

UC Santa Barbara

UC Santa Barbara Electronic Theses and Dissertations

Title

Special properties of solutions to nonlocal dispersive equations

Permalink

<https://escholarship.org/uc/item/04f405r5>

ISBN

9798171614119

Author

Hong, Christian

Publication Date

2026-08-18

Peer reviewed|Thesis/dissertation

University of California
Santa Barbara

Special properties of solutions to nonlocal dispersive equations

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Applied Mathematics

by

Christian Hong

Committee in charge:

Professor Gustavo Ponce, Co-Chair
Professor Hector Ceniceros, Co-Chair
Professor Katy Craig
Professor Carlos Garcia-Cervera

September 2026

The Dissertation of Christian Hong is approved.

Professor Katy Craig

Professor Carlos Garcia-Cervera

Professor Gustavo Ponce, Committee Co-Chair

Professor Hector Cenicerros, Committee Co-Chair

August 2026

Special properties of solutions to nonlocal dispersive equations

Copyright © 2026

by

Christian Hong

Acknowledgements

I would like to thank my advisor, Professor Gustavo Ponce, for his guidance and support. Thank you for your patience and thank you for teaching me so much about dispersive equations.

I am also grateful to my second advisor, Professor Hector Ceniceros, for providing an opportunity to pursue research in an unexpected area. Without your guidance, I never would have developed my newfound interest in polymer theory and data science.

I would also like to thank my other committee members, Professor Katy Craig and Professor Carlos Garcia-Cervera. I greatly appreciate your instruction and coursework that helped broaden my horizons as a graduate student.

I also wish to thank many of the friends I made here as a student at UCSB. Thank you Alex and Will. I enjoyed the time and research we spent together on our project, and I could not imagine how that would have went without your knowledge and company. Thank you Connor for all the math conversations we had together. Thank you Dörde and Jack, my fellow cohort members, for your time and company during my entire stay as a PhD student. I also should not forget to thank you two for teaching me the joys of coffee.

Finally, I would like to thank my family. Thank you for your love and support and thank you for always encouraging me to pursue my interests. I would never had considered pursuing this career path without the care and belief you all had in me.

Curriculum Vitæ

Christian Hong

Education

- 2026 Ph.D. in Applied Mathematics (Expected), University of California, Santa Barbara.
- 2021 M.A. in Applied Mathematics, University of California, Santa Barbara.
- 2018 B.A. In Applied Mathematics and Computer Science, University of California, Berkeley

Publications

C. Hong, G. Ponce, *On special properties of solutions to the Benjamin-Bona-Mahony equation*, *Journal of Differential Equations* **393** (2024) 321-342.

C. Hong, *On local energy decay for solutions to the b-family of peakon equations*, *Non-linear Analysis* **253** (2025) 113739.

C. Hong, F. Linares, and G. Ponce, *On special properties of solutions to Camassa-Holm equation and related models*, to appear in *Advanced Studies in Pure Mathematics*.

A. Sabater, W. Sheppard, Christian Hong, Y. Xuan, H. Ceniceros, G. Fredrickson, *Automating the Discovery and Stability of Bulk Morphologies in $A(BA')_3$ Miktoarm Star Copolymers*, submitted to *Journal of Chemical Physics*

Abstract

Special properties of solutions to nonlocal dispersive equations

by

Christian Hong

Wave propagation phenomena often involve the study of dispersive partial differential equations, which is defined by the linear terms of the system considered. An interesting case arises when a model also incorporates nonlocal operators. In this work, we focus on the Benjamin-Bona-Mahony (BBM) equation and the b -family of peakon equations, which contain the nonlocal operator $(1 - \partial_x^2)^{-1}$, with the goal of studying unique continuation properties, smoothing properties, and local energy decay properties of the respective aforementioned systems.

Such properties can often be derived from dispersive estimates. However, the BBM equation has very weak dispersion relation and the b -family equations are not dispersive, so such techniques are unavailable. We instead derive the desired properties via methods that rely heavily upon the nonlinear terms of the respective equations.

Contents

Curriculum Vitae	v
Abstract	vi
1 Introduction	1
1.1 The Benjamin-Bona-Mahony (BBM) Equation	2
1.2 The b -Family of Peakon Equations	6
1.3 Main Results	14
1.4 Permissions and Attributions	27
2 Unique Continuation and Smoothing Effects for the BBM Equation	28
2.1 Preliminary estimates	28
2.2 Proof of the Unique Continuation Results for the BBM Equation	33
2.3 Proof of the Regularity Results for the BBM Equation	39
3 Local Energy Decay for a Family of Peakon Equations	43
3.1 Preliminary Estimates	43
3.2 Long-time L^2 Decay for the Peakon Equations	46
3.3 Long-time L^1 Decay for the Peakon Equations	52
3.4 Long-time Decay for the Peakon Equations on the Half-line	60
Bibliography	64

Chapter 1

Introduction

Nonlinear dispersive equations arise in several physical contexts, including wave propagation. The canonical equations of this type include the Schrödinger equation, the Korteweg-de-Vries (KdV) equation, and the Benjamin-Ono (BO) equation, among others. The BO and KdV equations are both examples of completely integrable systems. The BO equation is also a standard example of a nonlocal dispersive system (the dispersive relation is modeled by the operator $\mathcal{H}\partial_{xx}^2$ where $\mathcal{H}$ is the Hilbert transform). It turns out that there are many other interesting systems that model water waves that involve nonlocal operators. Examples include:

The Benjamin-Bona-Mahony (BBM) equation

$$\partial_t u + \partial_x (1 - \partial_x^2)^{-1} \left(u + \frac{u^2}{2} \right) = 0,$$

Camassa-Holm (CH) equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_x (1 - \partial_x^2)^{-1} \left(u^2 + \frac{1}{2} (\partial_x u)^2 \right) = 0,$$

and b-family of peakon equations

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_x (1 - \partial_x^2)^{-1} \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) = 0, \quad b \in (0, 3).$$

1.1 The Benjamin-Bona-Mahony (BBM) Equation

The Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x u + u \partial_x u + \partial_x^3 u = 0, \quad (1.1.1)$$

was first derived as a model of long waves in shallow water [1]. In [2] Benjamin-Bona-Mahony proposed the (BBM) equation

$$\partial_t u + \partial_x u + u \partial_x u - \partial_{xxt}^3 u = 0, \quad (1.1.2)$$

as a regularized version of the KdV equation.

Thus, for the BBM equation, its accuracy as an approximation to the KdV, the existence and stability of traveling waves, their interaction, the well-posedness and the asymptotic behavior of solutions, among many other aspects, have been extensively studied. See [3], [4], [5] and [6] and references therein.

We shall be concerned with the initial value problem (IVP) associated to the formally equivalent form of the BBM

$$\begin{cases} \partial_t u + (1 - \partial_x^2)^{-1} (\partial_x u + u \partial_x u) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.1.3)$$

The solution of the IVP (1.1.3) can then be expressed in the equivalent integral form :

$$u(x, t) = U(t)u_0(x) - \int_0^t U(t - t') \partial_x J^{-2} \left(\frac{u^2}{2} \right) (x, t') dt', \quad (1.1.4)$$

where $\{U(t) : t \in \mathbb{R}\}$ denotes the unitary group

$$U(t)f(x) = e^{t\partial_x J^{-2}} f(x) = (e^{-2\pi i t \xi (1+4\pi^2 \xi^2)^{-1}} \widehat{f}(\xi))^\vee(x) \quad (1.1.5)$$

with

$$J^s f(x) = (1 - \partial_x^2)^{s/2} f(x) = ((1 + 4\pi^2 \xi^2)^{s/2} \widehat{f}(\xi))^\vee(x), \quad s \in \mathbb{R}. \quad (1.1.6)$$

In [6], it was shown that the IVP (1.1.3) is globally well-posed in classical Sobolev spaces

$$H^s(\mathbb{R}) = J^{-s} L^2(\mathbb{R}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R}) \quad \text{with } s \geq 0. \quad (1.1.7)$$

Theorem 1.1.1 ([6]) *Let $s \geq 0$. For any $u_0 \in H^s(\mathbb{R})$ and for any $T > 0$ there exists a unique solution $u = u(x, t)$ of the IVP (1.1.3) satisfying*

$$u \in C([-T, T] : H^s(\mathbb{R})). \quad (1.1.8)$$

Moreover, the map data solution, $u_0 \rightarrow u$, is locally analytic from $H^s(\mathbb{R})$ to $C([-T, T] : H^s(\mathbb{R}))$.

This strong notion of well-posedness includes existence, uniqueness, persistence properties, i.e if $u_0 \in X$ then the solution $u = u(x, t)$ describes a continuous curve on X so $u \in C([0, T] : X)$, and regularity of the solution upon the initial data.

In [6], it was also proven that the result in Theorem 1.1.1 is the best possible in the scale of Sobolev spaces $H^s(\mathbb{R})$. More precisely, it was verified that in $H^s(\mathbb{R})$, for any

$s < 0$, the map data-solution $u_0 \rightarrow u(x, t)$ is not C^2 . Hence, well-posedness cannot be obtained in $H^s(\mathbb{R})$, $s < 0$, by an argument based on the contraction principle. This ill-posedness was strengthened in [7].

In [8], the well-posedness in $H^s(\mathbb{R})$, $s \geq 0$, established in [6] was extended to the L^p -Sobolev spaces :

$$L_s^p(\mathbb{R}) = J^{-s}L^p(\mathbb{R}) = (1 - \partial_x^2)^{-s/2}L^p(\mathbb{R}), \quad p \in [1, \infty), \quad (1.1.9)$$

with

$$s \geq \max \left\{ \frac{1}{p} - \frac{1}{2}; 0 \right\} \quad \text{if } p \in (1, \infty), \quad \text{and} \quad s > \frac{1}{2}, \quad \text{if } p = 1. \quad (1.1.10)$$

Moreover, it was shown in [8] that the well-posedness results obtained are optimal in the scale described in (1.1.9)-(1.1.10).

We will also consider the initial periodic boundary value problem (IPBVP) associated to the BBM equation

$$\begin{cases} \partial_t u + (1 - \partial_x^2)^{-1}(\partial_x u + u \partial_x u) = 0, & (t, x) \in \mathbb{R} \times \mathbb{S}, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.1.11)$$

The following local well-posedness result was established in [9] (see also [6] and [10]):

Theorem 1.1.2 ([9]) *Let $s \geq 0$. For any $u_0 \in H^s(\mathbb{S})$ and for any $T > 0$ there exists a unique solution $u = u(x, t)$ of the IPBVP (1.1.11) such that*

$$u \in C([-T, T] : H^s(\mathbb{S})). \quad (1.1.12)$$

Furthermore, the map data solution, $u_0 \rightarrow u$, is locally real analytic from $H^s(\mathbb{S})$ to $C([-T, T] : H^s(\mathbb{S}))$.

It was also shown in [9] that if $u_0 \in H^1(\mathbb{S})$ the quantities

$$\int_{\mathbb{S}} u(x, t) dx, \quad \int_{\mathbb{S}} ((\partial_x u)^2 + u^2)(x, t) dx, \quad \int_{\mathbb{S}} (u^3 + 3u^2)(x, t) dx \quad (1.1.13)$$

are time independent. This was used in the proof of the next result achieved in [9]:

Theorem 1.1.3 ([9]) *Let $u_0 \in H^1(\mathbb{S})$ be such that*

$$\int_{\mathbb{S}} u_0(x) dx \geq 0, \quad (1.1.14)$$

and

$$\|u_0\|_{\infty} < 3. \quad (1.1.15)$$

Assume that the corresponding solution $u = u(x, t)$ of the IPBVP (1.1.11) provided by Theorem 1.1.2 satisfies

$$u(x, t) = 0 \quad \forall (x, t) \in \omega \times (0, T), \quad (1.1.16)$$

where $\omega \subset \mathbb{S}$ is a non-empty open set. Then $u(x, t) \equiv 0$.

Remark 1.1.1 *1. Notice that Theorem 1.1.3 contain the global hypothesis (1.1.14) and (1.1.15).*

2. The proof used the conservation laws stated in (1.1.13). This explains the hypothesis $u_0 \in H^1(\mathbb{S})$.

3. As was remarked in [9], Theorem 1.1.3 fails if one assumes that $u_0 \in H^s(\mathbb{S}) \cap L^\infty(\mathbb{S})$, $s \in [0, 1/2)$ and removes assumption (1.1.14). Let $a, b \in \mathbb{S}$ and $a < b$.

Then consider the example:

$$u(x, t) = u_0(x) = \begin{cases} -2, & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases} \quad (1.1.17)$$

The above result describes an example of unique continuation for periodic solutions to the (1.1.3). Roughly, the unique continuation problem (UCP) can be stated as follows. Suppose that u_1, u_2 are solutions to the IVP, or the IPBVP, of an associated PDE such that

$$u_1 \equiv u_2, \quad (x, t) \in (a, b) \times [t_1, t_2].$$

Can we conclude that $u_1 \equiv u_2$? A particular case that we are interested in is when $u_2 = c_0$, a constant solution. The UCP is often proven for dispersive systems via Carleman estimates [5]. However, the linear terms in (1.1.3) are not conducive to such estimates due to the corresponding characteristic curves. Further discussion of the UCP of (1.1.3) can be found in [9].

1.2 The b -Family of Peakon Equations

In this work, we consider a family of equations related to the following system:

$$\partial_t u - \partial_{xxt}^3 u + 3u\partial_x u - u\partial_x^3 u - 2\partial_x u \partial_x^2 u = 0, \quad (t, x) \in \mathbb{R}^2, \quad (1.2.1)$$

which is known as the Camassa-Holm (CH) equation.

The CH equation was first noticed by Fuchssteiner and Fokas [11] when they studied hereditary symmetries. It was later derived by Camassa and Holm [12] as an alternative integrable model for shallow water waves in a channel. This model was also deduced by

Dai [13] in the context of elastic rods. Among the notable properties of this system are the existence of a bi-Hamiltonian structure and the existence of peaked traveling wave solutions called peakons. See (1.2.4) below.

As discussed in [12], the CH equation is completely integrable and possesses a Lax pair, which gives infinitely many conservation laws. The first two laws are given by

$$I(u) = \int u \, dx, \quad E(u) = \int (u^2 + (\partial_x u)^2) \, dx. \quad (1.2.2)$$

Li and Oliver [14] and Rodriguez [15] showed that the initial value problem (IVP) associated to the CH equation is (strongly) locally well-posed in $H^s(\mathbb{R})$, $s > 3/2$. The second conservation law hints at the existence of global solutions in $H^1(\mathbb{R})$ in the weak sense. Constantin and Molinet [16] managed to prove one can get such solutions that are continuous in time with respect to the $H^1(\mathbb{R})$ norm under a nonnegativity condition on the quantity $m = u - \partial_x^2 u$. McKean [17] then showed that solution blow-up occurs if and only if m has a positive part to the left of some negative part of itself. In addition, one can establish a type of local well posedness for solutions in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, which is a class that includes our peakon solutions. This was proven by Linares, Ponce, and Sideris in [18].

Constantin [19] showed that classical solutions will only blow-up via wave-breaking, which means that the slope of a solution may diverge at a point. Moreover, Bressan and Constantin [20] managed to construct global conservative solutions in $H^1(\mathbb{R})$ for any initial condition. Such solutions can experience wave breakdown, but can be extended globally in time after blow up.

