subset $G \in \mathscr{G}$ must have nonnegative sum. As $n = ck$, Baranyai's theorem asserts that the complete hypergraph on ck vertices $\binom{X}{k}$ has a partition $\mathscr{P}$ into $\binom{ck-1}{k-1}$ perfect matchings. Each perfect matching in $\mathscr{P}$ contains a nonnegative k-subset, and as $\mathscr{P}$ is a partition, the corresponding k-subsets are distinct. Hence, there are at least $\binom{ck-1}{k-1}$ k-element subsets with nonnegative sum.

We do not need the full power of Baranyai's theorem to prove the statement, however. The Greene-Kleitman type argument in Section 4.6.2 not only yields the same conclusion, but also allows us to prove that (ck,k) is strongly good when $c \neq 2$. Let $\mathscr{G} \subset \binom{X}{k}$ be a perfect matching and let $\pi \in S_{ck}$ be a permutation. As in Lemma 4.5.7, the family $\pi(\mathscr{G}) := \{\pi(G) : G \in \mathscr{G}\}$ is also a perfect matching. Recall that $\mathscr{F}_k \subset \binom{X}{k}$ from Definition 7.2.2 is the family of nonnegative k-subsets in $x_1,\ldots,x_{ck}$. We showed in the preceding paragraph that any perfect matching contains a nonnegative k-subset so $|\mathscr{F}_k \cap \pi(\mathscr{G})| \geq 1$ for any $\pi \in S_{ck}$. On the other hand, given $G \in \mathscr{G}$ and $F \in \mathscr{F}_k$, there are $k!(ck-k)!$ permutations $\pi \in S_{ck}$ such that $\pi(G) = F$. Consequently,

$$\binom{ck}{k} |\mathscr{F}_k| k!(ck-k)! = |\mathscr{G}||\mathscr{F}_k|\{\pi \in S_{ck} : \pi(G) = F\} = \sum_{\pi\in S_{ck}} |\mathscr{F}_k \cap \pi(\mathscr{G})| \geq (ck)! \tag{7.4.1}$$

Rearranging (7.4.1) yields that $|\mathscr{F}_k| \geq \binom{ck-1}{k-1}$, as desired.

If there are exactly $\binom{ck-1}{k-1}$ nonnegative k-subsets, then equality holds everywhere in (7.4.1). Hence, for each $\pi \in S_{ck}$, we have $|\mathscr{F}_k \cap \pi(\mathscr{G})| = 1$. This proves that every perfect matching contains exactly one nonnegative k-subset in $\mathscr{F}_k$, which implies that $\mathscr{F}_k \subset \binom{X}{k}$ is intersecting. By the Erdős-Ko-Rado theorem, Theorem 2.2.2, if $c \neq 2$, then $\mathscr{F}_k$ must be the star on x_1. Hence, (ck,k) is strongly good if $c \neq 2$. □

Our next lemma shows that if a set of n' real numbers has at least $\binom{n'-1}{d-1}$ nonnegative d-subsets and (d,k) is a good pair, then it has at least $\binom{n'-1}{k-1}$ nonnegative k-subsets.

Lemma 7.4.2. *Suppose $x_1 \geq \cdots \geq x_{n'} \in \mathbb{R}$ has $|\mathscr{F}_d| \geq \binom{n'-1}{d-1}$. If (d,k) is a good pair, then $|\mathscr{F}_k| \geq \binom{n'-1}{k-1}$. Moreover, if $|\mathscr{F}_k| = \binom{n'-1}{k-1}$, $d \geq 2k-1$, and the pair (d,k) is strongly good, then $\mathscr{F}_k$ is the star on x_1.*

Proof. Count pairs (A,B) where $A \in \mathscr{F}_d$, $B \in \mathscr{F}_k$, and $B \subset A$. Since (d,k) is a good pair, each $A \in \mathscr{F}_d$ contains at least $\binom{d-1}{k-1}$ sets $B \in \mathscr{F}_k$. On the other hand, each $B \in \mathscr{F}_k$ is contained in at most $\binom{n'-k}{d-k}$ sets $A \in \mathscr{F}_d$. Putting all this together, we have

$$\binom{n'-1}{d-1}\binom{d-1}{k-1} \leq |\{(A,B) : A \in \mathscr{F}_d, B \in \mathscr{F}_k, B \subset A\}| \leq |\mathscr{F}_k|\binom{n'-k}{d-k}. \tag{7.4.2}$$

Hence, $|\mathscr{F}_k| \geq \binom{n'-1}{k-1}$.

