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SIGN-CHANGING CRITICAL POINTS VIA SANDWICH PAIR THEOREMS

LINGHUA CHEN, MARTIN SCHECHTER, AND WENMING ZOU

Abstract. The Sandwich Pair theorems have presented very efficient ways to determine existence of critical points or critical sequences for nonlinear differentiable functionals. In this paper, under rather weak hypotheses new relationships are established between sign-changing critical points and Sandwich Pairs or Linking Sandwich Pairs. The abstract results are demonstrated by applications on semi-linear elliptic equations.

1. Introduction

A successful method for obtaining critical points or critical sequences of C^1 functionals on Banach spaces E is to construct subsets A and B of E such that A links B, in the sense of Rabinowitz [7] (or Schechter [12]), with the value-splitting condition

$$a_0 := \sup_A G \leq b_0 := \inf_B G. \tag{1.1}$$

Assume E is a Banach space. And let Φ be the set of all continuous maps $\Gamma(t, u)$ from $[0, 1] \times E$ to E such that

(1) $\Gamma(0, \cdot) = Id$ on E,
(2) For each $t \in [0, 1)$, $\Gamma(t, \cdot)$ is a homeomorphism of E to itself,
(3) $\Gamma(1, E)$ is a single point in E and $\Gamma(t, A)$ converges uniformly to $\Gamma(1, E)$ as $t \to 1$ for each bounded subset $A \subset E$,

Given two subsets A and B of E, we say A **links** B if

(1) $A \cap B = \emptyset$,
(2) for each $\Gamma \in \Phi$, $\Gamma([0, 1] \times A) \cap B \neq \emptyset$.

If A links B with condition (1.1), it is guaranteed the existence of a Palais-Smale sequence, (PS) sequence in short, namely a sequence $u_k \in E$ such that

$$G(u_k) \to c \in [a_0, b_0], \quad G'(u_k) \to 0.$$

In some applications, however, it would be too demanding for a functional to satisfy (1.1). To meet that inequality, one has to impose strong restrictions. While there is also discussion for functionals only semi-bounded on linking subsets without (1.1), c.f. [14], and unfortunately (PS) sequences do not always exist there.

Nevertheless, there are different kinds of subsets that generate critical sequences, require only semiboundedness of functionals on them, c.f. Theorem 2.3 in [5] (also Theorem 2.13 in Silva [15]). There, a generalization of Saddle Point theorem was presented as:

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Theorem 1.1. *Assume $E = Y \oplus M$ is a real Hilbert space with $Y = M^{\perp}$, and $\dim Y < \infty$; let a functional $G \in C^1(E, \mathbb{R})$ satisfy*

$$a_0 := \sup_Y G \neq \infty, \qquad b_0 := \inf_M G \neq -\infty,$$

and let every sequence $u_k \in E$ such that

$$G(u_k) \to a \in [a_0 \wedge b_0, a_0 \vee b_0], \qquad (1 + \|u_k\|)G'(u_k) \to 0 \tag{1.2}$$

possess a weakly convergent subsequence. Then G has a critical point.

In recent years, the second author discovered new pairs of sets beyond merely complementing subspaces, namely, the so called *Sandwich Pairs* (cf. [11]). He does not require (1.1), but still derives (PS) sequences based on such structures. Unlike linking, the order of sets in a sandwich pair is immaterial, and intersection of them is also allowed. This approach soon enjoys both prosperity in abstract results [8, 10, 11] and efficiency in applications on differential equations [6, 9].

Definition 1.2. Two subsets A, B of a Banach Space E form a **Sandwich Pair**, if for any $G \in C^1(E, \mathbb{R})$, the inequalities

$$-\infty < b_0 := \inf_B G \leq a_0 := \sup_A G < +\infty,$$

always imply existence of a (PS) sequence for G at some level $c \in [b_0, a_0]$.

The fundamental Sandwich Pairs Theorem in [11] states as

Theorem 1.3. *Let N be a finite dimensional subspace of a Banach space E. Assume F a Lipschitz continuous map of E onto N, with Lipschitz constant K, such that $F|_N = Id$ and*

$$\|F(g) - F(h)\| \leq K\|g - h\|, \quad g, h \in E.$$

Let p be any point of N. Then N and $B = F^{-1}(p)$ form a sandwich pair.

On the other hand, a sequence with property (1.2) is called weak (PS) sequence (or Cerami sequence [3]), and obviously has better properties than an ordinary one. Though it might be harder to obtain in abstract theorems, but its convergence turns out to be easier to check in applications. There are indeed applications where ordinary (PS) sequences do not guarantee a convergent subsequence, while a weak one does. Under the setting of Linking or Sandwich Pairs, existence of weak (PS) sequence or its variants has been investigated. For the linking case, in [13], with the only additional assumption A being compact, it was proved the existence of Cerami sequence.

Recently in [10], a new concept, *Linking Sandwich Pair* was introduced by the second author. Such pairs of subsets possess advantages from both linking sets and sandwich pairs. Only requiring semi-boundedness, they generate sequences with property alike (1.2), that is, $u_k \in E$ such that for $\nu_k \in \mathbb{R}$, $\nu_k \to \infty$, and some constant $C \in \mathbb{R}$

$$G(u_k) \to a \in \mathbb{R}, \qquad (\nu_k + \|u_k\|)\|G'(u_k)\| \leq C. \tag{1.3}$$

Definition 1.4. Two subsets A, B of a Banach Space E form a **Linking Sandwich Pair**, if for any $G \in C^1(E, \mathbb{R})$, the inequalities

$$a_0 := \sup_A G \neq \infty, \qquad b_0 := \inf_B G \neq -\infty,$$

always imply existence of sequence satisfying (1.3).

The sequence with property (1.3) looks somewhat different from weak (PS) sequences, but it works at similar efficiency in most applications.

We here quote Theorem 12 in [10] as a practical example of Linking Sandwich Pair:

Theorem 1.5. *Let N be a finite dimensional subspace of a Banach space E, and let p be any point in N. Let F be a continuouse map of E onto N such that $F = I$ on N and*

$$d(N \cap \partial \mathbf{B}_r, F^{-1}(p)) \to \infty \text{ as } r \to \infty.$$

If $R > \|p\|$, then $A^R := N - \mathbf{B}_R$ and $F^{-1}(p)$ form a linking sandwich pair.

In the meanwhile, there has been increasing concerns to more information than mere existence of the critical points or critical sequences. Numerous works have been devoted to the sign-changing critical point theory. For instance, [16, 17], *etc.* include plentiful results based on linking subsets.

However, the investigation of sign-changing critical points from Sandwich Pairs has just initiated. To the best of our knowledge, only one special case has been considered so far, i.e., complemental subspaces with one of them finite dimensional, cf. [18], or Theorem 3.4 in [17] by the third author.