Using the fact that $(1 - \partial_x^2)^{-1} f = \frac{1}{2} e^{-|x|} * f$, one gets the conservative nonlocal form of the CH equation. For convenience, we will write $G = (1 - \partial_x^2)^{-1}$ when in the context of peakon equations.

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 + G \left(u^2 + \frac{1}{2} (\partial_x u)^2 \right) \right) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (1.2.3)$$

This weaker formulation justifies the validity of peakon solutions, which are peaked traveling wave equations to our equation. These peakons take the general form

$$u(t, x) = cQ(x - ct), \quad Q(x) = ce^{-|x|}, \quad c \in \mathbb{R}. \quad (1.2.4)$$

Notice that $Q \notin H^{3/2}(\mathbb{R})$. Given this lack of regularity, Linares, Ponce, and Sideris obtained a well-posedness result on a space that contains the peakon solutions. It is stated as follows:

Theorem 1.2.1 ([18]) *Given $u_0 \in X \equiv H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, there exists $T > 0$ such that $T = T(\|u_0\|_X)$ and a unique solution $u = u(x, t)$ of the IVP associated to the CH equation (1.2.3) such that*

$$\begin{aligned} u \in & C([-T, T]: H^1(\mathbb{R})) \cap C^1((-T, T): L^2(\mathbb{R})) \\ & \cap L^\infty([-T, T]: W^{1,\infty}(\mathbb{R})) \equiv Z_T \cap L^\infty([-T, T]: W^{1,\infty}(\mathbb{R})). \end{aligned} \quad (1.2.5)$$

Moreover, the map $u_0 \mapsto u$, taking the data to the solution, is locally continuous from X into Z_T .

We remark that the notion of well-posedness here is not as strong as those stated in previous theorems. For other “well-posedness” results concerning the IVP (1.2.3) we refer to [18] and references therein.

As mentioned before, solutions to the CH equation may experience blow-up in finite time via wave breaking and still be extended to $H^1(\mathbb{R})$ global solutions. Such behavior does not occur for similar models. For example, some solutions to the mass critical gKdV equation can blow up in $H^1(\mathbb{R})$ by having mass escape to infinity in finite time. This

behavior is fully described by Martel and Merle [21]. Other examples, such as the KdV equation, possess global solutions in $H^1(\mathbb{R})$ without blow up.

Thanks to its complete integrability, the CH equation demonstrates elastic collisions between peakons, much like soliton behavior for the KdV equation. The stability of single peakons was established by Constantin and Strauss [22]. As hypothesized during its derivation, asymptotic stability of trains of peakons solutions have been established by Molinet in [23], which aligns with asymptotic stability of solitons for the KdV equation. The speed of these peakons is determined by their amplitude, which is bounded by their $H^1(\mathbb{R})$ through the Sobolev embedding. This implies that solutions should decay in the $H^1(\mathbb{R})$ norm in the shrinking region $|x| \gtrsim \beta t$ for some $\beta > 0$.

A train of peakons should decay around the origin since they will carry their mass away from the origin at constant speed. This, combined with asymptotic stability results, implies decay on some expanding intervals centered at the origin. Alejo, Cortez, Kwak, and Muñoz [24] proved this inside the interval approximately of the size $|x| \lesssim t^{1/2}$. The relevant theorem is

Theorem 1.2.2 [24] *Let u be a nontrivial global solution to (1.2.3), such that $u \in C([0, \infty] : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^1(\mathbb{R}))$. Then we have*

$$\lim_{t \rightarrow +\infty} \|u(t)\|_{H^1(\Lambda_{1/2}(t))} = 0 \quad (1.2.6)$$

where

$$\Lambda_c(t) := \left(-\frac{|t|^c}{\log |t|}, \frac{|t|^c}{\log |t|} \right). \quad (1.2.7)$$

Remark 1.2.1 *Note that (1.2.2) and the global existence of solutions with nonnegative initial data [19] imply that the hypotheses, $u \in C([0, \infty] : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^1(\mathbb{R}))$, are satisfied for $m_0(x) \geq 0$, and $m_0 \in L^1(\mathbb{R})$.*

Remark 1.2.2 *The above theorem was also proved for the L^2 -norm for the Degasperis-Procesi (DP) equation, which was first discovered by Degasperis and Gaeta [25]. This has the form*

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 + \frac{3}{2} G u^2 \right) = 0. \quad (1.2.8)$$

This equation also has the same peakon solutions, is a bi-Hamiltonian system, and is completely integrable. In addition, (1.2.8) allows for shock peakon solutions, which are like peakons except they are discontinuous at a point. This behavior is possible since weak solutions to (1.2.8) may simply be $L^2(\mathbb{R})$ instead of $H^1(\mathbb{R})$. These solutions take the form

$$u(t, x) = \frac{1}{t + k} \operatorname{sgn}(x) e^{-|x|}, \quad k > 0. \quad (1.2.9)$$

In [24], one finds a similar result by accepting a growth in time of the $L^1(\mathbb{R})$ norm of solutions, as opposed to requiring a global bound. In particular, one allows polynomial growth. The result is

Theorem 1.2.3 *Let $u \in C([0, \infty) : H^1(\mathbb{R}))$ be a solution to (1.2.3) or (1.2.8) such that*

$$\int |u(t, x)| dx \lesssim \langle t \rangle^a. \quad (1.2.10)$$

For $0 \leq a < 1$, and $c := 1 - a$, we have

1. For solutions to CH (1.2.3)

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{H^1(\Lambda_c(t))} = 0. \quad (1.2.11)$$

2. For solutions to DP (1.2.8)

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{L^2(\Lambda_c(t))} = 0. \quad (1.2.12)$$

When $a < 1/2$, this result holds for a larger expanding interval than described in Theorem 1.2.2, but the wider interval is exchanged for a $\liminf$ result rather than a true limit. The proof extends to the b -family of peakon equations, which will be the focus of the results in this manuscript.

The CH equation and the DP equation belong to the b -family of peakon equations

$$\partial_t u - \partial_{xxt}^3 u + (b+1)u\partial_x u - b\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \quad (1.2.13)$$

or equivalently

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2 + G \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) \right) = 0, \quad b \in (0, 3). \quad (1.2.14)$$

The CH equation is represented when $b = 2$ while the DP equation is represented when $b = 3$. Note that all of these equations admit peakon solutions, but only the DP equation admits shock peakon solutions. A caveat in this relation is that only the CH equation and the DP equation are integrable. A question we seek to answer in this thesis is whether results similar to those in [24] apply to solutions of (1.2.14) in general since the integrability of these equations are not directly applied in the proof of Theorem 1.2.2. This hypothesis is not unfounded since it was shown in [26] that peakon solutions are stable for $b > 1$. The same paper found numerical evidence of the formation of traveling wave-like behavior for all $0 < b < 3$, although the system behaved more like the viscous Burger's equation when $b < 1$. This also begs the question whether we will have decay in the region $|x| \gtrsim \beta t$. Since (1.2.14) is not integrable for $b \neq 2, 3$, one does

not have access to inverse scattering techniques to investigate this behavior. In addition, we are also interested in a similar result that does not require any assumptions on the longtime behavior of the $L^1(\mathbb{R})$ norm of the solution. This is a natural question since there exist global solutions to the CH equation that are bounded in $H^1(\mathbb{R})$, but have varying behavior in the $L^1(\mathbb{R})$ norm. This also applies to the b -family via the peakon solutions.

Since the b -family equations are not integrable in general, we have to take advantage of different conserved quantities than what were used in the theorems mentioned above. Consider the quantity $m(t, x) = (1 - \partial_x^2)u(t, x)$. We can rewrite (1.2.14) in terms of m . As stated in [27], This gives us

$$\partial_t m + u \partial_x m + b m \partial_x u = 0. \quad (1.2.15)$$

It can be shown that the sign of $m(t, x)$ is determined by the sign of $m(0, x)$. In particular, if $m(0, x)$ is nonnegative, $m(t, x)$ remains nonnegative. Using the Green's function of G , we learn that $m(0, x) = m_0(x) \geq 0$ implies $u(t, x) \geq 0$ since $\frac{1}{2}e^{-|x|}$ is nonnegative.

Like the $b = 2$ CH case, m is an important quantity for the analysis of these equations. For example, it was shown that $u \in C([0, \infty) : H^2(\mathbb{R}))$ is a global solution to (1.2.3) if m_0 is signed. This is similar to the behavior of (1.2.1). Breakdown of solutions even occur in the same way, with the slope of our solutions diverging. The proof of these results can be found in [27]. Moreover, $\int m \, dx$ is a conserved quantity for solutions of (1.2.14). This means that for all u that solves (1.2.14), we have

$$\frac{d}{dt} H(t) = \frac{d}{dt} \int_{-\infty}^{\infty} m \, dx = \frac{d}{dt} \int_{-\infty}^{\infty} (u - \partial_x^2 u)(x, t) \, dx = 0. \quad (1.2.16)$$

Note that (1.2.3) and (1.2.8) have other integrals of motion that are very useful.

In particular, solutions to the CH equation (1.2.3) and the DP equation (1.2.8) have constants of motion that are equivalent to the $H^1(\mathbb{R})$ norm. Since the b -family lacks access to these conservation laws, it is difficult to obtain the exact same results for $b \neq 2, 3$. This is in spite of the fact that [26] found very similar behavior numerically when $b > 1$. The techniques used in this paper seem unable to take advantage of this fact, so it is possible that further results can be found via alternative methods in the $b > 1$ case.

These local decay properties are closely linked to the soliton resolution conjecture for systems modeling wave propagation. The soliton resolution conjecture states that “As $t \rightarrow \infty$, every reasonable solution decomposes into a finite sum of solitary waves plus a dissipative radiation term (an approximately linear solution),” where “solitary waves” refers to traveling wave solutions that maintain their shape such as the peakon solutions described earlier. This is an open problem for many models.

In view of this conjecture, it is interesting to determine the time asymptotic behavior of the L^2 norm (as well as the H^s norm depending on the system). The goal should be to prove that for a given solution $u(x, t)$, there exists some $C > 0$ such that

$$\lim_{t \rightarrow \infty} \int_{\Omega(t)} |u(x, t)|^2 dx = 0, \quad \Omega(t) = (-Ct, Ct).$$

This is in contrast with results on several models that restrict to intervals that grow sublinearly.

For further comparison with the BBM model, we remark that the UCP for the peakon family runs into similar issues as discussed for the BBM equation. In fact, the peakon equations lack interesting linearized variants due to a lack of dispersion.

1.3 Main Results

First, we will discuss our results for the BBM equation.

1.3.1 UCP Properties for the BBM Equation

For our purpose, we shall need a local well-posedness result for the IVP (1.1.3) in Hölder spaces, i.e. for $k \in 0, 1, 2, \dots$ and $\theta \in [0, 1]$

$$C^{k,\theta}(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : f \in C^k \cap W^{k,\infty}(\mathbb{R}), f^{(k)} \in C^\theta(\mathbb{R})\}, \quad (1.3.1)$$

where for $k \in \mathbb{Z}, k \geq 0$,

$$W^{k,\infty}(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : \partial_x^j f \in L^\infty(\mathbb{R}), j = 0, 1, \dots, k\}, \quad (1.3.2)$$

with

$$\|f\|_{C^{k,\theta}} = \sum_{j=0}^k \|f^{(j)}\|_\infty + \sup_{h \neq 0} \left\| \frac{f^{(k)}(\cdot + h) - f^{(k)}(\cdot)}{|h|^\theta} \right\|_\infty. \quad (1.3.3)$$

Theorem 1.3.1 *Let $k \in \mathbb{Z}, k \geq 0$, and $\theta \in [0, 1]$. For any $u_0 \in C^{k,\theta}(\mathbb{R})$ there exist $T = T(\|u_0\|_{C^{k,\theta}}) > 0$ and a unique solution $u = u(x, t)$ of the IVP (1.1.3) such that*

$$u \in C^\infty((-T, T) : C^{k,\theta}(\mathbb{R})) \equiv X_T^{k,\theta}(\mathbb{R}). \quad (1.3.4)$$

Moreover, the map data solution, $u_0 \rightarrow u$, is locally analytic from $C^{k,\theta}(\mathbb{R})$ to $X_T^{k,\theta}(\mathbb{R})$.

Observe that the constant solution $u(x, t) \equiv c_0$ belongs to the class (1.3.4). Here we shall not address the question on the possible global in time extension of the local solution obtained in Theorem 1.3.1.

Next, we have our first result concerning unique continuation properties of solutions to the IVP (1.1.3):

Theorem 1.3.2 *Let $u = u(x, t)$ be the solution of the IVP (1.1.3) provided by Theorem 1.3.1. If there exist $t_0 \in (-T, T)$ and $a, b \in \mathbb{R}$ with $a < b$ such that*

$$\begin{cases} (i) & u(x, t_0) = -1, \quad x \in (a, b), \\ (ii) & \partial_t u(b, t_0) \geq \partial_t u(a, t_0), \end{cases} \quad (1.3.5)$$

then $u(x, t) = -1$ for all $(x, t) \in \mathbb{R} \times (-T, T)$.

Corollary 1.3.3 *There does not exist any solution $u = u(x, t)$ of the IVP (1.1.3) in the class (1.1.8) provided by Theorem 1.1.1 satisfying the conditions in (1.3.5).*

Remark 1.3.1 1. *In Corollary 1.3.3 the solution*

$$u \in C([-T, T] : H^s(\mathbb{R}))$$

satisfies (1.3.5) (i), almost everywhere in (a, b) , if $s \in [0, 1/2]$, and (1.3.5) (ii) pointwise for all $s \geq 0$, since in this case $\partial_t u \in C(\mathbb{R} \times [-T, T])$.

2. *The condition (1.3.5) is weaker than the usual one for unique continuation. That assumes the existence of a (non-empty) open set $\Omega \subset \mathbb{R} \times (-T, T)$ such that the solution $u = u(x, t)$ satisfies $u(x, t) = -1$ if $(x, t) \in \Omega$.*

3. *For previous unique continuation results for the BBM, we refer to [9] and references therein.*

4. *In Theorem 1.3.2 the condition (ii) in (1.3.5) can be restated as :*

$$\partial_t u(x, t_0) \quad \text{is not strictly decreasing for} \quad x \in [a, b].$$

5. In the case of the KdV equation, see (1.1.1), formally the condition (1.3.5) part (i), and the equation imply equality in (1.3.5) part (ii).
6. Example (1.1.17) from earlier shows that Corollary 1.3.3 fails for the value -2 in (1.3.5) part (i). Let $a, b \in \mathbb{R}$, $a < b$, and

$$u(x, t) = u_0(x) = \begin{cases} -2, & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases} \quad (1.3.6)$$

Then $u \in C(\mathbb{R} : H^s(\mathbb{R}))$, $s \in [0, 1/2)$, solves the IVP (1.1.3) with $\partial_t u(x, t) = 0$, $\forall (x, t) \in \mathbb{R}^2$

In fact, following result shows that the value -1 in (1.3.5) is the only one for which Theorem 1.3.2 and Corollary 1.3.3 hold.

