LBL-20242
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

Submitted for publication

PARTICLES, FIELDS
AND THE GAUSSIAN APPROXIMATION

R. Ingermanson

September 1985

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

LBL-20242

DISCLAIMER

September 17, 1985 LBL-20242

PARTICLES, FIELDS AND THE GAUSSIAN APPROXIMATION *

Randall Ingermanson

Theoretical Physics, 50A-3115
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720, U.S.A.

ABSTRACT

We review the Gaussian effective potential in two entirely different systems of notation. We prove the equivalence of the two systems and show how to translate from one to the other. We discuss the extension of the Gaussian approximation (in both notations) to states in which the classical field is not constant, and discuss prospects for future work on quantum solitons.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy Physics and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation, under Research Grant No. PHY-81-18547.

1 Introduction

Recently, the "Gaussian approximation" has proved useful for computing nonperturbatively the effective potential of certain bosonic field theories. Two notations are in common use. The notation used by Stevenson [1] employs an expansion of the field in terms of creation and destruction operators. We will call this system the "particle notation". The other system of notation was used by Barnes and Ghandour [2]. In this paper, a functional notation was adopted, in which the basic variables were the fields. We will call this system the "field notation".

Both systems of notation have been used extensively in the literature. The particle notation has been used by Stevenson [1] to analyse ϕ^4 theory and by Bardeen and Moshe [3] to study the large-N $(\vec{\phi}\cdot\vec{\phi})^2$ model at finite temperature. The field notation has been used to study a number of issues in pure Yang-Mills theory [4, 5, 6]. Consoli and Ciancitto [7] have examined ϕ^4 theory using *both* systems. A fairly complete list of references to papers written before 1980 is given by Stevenson [1]. A survey of these papers shows that no standardization of notation has developed; consequently, similar ideas have been expressed in several different guises.

Leaving aside the question of which notation is "better", we will first ask how these two systems are related. Actual calculations look superficially quite different in the two notations. Nevertheless, the answers always agree, because the two systems are completely equivalent. This fact is non-trivial. Obviously, the full quantum theory should give results which are independent of the notation. But the Gaussian approximation, (described below), is not the full quantum theory. The non-trivial statement is that the "Gaussian Effective Potential" advocated by *e.g.*, Stevenson [1], (using particle notation), is precisely the same as the effective potential using "Gaussian Wavefunctionals" espoused by *e.g.*, Barnes and Ghandour [2], (using field notation). The first part of this paper will be devoted to proving this equivalence.

We will also sketch the procedure for renormalization and for the explicit computation of the Gaussian effective potential. These ideas have been applied to particular models in refs. [1–3, 7–9]. A detailed calculation for an incompletely studied class of models will be presented in a future paper.

But we can do more. We need not restrict ourselves to computing the

effective potential (*i.e.*, the energy of states in which the classical field is constant.) We can also apply the same approximation scheme to states in which the classical field is some arbitrary function of $\vec{x}$. This topic has arisen in the older literature [8, 9, 10], but only two-dimensional models seem to have received serious attention. The best-known discussion appears in Coleman's lectures [10], under the name of the "coherent state approximation". In the latter half of this paper, we show that this approximation is nothing more than a special case of the Gaussian approximation.

Finally, there is a class of states for which the Gaussian approximation can and should be improved upon: namely, soliton states (if they exist in the theory). The reason that the Gaussian approximation is unsuitable is that the zero modes cannot be adequately described by a Gaussian wavefunction. Nevertheless, there is a simple, workable extension of the Gaussian method, which we will describe only briefly in this paper. A lengthy calculation will be given in a future work.

This paper is organized as follows. In Section 2, we discuss the free scalar field (for which the Gaussian approximation is exact) in both notations. Section 3 is a straightforward extension to the translationally-invariant static states of an interacting field theory, with an outline of the renormalization procedure. In Section 4, we discuss arbitrary static states, explain the relation to coherent states and make some general remarks on future extensions to solitons. Section 5 contains a brief conclusion.

2 The Free Scalar Field

In this section, we will consider the Lagrangian

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 \qquad (2.1)$$

in $D+1$ dimensions. It should be understood that the inclusion of internal degrees of freedom throughout this paper is trivial.