In this paper, under rather weak conditions, we confirm the existence of sign-changing critical sequences generating from Sandwich Pairs or Linking Sandwich Pairs; our abstract theorems are very handy and able to cope with a large class of semi-linear elliptic differential equations. Additionally, the novelty is also reflected within the proofs for the abstract results. Unlike the special case in [18] or [17], we do not assume any compactness conditions on the functional, e.g. (PS) or weak (PS) condition. Instead, we directly derive sign-changing critical sequences in the abstract theorems, and this approach leaves more flexibility for the applications to verify their convergence. A byproduct of our results is that a sign-changing critical point is automatically nontrivial, and thus frees us from imposing extra restrictions to rule out the zero solution in applications.

This paper is organized as follows. In Section 2 and 3, we present abstract theorems generating sign-changing critical sequence based on Sandwich Pair structure; and Sections 4 and 5 are devoted to applications to semi-linear elliptic partial differential equations.

2. Sign-Changing Critical Sequence via Ordinary Sandwich Pair

Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert space, with a closed convex cone $\mathcal{P} \subset E$. In the sequel a functional G is always assumed to be of $C^1(E,\mathbb{R})$. And for some real number $\mu_0 > 0$, we define

$$\begin{aligned} \pm\mathcal{D}(\mu_0) :=& \{u \in E : dist(u, \pm\mathcal{P}) < \mu_0\}, \\ \mathcal{W} :=& -\mathcal{D}(\mu_0) \cup \mathcal{D}(\mu_0), \\ \mathcal{S} :=& E - \mathcal{W}, \\ \tilde{E} :=& \{u \in E : G'(u) \neq 0\}. \end{aligned}$$

To obtain sign-changing critical sequence, we need some assumptions for G:

A1: G' is of the form $G'(u) = u - J(u)$, and $J(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu)$ for some $\mu \in (0, \mu_0)$.

A2: $G(u)$ maps bounded sets in E to bounded sets.

For some $a^\star \in \mathbb{R}$ and certain nonempty subset $B \subset E$, we define

$$B^\star := B \cap G^{a^\star},$$

where $G^{a^\star} := \{u \in E : G(u) \le a^\star\}$. Note that as long as $a^\star > \inf_B G$, there holds $B^\star \ne \emptyset$, and $\inf_{B^\star} G = \inf_B G$. Then we also assume

A3: for that $B^\star$ above, $B^\star \ne \emptyset$ always implies $dist(B^\star, \mathcal{P}) > 0$.

Under **A3**, we shall choose $\mu_0 \in (0, dist(B^\star, \mathcal{P}))$ in **A1**.

We begin with Sandwich Pairs generating sign-changing ordinary (PS) sequences. Although the conclusion in this section is not as strong as later ones, its proof remains succinct and heuristic.

Firstly we quote Lemma 2.11 in [17], as an indispensable result for finding sign-changing critical point, which reads

Lemma 2.1. *Assume **A1**. Then there exists a locally Lipschitz continuous map $L : \tilde{E} \to E$, such that $L(\pm\mathcal{D}(\mu_0) \cap \tilde{E}) \subset \pm\mathcal{D}(\mu)$ for some $\mu \in (0, \mu_0)$. And $V(u) = u - L(u)$ is a pseudo-gradient vector field of G.*

Now we are to state the main result of this section.

Theorem 2.2. *Let N be a finite dimensional subspace of E with $\dim N > 1$; assume F is a Lipschitz continuous map of E onto N such that $F|_N = Id$ and*

$$\|F(g) - F(h)\| \le K\|g - h\|, \quad g, h \in E.$$

*Let p be any point in N, such that $F^{-1}(p)$ and G satisfy **A1** to **A3** with $B = F^{-1}(p)$, $a^\star = 1 + \sup_N G$ in **A3**. Moreover, suppose that*

$$-\infty < b_0 := \inf_{B^\star} G \le a_0 := \sup_N G < +\infty.$$

Then there is a sequence $u_k \in \mathcal{S}$ such that

$$G(u_k) \to c \in [b_0, a_0], \qquad G'(u_k) \to 0.$$

Proof. By negation, suppose the theorem were not true, then

$$\|G'(u)\| \ge 3\delta$$

would hold for some $\delta \in (0, 1/3)$ small and all elements in

$$Q := \{u \in \mathcal{S} : b_0 - 3\delta \le G(u) \le a_0 + 3\delta\}.$$

Define

$$\begin{aligned} Q_0 :=& \{u \in Q : b_0 - 2\delta \le G(u) \le a_0 + 2\delta\}, \\ Q_1 :=& \{u \in Q : b_0 - \delta \le G(u) \le a_0 + \delta\}, \\ Q_2 :=& E - Q_0, \\ \eta(u) :=& dist(u, Q_2) / \left(dist(u, Q_1) + dist(u, Q_2)\right). \end{aligned}$$

Via standard procedure one can check that $\eta(u)$ is locally Lipschitz continuous on E. Let $V(u)$ be a pseudo-gradient vector field of G with the form $V(u) = u - L(u)$, where $L(u)$ is locally Lipschitz continuous, and satisfies that

$$L\left(\tilde{E} \cap \pm\mathcal{D}(\mu_0)\right) \subset \pm\mathcal{D}(\mu),$$

for some $\mu_0 > 0$ small and $\mu \in (0, \mu_0)$. According to Lemma 2.1, such $V(u)$ can be readily constructed.

Set

$$Y(u) := V(u)/\|V(u)\|, \qquad u \in \tilde{E}.$$

Clearly, $\|Y(u)\| \equiv 1$. And by properties of the pseudo-gradient vector field $Y(u)$, we could reqire that

$$2\delta \leq \langle G'(u), Y(u)\rangle, \qquad u \in Q.$$

Consider the Cauchy initial value problem

$$\frac{\partial}{\partial t}\sigma(t,u) = -\eta(\sigma(t,u))Y(\sigma(t,u)), \qquad \sigma(0,u) = u \in E.$$

By ODE theory in Banach space, the above problem has a unique solution in $[0,+\infty) \times E$. According to results in [16] or [17], there holds the relationship

$$\sigma(t, \pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu_0), \qquad \forall t \geq 0. \tag{2.1}$$

We here omit the details since it would be repeating *Step 2* in the proof of Theorem 3.4 from [17].