If $|\mathscr{F}_k| = \binom{n'-1}{k-1}$, then (7.4.2) implies that $|\mathscr{F}_d| = \binom{n'-1}{d-1}$, that each $A \in \mathscr{F}_d$ contains exactly $\binom{d-1}{k-1}$ sets $B \in \mathscr{F}_k$, and that if $B \in \mathscr{F}_k$, then every d-set that contains B lies in $\mathscr{F}_d$. Clearly, $\{x_1,\ldots,x_k\} \in \mathscr{F}_k$ as it is the k-set with largest sum. Let $B' \subset \{x_1,\ldots,x_{n'}\}$ be any k-subset containing x_1. We will show that $B' \in \mathscr{F}_k$. Since $d \geq 2k-1$, there exists a d-subset A' that contains $B' \cup \{x_1,\ldots,x_k\}$. Observe that $A' \in \mathscr{F}_d$ because it contains $\{x_1,\ldots,x_k\} \in \mathscr{F}_k$. Now A' contains exactly $\binom{d-1}{k-1}$ sets $B \in \mathscr{F}_k$ and since (d,k) is strongly good, these sets form the star on x_1. Hence, $B' \in \mathscr{F}_k$ as it contains x_1 and lies in A' so $\mathscr{F}_k$ is the star on x_1. □

Lemma 7.4.2 has two corollaries; the first is Theorem 7.3.1.

Proof of Theorem 7.3.1. Suppose, for a contradiction, that (n,k) is a good pair, but that $(n+k,k)$ is not a good pair. Consequently, there exist $x_1 \geq \cdots \geq x_{n+k} \in \mathbb{R}$ with nonnegative sum and $|\mathscr{F}_k| < \binom{n+k-1}{k-1}$. By the Pascal rule, there are greater than $\binom{n+k-1}{k}$

negative k-subsets. As $\sum_{i=1}^{n+k} x_i \geq 0$, the complement of a negative k-subset must be a positive n-subset. Hence, $|\mathscr{F}_n| > \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$. We may now apply Lemma 7.4.2 with $n' = n+k$ and $d = n$ to conclude that $|\mathscr{F}_k| \geq \binom{n+k-1}{k-1}$, which contradicts our initial assumption. Hence, if (n,k) is good, then $(n+k,k)$ is also good. If (n,k) is strongly good and $n \geq 2k-1$, then a similar argument yields that $\mathscr{F}_k$ is the star on x_1. □

The second corollary of Lemma 7.4.2 is a result of Manickam and Singhi that shows that if (n,k) is a good pair, then (cn,k) is a good pair for any $c \in \mathbb{Z}^+$.

Corollary 7.4.3 (Manickam-Singhi, [89])**.** *If (n,k) is a good pair, then (cn,k) is a good pair for any $c \in \mathbb{Z}^+$. If (n,k) is a strongly good pair and $n \geq 2k-1$, then (cn,k) is strongly good for any $c \in \mathbb{Z}^+$.*

Proof. Let $x_1 \geq \cdots \geq x_{cn} \in \mathbb{R}$ have nonnegative sum. By Lemma 7.4.1, the pair (cn,n) is good so $|\mathscr{F}_n| \geq \binom{cn-1}{n-1}$. If the pair (n,k) is good, then we may apply Lemma 7.4.2 with $n' = cn$ and $d = n$ to conclude that $|\mathscr{F}_k| \geq \binom{cn-1}{k-1}$. If $|\mathscr{F}_k| = \binom{cn-1}{k-1}$, the pair (n,k) is strongly good, and $n \geq 2k-1$, then we may apply Lemma 7.4.2 again to conclude that $\mathscr{F}_k$ is the star on x_1. □