This field theory has the exact solution at $t = 0$ (see, *e.g.*, Itzykson and Zuber [11])

$$\begin{aligned} \phi(\vec{x}) &= \int d\tilde{k}\,[a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \text{h.c.}] \qquad (2.2)\\ \Pi(\vec{x}) &= -i\int d\tilde{k}\,\omega_k[a(\vec{k})e^{i\vec{k}\cdot\vec{x}} - \text{h.c.}] \end{aligned}$$

where

$$\omega_k \equiv \sqrt{\vec{k}^2 + m^2} \quad , \quad d\tilde{k} \equiv \frac{d^D\vec{k}}{(2\pi)^D}\frac{1}{2\omega_k}$$

and

$$\begin{aligned} [a(\vec{k}), a^\dagger(\vec{k}')] &= (2\pi)^D 2\omega_k\,\delta^D(\vec{k}-\vec{k}') \qquad (2.3)\\ [a(\vec{k}), a(\vec{k}')] &= 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')]. \end{aligned}$$

This solution satisfies the canonical commutation relation

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^D(\vec{x}-\vec{y}).$$

One can invert (2.2) to obtain

$$a(\vec{k}) = \int d^D\vec{x}\,e^{-i\vec{k}\cdot\vec{x}}[\omega_k\phi(\vec{x}) + i\Pi(\vec{x})]. \qquad (2.4)$$

The vacuum $|0\rangle$ is defined as the state which satisfies

$$\begin{aligned} \langle 0|0\rangle &= 1\\ a(\vec{k})\,|0\rangle &= 0. \end{aligned}$$

It is clear that $|0\rangle$ is the ground state of the system when we compute the Hamiltonian

$$H = \int d\tilde{k}\,\omega_k a^\dagger(\vec{k})a(\vec{k}) + \tfrac{1}{2}\delta^D(0)\int d^D\vec{k}\,\omega_k.$$

Obviously, $|0\rangle$ is the lowest energy eigenstate of H. It has energy

$$E_{vac} = \langle 0|H|0\rangle = \tfrac{1}{2}\delta^D(0)\int d^D\vec{k}\,\omega_k.$$

E_{vac} is usually shifted to zero, by normal-ordering H, but this is unnecessary cosmetic surgery.

It is convenient to rewrite

$$\delta^D(0) = \int \frac{d^D\vec{x}}{(2\pi)^D}$$

so that

$$E_{vac} = \int d^D\vec{x}\int\frac{d^D\vec{k}}{(2\pi)^D}\tfrac{1}{2}\omega_k. \qquad (2.5)$$

The above summarizes the well-known results for a free field theory in particle notation. The corresponding formulae in field notation are much less well known; we now turn to their derivation. We rewrite (2.1) as

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{1}{2}\dot{\phi}^2 - V(\phi)\\ V(\phi) &\equiv \tfrac{1}{2}(\partial_i\phi)^2 + \tfrac{1}{2}m^2\phi^2. \end{aligned}$$

The canonical momentum is

$$\Pi_x = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}_x} = \dot{\phi}_x.$$

(Here, ϕ_x means $\phi(\vec{x})$.) Then the classical Hamiltonian is

$$\begin{aligned} H &= \int_x (\Pi_x \dot{\phi}_x - \mathcal{L}_0) \\ &= \int_x \left[\tfrac{1}{2}\Pi_x^2 + V(\phi_x)\right]. \end{aligned}$$

To quantize the system, we assume the commutation relation

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^D(\vec{x} - \vec{y}).$$

We can realize this relation by the functional derivative

$$\Pi_x = -i\frac{\delta}{\delta \phi_x}.$$

Rewriting H as

$$H = \int_x \left[\tfrac{1}{2}\Pi_x^2 + \tfrac{1}{2}\phi_x(-\nabla_x^2 + m^2)\phi_x\right],$$

we see that our model is the functional equivalent of the standard quantum harmonic oscillator. It is natural to expect that the lowest energy eigenstate will be a Gaussian wavefunctional of the form

$$\Psi[\phi; f] = N_f \exp\left[-\tfrac{1}{2}\int_{x,y} \phi_x f_{xy} \phi_y\right]. \tag{2.6}$$

N_f is a normalization constant which depends on f_{xy}. It is trivial to compute N_f in terms of f_{xy}; however, it is also unnecessary. The real question is: what is f_{xy}? (Without loss of generality, we can assume that $f_{xy} = f_{yx}$, because of the symmetric form of (2.6).) We compute

$$\begin{aligned} \Pi_x^2 \Psi &= \Pi_x \left(i \int_y f_{xy}\phi_y\right) \Psi \\ &= \left[f_{xx} - \int_{y,z} f_{xy}\phi_y f_{zx}\phi_z\right] \Psi. \end{aligned}$$

So

$$H\Psi = \tfrac{1}{2}\left(\int_x f_{xx} + \int_{y,z} \phi_y \left[\delta_{yz}(-\nabla_z^2 + m^2) - \int_x f_{yx} f_{xz}\right] \phi_z\right) \Psi.$$