Theorem 1.3.4 *Given any $c_0 \in \mathbb{R} - \{-1\}$ and $a, b \in \mathbb{R}, a < b$, there exists $u_0 \in C_0^\infty(\mathbb{R}) - \{0\}$ such that the corresponding solution $u = u(x, t)$ of the IVP (1.1.3) provided by Theorem 1.1.1, with $s > 1/2$, (or by Theorem 1.3.1) satisfies that*

$$(i) \ u_0(x) = c_0 \quad \text{and} \quad (ii) \ \partial_t u(x, 0) = 0, \quad x \in (a, b). \quad (1.3.7)$$

With a similar approach one can get the following general result :

Theorem 1.3.5 *Let $c_0 \in \mathbb{R}$. Let $u = u(x, t)$ be a solution of the IVP (1.1.3) provided by Theorem 1.3.1 such that for some $t_0 \in (-T, T)$ and some $a, b \in \mathbb{R}, a < b$, one has*

$$u(y, t_0) = c_0, \quad y \in \{a, b\}. \quad (1.3.8)$$

If one of the following conditions hold with $f(y) = y + y^2/2$:

$$\left\{ \begin{array}{ll} f(u(x, t_0)) \leq f(c_0) & \text{if } x \in [a, b], \\ f(u(x, t_0)) \geq f(c_0) & \text{if } x \notin [a, b], \\ \partial_t u(b, t_0) \geq \partial_t u(a, t_0), \end{array} \right. \quad (1.3.9)$$

or

$$\left\{ \begin{array}{ll} f(u(x, t_0)) \leq f(c_0) & \text{if } x \notin [a, b], \\ f(u(x, t_0)) \geq f(c_0) & \text{if } x \in [a, b], \\ \partial_t u(a, t_0) \geq \partial_t u(b, t_0), \end{array} \right. \quad (1.3.10)$$

then $u(x, t) \equiv c_0$.

- Remark 1.3.2** 1. In the case $c_0 = -1$, since $f(y) = y + y^2/2$ realizes its minimum at $y = -1$, one sees that the condition (1.3.9) agrees with (1.3.5) in Theorem 1.3.2.
2. For $c_0 = 0$, using that $f(0) = 0$, from Theorem 1.3.5, one obtains the following conclusion : if the solution $u(x, t)$ of the IVP (1.1.3), provided by Theorem 1.3.2, satisfies

$$u(y, t_0) = \partial_t u(y, t_0) = 0, \quad y \in \{a, b\}, \quad (1.3.11)$$

for some $t_0 \in (-T, T)$ and $a < b$, with $u(x, t_0)$ having different sign in $[a, b]$ and $[a, b]^c$, then $u(x, t)$ is identically zero. In this case, the solution $u(x, t)$ belongs to the class described in Theorem 1.1.1.

3. Previous unique continuation results (see Theorem 1.1.3) assume some additional constraints on the solution.

Our results in Theorem 1.3.2 and Theorem 1.3.4 extend to the IPBVP (1.1.11):

Theorem 1.3.6 *Let $u = u(x, t)$ be a solution of the IPBVP (1.1.11) provided by Theorem 1.1.2. If there exist $t_0 \in (-T, T)$ and $a, b \in \mathbb{S} \cong [0, 1]$ with $a < b$ such that*

$$\begin{cases} (i) & u(x, t_0) = -1, \quad \text{a.e. } x \in (a, b), \\ (ii) & \partial_t u(b, t_0) \geq \partial_t u(a, t_0), \end{cases} \quad (1.3.12)$$

then $u(x, t) = -1$ for all $(x, t) \in \mathbb{S} \times [-T, T]$, if $s > 1/2$, and a.e. for $(x, t) \in \mathbb{S} \times [-T, T]$, if $s \in [0, 1/2]$.

Remark 1.3.3 *One can easily see that if the solution $u \in C([-T, T] : L^2(\mathbb{S}))$, then $\partial_t u \in C(\mathbb{S} \times [-T, T])$. Hence, the pointwise condition (1.3.12) is well-defined.*

The following theorem and its remark confirm that the value -1 is the only one for which Theorem 1.3.6 holds.

Theorem 1.3.7 *Given any $c_0 \in \mathbb{R} - \{-1\}$ there exist $a, b \in [0, 1] \cong \mathbb{S}$ and $u_0 \in C^\infty(\mathbb{S}) - \{c_0\}$ such that the corresponding solution $u = u(x, t)$ of the IPBVP (1.1.11) provided by Theorem 1.1.2 satisfies that*

$$u_0(x) = c_0, \quad x \in (a, b) \subset [0, 1] \cong \mathbb{S}, \quad (1.3.13)$$

and

$$\partial_t u(a, 0) = \partial_t u(b, 0). \quad (1.3.14)$$

Remark 1.3.4 *The following example is an extension of that attained in [9]. Let $f(x) = x + x^2/2$, $c_0 \in \mathbb{R} - \{-1\}$ and $a, b \in (0, 1) \cong \mathbb{S}$, $a < b$. Let*

$$f^{-1}(f(c_0)) = \{c_0, c'_0\}, \quad c_0 \neq c'_0.$$

Then

$$u(x, t) = u_0(x) = \begin{cases} c_0, & x \in [a, b], \\ c'_0, & x \notin [a, b]. \end{cases} \quad (1.3.15)$$

is a solution of the IPBVP (1.1.11), with $u \in C([-T, T] : H^s(\mathbb{S}))$, $s \in [0, 1/2)$, and $\partial_t u(x, t) \equiv 0$.

We also have:

Theorem 1.3.8 *Under the appropriate modifications the results in Theorem 1.3.5 still hold for solutions of the IPBVP (1.1.11) provided by Theorem 1.1.2 with $s > 1/2$.*

1.3.2 UCP Properties for the CH Equation

We now turn to the UCP for the CH equation.

The argument used in this manuscript for the BBM equations [28] allows us to get a slight improvement of the unique continuation results concerning the Camassa-Holm (CH) equation [29].

In [29] this unique continuation principle was derived :

Theorem 1.3.9 ([29]) *Let $u = u(x, t)$ be a solution of the IVP associated to the CH equation (1.2.3) in the class described in Theorem 1.2.1. If there exists an open set $\Omega \subset \mathbb{R} \times [0, T]$ such that*

$$u(x, t) = 0, \quad (x, t) \in \Omega, \quad (1.3.16)$$

then $u \equiv 0$.

Remark 1.3.5 *This results requires the application of Theorem 1.2.1.*

Our argument here shows the following:

Theorem 1.3.10 *Let $u = u(x, t)$ be a solution of the IVP associated to the CH equation (1.2.3) in the class described in Theorem 1.2.1. If there exist $t_0 \in (-T, T)$ and $a, b \in \mathbb{R}, a < b$, such that*

$$u(x, t_0) = 0, \quad x \in [a, b], \quad (1.3.17)$$

and

$$\partial_t u(x, t_0) \quad \text{is not strictly decreasing on} \quad [a, b], \quad (1.3.18)$$

then $u \equiv 0$.

One observes that from hypothesis (1.3.17) it follows that $\partial_t u(\cdot, t_0)$ is continuous in $[a, b]$ such that the pointwise evaluation in (1.3.18) makes sense.

We remark that the results in Theorem 1.3.10 expands to all the b-equations (see [29]) for both their IVP and their IPBVP.

1.3.3 Regularity Properties of Solutions to the BBM Equation

Next, we shall examine some special regularity properties of solutions to the BBM equation.

In [6] and [8] the integral form of the equation written in (1.1.4) was considered. Here, we shall work with the following equivalent integral version (for the IVP and the IPBVP) of (1.1.3) :

$$u(x, t) = u_0(x) - \int_0^t \partial_x J^{-2} \left(u + \frac{u^2}{2} \right) (x, t') dt'. \quad (1.3.19)$$

To simplify our exposition in the case of the IVP we shall restrict ourselves to consider only solutions provided by Theorem 1.1.1 and Theorem 1.3.1. It will be clear that the results here extend to the $L_s^p(\mathbb{R})$ -solutions obtained in [8] .

In this direction we have:

Theorem 1.3.11 (i) Let $u_0 \in H^s(\mathbb{R})$, $s \geq 0$. Let

$$u \in C([-T, T] : H^s(\mathbb{R}))$$

be the corresponding solution of the IVP (1.1.3) provided by Theorem 1.1.1. Then

$$u - u_0 \in C^\infty([-T, T] : H^{(2s+1/2)^-}(\mathbb{R})), \quad \text{if } s \in [0, 1/2], \quad (1.3.20)$$

and

$$u - u_0 \in C^\infty([-T, T] : H^{s+1}(\mathbb{R})), \quad \text{if } s > 1/2. \quad (1.3.21)$$

(ii) Let $u_0 \in C^{k,\theta}(\mathbb{R})$, $k \in \mathbb{Z}$, $k \geq 0$, $\theta \in [0, 1]$. Let

$$u \in C([-T, T] : C^{k,\theta}(\mathbb{R}))$$

be the corresponding solution of the IVP (1.1.3) provided by Theorem 1.3.1. Then

$$u - u_0 \in C^\infty([-T, T] : C^{k+1,\theta}(\mathbb{R})). \quad (1.3.22)$$

Theorem 1.3.12 The results in Theorem 1.3.11 part (i) holds for solutions of the IP-BVP provided by Theorem 1.1.2.

Remark 1.3.6 1. Concerning the time regularity, our results show that the given solution $u \in C(\mathbb{R} : X)$ belongs to $C^\infty(\mathbb{R} : X)$.

In particular, for the explicit solutions described in (1.1.17) and (1.3.15), one has that $u \in C(\mathbb{R} : H^2(\mathcal{T}))$, $s \in [0, 1/2]$, for $\mathcal{T} = \mathbb{R}$ or $\mathbb{S}$, with $\partial_t u(x, t) = u(x, t) - u_0(x) \equiv 0$, $(x, t) \in \mathcal{T} \times \mathbb{R}$.

2. The arguments below also show that the group $\{U(t) : t \in \mathbb{R}\}$ defined in (1.1.5)

satisfies: if $u_0 \in H^s(\mathbb{R})$, then

$$U(t)u_0 - u_0 \in C^\infty(\mathbb{R} : H^{s+1}(\mathbb{R})).$$

In the case $u_0 \in C^{k,\theta}(\mathbb{R})$, one has

$$\partial_x(U(t)u_0 - u_0) \in C^\infty(\mathbb{R} : C^{k+1,\theta}(\mathbb{R})).$$

3. In Theorem 1.1.1 and Theorem 1.1.2 one can use the continuous dependence of the solution upon the data to see that the solutions of (1.1.2) are the limit of classical solutions of the equations (1.1.3) and (1.1.2). In the case $k \in \mathbb{Z}, k \geq 0$ with $\theta = 0$ the solutions in Theorem 1.3.1 satisfy the same property. However, this does not hold when $k = 0, 1, 2$ and $\theta \in (0, 1]$, since in this setting the initial data is not the limit of smooth functions.

Next, we recall the definition of $H^s(\Omega)$, $s \geq 0$, and $\Omega \subset \mathbb{R}$ open set. Given $f : \Omega \rightarrow \mathbb{R}$ one says that $f \in H^s(\Omega)$ if there exists $F \in H^s(\mathbb{R})$ such that $F|_\Omega = f$. The norm of f is the infimum of the norm in $H^s(\mathbb{R})$ among all the possible extensions. Similarly for $C^{k,\theta}(\Omega)$.

Theorem 1.3.13 *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 0$, and $\Omega \subseteq \mathbb{R}$ be an open set.*

(a) *If $s \leq 1/2$ and $u_0 \in H^{s+\eta}(\Omega)$, $\eta \in (0, s + 1/2)$, then the corresponding solution u of the IVP (1.1.3) provided by Theorem 1.1.1 satisfies:*

$$u|_\Omega \in C(\mathbb{R} : H^{s+\eta}(\Omega)). \quad (1.3.23)$$

(b) *If $s \leq 1/2$ and $u_0 \notin H^{s+\eta}(\Omega)$, $\eta \in (0, s + 1/2)$, then the corresponding solution u*

of the IVP (1.1.3) provided by Theorem 1.1.1 satisfies:

$$u|_{\Omega} \notin H^{s+\eta}(\Omega), \quad \forall t \in \mathbb{R}. \quad (1.3.24)$$

(c) If $s > 1/2$ and $u_0 \in H^{s+\eta}(\Omega)$, $\eta \in (0, 1]$, then the corresponding solution u of the IVP (1.1.3) provided by Theorem 1.1.1 satisfies:

$$u|_{\Omega} \in C(\mathbb{R} : H^{s+\eta}(\Omega)). \quad (1.3.25)$$

(d) If $s > 1/2$ and $u_0 \notin H^{s+\eta}(\Omega)$, $\eta \in (0, 1]$, then the corresponding solution u of the IVP (1.1.3) provided by Theorem 1.1.1 satisfies:

$$u|_{\Omega} \notin H^{s+\eta}(\Omega), \quad \forall t \in \mathbb{R}. \quad (1.3.26)$$

Theorem 1.3.14 Let $u_0 \in C^{k,\theta}(\mathbb{R})$, $k \in \mathbb{Z}$, $k \geq 0$, $\theta \in [0, 1]$ and $\Omega \subseteq \mathbb{R}$ be an open set.

(a) If $u_0 \in C^{j,\eta}(\Omega)$, $j \in \{k; k+1\}$, $\eta \in [0, 1]$, and $j + \eta \leq k + 1 + \theta$, then the corresponding solution u of the IVP (1.1.3) provided by Theorem 1.3.1 satisfies:

$$u|_{\Omega} \in C^{\infty}(\mathbb{R} : C^{j,\eta}(\Omega)). \quad (1.3.27)$$

(b) If $u_0 \notin C^{j,\eta}(\Omega)$, $j \in \{k; k+1\}$, $\eta \in [0, 1]$, and $j + \eta \leq k + 1 + \theta$, then the corresponding solution u of the IVP (1.1.3) provided by Theorem 1.3.1 satisfies:

$$u|_{\Omega} \notin C(\mathbb{R} : C^{j,\eta}(\Omega)), \quad \forall t \in [-T, T]. \quad (1.3.28)$$

Next, we introduce the notation : given $f : \mathbb{R} \rightarrow \mathbb{R}$ for $k \in \mathbb{Z}, k \geq 0$ and $\theta \in [0, 1]$ define

$$\Upsilon_{k,\theta}(f) = \{x \in \mathbb{R} : f \text{ is not } C^{k,\theta} \text{ at } x\}. \quad (1.3.29)$$

Corollary 1.3.15 *If $u_0 \in C^{k,\theta}(\mathbb{R})$, $k \in \mathbb{Z}, k \geq 0$, and $\theta \in [0, 1]$, then the corresponding solution $u \in C([-T, T] : C^{k,\theta}(\mathbb{R}))$ of the IVP (1.1.3) obtained in Theorem 1.3.1 satisfies for all $t \in [-T, T]$*

$$\Upsilon_{j,\eta}(u(t)) = \Upsilon_{j,\eta}(u_0), \quad j \in \{k; k+1\}, \quad \eta \in [0, 1], \quad j + \eta \leq k + 1 + \theta. \quad (1.3.30)$$

Remark 1.3.7 1. *Roughly, the reversibility in time of the equation and the above results tells us that in the H^s and $C^{k,\theta}$ settings the regularity (and singularities) of u_0 are preserved by the solution $u(\cdot, t)$ in its time interval of definition.*

2. *The "gain" of regularity described in Theorem 1.3.11 and Theorem 1.3.12 are quite different to those known in solutions of the IVP associated for the KdV equation (1.1.1). In this case, as a consequence of Kato smoothing effect [30] one has : if $u(x, 0) = u_0(x) \in L^2(\mathbb{R})$, then the corresponding solution $u = u(x, t)$ of (1.1.1) satisfies*

$$u(\cdot, t) \in C_{loc}^{0,1/2}(\mathbb{R}), \quad a.e. \quad t \in \mathbb{R},$$

see, [31], [32].

