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Fixed points of analytic actions of supersoluble Lie groups on compact surfaces

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Abstract

We show that every real analytic action of a connected supersoluble Lie group on a compact surface with nonzero Euler characteristic has a fixed point. This implies that E. Lima's fixed point free C^∞ action on S^2 of the affine group of the line cannot be approximated by analytic actions. An example is given of an analytic, fixed point free action on S^2 of a solvable group that is not supersoluble.

Introduction

Let M denote a compact connected surface, with possibly empty boundary ∂M, endowed with a (real) analytic structure. T_pM is the tangent space to M at $p \in M$. The Euler characteristic of M is denoted by $\chi(M)$.

Let G be a Lie group with Lie algebra $\mathcal{L}(G) = \mathcal{G}$; all groups are assumed connected unless the contrary is indicated. An *action* of G on M is a homomorphism α from G to the group $\mathsf{H}(M)$ of homeomorphisms of M such that the evaluation map

$$\mathsf{ev}^\alpha = \mathsf{ev}\colon\ G \times M \to M,\ (g, x) \to \alpha(g)(x)$$

is continuous. We usually suppress notation for α, denoting $\alpha(g)(x)$ by $g(x)$. The action is called C^r, $r \in \{1, 2 \ldots ; \omega\}$ if ev is a C^r map, where C^ω means analytic.

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The set $\mathcal{A}(G, M)$ of actions of G on M is embedded in the space of continuous maps $G \times M \to M$ by the correspondence $\alpha \mapsto \mathsf{ev}^\alpha$. We endow $\mathcal{A}(G, M)$ with the topology of uniform convergence on compact sets.

A point $p \in M$ is a *fixed point* for an action α of G if $\alpha(g)(p) = p$ for all $g \in G$. The set of fixed points is denoted by $\mathsf{Fix}\,(G)$ or $\mathsf{Fix}\,(\alpha(G))$.

In this paper we consider the problem of finding conditions on solvable group actions that guarantee existence of a fixed point.

When $\chi(M) \neq 0$, every flow (action of the real line $\mathbf{R}$) on M has a fixed point; this was known to Poincaré for flows generated by vector fields, and for continuous actions it is a well known consequence of Lefschetz's fixed point theorem. E. Lima [4] showed that every abelian group action on M has a fixed point, and J. Plante [6] extended this to nilpotent groups.

These results do not extend to solvable groups: Lima [4] constructed a fixed point free action on the 2-sphere of the solvable group A of homeomorphisms of $\mathbf{R}$ having the form $x \mapsto ax + b,\ a > 0, b \in \mathbf{R}$; and Plante [6] constructs fixed point free action of A on all compact surfaces. These actions are not known to be analytic; but Example 3 below describes a fixed point free, analytic action of a 3-dimensional solvable group on S^2.

Recall that G is *supersoluble* if every element of $\mathcal{G}$ belongs to a codimension one subalgebra (see Barnes [1]). Our main result is the following theorem:

Theorem 1 *Let G be a connected supersoluble Lie group and M a compact surface M such that $\chi(M) \neq 0$. Then every analytic action of G on M has a fixed point.*

Since the group A described above is supersoluble, Lima's C^∞ action cannot be improved to a fixed point free analytic action. The following result shows it cannot be approximated by analytic actions:

Corollary 2 *Let G and M be as in Theorem 1. If $\alpha \in \mathcal{A}(G, M)$ has no fixed point, then α has a neighborhood in $\mathcal{A}(G, M)$ containing no analytic action.*

Proof By Theorem 1 and compactness of M, it suffices to prove the following: For all convergent sequences $\beta_n \to \beta$ in $\mathcal{A}(G, M)$ and $p_n \to p$ in M, with $p_n \in \mathsf{Fix}\,(\beta_n(G))$, we have $p \in \mathsf{Fix}\,(\beta(G))$. Being a connected locally compact group, G is generated by a compact neighborbood K of the identity. Then $\beta_n(g) \to \beta(g)$ uniformly for $g \in K$, so $\beta(g)(p) = p$ for all $g \in K$. Since K generates G, this implies that $p \in \mathsf{Fix}\,(\beta(G))$. ∎

In Theorem 1, the hypothesis that G is connected is essential: the abelian group of rotations of S^2 generated by reflections in the three coordinate axes is a well known counterexample. And every Lie group with a nontrivial homomorphism to the group of integers acts analytically without fixed point on every compact surface admitting a fixed point free homeomorphism, thus on every surface except the disk and the projective plane.