Thus, Ψ is an energy eigenstate if and only if

$$f_{xy} = \sqrt{-\nabla_x^2 + m^2}\,\delta_{xy}, \tag{2.7}$$

in which case the energy is

$$\begin{aligned} E_{vac} &= \tfrac{1}{2}\int_x f_{xx} \\ &= \tfrac{1}{2}\int d^D\vec{x} \int \frac{d^D\vec{p}}{(2\pi)^D}\sqrt{\vec{p}^2 + m^2} \\ &= \int d^D\vec{x} \int \frac{d^D\vec{p}}{(2\pi)^D}\tfrac{1}{2}\omega_p. \end{aligned} \tag{2.8}$$

The claim that Ψ represents the vacuum state is justified by observing that Ψ has no nodes. Under very general conditions, this characterizes the ground state wave-functional of a quantum system. (Feynman [6] gives a simple proof of this in his paper on Yang-Mills theory in $2+1$ dimensions.) Note that E_{vac} in (2.8) is the same as in (2.5). As we already noted, this is unremarkable; all our results so far are exact, so they are unaffected by our choice of notation.

We have constructed the vacuum explicitly (in field notation). What about excitations? We define the operator

$$A(\vec{k}) = \int_x e^{-i\vec{k}\cdot\vec{x}} \left[\int_y f_{xy}\phi_y + \frac{\delta}{\delta\phi_x}\right]. \tag{2.9}$$

It is trivial to verify that

$$A(\vec{k})\Psi = 0.$$

Substituting (2.7) into (2.9) and integrating by parts, we can rewrite it as

$$A(\vec{k}) = \int_x e^{-i\vec{k}\cdot\vec{x}}[\omega_k\phi_x + i\Pi_x]. \tag{2.10}$$

This should be compared to the formula (2.4) for $a(\vec{k})$, which we found in the particle notation. Clearly, the $A(\vec{k})$'s and their adjoints the $A^\dagger(\vec{k})$'s satisfy the usual algebra of creation and destruction operators given in (2.3). Moreover, $A^\dagger(\vec{k})$ creates a one-particle state of momentum $\vec{k}$ and energy ω_k when applied to the vacuum.

We have written the exact solution to the free theory in both the particle and the field notations and have seen how to translate from one system to the other. We now turn to the case of an interacting theory.

3 Interacting Fields: The Effective Potential

In this section we extend the Lagrangian (2.1) to be of the form

$$\mathcal{L} = \mathcal{L}_0 - U(\phi)$$

where U is some arbitrary potential. It is now impossible, in general, to compute the expansion of $\phi(\vec{x})$ in terms of creation and destruction operators or to compute the ground state wave-functional. A tractable approach is to force ϕ into the mold of a free field as much as possible. This basic idea has come to be called the Gaussian approximation. In this section, we will discuss the Gaussian effective potential in both the field and the particle notations and show that the two systems are equivalent. We consider the particle notation first, since it is more familiar.

We begin by approximating ϕ and Π by the linear expansions at $t = 0$:

$$\begin{aligned} \phi(\vec{x}) &= \Phi + \int d\tilde{k}\,[a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \text{h.c.}] \\ \Pi(\vec{x}) &= -i\int d\tilde{k}\,\omega_k[a(\vec{k})e^{i\vec{k}\cdot\vec{x}} - \text{h.c.}] \quad . \end{aligned} \tag{3.1}$$

Φ here is a constant, the expectation value of ϕ_x. We will define the effective potential as the energy of the state in which $\langle\phi_x\rangle = \Phi$. Because we expect the mass to get renormalized, we define $a^\dagger(\vec{k})$ and $a(\vec{k})$ to be the creation and destruction operators for particles of mass M, rather than of the bare mass m. So from now on, we will define ω_k by

$$\omega_k = \sqrt{\vec{k}^2 + M^2}$$

One could now go on to study any particular model by the following general procedure:

1. Define $|\Phi\rangle$ to be the normalized state which is destroyed by $a(\vec{k})$ and which therefore satisfies $\langle\Phi|\,\phi_x\,|\Phi\rangle = \Phi$.

2. Expand the Hamiltonian H by using (3.1) and compute the energy

$$E[\Phi; M] \equiv \langle\Phi|\,H\,|\Phi\rangle\,.$$

3. Consider Φ fixed and choose the value of M which minimizes the energy. Denote this value of M by $\mu(\Phi)$. That is,

$$\left.\frac{\partial E[\Phi; M]}{\partial M}\right|_{M=\mu(\Phi)} = 0.$$

[One should be alert for the possibility that $M = 0$ or $M \to \infty$ minimizes E.]