The previous results are complemented by some found in [33]. There, the propagation of local regularities in $C^{k,\theta}$ of the initial data $u_0 \in H^s(\mathbb{R})$ to the corresponding solution $u = u(x, t)$ of the IVP (1.1.3) was also studied.

Finally, we observe that in [34], the system of BBM equations

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt}^3 u + \partial_x (Au^2 + Buv + Cv^2) = 0, \\ \partial_t v + \partial_x v - \partial_{xxt}^3 v + \partial_x (Du^2 + Euv + Fv^2) = 0, \end{cases} \quad (1.3.31)$$

with $A, B, C, D, E, F \in \mathbb{R}$ was considered. Under appropriate algebraic conditions on the coefficients A, B, C, D, E, F , it was shown in [34] that the associated IVP is globally well posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \geq 0$. It should be clear from our proofs below that the results stated above extend, under appropriate modifications, to the solution of the system (1.3.31).

1.3.4 Asymptotic Behavior of Solutions to the b -family of Peakon Equations

Next, we will discuss our results for the peakon equations.

If one weakens their assumptions on regularity of their solutions but without any hypotheses on the asymptotic behavior of the $L^1(\mathbb{R})$ norm of the solution, one can still get a weak decay result for solutions. In particular, one can get $\liminf_{t \rightarrow \infty} \|u(t)\|_{H^1(\Lambda_{2/3}(t))} = 0$ on expanding intervals.

Theorem 1.3.16 *Let $u \in C([0, \infty) : H^1(\mathbb{R})) \cap L^\infty([0, \infty) : H^1(\mathbb{R}))$ be a global solution to (1.2.14). Then we have*

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{H^1(\Lambda_{2/3}(t))} = 0. \quad (1.3.32)$$

If we instead restrict the L^1 norm of $m_0(x)$, we get a similar result to Theorem 1.2.2. This result is

Theorem 1.3.17 *Let $u \in C([0, \infty) : H^2(\mathbb{R}))$ be a global solution to (1.2.14) such that $m_0 \geq 0$, and $m_0 \in L^1(\mathbb{R})$, then*

$$\lim_{t \rightarrow \infty} \|m(t)\|_{L^1(\Lambda_{1/2}(t))} = \lim_{t \rightarrow \infty} \|u(t)\|_{L^1(\Lambda_{1/2}(t))} = L \quad (1.3.33)$$

for some $L \in \mathbb{R}$.

Remark 1.3.8 *There is a similar decay result for the Zakharov-Kuznetsov equation that was proven by Mendez, Muñoz, Poblete, and Pozo [35]. Our argument was inspired by their approach.*

Remark 1.3.9 *Note that we require u to be in $H^2(\mathbb{R})$ for all time. This is because we have to make use of $m = u - u_{xx}$ to gain control over the $H^1(\mathbb{R})$ norm for u . For the CH equation, this is not necessary because $\|u\|_{H^1}$ is a conserved quantity for the CH equation. Moreover, we do not know what the limit L will be.*

Finally, we can show that signed solutions to (1.2.14) decay to zero on $(\sigma t, \infty)$ and $(-\infty, -\sigma t)$. In essence, this implies the traveling waves we expect from these PDEs to have a velocity capped by some quantity related to the initial data.

Theorem 1.3.18 *Let $u \in C([0, \infty) : H^2(\mathbb{R}))$ be a global solution to (1.2.14) such that $m_0 \geq 0$, and $m_0 \in L^1(\mathbb{R})$. Then there exists $\sigma > 0$ such that*

$$\lim_{t \rightarrow \infty} \|u\|_{W^{1,p}(\bar{\lambda}(t))} \rightarrow 0,$$

where

$$\bar{\lambda}(t) = (\sigma t, \infty).$$

One can show a similar result and proof for $\tilde{\lambda}(t) = (-\infty, -\sigma t)$ when $m_0 \leq 0$.

We once again require m to be signed in order to easily bound the $L^1(\mathbb{R})$ norm of m . We follow a similar proof for the gBBM equation shown in [36].

1.4 Permissions and Attributions

1. The content of chapter 2 is the result of a collaboration with my advisor Gustavo Ponce, and has previously appeared in the Journal of Differential Equations [28]. It is reproduced here with the permission of Elsevier: <https://www.sciencedirect.com/science/article/pii/S0022039624000913>.
2. The content of chapter 3 has previously appeared in Nonlinear Analysis [37]. It is reproduced here with the permission of Elsevier: <https://www.sciencedirect.com/science/article/pii/S0362546X2400258X>.
3. Large parts of this introduction have also previously appeared in the Journal of Differential Equations and Nonlinear Analysis [28, 37]. The same permissions from Elsevier apply.

Chapter 2

Unique Continuation and Smoothing Effects for the BBM Equation

2.1 Preliminary estimates

In this section, we shall recall some results that will be used to prove properties for the BBM equation.

Lemma 2.1.1 *Let $p \in [1, \infty]$ and $s \geq 0$, with s not an odd integer in the case $p = 1$. Then*

$$\|J_x^s(fg)\|_p \leq c(\|f\|_{p_1}\|J_x^s g\|_{p_2} + \|g\|_{q_1}\|J_x^s f\|_{q_2}), \quad (2.1.1)$$

and

$$\|D_x^s(fg)\|_p \leq c(\|f\|_{p_1}\|D_x^s g\|_{p_2} + \|g\|_{q_1}\|D_x^s f\|_{q_2}) \quad (2.1.2)$$

with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$.

For the proof of Lemma 2.1.1 with $p \in (1, \infty)$ we refer to [38], [39] and [40]. The case $r = p_1 = p_2 = q_1 = q_2 = \infty$ was established in [41], see also [42]. The case $p = p_2 = q_2 = 1$ was settled in [43].

Also, we shall use the following estimate:

Proposition 2.1.2 *Let $k \in \mathbb{Z}$, $k \geq 0$ and $\theta \in [0, 1]$. Then*

$$\|fg\|_{C^{k,\theta}} \leq c (\|f\|_\infty \|g\|_{C^{k,\theta}} + \|f\|_{C^{k,\theta}} \|g\|_\infty). \quad (2.1.3)$$

and

$$\|f * g\|_{C^{k,\theta}} \leq c \|f\|_1 \|g\|_{C^{k,\theta}}. \quad (2.1.4)$$

Next, we recall some properties of the Bessel potential. To simplify the exposition we restrict ourselves to the one dimensional case, $n = 1$. For $s > 0$

$$J^{-s}f(x) = (1 - \partial_x^2)^{-s/2}f(x) = (G_s * f)(x) \quad (2.1.5)$$

where

$$G_s(x) = c_s \int_0^\infty \frac{e^{-\frac{\pi x^2}{y} - \frac{y}{4\pi}} dy}{y^{1+(1-s)/2}} \quad (2.1.6)$$

Thus, for any $p \in [1, \infty]$

$$\begin{cases} (a) & G_s \in L^p(\mathbb{R}), & s > 1 - 1/p, \\ (b) & \partial_x G_s \in L^p(\mathbb{R}), & s > 2 - 1/p. \end{cases} \quad (2.1.7)$$

In particular, for $s = 2$ (and $n = 1$), one sees that

$$\partial_x G_2 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad s \in [0, 1/2) \quad \text{and} \quad G_2 \in C^{0,1}(\mathbb{R}), \quad (2.1.8)$$

since in this case $G_2(x) = e^{-|x|}/2$ and $\partial_x G_2(x) = -\text{sgn}(x)e^{-|x|}/2$.

Gathering this information one gets:

Lemma 2.1.3 *Let*

$$\phi(D_x) = \partial_x(1 - \partial_x^2)^{-1} = \partial_x J^{-2}. \quad (2.1.9)$$

Then for $p \in [1, \infty]$

$$\|J_x^\eta \phi(D_x) f\|_p \leq c_{p,\theta} \|f\|_q, \quad (2.1.10)$$

with $\eta \in [0, 1 - 1/q + 1/p)$ and $q \in [1, p]$.

For $p \in (1, \infty)$, (2.1.10) holds for $\eta = 1$ and $q = p$. Also for $k \in \mathbb{Z}, k \geq 0$ and $\theta \in [0, 1]$

$$\|\phi(D_x) f\|_{C^{k,\theta}} \leq c_{k,\theta} \|f\|_{C^{k,\theta}}, \quad (2.1.11)$$

Moreover, in all these cases, (2.1.10) still holds with J^θ replaces by D_x^θ .

Proof: [Proof of Lemma 2.1.3]

We observe that for $\theta \in [0, 1]$

$$J^\theta \phi(D_x) f(x) = (\partial_x G_{2-\theta} * f)(x).$$

(2.1.10) follows by combining (2.1.7)(b) and Young's inequality. In the case $p \in (1, \infty)$ and $\theta = 1$, the estimate in (2.1.10) is a consequence of Mihlin-Hörmander multiplier theorem. Finally, in the one-dimensional case, $n = 1$, for any $s > 0$ and any $p \in [1, \infty]$

$$D_x^s J^{-s} : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$$

is bounded, see [44] Chapter VI, one concludes the proof of the lemma. ■

Remark 2.1.1 *Combining (2.1.7) (a) and*

$$\partial_x \phi(D_x) = J^{-2} - I, \quad (2.1.12)$$

one gets that

$$\|\partial_x \phi(D_x) f\|_p \leq c \|f\|_p, \quad p \in [1, \infty], \quad (2.1.13)$$

and

$$\|\partial_x \phi(D_x) f\|_{C^{k,\theta}} \leq c \|f\|_{C^{k,\theta}}, \quad k \in \mathbb{Z}, k \geq 0, \theta \in [0, 1], \quad (2.1.14)$$

which will be also used on the next section.

In the periodic case, one has that for $h \in L^2(\mathbb{S})$, then

$$\phi(D_x) h(x) = \partial_x (1 - \partial_x^2)^{-1} h(x) = (\partial_x \mathcal{G} * h)(x)$$

where

$$\mathcal{G}(x) = \frac{\cosh(x - \lfloor x \rfloor - 1/2)}{2 \sinh(1/2)}, \quad x \in \mathbb{R}/\mathbb{Z} \cong \mathbb{S}, \quad (2.1.15)$$

and $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Thus, here $\mathcal{G}(x)$ plays the role of the (Green) function $G_2(x) = e^{-|x|}/2$ on the line (for $(1 - \partial_x^2)$). All the estimates in (2.1.7)-(2.1.14) still holds with $\mathcal{G}(x)$ instead of G_2 .

Next, we recall the following result and its proof (since we shall use some modifications of it) obtained in [29]. This will be a key ingredient in the proof of our unique continuation results.

Lemma 2.1.4 ([29]) (a) *Let $f \in L^p(\mathbb{R})$ with $p \in [1, \infty]$ such that there exists a non-empty interval $[a, b]$ such that*

$$f(x) \leq 0 \text{ a.e. } x \in [a, b] \text{ and } f(x) \geq 0, \text{ a.e. } x \in [a, b]^c. \quad (2.1.16)$$

Then the function

$$F(x) = \phi(D_x) f(x) = \partial_x (1 - \partial_x^2)^{-1} f(x) = -\operatorname{sgn}(\cdot) \frac{e^{-|\cdot|}}{2} * f(x) \quad (2.1.17)$$

satisfies that $F(b) \geq F(a)$ with the equality holding only when $f(x) = 0$, a.e. $x \in \mathbb{R}$.

(b) The results in part (a) still hold in the periodic setting, i.e. for $f \in L^p(\mathbb{S})$.

Remark 2.1.2 We observe that the value of F defined in (2.1.17) does not change if one replaces in its definition f by $f + \alpha$ with $\alpha \in \mathbb{R}$.

Proof:

(a) By hypothesis F is continuous so its evaluation makes sense at any point. Now, we observe that

$$-\operatorname{sgn}(a-y)e^{-|a-y|} < -\operatorname{sgn}(b-y)e^{-|b-y|} \quad \text{if } y \notin (a, b), \quad (2.1.18)$$

and

$$-\operatorname{sgn}(a-y)e^{-|a-y|} > -\operatorname{sgn}(b-y)e^{-|b-y|} \quad \text{if } y \in (a, b). \quad (2.1.19)$$

This combined with the definition of F in (2.1.17) yields the desired result.

(b) It follows by just checking that if $0 < a < b < 1$, then

$$\partial_x \mathcal{G}(b-y) > \partial_x \mathcal{G}(a-y), \quad y \in [0, 1] - [a, b]. \quad (2.1.20)$$

and

$$\partial_x \mathcal{G}(b-y) < \partial_x \mathcal{G}(a-y), \quad y \in [a, b], \quad (2.1.21)$$

■

with

$$\partial_x \mathcal{G}(x) = \frac{\sinh(x - \lfloor x \rfloor - 1/2)}{2 \sinh(1/2)}. \quad (2.1.22)$$

2.2 Proof of the Unique Continuation Results for the BBM Equation

Proof: [Proof of Theorem 1.3.1] We recall the notation in (2.1.9)

$$\phi(D_x) = \partial_x(1 - \partial^2)^{-1} = \partial_x J^{-2}.$$

Hence,

$$\partial_t u + \phi(D_x)\left(u + \frac{u^2}{2}\right) = 0, \quad (2.2.1)$$

and

$$u(t) - u_0 = \int_0^t \phi(D_x)\left(u + \frac{u^2}{2}\right)(t') dt'. \quad (2.2.2)$$

Thus, for $u_0 \in C^{k,\theta}(\mathbb{R})$ we define the operator

$$\Psi_{u_0} v(x, t) = u_0(x) + \int_0^t \phi(D_x)\left(v + \frac{v^2}{2}\right)(t') dt' \quad (2.2.3)$$

on

$$X_T^{k,\theta}(\mathbb{R}) = \{v : \mathbb{R} \times [-T, T] \rightarrow \mathbb{R} : v \in C([-T, T] : C^{k,\theta}(\mathbb{R}))\}. \quad (2.2.4)$$

A combination of (2.1.3), (2.1.4) and (2.1.14) leads to the estimates

$$\sup_{[-T, T]} \|\Psi_{u_0} v\|_{C^{k,\theta}} \leq c \left(\|u_0\|_{C^{k,\theta}} + T \sup_{[-T, T]} \|v\|_{C^{k,\theta}} \left(1 + \sup_{[-T, T]} \|v\|_{C^{k,\theta}}\right) \right).$$

and

$$\sup_{[-T, T]} \|\Psi_{u_0}(v - w)\|_{C^{k,\theta}} \leq c T \sup_{[-T, T]} \|v - w\|_{C^{k,\theta}} \left(1 + \sup_{[-T, T]} (\|v\|_{C^{k,\theta}} + \|w\|_{C^{k,\theta}})\right).$$

Once that the above estimates are available by restricting the size of the time interval $[-T, T]$ with $T = T(\|u_0\|_{C^{k,\theta}}) > 0$ one can apply the contraction principle to the operator Ψ_{u_0} in the domain $X_T^{k,\theta}(\mathbb{R})$ to obtain the desired solution $u(x, t)$ with

$$u(x, t) = u_0(x) + \int_0^t \phi(D_x) \left(u + \frac{u^2}{2} \right) (t') dt', \quad t \in [-T, T]. \quad (2.2.5)$$

To get the regularity of the solution $u(x, t)$ stated in (1.3.4) we take the time derivatives in (2.2.5) and use Proposition 2.1.2 to see that $u \in C^1([-T, T] : C^{k,\theta}(\mathbb{R}))$. Next, we use the equation to deduce that

$$\begin{aligned} \partial_{tt}^2 u &= -\phi(D_x)(\partial_t u + u \partial_t u) \\ &= \phi(D_x) \left(\phi(D_x)(u + u^2/2) + u \phi(D_x)(u + u^2/2) \right). \end{aligned} \quad (2.2.6)$$

Hence, Proposition 2.1.2 shows that $u \in C^2([-T, T] : C^{k,\theta}(\mathbb{R}))$. An iteration of this argument yields the desired result. ■

Proof: [Proof of Theorem 1.3.2]

The idea is to reduce the proof to that given for Lemma 2.1.4.