In Q, we estimate that

$$\frac{\partial}{\partial t}G \circ \sigma(t,u) = -\eta(\sigma)\langle G'(\sigma), Y(\sigma)\rangle \leq -2\delta\eta(\sigma(t,u)) \leq 0.$$

Now we are able to claim that there exists $T > 0$ so large that

$$\sigma(T, G^{a_0+\delta}) \subset G^{b_0-\delta} \cup \mathcal{W}. \tag{2.2}$$

We take $T > (a_0 - b_0 + 2\delta)/2\delta$. For $u \in G^{a_0+\delta}$, if there exists $t_1 \in [0,T]$ such that $\sigma(t_1,u) \notin Q_1$, then either $G(\sigma(t_1,u)) < b_0 - \delta$, or $\sigma(t_1,u) \notin \mathcal{S}$. In other words, by (2.1), (2.2) holds. If for all $t \in [0,T]$, there holds $\sigma(t,u) \in Q_1$, then $\eta(\sigma(t,u)) \equiv 1$, and hence

$$\frac{\partial}{\partial t}G \circ \sigma(t,u) \leq -2\delta.$$

Then by our choice of T, we have

$$G \circ \sigma(T,u) \leq G(u) - 2\delta T \leq a_0 + \delta - 2\delta T < b_0 - \delta.$$

In sum, claim (2.2) has been verified. Define $\Omega := \{u \in N : \|u - p\| < KT + \delta\}$. For $v \in \partial\Omega$, because of

$$KT + \delta = \|v - p\| \leq \|v - F \circ \sigma(t,v)\| + \|F \circ \sigma(t,v) - p\|,$$
$$\|v - F \circ \sigma(t,v)\| \leq K\|v - \sigma(t,v)\| \leq K\|\int_0^T \sigma(t,v)dt\| \leq KT,$$

there holds

$$\|F \circ \sigma(t,v) - p\| \geq \|v - p\| - KT = \delta > 0.$$

Define $H(t,\cdot) := F \circ \sigma(t,\cdot)$, a continuous map from $\bar{\Omega}$ to N, with $t \in [0,T]$. From the above arguments, $H(t,v) \neq p$, for all $v \in \partial\Omega$. Hence the Brouwer degree $d(H(t,\cdot),\Omega,p)$ is well-defined. Consequently by homotopic invariance

$$deg(H(T,\cdot),\Omega,p) = deg(H(0,\cdot),\Omega,p) = deg(Id,\Omega,p) = 1.$$

This implicates there exists $v \in \bar{\Omega}$ such that $F \circ \sigma(T,v) = p$. Since $a^\star := a_0 + 1$, $\delta \in (0, 1/3)$ and

$$\sup_{\Omega} G \leq \sup_{N} G =: a_0 < a_0 + \delta,$$

there holds $\sigma(T,v) \in B^\star := F^{-1}(p) \cap G^{a^\star} \subset \mathcal{S}$. Then $\sigma(T,v) \in \mathcal{S}$ and $b_0 := \inf_{B^\star} G$ contradict against (2.2). Hence our assumption in the beginning is not true. In an other word, a critical sequence must exist within $\mathcal{S}$. □

3. Sign-Changing Critical Sequence via Linking Sandwich Pair

Next we consider linking sandwich pairs. With notions same to the previous section, our arguments consist of the following theorems.

Let A and B be subsets of E such that A is bounded in $\mathbf{B}_0(R) := \{u \in E : \|u\| \leq R\}$, A links B; additionally, $d := dist(A, B) > 0$. Define

$$\Gamma^\star(t,u) := (1-t)u, \qquad (t,u) \in [0,1] \times E,$$
$$a^\star := 1 + \sup_{\Gamma^\star([0,1]\times A)} G,$$
$$\Phi^\star := \left\{\Gamma \in \Phi : \Gamma([0,1] \times A) \subset G^{a^\star}\right\}.$$

Clearly, by definition of $a^\star$, $\Gamma^\star \in \Phi^\star$, so $\Phi^\star \neq \emptyset$

Next, suppose there is $\psi(t)$ a real function defined on $[0, \infty)$, Lipschitz continuous, non-increasing and positive, such that

$$\int_1^\infty \psi(t)dt = \infty.$$

Now we quote Theorem 2.8 in [16],

Theorem 3.1. *Under above hypotheses, we denote*

$$a := \inf_{\Gamma \in \Phi^\star} \sup_{\Gamma([0,1]\times A)\cap \mathcal{S}} G.$$

If G satisfies (1.1)*, and* ***A1*** *to* ***A3****, then there is a sequence $u_k \in \mathcal{S}$ such that*

$$G(u_k) \to a, \qquad G'(u_k)/\psi(\|u_k\|) \to 0.$$

Then we proceed to

Theorem 3.2. *Assume G satisfies* ***A1*** *to* ***A3****, with*

$$-\infty < b_0 := \inf_{B^\star} G \leq a_0 := \sup_{A\cap\mathcal{S}} G < \infty,$$

where $B^\star := B \cap G^{a^\star}$. Moreover, suppose there exists constant $\delta_0 \in (0, 1/3)$ small such that

$$a_0 - b_0 + 2\delta_0 < \frac{1}{4}\int_{R+\nu}^{R+\nu+T} \psi(t)dt, \tag{3.1}$$

for some $T \in (0, d)$ and some $\nu > R$. Then there is a sequence $u_k \in \mathcal{S}$ such that

$$G(u_k) \to c \in [b_0, a^\star], \qquad \|G'(u_k)\| \leq \psi(dist(u_k, \tilde{B})),$$

where $\tilde{B} := B \cap \mathbf{B}_0(\nu)$.

Proof. Firstly, because $(\Gamma([0,1] \times A) \cap \mathcal{S}) \cap B^\star \neq \emptyset$, we have $a \geq b_0$. Moreover, according to definition of $\Phi^\star$, $a \leq a^\star$. Therefore the a defined above is actually finite, and more precisely, bounded in the interval $[b_0, a^\star]$. Then our arguments are two-fold:

Case I. $a_0 < a$.

Define for each $\Gamma \in \Phi^\star$,

$$S_\Gamma := \{v = \Gamma(s,u) : s \in (0,1], u \in A; v \in \mathcal{S} - A; G(v) \geq a_0\}$$
$$B' := \bigcup_{\Phi^\star} \mathcal{S}_\Gamma \subset \mathcal{S}.$$

Since $a_0 < a$, we have for each $\Gamma \in \Phi^\star$, there are $s \in [0, 1]$ and $u \in A$ such that $G \circ \Gamma(s,u) > a_0 := \sup_{A\cap\mathcal{S}} G$, and $\Gamma(s,u) \in \mathcal{S}$. Hence, $S_\Gamma \neq \emptyset$, and $B' \neq \emptyset$ follows.

Clearly, this $\Gamma(s,u)$ lies out of A. Then, by definition, $A \cap B' = \emptyset$, and for each $\Gamma \in \Phi^\star$, there are $s \in [0,1]$ and $u \in A$ such that $\Gamma(s,u) \in S_\Gamma \neq \emptyset$. We conclude that A links B'. Obviously,

$$\sup_{A\cap\mathcal{S}} G =: a_0 \le b_0' := \inf_{B'\cap G^{a^\star}} G.$$

Then we replace B by B' and employ Theorem 3.1, there exists a sequence $u_k \in \mathcal{S}$ such that

$$G(u_k) \to a \in [b_0, a^\star], \qquad G'(u_k)/\psi(\|u_k\|) \to 0,$$

which implies $\|G'(u_k)\| \le \psi(dist(u_k, \tilde{B}))$. for sufficiently large k.