4. Define the Gaussian effective potential $\mathcal{V}(\Phi)$ by

$$\mathcal{V}(\Phi) \equiv E[\Phi; \mu(\Phi)].$$

5. The vacuum is now determined by finding the minimum energy state. Define $\overline{\Phi}$ to be the expectation value of ϕ in the vacuum state:

$$\left.\frac{\partial \mathcal{V}}{\partial \Phi}\right|_{\Phi=\overline{\Phi}} = 0.$$

Then the vacuum is just $\left|\overline{\Phi}\right\rangle$, in the Gaussian approximation. [Caution: the Gaussian effective potential is not convex [1], so a local minimum may not be a global minimum. In addition, there may be degenerate minima.]

6. Renormalize the model by expanding in a power series in $\Phi - \overline{\Phi}$:

$$\mathcal{V}(\Phi) = E_{vac} + \int d^D\vec{x}\,\left[\tfrac{1}{2}m_R^2(\Phi - \overline{\Phi})^2 + \ldots\right]$$

where m_R is the renormalized mass.

Essentially this procedure was used by Stevenson [1] and by Consoli and Ciancitto [7] to study the $\lambda\phi^4$ model in four dimensions. The reader is referred to these interesting papers for more details. In fact, an even earlier paper on the same topic was written by Barnes and Ghandour [2], using the less-familiar field notation. We will now discuss the Gaussian effective potential in this notation and prove the equivalence of the two systems.

We begin by defining the set of states (at $t = 0$)

$$\Psi[\phi; \Phi, f] = N_f \exp\left[-\tfrac{1}{2}\int_{x,y} \xi_x f_{xy} \xi_y\right] \tag{3.2}$$

where

$$\xi_x \equiv \phi_x - \Phi.$$

Note that we automatically have $\langle\Psi|\,\phi_x\,|\Psi\rangle = \Phi$. Furthermore, expectation values of any functional of ϕ_x are obtained by computing Gaussian path integrals, an easy operation. We have not yet specified f_{xy}. It should be chosen so as to minimize the energy

$$E[\Phi; f] \equiv \langle\Psi|\,H\,|\Psi\rangle\,.$$

One can show [2] that f_{xy} has the form

$$f_{xy} = \sqrt{-\nabla_x^2 + \mathcal{M}^2}\delta_{xy}$$

where $\mathcal{M}$ is some constant which depends on the particular Lagrangian under consideration.

We can again define destruction operators as in (2.9):

$$A(\vec{k}) \equiv \int_x e^{-i\vec{k}\cdot\vec{x}} \left[\int_y f_{xy}\xi_y + \frac{\delta}{\delta\xi_x} \right]. \tag{3.3}$$

By performing exactly the same manipulations that we did in the free-field case, one can show that $A(\vec{k})$ and $A^\dagger(\vec{k})$ satisfy the correct commutation relations to justify calling them creation and destruction operators of mass $\mathcal{M}$. Furthermore, we compute

$$\int d\tilde{k} \left[A(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \text{h.c.} \right] = \xi_x = \phi_x - \Phi$$
$$-i\int d\tilde{k}\,\omega_k \left[A(\vec{k})e^{i\vec{k}\cdot\vec{x}} - \text{h.c.} \right] = -i\frac{\delta}{\delta\xi_x} = \Pi_x$$

which was our starting point (3.1) in the particle notation. This guarantees that the two notations are equivalent.

There are two final points that are worth mentioning.

At least two authors [2, 8] have observed that the computation of the effective potential in the Gaussian approximation amounts to summing a particular class of diagrams, those with no overlapping divergences. Since the large-N expansion of the $O(N)$ scalar model sums only these diagrams, in leading order, this is some indication of why the first term in the large-N expansion is easy to compute; it is the Gaussian approximation.

Typically, the renormalized mass turns out to be $m_R = \mu(\overline{\Phi})$. This gives us some intuition for the rather mysterious function $\mu(\Phi)$: it is the mass of the one-particle excitations of the state in which the classical field is Φ. (We can imagine reaching this state by applying an external source to the vacuum.)

4 Interacting Fields: Static Configurations

Virtually all papers on the Gaussian approximation discuss only the effective potential, and are aimed at discovering the true vacuum, á lá Coleman

and Weinberg [12]. In this section, we will consider the extension of the above methods to states in which the classical field is not spatially constant. As usual, we will work in the particle notation first. These ideas have been discussed in general by Jackiw [13], and have been applied to specific models by Chang [8] and by Weinstein, Drell and Yankielowicz [9].