We now show that if $\gcd(r,k) = 1$ and $(\alpha k + r, k)$ is a (strongly) good pair, then (n,k) is a (strongly) good pair for any $n \geq (\alpha k + r)(k-1)$ and so $f(k) \leq (\alpha k + r)(k-1)$. Hence, if $\gcd(r,k) = 1$ and we could show $(\alpha k + r, k)$ is a good pair for some $\alpha < 33$, then we could improve the current best upper bound $f(k) \leq \min\{33k^2, 2k^3\}$ in [1]. Two natural candidates for this approach would be the pairs $(4k-1,k)$ and $(4k+1,k)$, but we were not able to show that these were good pairs in general.

Corollary 7.4.4. *If $\gcd(r,k) = 1$ and $(\alpha k + r, k)$ is a (strongly) good pair, then (n,k) is a (strongly) good pair for any $n \geq (\alpha k + r)(k-1)$, and hence $f(k) \leq (\alpha k + r)(k-1)$.*

Proof If $(\alpha k + r, k)$ is a (strongly) good pair, then $((\alpha k + r)i, k)$ is a (strongly) good pair, where $i \in [k-1]$, by Corollary 7.4.3. We have $(\alpha k + r)i \equiv ri \pmod{k}$ and since $\gcd(r,k) = 1$, we have hit all nonzero congruence classes modulo k. Hence, (n,k) is a (strongly) good pair for any $n \geq (\alpha k + r)(k-1)$ by Theorem 7.3.1 and Lemma 7.4.1. As a result, $f(k) \leq (\alpha k + r)(k-1)$. □

7.5 The MMS Conjecture when k is Three

It is not difficult to see that $f(2) = 6$ and that $(n,2)$ is a strongly good pair if and only if $n = 6$ or $n \geq 8$. In this section, we prove Theorem 7.3.2 in the case that $k = 3$. That is, we show $f(3) = 11$ and that $(n,3)$ is a strongly good pair if $n \geq 11$. By Theorem 7.3.1, it suffices to show that the pairs $(11,3)$ and $(13,3)$ are strongly good. This is best possible since when $n \in \{4,5,7,8,10\}$, the pair $(n,3)$ is not good. In our proof, we will use the full version of the Kruskal-Katona theorem, Theorem 2.1.4. We first observe that we can get a lower bound on $|\mathscr{F}_k|$ given that it contains a certain sum.

Lemma 7.5.1. *Define $F_1(i_1) = i_1$ and $F_k(i_1,\ldots,i_k)$ recursively by*

$$F_k(i_1,\ldots,i_k) = \binom{i_k}{k} - \binom{i_k - i_1}{k} - \sum_{j=1}^{k-2} F_j(i_1,\ldots,i_j)\binom{i_k - i_{j+1}}{k-j}.$$

If $\sum_{l=1}^{k} x_{i_l} \in \mathscr{F}_k$, where $i_1 < \cdots < i_k$, then $|\mathscr{F}_k| \geq F_k(i_1,\ldots,i_k)$.

Proof. We assume $x_1 \geq \cdots \geq x_n$. Hence, if $\sum_{l=1}^{k} x_{i_l} \in \mathscr{F}_k$ then $\mathscr{F}_k$ contains $\sum_{l=1}^{k} x_{j_l}$, where $j_l \leq i_l$ for $l \in [k]$ and $j_1 < \cdots < j_k$. By induction, $F_k(i_1,\ldots,i_k)$ counts the number of k-tuples $(j_1,\ldots,j_k)$ satisfying $j_l \leq i_l$ for $l \in [k]$ and $j_1 < \cdots < j_k$. Hence, if $\sum_{l=1}^{k} x_{i_l}$ lies in $\mathscr{F}_k$, where $i_1 < \cdots < i_k$, then $|\mathscr{F}_k| \geq F_k(i_1,\ldots,i_k)$. □