Case II. $a_0 = a$.

We assert that there exists sequence $u_k \in \mathcal{S}$, such that $G(u_k) \to c \in [b_0, a_0]$, and $\|G'(u_k)\| \le \psi(dist(u, \tilde{B})$.

By negation, if it were not true, there would be a $\delta \in (0, \delta_0)$ small such that

$$\|G'(u)\| > \psi(dist(u, \tilde{B}))$$

holds for all elements in the set

$$Q := \{u \in \mathcal{S} : b_0 - 3\delta \le G(u) \le a_0 + 3\delta\}$$

Define

$$\begin{aligned} Q_0 :=& \{u \in Q : b_0 - 2\delta \le G(u) \le a_0 + 2\delta\}, \\ Q_1 :=& \{u \in Q : b_0 - \delta \le G(u) \le a_0 + \delta\}, \\ Q_2 :=& E - Q_0, \\ \eta(u) :=& dist(u, Q_2)/\left(dist(u, Q_1) + dist(u, Q_2)\right). \end{aligned}$$

Similar to the proof of Theorem 2.2, let $V(u)$ be a pseudo-gradient vector field of G with the form $V(u) = u - L(u)$. $L(u)$ is locally Lipschitz continuous, and

$$L(\tilde{E} \cap \pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu),$$

for some $\mu_0 > 0$ small and $\mu \in (0, \mu_0)$. Set

$$Y(u) := V(u)/\|V(u)\|, \qquad u \in \tilde{E}.$$

Clearly, $\|Y(u)\| \equiv 1$. And by properties of pseudo-gradient vector field $Y(u)$, we could require that, there holds in $\tilde{E}$

$$\frac{1}{4}\|G'(u)\| \le \langle G(u), Y(u)\rangle.$$

Then the following Cauchy initial value problem

$$\frac{\partial}{\partial t}\sigma(t,u) = -\eta(\sigma(t,u))Y(\sigma(t,u)), \qquad \sigma(0,u) = u \in E.$$

has a unique solution in $[0, +\infty) \times E$. According to the same argument in the previous section, we have that

$$\sigma(t, \pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu_0), \qquad \forall t \ge 0. \tag{3.2}$$

Since $\|\sigma(t,u) - u\| \le t$ and, by assumption $d := dist(A, B) > 0$, then for each $T \in (0, d)$ there holds

$$\sigma([0,T], A) \cap B = \emptyset. \tag{3.3}$$

Additionally, note that

$$\begin{aligned}\frac{\partial}{\partial t}G\circ\sigma(t,u) =& -\eta(\sigma)\langle G'(\sigma),Y(\sigma)\rangle\\ \le& -\frac{1}{4}\eta(\sigma)\|G'(\sigma)\|\\ \le& -\frac{1}{4}\eta(\sigma)\psi(dist(\sigma,\tilde{B})).\end{aligned} \tag{3.4}$$

By definition of a, we may choose $\Gamma_0\in\Phi^\star$ such that

$$\sup_{\Gamma_0([0,1]\times A)\cap\mathcal{S}} G(u) < a+\delta.$$

Now only consider those $v\in\Gamma_0([0,1]\times A)\cap\mathcal{S}$ such that $\sigma(T,v)\in\mathcal{S}$. If there exists $t\in[0,T]$ such that $\sigma(t,v)\notin Q_1$, we must have either $G\circ\sigma(T,v)\le b_0-\delta$, or $\sigma(T,v)\notin\mathcal{S}$. If for each $t\in[0,T]$, there is always $\sigma(t,v)\in Q_1$, we have

$$\begin{aligned}G\circ\sigma(T,v)=&G(v)+\int_0^T dG\circ\sigma(t,v)\\ \le& a_0+\delta-\frac{1}{4}\int_0^T\psi(dist(\sigma(t,v),\tilde{B})dt\\ \le& a_0+\delta-(a_0-b_0+2\delta_0))\\ <& b_0-\delta,\end{aligned}$$

from assumption (3.1) on $\psi(t)$, along with the fact

$$dist(\sigma(t,v),\tilde{B})\le\|\sigma(t,u)\|+\nu\le R+t+\nu.$$

In sum, we conclude that

$$G(\sigma(T,\Gamma_0([0,1]\times A)\cap\mathcal{S})\cap\mathcal{S}) < b_0-\delta.$$

Combine with property (3.2) of the flow $\sigma(t,v)$, we have

$$\sigma(T,\Gamma_0([0,1]\times A))\subset G^{b_0-\delta}\cup\mathcal{W}.$$

By definitions of $a^\star$, b_0, $B^\star$ and the fact $B\cap G^{a^\star}\subset\mathcal{S}$, there must hold

$$\sigma(T,\Gamma_0([0,1]\times A))\cap B^\star=\emptyset. \tag{3.5}$$

Then consider the map

$$\Gamma_1(t,u):=\begin{cases}\sigma(2Tt,u), & t\in[0,1/2),\\ \sigma(T,\Gamma_0(2t-1,u)), & t\in[1/2,1].\end{cases}$$

Clearly $\Gamma_1\in\Phi^\star$, while (3.3) and (3.5) imply $\Gamma_1([0,1]\times A)\cap B^\star=\emptyset$. On the other hand, $\Gamma_1([0,1]\times A)\subset G^{a+\delta}\subset G^{a^\star}$ and $B^\star:=B\cap G^{a^\star}$ implicate $\Gamma_1([0,1]\times A)\cap B=\emptyset$, which contradicts the fact that A links B. Hence, there exists a sequence $u_k\in\mathcal{S}$ such that

$$G(u_k)\to c\in[b_0,a_0],\qquad \|G'(u_k)\|\le\psi(dist(u_k,\tilde{B})).$$

The existence of the desired sequence is confirmed. □

Theorem 3.3. *Let G satisfy **A1** to **A3** and let A_k, B be such subsets of E that $d_k:=dist(A_k,B)\to\infty$, $A_k\subset\mathbf{B}_0(R_k)$, A_k links B, and assume that for each $k\in\mathbb{N}$*

$$\sup_{\Gamma^\star([0,1]\times A_k)} G\le C^\star<\infty.$$

We define

$$a^\star:=C^\star+1,\qquad B^\star:=B\cap G^{a^\star},$$

and suppose that there exist real numbers $b_0 \le a_0 \le a^\star$, such that

$$\sup_{A_k \cap \mathcal{S}} G(u) \le a_0 < \infty, \qquad \inf_{B} G(u) =: b_0 > -\infty;$$