The states we want to consider are the states $|\Phi\rangle$ such that

$$\langle\Phi|\,\phi_x\,|\Phi\rangle = \Phi_x$$
$$\dot{\Phi}_x = 0.$$

A semi-classical approach to this problem would be the following: let

$$K_{xy} \equiv \left.\frac{\delta^2 L}{\delta\phi_x\delta\phi_y}\right|_{\phi_x=\Phi_x}, \tag{4.1}$$

where $L = \int_x \mathcal{L}$. (For a free field, K_{xy} is just the kernel of the Klein-Gordon operator $-(\partial_\mu^2 + m^2)\delta_{xy}^D$.) Let $U_n(\vec{x},t)$ be the solutions of the generalized Klein-Gordon equation

$$\int_y K_{xy}U_n(\vec{y},t) = 0.$$

n here is an index that might be partly discrete and partly continuous. (For the free field, the U_n's are the plane-waves $e^{i\vec{k}\cdot\vec{x}-i\omega_k t}$, indexed by $\vec{k}$.)

If the model has no derivative couplings, then K_{xy} is second-order in time derivatives, and it is trivial to separate variables. We find

$$U_n(\vec{x},t) = u_n(\vec{x})e^{\pm i\omega_n t}$$

where the $u_n(\vec{x})$ are eigenfunctions of the spatial part of K_{xy}, namely

$$k_{xy} \equiv K_{xy} + \partial_i^2\delta_{xy}.$$

This leaves the equation

$$\int_y k_{xy}u_n(\vec{y}) = -\omega_n^2 u_n(\vec{x}).$$

By analogy with (2.2) and (3.1), we then approximate the field and its conjugate momentum by the linear expansions at $t = 0$:

$$\phi(\vec{x}) = \Phi(\vec{x}) + \sum_n \frac{1}{\sqrt{2\omega_n}}\left[a_n u_n(\vec{x}) + \text{h.c.}\right] \tag{4.2}$$
$$\Pi(\vec{x}) = -i\sum_n \sqrt{\frac{\omega_n}{2}}\left[a_n u_n(\vec{x}) - \text{h.c.}\right].$$

If

$$[a_n, a_m^\dagger] = \delta_{nm} \qquad (4.3)$$
$$[a_n, a_m] = 0 = [a_n^\dagger, a_m^\dagger],$$

then

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^D(\vec{x} - \vec{y})$$

where we assume the orthonormality properties

$$\int_x u_n(\vec{x}) u_m^\dagger(\vec{x}) = \delta_{nm} \qquad (4.4)$$
$$\sum_n u_n(\vec{x}) u_n^\dagger(\vec{y}) = \delta^D(\vec{x} - \vec{y}).$$

[For convenience, we have changed the normalization of the creation and destruction operators from that in Sections 2 and 3.]

This semi-classical procedure would work quite well, except for one problem. Quantum effects change the classical action $\int dt\, L$ to some effective action $\Gamma[\Phi] = \int dt\, L^Q$ *which we do not know.*

Rather than defining K_{xy} as in (4.1), we should define some operator K_{xy}^Q which reflects the effects of quantum fluctuations

$$K_{xy}^Q = \frac{\delta^2 L^Q}{\delta\Phi_x \delta\Phi_y}.$$

But then the eigenfunctions $u_n(\vec{x})$ must be replaced by some unknown eigenfunctions $u_n^Q(\vec{x})$. Thus, our whole scheme is plagued by the usual problem of quantum effects feeding back and changing the starting point. The right approach to take is the following.

Assume that ϕ_x and Π_x can be expanded as in (4.2), where the functions $u_n(\vec{x})$ and the constants ω_n are unknown. (We will see that this is the Gaussian approximation.) $u_n(\vec{x})$ and ω_n will be determined variationally; they are to be chosen at the end of the calculation so as to minimize the energy

$$E[\Phi; \{u_n\}, \{\omega_n\}] = \langle\Phi|\, H\, |\Phi\rangle\,.$$

(H should be expanded in terms of the a_n's, ω_n's and $u_n(\vec{x})$'s.) $|\Phi\rangle$ is the state annihilated by the a_n's.

The computation of $E[\Phi; \{u_n\}, \{\omega_n\}]$ is rather easy, both in principle and in practice. What is not so easy is to solve the variational problem

$$\frac{\delta E}{\delta u_n(\vec{x})} = 0, \qquad (4.5)$$

which determines $u_n(\vec{x})$ and ω_n. We will return to this difficulty after deriving the analogous formulae in field notation.