Lemma 7.5.2. *The pair $(11,3)$ is strongly good.*

Proof. We have $x_1 \geq \cdots \geq x_{11}$ satisfying $\sum_{i=1}^{11} x_i \geq 0$. We first show that $|\mathscr{F}_3| > \binom{10}{2}$ if $x_1 + x_{11} \leq 0$. Since $\sum_{i=1}^{11} x_i \geq 0$, we have $\sum_{i=2}^{10} x_i \geq 0$. Let $\mathscr{S}$ be the family of nonnegative 3-subsets among $x_2,\ldots,x_{10}$. By Lemma 7.4.1, $|\mathscr{S}| \geq \binom{8}{2} = 28$ and Theorem 2.1.4 yields that $|\partial\mathscr{S}| \geq 20$. As we can add x_1 to any 2-subset in $\partial\mathscr{S}$ to form a nonnegative 3-set, we have $|\mathscr{F}_3| \geq 28 + 20 > \binom{10}{2}$. Hence, we may assume that $x_1 + x_{11} > 0$.

We may assume $x_1 + x_{10} + x_{11} < 0$ as otherwise $\mathscr{F}_3$ contains the star on x_1. Since $\sum_{i=1}^{11} x_i = 0$, we must have $\sum_{i=2}^{7} x_i \geq (6/8)\sum_{i=2}^{9} x_i > 0$. Let $\mathscr{W}$ be the family of nonnegative 3-subsets among $x_2,\ldots,x_7$. Lemma 7.4.1 yields that $|\mathscr{W}| \geq 10$. Note that $x_1 + x_2 + x_{11} \in \mathscr{F}_3$ since x_2 and $x_1 + x_{11}$ are nonnegative. If $x_1 + x_8 + x_{10} \geq 0$, then Lemma 7.5.1 implies that $|\mathscr{F}_3| \geq 35 + 10 + 1 > \binom{10}{2}$ by taking into account the sets in $\mathscr{W}$ and $x_1 + x_2 + x_{11}$. Hence, by Lemma 7.5.1, we may assume that

$$x_1 + x_8 + x_{10} < 0, \;\; x_2 + x_7 + x_9 < 0, \;\; x_3 + x_4 + x_{11} < 0. \tag{7.5.1}$$

Equation (7.5.1) and $\sum_{i=1}^{11} x_i \geq 0$ yield that $x_5+x_6>0$ so $x_5>0$.

By (7.5.1), we have $x_1+x_9+x_{10} \leq x_1+x_8+x_{10} < 0$. Since $\sum_{i=1}^{11} x_i \geq 0$, either $x_2+x_{11}>0$ or $\sum_{i=3}^{8} x_i > 0$. In the first case, $x_2+x_5+x_{11}>0$ since $x_5>0$, so $|\mathscr{F}_3| > \binom{10}{2}$ by Lemma 7.5.1. We now show that if $\sum_{i=3}^{8} x_i \geq 0$, then $|\mathscr{F}_3| > \binom{10}{2}$. Let $\mathscr{T}$ be the family of nonnegative 3-subsets among $x_3,\ldots,x_8$. By Lemma 7.4.1, $|\mathscr{T}| \geq \binom{5}{2} = 10$ and Theorem 2.1.4 yields that $|\partial\mathscr{T}| \geq 10$. We can add either x_1 or x_2 to any 2-subset in $\partial\mathscr{T}$ to form a nonnegative 3-subset. The family $\mathscr{F}_3$ also contains

$$x_1+x_2+x_i,\ \ i\in\{3,\ldots,11\};\quad x_1+x_b+x_c,\ 3\leq b\leq 5,\ \ c\in\{9,10,11\}, \tag{7.5.2}$$

since $x_1+x_{11}>0$ and $x_5>0$. Taking into account the sets in $\mathscr{T}$, the sets formed by adding x_1 or x_2 to a set in $\partial\mathscr{T}$, and the sets in (7.5.2), $|\mathscr{F}_3| \geq 10+10+10+18 > \binom{10}{2}$.

