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SPLICING SKEW SHAPED POSITROIDS

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Abstract. Skew shaped positroids (or skew shaped positroid varieties) are certain Richardson varieties in the flag variety that admit a realization as explicit subvarieties of the Grassmannian $\text{Gr}(k, n)$. They are parametrized by a pair of Young diagrams $\mu \subseteq \lambda$ fitting inside a $k \times (n - k)$ -rectangle. For every $a = 1, \dots, n - k$, we define an explicit open set U_a inside the skew shaped positroid $S_{\lambda/\mu}^\circ$, and show that U_a is isomorphic to the product of two smaller skew shaped positroids. Moreover, U_a admits a natural cluster structure and the aforementioned isomorphism is quasi-cluster isomorphism in the sense of Fraser. Our methods depend on realizing the skew shaped positroid as an explicit braid variety, and generalize the work of the first and third authors for open positroid cells in the Grassmannian.

Keywords. Cluster algebras, positroid varieties

Mathematics Subject Classifications. 13F60, 14M15

1. Introduction

In this paper, we study cutting and gluing operations for skew shaped positroid varieties, extending the constructions in [GS26] for the maximal positroid cell.

1.1. Main results

Let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n - k)$ rectangle. The skew diagram λ/μ defines the *skew shaped positroid* (or skew shaped positroid variety) $S_{\lambda/\mu}^\circ$ which can be thought of as the

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intersection of a Schubert cell in the complete flag variety $\mathcal{Fl}(n)$ labeled by μ and an opposite Schubert cell labeled by λ , see Definition 3.16. Skew shaped positroids can be thought of as special cases of open Richardson [KL79] or positroid varieties [KLS13]. Crucially, even though the skew shaped positroid is a Richardson variety in the full flag variety, it can also be defined inside the Grassmannian $\text{Gr}(k, n)$. Thus we can, and for the most part will, think of $S_{\lambda/\mu}^\circ$ as being a subvariety of the Grassmannian $\text{Gr}(k, n)$. When $\mu = \emptyset$ the skew shaped positroid is an open Schubert variety, considered in [SSBW19].

Remark 1.1. While the skew shaped positroid $S_{\lambda/\mu}^\circ$ is defined in a similar way to the skew Schubert varieties of [SSBW19], when $\mu \neq \emptyset$ the variety $S_{\lambda/\mu}^\circ$ is generally not a skew Schubert variety. The variety $S_{\lambda/\mu}^\circ$ is closely related to Grassmannian Richardson varieties, see Remark 3.18.

Skew shaped positroids admit a cluster structure coming from source labels of plabic graphs, established in work of Galashin–Lam [GL23] following earlier work of Leclerc [Lec16] and Serhiyenko–Sherman–Bennett–Williams [SSBW19]; see Section 4 for more details.

Using the lattice path labeling of a skew Young diagram described in Section 3.3, we explicitly define the Grassmann necklace of $S_{\lambda/\mu}^\circ$ based on the boundary ribbon of the skew diagram λ/μ in Section 3.4.

The main construction of the paper concerns cutting the skew diagram λ/μ into two. Choose an integer a such that $1 \leq a \leq n - k$. Let $\lambda^{a,R}$ be the partition obtained by considering only the first $a - 1$ columns (from the right) of λ , and define $\mu^{a,R}$ similarly. Note that $\mu^{a,R} \subseteq \lambda^{a,R}$ are partitions fitting inside a $k \times (a - 1)$ -rectangle, so we can consider the skew shaped positroid $S_{\lambda^{a,R}/\mu^{a,R}}^\circ \subseteq \text{Gr}(k, k + a - 1)$.

Now let $\lambda^{a,L}$ be the partition obtained by considering the $a, \dots, n - k$ -columns (from the right) of λ , and similarly for $\mu^{a,L}$. Note that $\mu^{a,L} \subseteq \lambda^{a,L}$ fit inside a $k \times (n - k - a + 1)$ -rectangle, so we can consider the skew shaped positroid $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \subseteq \text{Gr}(k, n - a + 1)$. We are ready to state the main result of the paper.

Theorem 1.2. *Let $n > 0$ and $0 < k < n$. Let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n - k)$ -rectangle. Choose $1 \leq a \leq n - k$, and let U_a be the principal open set in $S_{\lambda/\mu}^\circ$ defined by the non-vanishing of the cluster variables in the a -th column of λ/μ (from the right), see Definition 5.1. Then, we have an isomorphism*

$$\Phi_a : U_a \xrightarrow{\sim} S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ.$$

Furthermore, Φ_a is a quasi-cluster isomorphism.

Let us clarify the last sentence in the statement of Theorem 1.2. The open subvariety $U_a \subseteq S_{\lambda/\mu}^\circ$ admits a natural cluster structure obtained from that on $S_{\lambda/\mu}^\circ$ by freezing the variables appearing in the a -th column of λ/μ , see Proposition 5.7. On the other hand, the product $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ$ has a natural cluster structure induced from the cluster structure on each of its factors. The term *quasi-cluster isomorphism* means that the map $\Phi_a^* : \mathbb{C}[S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ] \rightarrow \mathbb{C}[U_a]$ sends cluster variables to cluster variables, up to nicely behaved monomials in frozen variables, see Section 4.1. In particular, Φ_a^* preserves cluster monomials, and the map Φ_a sends cluster tori to cluster tori.

In [Mul13], Muller defines a *cluster chart* of a cluster variety to be the open set obtained by localizing some cluster variables in a single cluster, provided some technical conditions are satisfied, see e.g. [Mul13, Lemma 3.2]. Theorem 1.2, then, provides examples of cluster charts in skew shaped positroids that are themselves isomorphic to products of skew shaped positroids. We expect that this will allow for inductive methods to, for example, compute the cohomology [Scr25] or deep locus [CGSS26] of skew shaped positroids. We remark, however, that outside of some special cases, the open sets U_a do not cover the skew shaped positroid $S_{\lambda/\mu}^\circ$. Nevertheless, from the cluster structure on U_a one may obtain the cluster structure on $S_{\lambda/\mu}^\circ$ by un-freezing the cluster variables in the a -th column. Thus, while U_a is in general only a proper open set in $S_{\lambda/\mu}^\circ$, the cluster structure on U_a already has enough information to recover the entire variety $S_{\lambda/\mu}^\circ$. Note, however, that one cannot obtain the cluster structure on $S_{\lambda/\mu}^\circ$ from that on $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ$ because, in general, the quiver defining the cluster structure on $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ$ contains less arrows than that of U_a , see [GKSS25, Example 6.3]. In order to obtain $S_{\lambda/\mu}^\circ$ from $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ$, one needs to add more frozen variables to $S_{\lambda^{a,R}/\mu^{a,R}}^\circ$ and use the process of *amalgamation*, see [FG06, SW21]. We will not pursue this direction.

We describe the construction of the map Φ_a and prove Theorem 1.2 in Section 5. Instead of going into details in the introduction, we chose to illustrate the complexity of Φ_a in an example.

Example 1.3. Let $n = 12$, $k = 5$ so that $n - k = 7$. For the Young diagrams $\lambda = (7, 7, 5, 3, 1)$ and $\mu = (3, 1)$ the corresponding skew diagram is shown in Figure 1.1. Points in the skew shaped

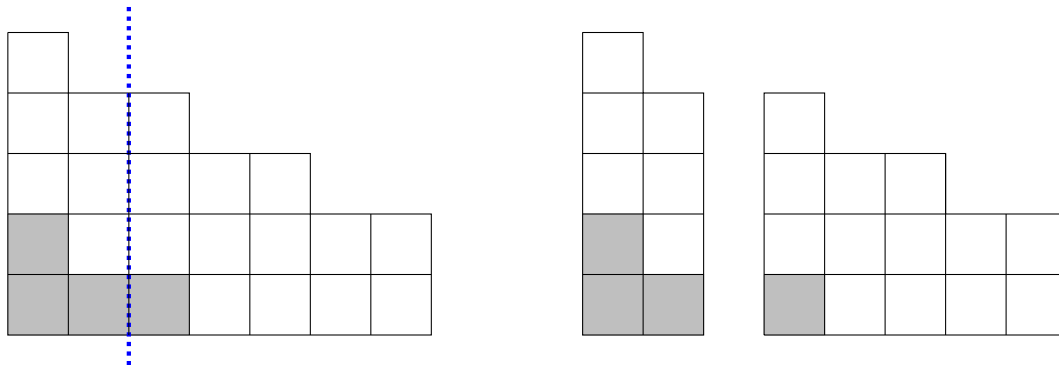


Figure 1.1: On the left is the diagram for λ/μ with corresponding matrix $(v_1, \dots, v_{12})$, where $\lambda = (7, 7, 5, 3, 1)$ and $\mu = (3, 1)$. Performing the cut at $a = 6$ is equivalent to cutting the diagram along the boundary of the 5-th and 6-th column from the right, as indicated by the blue dotted line. The resulting skew diagrams under the map Φ_6 are shown on the right, corresponding to the diagrams for $\lambda^{6,L}/\mu^{6,L}$ with matrix $(w_1, \dots, w_7)$ and $\lambda^{6,R}/\mu^{6,R}$ with matrix $(u_1, \dots, u_{10})$, respectively.

positroid $S_{\lambda/\mu}^\circ \subset \text{Gr}(5, 12)$ can be represented by 5×12 matrices with columns $(v_1, \dots, v_{12})$.

We choose $a = 6$ and cut the skew diagram as shown in Figure 1.1. We get two skew diagrams

$$\lambda^{6,L}/\mu^{6,L} = (2, 2, 2, 2, 1)/(2, 1), \quad \lambda^{6,R}/\mu^{6,R} = (5, 5, 3, 1)/(1)$$

and the corresponding skew shaped positroids

$$S_{\lambda^{6,L}/\mu^{6,L}}^\circ \subset \text{Gr}(5, 7), \quad S_{\lambda^{6,R}/\mu^{6,R}}^\circ \subset \text{Gr}(5, 10).$$

We denote the corresponding matrices by $V^L = (w_1, \dots, w_7)$ and $V^R = (u_1, \dots, u_{10})$. The open subset $U_6 \subset S_{\lambda/\mu}^\circ$ is defined by the inequalities

$$\Delta_{5,7,10,11,12} \neq 0, \quad \Delta_{5,7,8,11,12} \neq 0, \quad \Delta_{5,7,8,9,12} \neq 0.$$

Now the map $\Phi_6 : U_6 \rightarrow S_{\lambda^{6,L}/\mu^{6,L}}^\circ \times S_{\lambda^{6,R}/\mu^{6,R}}^\circ$ is defined by the following equations:

$$w_1 = v_5, w_2 = v_7, w_3 = v_8, w_4 = v_9, w_5 = v_{10}, w_6 = v_{11}, w_7 = v_{12},$$

while

$$u_1 = v_1, \quad u_2 = v_2, \quad u_3 = v_3, \quad u_4 = v_4, \quad u_5 = v_5, \quad u_6 = v_6,$$

$$u_7 \in \langle v_5, v_7 \rangle \cap \langle v_8, v_{10}, v_{11}, v_{12} \rangle, \quad u_8 \in \langle v_5, v_7, v_8 \rangle \cap \langle v_{10}, v_{11}, v_{12} \rangle,$$

$$u_9 \in \langle v_5, v_7, v_8, v_9 \rangle \cap \langle v_{11}, v_{12} \rangle, \quad u_{10} = v_{12},$$

We prove that all intersections in question are one-dimensional, and provide explicit formulas for the vectors u_7, u_8, u_9 , see Section 5.5 for all details.

In general, the construction of the map Φ_a has a similar flavor. The vectors $(w_1, \dots, w_{n-a+1})$ corresponding to $S_{\lambda^{a,L}/\mu^{a,L}}^\circ$ are chosen as a specific subset of vectors $(v_1, \dots, v_n)$ described in Section 5.3. The vectors $(u_1, \dots, u_{a+k-1})$ corresponding to $S_{\lambda^{a,R}/\mu^{a,R}}^\circ$ are instead obtained by a rather intricate procedure that involves intersecting certain subspaces spanned by v_i , see Section 5.4.

Sometimes, the cluster variables in the a -th column of the skew diagram λ/μ will all be frozen, and we obtain the following result.

Corollary 1.4. *Assume that all boxes in the a -th column from the right in λ/μ belong to the boundary ribbon $R(\lambda/\mu)$ (see Definition 3.19). Then U_a coincides with the whole skew shaped positroid $S_{\lambda/\mu}^\circ$, and we have the isomorphism*

$$S_{\lambda/\mu}^\circ \simeq S_{\lambda^{a,R}/\mu^{a,R}}^\circ \times S_{\lambda^{a,L}/\mu^{a,L}}^\circ.$$

Remark 1.5. One can define and study the compositions of the splicing maps Φ_a as follows. Suppose that $1 \leq a < b \leq n - k$ and λ/μ is a skew diagram, then we can draw two vertical cuts, one between a -th and $(a - 1)$ -th columns, and another between b -th and $(b - 1)$ -th columns. After performing these cuts we obtain three pieces: left $\lambda^{a,L}/\mu^{a,L}$, middle $\lambda^{a,M}/\mu^{a,M}$, and right $\lambda^{a,R}/\mu^{a,R}$.

Now, we can make an initial cut at a , then follow with a cut at b to get the following maps:

$$S_{\lambda/\mu}^\circ \xleftarrow{\Phi_a^{-1}} S_{\lambda^{a,L}/\mu^{a,L} \cup \lambda^{a,M}/\mu^{a,M}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ \xleftarrow{\Phi_{b-a+1}^{-1} \times \text{Id}} S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,M}/\mu^{a,M}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ.$$

On the other hand, if we make the initial cut at b , then follow with a cut at a we then get the maps:

$$S_{\lambda/\mu}^{\circ} \xleftarrow{\Phi_b^{-1}} S_{\lambda^{a,L}/\mu^{a,L}}^{\circ} \times S_{\lambda^{a,M}/\mu^{a,M} \cup \lambda^{a,R}/\mu^{a,R}}^{\circ} \xleftarrow{\text{Id} \times \Phi_a^{-1}} S_{\lambda^{a,L}/\mu^{a,L}}^{\circ} \times S_{\lambda^{a,M}/\mu^{a,M}}^{\circ} \times S_{\lambda^{a,R}/\mu^{a,R}}^{\circ}.$$

The image for both maps is the principal open subset $U_{a,b} \subset S_{\lambda^{a,R}/\mu^{a,R}}^{\circ}$ defined by non-vanishing of all cluster variables in a -th and b -th columns of λ/μ . However, we expect that the maps do not coincide but differ by some explicit monomials in frozen cluster variables on $U_{a,b}$.

See [Scr25, Section 4] for detailed computations for $k = 2$. We plan to study the general case in future work.¹

1.2. Braid varieties

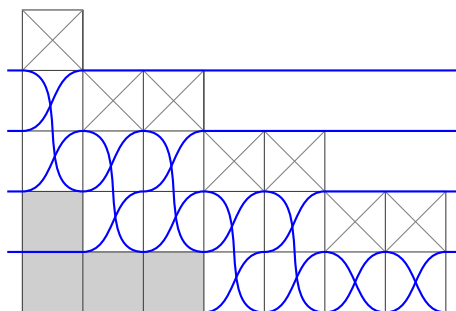
The key insight for the construction of the map Φ_a comes from the construction of **braid varieties** [CGGS26, CGGS24]. A braid variety $X(\beta)$ is an affine algebraic variety which can be associated to a positive braid β on k strands. It parametrizes sequences of complete flags in $\mathbb{C}^k$ with relative positions governed by the crossings in β , see Definition 2.1 for more details. Braid varieties have a cluster structure constructed in [CGG⁺25b, GLSBS22, GLSB26].

Given a skew diagram λ/μ , we define a k -strand braid $\beta_{\lambda/\mu}$, see Section 3.5. The crossings in $\beta_{\lambda/\mu}$ are in bijection with the boxes in λ/μ , except for the top boxes in each column.

Example 1.6. For $\lambda = (7, 7, 5, 3, 1), \mu = (3, 1)$ we get the braid

$$\beta_{\lambda/\mu} = (\sigma_4\sigma_3)(\sigma_3\sigma_2)(\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_2\sigma_1)(\sigma_1)(\sigma_1).$$

The braid is shown in blue and the crossed out boxes are not part of it.



Remark 1.7. Note that all boxes in the k -th (topmost) row of λ/μ are crossed out and not used in the braid $\beta_{\lambda/\mu}$. In fact, the topmost strand of $\beta_{\lambda/\mu}$ goes right below the topmost row of λ/μ .

We construct an explicit isomorphism between the skew shaped positroid and a certain braid variety. This is inspired by the constructions of [CGGS26, CLSBW23, STWZ19] but we do not check if our isomorphism coincides with either of these.

¹There is a slight discrepancy for the notation of the splicing map between [Scr25] and our paper. In [Scr25], the splicing (or cutting) map is denoted Φ_{ij}^{-1} , whereas in our paper it is denoted Φ_a .

Theorem 1.8. *We have an isomorphism of algebraic varieties*

$$S_{\lambda/\mu}^\circ \cong X(\underline{w_0}\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s. \quad (1.1)$$

Here $\underline{w_0}$ is the half-twist braid on k strands and s is the number of crossed out boxes. Moreover, (1.1) is a quasi-cluster isomorphism.

We can visualize Theorem 1.8 by the diagram

$$\Omega(V) = \left[\mathcal{F}^W \xrightarrow{\underline{w_0}} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_{\lambda/\mu}} \mathcal{F}^0 \right], \quad (1.2)$$

where $\mathcal{F}^W$ and $\mathcal{F}^{W^{\text{op}}}$ are a specific pair of opposite flags, and dotted arrows encode sequences of flags in $\mathbb{C}^k$. All the subspaces in all these flags can be defined as spans of some explicit subsets of columns of V , see Figure 3.5 for an example. Our construction of both the isomorphism Ω and its inverse uses only elementary linear algebra, avoiding the use of microlocal sheaves [STWZ19] or the Chekanov–Eliashberg DGA [CGGS26], or marking points in the braid closure of $\underline{w_0}\beta_{\lambda/\mu}$ [CLSBW23].

Next, we can describe the analogue of the map Φ_a for braid varieties. We factor the braid $\beta_{\lambda/\mu}$ as $\beta_{\lambda/\mu} = \beta_L \beta_R$ where β_L and β_R are two braids corresponding to $\lambda^{a,L}/\mu^{a,L}$ and $\lambda^{a,R}/\mu^{a,R}$, respectively. This allows us to expand (1.2) to the diagram

$$\Omega(V) = \left[\mathcal{F}^W \xrightarrow{\underline{w_0}} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_L} \mathcal{F}^a \xrightarrow{\beta_R} \mathcal{F}^0 \right], \quad (1.3)$$

where $\mathcal{F}^a$ is a specific flag on the boundary between β_L and β_R , refer to (5.2) for further details.

The key observation (Lemma 5.5) is that the open subset $U_a \subset S_{\lambda/\mu}^\circ$ corresponds under the isomorphism (1.1) to the set of sequences of complete flags (1.3) such that the flag $\mathcal{F}^a$ is transversal to $\mathcal{F}^W$. This allows us to define a map

$$(\Omega^L, \Omega^R) : U_a \rightarrow X(\underline{w_0}\beta_R) \times X(\underline{w_0}\beta_L)$$

given by

$$\Omega^L(V) = \left[\mathcal{F}^W \xrightarrow{\underline{w_0}} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_L} \mathcal{F}^a \right], \quad \Omega^R(V) = \left[\mathcal{F}^W \xrightarrow{\underline{w_0}} \mathcal{F}^a \xrightarrow{\beta_R} \mathcal{F}^0 \right]. \quad (1.4)$$

By using (1.1) again and taking into account the $\mathbb{C}^\times$ factors, we arrive at our construction of the map Φ_a . This argument shows that Φ_a is indeed well defined, and is an isomorphism of algebraic varieties. In Section 5, we prove that this construction of Φ_a is consistent with the linear algebra construction of $V^L = (w_1, \dots, w_{n-a+1})$ and $V^R = (u_1, \dots, u_{a+k-1})$, alluded to in Example 1.3, and provide a precise definition of the latter.

To prove that Φ_a is a quasi-cluster isomorphism of cluster structures, we use the triangularity of u_i with respect to v_i and compare the corresponding minors in Lemma 5.21. The computation of exchange ratios is then fairly straightforward (yet tedious), and is done in Lemmas 5.22 and 5.23. This step does not require braid varieties.

To summarize, the most technically challenging part of the paper is to construct the element $V^R = (u_1, \dots, u_{k+a-1})$ and verify that it indeed represents an element in the skew shaped positroid $S_{\lambda^{a,R}/\mu^{a,R}}^\circ$.