Moreover, if there are real functions $\psi_k(t)$, Lipschitz continuous on $[0,\infty)$, positive and non-increasing, with $\int_1^\infty \psi_k(t)dt = \infty$. And real numbers $\delta_0 \in (0, 1/3)$, $\nu_k > R_k$ and $T_k \in (0, d_k)$ such that for each $k \in \mathbb{N}$

$$a_0 - b_0 + 2\delta_0 < \frac{1}{4} \int_{R_k+\nu_k}^{R_k+\nu_k+T_k} \psi_k(t)dt. \tag{3.6}$$

Then there exists a sequence $u_k \in \mathcal{S}$, such that

$$G(u_k) \to c \in [b_0, a^\star], \qquad \|G'(u_k)\| \le \psi_k(dist(u_k, \tilde{B}_k),$$

where $\tilde{B}_k := B \cap \mathbf{B}_0(\nu_k)$.

Remark 3.4. Condition (3.6) will be guaranteed if there holds $d_k \sim O(\nu_k)$, and hence we are able to choose $T_k \sim O(\nu_k)$. This is indeed true in the succeeding theorems.

Proof. of Theorem 3.3. Firstly we denote

$$a_k := \inf_{\Gamma \in \Phi^\star} \sup_{\Gamma([0,1]\times A_k)\cap \mathcal{S}} G(u).$$

Clearly $a_k \in [b_0, a^\star]$, bounded in a finite interval uniformly with respect to k.

It follows from Theorem 3.2 that for each $k \in \mathbb{N}$ there is a sequence $u_l^k \in \mathcal{S}$ such that

$$G(u_l^k) \to c_k \in [b_0, a^\star], \qquad \|G'(u_l^k)\| \le \psi_k(dist(u_l^k, \tilde{B}_k), \qquad \text{as } l \to \infty.$$

Note that it is possible that $\sup_{A_k \cap \mathcal{S}} G$ could tends to $-\infty$ as $k \to \infty$. In such cases we will have $\sup_{A_k\cap\mathcal{S}} G \le \inf_{B^\star} G$ for k large enough, which turns out Theorem 3.1 is applicable, still assures $c_k \in [b_0, a^\star]$.

As the values $\{c_k\}$ are confined within a compact interval independent of k, we can consider the diagonal sequence of $\left\{u_l^k \in \mathcal{S} : (k,l) \in \mathbb{N}\times\mathbb{N}\right\}$, still denoted as u_k. Then up to a subsequence if necessary, there holds

$$G(u_k) \to c \in [b_0, a^\star], \qquad \|G'(u_k)\| \le \psi_k(dist(u_k, \tilde{B}_k),$$

as desired. □

Theorem 3.5. *Under same hypotheses of Theorem 3.3, there exist $\gamma > 0$, $\nu_k \to \infty$ and $u_k \in \mathcal{S}$ such that*

$$G(u_k) \to c \in [b_0, a^\star], \qquad (\nu_k + \|u_k\|)\|G'(u_k)\| \le \gamma.$$

Proof. Applying Theorem 3.3, we have already a sequence $u_k \in \mathcal{S}$ such that

$$G(u_k) \to c \in [b_0, a^\star], \qquad \|G'(u_k)\| \le \psi_k(dist(u_k, \tilde{B}_k).$$

Then take $\psi_k(t) = \gamma/(2\nu_k + t)$, with $\gamma > 0$ so large that (3.6) holds. Then we deduce that for each $k \in \mathbb{N}$

$$G(u_k) \to c \in [b_0, a^\star], \qquad (2\nu_k + (dist(u_k, \tilde{B}_k))\|G'(u_k)\| \le \gamma, \qquad \text{as } k \to \infty.$$

Note that for each $w \in \tilde{B}_k := B \cap \mathbf{B}_0(\nu_k)$, there holds

$$\|u_k\| \le \|w\| + \|u_k - w\| \le \nu_k + \|u_k - w\|.$$

Hence

$$\|u_k\| \le \nu_k + dist(u_k, \tilde{B}_k),$$

It yields that $G(u_k) \to c \in [b_0, a^\star]$, and $(\nu_k + \|u_k\|)\|G'(u_k)\| \le \gamma$. □

With above theorems in hand, we are now able to illustrate practical examples of Linking Sandwich Pairs which generate Cerami-like sequence within $\mathcal{S}$.

Let N be a closed subspace of a Hilbert space E, with complement $M := N^{\perp}$, one of N and M is finite dimensional. $\dim N > 1$ and $\mathcal{P} \subset E$ is a closed convex cone with its vertex on origin point. Define $\pm\mathcal{D}(\mu)$, $\mathcal{W}$ and $\mathcal{S}$ as before. Set $A^R := N - \mathbf{B}_0(R)$.

Theorem 3.6. *Under above hypotheses, suppose $F(u) \in C(E, N)$ is a Lipschitz continuous map with $F|_N = Id$. Let $p \in N$ be a fixed point, such that*

$$-\infty < b_0 := \inf_{F^{-1}(p)\cap G^{a^\star}} G \le \sup_{A^R} G \le \sup_N G =: a_0 < +\infty.$$

*Moreover assume G satisfies **A1**, **A2** and **A3** with $B := F^{-1}(p)$, $a^\star := 1 + a_0$. Then A^R and $F^{-1}(p)$ form a linking Sandwich Pair, and generate sign-changing critical sequence.*

Proof. Take $A_{R_k} := N \cap \partial\mathbf{B}_0(R_k)$ with $R_k \nearrow \infty$, and $\nu_k := 3R_k$. Under assumptions of the theorem and F is Lipschitz continuous, it is straightforward to check that for some $\alpha \in (0, 1)$,

$$d_k := dist(A_{R_k}, F^{-1}(p)) \ge \alpha R_k. \tag{3.7}$$

Indeed, suppose the contrary that for any $\alpha > 0$, there exit $x \in F^{-1}(p)$ and $k \in \mathbb{N}$, such that $dist(A_{R_k}, x) < \alpha R_k$. Then for each $n \in \mathbb{N}$, we can set $\alpha_n = 1/n$ and find $x_n \in F^{-1}(p)$ and $k_n \in \mathbb{N}$, such that R_{k_n} is greater than $\|p\|$, and $dist(A_{R_{k_n}}, x_n) < 2R_{k_n}/n$. Hence there is $y_n \in A_{R_{k_n}} \subset N$, with $\|y_n - x_n\| < 3R_{k_n}/n$. Hence,

$$\|y_n - p\| = \|F(y_n) - F(x_n)\| \le K\|y_n - x_n\| < 3KR_{k_n}/n,$$

where K stands for the Lipschitz constant of F. Obviously the left hand side grows to infinity at a rate of $O(R_{k_n})$ as $n \to \infty$, while right hand side at most $O(R_{k_n}/n)$, which is contradictory.