The preceding arguments show that if $x_1+x_{10}+x_{11}<0$, then $|\mathscr{F}_3| > \binom{10}{2}$. Hence, $|\mathscr{F}_3| \geq \binom{10}{2}$ and $|\mathscr{F}_3| = \binom{10}{2}$ if and only if $\mathscr{F}_3$ is the star on x_1. Consequently, $(11,3)$ is a strongly good pair. □

Lemma 7.5.3. *The pair* $(13,3)$ *is strongly good.*

Proof. We have $x_1 \geq \cdots \geq x_{13}$ satisfying $\sum_{i=1}^{13} x_i \geq 0$. As in the proof of Lemma 7.5.2, if $x_1+x_{13} \leq 0$, then Theorem 2.1.4 implies $|\mathscr{F}_3| > \binom{12}{2}$. Hence, we may assume that $x_1+x_{13}>0$.

We may assume $x_1+x_{12}+x_{13}<0$ as otherwise $\mathscr{F}_3$ contains the star on x_1. Let $\mathscr{T}$ be the family of nonnegative 3-subsets among $x_2,\ldots,x_{10}$. As in the proof of Lemma 7.5.2, we may conclude that $|\mathscr{T}| \geq \binom{8}{2} = 28$. Note that $x_1+x_2+x_i \in \mathscr{F}_3$ for $i\in\{11,12,13\}$ since x_2 and x_1+x_{13} are nonnegative. If $x_1+x_9+x_{10} \geq 0$, then Lemma 7.5.1 implies that $|\mathscr{F}_3| \geq 36+28+3 > \binom{12}{2}$ by taking into account the sets in $\mathscr{T}$ and the sets $x_1+x_2+x_i$ for $i\in\{11,12,13\}$. Lemma 7.5.1 thus implies

$$x_1+x_9+x_{10}<0,\ \ x_3+x_8+x_{12} \leq x_2+x_7+x_{11}<0,\ \ x_4+x_5+x_{13}<0. \tag{7.5.3}$$

Equation (7.5.3) and $\sum_{i=1}^{13} x_i \geq 0$ yield that $x_6>0$ so $x_1+x_6+x_{13}>0$. Consequently Lemma 7.5.1 implies that $|\mathscr{F}_3| \geq 45+28 > \binom{12}{2}$ by taking into account the sets in $\mathscr{T}$. We conclude that $(13,3)$ is a strongly good pair. □

7.6 The MMS Conjecture when k is Four

The preceding arguments yield improved upper bounds on $f(k)$ for $k > 3$. For example, in this section, we prove Theorem 7.3.2 in the case that $k = 4$. That is, we show $f(4) \leq 24$ and that $(n,4)$ is a strongly good pair if $n = 22$ or $n \geq 24$. Taking into account the counterexamples in Section 7.1, this shows $14 \leq f(4) \leq 24$. By Lemma 7.4.1 and Theorem 7.3.1, it suffices to prove that $(22,4)$, $(25,4)$, and $(27,4)$ are strongly good pairs.

Lemma 7.6.1. *The pair* $(22,4)$ *is strongly good.*

Proof. We have $x_1 \geq \cdots \geq x_{22}$ satisfying $\sum_{i=1}^{22} x_i \geq 0$. As in the proof of Lemma 7.5.2, if $x_1 + x_{22} \leq 0$, then Theorem 2.1.4 implies $|\mathscr{F}_4| > \binom{21}{3}$. Hence, we may assume that $x_1 + x_{22} > 0$.

We may assume $x_1 + x_{20} + x_{21} + x_{22} < 0$ as otherwise $\mathscr{F}_4$ contains the star on x_1. Let $\mathscr{S}$ be the family of nonnegative 4-subsets among $x_2, \ldots, x_{17}$. As in the proof of Lemma 7.5.2, we may conclude that $|\mathscr{S}| \geq \binom{15}{3}$. Consequently, if there are greater than $\binom{21}{3} - \binom{15}{3} = 875$ nonnegative 4-sets on x_1, then $|\mathscr{F}_4| > \binom{21}{3}$. Hence, by Lemma 7.5.1, we may assume that

$$x_1 + x_{11} + x_{14} + x_{22} < 0, \quad x_2 + x_9 + x_{15} + x_{21} < 0, \quad x_3 + x_7 + x_{16} + x_{19} < 0, \quad (7.6.1)$$
$$x_4 + x_{10} + x_{12} + x_{18} < 0, \quad x_5 + x_6 + x_{13} + x_{20} < 0.$$