In the forthcoming paper [GKSS26], we use the cluster structure on braid varieties [CGG⁺25b, GLSBS22, SW21] to study more general splicing maps for braid varieties and to prove that these are quasi-cluster isomorphism. This would give an alternative proof that Φ_a is a quasi-cluster isomorphism.

2. Background

2.1. Flags and framed flags

We will work with the variety $\mathcal{F}\ell(k)$ of complete flags in the k -dimensional space $\mathbb{C}^k$. Given a matrix $M = (m_1, m_2, \dots, m_k) \in \mathrm{GL}(k)$ with columns $m_1, \dots, m_k$, its associated flag is

$$\mathcal{F}(M) := (\{0\} \subseteq \langle m_1 \rangle \subseteq \langle m_1, m_2 \rangle \subseteq \dots \subseteq \langle m_1, \dots, m_k \rangle = \mathbb{C}^k).$$

Note that $\mathcal{F}(M) = \mathcal{F}(MU)$ for any upper triangular matrix U , so that we have an identification

$$\mathcal{F}\ell(k) = \mathrm{GL}(k)/\mathrm{B}(k),$$

where $\mathrm{B}(k) \subseteq \mathrm{GL}(k)$ is the Borel subgroup of upper triangular matrices.

Let $\mathrm{U}(k) \subseteq \mathrm{B}(k)$ be the subgroup of upper triangular matrices with 1's on the diagonal. The variety of *framed flags* is the space

$$\mathcal{F}\ell^{\mathrm{fr}}(k) := \mathrm{GL}(k)/\mathrm{U}(k).$$

An element of $\mathcal{F}\ell^{\mathrm{fr}}(k)$ is known as a framed flag – it is the data of a flag $\mathcal{F} \in \mathcal{F}\ell(k)$, together with a choice of a nonzero element in $\mathcal{F}_i/\mathcal{F}_{i-1}$ for all $i = 1, \dots, k$ (referred to as a framing). Forgetting the framing gives rise to a natural map $\pi : \mathcal{F}\ell^{\mathrm{fr}}(k) \rightarrow \mathcal{F}\ell(k)$. As above, every element $M \in \mathrm{GL}(k)$ gives rise to a framed flag, that we will denote by $\mathcal{F}^{\mathrm{fr}}(M)$.

2.2. Relative position

Let $w \in S_k$. Abusing the notation, we denote by $w \in \mathrm{GL}(k)$ the corresponding permutation matrix, that has a 1 in the $(w(i), i)$ position for $i = 1, \dots, k$ and zeros elsewhere. In particular, we have the flag (resp. framed flag) $\mathcal{F}(w)$ (resp. $\mathcal{F}^{\mathrm{fr}}(w)$). The *standard flag* $\mathcal{F}^{\mathrm{std}}$ is the flag $\mathcal{F}(e)$, where $e \in S_k$ is the identity, and the *antistandard flag* $\mathcal{F}^{\mathrm{ant}}$ is the flag $\mathcal{F}(w_0)$, where $w_0 \in S_k$ is the longest element. The standard and antistandard framed flags are defined analogously.

Note that we have an action of the group $\mathrm{GL}(k)$ both on $\mathcal{F}\ell(k)$ and $\mathcal{F}\ell^{\mathrm{fr}}(k)$. We say that two flags $\mathcal{F}^1, \mathcal{F}^2 \in \mathcal{F}\ell(k)$ are in relative position $w \in S_k$ if there exists an element $g \in \mathrm{GL}(k)$ such that

$$g\mathcal{F}^1 = \mathcal{F}^{\mathrm{std}}, \quad g\mathcal{F}^2 = \mathcal{F}(w).$$

We will write $\mathcal{F}^1 \xrightarrow{w} \mathcal{F}^2$ to express that $\mathcal{F}^1$ and $\mathcal{F}^2$ are in relative position w . It is easy to verify that $\mathcal{F}^1 \xrightarrow{w} \mathcal{F}^2$ if and only if for every $i, j = 1, \dots, k$ we have

$$\dim(\mathcal{F}_i^1 \cap \mathcal{F}_j^2) = |\{1, \dots, i\} \cap \{w(1), \dots, w(j)\}|.$$

In particular, for a simple reflection $s_j \in S_k$, we have that $\mathcal{F}^1 \xrightarrow{s_j} \mathcal{F}^2$ if and only if $\mathcal{F}_j^1 \neq \mathcal{F}_j^2$, but $\mathcal{F}_i^1 = \mathcal{F}_i^2$ for every $i \neq j$.

The notion of relative position for framed flags is slightly more complicated. We say that two framed flags $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ are in *strong relative position* w if there exists $g \in \text{GL}(k)$ such that

$$g(\mathcal{F}^{\text{fr}})^1 = (\mathcal{F}^{\text{fr}})^{\text{std}}, \quad g(\mathcal{F}^{\text{fr}})^2 = \mathcal{F}^{\text{fr}}(w).$$

As before, we write $(\mathcal{F}^{\text{fr}})^1 \xrightarrow{w} (\mathcal{F}^{\text{fr}})^2$ if the framed flags $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ are in strong relative position w .

In particular, consider two framed flags $\mathcal{F}^{\text{fr}}, (\mathcal{F}^{\text{fr}})'$ with underlying flags $\mathcal{F} = \pi(\mathcal{F}^{\text{fr}})$, $\mathcal{F}' = \pi((\mathcal{F}^{\text{fr}})')$. Then $\mathcal{F}^{\text{fr}} \xrightarrow{s_i} (\mathcal{F}^{\text{fr}})'$ if and only if $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$ and the framing on $(\mathcal{F}^{\text{fr}})'$ is determined by the framing on $\mathcal{F}^{\text{fr}}$ as follows. Let $\mathcal{L}_j = \mathcal{F}_j / \mathcal{F}_{j-1}$ and $\mathcal{L}'_j = \mathcal{F}'_j / \mathcal{F}'_{j-1}$ for $j = 1, \dots, n$. Then we have canonical isomorphisms

$$\mathcal{L}_i \simeq \mathcal{L}'_{i+1}, \quad \mathcal{L}_{i+1} \simeq \mathcal{L}'_i, \quad \mathcal{L}_j \simeq \mathcal{L}'_j \quad (j \neq i, i+1). \quad (2.1)$$

Indeed, the following composition is an isomorphism whenever $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$:

$$\mathcal{L}_i = \mathcal{F}_i / \mathcal{F}_{i-1} \hookrightarrow \mathcal{F}_{i+1} / \mathcal{F}_{i-1} = \mathcal{F}'_{i+1} / \mathcal{F}'_{i-1} \rightarrow \mathcal{F}'_{i+1} / \mathcal{F}'_i = \mathcal{L}'_{i+1}.$$

The other isomorphisms are constructed similarly. The framing on $(\mathcal{F}^{\text{fr}})'$ is then determined by the framing on $\mathcal{F}^{\text{fr}}$ using these isomorphisms.

We also say that the framed flags $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ are in *weak relative position* w if there exists $g \in \text{GL}(k)$ and a diagonal matrix t such that

$$g(\mathcal{F}^{\text{fr}})^1 = (\mathcal{F}^{\text{fr}})^{\text{std}}, \quad g(\mathcal{F}^{\text{fr}})^2 = t\mathcal{F}^{\text{fr}}(w).$$

We write $(\mathcal{F}^{\text{fr}})^1 \xRightarrow{w} (\mathcal{F}^{\text{fr}})^2$ to express this relationship. Note that $(\mathcal{F}^{\text{fr}})^1 \xrightarrow{w} (\mathcal{F}^{\text{fr}})^2$ implies $(\mathcal{F}^{\text{fr}})^1 \xRightarrow{w} (\mathcal{F}^{\text{fr}})^2$ but the converse is not true. Also, note that $(\mathcal{F}^{\text{fr}})^1 \xRightarrow{w} (\mathcal{F}^{\text{fr}})^2$ if and only if the corresponding flags $\pi((\mathcal{F}^{\text{fr}})^1)$ and $\pi((\mathcal{F}^{\text{fr}})^2)$ are in relative position w .

2.3. Transversality

We say that two flags $\mathcal{F}^1$ and $\mathcal{F}^2$ are transversal, and write $\mathcal{F}^1 \pitchfork \mathcal{F}^2$, if $\mathcal{F}^1 \xrightarrow{w_0} \mathcal{F}^2$. Note that this is the case if and only if

$$\mathcal{F}_i^1 \cap \mathcal{F}_{k-i}^2 = \{0\}, \quad \text{equivalently} \quad \mathcal{F}_i^1 + \mathcal{F}_{k-i}^2 = \mathbb{C}^k$$

for every $i = 1, \dots, k$. By definition, if $\mathcal{F}^1 \pitchfork \mathcal{F}^2$ then there exists a matrix $g \in \text{GL}(k)$ such that $g\mathcal{F}^1 = \mathcal{F}^{\text{std}}$ and $g\mathcal{F}^2 = \mathcal{F}^{\text{ant}}$, and this matrix is unique up to left multiplication by an element of $\text{B}(k) \cap w_0 \text{B}(k) w_0$, i.e., a diagonal matrix. Equivalently, one can say that there exists a basis $u_1, \dots, u_k$ of $\mathbb{C}^k$ such that

$$\mathcal{F}^1 = (\langle u_1 \rangle \subseteq \langle u_1, u_2 \rangle \subseteq \dots \subseteq \mathbb{C}^k) \quad \text{and} \quad \mathcal{F}^2 = (\langle u_k \rangle \subseteq \langle u_k, u_{k-1} \rangle \subseteq \dots \subseteq \mathbb{C}^k),$$

such a basis is unique up to scalar multiplication.

For framed flags, we say that $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ are transversal, and again write $(\mathcal{F}^{\text{fr}})^1 \pitchfork (\mathcal{F}^{\text{fr}})^2$, if $(\mathcal{F}^{\text{fr}})^1 \xrightarrow{w_0} (\mathcal{F}^{\text{fr}})^2$ (note that we are using *weak relative position* here). In this case, there exists a matrix $g \in \text{GL}(k)$ and a diagonal matrix t such that $g(\mathcal{F}^{\text{fr}})^1 = (\mathcal{F}^{\text{fr}})^{\text{std}}$ and $g(\mathcal{F}^{\text{fr}})^2 = t(\mathcal{F}^{\text{fr}})^{\text{ant}}$. Since $\text{U}(k) \cap w_0 \text{B}(k) w_0 = \{1\}$, such a matrix g is unique. If we have framed flags $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ that are in strong relative position w_0 , we say that $(\mathcal{F}^{\text{fr}})^1$ and $(\mathcal{F}^{\text{fr}})^2$ are strongly transversal.

2.4. Braid varieties

Now we are ready to define braid varieties.

Definition 2.1. Let $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \text{Br}_k^+$ be a positive braid word. The braid variety $X(\beta)$ is defined as the space of configurations of flags

$$\mathcal{F}^0 \xrightarrow{s_{i_1}} \mathcal{F}^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_r}} \mathcal{F}^r. \quad (2.2)$$

such that $\mathcal{F}^0$ is standard and $\mathcal{F}^r$ is antistandard. We will often abbreviate (2.2) as $\mathcal{F}^0 \xrightarrow{\beta} \mathcal{F}^r$.

Following [CGGS24, CZ22] we will often use the following equivalent version of Definition 2.1. Given the braid diagram of β drawn horizontally, we encode a point in the braid variety $X(\beta)$ by labeling all regions by subspaces of $\mathbb{C}^k$ as follows:

- (a) The unbounded region below β is labeled by 0, and the unbounded region above β is labeled by $\mathbb{C}^k$.
- (b) If a subspace W labels a region directly above one labeled by a subspace V , then $V \subset W$ and $\dim W = \dim V + 1$.
- (c) At a crossing in β , we have a configuration of subspaces

$$\begin{array}{c} D \\ \diagup \quad \diagdown \\ B \quad \quad C \\ \diagdown \quad \diagup \\ A \end{array}$$

where $A \subset B \subset D$, $A \subset C \subset D$ and $B \neq C$.

- (d) On the left boundary of β we get the flag $\mathcal{F}^0 = \mathcal{F}^{\text{std}}$ and on the right boundary of β we get the flag $\mathcal{F}^r = \mathcal{F}^{\text{ant}}$.

See Figure 3.5 for an example of such labeling. The conditions (a) and (b) imply that every vertical cross-section of the diagram corresponds to a complete flag in $\mathbb{C}^k$, and the condition (c) ensures that the neighboring flags are in a correct relative position.

Furthermore, we can think of elements of $X(\beta)$ as configurations of framed flags

$$(\mathcal{F}^{\text{fr}})^0 \xrightarrow{s_{i_1}} (\mathcal{F}^{\text{fr}})^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_r}} (\mathcal{F}^{\text{fr}})^r. \quad (2.3)$$

such that $(\mathcal{F}^{\text{fr}})^0$ is transversal to $(\mathcal{F}^{\text{fr}})^r$, modulo simultaneous left action of $\text{GL}(k)$. Indeed, if $(\mathcal{F}^{\text{fr}})^0$ is transversal to $(\mathcal{F}^{\text{fr}})^r$ then there exists a unique matrix $g \in \text{GL}(k)$ such that $g(\mathcal{F}^{\text{fr}})^0$ is the standard framed flag and $g(\mathcal{F}^{\text{fr}})^r$ projects to the antistandard flag. See [CGG⁺25a, GLSB26, GLSBS22].

Lemma 2.2. *Let $\underline{w}_0$ be the positive braid lift of w_0 . The braid variety $X(\underline{w}_0\beta)$ can be equivalently described as the space of following data, up to simultaneous left action of $\text{GL}(k)$:*

- (a) Configuration of framed flags $(\mathcal{F}^{\text{fr}})^0 \xrightarrow{\underline{w}_0} (\mathcal{F}^{\text{fr}})^1 \xrightarrow{\beta} (\mathcal{F}^{\text{fr}})^2$ such that $(\mathcal{F}^{\text{fr}})^0$ is transversal to $(\mathcal{F}^{\text{fr}})^2$.
- (b) Configuration of framed flags $(\mathcal{F}^{\text{fr}})^1 \xrightarrow{\beta} (\mathcal{F}^{\text{fr}})^2$ and a framed flag $(\mathcal{F}^{\text{fr}})^0$ which is strongly transversal to $(\mathcal{F}^{\text{fr}})^1$ and weakly transversal to $(\mathcal{F}^{\text{fr}})^2$.
- (c) Configuration of flags $\mathcal{F}^1 \xrightarrow{\beta} \mathcal{F}^2$ and a choice of basis $u_1, \dots, u_k$ such that

$$\mathcal{F}^1 = \{\langle u_1 \rangle, \langle u_1, u_2 \rangle, \dots, \langle u_1, u_2, \dots, u_k \rangle\}$$

and $\mathcal{F}^2$ is transversal to the framed flag

$$\{\langle u_k \rangle, \langle u_k, u_{k-1} \rangle, \dots \subseteq \langle u_k, \dots, u_1 \rangle\}.$$

Proof. Part (a) follows from the definition. Part (b) follows from (a) since $(\mathcal{F}^{\text{fr}})^0 \xrightarrow{\underline{w}_0} (\mathcal{F}^{\text{fr}})^1$ is equivalent to strong transversality of $(\mathcal{F}^{\text{fr}})^0$ and $(\mathcal{F}^{\text{fr}})^1$. Part (c) follows from (b) since for strongly transversal $(\mathcal{F}^{\text{fr}})^0$ and $(\mathcal{F}^{\text{fr}})^1$ there is a unique basis $u_1, \dots, u_k$ such that

$$\mathcal{F}^1 = \{\langle u_1 \rangle, \langle u_1, u_2 \rangle, \dots, \langle u_1, u_2, \dots, u_k \rangle\}$$

and

$$\mathcal{F}^0 = \{\langle u_k \rangle, \langle u_k, u_{k-1} \rangle, \dots, \langle u_k, u_{k-1}, \dots, u_1 \rangle\},$$

where we denote by $\mathcal{F}^0$ and $\mathcal{F}^1$ the standard projections of the framed flags $(\mathcal{F}^{\text{fr}})^0, (\mathcal{F}^{\text{fr}})^1$, respectively. Finally, if we are given the data as in (c) then the basis $u_1, \dots, u_k$ naturally endows $\mathcal{F}^1$ with a framing, that we extend through the configuration $\mathcal{F}^1 \xrightarrow{\beta} \mathcal{F}^2$ to obtain a framing on $\mathcal{F}^2$, and we recover the data of (b). $\square$

Remark 2.3. In what follows, we only record the framing on $(\mathcal{F}^{\text{fr}})^0$ and treat all other flags as unframed. As above, the strong relative position requirement induces a framing on all other flags using (2.1).

3. Skew shaped positroid varieties

3.1. Positroid varieties

First, we recall equivalent ways of describing positroid varieties in the Grassmannian: Grassmann necklaces and bounded affine permutations.

Definition 3.1. A (k, n) -*source Grassmann necklace* $\mathcal{I} = (I_1, I_2, \dots, I_n)$ is an n -tuple of k -element subsets $I_i \subseteq [n]$ where

- if $i \in I_i$ then there exists $j \in [n]$ such that $I_{i-1} = (I_i \setminus \{i\}) \cup \{j\}$, and
- if $i \notin I_i$ then $I_i = I_{i-1}$.

We denote by $\text{GN}(k, n)$ the set of (k, n) -Grassmann necklaces.

Remark 3.2. Note that if $i \in I_i$ then it may be that $I_{i-1} = I_i = (I_i \setminus \{i\}) \cup \{i\}$. Finally, note that $I_{i-1} \setminus I_i$ is either empty or a singleton.

Remark 3.3. Grassmann necklaces were first defined in [Pos06]. Our definition is slightly different, as we use I_{i-1} instead of I_{i+1} , this difference is indicated by the word **source** in the name.

Definition 3.4 ([KLS13]). A bijection $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is called a (k, n) -*bounded affine permutation* if it satisfies the following conditions:

1. $f(i + n) = f(i) + n$ for every $i \in \mathbb{Z}$.
2. For every $i \in \mathbb{Z}$, $i \leq f(i) \leq i + n$.
3. We have:

$$\sum_{i=1}^n (f(i) - i) = nk.$$

Note that, by (1), f is uniquely determined by the values $f(1), \dots, f(n)$, which are pairwise distinct modulo n . We often simply denote $f = [f(1), \dots, f(n)]$. Note also that, upon the presence of (1) and (2), (3) is equivalent to $|\{i \in [n] : f(i) > n\}| = k$. We denote the set of (k, n) -bounded affine permutations by $\text{BA}(k, n)$. We say that f is a **bounded n -affine permutation** if there exists k such that f is a (k, n) -bounded affine permutation.

Lemma 3.5. The sets $\text{GN}(k, n)$ and $\text{BA}(k, n)$ are in natural bijection.

In the proof, we will need the following definition.

Definition 3.6. We define the cyclic order $\leq_i$ on the set $[n]$:

$$i <_i i + 1 <_i i + 2 <_i \dots <_i n <_i 1 <_i \dots <_i i - 1.$$

Proof of Lemma 3.5. This is essentially a combination of [KLS13, Corollary 3.13] and [MS17, Remark 2.4]. For the reader's convenience and to fix notation, we provide the bijection $\varphi : \text{GN}(k, n) \rightarrow \text{BA}(k, n)$. Let us first describe $\bar{f}_{\mathcal{I}} := \varphi(\mathcal{I}) \pmod{n}$ through its inverse $\bar{f}_{\mathcal{I}}^{-1}$: we have $\bar{f}_{\mathcal{I}}^{-1}(i) = i$ if $I_{i-1} = I_i$, and $\bar{f}_{\mathcal{I}}^{-1}(i) = j$ if $I_{i-1} \setminus I_i = \{j\}$. To lift this to a bounded affine permutation, we set $f_{\mathcal{I}}(i) = i + n$ if $I_{i-1} = I_i$ and $i \in I_i$, and $f_{\mathcal{I}}(i) = i$ if $i \notin I_i$. On the other hand, if $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is a k -bounded affine permutation, let $I = \{i \in [n] \mid f(i) = i + n\}$, and $\bar{f} = f \pmod{n}$. Then I_i consists of I together with the elements $\{a \in [n] \mid a <_i \bar{f}(a)\}$. The collection $\mathcal{I}_f = (I_1, \dots, I_n)$ is a Grassmann necklace, and $f \mapsto \mathcal{I}_f, \mathcal{I} \mapsto f_{\mathcal{I}}$ are inverse bijections. $\square$

Now, we associate a Grassmann necklace and a bounded affine permutation to an element V in the Grassmannian $\text{Gr}(k, n)$ following [KLS13, Pos06]. We represent V by a $k \times n$ matrix of rank k , up to row operations, and denote by $v_1, \dots, v_n \in \mathbb{C}^k$ the columns of V . Furthermore, we define v_i for all $i \in \mathbb{Z}$ by setting $v_{i+n} := (-1)^{k-1} v_i$. Given an (ordered) k -element subset $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$, we denote by $\Delta_J = \Delta_J(V) = v_{j_1} \wedge \dots \wedge v_{j_k}$ the maximal minor of V in the columns of J .