The equation in (1.1.3) holds pointwise, therefore evaluating it at (a, t_0) and (b, t_0) , subtracting these values, using the remark after Lemma 2.1.4 and the hypothesis one

gets

$$\begin{aligned}
0 &= \partial_t u(b, t_0) - \partial_t u(a, t_0) \\
&+ \int_{-\infty}^{\infty} \frac{-\operatorname{sgn}(b-y)e^{-|b-y|} + \operatorname{sgn}(a-y)e^{-|a-y|}}{2} (u + \frac{u^2}{2})(y, t_0) dy \\
&= \partial_t u(b, t_0) - \partial_t u(a, t_0) \\
&+ \int_{-\infty}^{\infty} \frac{-\operatorname{sgn}(b-y)e^{-|b-y|} + \operatorname{sgn}(a-y)e^{-|a-y|}}{4} (u+1)^2(y, t_0) dy \\
&= \partial_t u(b, t_0) - \partial_t u(a, t_0) \\
&+ \int_{-\infty}^a (-e^{-|b-y|} + e^{-|a-y|}) \frac{(u+1)^2}{4}(y, t_0) dy \\
&+ \int_a^b (-e^{-|b-y|} - e^{-|a-y|}) \frac{(u+1)^2}{4}(y, t_0) dy \\
&+ \int_b^{\infty} (e^{-|b-y|} - e^{-|a-y|}) \frac{(u+1)^2}{4}(y, t_0) dy \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{2.2.7}$$

By hypothesis $A_1 \geq 0$ and $A_3 = 0$. Since the integrands in A_2 and A_4 are non-negative, see (2.1.18), the equality in (2.2.7) only holds if $A_1 = A_2 = A_4 = 0$. But this implies that

$$u(x, t_0) \equiv -1 \quad \forall x \in \mathbb{R} \quad \text{and} \quad \partial_t u(b, t_0) = \partial_t u(a, t_0).$$

■

Proof: Proof of Corollary 1.3.3 It follows directly from the proof of Theorem 1.3.2 given above. ■

Proof: [Proof of Theorem 1.3.4]

We look for $u_0 \in C^\infty(\mathbb{R}) - \{0\}$ such that

$$u_0(x) = c_0 \neq -1, \quad \partial_t u(x, 0) = 0, \quad x \in [a, b], \quad a < b.$$

Fixing $a = 0$ and recalling that $f(x) = x + x^2/2$ one has that if $x \in [0, b]$

$$\begin{aligned}
\partial_t u(x, 0) &= -\phi(D_x)f(u_0(x)) = \int_{-\infty}^{\infty} e^{-|x-y|} \operatorname{sgn}(x-y) f(u_0(y)) dy \\
&= e^{-x} \int_{-\infty}^0 e^y f(u_0(y)) dy + e^{-x} f(c_0) \int_0^x e^y dy \\
&\quad - e^x f(c_0) \int_x^b e^{-y} dy - e^x \int_b^{\infty} e^{-y} f(u_0(y)) dy \\
&= e^{-x}(\alpha + f(c_0)(e^x - 1)) - e^x(\beta + f(c_0)(e^{-x} - e^{-b})) \\
&= e^{-x}(\alpha - f(c_0)) + e^x(e^{-b} f(c_0) - \beta),
\end{aligned} \tag{2.2.8}$$

where

$$\alpha = \int_{-\infty}^0 e^y f(u_0(y)) dy, \quad \beta = \int_b^{\infty} e^{-y} f(u_0(y)) dy. \tag{2.2.9}$$

Since $c_0 \neq -1$, then $f(c_0) \in (-1/2, \infty)$ with $f(-1) = -1/2$ so

$$\alpha \in (-1/2, \infty) \quad \beta \in \left(\frac{-e^{-b}}{2}, \infty\right).$$

Thus, for any fixed $b > 0$, we can find $u_0 \in C^\infty(\mathbb{R})$ such that $u_0(0) = u_0(b) = c_0$ and $u_0(x) = 0$ if $|x| \geq M > 0$ for large enough $M = M(c_0) > 0$. This finishes the proof of Theorem 1.3.4. ■

Proof: [Proof of Theorem 1.3.5]

Using the function $f(x) = x + x^2/2$ and the remark after Lemma 2.1.4 we write the equation in (1.1.3) as

$$\partial_t u + \partial_x(1 - \partial_x^2)^{-1}(f(u) - f(c_0)) = 0. \tag{2.2.10}$$

At this point the proof follows the argument given in the proof of Theorem 1.3.4. ■

Proof: [Proof of Theorem 1.3.6]

It follows a similar argument to that provided in details in the proof of Theorem 1.3.4 but using (2.1.20) and (2.1.21) instead of (2.1.18) and (2.1.19). ■

Proof: [Proof of Theorem 1.3.7]

We re-write the equation in (1.1.11) as

$$\partial_t u + \phi(D_x)(f(u) - f(c_0)) = 0, \quad (2.2.11)$$

with $f(x) = x + x^2/2$.

Let $a, b \in [0, 1) \cong \mathbb{S}$ with $0 < a < b < 1$. Since $c_0 \neq -1$ we can find $\psi_1, \psi_2 \in C_0^\infty(\mathbb{S}) - \{0\}$ with $\text{supp } \psi_1 \subset (0, a)$ and $\text{supp } \psi_2 \subset (b, 1)$ such that

$$\begin{aligned} f(\psi_1(x)) &< f(c_0), & x \in (0, a), \\ f(\psi_2(x)) &> f(c_0), & x \in (b, 1). \end{aligned}$$

Let

$$v_j(x) = c_0 + \psi_j(x), \quad j = 1, 2.$$

and define the operator

$$Q : H^s(\mathbb{S}) \rightarrow \mathbb{R}, \quad s > 1/2,$$

as

$$\begin{aligned} Q(u_0) &\equiv \partial_t u(b, 0) - \partial_t u(a, 0) \\ &= - \int_0^1 (\partial_x \mathcal{G}(b - y) - \partial_x \mathcal{G}(a, 0))(f(u(x, 0)) - f(c_0)) dy \end{aligned} \quad (2.2.12)$$

where $u \in C^\infty((-T, T) : H^s(\mathbb{S}))$ is the solution of the IVP (1.1.3) provided by Theorem 1.1.2.

Hence, from (2.1.20), (2.1.21) and (2.1.22) one has that

$$Q(v_1) > 0 \quad \text{and} \quad Q(v_2) < 0. \quad (2.2.13)$$

The operator Q defined in (2.2.12) is continuous. Therefore, by considering

$$Q(\lambda v_1 + (1 - \lambda)v_2), \quad \lambda \in [0, 1],$$

one has that there exists $\lambda_0 \in (0, 1)$ such that for $v_0 = \lambda_0 v_1 + (1 - \lambda_0)v_2$ one has $Q(v_0) = 0$, i.e.

$$\partial_t v(b, 0) - \partial_t v(a, 0) = 0, \quad (2.2.14)$$

with $v(x, t)$ the solution corresponding to the data v_0 which satisfies $v_0(x) = c_0$, $x \in [a, b]$.

This yields the desired result. ■

Proof: [Proof of Theorem 1.3.10]

It follows an argument similar to that provided in details in the proof of Theorem 1.3.2. Hence, it will be omitted. ■

Proof: [Proof of Theorem 1.3.8]

It follows a similar argument to that provided in details in the proof of Theorem 1.3.4 but using (2.1.20) and (2.1.21) instead of (2.1.18) and (2.1.19). ■

2.3 Proof of the Regularity Results for the BBM Equation

Proof: [Proof of Theorem 1.3.11]

We recall the notation in (2.1.9)

$$\phi(D_x) = \partial_x(1 - \partial^2)^{-1} = \partial_x J^{-2}.$$

Hence,

$$\partial_t u + \phi(D_x)\left(u + \frac{u^2}{2}\right) = 0, \quad (2.3.1)$$

and

$$u(t) - u_0 = \int_0^t \phi(D_x)\left(u + \frac{u^2}{2}\right)(t') dt'. \quad (2.3.2)$$

First, we consider the case $s \in [0, 1/2)$.

Applying the operator $J_x^{(2s+1/2)^-}$ to the integral term in (2.3.2), and combining Proposition 2.1.2, (2.1.10) with $(\eta, p, q) = ((s + 1/2)^-, 2, 2)$ and $(\eta, p, q) = ((1/r)^-, 2, q)$ with $1 + 1/2 = 1/r + 1/q$ and $s + 1/2 = 1/r + s_2$, Lemma 2.1.1 and Sobolev embedding theorem, one gets

$$\begin{aligned} & \|J_x^{(2s+1/2)^-}(u(t) - u_0)\|_2 \\ & \leq c \int_0^t (\|J_x^s u(t')\|_2 + \|J_x^{s_2}(u^2)(t')\|_q) dt' \\ & \leq c \int_0^t (\|J_x^s u(t')\|_2 + \|u(t')\|_r \|J_x^{s_2} u(t')\|_l) dt' \\ & \leq c \int_0^t (\|J_x^s u(t')\|_2 + \|J_x^s u(t')\|_2^2) dt', \end{aligned} \quad (2.3.3)$$

with $1/q = 1/r + 1/l$, $s - 1/2 = -1/r$ and $s - 1/2 = s_2 - 1/l$ such that

$$1/r + s_2 = 1/r + 2s + 1/q - 1 = 2s + 1/2.$$

This yields the desired result (1.3.20).

Next, we consider the case in (1.3.21), i.e. $u_0 \in H^s(\mathbb{R})$ with $s > 1/2$.

Applying the operator $J_x^{(2s+1)}$ to the integral term in (2.3.2), and combining (2.1.10) with $(\eta, p, q) = (1, 2, 2)$, Lemma 2.1.3 and Sobolev embedding theorem, one gets

$$\begin{aligned} \|J_x^{s+1}(u(t) - u_0)\|_2 &\leq c \int_0^t (\|J_x^s u(t')\|_2 + \|J_x^s(u^2)(t')\|_2) dt' \\ &\leq c \int_0^t (\|J_x^s u(t')\|_2 + \|u(t')\|_\infty \|J_x^s u(t')\|_2) dt' \\ &\leq c \int_0^t (\|J_x^s u(t')\|_2 + \|J_x^s u(t')\|_2^2) dt', \end{aligned} \quad (2.3.4)$$

which yields (1.3.21).

To prove the regularity in the t -variable, we first show that if $u, v \in C([-T, T] : H^s(\mathbb{R}))$, then

$$\begin{aligned} (i) \quad &\phi(D_x)u \in C([-T, T] : H^s(\mathbb{R})), \\ (ii) \quad &\phi(D_x)(uv) \in C([-T, T] : H^s(\mathbb{R})). \end{aligned} \quad (2.3.5)$$

We recall that

$$\phi(D_x)f(x) = \partial_x G_2 * f(x),$$

with $\partial_x G_2(x) = -\text{sgn}(x)e^{-|x|}/2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus

$$\|J_x^s \phi(D_x)u\|_2 \leq \|J_x^s u\|_2$$

and using Lemma 2.1.1

$$\|J_x^s \phi(D_x)(uv)\|_2 = \|\phi(D_x)J_x^s(uv)\|_2 \leq \|J_x^s(uv)\|_1 \leq c\|J_x^s u\|_2 \|J_x^s v\|_2$$

which yields (2.3.5).

Since $u \in C([-T, T] : H^s(\mathbb{R}))$ the equation in (2.3.2) and (2.3.5) shows that $u \in$

$C^1([-T, T] : H^s(\mathbb{R}))$. Next, we use (2.3.1) to deduce that

$$\begin{aligned}\partial_{tt}^2 u &= -\phi(D_x)(\partial_t u + u\partial_t u) \\ &= \phi(D_x)\left(\phi(D_x)(u + u^2/2) + u\phi(D_x)(u + u^2/2)\right).\end{aligned}\tag{2.3.6}$$

Applying (2.3.5) one gets that $u \in C^2([-T, T] : H^s(\mathbb{R}))$. An iteration of this argument yields the proof of Theorem 1.3.11 part(i).

To prove part (ii) one just needs to show that

$$\partial_x^{k+1}(u(x, t) - u_0(x)) \in C^{0, \theta}(\mathbb{R}).$$

Applying the operator ∂_x^{k+1} to the integral term in (2.3.2), and combining (2.1.14) and Proposition 2.1.2 one gets

$$\begin{aligned}\|\partial_x^{k+1}(u(t) - u_0)\|_{C^{0, \theta}(\mathbb{R})} &\leq c \int_0^t (\|\partial_x^k u(t')\|_{C^{0, \theta}(\mathbb{R})} + \|u^2(t')\|_{C^{k, \theta}(\mathbb{R})}) dt' \\ &\leq c \int_0^t (\|u(t')\|_{C^{k, \theta}(\mathbb{R})} + \|u(t')\|_{\infty} \|u(t')\|_{C^{k, \theta}(\mathbb{R})}) dt'\end{aligned}\tag{2.3.7}$$

which basically yields (1.3.22). This completes the proof of Theorem 1.3.11. ■

Proof: [Proof of Theorem 1.3.12]

The proof is similar to that provided in details for Theorem 1.3.11. Therefore, it will be omitted. ■

Proof: [Proof of Theorem 1.3.13] By writing the solution $u = u(x, t)$ as

$$u(x, t) = (u(x, t) - u_0(x)) + u_0(x),\tag{2.3.8}$$

with $u_0 \in H^s(\mathbb{R})$, from Theorem 1.3.11 one has

$$u - u_0 \in C^\infty([-T, T] : H^{s'}(\mathbb{R})),$$

with

$$s' = \begin{cases} (2s + 1/2)^-, & s \in [0, 1/2], \\ s + 1, & s > 1/2. \end{cases} \quad (2.3.9)$$

Therefore, the regularity (and singularities) of u_0 at the level of $H^\mu(\Omega)$, with $\mu \in (s, s']$, are the same as those of $u = u(x, t)$ for all $t \in [-T, T]$. ■

Proof: [Proof of Theorem 1.3.14] The proof is similar to that provided in details for Theorem 1.3.11. Therefore, it will be omitted. ■

Proof: [Proof of Corollary 1.3.15]

Combining Sobolev embedding theorem, the expression in (2.3.8) and the results obtained in Theorem 1.3.11, one gets the desired result. ■

Chapter 3

Local Energy Decay for a Family of Peakon Equations

3.1 Preliminary Estimates

We need to bound the $H^1(\mathbb{R})$ norm on u in time for solutions to the b -family of peakon equations (1.2.14). However, with the exception of $b = 2$, the $H^1(\mathbb{R})$ norm is not a conserved quantity. We can take advantage of $m = u - u_{xx}$ and $(1 - \partial_x^2)^{-1} = G$ to get a bound on the $H^1(\mathbb{R})$ norm instead. To do this, note that

$$u = Gm = \frac{1}{2}e^{-|x|} * m, \quad \partial_x u = \partial_x Gm = -\frac{\operatorname{sgn}(x)}{2}e^{-|x|} * m. \quad (3.1.1)$$

One special case of Young's convolution inequality tells us that for $1 \leq p \leq \infty$, $f \in L^1$, and $g \in L^p$

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

We have $\|\frac{1}{2}e^{-|x|}\|_p = \|\frac{\operatorname{sgn}(x)}{2}e^{-|x|}\|_p \leq 1$. This then implies

$$\|u\|_p \leq \|m\|_1, \quad \|\partial_x u\|_p \leq \|m\|_1, \quad 1 \leq p \leq \infty. \quad (3.1.2)$$

Recall that $\int m \, dx$ is a conserved quantity, and m remains signed if $m(0, x)$ is signed. This gives a condition to bound the $L^1(\mathbb{R})$ norm of m , which can then be used to bound the $H^1(\mathbb{R})$ norm of u for all time. Moreover, a decay on $\|m\|_1$ is equal to a decay in $\|u\|_{H^1(\mathbb{R})}$.