Since $\sum_{i=1}^{22} x_i \geq 0$, equation (7.6.1) implies that $x_8 + x_{17} > 0$. We consequently have that $x_1 + x_8 + x_{17} + x_{22} > 0$ since $x_1 + x_{22} > 0$, which implies that $|\mathscr{F}_4| > \binom{21}{3}$ by Lemma 7.5.1. We conclude that $(22,4)$ is a strongly good pair. □

Lemma 7.6.2. *The pair* $(25,4)$ *is strongly good.*

Proof. We have $x_1 \geq \cdots \geq x_{25}$ with $\sum_{i=1}^{25} x_i \geq 0$. We may assume $x_1 + x_{23} + x_{24} + x_{25} < 0$, as otherwise $\mathscr{F}_4$ contains the star on x_1. Let $\mathscr{S}$ be the family of all nonnegative 4-sets in $x_2, \ldots, x_{21}$. As in the proof of Lemma 7.5.2, we may conclude that $|\mathscr{S}| \geq \binom{19}{3}$. Consequently, if there are greater than $\binom{24}{3} - \binom{19}{3} = 1055$ nonnegative 4-sets on x_1 then

$|\mathscr{F}_4| > \binom{24}{3}$. Hence, by Lemma 7.5.1, we may assume that

$$x_1 + x_7 + x_{18} + x_{25} < 0, \quad x_2 + x_8 + x_{19} + x_{24} < 0, \quad x_3 + x_9 + x_{16} + x_{23} < 0, \quad (7.6.2)$$
$$x_4 + x_{10} + x_{15} + x_{22} < 0, \quad x_5 + x_{11} + x_{14} + x_{21} < 0, \quad x_6 + x_{12} + x_{13} + x_{20} < 0.$$

Since $\sum_{i=1}^{25} x_i \geq 0$, equation (7.6.2) implies that $x_{17} > 0$. Hence, $|\mathscr{F}_4| \geq \binom{17}{4} > \binom{24}{3}$. We conclude that $(25,4)$ is a strongly good pair. □

Lemma 7.6.3. *The pair* $(27,4)$ *is strongly good.*

Proof. We have $x_1 \geq \cdots \geq x_{27}$ with $\sum_{i=1}^{27} x_i \geq 0$. We may assume $x_1 + x_{25} + x_{26} + x_{27} < 0$, as otherwise $\mathscr{F}_4$ contains the star on x_1. Let $\mathscr{S}$ be the family of all nonnegative 4-sets in $x_2, \ldots, x_{21}$. As in the proof of Lemma 7.5.2, we may conclude that $|\mathscr{S}| \geq \binom{19}{3}$. Consequently, if there are greater than $\binom{26}{3} - \binom{19}{3} = 1631$ nonnegative 4-sets on x_1 then $|\mathscr{F}_4| > \binom{26}{3}$. Hence by Lemma 7.5.1, we may assume that

$$x_1 + x_{12} + x_{17} + x_{27} < 0, \quad x_2 + x_8 + x_{22} + x_{26} < 0, \quad x_3 + x_{11} + x_{16} + x_{23} < 0, \quad (7.6.3)$$
$$x_4 + x_{10} + x_{13} + x_{25} < 0, \quad x_5 + x_7 + x_{15} + x_{24} < 0, \quad x_6 + x_9 + x_{14} + x_{21} < 0.$$

Since $\sum_{i=1}^{27} x_i \geq 0$, equation (7.6.3) implies that $x_{18} + x_{19} + x_{20} > 0$. We consequently have $|\mathscr{F}_4| \geq \binom{18}{4} > \binom{26}{3}$. We conclude that $(27,4)$ is a strongly good pair. □

7.7 Open Problems

Like the preceding combinatorial questions in this thesis, the Manickam-Miklós-Singhi conjecture, Conjecture 7.1.1, has a vector space analog about which we know distressingly little. In this section, we show that the methods discussed in Section 7.4 may also be useful for attacking the vector space analog of Conjecture 7.1.1.