Definition 3.7. ([MS17]) Note that the order $\leq_i$ naturally extends to a partial order on the set of k -element subsets of $[n]$:

$$\{j_1 <_i j_2 <_i \dots <_i j_k\} \leq_i \{\ell_1 <_i \ell_2 <_i \dots <_i \ell_k\} \quad \text{if} \quad j_s \leq_i \ell_s \text{ for every } s = 1, \dots, k.$$

Given this ordering, the **source Grassmann necklace** associated to V is a collection of k -element sets $\mathcal{I}_V = (I_1, I_2, \dots, I_n)$ where I_i is defined as

$$I_i := \max_{\leq_{i+1}} \left\{ J \in \binom{[n]}{k} \mid \Delta_J(V) \neq 0 \right\}.$$

In other words, I_i is the maximal element, with respect to the $\leq_{i+1}$ ordering, among all k -element subsets J such that $\Delta_J(V) \neq 0$.

Remark 3.8. Note that the order $\leq_{i+1}$ is the cyclic ordering where i is the *largest* element and $i + 1$ is the *smallest*.

Remark 3.9. Here we have used the source Grassmann necklace. Other works in the literature use *target* Grassmann necklaces, see e.g. [MS17, FSB22]. These are related to source and target labelings for plabic graphs, see Section 4.

Definition 3.10. ([KLS13]) Let $V = (v_1 | \dots | v_n) \in \text{Gr}(k, n)$. The bounded affine permutation f_V is defined as

$$f_V(i) := \min\{j \geq i \mid v_i \in \text{span}(v_{i+1}, v_{i+2}, \dots, v_j)\}.$$

Definition 3.11. ([KLS13, Theorem 5.1]) Given a bounded affine permutation f (or, equivalently, the corresponding Grassmann necklace $\mathcal{I} = \varphi^{-1}(f)$) we define the **open positroid variety** by

$$\Pi_f^\circ = \Pi_{\mathcal{I}}^\circ := \{V \in \text{Gr}(k, n) : f_V = f\} = \{V \in \text{Gr}(k, n) : \mathcal{I}_V = \mathcal{I}\}.$$

3.2. Skew shaped positroids

We fix $0 < k < n$ and consider a partition λ whose Young diagram fits inside the $(n - k) \times k$ -rectangle, that is, $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \leq n - k$. We draw partitions using the French convention, i.e., the boxes are lower left justified. Given partitions λ and μ with $\mu_i \leq \lambda_i$ for all $1 \leq i \leq k$, the **skew diagram** λ/μ denotes the set-theoretic difference of the Young diagrams of λ and μ . We also define

$$\bar{\mu}_a := \mu_{n-k+1-a}^t, \quad \bar{\lambda}_a := \lambda_{n-k+1-a}^t.$$

In other words, $\bar{\lambda}_a$ is the height of the a -th column of λ , reading from right-to-left.

Recall that $w \in S_n$ is a k -Grassmannian permutation if $w(1) < \dots < w(k)$ and $w(k+1) < \dots < w(n)$.

Definition 3.12. Given a partition $\lambda \subseteq (n - k)^k$ we define the k -Grassmannian permutation

$$w_\lambda = [n - k + 1 - \lambda_1, \dots, n - \lambda_k, 1 + \lambda_{n-k}^t, \dots, k + \lambda_1^t]. \quad (3.1)$$

Remark 3.13. Note that the permutation w_λ as defined by (3.1) differs from its namesake element in [CGGS26, (2.1)], that here we call $\bar{w}_\lambda$. In fact, the two elements w_λ and $\bar{w}_\lambda$ are related as follows. For $\lambda \subseteq (n - k)^k$, consider the complementary partition

$$\lambda' = (n - k - \lambda_k, \dots, n - k - \lambda_1) \subseteq (n - k)^k.$$

Then, $w_\lambda = \bar{w}_{\lambda'}$.

Let us now obtain reduced decompositions for w_λ and w_μ . For this, we use [CGGS26, (2.2)], together with Remark 3.13. Note that if λ' is as defined in Remark 3.13 then $(\lambda')^t = (k - \lambda_{n-k}^t, k - \lambda_{n-k-1}^t, \dots, k - \lambda_1^t)$. Thus,

$$\begin{aligned} w_\lambda = \bar{w}_{\lambda'} &= (s_{n-(\lambda')_{n-k}^t} \cdots s_{n-1}) \cdots (s_{k+1-(\lambda')_1^t} \cdots s_k) \\ &= (s_{n-k+\lambda_1^t} \cdots s_{n-1}) \cdots (s_{1+\lambda_{n-k}^t} \cdots s_k) \end{aligned} \quad (3.2)$$

More precisely, we can write $w_\lambda = C'_{\lambda, n-k} \cdots C'_{\lambda, 1}$, where

$$C_{\lambda, j} = (s_{j+\lambda_{n-k-j+1}^t} \cdots s_{k+j-1}).$$

If some $\lambda_j^t = k$, we just skip these factors so $C_{\lambda, j} = 1$.

We can picture the permutation w_λ as follows. Starting on the southeast corner of the rectangle, label the vertical steps on the east side of the rectangle by $1, \dots, k$, and the horizontal steps on the north side of the rectangle by $k+1, \dots, n$. Similarly, starting on the southeast corner of the rectangle move towards the northwest corner following the shape of the partition λ , labeling these steps consecutively – this is referred to as a **λ -labeling**. The permutation w_λ then starts at the rectangle labels and moves west or south to the λ labels. See Figure 3.1.

Remark 3.14. Note that (3.2) is a reduced decomposition of w_λ , that has length $k(n - k) - |\lambda| = |\lambda'|$.

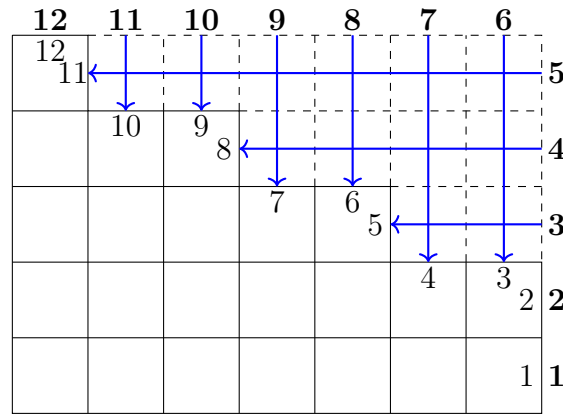


Figure 3.1: The permutation w_λ for $\lambda = (7, 7, 5, 3, 1)$ considered inside the rectangle (7^5) . The factors $C_{\lambda,j}$ correspond to the columns of the complement of λ inside the rectangle.

Lemma 3.15. *If $\mu \subseteq \lambda$ then there is a length-additive decomposition*

$$w_\mu = w_{\lambda/\mu} w_\lambda,$$

where

$$w_{\lambda/\mu} = (s_{n-k+\mu_1^t} \cdots s_{n-k+\lambda_1^t-1}) \cdots (s_{1+\mu_{n-k}^t} \cdots s_{\lambda_{n-k}^t})$$

is a reduced decomposition. In particular, $w_\mu \geq w_\lambda$ in Bruhat order.

See Figure 3.2 for a pictorial interpretation of Lemma 3.15. The permutation $w_{\lambda/\mu}$ starts at the λ labels and moves west or south to the μ labels.

Definition 3.16. Recall that $B(n)$ is the Borel subgroup of upper triangular matrices in $GL(n)$, and $B_-(n)$ the opposite Borel subgroup of lower triangular matrices. Given two partitions $\mu \subseteq \lambda \subseteq (n - k)^k$, we define the *skew shaped positroid* as the intersection of $B(n)w_\mu B(n)/B(n)$ and $B_-(n)w_\lambda B(n)/B(n)$. This is a subvariety of $\mathcal{F}\ell(n) = GL(n)/B(n)$ which will be denoted by $S_{\lambda/\mu}^\circ$.

Since $w_\lambda \leq w_\mu$ and w_μ is k -Grassmannian, we have that the function $f_{\lambda/\mu} = w_\lambda t_k w_\mu^{-1}$ is a (k, n) -bounded affine permutation, where $t_k = [1 + n, \dots, k + n, k + 1, \dots, n]$. We denote the corresponding positroid variety by $\Pi_{w_\lambda, w_\mu}^\circ$.

Remark 3.17. The skew shaped positroid $S_{\lambda/\mu}^\circ$ is, by definition, the open Richardson variety $R_{w_\lambda, w_\mu}^\circ \subset \mathcal{F}\ell(n)$. It is known [KLS13] that under the projection map $\mathcal{F}\ell(n) \rightarrow \mathrm{Gr}(k, n)$, the open Richardson variety $R_{w_\lambda, w_\mu}^\circ$ is isomorphic to the open positroid variety Π_{w_λ, w_μ} .

Remark 3.18. The skew shaped positroid $S_{\lambda/\mu}^\circ$ differs from the skew Schubert varieties studied in [SSBW19]. The variety $S_{\lambda/\mu}^\circ$ is closely related to Grassmannian Richardson varieties [ES23] as follows. Let $P = P(k, n) \subseteq GL(n)$ be the parabolic subgroup of block-upper triangular matrices, with diagonal blocks of sizes k and $n - k$. It is known and easy to show that

$$\mathrm{Gr}(k, n) = \mathrm{GL}(n)/\mathrm{P}.$$

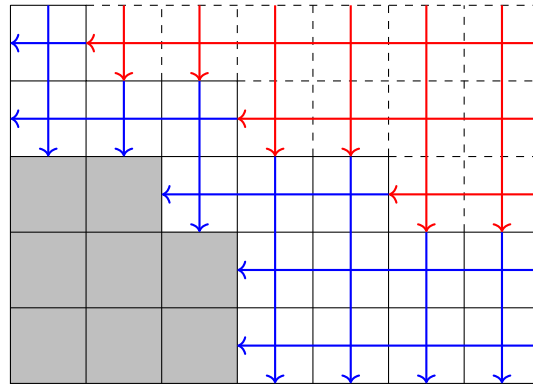


Figure 3.2: The length-additive decomposition $w_\mu = w_{\lambda/\mu} w_\lambda$.

For an element $w \in S_n$, the Grassmannian Schubert cell (resp. opposite Grassmannian Schubert cell) is the subvariety

$$B(n)wP/P \subseteq \text{Gr}(k, n) \quad (\text{resp.} \quad B_-(n)wP/P).$$

These depend only on the coset $w(S_k \times S_{n-k})$ and not on w itself. In particular, we can take minimal-length coset representatives, which are precisely the k -Grassmannian permutations. If w_λ, w_μ are k -Grassmannian permutations, the Grassmannian Richardson variety is

$$GR_{w_\lambda, w_\mu}^\circ := (B_-(n)w_\lambda P/P) \cap (B(n)w_\mu P/P) \subseteq \text{Gr}(k, n).$$

The variety $GR_{w_\lambda, w_\mu}^\circ$ is nonempty if and only if $w_\lambda \leq w_\mu$ in Bruhat order, in which case it is an affine, smooth algebraic variety of dimension $\ell(w_\mu) - \ell(w_\lambda)$. In fact, $w_\lambda \leq w_\mu$ if and only if $\mu \subseteq \lambda$, and in this case $\ell(w_\mu) - \ell(w_\lambda) = |\lambda/\mu|$. We will denote $GR_{\lambda/\mu}^\circ := GR_{w_\lambda, w_\mu}^\circ$.

The skew shaped positroid $S_{\lambda/\mu}^\circ$ is a dense subvariety of the Grassmannian Richardson variety $GR_{\lambda/\mu}^\circ$. Moreover, the Grassmannian Richardson variety $GR_{\lambda/\mu}^\circ$ may be expressed as the union of positroids

$$GR_{\lambda/\mu}^\circ = \bigcup_{\substack{w \leq w_\mu \\ w(S_k \times S_{n-k}) = w_\lambda(S_k \times S_{n-k})}} \Pi_{w, w_\mu}^\circ$$

and $\dim(\Pi_{w, w_\mu}^\circ) \leq \dim(GR_{\lambda/\mu}^\circ)$, with equality if and only if $w = w_\lambda$. Indeed, if $w(S_k \times S_{n-k}) = w_\lambda(S_k \times S_{n-k})$, then the opposite Schubert cell in the flag variety $B_-(n)wB(n)/B(n)$ projects to the opposite Schubert cell $B_-(n)w_\lambda P/P$ in the Grassmannian, while the Schubert cell $B(n)w_\mu B(n)/B(n)$ projects to $B(n)w_\mu P/P$ so $\Pi_{w, w_\mu}^\circ \subseteq GR_{\lambda/\mu}^\circ$. We also have $\ell(w_\lambda) \leq \ell(w)$, with equality if and only if $w = w_\lambda$. We thank Melissa Sherman-Bennett for discussions pertaining to this remark.

3.3. Lattice path labeling of a skew Young diagram

Given $1 \leq a \leq n - k$, $1 \leq i \leq k$ we define the square $\square_{a,i}$ as the i -th box from the bottom in a -th column from the right. It is clear that

$$\square_{a,i} \in \lambda/\mu \Leftrightarrow \bar{\mu}_a < i \leq \bar{\lambda}_a.$$

We denote

$$d_i = n - k + 1 - \lambda_i \quad (3.3)$$

so that $\square_{d_i, i}$ is the rightmost box in the i -th row of λ .

Definition 3.19. The *boundary ribbon* $R(\lambda)$ of the diagram λ consists of those boxes in λ such that the box immediately northeast of it does not belong to λ . More precisely, $R(\lambda)$ consists of the boxes $\square_{a, i}$ for $d_i \leq a \leq d_{i+1}$ which correspond to the horizontal steps, and of boxes $\square_{a, i}$ for $\bar{\lambda}_{a-1} \leq i \leq \bar{\lambda}_a$ which correspond to the vertical steps. The ribbon $R(\lambda)$ has exactly $\lambda_1 + \bar{\lambda}_{n-k} - 1$ boxes in total.

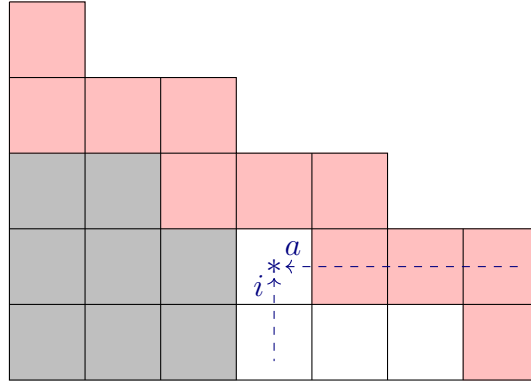
Given $\mu \subseteq \lambda$, we decompose

$$R(\lambda) = R(\lambda/\mu) \cup \bar{R}(\lambda/\mu),$$

where

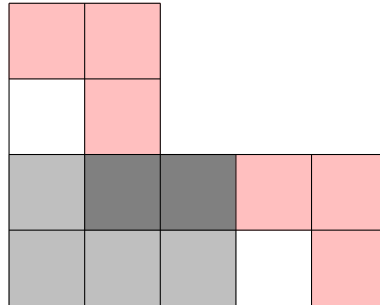
$$R(\lambda/\mu) = R(\lambda) \cap (\lambda/\mu), \quad \bar{R}(\lambda/\mu) = R(\lambda) \cap \mu.$$

Example 3.20. Throughout the paper, we consider the running example with $\lambda = (7, 7, 5, 3, 1)$ and $\mu = (3, 3, 2)$. Here $n - k = 7$ and $k = 5$, so $n = 12$.



The boundary ribbon $R(\lambda) = R(\lambda/\mu)$ is marked in pink, $\bar{R}(\lambda/\mu)$ is empty and the box $\square_{4,2}$ is labeled by $*$.

Example 3.21. For $n = 9$, $k = 4$, $\lambda = (5, 5, 2, 2)$ and $\mu = (3, 3)$ we get the following picture:



The boundary ribbon $R(\lambda/\mu)$ is marked in pink and $\bar{R}(\lambda/\mu)$ is marked in dark gray. Note that $R(\lambda/\mu)$ is disconnected.

Definition 3.22. Given a square $\square_{a,i} \in (n-k)^k$, we define $\mu_{a,i}$ as the minimal Young diagram such that $\mu \subseteq \mu_{a,i}$ and $\square_{a,i} \in \mu_{a,i}$. Note that $\mu_{a,i} = \mu$ if and only if $\square_{a,i} \in \mu$, and if $\square_{a,i} \in \lambda/\mu$ then $\mu_{a,i} \subseteq \lambda$. We will identify $\mu_{a,i}$ with its **boundary path** $P_{a,i}$ from the southeast to the northwest corner of the $(n-k) \times k$ -rectangle.

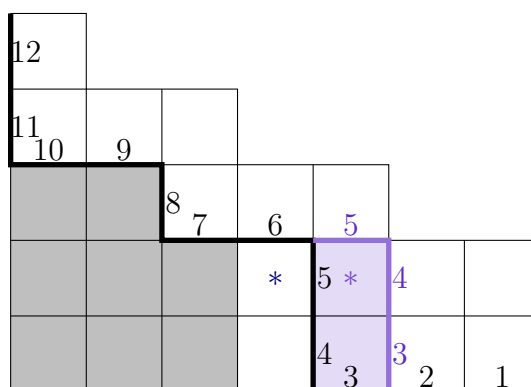
Remark 3.23. In [RW24, eq. (13.1)] Rietsch and Williams define “shapes”

$$\text{sh}(a, i) = \text{Rect}(a, i) \cup \mu$$

where $\text{Rect}(a, i)$ is the rectangle with northeast corner at $\square_{a,i}$ and southwest corner at $\square_{n-k,1}$. Clearly, the boundary of $\text{sh}(a, i)$ is our path $P_{a,i}$, so that $\text{sh}(a, i) = \mu_{a,i}$.

Definition 3.24. Given a square $\square_{a,i} \in \lambda/\mu$, we define its **label** $I'(a, i) \subset \{1, \dots, n\}$ as follows. Label all the steps of the path $P_{a,i}$ with numbers from 1 to n , starting at the southeast corner and increasing as we go west and north. The labels of the **vertical steps** in $P_{a,i}$ are the elements of $I'(a, i)$. Identifying μ (resp. λ) with its boundary path P_μ (resp. P_λ), we define the label I_μ (resp. I_λ) to be the set consisting of the labels of the vertical steps of P_μ (resp. P_λ).

Example 3.25. In our running example for $(a, i) = (4, 2)$ we show the path $P_{a,i}$ as follows:



Therefore, $I'(4, 2) = \{4, 5, 8, 11, 12\}$ which is labeled by *. We also draw an another path $P_{3,2}$ for $\square_{3,2}$ with $I'(3, 2) = \{3, 4, 8, 11, 12\}$, labeled by *. We can see that $P_{4,2}$ and $P_{3,2}$ are the same except for the **purple** rectangle region. We also note that $I_\mu = \{5, 6, 8, 11, 12\}$.

It should be noted that $I'(a, i)$ is a k -element subset of $\{1, \dots, n\}$. A nice way to describe this k -element set is given by starting with the k -element set $I_\mu = \{b_1 < b_2 < \dots < b_k\}$ and expressing the ordered subset $I(a, i)$ as

$$I(a, i) = (a, a+1, \dots, a+i-1, b_{i+1}, \dots, b_k), \quad (3.4)$$

where $1 \leq a \leq n-k$ and $1 \leq i \leq k$. From this set $I(a, i)$, we describe an ordered set

$$I'(a, i) = \{a_1, \dots, a_i, b_{i+1}, \dots, b_k\}, \quad (3.5)$$

where $a_j = \min\{a+j-1, b_j\}$. Then we label $\square_{a,i}$ with $I'(a, i)$, and the diagram μ with I_μ .

Example 3.26. In Example 3.20 we label all squares $\square_{a,i} \in \lambda/\mu$ by $I'(a, i)$. We also put the label I_μ inside μ .

5, 6, 8, 10, 11							
5, 6, 8, 10, 12	5, 6, 8, 9, 12	5, 6, 7, 8, 12					
		5, 6, 7, 11, 12	4, 5, 6, 11, 12	3, 4, 5, 11, 12			
	5, 6, 8, 11, 12		4, 5, 8, 11, 12	3, 4, 8, 11, 12	2, 3, 8, 11, 12	1, 2, 8, 11, 12	
			4, 6, 8, 11, 12	3, 6, 8, 11, 12	2, 6, 8, 11, 12	1, 6, 8, 11, 12	

It will be convenient to shorten the labeling $I'(a, i)$ as follows.