3.1.1 Estimates on $\lambda(t)$ and G

We will be taking advantage of a couple of weight functions. Let $b \in [0, 1)$. We will define the following weight function $\phi(x)$.

$$\phi(x) = \begin{cases} e^x - 1 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}, \quad \phi'(x) = e^{-|x|}, \quad \phi''(x) = -\operatorname{sgn}(x)e^{-|x|}. \quad (3.1.3)$$

We will also use the following functions $\lambda_c(t), \mu_c(t)$. $\lambda_c(t)$ is related to the growing interval where we measure decay, while $\mu_c(t)$ is necessary to prove Theorem 1.3.16.

$$\lambda_c(t) = \frac{t^c}{\log t}, \quad \mu_c(t) = t^{1-c} \log^2 t, \quad \mu_c(t) \lambda_c(t) = t \log t. \quad (3.1.4)$$

In the case $c = 0$, we adopt the convention $\lambda_c(t) = 1$. When $c > 0$, we have

$$\lambda'_c(t) = \frac{t^{c-1}}{\log t} \left(c - \frac{1}{\log t} \right), \quad \frac{\lambda'_c(t)}{\lambda_c(t)} = \frac{1}{t} \left(c - \frac{1}{\log t} \right), \quad \frac{\mu'_c(t)}{\mu_c(t)} = \frac{1-c}{t} + \frac{2 \log t}{t}. \quad (3.1.5)$$

We will also take advantage of the fact that $(1 - \partial_x^2)^{-1} f = Gf = \frac{1}{2} e^{-|x|} * f$. Since $\frac{1}{2} e^{-|x|} > 0$, we get

$$f \geq g \implies Gf \geq Gg, \quad f \geq 0 \implies Gf \geq 0. \quad (3.1.6)$$

We will also need the following Lemma

Lemma 3.1.1

$$G\phi'(x/\lambda) = \frac{\lambda^2}{\lambda^2 - 1} e^{-|x|/\lambda} - \frac{\lambda}{\lambda^2 - 1} e^{-|x|} \approx \phi'(x/\lambda) \quad (3.1.7)$$

when $\lambda \gg 1$.

Proof: Let's consider when $x \leq 0$

$$\begin{aligned} 2G\phi'(x/\lambda) &= \int_{-\infty}^{\infty} e^{-|x-y|} e^{-|y|/\lambda} dy \\ &= \int_{-\infty}^x e^{y-x} e^{y/\lambda} dy + \int_x^0 e^{x-y} e^{y/\lambda} dy + \int_0^{\infty} e^{x-y} e^{-y/\lambda} dy \\ &= e^{-x} \int_{-\infty}^x e^{y+y/\lambda} dy + e^x \int_x^0 e^{-y+y/\lambda} dy + e^x \int_0^{\infty} e^{-y-y/\lambda} dy \\ &= e^{-x} \frac{1}{1+1/\lambda} e^{y+y/\lambda} \Big|_{-\infty}^x + e^x \frac{1}{1/\lambda-1} e^{-y+y/\lambda} \Big|_x^0 \\ &\quad + e^x \frac{1}{-1-1/\lambda} e^{-y-y/\lambda} \Big|_0^{\infty} \\ &= \frac{1}{1+1/\lambda} e^{x/\lambda} + \frac{1}{1/\lambda-1} (e^x - e^{x/\lambda}) + \frac{1}{1+1/\lambda} e^x \\ &= \frac{2\lambda^2}{\lambda^2-1} e^{x/\lambda} - \frac{2\lambda}{\lambda^2-1} e^x \end{aligned} \quad (3.1.8)$$

We get a similar computation for $x > 0$. Note that since $\lambda > 1$, $e^{x/\lambda}$ dominates e^x , so we get the result. Moreover, as $\lambda \rightarrow \infty$, $G\phi'(x/\lambda) \approx \frac{\lambda^2}{\lambda^2-1} \rightarrow \phi'(x/\lambda)$ uniformly. ■

3.2 Long-time L^2 Decay for the Peakon Equations

3.2.1 Functional Estimate

Consider the functional

$$\mathcal{M}_q(t) = \frac{1}{\mu_c(t)} \int \phi \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) u(t, x) dx \quad (3.2.1)$$

where $0 < c < 1$, and $q > 1$. We will find a tighter bound on b as we prove the upcoming lemma. To use this functional, we will need to show that it is bounded above after sufficient time. Let us assume that $u \in C([0, \infty) : H^1(\mathbb{R}))$. Since $\phi \in L^\infty(R)$, and $\phi' \in H^1(\mathbb{R})$, we know that $\mathcal{M}_q(t) < +\infty$ for all $t > 2$. In fact,

$$\begin{aligned} |\mathcal{M}_q(t)| &\leq \frac{1}{\mu_c(t)} \left\| \phi \left(\frac{\cdot}{\lambda_c(t)} \right) \right\|_{L^\infty} \left\| \phi' \left(\frac{\cdot}{\lambda_c^q(t)} \right) \right\|_{L^2} \|u(t)\|_{L^2} \\ &= \frac{\lambda^{q/2}(t)}{\mu_c(t)} \|\phi\|_{L^\infty} \|\phi'\|_{L^2} \|u(t)\|_{L^2} \\ &= C \frac{t^{cq/2}}{t^{1-c} \log^2 t \log^{q/2} t} \|u(t)\|_{L^2} = C \frac{1}{t^{(2-2c-cq)/2} \log^{(4+q)/2} t} \|u(t)\|_{L^2}. \end{aligned} \quad (3.2.2)$$

$\|u(t)\|_{L^2} \leq \|u(t)\|_{H^1}$, which is bounded by hypothesis, so if we let $c \leq \frac{2}{2+q}$, we get

$$\sup_{t \gg 1} |\mathcal{M}_q(t)| = M < +\infty. \quad (3.2.3)$$

We will use this fact in conjunction with the derivative of $\mathcal{M}_q(t)$. If we take the derivative of $\mathcal{M}_q(t)$ with respect to time, (1.1) and integrating by parts tells us that

$$\begin{aligned}
\frac{d}{dt} \mathcal{M}_q(t) &= -\frac{\mu'_c(t)}{\mu_c^2(t)} \mathcal{M}_q(t) - \frac{1}{\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) u(t, x) dx \\
&\quad - \frac{q}{\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c^q(t)} \phi \left(\frac{x}{\lambda_c(t)} \right) \phi'' \left(\frac{x}{\lambda_c^q(t)} \right) u(t, x) dx \\
&\quad + \frac{1}{\mu_c(t) \lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \left[\frac{u^2}{2} + G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \right] dx \\
&\quad + \frac{1}{\mu_c(t) \lambda_c^q(t)} \int \phi \left(\frac{x}{\lambda_c(t)} \right) \phi'' \left(\frac{x}{\lambda_c^q(t)} \right) \left[\frac{u^2}{2} + G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \right] dx \\
&= -M_1 - M_2 - M_3 + M_4 + M_5.
\end{aligned} \tag{3.2.4}$$

Now we can prove the following lemma

Lemma 3.2.1 *Let u be a nontrivial global solution of (1.2.14), such that $u \in C([0, \infty) : H^1(\mathbb{R})) \cap L^\infty([0, \infty) : H^1(\mathbb{R}))$. If $b < 2/3$, we get*

$$\int_{10}^{\infty} \frac{1}{t \log t} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) (u^2 + (\partial_x u)^2) dx dt < +\infty. \tag{3.2.5}$$

Proof: Note that $M_1 = \frac{\mu'_c(t)}{\mu_c^2(t)} \mathcal{M}_q(t)$ is integrable for $t \gg 1$. This is because $\frac{\mu'_c(t)}{\mu_c^2(t)} = \frac{1+\log t}{t^2 \log^2 t}$, which is integrable for $t \gg 1$, and $\mathcal{M}_q(t)$ is bounded for $t \gg 1$ as well. Similar reasoning tells us that

$$\begin{aligned}
M_5 &= \frac{1}{t^{1-c+q} \log^{2-q} t} \int \phi \left(\frac{x}{\lambda_c(t)} \right) \phi'' \left(\frac{x}{\lambda_c^q(t)} \right) \left[\frac{u^2}{2} + G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \right] dx \\
&\leq C \frac{1}{t^{1-c+q} \log^{2-q} t} \|\phi \phi''\|_{L^\infty} \|u\|_{H^1}^2.
\end{aligned} \tag{3.2.6}$$

The last inequality follows from the fact that $Gf = \frac{1}{2} e^{-|x|} * f$. M_5 is then integrable because $q > 1$.

Young's inequality implies that

$$\begin{aligned}
|M_2| &\leq \left| \frac{\lambda'_c(t)}{2\epsilon\mu_c(t)} \right| \int \left(\frac{x}{\lambda_c(t)} \right)^2 \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] \frac{dx}{\lambda_c(t)} \\
&\quad + \left| \frac{\epsilon\lambda'_c(t)}{2\mu_c(t)\lambda_c(t)} \right| \int \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] u^2 dx \\
&= \left| \frac{(\lambda'_c(t))^2}{2\mu_c(t)} \right| \int \left(\frac{x}{\lambda_c(t)} \right)^2 \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] \frac{dx}{\lambda_c(t)} \\
&\quad + \left| \frac{1}{2\mu_c(t)\lambda_c(t)} \right| \int \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] u^2 dx
\end{aligned} \tag{3.2.7}$$

when $\epsilon = \frac{1}{\lambda'_c(t)}$.

Similar reasoning tells us that

$$\begin{aligned}
|M_3| &\leq \left| \frac{q}{2\mu_c(t)} \frac{\lambda'_c(t)}{\epsilon\lambda_c(t)} \right| \int \left(\frac{x}{\lambda_c^q(t)} \right)^2 \left| \phi \left(\frac{x}{\lambda_c(t)} \right) \right| \left[\phi'' \left(\frac{x}{\lambda_c^q(t)} \right) \right]^2 dx \\
&\quad + \left| \frac{\epsilon q}{2\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \right| \int \left| \phi \left(\frac{x}{\lambda_c(t)} \right) \right| u^2 dx \\
&= \left| \frac{q}{2\mu_c(t)} \frac{\lambda'_c(t)}{t^{1-c}\lambda_c(t)} \right| \int \left(\frac{x}{\lambda_c^q(t)} \right)^2 \left| \phi \left(\frac{x}{\lambda_c(t)} \right) \right| \left[\phi'' \left(\frac{x}{\lambda_c^q(t)} \right) \right]^2 dx \\
&\quad + \left| \frac{qt^{1-c}}{2\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \right| \int \left| \phi \left(\frac{x}{\lambda_c(t)} \right) \right| u^2 dx
\end{aligned} \tag{3.2.8}$$

when $\epsilon = t^{1-c}$.

In (3.2.7), say

$$M_2^{(1)} = \frac{(\lambda'_c(t))^2}{2\mu_c(t)} \int \left(\frac{x}{\lambda_c(t)} \right)^2 \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] \frac{dx}{\lambda_c(t)}. \tag{3.2.9}$$

Note that $|\phi'(x)| \leq 1$. We can then use (3.1.3), (3.1.5), and the change of variables

$y = \frac{x}{\lambda_c(t)}$ to get

$$\begin{aligned}
M_2^{(1)} &\leq \frac{(\lambda'_c(t))^2}{2\mu_c(t)} \int \left(\frac{x}{\lambda_c(t)} \right)^2 \left| \phi' \left(\frac{x}{\lambda_c(t)} \right) \right| \frac{dx}{\lambda_c(t)} \\
&= \frac{(\lambda'_c(t))^2}{2\mu_c(t)} \int y^2 |\phi'(y)| dy \\
&= C \frac{(\lambda'_c(t))^2}{2\mu_c(t)} \lesssim C \frac{1}{t^{3-3c} \log^4 t},
\end{aligned} \tag{3.2.10}$$

which is integrable for $t \gg 1$ when $c < 2/3$.

For (3.2.8), note that $|\phi(x)| \leq 1$, and $\int u^2 dx \leq \|u\|_{H^1}^2$. This gives us

$$\begin{aligned}
|M_3| &\leq \left| \frac{q}{2\mu_c(t)} \frac{\lambda'_c(t)}{t^{1-c}\lambda_c(t)} \right| \int \left(\frac{x}{\lambda_c^q(t)} \right)^2 \left[\phi'' \left(\frac{x}{\lambda_c^q(t)} \right) \right]^2 dx \\
&\quad + \left| \frac{qt^{1-c}}{2\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \right| \int u^2 dx \\
&\leq C \left| \frac{q\lambda_c^q(t)}{2\mu_c(t)} \frac{\lambda'_c(t)}{t^{1-c}\lambda_c(t)} \right| + C \left| \frac{qt^{1-c}}{2\mu_c(t)} \frac{\lambda'_c(t)}{\lambda_c(t)} \right| \\
&\lesssim C \frac{1}{t^{3-c(2+q)} \log^{2+q} t} + C \frac{1}{t \log^2 t},
\end{aligned} \tag{3.2.11}$$

which is integrable for $t \gg 1$ because $c \leq \frac{2}{2+q}$.