Now apply Theorem 3.5 for $A_k := A_{R_k}$, $B := F^{-1}(p)$ and real functions $\psi_k(t)$ satisfying (3.6). Since $\sup_N G =: a_0 < +\infty$ and $\Gamma^\star([0, 1] \times A_k) \subset N$, there holds $\sup_{\Gamma^\star([0,1]\times A_k)\cap\mathcal{S}} G \le a_0 < +\infty$. The required sequence in $\mathcal{S}$ is then obtained. □

Remark 3.7. The map F can actually be replaced by any continuous one with property (3.7), provided $F|_N = Id$, and $F^{-1}(p) \cap G^{a^\star} \subset \mathcal{S}$ for some $p \in N$ and $a^\star \in \mathbb{R}$.

4. Application I - Sign-changing Solution via Sandwich Pair

After developing the previous theorems, we are now in a position to examine the utility and applications of them to questions concerning the existance of sign-chaging solutions of elliptic partial differential equations. This material forms the core of this and the following section. Although the examples below remain relatively simple and illustrative, we believe that those abstract results are capable to deal with even more complicated equations.

Let Ω be a subset of $\mathbb{R}^N$, $N \ge 3$. Consider the Dirichlet Problem:

$$-\Delta u = f(x, u), \quad x \in \Omega; \qquad u|_{\partial\Omega} = 0. \tag{4.1}$$

Let E be the Sobolev space $H_0^1(\Omega)$ with inner product

$$\langle u, v\rangle := \int_\Omega \nabla u \nabla v dx,$$

for $u, v \in E$ and norm $\|u\| := \langle u, u\rangle^{1/2}$. The corresponding energy functional to (4.1) is

$$G(u) = \|u\|^2 - 2\int_\Omega F(x,u)dx,$$

where $F(x,u) := \int_0^u f(x,s)ds$ stands for the primitive of $f(x,t)$. We propose following preassumptions for problem (4.1).

B1: Ω is a bounded subset, with boundary smooth enough for Sobolev Embedding Theorem to hold.

The role of **B1** is to ensure the compact embedding $E \hookrightarrow\hookrightarrow L^s(\Omega)$, for $s \in [2, \frac{2N}{N-2})$. Furthermore we propose the following growth restriction and geometric condition to construct the Sandwich Pair structure and guarantee the sign-changingness of critical sequences.

B2: $f(x,t) : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and there exist positive constants $a_1 > 0$, $a_2 > 0$, and functions $W_1(x)$, $W_2(x) \in L^1(\Omega)$, such that

$$a_1 t^2 - W_1(x) \le 2F(x,u) \le a_2 t^2 + W_2(x),$$

for every $(x,t) \in \Omega \times \mathbb{R}$;

B3: $f(x,t)t \ge 0$ a.e. $(x,t) \in \Omega \times \mathbb{R}$, $f(x,t) = o(|t|)$ uniformly for $x \in \Omega$ as $|t| \to 0$.

For the boundedness of critical sequence, we assume

B4:

$$\begin{aligned} &2F(x,t) - tf(x,t) \le 0, && \forall (x,t) \in \Omega\times\mathbb{R},\\ &\limsup_{|t|\to\infty}(2F(x,t) - tf(x,t))/|t| < 0, && \text{a.e. } x \in \Omega. \end{aligned}$$

Remark 4.1. There do exist functions satisfying **B2** to **B4**, for instance

$$f(x,t) = \begin{cases} 4t-1 & t > 1\\ 3|t|t & t \in [-1,1]\\ 4t+1 & t < -1. \end{cases}$$

We thence derive that

$$2F(x,t) = \begin{cases} 2|t|^3 & |t| \in [0,1]\\ 4t^2 - 2|t| & |t| > 1. \end{cases}$$

Therefore,

$$2t^2 - 2 \le 2F(x,t) \le 4t^2, \text{ for all } (x,t) \in \Omega\times\mathbb{R},$$

and

$$2F(x,t) - tf(x,t) = \begin{cases} -|t| & |t| > 1\\ -|t|^3 & |t| \in [0,1], \end{cases}$$

which meets all requirements in **B2** to **B4**.

From classic elliptic theory, (c.f. e.g. [4]), for the corresponding eigenvalue problem, there is

Lemma 4.2. *Under **B1**, the eigenvalue problem*

$$-\Delta u = \lambda u, \quad x \in \Omega; \quad u|_{\partial\Omega} = 0 \tag{4.2}$$

possesses a sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$$

with finite multiplicity for each λ_k. The pricipal eigenvalue λ_1 is simple with a positive eigenfunction ϕ_1, and the eigenfunctions corresponding to λ_k ($k \ge 2$) are sign-changing.

Now we state the main result of this section.

Theorem 4.3. *Under hypotheses* ***B1*** *to* ***B4****, and assuming* $l > 1$*,* $\lambda_l \le a_1$*,* $a_2 < \lambda_{l+1}$ *in* ***B2****, then problem* (4.1) *possesses at least one sign-changing solution.*

Denote N as the eigenspace corresponding to $\lambda_1, \cdots, \lambda_l$, and $M = N^{\perp}$ in E. Let

$$\mathcal{P} := \{u \in E : u \ge 0, \text{ a.e. } x \in \Omega\}.$$

Then $\mathcal{P}$ $(-\mathcal{P})$ is the positive (negative) cone in E. We are to apply to Theorem 2.2. Take $v_0 \in N \cap \mathcal{S}$, eigenfunction corresponding to λ_l such that $\|v_0\|_2 = 1$ and for $R > 0$ we define

$$A^R := \{v \in N : \|v\| \ge R\}.$$

Set for some $\rho > 0$ and $u = v + sv_0 + w \in E = (N \ominus \{v_0\}) \oplus \{v_0\} \oplus M$

$$F_0(v + sv_0 + w) = \begin{cases} v + (s + \|w\|_2)v_0 & \text{if } \|w\|_2 \le \rho; \\ v + (s + \rho)v_0 & \text{if } \|w\|_2 > \rho. \end{cases}$$

Obviously $F_0 \in C(E, N)$ is Lipschitz continuous and $F_0|_N = Id$

$$B_0 := F_0^{-1}(\rho v_0) = \{u \in M : \|u\|_2 \ge \rho\} \cup \{(1-t)w + t\rho v_0 : t \in [0,1], w \in M, \|w\|_2 = \rho\}.$$

Then each element of B_0 is sign-changing, because $M \oplus \{v_0\} \cap (-\mathcal{P} \cup \mathcal{P}) = \{0\}$.