It is easy to guess the form of the wave-functional which "ought" to correspond to the state $|\Phi\rangle$. By analogy to (2.6) and (3.2), we try

$$\Psi[\phi; \Phi, f] = N_f \exp\left[-\tfrac{1}{2}\int_{x,y} \xi_x f_{xy} \xi_y\right].$$

where

$$\xi_x \equiv \phi_x - \Phi_x.$$

Again, the problem of computing the energy functional

$$E[\Phi; f] = \langle\Psi|\, H\, |\Psi\rangle$$

amounts to a simple Gaussian path integration. Fixing Φ_x, we then face the non-trivial task of finding the function f_{xy} which minimizes $E[\Phi; f]$; *i.e.*, we must solve the equation

$$\frac{\delta E}{\delta f_{xy}} = 0. \qquad (4.6)$$

In practice, this can be reduced to an equation of the general form

$$\int_y f_{xy} f_{yz} = \left[-\nabla_x^2 + m^2 + F(\vec{x})\right] \delta_{xy}, \qquad (4.7)$$

where $F(\vec{x})$ is some function that depends on the detailed form of the interaction term in $\mathcal{L}$.

Formally, we can easily solve (4.7). Let $v_n(\vec{x})$ and λ_n^2 be the eigenfunctions and eigenvalues of the differential operator in (4.7):

$$\left[-\nabla_x^2 + m^2 + F(\vec{x})\right] v_n(\vec{x}) = \lambda_n^2 v_n(\vec{x}). \qquad (4.8)$$

This equation is real, so $v_n(\vec{x})$ and $v_n^\dagger(\vec{x})$ have the same eigenvalue λ_n. It is useful to denote $v_n^\dagger(\vec{x}) = v_{-n}(\vec{x})$. Assuming that the $v_n(\vec{x})$'s satisfy orthonormalization conditions analogous to (4.4), it is simple to check that (4.7) has the formal solution

$$f_{xy} = \tfrac{1}{2}\sum_n \lambda_n v_n(\vec{x}) v_n^\dagger(\vec{y}) + \text{h.c.} \qquad (4.9)$$

(This solution agrees with our results for the free-field case when we identify: $\lambda_n \to \omega_k$, $v_n(\vec{x}) \to e^{i\vec{k}\cdot\vec{x}}$, $\sum_n \to \int \frac{d^D\vec{k}}{(2\pi)^D}$ and $\delta_{nm} \to (2\pi)^D \delta^D(\vec{k} - \vec{k}')$.)

What is the connection of this formal solution to the (equally formal) approach we devised for solving the problem in particle notation? Based

on the exact free-field results, it is tempting to guess that $u_n(\vec{x}) \leftrightarrow v_n(\vec{x})$ and $\omega_n \leftrightarrow \lambda_n$. We can prove this conjecture without solving any differential equations. The proof is similar to the earlier proofs of equivalence in Sections 2 and 3. The basic strategy has three steps:

1. In the field notation, find destruction operators A_n which annihilate $\Psi[\phi;\Phi,f]$.

2. Compute ϕ_x and Π_x in terms of the A_n's, λ_n's and $v_n(\vec{x})$'s and show that they have precisely the same form as in (4.2), with $A_n \to a_n$, $\lambda_n \to \omega_n$ and $v_n(\vec{x}) \to u_n(\vec{x})$.

3. Observe that, because of (4.9), the variational equation (4.6) is guaranteed to give exactly the same eigenfunctions and eigenvalues as the variational equation (4.5).

To carry out step 1), we define the destruction operators

$$A_n \equiv \int_x \frac{v_n^\dagger(\vec{x})}{\sqrt{2\lambda_n}} \left[\int_y f_{xy}\xi_y + \frac{\delta}{\delta \xi_x} \right].$$

analogous to (2.9) and (3.3). One can quickly verify that

$$\begin{aligned} A_n\Psi &= 0 \\ [A_n, A_m^\dagger] &= \delta_{nm} \\ [A_n, A_m] &= 0 = [A_n^\dagger, A_m^\dagger]. \end{aligned}$$

To implement step 2), it requires only a short computation to prove that

$$\begin{aligned} \sum_n \frac{1}{\sqrt{2\lambda_n}} [A_n v_n(\vec{x}) + \text{h.c.}] &= \xi_x = \phi_x - \Phi_x \\ -i\sum_n \sqrt{\frac{\lambda_n}{2}} [A_n v_n(\vec{x}) - \text{h.c.}] &= -i\frac{\delta}{\delta \xi_x} = \Pi_x \end{aligned}$$

which is the same basic form as (4.2). The observation 3) then completes the proof.