If we combine our estimates for M_1 , M_2 , M_3 , and M_5 with (3.2.4), we get

$$\begin{aligned}
&\frac{d}{dt} \mathcal{M}_q(t) + M_1 + \frac{C}{t^{3-3c} \log^4 t} + M_3 + M_5 \\
&\geq - \left| \frac{1}{2\mu_c(t)\lambda_c(t)} \right| \int \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] u^2 dx + M_4 \\
&= -N + M_4
\end{aligned} \tag{3.2.12}$$

If we assume $t \gg 1$, we get

$$\begin{aligned}
-N + M_4 &= - \left| \frac{1}{2\mu_c(t)\lambda_c(t)} \right| \int \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] u^2 dx \\
&\quad + \frac{1}{\mu_c(t)\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \frac{u^2}{2} dx \\
&\quad + \frac{1}{\mu_c(t)\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \left[G \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) \right] dx \\
&= \underbrace{\frac{1}{\mu_c(t)\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \left[G \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) \right] dx}_{(i)} \\
&= \underbrace{\frac{1}{\mu_c(t)\lambda_c(t)} \int G \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \right] \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) dx}_{(ii)} \\
&\approx \frac{1}{t \log t} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) dx
\end{aligned} \tag{3.2.13}$$

which is the quantity we need to estimate for our lemma. (i) and (ii) are equal because of Fubini's theorem since $Gf = \frac{1}{2}e^{-|\cdot|} * f$, $u^2 + (\partial_x u)^2$ is $L^1(\mathbb{R})$, and $\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right)$ is a Schwartz function. For the last step, we used the fact that

$$\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) = e^{-|\frac{x}{\lambda}|} e^{-|\frac{x}{\lambda^q}|} = e^{-|\frac{x}{\lambda^{\frac{q-1}{q}+1}}|}, \quad \frac{\lambda^{q-1}+1}{\lambda^q} \approx \frac{1}{\lambda} \tag{3.2.14}$$

and Lemma 3.1.1.

Now if we look back at (3.6), we showed that

$$\begin{aligned}
&\frac{d}{dt} \mathcal{M}_q(t) + M_1 + C \frac{1}{t^{3-3c} \log^4 t} + M_3 + M_5 \geq \\
&\frac{C}{t \log t} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \right) dx
\end{aligned} \tag{3.2.15}$$

The left hand side of the above inequality is integrable for $t \gg 1$. Since the right hand side is nonnegative, the right hand side of our inequality must be integrable when

$t \gg 1$ as well.

Note that $\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2 \geq \min\{b, \frac{3-b}{2}\}(u^2 + (\partial_x u)^2)$, so we know that

$$\frac{C}{t \log t} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) (u^2 + (\partial_x u)^2) dx$$

is integrable for $t \gg 1$, which proves the lemma. ■

Now we can prove the weak decay estimate

3.2.2 Proof of Theorem 1.3.16

Lemma 3.1 states that

$$\int_{10}^{\infty} \frac{1}{t \log t} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) (u^2 + (\partial_x u)^2) dx dt < +\infty. \quad (3.2.16)$$

However, $t \ln t$ is not integrable for any $t > 1$. This implies that

$$\liminf_{t \rightarrow +\infty} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) (u^2 + (\partial_x u)^2) dx = 0. \quad (3.2.17)$$

Recall that $\lambda_c(t) = \frac{t^c}{\log t}$. This means that when $|x| \in (-\Lambda_c(t), \Lambda_c(t))$, there exists $\delta > 0$ such that $\phi' \left(\frac{x}{\lambda_c(t)} \right) \phi' \left(\frac{x}{\lambda_c^q(t)} \right) \geq \delta$. This then implies that

$$\liminf_{t \rightarrow +\infty} \|u\|_{H^1(\Lambda_c(t))} = 0, \quad (3.2.18)$$

which gives us the weak decay estimate.

3.3 Long-time L^1 Decay for the Peakon Equations

3.3.1 First Functional Estimate

Consider the functional

$$\mathcal{I}(t) = \int \phi \left(\frac{x}{\lambda_c(t)} \right) u(t, x) dx. \quad (3.3.1)$$

Note that (1.2.2) implies that $\mathcal{I}(t)$ is bounded in time. Now we wish to prove

Lemma 3.3.1

$$\int_t^\infty \frac{1}{\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) dx < \infty, \quad (3.3.2)$$

for $t \gg 1$.

Proof: First, take the time derivative of (3.3.1). Applying (1.2.14) and integrating by parts gives us

$$\begin{aligned} \frac{d}{dt} \mathcal{I}(t) &= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \phi' \left(\frac{x}{\lambda_c(t)} \right) u dx + \int \phi \left(\frac{x}{\lambda_c(t)} \right) u_t dx \\ &= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \phi' \left(\frac{x}{\lambda_c(t)} \right) u dx - \int \phi \left(\frac{x}{\lambda_c(t)} \right) \partial_x \left(\frac{u^2}{2} \right) dx \\ &\quad - \int \phi \left(\frac{x}{\lambda_c(t)} \right) \partial_x \left(G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \right) dx \\ &= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \phi' \left(\frac{x}{\lambda_c(t)} \right) u dx + \frac{1}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 dx \\ &\quad + \frac{x}{\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) dx. \end{aligned} \quad (3.3.3)$$

If we let $c = 1/2$, Young's inequality, a change of variables, and (3.1.5) tells us that

$$\begin{aligned}
\left| \frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \phi' \left(\frac{x}{\lambda_c(t)} \right) u \, dx \right| &\leq \frac{(\lambda'_c(t))^2}{2\delta} \int \left(\frac{x}{\lambda_c(t)} \right)^2 \phi' \left(\frac{x}{\lambda_c(t)} \right) \frac{dx}{\lambda_c(t)} \\
&\quad + \frac{\delta}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 \, dx \\
&\lesssim \frac{C}{\delta t \log^2 t} + \frac{\delta}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 \, dx,
\end{aligned} \tag{3.3.4}$$

where $\delta > 0$ is a constant that will be chosen later. Plugging this into (3.3.3) gives

us

$$\begin{aligned}
\frac{d}{dt} \mathcal{I}(t) &\gtrsim -\frac{C}{t \log^2 t} - \frac{\delta}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 \, dx \\
&\quad + \underbrace{\frac{1}{\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) G \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \, dx}_{(i)} \\
&= -\frac{C}{t \log^2 t} - \frac{\delta}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 \, dx \\
&\quad + \underbrace{\frac{1}{\lambda_c(t)} \int G \left[\phi' \left(\frac{x}{\lambda_c(t)} \right) \right] \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \, dx}_{(ii)} \\
&\gtrsim -\frac{C}{t \log^2 t} - \frac{\delta}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) u^2 \, dx \\
&\quad + \frac{C(\lambda)}{\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \left(\frac{b}{2} u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \, dx \\
&\gtrsim -\frac{C}{t \log^2 t} + \frac{1}{2\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \left(\epsilon u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) \, dx, \quad \epsilon > 0.
\end{aligned} \tag{3.3.5}$$

We can equate (i) and (ii) via the Fourier transform since $Gf = \frac{1}{2}e^{-|x|} * f$, $u^2 + (\partial_x u)^2$ is $L^1(\mathbb{R})$, and $\phi' \left(\frac{x}{\lambda_c(t)} \right)$ is a positive Schwartz function. The second to last inequality comes from applying (3.1.1). Recall that $C(\lambda) = \frac{\lambda^2}{\lambda^2 - 1}$. The last inequality comes from an appropriate choice of δ as long as λ is large enough. Since $\mathcal{I}(t)$ is bounded, and $\frac{1}{t \log^2 t}$

is integrable for $t \gg 1$, we get

$$\int_t^\infty \frac{1}{\lambda_c(t)} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) \left(\epsilon u^2 + \frac{3-b}{2} (\partial_x u)^2 \right) dx < \infty, \quad (3.3.6)$$

where the integrand clearly is bounded below by a constant multiple of the left hand side of (3.3.2). This proves the lemma. Note that $\frac{1}{\lambda_c(t)}$ is not integrable in time. This implies that

$$\liminf_{t \rightarrow \infty} \int \phi' \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x^2 u)^2) dx = 0. \quad (3.3.7)$$

Otherwise the previous integrand could not be integrable. ■

3.3.2 Second Functional Estimate

To complete the proof of Theorem 1.3.11, we will need to consider the following functional

$$\mathcal{E}(t) := \int \psi \left(\frac{x}{\lambda_c(t)} \right) m(t, x) dx, \quad (3.3.8)$$

where $\psi(x) = \text{sech}^2 x$. $\mathcal{E}(t) < +\infty$ for all t since $\int m dx$ is a constant of motion, and $\text{sech}^2(x)$ is a Schwartz function. Note that $\psi'(x) = -2\text{sech}^2 x \tanh x$, so $|\psi'(x)| \leq 2\psi(x)$.

First we need to compute the derivative of the above functional. If we apply (1.2.15), we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) m \, dx + \int \psi \left(\frac{x}{\lambda_c(t)} \right) m_t \, dx \\
&= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) (u - \partial_x^2 u) \, dx \\
&\quad - \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u \partial_x u - u \partial_x^3 u) \, dx \\
&\quad - b \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u \partial_x u - \partial_x u \partial_x^2 u) \, dx
\end{aligned} \tag{3.3.9}$$

Now consider the following lemma

Lemma 3.3.2

$$\left| \frac{d}{dt}\mathcal{E}(t) \right| \lesssim \frac{1}{t \log^2 t} + \frac{\log t}{t^{3/2}} + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) \, dx. \tag{3.3.10}$$

Proof:

If we look at (3.3.9), we can integrate by parts to get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) (u - \partial_x^2 u) \, dx \\
&\quad - \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u \partial_x u - u \partial_x^3 u) \, dx \\
&\quad - b \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u \partial_x u - \partial_x u \partial_x^2 u) \, dx \\
&= -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) u \, dx - \frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) \partial_x^2 u \, dx \\
&\quad + \frac{1}{2\lambda_c(t)} \int \psi' \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) \, dx - \frac{1}{\lambda_c(t)} \int \psi' \left(\frac{x}{\lambda_c(t)} \right) u \partial_x^3 u \, dx \\
&\quad + \frac{b}{2\lambda(t)} \int \psi' \left(\frac{x}{\lambda_c(t)} \right) (u^2 - (\partial_x u)^2) \, dx.
\end{aligned} \tag{3.3.11}$$

First, we can bound the first two terms with the $-\frac{\lambda'_c(t)}{\lambda_c(t)}$ factor. The triangle inequality

and integration by parts implies

$$\begin{aligned}
& \left| -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) (u - \partial_x^2 u) dx \right| \\
& \leq \left| -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) u dx \right| \\
& + \left| \frac{\lambda'_c(t)}{(\lambda_c(t))^2} \int \left[\frac{x}{\lambda_c(t)} \psi'' \left(\frac{x}{\lambda_c(t)} \right) + \psi' \left(\frac{x}{\lambda_c(t)} \right) \right] \partial_x u dx \right|.
\end{aligned} \tag{3.3.12}$$

Note that $y\psi''(y)$ and $\psi'(y)$ are bounded since $\psi(y) = \text{sech}^2 y$. $\partial_x u$ is integrable by (3.1.2). These facts combined with (3.1.5) imply

$$\begin{aligned}
& \left| \frac{\lambda'_c(t)}{(\lambda_c(t))^2} \int \left[\frac{x}{\lambda_c(t)} \psi'' \left(\frac{x}{\lambda_c(t)} \right) + \psi' \left(\frac{x}{\lambda_c(t)} \right) \right] \partial_x u dx \right| \\
& \leq \left| \frac{\lambda'_c(t)}{(\lambda_c(t))^2} \int \frac{x}{\lambda_c(t)} \psi'' \left(\frac{x}{\lambda_c(t)} \right) \partial_x u dx \right| + \left| \frac{\lambda'_c(t)}{(\lambda_c(t))^2} \int \psi' \left(\frac{x}{\lambda_c(t)} \right) \partial_x u dx \right| \\
& \leq \frac{\lambda'_c(t)}{(\lambda_c(t))^2} \tilde{D} \|\partial_x u\|_1
\end{aligned} \tag{3.3.13}$$

If we plug this inequality into (3.3.12) and let $c = 1/2$, we get

$$\begin{aligned}
& \left| -\frac{\lambda'_c(t)}{\lambda_c(t)} \int \frac{x}{\lambda_c(t)} \psi' \left(\frac{x}{\lambda_c(t)} \right) (u - \partial_x^2 u) dx \right| \\
& \leq (\lambda'_c(t))^2 \int \left(\frac{x}{\lambda_c(t)} \right)^2 \psi \left(\frac{x}{\lambda_c(t)} \right) \frac{dx}{\lambda_c(t)} + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) u^2 dx + \frac{D \log t}{t^{3/2}} \\
& \leq \frac{C}{t \log^2 t} + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) u^2 dx + \frac{D \log t}{t^{3/2}}.
\end{aligned} \tag{3.3.14}$$

Recall that $|\psi'(x)| \leq 2\psi(x)$. Now if we plug in (3.3.14) into (3.3.11) and apply the triangle inequality, we get

$$\begin{aligned}
\left| \frac{d}{dt} \mathcal{E}(t) \right| &\leq \frac{E}{t \log^2 t} + \frac{D \log t}{t^{3/2}} + \frac{C}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) dx \\
&\quad + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) u \partial_x^2 u dx \\
&\leq \frac{1}{t \log^2 t} + \frac{D \log t}{t^{3/2}} + \frac{C}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) dx \\
&\quad + \frac{1}{(\lambda_c(t))^2} \int \psi \left(\frac{x}{\lambda_c(t)} \right) u \partial_x u dx + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (\partial_x u)^2 dx \\
&\lesssim \frac{1}{t \log^2 t} + \frac{\log t}{t^{3/2}} + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) dx
\end{aligned} \tag{3.3.15}$$

where the last inequality simply comes from Young's inequality. ■

Now we can prove Theorem 1.3.11

3.3.3 Proof of Theorem 1.3.11

Proof:

Say $t_0 \gg 1$. The right hand side of (3.3.2) is integrable based on Lemma 3.3.1 when we choose $c = 1/2$ for $\lambda_c(t) = \frac{t^c}{\log t}$ and integrate from t_0 to ∞ . This tells us that

$$\begin{aligned}
|\mathcal{E}(t_0)| &= \left| L + \int_{t_0}^{\infty} \frac{d}{dt} \mathcal{E}(t) dt \right| \leq L + \int_{t_0}^{\infty} \left| \frac{d}{dt} \mathcal{E}(t) \right| dt \\
&\lesssim \int_{t_0}^{\infty} \frac{1}{t \log^2 t} + \frac{\log t}{t^{3/2}} + \frac{1}{\lambda_c(t)} \int \psi \left(\frac{x}{\lambda_c(t)} \right) (u^2 + (\partial_x u)^2) dx dt,
\end{aligned} \tag{3.3.16}$$

which then implies there exists a limit for $\mathcal{E}(t)$ as $t \rightarrow \infty$ by Dominated Convergence since $\mathcal{E}(t) \geq 0$. However, note that we do not know whether or not the limit is 0.