The following example shows that supersolubility is essential:

Example 3
Let Q be the 3-dimensional Lie group obtained as the semidirect product of the real numbers $\mathbf{R}$ acting on the complex numbers $\mathbf{C}$ by $t \cdot z = e^{it}z$; this group is solvable but not supersoluble. Identify Q with the space $\mathbf{R} \times \mathbf{C} \approx \mathbf{R}^3$ and note that left multiplication defines a linear action of Q on $\mathbf{R}^3$. The induced action on the 2-sphere S of oriented lines in $\mathbf{R}^3$ through the origin has no fixed point, and $\chi(S) = 2$. Geometrically, one can see this as the universal cover of the proper euclidean motions of the plane, acting on two copies of the plane joined along a circle at infinity.

We thank F.-J. Turiel for pointing out a small error in an earlier version of our manuscript. He has also obtained some interesting results complementary to ours in [8].

Proof of Theorem 1

We assume given an action $\alpha\colon\ G \to \mathsf{H}(M)$. The *orbit* of $p \in M$ is $G(p) = \{g(x)\colon\ g \in G\}$. The *isotropy group* of $p \in M$ is the closed subgroup $I_p = \{g \in G\colon\ \alpha(g)(p) = p\}$. The *evaluation map* $\mathsf{ev}_p\colon\ G \to M$ at $p \in M$ is defined by $g \mapsto g(p)$.

Suppose that the action is C^r, $r \geq 1$. Then ev_p induces a bijective C^r immersion $i_p\colon\ G/I(p) \to G(p)$. The tangent space $E(p) \subset T_pM$ to this immersed manifold at p is the image of T_eG under the differential of ev_p at the identity $e \in G$.

For $j = 0, 1, 2$, let $V_j = V_j(G) \subset M$ denote the union of the j-dimensional orbits. Then $M = V_2 \cup V_1 \cup V_0$. Each V_j is invariant, V_2 is open, $V_1 \cup V_0$ is compact, and $V_0 = \mathsf{Fix}\,(G)$.

Lemma 4 (Plante) *Assume that G is solvable and that $G(p)$ is a compact 1-dimensional orbit. Then there is a closed normal subgroup $H \subset G$ of codimension 1 such that every point of $G(p)$ has isotropy group H.*

Proof Choose a homeomorphism $f\colon\ G(p) \approx S^1$ (the circle). Let $\beta\colon\ G \to \mathsf{H}(S^1)$ be the action defined by $\beta(g) = f \circ \alpha(g) \circ f^{-1}$. Because G is solvable, by a result of Plante ([6], Theorem 1.2) there exists a homeomorphism h of S^1 conjugating $\beta(G)$ to the rotation group SO(2). Since $\beta(G)$ is abelian and acts transitively on S^1, all points of S^1 have the same isotropy group for β; this isotropy group is the required H. ∎

Analyticity is used to establish the following useful property:

Lemma 5 *Assume that G acts analytically and that $\mathsf{Fix}\,(G) = \varnothing$. Then either $V_1 = M$ and $\chi(M) = 0$, or else V_1 is the (possibly empty) union of a finite family of orbits, each of which is a smooth Jordan curve contained in ∂M or in $M \setminus \partial M$.*

Proof Since there are no orbits of dimension 0, V_1 is a compact set comprising the points p such that $\dim E_p \leq 1$. It is easy to see that V_1 is a local analytic variety.

If $V_1 = M$ then the map $p \mapsto E_p$ is a continuous field of tangent lines to M, tangent to ∂M at boundary points. The existence of such a field implies that $\chi(M) = 0$.