Definition 3.27. Given a square $\square_{a,i} \in \lambda/\mu$, we define its *short path* $\bar{P}_{a,i}$ as the part of $P_{a,i}$ which stops after reaching the northeast corner of $\square_{a,i}$ (but, in particular, includes the eastern boundary of $\square_{a,i}$). We define the *short label* $J(a, i)$ by reading the numbers at the vertical steps of $\bar{P}_{a,i}$.

Clearly, we have $|J(a, i)| = i$. Furthermore, the following result is clear from definitions and allows us to go back and forth between the short labels $J(a, i)$ and longer labels $I'(a, i)$.

Proposition 3.28. *The short label $J(a, i)$ consists of the first i elements in $I'(a, i)$. The remaining $(k - i)$ elements of $I'(a, i)$ are the same as the last $(k - i)$ elements of the fixed subset I_μ .*

Example 3.29. In our running example we can label all squares $\square_{a,i} \in \lambda/\mu$ by $J(a, i)$:

5, 6, 8, 10, 11							
5, 6, 8, 10	5, 6, 8, 9	5, 6, 7, 8					
		5, 6, 7	4, 5, 6	3, 4, 5			
			4, 5	3, 4	2, 3	1, 2	
			4	3	2	1	

The following gives us a recursive description of $J(a, i)$.

Lemma 3.30.

- (a) Assume that the boxes $\square_{a,i}$ and $\square_{a+1,i}$ are both in the diagram λ/μ . Then $J(a+1, i)$ is obtained from $J(a, i)$ by swapping $a + \bar{\mu}_a$ by $a + i$.
- (b) Assume that the boxes $\square_{a,i}$ and $\square_{a,i+1}$ are both in the diagram λ/μ . Then $J(a, i+1)$ is obtained from $J(a, i)$ by adding the element $a + i$.
- (c) In the above assumptions, $J(a, i) \subset J(a, i+1)$ and $J(a+1, i) \subset J(a, i+1)$.

Proof. Part (a) follows from the fact that $J(a, i) \setminus I(a, i) = \{a + \bar{\mu}_a, a + 1 + \bar{\mu}_a, \dots, a + i - 1\}$ and $J(a+1, i) \setminus I(a, i) = \{a + 1 + \bar{\mu}_a, a + 2 + \bar{\mu}_a, \dots, a + i\}$ (see Example 3.25). Part (b) follows from the observation that the short path $\bar{P}_{a,i+1}$ is obtained from $\bar{P}_{a,i}$ by adding an additional vertical step at the end. Thus, we must include the label of this vertical step, which equals the length of $\bar{P}_{a,i} + 1 = (a + i - 1) + 1$. The first claim of part (c) follows from (b), and the second claim of (c) follows from (a). $\square$

Corollary 3.31. We have $J(a, i) \subset J(a, \bar{\lambda}_a)$.

We will also need the analogues of the labeling $J(a, i)$ for the rightmost boxes in each row of μ .

Definition 3.32. Suppose that $\square_{a,i} \in \mu$ but $\square_{a-1,i} \in \lambda/\mu$. We define $\tilde{J}(a, i) = \{b_1, \dots, b_i\}$ where $I_\mu = \{b_1, \dots, b_k\}$ as above.

Lemma 3.33. Suppose that $\square_{a+1,i} \in \mu$ but $\square_{a,i} \in \lambda/\mu$. Then:

- (a) Assume that $\square_{a,i}, \square_{a,i+1}$ are both in the diagram λ/μ . Then $J(a, i+1) = J(a, i) \cup \tilde{J}(a+1, i)$.
- (b) $\tilde{J}(a+1, i)$ is obtained from $J(a, i)$ by swapping $(a + \bar{\mu}_a)$ for $(a + i)$.

Proof. This is completely analogous to Lemma 3.30. $\square$

3.4. Grassmann necklace for a skew shaped positroid

We have seen in Remark 3.17 that the skew shaped positroid is in fact a positroid variety. In this subsection, we use the combinatorics from the previous subsection to explicitly describe the corresponding Grassmann necklace. We define a Grassmann necklace associated to a skew diagram λ/μ using the boundary ribbon $R(\lambda) = R(\lambda/\mu) \cup \bar{R}(\lambda/\mu)$ from Definition 3.19. Recall the definition of labels $I'(a, i)$ and I_μ from Definition 3.24.

Definition 3.34. Given a skew diagram λ/μ , we define the corresponding Grassmann necklace $\mathcal{I}_{\lambda/\mu}$ as follows:

- If $\lambda_1 < n - k$, we define $\mathcal{I}_{\lambda/\mu, 1} = \dots = \mathcal{I}_{\lambda/\mu, n-k-\lambda_1} = I_\mu$.
- For a box $\square_{a,i} \in R(\lambda/\mu)$, we define $\mathcal{I}_{\lambda/\mu, a+i-1} = I'(a, i)$.
- For a box $\square_{a,i} \in \bar{R}(\lambda/\mu)$ we define $\mathcal{I}_{\lambda/\mu, a+i-1} = I_\mu$.

- If $\bar{\lambda}_1 < k$, we define $\mathcal{I}_{\lambda/\mu, n-k+\bar{\lambda}_1} = \cdots = \mathcal{I}_{\lambda/\mu, n-1} = I_\mu$.
- Finally, $\mathcal{I}_{\lambda/\mu, n} = I_\mu$.

Lemma 3.35. *The collection of subsets $\mathcal{I}_{\lambda/\mu}$ is in fact a Grassmann necklace.*

Proof. We first notice that $I'(a, i)$ and I_μ all have the same cardinality k . For the steps between I_μ and I_μ , no additional verification is required. We now consider several cases:

Case 1a: Horizontal step where both $\square_{a,i}, \square_{a+1,i}$ are in $R(\lambda/\mu)$. Here $i = \bar{\lambda}_a$, and by Lemma 3.30 we have $I'(a+1, \bar{\lambda}_a) = I'(a, \bar{\lambda}_a) \setminus \{a + \bar{\mu}_a\} \cup \{a + \bar{\lambda}_a\}$. Therefore

$$\mathcal{I}_{\lambda/\mu, a+\bar{\lambda}_a} = \mathcal{I}_{\lambda/\mu, a+\bar{\lambda}_a-1} \setminus \{a + \bar{\mu}_a\} \cup \{a + \bar{\lambda}_a\}. \quad (3.6)$$

Case 1b: Horizontal step where $\square_{a,i} \in R(\lambda/\mu)$ but $\square_{a+1,i} \in \bar{R}(\lambda/\mu)$. In this case, we similarly get $I_\mu = I'(a, \bar{\lambda}_a) \setminus \{a + \bar{\mu}_a\} \cup \{a + \bar{\lambda}_a\}$ and (3.6) holds.

Case 1c: From $I'(n-k, \bar{\lambda}_{n-k})$ to I_μ . If $\bar{\mu}_{n-k} < \bar{\lambda}_{n-k}$ then $\mathcal{I}_{\lambda/\mu, n-k+\bar{\lambda}_{n-k}-1} = I'(n-k, k)$ and (similarly to Case 1) we get

$$\mathcal{I}_{\lambda/\mu, n-k+\bar{\lambda}_{n-k}} = I_\mu = \mathcal{I}_{\lambda/\mu, n-k+\bar{\lambda}_{n-k}-1} \setminus \{n-k + \bar{\mu}_{n-k}\} \cup \{n-k + \bar{\lambda}_{n-k}\}. \quad (3.7)$$

Case 2a: Vertical step where both $\square_{a,i}, \square_{a,i+1}$ are in $R(\lambda/\mu)$. In this case by Lemma 3.30 we have $I'(a, i+1) = I'(a, i) \setminus \{b_{i+1}\} \cup \{a+i\}$ and

$$\mathcal{I}_{\lambda/\mu, a+i} = \mathcal{I}_{\lambda/\mu, a+i-1} \setminus \{b_{i+1}\} \cup \{a+i\}. \quad (3.8)$$

Case 2b: Vertical step where $\square_{a,i} \in \bar{R}(\lambda/\mu)$ but $\square_{a,i+1} \in R(\lambda/\mu)$. In this case $I'(a, i+1) = I_\mu \setminus \{b_{i+1}\} \cup \{a+i\}$ and (3.8) holds.

Case 2c: From I_μ to $I'(n-k-\lambda_1, 1)$. If $\mu_1 < \lambda_1$, then $\square_{n-k-\lambda_1+1, 1} \in R(\lambda/\mu)$ and $I'(n-k-\lambda_1+1, 1) = \{n-k-\lambda_1+1, b_2, \dots, b_k\}$. Therefore

$$\mathcal{I}_{\lambda/\mu, n-k-\lambda_1+1} = \mathcal{I}_{\lambda/\mu, n-k-\lambda_1} \setminus \{b_1\} \cup \{n-k-\lambda_1+1\}. \quad (3.9)$$

□

Let us denote $f_{\lambda/\mu}$ to be the bounded affine permutation corresponding to $\mathcal{I}_{\lambda/\mu}$ via the bijection from Lemma 3.5.

Lemma 3.36. *The bounded affine permutation $f_{\lambda/\mu}$ satisfies*

$$f_{\lambda/\mu}(a + \bar{\mu}_a) = a + \bar{\lambda}_a, \quad 1 \leq a \leq n-k \quad (3.10)$$

$$f_{\lambda/\mu}(b_i) = f_{\lambda/\mu}(n-k-\mu_i+i) = (n-k-\lambda_i+i) + n, \quad 1 \leq i \leq k. \quad (3.11)$$

Proof. It is easy to see that

$$b_i = n-k-\mu_i+i.$$

Furthermore, in (3.8) we have $b_{i+1} = (n-k-\mu_{i+1}) + i + 1$ and $a = n-k-\lambda_{i+1}+1$, so $b_{i+1} > a+i$. Now the statement mostly follows from the proof of Lemma 3.35 and Lemma 3.5, with cases 1c and 2c corresponding to $a = n-k$ and $i = 1$ respectively. It remains to consider the steps in $\mathcal{I}_{\lambda/\mu}$ from I_μ to I_μ which split into several cases:

Case 1: If $\mu_1 \leq \lambda_1 < n - k$, then $b_1 = n - k - \mu_1 + 1$. Therefore for $1 \leq j \leq n - k - \lambda_1$ we have $j \notin I_\mu$, so $f(j) = j$.

Case 2: If $\bar{\lambda}_{n-k} < k$ and $n - k + \bar{\lambda}_{n-k} \leq j \leq n$, then $j \in I_\mu$, so $f(j) = j + n$.

Case 3: If we have a horizontal step with both $\square_{a,i}, \square_{a+1,i} \in \overline{R}(\lambda/\mu)$ then $i = \bar{\mu}_a = \bar{\lambda}_a$ while $\mathcal{I}_{\lambda/\mu, a+i-1} = \mathcal{I}_{\lambda/\mu, a+i} = I_\mu$. Since $a + i = a + \bar{\mu}_a \notin I_\mu$, we get $f(a + \bar{\mu}_a) = a + \bar{\mu}_a = a + \bar{\lambda}_a$, in agreement with (3.10).

Case 4: If we have a vertical step with both $\square_{a,i}, \square_{a,i+1} \in \overline{R}(\lambda/\mu)$ then $a + i - 1 = b_i \in I_\mu$, so $f(b_i) = b_i + n$. On the other hand, in this case $b_i = n - k - \mu_i + i = n - k - \lambda_i + i$, in agreement with (3.11). $\square$

Example 3.37. In our running example we have

$$\mathcal{I}_{\lambda/\mu} = \left\{ \begin{array}{l} \{1,6,8,11,12\}, \{1,2,8,11,12\}, \{2,3,8,11,12\}, \{3,4,8,11,12\}, \\ \{3,4,5,11,12\}, \{4,5,6,11,12\}, \{5,6,7,11,12\}, \{5,6,7,8,12\}, \\ \{5,6,8,9,12\}, \{5,6,8,10,12\}, \{5,6,8,10,11\}, \{5,6,8,11,12\} \end{array} \right\}.$$

Therefore the affine permutation $f = f_{\lambda/\mu}$ has the following values:

$$f(1) = 3, f(2) = 4, f(3) = 6, f(4) = 7, f(7) = 9, f(9) = 10, f(10) = 12 \quad (3.12)$$

and

$$f(6) = 2 + 12, f(8) = 5 + 12, f(11) = 8 + 12, f(12) = 11 + 12, f(5) = 1 + 12. \quad (3.13)$$

Note that the equations (3.12) correspond to the horizontal steps in $R(\lambda/\mu)$ and (3.13) correspond to the vertical steps in it. Also, (3.13) describes the values $f(b_i)$ for $i = 1, \dots, k$.

Example 3.38. In Example 3.21 we get

3, 4, 7, 8	3, 4, 6, 7			
3, 4, 7, 9	3, 4, 6, 9			
			2, 3, 8, 9	1, 2, 8, 9
	3, 4, 8, 9		2, 4, 8, 9	1, 4, 8, 9

The Grassmann necklace is

$$\mathcal{I}_{\lambda/\mu} = \left\{ \begin{array}{l} \{1, 4, 8, 9\}, \{1, 2, 8, 9\}, \{2, 3, 8, 9\}, \{3, 4, 8, 9\}, \{3, 4, 8, 9\}, \\ \{3, 4, 6, 9\}, \{3, 4, 6, 7\}, \{3, 4, 7, 8\}, \{3, 4, 8, 9\} \end{array} \right\}.$$

The bounded affine permutation is given by

$$f(1) = 3, f(2) = 4, f(5) = 5, f(6) = 8, f(7) = 9,$$

and

$$f(3) = 1 + 9, f(4) = 2 + 9, f(8) = 6 + 9, f(9) = 7 + 9.$$

Let us define a wiring diagram for the bounded affine permutation $f_{\lambda/\mu}$.

Given partitions $\mu \subseteq \lambda$, draw both the λ and μ -labeling in the same picture. For each horizontal step of μ , labeled by i in the μ -labeling, draw a vertical arrow going up until reaching the boundary of λ , with λ -label j . Then, $f(i) = j$. If on the other hand i labels a vertical step of μ , draw a horizontal arrow going right until reaching the boundary of λ with label j , and $f(i) = j + n$.

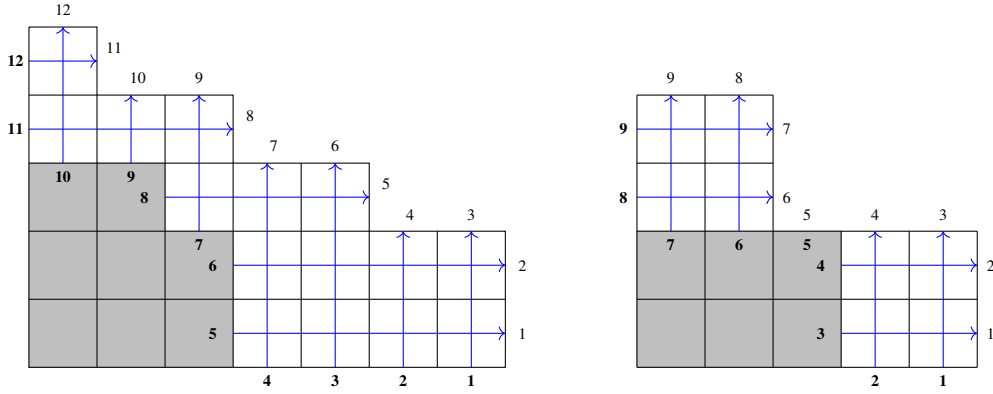


Figure 3.3: The wiring diagram associated to the bounded affine permutation $f_{\lambda/\mu}$. For horizontal arrows, we have to add n to the value of the target. Note that 5 is a fixed point of $f_{\lambda/\mu}$ for the right-hand figure.

Lemma 3.39. We have $f_{\lambda/\mu} = w_{\lambda} t_k w_{\mu}^{-1}$, where $t_k = [1 + n, \dots, k + n, k + 1, \dots, n]$.

Proof. We have $w_{\lambda}(i) = n - k - \lambda_i + i$ for $1 \leq i \leq k$ and $w_{\lambda}(k + a) = a + \bar{\lambda}_a$ for $1 \leq a \leq n - k$. Therefore

$$w_{\lambda} t_k w_{\mu}^{-1}(a + \bar{\mu}_a) = w_{\lambda} t_k(k + a) = w_{\lambda}(k + a) = a + \bar{\lambda}_a$$

and

$$w_{\lambda} t_k w_{\mu}^{-1}(n - k - \mu_i + i) = w_{\lambda} t_k(i) = w_{\lambda}(i + n) = n - k - \lambda_i + i.$$

This agrees with $f_{\lambda/\mu}$ by Lemma 3.36. □

Corollary 3.40. Given a skew diagram λ/μ , we have that $f_{\lambda/\mu} \pmod{n} = w_{\lambda/\mu}^{-1}$.

Proof. Indeed, we have that $f_{\lambda/\mu} = w_\lambda t_k w_\mu^{-1}$, while $w_{\lambda/\mu} = w_\mu w_\lambda^{-1}$ and $t_k \pmod{n}$ is the identity. $\square$

The following statement can be easily deduced from the results of [KLS13].

Theorem 3.41. *The skew shaped positroid $S_{\lambda/\mu}^\circ$ is isomorphic to the open positroid variety $\Pi_{\mathcal{I}_{\lambda/\mu}}^\circ$.*

Proof. By definition, the skew shaped positroid $S_{\lambda/\mu}^\circ$ is the open Richardson variety $R_{w_\lambda, w_\mu}^\circ$. Since the permutation w_μ is k -Grassmannian, the natural projection from the flag variety $\mathcal{F}\ell(n)$ to $\text{Gr}(k, n)$ identifies $R_{w_\lambda, w_\mu}^\circ$ with its image, cf. [KLS13]. By [KLS13], this projection identifies $R_{w_\lambda, w_\mu}^\circ$ with the positroid Π_f° where $f = w_\lambda t_k w_\mu^{-1}$ which agrees with $f_{\lambda/\mu}$ by Lemma 3.39. $\square$

Remark 3.42. Let V be a $(k \times n)$ -matrix representing a point in the positroid $\Pi_{\mathcal{I}_{\lambda/\mu}}^\circ$. Since I_μ is an element of the corresponding Grassmann necklace, we have that $\Delta_{I_\mu}(V) \neq 0$. By rescaling, we may and will assume that $\Delta_{I_\mu}(V) = 1$ for every element $V \in \Pi_{\mathcal{I}_{\lambda/\mu}}^\circ$.

3.5. Skew shaped positroid as a braid variety

Following [CGGS26, Lemma 2.5] we define a k -strand braid $\beta_{\lambda/\mu}$ associated to a skew diagram λ/μ via the following procedure.

1. Cross out the top box of every column of λ .
2. For each column of $(n - k)^k$, put strands starting on the lower left corner of each box in this column, so that there are precisely k strands. We join the strands to corners in the right border of this column, as follows:
 - (a) Join the strand starting in the lower left corner of the crossed box belonging to λ/μ (from Step 1) in this column to the lower right corner of the lowest box in the same column belonging to λ/μ . If the crossed box also belongs to μ , then simply join the strand, in a straight line, to the lower right corner of the same box.
 - (b) If a strand starts in the lower left corner of a non-crossed box in λ/μ , join it to the upper right corner of the same box.
 - (c) If a strand starts on the lower left corner of any box which is not of the above type, join it (in a straight line) to the lower right corner of the same box.
3. The braid $\beta_{\lambda/\mu}$ is defined to be the concatenation of the $(n - k)$ -braids from Step 2. In formulas,

$$\beta_{\lambda/\mu} = C_1 C_2 \cdots C_{n-k}, \quad C_j = \sigma_{\lambda_j^t - 1} \sigma_{\lambda_j^t - 2} \cdots \sigma_{\mu_j^t + 1}, \quad (3.14)$$

where C_j is the empty braid if $\mu_j^t + 1 > \lambda_j^t - 1$.

See Example 1.6 and Figure 3.4 for examples of the braid $\beta_{\lambda/\mu}$.

Definition 3.43. We say that a box $\square \in \lambda/\mu$ is a **braid box** if it is not the top row of its column. By definition, the braid boxes of λ/μ are in bijection with the crossings of $\beta_{\lambda/\mu}$.

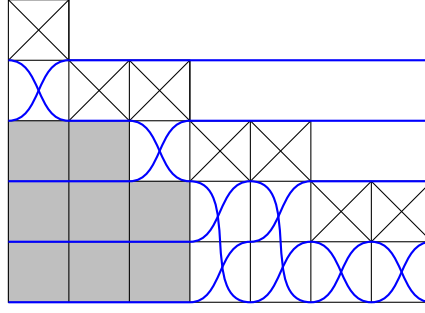


Figure 3.4: The braid $\beta_{\lambda/\mu}$ (in blue) for $\lambda = (7, 7, 5, 3, 1)$ and $\mu = (3, 3, 2)$.