Recall that $m \geq 0$. With the definition of $\mathcal{E}(t)$, we have

$$\mathcal{E}(t) = \int \psi \left(\frac{x}{\lambda_c(t)} \right) m(t, x) dx \rightarrow L. \quad (3.3.17)$$

We also have $1 \geq \psi(x/\lambda_c(t)) = e^{-|x/\lambda_c(t)|} \geq e^{-1}$ for all $x \in (-\Lambda_{1/2}(t), \Lambda_{1/2}(t))$. This implies that

$$e\mathcal{E}(t) \geq \|m(t)\|_{L^1(\Lambda_{1/2}(t))} \geq \mathcal{E}(t). \quad (3.3.18)$$

Note that we essentially get the same estimate for $\|u\|_{L^1(\Lambda_{1/2})}$. To see this, recall that $m = u - \partial_x^2 u$. This tells us that

$$\begin{aligned} \mathcal{E}(t) &= \int \psi \left(\frac{x}{\lambda_c(t)} \right) m dx \\ &= \int \psi \left(\frac{x}{\lambda_c(t)} \right) u dx - \int \psi \left(\frac{x}{\lambda_c(t)} \right) \partial_x^2 u dx \\ &= \int \psi \left(\frac{x}{\lambda_c(t)} \right) u dx - \frac{1}{\lambda^2(t)} \int \psi'' \left(\frac{x}{\lambda_c(t)} \right) u dx. \end{aligned} \quad (3.3.19)$$

Since $|\psi''(x)| \leq \psi(x)$, $\lambda_c(t) \rightarrow \infty$ as $t \rightarrow \infty$, and $\mathcal{I}(t)$ is bounded, the second term in the last inequality goes to 0. u is signed since $u = Gm$, so we get

$$\begin{aligned} L = \lim_{t \rightarrow \infty} \mathcal{E}(t) &= \lim_{t \rightarrow \infty} \int \psi \left(\frac{x}{\lambda_c(t)} \right) u dx - \lim_{t \rightarrow \infty} \frac{1}{\lambda^2(t)} \int \psi'' \left(\frac{x}{\lambda_c(t)} \right) u dx \\ &= \lim_{t \rightarrow \infty} \int \psi \left(\frac{x}{\lambda(t)} \right) u dx. \end{aligned} \quad (3.3.20)$$

By a similar argument as before, this implies that

$$e\mathcal{E}(t) \geq \|u(t)\|_{L^1(\Lambda_{1/2}(t))} \geq \mathcal{E}(t) \quad (3.3.21)$$

,

which then implies that

$$\lim_{t \rightarrow \infty} \|u(t)\|_{L^1(\Lambda_{1/2}(t))} = \lim_{t \rightarrow \infty} \|m(t)\|_{L^1(\Lambda_{1/2}(t))} = L. \quad (3.3.22)$$

It is difficult to get any further results from here without further assumptions. The proof in [24] makes use of the fact that there exists a subsequential limit for a similar functional. From there, integrability would imply that limit, 0, would be the only possible limit. However, we do not know that 0 is the limit. In fact, we can't even determine the limit for the H^1 norm on $\Lambda_{1/2}(t)$ because the L^1 norm of m only gives an upper bound for the H^1 norm of u .

■

3.4 Long-time Decay for the Peakon Equations on the Half-line

3.4.1 Proof of Functional Estimate

We will follow the method of proof used in [36]. Consider the following functional

$$\mathcal{I}(t) = \int \phi \left(\frac{x - \sigma t}{L} \right) m(t, x) dx \quad (3.4.1)$$

where $\sigma, L > 0$. By our assumptions, this functional is bounded in time, and if we take $\phi(x) = \frac{1}{2}(\tanh x + 1)$, our functional is nonnegative as well given m is nonnegative. Say $\|m\|_{L^1} = M$. To prove the result, we want to show

Lemma 3.4.1 *There exists some $\sigma > 0$ such that.*

$$\frac{d}{dt} \mathcal{I}(t) \lesssim -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m dx. \quad (3.4.2)$$

Proof:

Following a similar method to the proof of Lemma 3.3.2, we first take the derivative of $\mathcal{I}(t)$.

$$\begin{aligned} \frac{d}{dt} \mathcal{I}(t) &= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m dx + \int \phi \left(\frac{x - \sigma t}{L} \right) m_t dx \\ &= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m dx - \int \phi \left(\frac{x - \sigma t}{L} \right) (u \partial_x m + b \partial_x u m) dx. \end{aligned} \quad (3.4.3)$$

We then integrate by parts to get

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) um \, dx \\
&\quad + (1 - b) \int \phi \left(\frac{x - \sigma t}{L} \right) \partial_x um \, dx \\
&= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) um \, dx \\
&\quad + (1 - b) \int \phi \left(\frac{x - \sigma t}{L} \right) u \partial_x u \, dx - (1 - b) \int \phi \left(\frac{x - \sigma t}{L} \right) \partial_x u \partial_x^2 u \, dx \\
&= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) um \, dx \\
&\quad + \frac{b-1}{2L} \int \phi' \left(\frac{x - \sigma t}{L} \right) u^2 \, dx - \frac{b-1}{2L} \int \phi' \left(\frac{x - \sigma t}{L} \right) (\partial_x u)^2 \, dx.
\end{aligned} \tag{3.4.4}$$

Now we can use the fact that $u = (1 - \partial_x^2)^{-1}m = Gm$ from (3.1.1) to get

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) um \, dx \\
&\quad + \frac{b-1}{2L} \int \phi' \left(\frac{x - \sigma t}{L} \right) uGm \, dx - \frac{b-1}{2L} \int \phi' \left(\frac{x - \sigma t}{L} \right) \partial_x u \partial_x G(m) \, dx \\
&= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) um \, dx \\
&\quad + \frac{b-1}{2L} \int G \left(\phi' \left(\frac{x - \sigma t}{L} \right) u \right) m \, dx \\
&\quad - \frac{b-1}{2L} \int \partial_x G \left(\phi' \left(\frac{x - \sigma t}{L} \right) \partial_x u \right) m \, dx.
\end{aligned} \tag{3.4.5}$$

Note that $\|u\|_{L^\infty}, \|\partial_x u\|_{L^\infty} \leq M$ by Young's inequality. Then for large enough L , (3.1.1), (3.1.1), and (3.1.2) imply that

$$G \left(\phi' \left(\frac{x - \sigma t}{L} \right) u \right), \partial_x G \left(\phi' \left(\frac{x - \sigma t}{L} \right) \partial_x u \right) \lesssim M \phi' \left(\frac{x - \sigma t}{L} \right). \tag{3.4.6}$$

If we combine the above with (3.4.5) and (3.1.2), we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{1}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) u m \, dx \\
&\quad + \frac{b-1}{2L} \int G \left(\phi' \left(\frac{x - \sigma t}{L} \right) u \right) m \, dx \\
&\quad - \frac{b-1}{2L} \int \partial_x G \left(\phi' \left(\frac{x - \sigma t}{L} \right) \partial_x u \right) m \, dx \\
&\lesssim -\frac{\sigma}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx + \frac{M}{L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx \\
&\quad + 2M \frac{b-1}{2L} \int \phi' \left(\frac{x - \sigma t}{L} \right) m \, dx.
\end{aligned} \tag{3.4.7}$$

If we choose σ and L large enough, we get the result. ■

3.4.2 Proof of Theorem 1.3.18

Proof:

Consider the following functional

$$I_{t_0}(t) = \int \phi \left(\frac{x - \sigma t_0 + \frac{\sigma}{2}(t_0 - t)}{L} \right) m \, dx, \tag{3.4.8}$$

where $\phi(x) = \frac{1}{2}(\tanh x + 1)$, and $t_0 > 2$.

Lemma 3.4.1 implies that

$$\frac{d}{dt} I_{t_0}(t) \lesssim - \int \phi' \left(\frac{x - \sigma t_0 + \frac{\sigma}{2}(t_0 - t)}{L} \right) m \, dx. \tag{3.4.9}$$

Note that $\phi'(x) = \text{sech}^2 x$. Since the integrand is nonnegative, we get that $I_{t_0}(t)$ is decreasing on the interval $t \in [2, t_0]$. Moreover, since $\phi(x)$ tends to 0 as $x \rightarrow -\infty$, so $\phi \left(\frac{x - at - b}{L} \right) \rightarrow 0$ for positive a, b . This implies

$$\limsup_{t \rightarrow \infty} \int \phi \left(\frac{x - at - b}{L} \right) m(\delta, x) \, dx = 0 \tag{3.4.10}$$

when $a, b, \delta > 0$. Since $I_{t_0}(t)$ is decreasing on $[2, t_0]$, we get the following

$$0 \leq \int \phi\left(\frac{x - \sigma t_0}{L}\right) m(t_0, x) dx \leq \int \phi\left(\frac{x - \frac{\sigma}{2}t_0 - \sigma}{L}\right) m(2, x) dx \xrightarrow{t_0 \rightarrow \infty} 0. \quad (3.4.11)$$

Recall that $1 \geq \phi\left(\frac{x - \sigma t_0}{L}\right) \geq 1/2$ for $x \geq \sigma t_0$. Since $m(t, x) \geq 0$, we can apply (3.1.2) to get

$$\frac{1}{2} \|u(t)\|_{H^1(\tilde{\sigma}, \infty)} \leq \frac{1}{2} \|m(t)\|_{L^1(\tilde{\sigma}t, \infty)} \leq I_{t_0}(t) \xrightarrow{t \rightarrow \infty} 0 \quad (3.4.12)$$

In fact, (3.1.2) implies we get similar asymptotic decay for $u \in W_1^p(\tilde{\sigma}, \infty)$. This completes the proof. ■

Bibliography

- [1] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, *Philos. Mag.* **39** (1895), no. 240 422–443.
- [2] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, *Philos. Trans. Roy. Soc. London Ser. A* **272** (1972), no. 1220 47–78.
- [3] J. L. Bona, W. G. Pritchard, and L. R. Scott, *Numerical schemes for a model for nonlinear dispersive waves*, *J. Comput. Phys.* **60** (1985), no. 2 167–186.
- [4] P. J. Olver, *Euler operators and conservation laws of the BBM equation*, *Math. Proc. Cambridge Philos. Soc.* **85** (1979), no. 1 143–160.
- [5] J.-C. Saut and B. Scheurer, *Unique continuation for some evolution equations*, *J. Diff. Eqs.* **66** (1987) 118–139.
- [6] J. L. Bona and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*, *Discrete Contin. Dyn. Syst.* **23** (2009), no. 4 1241–1252.
- [7] A. Parker, *On the Camassa-Holm equation and a direct method of solutions. ii. solitons solutions*, *Proc. R. Soc. A.* **461** (2005) 3611–3632.
- [8] M. Wang, *Sharp well-posedness results for the BBM equation in L^p type sobolev spaces*, *Discrete Contin. Dyn. Syst.* **36** (2016) 5763–5788.
- [9] L. Rosier and B.-Y. Zhang, *Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain*, *J. Diff. Eqs.* **254** (2013) 141–178.
- [10] D. Roumégoux, *A symplectic non-squeezing theorem for BBM equation*, *Dyn. Partial Diff. Eqs.* **7** (2010) 289–305.
- [11] B. Fuchssteiner and A. S. Fokas, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*, *Phys. D* **4** (1981/82) 4–66.

- [12] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, *Phys. Rev. Lett.* **71** (1993) 1661–1664.
- [13] H. Dai, *Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod*, *Acta Mechanica* **127** (3, 1998) 193–207.
- [14] Y. A. Li and P. J. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, *Journal of Differential Equations* **162** (2000), no. 1 27–63.
- [15] G. Rodriguez, *On the Cauchy problem for the Camassa-Holm equation*, *Nonlinear Analysis* **46** (2001), no. 3 309–327.
- [16] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, *Comm. in Mathematical Physics* **211** (4, 2000) 45–61.
- [17] H. McKean, *Breakdown of the Camassa-Holm equation*, *Comm. Pure Appl. Math.* **57** (2004) 416–418.
- [18] F. Linares, G. Ponce, and T. Sideris, *Properties of solutions to the Camassa-Holm equation on the line in a class containing the peakons*, *Asymptotic Analysis for Nonlinear Dispersive and Wave Eqs* **81** (2019) 196–245.
- [19] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, *Annales de l'Institut Fourier* **50** (2000), no. 2 321–362.
- [20] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, *Archive for Rational Mechanics and Analysis* **183** (2, 2007) 215–239.
- [21] Y. Martel and F. Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation*, *Journal Math. Pures et Appl.* **79** (2000), no. 4 339–425.
- [22] A. Constantin and W. A. Strauss, *Stability of peakons*, *Comm. on Pure and Applied Mathematics* **53** (2000), no. 5 603–610.
- [23] L. Molinet, *A Liouville property with application to asymptotic stability for the Camassa-Holm equation*, *Archive for Rational Mechanics and Analysis* **230** (2018) 185–230.
- [24] M. A. Alejo, M. F. Cortez, C. Kwak, and C. Muñoz, *On the dynamics of zero-speed solutions for Camassa-Holm-type equations*, *International Mathematics Research Notices* **2021** (2, 2019) 6543–6585.
- [25] A. Degasperis and M. Procesi, *Asymptotic integrability, Symmetry and perturbation theory* **1** (1999), no. 1 23–37.

- [26] E. G. Charalampidis, R. Parker, P. G. Kevrekidis, and S. Lafortune, *The stability of the b-family of peakon equations*, *Nonlinearity* **36** (1, 2023) 1192.
- [27] G. Gui, Y. Liu, and L. Tian, *Global existence and blow-up phenomena for the peakon b-family of equations*, *Indiana University Mathematics Journal* **57** (2008), no. 3 1209–1234.
- [28] C. Hong and G. Ponce, *On special properties of solutions to the Benjamin-Bona-Mahony equation*, *Journal of Differential Equations* **393** (2024) 321–342.
- [29] F. Linares and G. Ponce, *Unique continuation properties for solutions to the Camassa-Holm equation and related models*, *Proc. Amer. Math. Soc.* **148** (2020) 3871–3879.
- [30] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, *Advances in Mathematics Supplementary Studies, Studies in Applied Math.* **8** (1983) 93–128.
- [31] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations i, ii*, *Geom. Funct. Anal.* **3** (1993) 107–156, 209–262.
- [32] C. E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, *J. Amer. Math. Soc.* **9** (1996) 573–603.
- [33] F. Linares, G. Ponce, and D. Smith, *On the regularity of solutions to a class of nonlinear dispersive equations*, *Math. Ann.* **369** (2017) 797–837.
- [34] H. Chen and C. A. Haidau, *Well-posedness for the coupled BBM systems*, *J. Appl. anal. and Comp.* **8** (2018) 890–914.
- [35] F. P. A. J. Mendez, C. Muñoz, and J. C. Pozo, *On local energy decay for large solutions of the Zakharov-Kuznetsov equation*, *Comm. in Partial Differential Equations* **46** (2021), no. 8 1440–1487.
- [36] C. Kwak and C. Muñoz, *Extended decay properties for generalized BBM equation*, *Nonlinear dispersive partial differential equations and inverse scattering* (2019) 397–411.
- [37] C. Hong, *On local energy decay for solutions to the b-family of peakon equations*, *Nonlinear Analysis* **253** (2025) 113739.
- [38] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, *Comm. Pure Appl. Math.* **41** (1988) 891–907.

- [39] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle*, *Comm. Pure Appl. Math.* **46** (1993) 527–620.
- [40] L. Grafakos and S. Oh, *The Kato-Ponce inequality*, *Comm. Partial Diff. Eqs.* **39** (2013), no. 6 1128–1157.
- [41] J. Bourgain and D. Li, *On an endpoint Kato-Ponce inequality*, *Diff. and Int. Eqs.* **27** (2014) 1037–1072.
- [42] L. Grafakos, D. Maldonado, and V. Naibo, *A remark on an end point Kato-Ponce inequality*, *Diff. and Int. Eqs.* **27** (2014) 415–424.
- [43] S. Oh and X. Wu, *On L^1 endpoint Kato-Ponce inequality*, *Math. Res. Lett.* **27** (2020) 1129–1163.
- [44] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, 1970.