Moreover, we assume $R > 0$ large enough such that $\|v\|_2 > 2\rho$ for all $v \in A^R$. This is possible since N is of finite dimension and thus all norms are equivalent.

Lemma 4.4. *Under hypotheses* ***B2*** *and* ***B3****, there holds* $d := dist(B^\star, -\mathcal{P} \cup \mathcal{P}) > 0$*, for* $a^\star > \inf_{B_0} G$*.*

Proof. First we shall verify $B^\star := B_0 \cap G^{a^\star}$ is bounded in $(E, \|\cdot\|)$.

For $\{(1-t)w + t\rho v_0 : t \in [0,1], w \in M, \|w\|_2 = \rho\}$, clearly by **B2** we can deduce

$$\begin{aligned} a^\star &\ge G((1-t)w + t\rho v_0) \\ &= \|(1-t)w + t\rho v_0\|^2 - 2\int_\Omega F(x, (1-t)w + t\rho v_0)dx \\ &\ge \|(1-t)w + t\rho v_0\|^2 - \int_\Omega (a_2|(1-t)w + t\rho v_0|^2 + W_2)dx \\ &\ge (1 - a_2/\lambda_{l+1})\|(1-t)w\|^2 + t^2(\|\rho v_0\|^2 - a_2\|\rho v_0\|_2^2) - \|W_2\|_1 \\ &\ge \left(1 - \frac{a_2}{\lambda_{l+1}}\right)\|(1-t)w + t\rho v_0\|^2 - a_2\left(\frac{1}{\lambda_l} - \frac{1}{\lambda_{l+1}}\right)\|t\rho v_0\|^2 - \|W_2\|_1. \end{aligned}$$

Hence the boundedness of $G^{a^\star} \cap \{(1-t)w + t\rho v_0 : t \in [0,1], w \in M, \|w\|_2 = \rho\}$ follows from $a_2 < \lambda_{l+1}$.

As for the other part of B_0, the arguments are analogous.

Then by negation, if we have $dist(B^\star, \mathcal{P}) = 0$, there exist sequences $u_k \in B^\star$ and $p_k \in \mathcal{P}$, such that $\|u_k - p_k\| \to 0$. By above arguments, u_k is bounded, and hence is p_k. We assume that $u_k \rightharpoonup u^*$ weakly in E, and $p_k \rightharpoonup p^*$ in $\mathcal{P}$. Then $u_k \to u^*$ strongly in $(E, \|\cdot\|_2)$.

By definitions of $B^\star$ and B_0, we have $u^* \in M \oplus \{v_0\}$; since B_0 is bounded away from 0 in $(E, \|\cdot\|_2)$ and $\|u_k - u^*\|_2 \to 0$, we have $u^* \ne 0$. However, because $u^* = p^*$, we get a contradiction since each element of B_0 is sign-changing. □

Remark 4.5. If there holds directly $F^{-1}(p) \subset \mathcal{S}$, we can assume $a_2 \le \lambda_{l+1}$ in **B2**.

For some $\mu_0 \in (0, d)$, denote

$$\begin{aligned} \pm\mathcal{D}(\mu_0) &:= \{u \in E : dist(u, \pm\mathcal{P}) < \mu_0\}, \\ \mathcal{W} &:= -\mathcal{D}(\mu_0) \cup \mathcal{D}(\mu_0), \\ \mathcal{S} &:= E - \mathcal{W}. \end{aligned}$$

Now we quote Lemma 2.29 in [17], of which the original idea comes from [2].

Lemma 4.6. *Assume $J' := Id - G'$, there exists a $\mu_0 \in (0, d)$ such that $J'(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu)$ for some $\mu \in (0, \mu_0)$, provided **B2** and **B3** hold.*

The above Lemma verifies assumption **A1**, and by Lemma 4.4, we have $B^\star \subset \mathcal{S}$. Moreover, via Sobolev embedding theorems, it is obvious that G satisfies assumption **A2**. Note that, by definitions, we derive

$$\begin{aligned} \|v\|^2 &\le \lambda_l \|v\|_2^2, && v \in N, \\ \|w\|^2 &\ge \lambda_{l+1} \|w\|_2^2, && w \in M. \end{aligned}$$

And hence from **B2**,

$$\begin{aligned} G(v) &\le B_1, && v \in N, \\ G(w) &\ge -B_2, && w \in M, \end{aligned}$$

where $B_i := \int_\Omega W_i(x)dx,\ i = 1, 2$. On $\{(1-t)w + t\rho v_0 : t \in [0, 1], w \in M, \|w\|_\star = \rho\}$, we have

$$\begin{aligned} G((1-t)w + t\rho v_0) &= \|(1-t)w + t\rho v_0\|^2 - 2\int_\Omega F(x, (1-t)w + t\rho v_0)dx \\ &\ge \|(1-t)w + t\rho v_0\|^2 - \int_\Omega (a_2|(1-t)w + t\rho v_0|^2 + W_2)dx \\ &\ge (1 - a_2/\lambda_{l+1})\|(1-t)w\|^2 + (1 - a_2/\lambda_l)\|t\rho v_0\|^2 - B_2 \\ &\ge -C - B_2 \\ &> -\infty. \end{aligned}$$

Now we estimate that

$$\inf_{B_0} G \ge \min\{-B_2, -C - B_2\} =: -B_3 > -\infty.$$

With all hypotheses of Theorem 2.2 verified, we set $a^\star := 1 + \sup_N G > \inf_{B_0} G$, and thus conclude there is $u_k \in \mathcal{S}$ such that

$$G(u_k) \to c \in [-B_3, B_1], \qquad G'(u_k) \to 0. \tag{4.3}$$

Lemma 4.7. *Under hypotheses of Theorem 4.3,* (4.3) *possesses a convergent subsequence.*

Proof. From properties of sequence (4.3), we know as $k \to \infty$

$$G(u_k) = \|u_k\|^2 - 2\int_\Omega F(x, u_k)dx \to c, \tag{4.4}$$

$$(G'(u_k), u_k)/2 = \|u_k\|^2 - \langle f(x, u_k), u_k\rangle = o(\|u_k\|). \tag{4.5}$$

Consequently,

$$\int_\Omega [f(x, u_k)u_k - 2F(x, u_k)]dx = o(\|u_k\|). \tag{4.6}$$

If $\rho_k := \|u_k\| \to \infty$, let $\tilde{u}_k := u_k/\rho_k$. Hence $\|\tilde{u}_k\| = 1$. Then there are a subsequence, still denoted as $\tilde{u}_k$, and $\tilde{u} \in E$ such that

$$\begin{aligned} \tilde{u}_k &\rightharpoonup \tilde{u}, && \text{in } E, \\ \tilde{u}_k &\to \tilde{u}, && \text{in } L^2(\Omega), \\ \tilde{u}_k &\to \tilde{u}, && \text{a.e. } \Omega. \end{aligned}$$