Next, we define a family of subspaces of $\mathbb{C}^k$ labeled by the braid boxes of λ/μ or, equivalently, by the regions of $\beta_{\lambda/\mu}$.

Definition 3.44. Let V be a $(k \times n)$ -matrix with columns $v_1, \dots, v_n$ representing an element of the open positroid variety $\Pi_{\mathcal{I}_{\lambda/\mu}}^\circ$. For a braid box (a, i) of λ/μ , we define the subspace $V(a, i) = \text{span}\{v_j : j \in J(a, i)\} \subset \mathbb{C}^k$.

Lemma 3.45. We have $\dim V(a, i) = i$.

Proof. We have $J(a, i) \subset J(a, \bar{\lambda}_a) \subset I'(a, \bar{\lambda}_a)$ by Corollary 3.31, and $\Delta_{I'(a, \bar{\lambda}_a)}(V) \neq 0$ by definition of the Grassmann necklace $\mathcal{I}_{\lambda/\mu}$. Therefore the vectors $\{v_j : j \in I'(a, \bar{\lambda}_a)\}$ are linearly independent, and the vectors $\{v_j : j \in J(a, i)\}$ are linearly independent as well. $\square$

Lemma 3.46. We have containment

$$V(a, i) \subset V(a, i+1), \quad V(a+1, i) \subset V(a, i+1).$$

Proof. This follows from Lemma 3.30. $\square$

Lemma 3.47. Suppose that $\square_{a,i}$ and $\square_{a+1,i}$ are any two adjacent squares chosen from the i -th row of $R(\lambda/\mu)$. Then $V(a, i) = V(a+1, i)$.

Proof. Assume that $\square_{a,i}$ and $\square_{a+1,i}$ are both in $R(\lambda/\mu)$. Lemma 3.30 implies that $J(a+1, i)$ is obtained from $J(a, i)$ by swapping $a + \bar{\mu}_a$ by $a + \bar{\lambda}_a$. By Lemma 3.36, we have that $v_{a+\bar{\mu}_a}$ is contained in the span of $v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a}$. By the definition of the Grassmann necklace (cf. Definition 3.7), the span of $v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a+1}$ is precisely $V(a, i)$, so we have a containment $V(a+1, i) \subset V(a, i)$. Since $V(a, i)$ and $V(a+1, i)$ have the same dimension i , we can conclude that $V(a, i) = V(a+1, i)$. $\square$

Example 3.48. In Example 3.29 we have $J(5, 4) = \{5, 6, 7, 8\}$ and $J(6, 4) = \{5, 6, 8, 9\}$. Since $f(7) = 9$, we have $v_7 \in \langle v_8, v_9 \rangle$, so $V(5, 4) = \langle v_5, v_6, v_7, v_8 \rangle \subset \langle v_5, v_6, v_8, v_9 \rangle = V(6, 4)$. On the other hand, $\dim V(5, 4) = \dim V(6, 4) = 4$, so $V(5, 4) = V(6, 4)$.

Lemma 3.49. Suppose that $\square_{a,i}$ is not in $R(\lambda/\mu)$ and $\square_{a+1,i} \in \lambda/\mu$. Then

$$V(a, i) \neq V(a+1, i).$$

Proof. By Lemma 3.30 we get $J(a, i) \cup J(a + 1, i) = J(a, i + 1)$, so $V(a, i) + V(a + 1, i) = V(a, i + 1)$. If $V(a, i) = V(a + 1, i)$ then $V(a, i + 1) = V(a, i)$ which contradicts Lemma 3.45. $\square$

Definition 3.50. Let $I_\mu = \{b_1, \dots, b_k\}$ as above, we define

$$W_i = \langle v_{b_{k-i+1}}, \dots, v_{b_k} \rangle, \quad W_i^{\text{op}} = \langle v_{b_1}, \dots, v_{b_i} \rangle.$$

Lemma 3.51. The subspaces W_i^{op} have the following properties:

- (a) $\dim W_i^{\text{op}} = i$.
- (b) Suppose that $\square_{a,i} \in \mu$ but $\square_{a-1,i} \in \lambda/\mu$. Then $W_i^{\text{op}} \subset V(a - 1, i + 1)$ and $W_i^{\text{op}} \neq V(a - 1, i)$.

Proof. Part (a) follows from $\Delta_{I_\mu} \neq 0$. For part (b), we first observe that

$$W_i^{\text{op}} = \text{span} \left\{ v_j : j \in \tilde{J}(a, i) \right\},$$

so by Lemma 3.33 we get $W_i^{\text{op}} + V(a - 1, i) = V(a - 1, i + 1)$, in particular $W_i^{\text{op}} \subset V(a - 1, i + 1)$. If $W_i^{\text{op}} = V(a - 1, i)$ then $\dim V(a - 1, i + 1) = i$, contradiction. $\square$

It is easy to see that the vector $v_{a+\bar{\mu}_a}$ vanishes if and only if $\bar{\mu}_a = \bar{\lambda}_a$, that is, the a -th column of the skew diagram λ/μ is empty (this includes the case $\bar{\mu}_a = \bar{\lambda}_a = 0$). Indeed, by Lemma 3.36 $f(a + \bar{\mu}_a) = a + \bar{\lambda}_a$ and by definition of the bounded affine permutation f the vector v_j vanishes if and only if $f(j) = j$. Below we will mostly focus on the case $\bar{\mu}_a < \bar{\lambda}_a$ where $v_{a+\bar{\mu}_a} \neq 0$.

Lemma 3.52. For all a , if $\mu_a < \lambda_a$ then the space

$$V(a, \bar{\mu}_a + 1) \cap \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle$$

is one-dimensional and spanned by $v_{a+\bar{\mu}_a}$.

Proof. We have $a + \bar{\mu}_a \in J(a, \bar{\mu}_a + 1)$, so $v_{a+\bar{\mu}_a} \in V(a, \bar{\mu}_a + 1)$. Furthermore, by Lemma 3.36 we have $f(a + \bar{\mu}_a) = a + \bar{\lambda}_a$, so

$$v_{a+\bar{\mu}_a} \in \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle.$$

Next, observe that

$$J(a, \bar{\lambda}_a) = J(a, \bar{\mu}_a) \cup \{a + \bar{\mu}_a + 1, \dots, a + \bar{\lambda}_a - 1\}$$

so

$$\begin{aligned} \dim [V(a, \bar{\mu}_a + 1) + \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle] &\geq \dim [V(a, \bar{\mu}_a + 1) + \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a-1} \rangle] \\ &= \dim V(a, \bar{\lambda}_a) = \bar{\lambda}_a \end{aligned}$$

Also, $\dim \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle \leq \bar{\lambda}_a - \bar{\mu}_a$. Therefore

$$\begin{aligned} \dim [V(a, \bar{\mu}_a + 1) \cap \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle] &= \dim V(a, \bar{\mu}_a + 1) + \dim \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle - \\ &\dim [V(a, \bar{\mu}_a + 1) + \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle] \leq (\bar{\mu}_a + 1) + (\bar{\lambda}_a - \bar{\mu}_a) - \bar{\lambda}_a = 1. \quad \square \end{aligned}$$

Lemma 3.53. *For all a , we have $v_{a+\bar{\mu}_a} \in W_{k-\bar{\mu}_a}$.*

Proof. We prove it by decreasing induction on a where $1 \leq a \leq n - k$. Note that

$$W_{k-\bar{\mu}_a} = \langle v_{b_{\bar{\mu}_a+1}}, \dots, v_{b_k} \rangle = \text{span} \{v_{b_i} \mid b_i > a + \bar{\mu}_a\}.$$

By Lemma 3.36 we have $f(a + \bar{\mu}_a) = a + \bar{\lambda}_a$, so

$$v_{a+\bar{\mu}_a} \in \langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle.$$

For all t such that $a + \bar{\mu}_a + 1 \leq t \leq a + \bar{\lambda}_a$, we either have $t \in I_\mu$ and then $v_t \in W_{k-\bar{\mu}_a}$, or $t = a' + \bar{\mu}_{a'}$ for $a' > a$ and by the assumption of induction $v_t \in W_{k-\bar{\mu}_{a'}} \subset W_{k-\bar{\mu}_a}$. Therefore $\langle v_{a+\bar{\mu}_a+1}, \dots, v_{a+\bar{\lambda}_a} \rangle \subset W_{k-\bar{\mu}_a}$ and we are done. $\square$

Lemma 3.54. *For all a , if $\mu_a < \lambda_a$ then the intersection*

$$V(a, \bar{\mu}_a + 1) \cap W_{k-\bar{\mu}_a}$$

is one-dimensional and spanned by $v_{a+\bar{\mu}_a}$.

Proof. Let $L = V(a, \bar{\mu}_a + 1) \cap W_{k-\bar{\mu}_a}$. We have $J(a, \bar{\mu}_a + 1) = \{b_1, \dots, b_{\bar{\mu}_a}, a + \bar{\mu}_a\}$, so

$$V(a, \bar{\mu}_a + 1) = W_{\bar{\mu}_a}^{\text{op}} + \langle v_{a+\bar{\mu}_a} \rangle.$$

By Lemma 3.53 we have $v_{a+\bar{\mu}_a} \in V(a, \bar{\mu}_a + 1) \cap W_{k-\bar{\mu}_a}$ and $\dim L \geq 1$. On the other hand, $W_{\bar{\mu}_a}^{\text{op}} \cap W_{k-\bar{\mu}_a} = 0$, so $L \cap W_{\bar{\mu}_a}^{\text{op}} = 0$. At the same time, $L + W_{\bar{\mu}_a}^{\text{op}} \subset V(a, \bar{\mu}_a + 1)$, therefore $\dim L + \dim W_{\bar{\mu}_a}^{\text{op}} \leq \dim V(a, \bar{\mu}_a + 1)$ and $\dim L \leq 1$. $\square$

Definition 3.55. We define $R^1(\lambda/\mu) \subset R(\lambda/\mu)$ as the set of boxes $\square_{a, \bar{\lambda}_a}$ for $a = 1, \dots, n - k$ (whenever $\square_{a, \bar{\lambda}_a} \in \lambda/\mu$). Equivalently, $R^1(\lambda/\mu)$ are the top boxes of columns of λ that are not in μ , that is, the boxes in λ/μ that are crossed out in the construction of $\beta_{\lambda/\mu}$.

We define $S_{\lambda/\mu}^{\circ, 1} \subset S_{\lambda/\mu}^\circ$ by the conditions

$$\Delta_{I'(a, i)} = 1, \square_{a, i} \in R^1(\lambda/\mu).$$

Theorem 3.56. *For any skew diagram λ/μ let $s = |R^1(\lambda/\mu)|$. Then there exist isomorphisms*

$$\tilde{\Omega} : S_{\lambda/\mu}^\circ \xrightarrow{\sim} X(\underline{w_0}\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s, \quad \tilde{\Omega}^1 : S_{\lambda/\mu}^{\circ, 1} \xrightarrow{\sim} X(\underline{w_0}\beta_{\lambda/\mu}).$$

Proof.

Step 1: Recall the equivalent descriptions of $X(\underline{w_0}\beta_{\lambda/\mu})$ from Lemma 2.2. First, we construct a map

$$\tilde{\Omega} : S_{\lambda/\mu}^\circ \rightarrow X(\underline{w_0}\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s \tag{3.15}$$

by its components. The map $S_{\lambda/\mu}^\circ \rightarrow (\mathbb{C}^\times)^s$ simply sends a matrix V to the tuple of minors $\Delta_{I'(a, i)}$ for $\square_{a, i} \in R^1(\lambda/\mu)$. To construct the map $\Omega : S_{\lambda/\mu}^\circ \rightarrow X(\underline{w_0}\beta_{\lambda/\mu})$, we label the regions of the braid diagram of $\beta_{\lambda/\mu}$ by subspaces of $\mathbb{C}^k$ as follows.

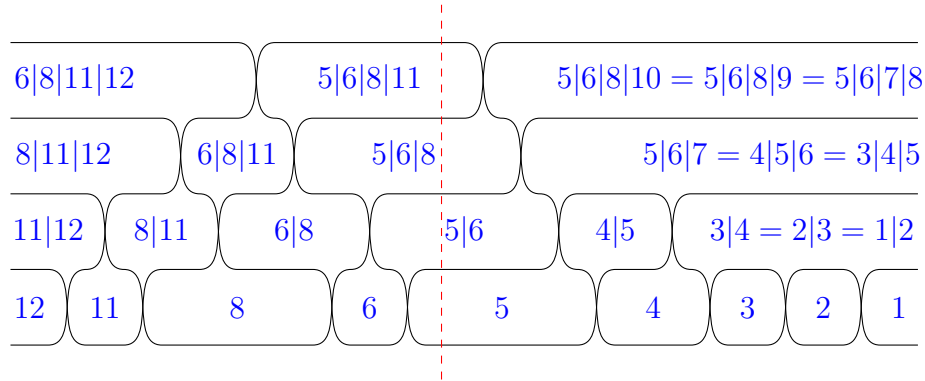


Figure 3.5: The crossings on the left of the dashed line form a reduced word for $\underline{w}_0$, while the crossings to the right of it form the braid $\beta_{\lambda/\mu}$ for $\lambda = (7, 7, 5, 3, 1)$, and $\mu = (3, 3, 2)$, see Figure 3.4. We have filled in the regions of the braid $\underline{w}_0\beta_{\lambda/\mu}$ with the subspaces W_i, W_i^{op} (on the $\underline{w}_0$ side) and $V(a, i)$ (on the $\beta_{\lambda/\mu}$ side) defined in Definitions 3.50 and 3.44, respectively.

We have a bijection between the braid boxes of λ/μ and the crossings of $\beta_{\lambda/\mu}$. Given a crossing, we label the region right of it by the subspace $V(a, i)$ where $\square_{a,i}$ is the corresponding box. Furthermore, we choose the basis $v_{b_1}, \dots, v_{b_k}$ for the flag on the left so that the regions on the left boundary of $\beta_{\lambda/\mu}$ are labeled by the subspaces W_i^{op} . We need to check that such labeling satisfies the conditions in Lemma 2.2(c).

All subspaces associated to the regions of $\beta_{\lambda/\mu}$ have correct dimension by Lemma 3.45. Furthermore, $\Delta_{I_\mu} \neq 0$ and $v_{b_1}, \dots, v_{b_k}$ are linearly independent and $\dim W_i^{\text{op}} = i$.

By Lemmas 3.46, 3.47 and 3.49 we ensure that the subspaces in the regions of $\beta_{\lambda/\mu}$ are in correct relative position. Note that for $i = k$ we get k -dimensional space which coincides with $\mathbb{C}^k$. By Lemma 3.51 the subspaces W_i^{op} are in correct relative position with the subspaces in regions of $\beta_{\lambda/\mu}$.

Finally, we need to check that the rightmost flag is transversal to $\mathcal{F}^W$ which is the opposite of $\mathcal{F}^{W^{\text{op}}}$. By construction, the rightmost flag $\mathcal{F}^0$ consists of subspaces $V(d_i, i) = \text{span } J(d_i, i)$ where $\square_{d_i,i}$ is the rightmost square in the i -th row of λ as in (3.3). For such squares, we have

$$I'(d_i, i) = J(d_i, i) \cup \{b_{i+1}, \dots, b_k\},$$

and $W_{k-i} = \langle v_{b_{i+1}}, \dots, v_{b_k} \rangle$. Since $\Delta_{I'(d_i,i)} \neq 0$, the subspaces $V(d_i, i)$ and W_{k-i} are indeed transversal. To sum up, we get the configuration

$$\Omega(V) = \left[\mathcal{F}^W \xrightarrow{\underline{w}_0} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_{\lambda/\mu}} \mathcal{F}^0 \right]. \quad (3.16)$$

Step 2: Next, we need to construct the inverse map

$$\Xi : X(\underline{w}_0\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s \rightarrow S_{\lambda/\mu}^\circ.$$

We use Lemma 2.2(c) to think of a point z in $X(\underline{w}_0\beta_{\lambda/\mu})$ as the data of a configuration of flags $\mathcal{F}^1 \xrightarrow{\beta} \mathcal{F}^2$ together with a choice of basis for $\mathcal{F}^1$. We denote this basis by $v_{b_1}, \dots, v_{b_k}$,

and the subspaces in regions of $\beta_{\lambda/\mu}$ by $L(a, i)$. Recall that for all $j \in \{1, \dots, n\}$ either $j \in \{b_1, \dots, b_k\}$ or $j = a + \bar{\mu}_a$ for some $a \in \{1, \dots, n - k\}$. The vectors $v_{b_1}, \dots, v_{b_k}$ are known, and we reconstruct the vectors $v_{a+\bar{\mu}_a}$. If $\bar{\mu}_a = \bar{\lambda}_a$ then we set $v_{a+\bar{\mu}_a} = 0$. From now on we assume $\bar{\mu}_a < \bar{\lambda}_a$. We claim that the intersection

$$L = L(a, \bar{\mu}_a + 1) \cap W_{k-\bar{\mu}_a}$$

is one-dimensional. Indeed,

$$\dim L(a, \bar{\mu}_a + 1) + \dim W_{k-\bar{\mu}_a} = (\bar{\mu}_a + 1) + (k - \bar{\mu}_a) = k + 1$$

and, since both $L(a, \bar{\mu}_a + 1)$ and $W_{k-\bar{\mu}_a}$ are vector subspaces of $\mathbb{C}^k$ we also obtain

$$\dim (L(a, \bar{\mu}_a + 1) + W_{k-\bar{\mu}_a}) \leq k$$

so $\dim L \geq 1$. On the other hand, $W_{\bar{\mu}_a}^{\text{op}} \subset L(a, \bar{\mu}_a + 1)$ and $W_{\bar{\mu}_a}^{\text{op}} \cap W_{k-\bar{\mu}_a} = 0$, hence $W_{\bar{\mu}_a}^{\text{op}} \cap L = 0$. Since $\dim(W_{\bar{\mu}_a}^{\text{op}} + L) \leq \dim L(a, \bar{\mu}_a + 1) = \bar{\mu}_a + 1$, we get $\dim L \leq 1$. Therefore $\dim L = 1$, and we can define $v_{a+\bar{\mu}_a}$ up to a scalar as a basis vector in L . To fix the scalar, observe that $\square_{a, \bar{\lambda}_a} \in R^1(\lambda/\mu)$ and

$$I'(a, \bar{\lambda}_a) = \{b_1, \dots, b_{\bar{\mu}_a}, a + \bar{\mu}_a, a + \bar{\mu}_a + 1, \dots, a + \bar{\lambda}_a - 1, b_{\bar{\lambda}_a+1}, \dots, b_k\}.$$

Therefore $\Delta_{I'(a, \bar{\lambda}_a)}$ involves $v_{a+\bar{\mu}_a}$ and other vectors which we have reconstructed earlier, so fixing the value of $\Delta_{I'(a, \bar{\lambda}_a)}$ determines the scalar for $v_{a+\bar{\mu}_a}$.

Altogether, the vectors v_i define a matrix

$$V = \Xi \left(z, \Delta_{I'(a, \bar{\lambda}_a)}, a = 1, \dots, n - k \right). \quad (3.17)$$

By construction, $v_{b_1}, \dots, v_{b_k}$ are linearly independent, so $\text{rank}(V) = k$ and we can interpret V as a point of $\text{Gr}(k, n)$. At this stage, we do not know if it belongs to $S_{\lambda/\mu}^\circ$, however, by Lemma 3.54 we get

$$\Xi \left(\tilde{\Omega}(V) \right) = V \text{ for } V \in S_{\lambda/\mu}^\circ. \quad (3.18)$$

Step 3: Given V as in (3.17), we can define the subspaces $V(a, i) = \text{span}(v_\alpha \mid \alpha \in J(a, i))$. We prove that $L(a, i) = V(a, i)$ by induction in i . For the base case, we can choose $i = \bar{\mu}_a + 1$ where $L(a, \bar{\mu}_a + 1) = W_{\bar{\mu}_a}^{\text{op}} + \text{span}(v_{a+\bar{\mu}_a}) = V(a, \bar{\mu}_a + 1)$ holds by construction. For the step of induction, assume that $\square_{a, i-1}$ and $\square_{a+1, i-1}$ are in λ/μ . From the braid diagram, we read $L(a, i-1) \subset L(a, i)$, $L(a+1, i-1) \subset L(a, i)$ and $L(a, i-1) \neq L(a+1, i-1)$, therefore $L(a, i) = L(a, i-1) + L(a+1, i-1)$. By the assumption of induction we have $L(a, i-1) = V(a, i-1)$ and $L(a+1, i-1) = V(a+1, i-1)$ and

$$L(a, i) = V(a, i-1) + V(a+1, i-1) = V(a, i).$$

The case when $\square_{a+1, i-1} \in \mu$ is treated similarly. Note that the same argument shows that $\dim V(a, i) = i$.