We can now write formal expressions for f_{xy} and f_{xy}^{-1}:

$$\begin{aligned} f_{xy} &= \tfrac{1}{2}\sum_n \omega_n u_n(\vec{x}) u_n^\dagger(\vec{y}) + \text{h.c.} \\ f_{xy}^{-1} &= \tfrac{1}{2}\sum_n \frac{1}{\omega_n} u_n(\vec{x}) u_n^\dagger(\vec{y}) + \text{h.c.} \end{aligned}$$

The difficulty still remains that the variational problem is too hard, in either notation. So, of what use is the knowledge that the two systems are completely equivalent? This question is rather unfair. We know more than the mere fact of equivalence; we also have shown how to translate from one system to the other. This is important for two reasons: 1) it allows us to compare papers not written in the same notation, and 2) it allows us to do future calculations using the best features of both systems.

We illustrate these two comments by returning to the pressing problem of how to tackle the variational equation. Because the full variational equation cannot be easily solved, it is natural to look for some simplification. After all, *any* choice of f_{xy} will give an upper bound to the true quantum energy. A very simple choice of f_{xy} is to set the $u_n(\vec{x})$'s equal to the plane-waves.

In the particle notation, this appears to cause a problem. The destruction operators $a(\vec{k})$ associated with the plane-waves do not annihilate the state $|\Phi\rangle$; they annihilate the vacuum state $|0\rangle$. So how can we satisfy the requirement that $\langle\Phi|\,\phi_x\,|\Phi\rangle = \Phi_x$? To restate the issue, how shall we represent the states $|\Phi\rangle$ in terms of the "plane-wave" operators $a^\dagger(\vec{k})$ and $a(\vec{k})$?

This simple puzzle is resolved very easily in the field notation. The choice of plane-waves for the $u_n(\vec{x})$'s implies that f_{xy} can be written in the general form

$$f_{xy}^P = \int \frac{d^D\vec{k}}{(2\pi)^D} \cos[\vec{k}\cdot(\vec{x}-\vec{y})]\omega(\vec{k}).$$

The wave-functionals we want to consider are then

$$\begin{aligned} \Psi[\phi;\Phi,f^P] &= N_{f^P} \exp\left[-\tfrac{1}{2}\int_{x,y}(\phi_x - \Phi_x) f_{xy}^P (\phi_y - \Phi_y)\right] \\ &= \exp\left[-\int_x \Phi_x \frac{\delta}{\delta\phi_x}\right] N_{f^P} \exp\left[-\tfrac{1}{2}\int_{x,y}\phi_x f_{xy}^P \phi_y\right]. \end{aligned} \quad (4.10)$$

This resolves the question we faced above, since we can translate (4.10) into particle notation:

$$|\Phi\rangle = \exp\left[-i\int_x \Phi_x \Pi_x^{(F)}\right]|0\rangle\,.$$

(Here, $\phi_x^{(F)}$ and $\Pi_x^{(F)}$ are of the free-field form (2.2).) This class of states is known as the set of coherent states. By construction, they satisfy

$$\langle\Phi|\,\phi_x^{(F)}\,|\Phi\rangle = \Phi_x. \quad (4.11)$$

The above formulae show that coherent states are just a special subset of the Gaussian states. (Usually, coherent states are characterized as the set of eigenstates of $a(\vec{k})$, which has no *a priori* connection to Gaussian states.)

In the literature, one sometimes sees a slight generalization

$$|\Phi, P\rangle = \exp\left[i \int_x P_x \phi_x^{(F)}\right] |\Phi\rangle .$$

Note that

$$\langle \Phi, P | \Pi_x^{(F)} | \Phi, P \rangle = P_x, \tag{4.12}$$

and that the analog of (4.11) continues to hold. It is trivial to translate (4.12) into field notation; the resulting wave-functional is still Gaussian.

Coherent states were used several years ago by Coleman [10] to study solitons in two dimensions (using particle notation.) He found the discouraging fact that, after renormalization, the energy of the coherent state soliton is just the classical energy. This was also found for the special case of the $(\phi^4)_2$ kink soliton by Weinstein, Drell and Yankielowicz [9], using essentially the field notation. Coleman's observation may well be true in higher dimensions (where renormalization is somewhat trickier to perform), though we know of no general proof of this conjecture. Assuming its validity for the newly popularized Skyrme model [14] , we need to look for some way to improve the coherent-state approximation. The work in this section makes it clear that we have two basic options. We can 1) try to find a better approximation to f_{xy} or 2) go beyond the Gaussian approximation.