In view of (4.4), we have

$$1 - 2\int_\Omega F(x, u_k)/\rho_k^2 dx \to 0. \tag{4.7}$$

However by **B2**, there is estimation

$$2F(x, u_k)/\rho_k^2 \le a_2 \tilde{u}_k^2 + W_2(x)/\rho_k^2.$$

Along with (4.7), we integrate the above inequality and then pass to the limit $k \to \infty$, we derive

$$1 \le a_2 \|\tilde{u}\|_2^2, \tag{4.8}$$

which implies $\tilde{u} \not\equiv 0$. By (4.6) and Fatou's Lemma, we have

$$\begin{aligned} 0 &= \limsup_{k\to\infty} \int_\Omega \left(2F(x, u_k) - u_k f(x, u_k)\right)/\rho_k dx \\ &\le \int_\Omega \limsup_{k\to\infty} \frac{2F(x, u_k) - u_k f(x, u_k)}{|u_k|} |\tilde{u}_k| dx \\ &= \int_{\{\tilde{u}\neq 0\}} |\tilde{u}| \limsup_{k\to\infty} \frac{2F(x, u_k) - u_k f(x, u_k)}{|u_k|} dx \\ &\le 0, \end{aligned}$$

the last inequality comes from **B4** and the fact

$$|u_k(x)| = \rho_k |\tilde{u}_k(x)| \to \infty, \qquad \text{for } x \in \{\tilde{u} \neq 0\}.$$

Hence $\tilde{u} \equiv 0$, which contradicts against (4.8). Thus $\rho_k := \|u_k\|$ is bounded in E. Then the existence of a convergent subsequence is then guaranteed by standard procedure, via subcritical growth of nonlinearity $f(x, t)$ and compact embedding, for which one can refer to Theorem 3.4.1 in [12]. □

Remark 4.8. We comment that the Sandwich Pair structure is also compatible with the Mountain Pass Geometry within cones $\pm\mathcal{P}$, hence the problem in this section may also possesses positive and negative solutions under conditions **B1** to **B4**. Details may refer to Chapter 6 of [17].

5. Application II - Sign-changing Solution via Linking Sandwich Pair

In this section we investigate the sign-changing solution of problem (4.1), under the setting of Linking Sandwich Pair. Taking the advantage of Theorem 3.6, we are able to loosen condition **B4** to a weaker version

B5: Suppose there holds

$$\text{either } \begin{cases} f(x,t)t - 2F(x,t) \geq -W(x) \in L^1(\Omega), \\ f(x,t)t - 2F(x,t) \to +\infty \quad \text{a.e. } x \in \Omega, \text{ as } |t| \to \infty; \end{cases}$$
$$\text{or } \begin{cases} f(x,t)t - 2F(x,t) \leq W(x) \in L^1(\Omega), \\ f(x,t)t - 2F(x,t) \to -\infty \quad \text{a.e. } x \in \Omega, \text{ as } |t| \to \infty. \end{cases}$$

Theorem 5.1. *We assume **B1** for Ω, **B2**, **B3** and **B5** for $f(x,t)$. Moreover let $l > 1$, $\lambda_l \leq a_1$ and $a_2 < \lambda_{l+1}$ in **B2**. Then problem* (4.1) *has at least one sign-changing solution.*

The proof of this theorem proceeds quite similar to the one of Theorem 4.3, and the only difference is to check the boundedness of the critical sequence. By Theorem 3.6, we obtain a Cerami-like critical sequence $u_k \in E$, and $\nu_k \in \mathbb{R}$, such that $\nu_k \to \infty$, and for some constants $c, C \in \mathbb{R}$

$$G(u_k) \to c, \qquad (\nu_k + \|u_k\|)\|G'(u_k)\| \leq C. \tag{5.1}$$

We state the boundedness of sequence (5.1) as the following

Lemma 5.2. *Under hypotheses of Theorem 5.1, the critical sequence $u_k \in \mathcal{S}$ satisfying* (5.1) *is bounded in E.*

Proof. From properties in (5.1), we know

$$\|u_k\|^2 - 2\int_\Omega F(x,u_k)dx \to c, \tag{5.2}$$
$$\left|\|u_k\|^2 - \langle f(x,u_k), u_k\rangle\right| \leq K. \tag{5.3}$$

Consequently,

$$\left|\int_\Omega [f(x,u_k)u_k - 2F(x,u_k)]dx\right| \leq K'. \tag{5.4}$$

If $\rho_k := \|u_k\| \to \infty$, let $\tilde{u}_k := u_k/\rho_k$. Hence $\|\tilde{u}_k\| = 1$. Then there are $\tilde{u} \in E$ and a subsequence, still denoted as $\tilde{u}_k$, such that

$$\tilde{u}_k \rightharpoonup \tilde{u}, \qquad \text{in } E,$$
$$\tilde{u}_k \to \tilde{u}, \qquad \text{in } L^2(\Omega),$$
$$\tilde{u}_k \to \tilde{u}, \qquad \text{a.e. } \Omega.$$

In view of (5.2), we have

$$1 - 2\int_\Omega F(x,u_k)/\rho_k^2 dx \to 0. \tag{5.5}$$

However from **B2**, there is estimation

$$2F(x,u_k)/\rho_k^2 \leq a_2\tilde{u}_k^2 + W_2(x)/\rho_k^2.$$

Along with (5.5) and passing to the limit, we derive

$$1 \leq a_2\|\tilde{u}\|_2^2,$$

which implies $\tilde{u} \not\equiv 0$.

Set $\Omega_0 := \{x \in \Omega : \tilde{u}(x) \neq 0\}$, and $\Omega_1 := \Omega - \Omega_0$. Then we have, by **B5**

$$\begin{aligned} \left| \int_\Omega [f(x,u_k)u_k - 2F(x,u_k)]dx \right| &= \left| \int_{\Omega_0} + \int_{\Omega_1} \right| \\ &\geq \left| \int_{\Omega_0} [f(x,u_k)u_k - 2F(x,u_k)]dx \right| - \|W\|_1 \\ &\to \infty. \end{aligned} \tag{5.6}$$

This contradicts against (5.4). Thus we conclude $\rho_k := \|u_k\|$ is bounded in E. □

Remark 5.3. We comment that **B5** is much less demanding than **B4**, admitting more general nonlinearity $f(x,t)$. The advantage of critical sequence in Theorem 5.1 involves (5.4). An ordinary (PS) sequence would only imply that

$$\|u_k\|^2 - \langle f(x,u_k), u_k \rangle = o(\|u_k\|),$$

and then

$$\left| \int_\Omega [f(x,u_k)u_k - 2F(x,u_k)]dx \right| = o(\|u_k\|),$$

which is insufficient to contradict against (5.6).

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