Recall that V is an n-dimensional vector space over a finite field $\mathbb{F}_q$. Suppose for each one-dimensional subspace $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, we assign a weight $f(v) \in \mathbb{R}$ such that $\sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} f(v) = 0$. Define the weight of a subspace $S \subset V$ to be the sum of the weights of all its one-dimensional subspaces,

$$f(S) := \sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix},\, v \subset S} f(v). \quad (7.7.1)$$

The family of k-dimensional subspaces with nonnegative weight will be denoted by

$$\mathscr{F}_{V,f,k} := \left\{ S \in \begin{bmatrix} V \\ k \end{bmatrix} : f(S) \geq 0 \right\}.$$

The vector space analog of Conjecture 7.1.1 states

Conjecture 7.7.1 (Manickam-Singhi, [89])**.** *Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$ and let $f : \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right] \to \mathbb{R}$ be a weighting of the one-dimensional subspaces of V such that $\sum_{v \in \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right]} f(v) = 0$. If $n \geq 4k$, then $|\mathscr{F}_{V,f,k}| \geq \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$.*

Unlike Conjecture 7.1.1, we do not have a good reason for the $n \geq 4k$ stipulation. The counterexamples in Section 7.1 do not generalize to the vector space case. In fact, there are no known counterexamples to Conjecture 7.7.1 for $n > 2k$, and it is easy to construct counterexamples to Conjecture 7.7.1 when $k < n < 2k$. Hence, it is possible that Conjecture 7.7.1 is true for $n \geq 2k$.

In this section, we show that the veracity of Conjecture 7.7.1 for $k = 2$ also boils down to proving its veracity in a few base cases. As in Section 7.4, given integers $n, k \in \mathbb{Z}^+$ we say the pair $[n, k]$ is *good* if whenever we are given an n-dimensional vector space V over $\mathbb{F}_q$ and a weighting $f : \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right] \to \mathbb{R}$ with $\sum_{v \in \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right]} f(v) = 0$, we have

$$|\mathscr{F}_{V,f,k}| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{7.7.2}$$

Similarly, the pair $[n, k]$ is *strongly good* if it is good and (7.7.2) holds with equality if and only if $\mathscr{F}_{V,f,k}$ is the star on the one-dimensional subspace $\hat{v} \in \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right]$ with largest weight,

$$\mathscr{F}_{V,f,k} := \left\{ S \in \begin{bmatrix} V \\ k \end{bmatrix} : \hat{v} \subset S \right\}, \quad f(\hat{v}) \geq f(v) \;\forall\, v \in \begin{bmatrix} V \\ 1 \end{bmatrix}.$$

The results and proofs of Lemma 7.4.2, Lemma 7.4.1, and Corollary 7.4.3 generalize straightforwardly to the vector space setting. The proof of Theorem 7.3.1, however, does not readily generalize because of the issues discussed in Section 4.6.1.2. We have not yet been able to surmount these difficulties to prove a vector space analog of Theorem 7.3.1. Nevertheless, we can still show that for $k = 2$, the veracity of Conjecture 7.7.1 boils down to proving its veracity in a few base cases.

Lemma 7.7.2. *If $m \in \mathbb{Z}^+$ is odd and $[m, 2]$ is a (strongly) good pair, then $[cm - 1, 2]$ and $[cm + 1, 2]$ are (strongly) good pairs for any $c \in \mathbb{Z}^+$.*

Proof. Since m is odd, we may assume that $c \geq 2$. Let V be a $(cm-1)$-dimensional vector space over $\mathbb{F}_q$ and let $f : \begin{bmatrix} V \\ 1 \end{bmatrix} \to \mathbb{R}$ be a weighting of the one-dimensional spaces such that $\sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} f(v) = 0$. If $|\mathscr{F}_{V,f,m}| \geq \begin{bmatrix} cm-2 \\ m-1 \end{bmatrix}$, then by the vector space analog of Lemma 7.4.2 with $n = cm-1$, $d = m$, and $k = 2$, we have $|\mathscr{F}_{V,f,2}| \geq \begin{bmatrix} cm-2 \\ 1 \end{bmatrix}$. Hence, we may assume that $|\mathscr{F}_{V,f,m}| < \begin{bmatrix} cm-2 \\ m-1 \end{bmatrix}$.