The first option requires us to solve a complicated partial differential equation. The second option appears more open-ended. Why should we go beyond the Gaussian approximation? So long as the potential is not too flat, a Gaussian wave-functional ought to give reasonably good results. This observation provides the key to the problem.

It is well-known that all solitons have zero modes. It costs no energy to translate a soliton spatially. In more than two dimensions, one can also rotate the soliton freely. These modes can *never* be represented accurately by a Gaussian wave-functional. They can be easily treated, however, by introducing collective coordinates. A good discussion of zero modes and collective coordinates is found in the review by Jackiw [13]. As a first approximation, it is common to quantize only the collective coordinates, ignoring vibrational modes. See, *e.g.*, the famous Skyrmion paper by Adkins, Nappi and Witten [15].

We propose that a reasonable scheme for improving on the semi-classical soliton calculations is to combine the collective coordinate approach with the coherent state approximation. One difficulty that must be resolved is that

there are more coordinates than degrees of freedom. We need to introduce constraints on the coordinates. Clearly, Dirac's system of constrained quantization [16] will be useful. It is interesting to note that the field notation seems to be much more convenient for this type of problem, a fact which was a major motivation for writing the present paper.

The above program has been carried out for a test case: the Sine-Gordon soliton, for which the exact answers were already known [17]. The results are strongly encouraging, and the calculation will be described in a forthcoming paper.

It is not so easy to perform the analogous calculation for the Skyrme model. The main problem there is the non-renormalizability of the model. We hope that this difficulty will not block all future progress on this important problem.

5 Conclusion

The purpose of this paper has been three-fold: 1) To review the essentials of the Gaussian effective potential, 2) to reveal the unity of ideas behind the diversity of notation, and 3) to advocate the extension of the Gaussian approximation (beyond the computation of the effective potential) to applications in soliton physics.

The Gaussian effective potential is, of course, an important topic. In a forthcoming paper, we will give a full account of the effective potential, renormalization, vacuum energy and vacuum stability in a sample model, the Sine-Gordon model (and also the sinh-Gordon model) in $D + 1$ dimensions. Much of this discussion is a prerequisite to soliton calculations, but it is interesting in its own right.

Another paper in preparation deals with our proposal to combine the Gaussian approximation with the collective coordinate approach. The paper will test this idea on the Sine-Gordon soliton, since we can compare it to the exact results of Dashen, Hasslacher and Neveu [17], to Coleman's paper on the bosonization of the massive Thirring model [18] and to the results of the coherent-state approximation reported in Coleman's lectures [10].

Acknowledgements

I thank Dae Sung Hwang for pointing out the need for this paper. I also thank Orlando Alvarez for his comments on the initial draft. This work was supported in part by the Director, Office of Energy Research, Office of High Energy Physics and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation, under Research Grant No. PHY-81-18547.

References

[1] P. M. Stevenson, Z. Phys. **C24** (1984), 87; Phys. Rev. **D30** (1984), 1712; University of Wisconsin-Madison preprint PRINT-85-0193.

[2] T. Barnes and G. I. Ghandour, Phys. Rev. **D22** (1980), 924.

[3] W. A. Bardeen and M. Moshe, Phys. Rev. **D28** (1983), 1372.

[4] R. Jackiw, Rev. Mod. Phys. **52** (1980), 661.

[5] G. C. Rossi and M. Testa, Nucl. Phys. **B163** (1980), 109; Nucl. Phys. **B176** (1980), 477.

[6] R. P. Feynman, Nucl. Phys. **B188** (1981), 479.

[7] M. Consoli and A. Ciancitto, Nucl. Phys. **B254** (1985), 653.

[8] J. S. Chang, Phys. Rep. **23C**, (1975), 301.

[9] M. Weinstein, S. D. Drell and S. Yankielowicz, Phys. Rev. **D14** (1976), 487.

[10] S. Coleman, in "New Phenomena in Subnuclear Physics" (A. Zichichi, Ed.) part A, p. 297, Plenum Press, New York, 1977.

[11] C. Itzykson and J. B. Zuber, "Quantum Field Theory", p. 115, McGraw-Hill, Inc., New York, 1980.

[12] S. Coleman and E. Weinberg, Phys. Rev. **D7** (1973), 1888.

[13] R. Jackiw, Rev. Mod. Phys. **49** (1977), 681.

[14] T. H. R. Skyrme, Proc. Roy. Soc. **A260** (1961), 127.

[15] G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. **B228**, (1983), 552.

[16] P. A. M. Dirac, Proc. Roy. Soc. London **A246** (1958), 326.

[17] R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. **D11** (1975), 3424.

[18] S. Coleman, Phys. Rev. **D11** (1975), 2088.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720