Step 4: By (3.18) we have $S_{\lambda/\mu}^\circ \subset \text{Im } \Xi$ and the corresponding inclusion of (Zariski) closures $\overline{S_{\lambda/\mu}^\circ} \subset \overline{\text{Im } \Xi}$. On the other hand, observe that

$$\dim X(\underline{w_0}\beta_{\lambda/\mu}) = \ell(\beta_{\lambda/\mu}) = |\lambda| - |\mu| - s, \quad \dim S_{\lambda/\mu}^\circ = |\lambda| - |\mu|$$

and both varieties are irreducible. Therefore $\overline{\text{Im } \Xi}$ is an irreducible closed subset of the Grassmannian $\text{Gr}(k, n)$ of dimension at most $|\lambda| - |\mu|$, hence $\overline{S_{\lambda/\mu}^\circ} = \overline{\text{Im } \Xi}$.

Step 5: Recall that by Definition 3.7 and Theorem 3.41 the skew shaped positroid $S_{\lambda/\mu}^\circ$ is characterized by the non-vanishing of the minors $\Delta_{I'(a,i)}$ for $\square_{a,i} \in R(\lambda/\mu)$, and vanishing of minors Δ_J with $J > I'(a,i)$ in the cyclic order $\leq_{a+i-1}$. The equations $\Delta_J(V) = 0$ hold on the closure $\overline{S_{\lambda/\mu}^\circ}$ and therefore for V as in (3.17) we get $\Delta_J(V) = 0$.

By Step 3, we have $V(d_i, i) = L(d_i, i)$ and by definition of $X(\underline{w_0}\beta_{\lambda/\mu})$ we have $L(d_i, i) \cap W_{k-i} = 0$. Therefore $V(d_i, i) \cap W_{k-i} = 0$ and $\Delta_{I'(d_i,i)}(V) \neq 0$. This implies $\Delta_{I'(a,i)}(V) \neq 0$ for $\square_{a,i} \in R(\lambda/\mu)$, and $\text{Im } \Xi = S_{\lambda/\mu}^\circ$.

Finally, by applying Step 3 again we conclude that $\tilde{\Omega}$ and Ξ are inverse to each other. $\square$

Example 3.57. Let us describe the map Ξ in our running example. Given the vectors $v_{b_1}, \dots, v_{b_k}$ and the subspaces $L(a, i)$ as in Figure 3.5, we know all vectors v_i right away except for v_7, v_9 and v_{10} . We also know that

$$L(5, 3) = \langle v_5, v_6, v_7 \rangle, \quad L(6, 4) = \langle v_5, v_6, v_8, v_9 \rangle = L(7, 4) = \langle v_5, v_6, v_8, v_{10} \rangle.$$

By Lemma 3.54, we can obtain the remaining vectors as intersections

$$\begin{aligned} v_7 &\in L(5, 3) \cap W_3 = \langle v_5, v_6, v_7 \rangle \cap \langle v_8, v_{11}, v_{12} \rangle, \\ v_9 &\in L(6, 4) \cap W_2 = \langle v_5, v_6, v_8, v_9 \rangle \cap \langle v_{11}, v_{12} \rangle, \\ v_{10} &\in L(7, 4) \cap W_2 = \langle v_5, v_6, v_8, v_{10} \rangle \cap \langle v_{11}, v_{12} \rangle. \end{aligned}$$

In fact, the vectors v_9 and v_{10} are proportional since $L(6, 4) = L(7, 4)$. To pin down the scaling factors, we first use the minor $\Delta_{5,6,8,10,11}$ to rescale v_{10} (since the vectors v_5, v_6, v_8, v_{11} are known), then $\Delta_{5,6,8,9,12}$ for v_9 and then $\Delta_{5,6,7,8,12}$ for v_7 .

Example 3.58. Alternatively, we can use Lemma 3.52 to reconstruct v_7, v_9 and v_{10} , but this turns out to be a more subtle and recursive process.

Let's start with v_{10} . Since $f(10) = 12$ by (3.12) we have $v_{10} \in \langle v_{11}, v_{12} \rangle$. On the other hand, clearly $v_{10} \in \langle v_5, v_6, v_8, v_{10} \rangle$. Observe that $\Delta_{I_\mu} = \Delta_{5,6,8,11,12} \neq 0$, so $\langle v_{11}, v_{12} \rangle + \langle v_5, v_6, v_8, v_{10} \rangle = \mathbb{C}^5$. Therefore the intersection $\langle v_{11}, v_{12} \rangle \cap \langle v_5, v_6, v_8, v_{10} \rangle$ is 1-dimensional and hence spanned by v_{10} .

Next, we determine v_9 . Since $f(9) = 10$, we get $v_9 \in \langle v_{10} \rangle$.

Finally, we determine v_7 . Since $f(7) = 9$, we get $v_7 \in \langle v_8, v_9 \rangle$ while clearly $v_7 \in \langle v_5, v_6, v_7 \rangle$. Note that $\dim \langle v_5, v_6, v_7, v_8 \rangle = 4$, so

$$\dim [\langle v_8, v_9 \rangle + \langle v_5, v_6, v_7 \rangle] \geq 4, \quad \dim [\langle v_8, v_9 \rangle \cap \langle v_5, v_6, v_7 \rangle] \leq 2 + 3 - 4 = 1.$$

Therefore the intersection is indeed one-dimensional and spanned by v_7 .

Remark 3.59. The flag $\mathcal{F}^0$ on the right boundary of $\Omega(V)$ constructed in Theorem 3.56 has the following concise description. Let

$$I_\lambda = \{d_i + i - 1 \mid i = 1, \dots, k\}$$

be the set of labels of the vertical steps in the λ -labeling. Then the j -dimensional subspace in the flag $\mathcal{F}^0$ is given by

$$\mathcal{F}_j^0 = V(d_j, j) = \text{span}\{v_{d_i+i-1} \mid i = 1, \dots, j\}. \quad (3.19)$$

This follows by induction from $V(d_j, j) = \langle v_{d_j+j-1} \rangle \oplus V(d_j, j-1)$ and Lemma 3.47. In particular, v_{d_i+i-1} form a basis of $\mathbb{C}^k$, and $\Delta_{I_\lambda} \neq 0$ on $S_{\lambda/\mu}^\circ$, in agreement with Remark 3.18. In our running example, $I_\lambda = \{1, 2, 5, 8, 11\}$ and we get

$$\mathcal{F}^0 = \{\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_5 \rangle \subset \langle v_1, v_2, v_5, v_8 \rangle \subset \langle v_1, v_2, v_5, v_8, v_{11} \rangle = \mathbb{C}^5\},$$

compare with Figure 3.5.

3.6. Skew shaped positroid as a braid variety on n strands

We have realized the skew shaped positroid variety $S_{\lambda/\mu}^\circ$ as a braid variety on k strands. It can also be realized as a braid variety on n strands, as follows. By definition, we have that the variety $S_{\lambda/\mu}^\circ$ is the open Richardson variety $R_{w_\lambda, w_\mu}^\circ$ inside the flag variety $\mathcal{F}\ell(n)$. Let $w_0^{(n)}$ denote the longest element in S_n . As shown in [CGG⁺25b, Theorem 3.14], we have an isomorphism

$$\begin{aligned} X \left(\beta(w_\mu) \beta(w_\lambda^{-1} w_0^{(n)}) \right) &\rightarrow R_{w_\lambda, w_\mu}^\circ \\ \left(\mathcal{F}^{\text{std}} \xrightarrow{\beta(w_\mu)} \mathcal{F}^r \xrightarrow{\beta(w_\lambda^{-1} w_0^{(n)})} \mathcal{F}^{\text{ant}} \right) &\mapsto \mathcal{F}^r. \end{aligned} \quad (3.20)$$

where $\beta(w_\mu)$ is a minimal braid lift of w_μ , and similarly for $\beta(w_\lambda^{-1} w_0^{(n)})$. Let us elaborate on this map, in particular, obtaining a $k \times n$ matrix V starting from the flag $\mathcal{F}^r$. The flag $\mathcal{F}^r$ belongs to the Schubert cell $B(n)w_\mu B(n)/B(n)$. Thus, we can find a matrix $M \in \text{GL}(n)$ such that $\mathcal{F}^r = \mathcal{F}(M)$ and $M = (m_{i,j})$ satisfies the following conditions:

- $m_{i, w_\mu(i)} = 1$ for every $i = 1, \dots, n$. These are the *pivots* of M .
- There are 0's below and to the right of every pivot of M .

Note that, since the permutation w_μ is k -Grassmannian, every non-pivot entry of the last $(n-k)$ -columns of M is zero. In other words, $M = (m_1 | m_2 | \dots | m_k | e_{w_\mu(k+1)} | \dots | e_{w_\mu(n)})$. The $k \times n$ -matrix associated to $\mathcal{F}^r$ is $(m_1 | \dots | m_k)^T$.

Now, by Lemma 3.15 we can find a decomposition $\beta(w_\mu) = \beta(w_{\lambda/\mu})\beta(w_\lambda)$, so that

$$\beta(w_\mu) \beta(w_\lambda^{-1} w_0^{(n)}) = \beta(w_{\lambda/\mu}) \cdot \beta(w_0^{(n)}). \quad (3.21)$$

We then have the diagram

$$\begin{array}{ccc}
 R_{e, w_{\lambda/\mu}}^{\circ} & \xleftarrow{\quad\quad\quad} X \left(\beta(w_{\lambda/\mu}) \beta(w_{\lambda}) \beta(w_{\lambda}^{-1} w_0^{(n)}) \right) & \xrightarrow{\quad\quad\quad} R_{w_{\lambda}, w_{\mu}}^{\circ} = S_{\lambda/\mu}^{\circ} \\
 \mathcal{F}^{r_1} & \xleftarrow{\quad\quad\quad} \left(\mathcal{F}^{\text{std}} \xrightarrow{\beta(w_{\lambda/\mu})} \mathcal{F}^{r_1} \xrightarrow{\beta(w_{\lambda})} \mathcal{F}^{r_2} \xrightarrow{\beta(w_{\lambda}^{-1} w_0^{(n)})} \mathcal{F}^{\text{ant}} \right) & \xrightarrow{\quad\quad\quad} \mathcal{F}^{r_2}.
 \end{array}$$

where both maps are isomorphisms.

Remark 3.60. Note that $\beta(w_{\lambda/\mu})$ is an n -strand braid while $\beta_{\lambda/\mu}$ is a k -strand braid, so these should not be confused.

4. Cluster structure on skew shaped positroids

In this section, we provide an initial seed construction for the skew shaped positroid following Galashin–Lam [GL23] and using the lattice path labelings as in Subsection 3.3. We first state the result in Section 4.2 and then give a proof in Section 4.3. The reader is invited to skip Section 4.3 at the first reading.

4.1. Background on cluster algebras

Let us now recall the definition of a cluster algebra [FZ02]. We restrict ourselves to the skew-symmetric case, as this is all that will be needed.

An *ice quiver* is a quiver (i.e., a directed graph, where we allow multiple arrows between vertices) with finite vertex set Q_0 that has no loops nor directed 2-cycles, and where a special subset Q_0^f of the vertices of Q is declared to be *frozen*. An element in $Q_0 \setminus Q_0^f$ is said to be a *mutable vertex* of Q .

Now let Q be an ice quiver, with $|Q_0| = n + m$, in such a way that there are n mutable vertices and m frozen ones. We consider a field $\mathcal{F}$ of transcendence degree $n + m$ over $\mathbb{C}$. A *seed* $\Sigma = (Q, \mathbf{x})$ consists of:

1. The ice quiver Q , and
2. A set $\mathbf{x} = \{x_i \mid i \in Q_0\}$ that is a transcendental basis for $\mathcal{F}$, i.e. $\mathcal{F} = \mathbb{C}(x_i \mid i \in Q_0)$.
The set $\mathbf{x}$ is known as the set of *cluster variables* of Σ .

Given a seed $\Sigma = (Q, \mathbf{x})$ and a mutable vertex $k \in Q_0$, the *mutation* of Σ in the direction k is the seed $\mu_k(\Sigma) = (\mu_k(Q), \mu_k(\mathbf{x}))$ where:

1. $\mu_k(\mathbf{x}) = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$, where $x'_k \in \mathcal{F}$ is defined via the equation

$$x_k x'_k = \prod_{i \rightarrow k} x_i + \prod_{k \rightarrow j} x_j.$$

2. $\mu_k(Q)$ has the same vertex set as Q , but the arrows change via the following three-step procedure:

- (a) Reverse all arrows incident with k .
- (b) For any pair of arrows $i \rightarrow k \rightarrow j$ in Q insert a new arrow $i \rightarrow j$, unless both i and j are frozen.
- (c) The previous two steps may have created 2-cycles. Remove all the arrows in a maximal collection of disjoint 2-cycles.

Two seeds Σ and Σ' are said to be *mutation equivalent* if there is a finite sequence of mutations taking one seed to the other. Since mutation at a fixed vertex k is an involutive operation, this is well-defined.

Definition 4.1. Let Σ be a seed. The *cluster algebra* $\mathcal{A}(\Sigma)$ is the subalgebra of the field $\mathcal{F}$ generated by the sets of cluster variables in all seeds mutation equivalent to Σ , as well as by x_k^{-1} for $k \in Q_0^f$. We say that a commutative algebra $\mathcal{A}$ *admits a cluster structure* if there exists a seed Σ such that $\mathcal{A} \cong \mathcal{A}(\Sigma)$. Similarly, we say that an affine algebraic variety X admits a cluster structure if the coordinate algebra $\mathbb{C}[X]$ admits a cluster structure.

Crucially for this paper, a commutative algebra $\mathcal{A}$ may admit more than one cluster structure, that is, there may exist two seeds Σ, Σ' which are not mutation equivalent but such that $\mathcal{A}(\Sigma) \cong \mathcal{A} \cong \mathcal{A}(\Sigma')$.

Let $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ be the cluster algebras associated to the seeds Σ and Σ' , respectively. Following Fraser [Fra16], see also [LS22, Section 5.2], we define a class of particularly well-behaved morphisms between cluster algebras associated to seeds that are not necessarily mutation-equivalent.

For a seed Σ and a mutable vertex i , we define the exchange ratio $\widehat{y}_i$ as the ratio

$$\widehat{y}_i = \frac{\prod_{j \rightarrow i} x_j^{\#\{j \rightarrow i\}}}{\prod_{i \rightarrow j} x_j^{\#\{i \rightarrow j\}}}.$$

Definition 4.2. [Fra16, FSB22] Let $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ be cluster algebras of rank $n + m$, both with m frozen variables. Let $\mathbf{x} = \{x_1, \dots, x_{n+m}\}$ be the cluster variables of Σ , and $\mathbf{x}' = \{x'_1, \dots, x'_{n+m}\}$ be the cluster variables of Σ' . An algebra isomorphism $f: \mathcal{A}(\Sigma) \rightarrow \mathcal{A}(\Sigma')$ is said to be a *quasi-cluster isomorphism* if the following conditions are satisfied:

1. For each frozen variable $x_j \in \mathbf{x}$, $f(x_j)$ is a Laurent monomial in the frozen variables of $\mathbf{x}'$.
2. For each mutable variable $x_i \in \mathbf{x}$, $f(x_i)$ coincides with x'_i , up to multiplication by a Laurent monomial in the frozen variables of $\mathbf{x}'$.
3. The exchange ratios are preserved, i.e., for each mutable variable x_i of Σ , $f(\widehat{y}_i) = \widehat{y}'_i$.

We remark that properties (1)–(3) are stable under mutations, i.e., if (1)–(3) holds for two seeds Σ, Σ' , then it also holds for $\mu_i(\Sigma), \mu_i(\Sigma')$ for every mutable vertex i , see [LS22, Section 5.2].

4.2. Cluster structure on skew shaped positroid: statement

The following statement is a consequence of the main result in [GL23] which we review in Section 4.3.

Theorem 4.3. *Consider a skew diagram λ/μ . The skew shaped positroid $S_{\lambda/\mu}^\circ$ admits a cluster structure, and an initial seed can be described as follows:*

- (a) **Cluster variables:** *For each box $\square_{a,i} \in \lambda/\mu$, we have a cluster variable $x_{a,i} := \Delta_{I'(a,i)}$. In particular, a variable $x_{a,i}$ is frozen if the box $\square_{a,i}$ belongs to the boundary ribbon $R(\lambda/\mu)$. Otherwise, a variable $x_{a,i}$ is mutable.*
- (b) **Quiver:** *The vertices of the quiver $Q_{\lambda/\mu}$ are in bijection with the boxes $\square_{a,i} \in \lambda/\mu$. There are three types of arrows:*

$$\square_{a,i} \rightarrow \square_{a+1,i}, \quad \square_{a,i} \rightarrow \square_{a,i-1}, \quad \square_{a,i} \rightarrow \square_{a-1,i+1}$$

provided at least one of these boxes corresponds to a mutable vertex.

Remark 4.4. In part of the literature, there is an extra frozen variable $x_\mu = \Delta_{I_\mu}$, and consequently the quiver $Q_{\lambda/\mu}$ has an extra frozen vertex. This yields a cluster structure on the affine cone over $S_{\lambda/\mu}^\circ$. We will work inside $\text{Gr}(k, n)$ (as opposed to its affine cone) by setting $\Delta_{I_\mu} = 1$, so we do not see this frozen variable, see Remark 3.42.

Recall the variety $S_{\lambda/\mu}^{\circ,1}$ is given by the conditions that $\Delta_{I'(a,i)} = 1$ for every box $\square_{a,i} \in R^1(\lambda/\mu)$. Note that these are all frozen variables for the cluster structure in $S_{\lambda/\mu}^\circ$ and we obtain the following result.

Corollary 4.5. *The skew shaped positroid $S_{\lambda/\mu}^{\circ,1}$ admits a cluster structure, obtained from the cluster structure on $S_{\lambda/\mu}^\circ$ from Theorem 4.3 by specializing the frozen variables $x_{a,i} = 1$, where $\square_{a,i} \in R^1(\lambda/\mu)$.*

Remark 4.6. We have $\Delta_{I_\lambda} \neq 0$ on $S_{\lambda/\mu}^\circ$, see Remark 3.59. By [GLS13, Theorem 1.3(i)] any invertible function on a cluster variety is equal to a monomial in frozen variables, and one may wonder how to write Δ_{I_λ} as such monomial explicitly.

To do this, we can consider the action of the torus $(\mathbb{C}^\times)^n$ on $\text{Gr}(k, n)$ which scales the column vectors v_i , and compare the weights of frozen variables and of Δ_{I_λ} . In our running example, we get

$$\Delta_{I_\lambda} = \Delta_{1,2,5,8,11} = \frac{\Delta_{1,2,8,11,12} \Delta_{3,4,5,11,12} \Delta_{5,6,7,8,12} \Delta_{5,6,8,10,11}}{\Delta_{3,4,8,11,12} \Delta_{5,6,7,11,12} \Delta_{5,6,8,10,12}} \quad \text{on } S_{\lambda/\mu}^\circ.$$

Note that the factors in the numerator correspond to the outside corners of the ribbon $R(\lambda/\mu)$ and the factors in the denominator correspond to the inside corners. We leave details to the reader.

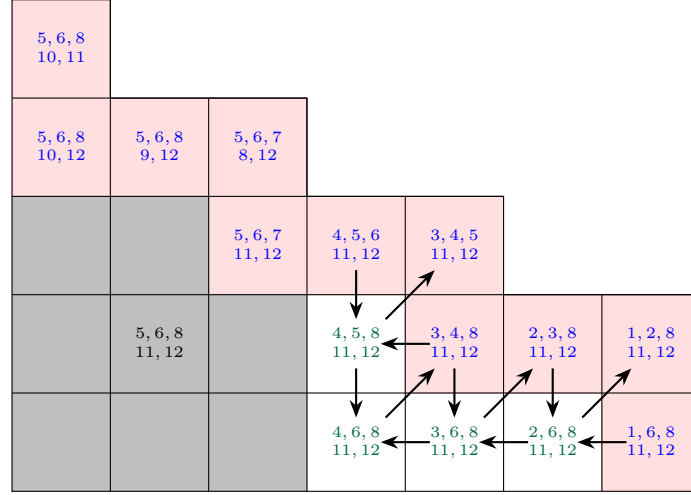


Figure 4.1: The quiver for $S_{\lambda/\mu}^\circ$ for our running example where $\lambda = (7, 7, 5, 3, 1)$, $\mu = (3, 3, 2)$ is shown superimposed on the skew diagram. The cluster variables are defined $x_{a,i} = \Delta_{I'(a,i)}$. Here we label the cluster variables only by the indices $I'(a, i)$. **Blue** indices correspond to frozen variables that arise from the boundary ribbon $R(\lambda/\mu)$, while **green** indices indicate the mutable vertices of the cluster seed.