Assume that $V_1 \neq M$. Note that $\dim_p V_1 \geq 1$ at each $p \in V_1$. Since M is connected and V_1 is a variety, V_1 must have dimension 1 at each point. The set of points where V_1 is not smooth is a compact, invariant 0-dimensional subvariety, i.e., a finite set of fixed points, hence empty. Since V_1 consists of 1-dimensional orbits, V_1 must be a compact, smooth invariant 1-manifold without boundary, i.e. each component of V_1 is a Jordan curve. Since ∂M is the union of invariant Jordan curves, any component of V_1 that meets ∂M is a component of ∂M. ∎

In view of Lemma 5, it suffices to prove the following more general result:

Proposition 6 *Let G be a connected supersoluble Lie group acting continuously on the compact connected surface M. Assume that*

(a) *there are no fixed points*

(b) *for each closed subgroup H, $V_1(H)$ is the union (perhaps empty) of finitely many disjoint Jordan curves.*

Then $\chi(M) = 0$.

By passing to a universal covering group we assume that G is simply connected. This implies that every closed subgroup is simply connected (see Hochschild [2], Theorem XII.2.2.)

We proceed by induction on $\dim G$, the case $G = \mathbf{R}$ having been covered in the introduction. Henceforth assume inductively that $\dim G = n \geq 2$ and that the proposition holds for all supersoluble groups of lower dimension. With this hypothesis in force, we first rule out the case that M is a disk:

Proposition 7 *If M is as in Proposition 6, then $\chi(M) \neq 1$*

Proof Suppose not; then M is a closed 2-cell. Since there are no fixed points, ∂M is an orbit, hence a component of V_1. Every component of V_1 bounds a unique 2-cell in M, and there are only finitely many such 2-cells. Let D be one that contains no other. Then D is invariant under G, and the action of G on D is fixed point free. Therefore we may assume that $M = D$, so that $V_1 = \partial M$.

By Lemma 4 there exists a closed normal subgroup H of codimension one with $\partial M \subset \mathsf{Fix}\,(H)$. Let $R \subset G$ be a 1-parameter subgroup transverse to H at the identity; then $RH = G$.

Because G is supersoluble, there is a codimension one subalgebra $\mathcal{K} \subset \mathcal{G}$ containing the Lie algebra $\mathcal{R}$ of R. Because G is simply connected and solvable $\mathcal{K}$ is the Lie algebra of a closed subgroup $K \subset G$ of dimension $n-1$, and $KH = G$. By the induction hypothesis there exists $p \in \mathsf{Fix}\,(K)$. Then $\dim G(p) \leq \dim G - \dim K = 1$. Therefore $p \in V_1 = \partial D$. We now have $p \in \mathsf{Fix}\,(K) \cap \mathsf{Fix}\,(H) = \mathsf{Fix}\,(G)$, a contradiction. ∎

We return now to the case of general M.

Denote the connected components of $M \setminus V_1$ by $U_i, \ldots, U_r,\ r \geq 1$. Each U_i is an open orbit, whose set theoretic boundary $\mathsf{bd}\,U_i$ is a (possibly empty) union of components of V_1. The closure $\overline{U}_i$ is a compact surface invariant under G, whose boundary as a surface is $\partial U_i = \mathsf{bd}\,U_i$.

We show that U_i is an open annulus. Let $H \subset G$ be the isotropy subgroup of $p \in U_i$. Evaluation at p is a surjective fibre bundle projection $G \to U_i$ with standard fibre H. Therefore there is an exact sequence of homotopy groups

$$\cdots \to \pi_j(G) \to \pi_j(U_i) \to \pi_{j-1}(H) \to \pi_{j-1}(G) \to \cdots \to \pi_0(G) = \{0\}$$

ending with the trivial group $\pi_0(G)$ of components of G. The component group $\pi_0(H)$ is solvable (see Raghunathan [7], Proposition III.3.10), so taking $j = 1$ shows that $\pi_1(U_i)$ is solvable. Therefore U_i is a sphere, torus, open 2-cell, or open annulus. If U_i is a torus then $U_i = M$, contradicting $\chi(M) \neq 0$. The sphere is ruled out by the exact sequence $\pi_2(G) \to \pi_2(U_i) \to \pi_1(H)$, because $\pi_2(G) = 0$ for every Lie group and $\pi_1(H) = 0$. Proposition 7 rules out the 2-cell.

It follows that $\overline{U}_i$ is a closed annulus, so $\chi(\overline{U}_i) = 0$. By the additivity property $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ of the Euler characteristic, any space M built by gluing annuli along their boundary circles must have $\chi(M) = 0$.

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