Embed V in W, a cm-dimensional vector space over $\mathbb{F}_q$ and extend f to a weighting $\hat{f} : \begin{bmatrix} W \\ 1 \end{bmatrix} \to \mathbb{R}$ by giving every one-dimensional $w \in \begin{bmatrix} W \\ 1 \end{bmatrix} \setminus \begin{bmatrix} V \\ 1 \end{bmatrix}$ a weight $\hat{f}(w) = 0$. Since $\sum_{w \in \begin{bmatrix} W \\ 1 \end{bmatrix}} \hat{f}(w) = 0$, the vector space analog of Lemma 7.4.1 yields that $|\mathscr{F}_{W,\hat{f},m}| \geq \begin{bmatrix} cm-1 \\ m-1 \end{bmatrix}$.

If $U \in \mathscr{F}_{W,\hat{f}.m}$ then $U \in \mathscr{F}_{V,f,m}$ or $U \cap V \in \mathscr{F}_{V,f,m-1}$. Also note that each S in $\mathscr{F}_{V,f,m-1}$ lies in $q^{(c-1)m}$ spaces $U \in \mathscr{F}_{W,\hat{f},m}$. By the q-Pascal identity,

$$\begin{bmatrix} cm-2 \\ m-1 \end{bmatrix} + q^{(c-1)m} \begin{bmatrix} cm-2 \\ m-2 \end{bmatrix} = \begin{bmatrix} cm-1 \\ m-1 \end{bmatrix} \leq |\mathscr{F}_{W,\hat{f},m}| = |\mathscr{F}_{V,f,m}| + q^{(c-1)m} |\mathscr{F}_{V,f,m-1}|. \tag{7.7.3}$$

By assumption, $|\mathscr{F}_{V,f,m}| < \begin{bmatrix} cm-2 \\ m-1 \end{bmatrix}$ so (7.7.3) implies that $|\mathscr{F}_{V,f,m-1}| > \begin{bmatrix} cm-2 \\ m-2 \end{bmatrix}$. Since m is odd, we have that $m-1$ is even, and hence $[m-1,2]$ is a good pair by Lemma 7.4.1. The vector space analog of Lemma 7.4.2 with $n = cm-1$, $d = m-1$, and $k = 2$ thus yields that $|\mathscr{F}_{V,f,2}| > \begin{bmatrix} cm-1 \\ 1 \end{bmatrix}$.

If $|\mathscr{F}_{V,f,2}| = \begin{bmatrix} cm-1 \\ 1 \end{bmatrix}$, then (7.7.3) and the vector space analog of Lemma 7.4.2 imply that $|\mathscr{F}_{V,f,m}| = \begin{bmatrix} cm-1 \\ m-1 \end{bmatrix}$. Hence, if $[m,2]$ is strongly good, then the vector space analog of Lemma 7.4.2 implies that $\mathscr{F}_{V,f,2}$ is the star on the one-dimensional subspace $\hat{v} \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ with largest weight. This shows that if m is odd and the pair $[m,2]$ is (strongly) good, then the pair $[cm-1,2]$ is also (strongly) good for any $c \in \mathbb{Z}^+$; a similar argument shows that the pair $[cm+1,2]$ is also (strongly) good. □

A corollary of the vector space analog of Lemma 7.4.1 and Lemma 7.7.2 is that if the pairs $[5,2]$ and $[7,2]$ are good, then the pair $[n,2]$ is good for any $n \geq 4$. If the pair $[5,2]$ is not good, but the pairs $[7,2]$, $[9,2]$ and $[11,2]$ are good, then the vector space analog of Lemma 7.4.1 and Lemma 7.7.2 yield that the pair $[n,2]$ is good for any $n \geq 6$.

Chapter 7, in part, is currently being prepared for submission for publication of the material. Chowdhury, Ameera. The dissertation author was the primary investigator and author of this material.

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