4.3. Cluster structure on skew shaped positroid: proof

Cluster structures on positroid varieties are intimately related with Postnikov’s plabic graphs [Pos06] which are planar, bicolored graphs in a disk with n edges that touch the boundary of the disk (we call these boundary edges).

Given a plabic graph G , Postnikov defines in [Pos06, §13] the collection of *trips* as follows. First, we label the boundary edges of G from 1 to n . A trip starts at a boundary edge i and follows the “rules of the road” by turning right at each black vertex and left at each white vertex. A plabic graph is called *reduced* if the trips in it satisfy certain technical conditions in [Pos06, Theorem 13.2]. If G is reduced, the trip starting at i ends at another boundary edge labeled by $\tau_G(i)$, see Figure 4.2 (left) for examples. If G is reduced then it is known that τ_G is a permutation in S_n , it is called the *trip permutation* for G .

The connected components of the complement to G will be called regions (also known as faces). The **source labeling** of regions by subsets of $\{1, \dots, n\}$ is defined as follows: for each trip path starting with a boundary edge i , we label all the regions to the right of it by i . It is known that whenever G is reduced, each region acquires exactly k labels this way, where k is the number of counter-clockwise trips.

Let f be a bounded affine permutation. For the positroid variety $\Pi_f^\circ \subset \text{Gr}(k, n)$, consider the family of all reduced plabic graphs G with trip permutation $f \pmod{n}$.² A plabic graph G in this family determines a quiver Q_G with vertices corresponding to the regions of G , and cluster variables given by the Plücker coordinates indexed by the source labels of regions defined

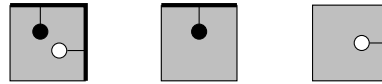
²For fixed points i of $f \pmod{n}$, we also need to carry the information of whether the trip permutation carries i to i clockwise or counterclockwise.

above. By the main result of [GL23], all these determine (possibly different) seeds for the cluster structure on Π_f° . We refer to [GL23] for more details.

To prove Theorem 4.3, we will produce a plabic graph with trip permutation $f \pmod n$ whose corresponding quiver and cluster variables coincide with those stated in Theorem 4.3.

Definition 4.7. Given $\mu \subseteq \lambda$, we define the plabic graph $G_{\lambda/\mu}$ as follows:

- (a) Replace each box in λ/μ with a box with a black-white edge, as in Figure 4.3 (left) and Figure 4.2 (left).
- (b) If $\square_{a,i}$ and $\square_{a-1,i}$ are both in λ/μ , join the white vertex of $\square_{a,i}$ to the black vertex of $\square_{a-1,i}$. Similarly, if $\square_{a,i}$ and $\square_{a,i-1}$ are both in λ/μ , join the white vertex of $\square_{a,i}$ to the black vertex of $\square_{a,i-1}$. See Figure 4.2 (left).
- (c) If a portion of the boundary of a box in λ/μ is contained in the northeast boundary of λ , we connect the corresponding vertices of $G_{\lambda/\mu}$ to the boundary, as in Figure 4.2.
- (d) If a portion of the boundary of a box in μ is contained in the northeast boundary of λ , we add new vertices and connect them to the boundary, as follows, where the thick sides are those belonging to the northeast boundary of λ :



We regard $G_{\lambda/\mu}$ as a subset of λ , which is homeomorphic to a disk. Below we will refer to the southwest boundary region R_{SW} which contains μ (except for vertices and edges in (d) above). Also note that $G_{\lambda/\mu}$ has exactly n boundary edges labeled from 1 to n as in Figure 4.2, and R_{SW} connects the boundary edges labeled by 1 and n .

Next, we introduce an equivalent model of $G_{\lambda/\mu}$ drawn on the lattice.

Definition 4.8. Given a skew diagram λ/μ where $I_\mu = \{b_1, \dots, b_k\}$, the *lattice source labeling trip* is given by the following:

1. Edge labeling:
 - The boundary edges e_i are labeled consecutively from 1 to n , starting from the southeast corner to the northwest corner.³
2. Construction of the lattice source labeling trip T_i :

The path T_i moves southwest until it reaches the boundary of μ at which point the direction of the path reverses to either the north or east.

³Note that this gives a *counterclockwise* enumeration of the boundary edges, that is not standard in the plabic graph literature. Had we drawn our Young diagrams in English notation, we would have obtained a clockwise orientation. We choose not to do so to keep in line with the rest of the paper.

(a) Initial turning rule:

- If an edge is horizontally oriented, then you make a left turn with respect to the orientation in the southern direction.
- If an edge is vertically oriented, then you make a right turn with respect to the orientation in the western direction.
- Continue until you reach the boundary of μ .

(b) Behavior at the boundary of μ :

- Upon reaching an edge that is contained in the boundary of μ , continue to follow the turning rule from step (a), until the next required step would be an edge contained in the interior of μ .
- At this point, reflect the direction of the new edge which would be contained in the interior of μ .
 - i. If the rule would require you to take the horizontal edge in the western direction, then instead choose the horizontal edge in the eastern direction.
 - ii. Similarly, if the rule would require you to take the vertical edge in the southern direction, then instead choose the vertical edge in the northern direction.

(c) Continuation of the path:

- Continue straight until you reach some edge e_j for $j \in [n] \setminus i$.

3. Box labeling:

- If the path T_i has a counterclockwise orientation, then label the boxes on the exterior of the path with the integer i , denote this set of boxes X_i .
- If the path T_i has a clockwise orientation, then label the boxes on the interior of the path with the integer i , denote this set of boxes N_i .

Lemma 4.9. *If the path T_i satisfies step (b)(i) in Definition 4.8, then $i \in I_\mu$.*

Proof. We can label all the edges between boxes in λ/μ by their (lattice) distance to the southeast corner. Specifically, we label the southern (horizontal) and the eastern (vertical) boundary of $\square_{a,j}$ by $a + j - 1$. Note that on the edges of the boundary path P_μ of λ/μ this labeling coincides with the one defined in Definition 3.24. Along the trip T_i we first move along the diagonal staircase (see Figure 4.2, right) where all edges have the same labels. In particular, the initial label i equals the label on the boundary of μ that the trip hits, and the result follows from the definition of I_μ . $\square$

Example 4.10. For our running example, the lattice trips T_4 and T_5 in λ/μ along with the lattice source labeling of 4 and 5, respectively, are shown on the right in Figure 4.2. The lattice trip T_4 is drawn in blue. Since the path T_4 is oriented clockwise, we label the boxes on the interior of the path with 4 and denote these boxes as N_4 . Similarly, T_5 is drawn in navy blue with counterclockwise orientation, so we label the boxes on the exterior of the path with 5, these boxes are denoted X_5 .

- Lemma 4.11.** (a) There is a bijection between the boxes in λ/μ and the regions in $G_{\lambda/\mu}$, except for the southwestern region R_{SW} .
- (b) There is a bijection between the trips in $G_{\lambda/\mu}$ and lattice trips T_i in λ/μ which agrees with the bijection for regions in (a).
- (c) Given a boundary edge i , the source labeling of regions for $G_{\lambda/\mu}$ and the lattice source labeling of boxes in λ/μ as in Definition 4.8 agree.
- (d) The southwestern region R_{SW} in $G_{\lambda/\mu}$ is labeled by I_μ .

Proof. Parts (a) and (b) are visually clear, see Figure 4.2. For each box in λ/μ there is a unique region for $G_{\lambda/\mu}$ containing its northeast corner. The steps in a lattice trip are obtained from the

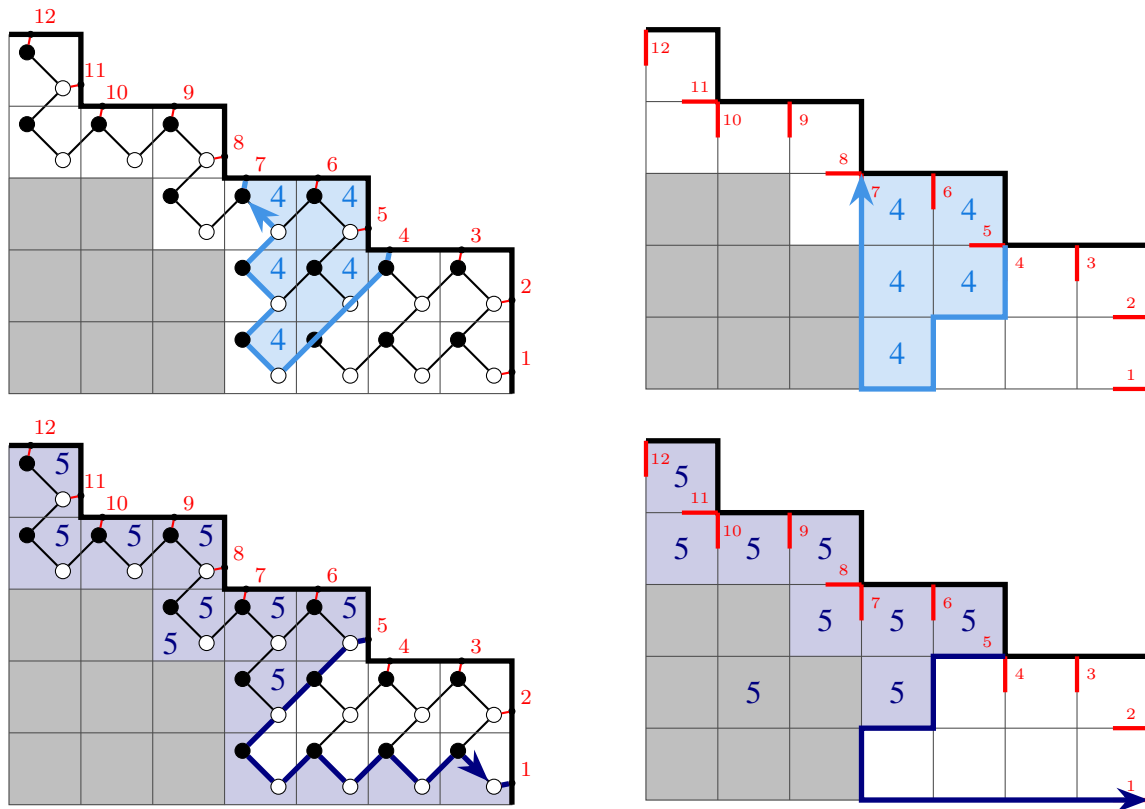


Figure 4.2: On the left, the plabic graph is shown superimposed on the skew diagram for λ/μ where $\lambda = (7, 7, 5, 3, 1)$ and $\mu = (3, 3, 2)$, with the path for T_4 highlighted in blue and the path for T_5 highlighted in navy blue. This path is determined using the standard “rules of the road” described in [Pos06], where the “driver” keeps to the right side of the path. When approaching a black dot, the driver continues straight, whereas approaching a white dot, the driver turns left. On the right, the lattice trip for T_4 and T_5 is depicted using lattice source labeling. The boundary edges e_i are shown in red on the right.

steps in the corresponding trip for $G_{\lambda/\mu}$ by a 45 degree rotation as in Figure 4.3. Part (c) is clear from (a) and (b).

For (d) we observe that R_{SW} is labeled by the sources of all counterclockwise trips in $G_{\lambda/\mu}$, while the counterclockwise lattice trips are labeled by the elements of I_μ , see Lemma 4.9. $\square$

Remark 4.12. A cautious reader might notice that the plabic graph $G_{\lambda/\mu}$ in the left of Figure 4.2 is not isomorphic to square grid (for example, most regions have 6 sides). Still, since we only use regions and their labels, we can use Lemma 4.11 to replace the data of $G_{\lambda/\mu}$ by its lattice analogue.



Figure 4.3: Identification between plabic graph on λ/μ with λ/μ . Note that the labeled edges a and b in the left-hand figure are compressed to the respective vertex point in the right-hand figure.

For the following result, we will need the notion of a decorated permutation, which consists of the data of a permutation $w \in S_n$ together with a decoration (either white or black) on its fixed points. It is easy to see that there is a bijection from the set of bounded affine permutations to that of decorated permutations, given by $f \mapsto f \pmod{n}$: the decoration keeps track of whether a fixed point of $f \pmod{n}$ satisfies $f(x) = x$ or $f(x) = x + n$. Each plabic graph defines a decorated trip permutation as in [Pos06, Section 13], whose underlying permutation coincides with the trip permutation of G .

Lemma 4.13. *The decorated trip permutation for the plabic graph $G_{\lambda/\mu}$ agrees with decorated trip permutation associated to the bounded affine permutation $f_{\lambda/\mu}$.*

Proof. First we verify the statement for the underlying permutations, i.e., we forget about the decorations. By Lemma 4.11 we can replace the trips in $G_{\lambda/\mu}$ by the lattice trips T_i . So it is sufficient to prove that $T = f_{\lambda/\mu} \pmod{n}$.

We follow the proof of Lemma 4.9 and label all the edges between boxes in λ/μ by their (lattice) distance to the southeast corner. Along the trip T_j , the initial label j equals the label on the boundary of μ that the trip hits. If we hit a vertical step of μ , we move horizontally, and if we hit a horizontal one we move vertically - this agrees with the definition of $f_{\lambda/\mu}$ in Figure 3.3. Therefore $T = f_{\lambda/\mu} \pmod{n}$.

To recover the decorations, note that the fixed points of $f_{\lambda/\mu} \pmod{n}$ correspond to special “lollipop” vertices in Definition 4.7(d). One can easily check that their decorations (as in [Pos06]) agree. $\square$

Corollary 4.14. *The positroid variety associated to the plabic graph $G_{\lambda/\mu}$ coincides with $S_{\lambda/\mu}^\circ$.*

Proof. One can check that the plabic graph $G_{\lambda/\mu}$ is reduced by verifying the conditions for trips in [Pos06, Theorem 13.2]. By [Pos06] the positroid variety associated to a reduced plabic graph is determined by the decorated trip permutation, and the result follows from Lemma 4.13. $\square$

Lemma 4.15. *For given λ/μ , the lattice source labeling at the square $\square_{a,i}$ in λ/μ coincides with $I'(a, i)$. The source labeling at μ is I_μ .*

Proof. In line with Lemma 4.9, we consider two cases for a trip T_l .

Case A: Let $l = b_m$ for some $1 \leq m \leq k$, i.e., let $l \in I_\mu$. Define $\square_{\hat{a}, \hat{i}}$ as the first box where its rightmost edge is in $P_\mu \cap T_l$. Since $l \in I_\mu$, the trip T_l has a counterclockwise orientation and we can group the boxes of X_l , determined in Definition 4.8 (3), into four types. The first type are boxes $\square_{a,i}$ directly above the diagonal path, denoted $\overline{X}_l$. Note that $\square_{a,i} \in \overline{X}_l$ then the i -th element (in increasing order) is b_m , since the vertical boundary of $\mu_{a,i}$ contains $\square_{a,i} \in \overline{X}_l$. The second type are boxes $\square_{a,i}$ above $\overline{X}_l$, where $a < \hat{a}$ and $i > \hat{i}$. For these boxes, we have $l \in I'(a, i)$ from Lemma 3.30. The third type are boxes $\square_{a,i} \in X_l$ such that $a \geq \hat{a}$. For these boxes, the first m terms of $I'(a, i)$ are $\{b_1, \dots, b_m\}$ since the boxes are above μ and after the l -th step in the path P_l , therefore, $l = b_m \in I'(a, i)$. The final type of boxes to consider in X_l are boxes $\square_{a,i}$ where $a < \hat{a}$ and $i < \hat{i}$. For these boxes, the last $k - m + 1$ terms in $I'(a, i)$ are given by $\{b_m, \dots, b_k\}$ since they are below $\square_{\hat{a}, \hat{i}}$ which gives the vertical step associated to $l = b_m$ and have that $l \in I'(a, i)$.

Case B: Let $l \neq b_m$ for some $1 \leq m \leq k$, i.e., let $l \notin I_\mu$. Then the trip T_l forms a clockwise-orientated path and the boxes are given by N_l , determined in Definition 4.8 (3). Here we have two types of boxes. The first type are squares $\square_{a,i} \in N_l$ that sit directly above the diagonal path formed from T_l which are denoted $\overline{N}_l$. Similar to Case A, we have that $\square_{a,i} \in \overline{N}_l$ if and only if the i -th term of $I'(a, i)$ (in increasing order) is l ; therefore, $l \in I'(a, i)$. The second type are boxes $\square_{a,i}$ above $\overline{N}_l$ up to the boundary of λ . For these boxes we use Lemma 3.30 to see that $l \in I'(a, i)$. $\square$

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. The part (b) is a consequence of Corollary 4.14; the quiver for $S_{\lambda/\mu}^\circ$ is the dual graph of $G_{\lambda/\mu}$ labeled by the source labeling (see [Sco06] and [Pos06, Section 14] for the arrow rules). Lemma 4.15 implies that the cluster variables from $G_{\lambda/\mu}$ by source labeling (or, equivalently, lattice source labeling) are $\Delta_{I'(a,i)}$, which proves the part (a). $\square$

4.4. Cluster variable comparison

From Theorem 3.56, the skew shaped positroid $S_{\lambda/\mu}^\circ$ is very closely related to the braid variety $X(w_0\beta_{\lambda/\mu})$, that also admits a cluster structure, cf. [CGG⁺25b, SW21]. In this section, we briefly compare the cluster variables in both constructions.

An initial seed for the cluster structure on $X(w_0\beta_{\lambda/\mu})$ is explicitly constructed in [SW21], see [CGG⁺25b, Proposition 5.20]. The cluster variables in $X(w_0\beta_{\lambda/\mu})$ are in bijection with the crossings of the braid $\beta_{\lambda/\mu}$. Let $\beta_{\lambda/\mu} = \sigma_{i_1} \cdots \sigma_{i_r}$. If the flag right after the j -th crossing of $\beta_{\lambda/\mu}$ is given by $(w_1 | \cdots | w_k)$ then, up to a sign, the corresponding cluster variable is the i_j -th principal minor of the matrix $(w_1 | \cdots | w_k)$. Note that this coincides with the determinant of $(w_1 | \cdots | w_{i_j} | e_{i_j+1} | \cdots | e_k)$, where $e_1, \dots, e_k$ is the standard basis of $\mathbb{C}^k$.

If $(v_1 | \dots | v_n)$ is a matrix representing an element of $S_{\lambda/\mu}^\circ$, we may normalize so that $v_{b_i} = e_i$, where $I_\mu = \{b_1 < \dots < b_k\}$. Recall that, for a box $\square_{a,i} \in \lambda/\mu$, the corresponding cluster variable is $\Delta_{I'(a,i)}$. From Proposition 3.28, $I'(a,i) = J(a,i) \cup \{b_{i+1}, \dots, b_k\}$. Thus,

$$\Delta_{I'(a,i)}(V) = \det(v_{j_1} | \dots | v_{j_i} | v_{b_{i+1}} | \dots | v_{b_k}) = \det(v_{j_1} | \dots | v_{j_i} | e_{i+1} | \dots | e_k).$$

where $J(a,i) = \{j_1, \dots, j_i\}$. Now, the space $V(a,i)$ is spanned by the vectors $j_1, \dots, j_i$, see Definition 3.44. Since a braid box $\square_{a,i} \in \lambda/\mu$ corresponds to a crossing between the i -th and $(i+1)$ -st strands of $\beta_{\lambda/\mu}$ we see that, upon setting the cluster variables corresponding to non-braid boxes equal to 1, the cluster variables in the seed of $S_{\lambda/\mu}^\circ$ given by Theorem 4.3 coincide, up to signs, with the cluster variables in the seed of $X(\underline{w_0}\beta_{\lambda/\mu})$ obtained in [SW21].

Moreover, it follows by construction that, upon deleting the frozen vertices corresponding to non-braid boxes of λ/μ , the quiver $Q_{\lambda/\mu}$ coincides with the *opposite* of the quiver $Q_{X(\underline{w_0}\beta_{\lambda/\mu})}$. Summarizing:

Theorem 4.16. *Consider the isomorphism $\tilde{\Omega}^1 : S_{\lambda/\mu}^{\circ,1} \rightarrow X(\underline{w_0}\beta_{\lambda/\mu})$. The dual map $\tilde{\Omega}^{1,*} : \mathbb{C}[X(\underline{w_0}\beta_{\lambda/\mu})] \rightarrow \mathbb{C}[S_{\lambda/\mu}^{\circ,1}]$ sends, up to signs, the cluster variables in the initial seed of $X(\underline{w_0}\beta_{\lambda/\mu})$ from [SW21] to the cluster variables in the initial seed of $S_{\lambda/\mu}^{\circ,1}$ given by Corollary 4.5.*

5. Splicing for skew shaped positroids

In this section, we generalize the splicing map introduced in [GS26] to a splicing map for $S_{\lambda/\mu}^\circ$ and investigate its cluster algebraic interpretation.

5.1. Freezing a column

Let $S_{\lambda/\mu}^\circ$ be the skew shaped positroid in the Grassmannian $\text{Gr}(k, n)$.

Definition 5.1. Given $1 \leq a \leq n - k$, we define the open subset $U_a \subset S_{\lambda/\mu}^\circ$ by the inequalities

$$U_a := \{V \in S_{\lambda/\mu}^\circ : \Delta_{I'(a,i)} \neq 0, \text{ for } \bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a\}. \quad (5.1)$$

Remark 5.2. Alternatively, we can define U_a by non-vanishing of the product of all minors in (5.1):

$$U_a := \left\{ V \in S_{\lambda/\mu}^\circ : \prod_{\bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a} \Delta_{I'(a,i)} \neq 0 \right\}.$$

Therefore U_a is a principal open subset defined by non-vanishing of a single function.

Given a skew diagram λ/μ , we cut it along the boundary of the $(a-1)$ -st and a -th column (from the right) where the $(a-1)$ -st column is contained in the right part of the diagram and the a -th column is contained in the left part of the diagram. We get two skew diagrams, $\lambda^{a,L}/\mu^{a,L}$ on the left and $\lambda^{a,R}/\mu^{a,R}$ on the right.

Remark 5.3. We can explicitly determine the resulting λ and μ for the new diagrams using

$$\begin{aligned}\lambda_i^{a,L} &= \begin{cases} n - k - a + 1, & \text{if } \lambda_i > n - k - a, \\ \lambda_i, & \text{otherwise} \end{cases} \\ \mu_i^{a,L} &= \begin{cases} n + k + a + 1, & \text{if } \mu_i > n - k - a \\ \mu_i, & \text{otherwise} \end{cases} \\ \lambda_i^{a,R} &= \begin{cases} \lambda_i - n + k + a - 1, & \text{if } \lambda_i > n - k - a \\ 0, & \text{otherwise} \end{cases} \\ \mu_i^{a,R} &= \begin{cases} \mu_i - n + k + a - 1, & \text{if } \mu_i > n - k - a \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

It is clear from definitions that the corresponding braids satisfy

$$\beta_{\lambda/\mu} = \beta_L \beta_R, \text{ where } \beta_L = \beta_{\lambda^{a,L}/\mu^{a,L}}, \beta_R = \beta_{\lambda^{a,R}/\mu^{a,R}}.$$

Definition 5.4. Given $1 \leq a \leq n - k$, we define the **flag at the cut** $\mathcal{F}^a$ as the following sequence of subspaces:

$$W_1^{\text{op}} \subset \cdots \subset W_{\bar{\mu}_a}^{\text{op}} \subset V(a, \bar{\mu}_a + 1) \subset \cdots \subset V(a, \bar{\lambda}_a) \subset V(d_{\bar{\lambda}_a+1}, \bar{\lambda}_a + 1) \subset \cdots \subset V(d_k, k) \quad (5.2)$$

where $(d_{\bar{\lambda}_a+1}, \bar{\lambda}_a + 1), \dots, (d_k, k)$ are the rightmost boxes in rows $\bar{\lambda}_a + 1, \dots, k$ as in (3.3).

One can check (see Example 5.6 below) that $\mathcal{F}^a$ is the flag corresponding to the boundary between β_L and β_R under the map in Theorem 3.56. In particular, (5.2) indeed defines a complete flag in $\mathbb{C}^k$.

Lemma 5.5. We have $V \in U_a$ if and only if the flag $\mathcal{F}^a$ is transversal to the flag $\mathcal{F}^W$.

Proof. The two flags are transversal if, for all i , we have

$$\mathcal{F}_i^a + W_{k-i} = \mathcal{F}_i^a + \langle v_{b_{j+1}}, \dots, v_{b_k} \rangle = \mathbb{C}^k.$$

We have the following cases:

Case 1: For $1 \leq i \leq \bar{\mu}_a$, we have $\langle v_{b_1}, \dots, v_{b_i} \rangle + \langle v_{b_{i+1}}, \dots, v_{b_k} \rangle = \mathbb{C}^k$, this is automatic.

Case 2: For $\bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a$, we have $I'(a, i) = J(a, i) \cup \{b_{i+1}, \dots, b_k\}$ and $V(a, i) = \text{span } J(a, i)$. Therefore $V(a, i) + W_{k-i} = \mathbb{C}^k$ if and only if $\Delta_{I'(a, i)} \neq 0$, these are precisely the conditions defining U_a .

Case 3: For $\bar{\lambda}_a + 1 \leq i \leq k$, the box $\square_{d_i, i}$ belongs to the boundary ribbon $R(\lambda/\mu)$, so $\Delta_{I'(d_i, i)} \neq 0$ and similarly to Case 2 we have $V(d_i, i) + W_{k-i} = \mathbb{C}^k$. $\square$

Example 5.6. In our running example, choose $a = 6$. The subset U_a is given by one condition $\Delta_{5,6,8,9,12} \neq 0$. The flag $\mathcal{F}^a$ is given by

$$\langle v_5 \rangle \subset \langle v_5, v_6 \rangle \subset \langle v_5, v_6, v_8 \rangle \subset \langle v_5, v_6, v_8, v_9 \rangle \subset \langle v_5, v_6, v_8, v_{10}, v_{11} \rangle,$$

while the flag $\mathcal{F}^W$ is given by

$$\langle v_{12} \rangle \subset \langle v_{12}, v_{11} \rangle \subset \langle v_{12}, v_{11}, v_8 \rangle \subset \langle v_{12}, v_{11}, v_8, v_6 \rangle \subset \langle v_{12}, v_{11}, v_8, v_6, v_5 \rangle.$$

Note that

$$\langle v_5 \rangle + \langle v_{12}, v_{11}, v_8, v_6 \rangle = \langle v_5, v_6 \rangle + \langle v_{12}, v_{11}, v_8 \rangle = \langle v_5, v_6, v_8 \rangle + \langle v_{12}, v_{11} \rangle = \mathbb{C}^5$$

since $\Delta_{5,6,8,11,12} \neq 0$. Furthermore, $\langle v_5, v_6, v_8, v_9 \rangle + \langle v_{12} \rangle = \mathbb{C}^5$ if and only if $\Delta_{5,6,8,9,12} \neq 0$ which is the defining condition for U_a . Finally, $\langle v_5, v_6, v_8, v_{10}, v_{11} \rangle = \mathbb{C}^5$ since $\Delta_{5,6,8,10,11} \neq 0$ on $S_{\lambda/\mu}^\circ$.

Next, we describe the cluster structure on U_a . Since it is a principal open subset in $S_{\lambda/\mu}^\circ$ (see Remark 5.2), we can consider U_a as an affine algebraic variety with the ring of regular functions obtained by localization:

$$\mathbb{C}[U_a] = \mathbb{C}[S_{\lambda/\mu}^\circ] \left[\Delta_{I'(a,i)}^{-1} \mid \bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a \right] \quad (5.3)$$

Since $\Delta_{I'(a,i)}$ are cluster variables in one seed, U_a is a cluster chart in the sense of Muller [Mul13]. More precisely, we have the following.

Proposition 5.7. *The open subset U_a is a cluster variety. The corresponding quiver Q' is obtained from $Q_{\lambda/\mu}$ by freezing all the vertices $\square_{a,i}$ with $\bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a$.*

Proof. The result follows from [Mul13] and we refer to that paper for more definitions and details.

Let $\mathcal{U}$ denote the upper cluster algebra for $Q_{\lambda/\mu}$, and $\mathcal{U}'$ denote the upper cluster algebra for Q' . Let $\mathcal{A}, \mathcal{A}'$ denote the respective cluster algebras.

The quiver $Q_{\lambda/\mu}$ is known to be locally acyclic by [MS16, Theorem 3.3], so by [Mul13, Theorem 4.1] we get

$$\mathcal{A} = \mathcal{U} = \mathbb{C}[S_{\lambda/\mu}^\circ].$$

The mutable part of the quiver Q' is the disjoint union of mutable parts of $Q_{\lambda^{a,L}/\mu^{a,L}}$ and $Q_{\lambda^{a,R}/\mu^{a,R}}$, hence by [Mul13, Lemma 3.10] Q' is also locally acyclic and $\mathcal{A}' = \mathcal{U}'$. On the other hand, by [Mul13, Theorem 3.1] we get inclusions

$$\mathcal{A}' \subset \mathcal{A} \left[\Delta_{I'(a,i)}^{-1} \mid \bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a \right] \subset \mathcal{U} \left[\Delta_{I'(a,i)}^{-1} \mid \bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a \right] \subset \mathcal{U}',$$

therefore by (5.3) we get

$$\mathcal{A}' = \mathcal{U}' = \mathbb{C}[U_a]. \quad \square$$

5.2. Splicing: definition

Recall Theorem 3.56 which relates the skew shaped positroid $S_{\lambda/\mu}^\circ$ to the braid variety $X(\underline{w_0}\beta_{\lambda/\mu})$ via the map Ω . Given $V \in S_{\lambda/\mu}^\circ$, we can draw the diagram (3.16) as follows:

$$\Omega(V) = \left[\mathcal{F}^W \xrightarrow{w_0} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_L} \mathcal{F}^a \xrightarrow{\beta_R} \mathcal{F}^0 \right]. \quad (5.4)$$

By Lemma 5.5, we have $V \in U_a$ if and only if $\mathcal{F}^a \pitchfork \mathcal{F}^W$ in $\Omega(V)$.

Definition 5.8. We define the *splicing map*

$$\Phi_a^{\text{flag}} : U_a \rightarrow X(\underline{w_0}\beta_L) \times X(\underline{w_0}\beta_R), \quad \Phi_a^{\text{flag}}(\Omega(V)) = (\Omega^L(V), \Omega^R(V))$$

as follows:

$$\Omega^L(V) = \left[\mathcal{F}^W \xrightarrow{w_0} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_L} \mathcal{F}^a \right], \quad \Omega^R(V) = \left[\mathcal{F}^W \xrightarrow{w_0} \mathcal{F}^a \xrightarrow{\beta_R} \mathcal{F}^0 \right]. \quad (5.5)$$

Note that the condition $\mathcal{F}^a \pitchfork \mathcal{F}^W$ ensures that $\Omega^L(V)$ and $\Omega^R(V)$ yield well defined points in the respective braid varieties, and the framings on the flag $\mathcal{F}^W$ in $\Omega(V)$, $\Omega^L(V)$ and $\Omega^R(V)$ agree.

Remark 5.9. The diagrams (5.5) are defined in terms of unframed flags. We can upgrade them to framed flags using the standard framing on $\mathcal{F}^W$ on the left, which can be propagated to the right using Remark 2.3.

Note that the framings for all the flags between $\mathcal{F}^{W^{\text{op}}}$ and $\mathcal{F}^a$ in $\Omega(V)$ and $\Omega^L(V)$ agree. However, Ω^R induces a possibly different framing on $\mathcal{F}^a$ and on all flags between $\mathcal{F}^a$ and $\mathcal{F}^0$.

Below we use Theorem 3.56 again to reinterpret Φ_a^{flag} as a map

$$\Phi_a : U_a \rightarrow S_{\lambda^a, L/\mu^a, L}^\circ \times S_{\lambda^a, R/\mu^a, R}^\circ,$$

and explore its cluster-theoretic properties. More precisely, we get the commutative diagram:

$$\begin{array}{ccc} U_a & \xrightarrow{\Phi_a} & S_{\lambda^a, L/\mu^a, L}^\circ \times S_{\lambda^a, R/\mu^a, R}^\circ \\ \downarrow \Omega & & \downarrow (\Omega, \Omega) \\ X(\underline{w_0}\beta_{\lambda/\mu}) & \xrightarrow{\Phi_a^{\text{flag}}} & X(\underline{w_0}\beta_L) \times X(\underline{w_0}\beta_R). \end{array} \quad (5.6)$$

If $\Phi_a(V) = (V^L, V^R)$ then $\Omega(V^L) = \Omega^L$, $\Omega(V^R) = \Omega^R$.

5.3. Splicing: Left diagram

We have

$$\Omega^L(V) = \left[\mathcal{F}^W \xrightarrow{w_0} \mathcal{F}^{W^{\text{op}}} \xrightarrow{\beta_L} \mathcal{F}^a \right],$$

in particular the pair of transversal flags $\mathcal{F}^{W^{\text{op}}} \pitchfork \mathcal{F}^W$ still corresponds to the basis $v_{b_1}, \dots, v_{b_k}$. We need to understand what happens to subspaces $V(a, i)$.

Lemma 5.10. Assume that $a \leq a' \leq n - k$, then the long labels $I'(a', i)$ starts with $b_1, \dots, b_{\bar{\mu}_a}$. Furthermore,

$$I'_L(a', i) = [I'(a', i) \setminus \{b_1, \dots, b_{\bar{\mu}_a}\} - a + 1] \cup \{1, \dots, \bar{\mu}_a\}, \quad (5.7)$$

$$I_{\mu^L} = [I_{\mu} \setminus \{b_1, \dots, b_{\bar{\mu}_a}\} - a + 1] \cup \{1, \dots, \bar{\mu}_a\}. \quad (5.8)$$

The notation $I - a$ defines the set obtained from shifting the elements of I by $-a$.

Proof. For $a + 1 \leq a' \leq n - k$ the path $P_{a', i}$ starts with the boundary of μ contained to the right of the cut, with vertical steps $b_1, \dots, b_{\bar{\mu}_a}$. The similar path $P_{a', i}^L$ in the left diagram starts with $\bar{\mu}_a$ vertical steps labeled by $1, \dots, \bar{\mu}_a$ instead. The remainders of the paths coincide, but the labels are shifted by $a - 1$ which is the number of horizontal steps in $P_{a', i}$ before the cut. $\square$

Definition 5.11. Given $V = (v_1 | \dots | v_n)$ we define $V^L = (w_1 | \dots | w_{n-a+1}) \in \text{Gr}(k, n - a + 1)$ where

$$w_1 = v_{b_1}, \dots, w_{\bar{\mu}_a} = v_{b_{\bar{\mu}_a}} \quad \text{and} \quad w_j = v_{j+a-1}, \quad \bar{\mu}_a < j \leq n - a + 1. \quad (5.9)$$

Corollary 5.12. The minors for V^L and V in the left diagram $\lambda^{a,L} / \mu^{a,L}$ coincide after relabeling. In particular, V^L corresponds to a well-defined point in the skew shaped positroid $S_{\lambda^{a,L} / \mu^{a,L}}^\circ$.

Example 5.13. In our running example, choose $a = 6$ and get the following left diagram

12356	
12357	12347
	12367

which defines a skew shaped positroid in $\text{Gr}(5, 7)$. If we denote

$$\mathbf{w}_1 = \mathbf{v}_5, \mathbf{w}_2 = \mathbf{v}_6, \mathbf{w}_3 = \mathbf{v}_8, w_4 = v_9, w_5 = v_{10}, w_6 = v_{11}, w_7 = v_{12},$$

then the labels would match the ones in the left side of Example 3.26. Note that $\bar{\mu}_a = 3$ and $(b_1, b_2, b_3) = (5, 6, 8)$.

Remark 5.14. If $\bar{\mu}_a = 0$, so that the cut is performed to the right of μ , we get $V^L = (w_1 | \dots | w_{n-a+1}) = (v_a | \dots | v_n)$ as in [GS26].

Remark 5.15. We have $\Delta_{I_{\mu^L}}(V^L) = \Delta_{I_{\mu}}(V) = 1$, which is consistent with Remark 3.42.

5.4. Splicing: Right diagram

We have

$$\Omega^R(V) = \left[\mathcal{F}^W \xrightarrow{w_0} \mathcal{F}^a \xrightarrow{\beta_R} \mathcal{F}^0 \right].$$

We define a point $V^R = (u_1 | \cdots | u_{k+a-1})$ in the skew shaped positroid $S_{\lambda^a, R / \mu^a, R}^\circ \subset \text{Gr}(k, a+k-1)$ in two steps. Observe that

$$I_{\mu^R} = \{b_1^R, \dots, b_k^R\} = \{b_1, \dots, b_{\bar{\mu}_a}, a + \bar{\mu}_a, \dots, a + k - 1\}.$$

As in Lemma 2.2, we can obtain the basis $u_{b_1}, \dots, u_{b_{\bar{\mu}_a}}, u_{a+\bar{\mu}_a}, \dots, u_{a+k-1}$ by intersecting two transverse flags $\mathcal{F}^a$ and $\mathcal{F}^W$.

Definition 5.16. For $i = 1, \dots, k$, we define $u_{b_i^R}$ as the unique vector such that

$$u_{b_i^R} \in \mathcal{F}_i^a \cap W_{k-i+1}$$

and

$$u_{b_i^R} \in v_{b_i} + \langle v_{b_{i+1}}, \dots, v_{b_k} \rangle, \quad i = 1, \dots, k. \quad (5.10)$$

The condition (5.10) ensures that v_{b_i} and $u_{b_i^R}$ induce the same framing on the flag $\mathcal{F}^W$. Note that we get

$$\begin{aligned} u_{b_1} &= v_{b_1}, \dots, u_{b_{\bar{\mu}_a}} = v_{b_{\bar{\mu}_a}}, \\ u_{a+i-1} &\in V(a, i) \cap W_{k-i+1}, \quad \bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a \end{aligned}$$

and

$$u_{a+i-1} \in V(d_i, i) \cap W_{k-i+1}, \quad \bar{\lambda}_a + 1 \leq i \leq k.$$

Lemma 5.17. Suppose that $\bar{\mu}_a + 1 \leq i \leq \bar{\lambda}_a$. Then

$$u_{a+i-1} \in \begin{cases} \frac{\Delta_{I'(a, i-1)}}{\Delta_{I'(a, i)}} v_{a+i-1} + V(a, i-1) & \text{if } i > \bar{\mu}_a + 1, \\ \frac{\Delta_{I_\mu}}{\Delta_{I'(a, i)}} v_{a+i-1} + \langle v_{b_1}, \dots, v_{b_{\bar{\mu}_a}} \rangle & \text{if } i = \bar{\mu}_a + 1. \end{cases} \quad (5.11)$$

Proof. Assume $i > \bar{\mu}_a + 1$ first. Recall that $V \in U_a$, so $\Delta_{I'(a, i)}(V) \neq 0$ and the subset $I'(a, i) = J(a, i) \cup \{b_{i+1}, \dots, b_k\}$ labels a basis in $\mathbb{C}^k$. If we expand v_{b_i} in this basis, we get

$$v_{b_i} = \sum_{\alpha \in J(a, i)} c_\alpha v_\alpha + c_{b_{i+1}} v_{b_{i+1}} + \cdots + c_{b_k} v_{b_k}, \quad (5.12)$$

and

$$u_{a+i-1} = v_{b_i} - c_{b_{i+1}} v_{b_{i+1}} - \cdots - c_{b_k} v_{b_k} = \sum_{\alpha \in J(a, i)} c_\alpha v_\alpha.$$

Recall that $a+i-1 \in J(a, i)$. By Cramer's Rule the coefficient c_{a+i-1} at v_{a+i-1} in (5.12) equals

$$\frac{\Delta_{I'(a, i) \cup b_i \setminus (a+i-1)}}{\Delta_{I'(a, i)}} = \frac{\Delta_{I'(a, i-1)}}{\Delta_{I'(a, i)}}.$$

The case $i = \bar{\mu}_a + 1$ is almost identical, but in this case $I'(a, i) \cup b_i \setminus (a+i-1) = I_\mu$. $\square$

Example 5.18. Following Example 5.6, we get

$$u_5 = v_5, u_6 = v_6, u_8 = v_8, u_9 \in V(6, 4) \cap W_2 = \langle v_5, v_6, v_8, v_9 \rangle \cap \langle v_{11}, v_{12} \rangle, u_{10} = v_{12}.$$

By Lemma 3.54 we get that u_9 is proportional to v_9 . By (5.11) we get

$$u_9 = \frac{\Delta_{5,6,8,11,12}}{\Delta_{5,6,8,9,12}} v_9.$$

Equation (5.10) immediately implies the following.

Lemma 5.19. *For all $1 \leq i \leq k$ we have $u_{b_i^R} \wedge \cdots \wedge u_{b_k^R} = v_{b_i} \wedge \cdots \wedge v_{b_k}$. In particular,*

$$\Delta_{I_{\mu^R}}(V^R) = \Delta_{I_\mu}(V) = 1.$$

Let us reconstruct the rest of the vectors u_i , following the proof of Theorem 3.56. Recall that all u 's not described by Definition 5.16 can be written as $u_{a'+\bar{\mu}_{a'}}$ for $1 \leq a' \leq a-1$.

Lemma 5.20. *We can choose $u_{a'+\bar{\mu}_{a'}} = v_{a'+\bar{\mu}_{a'}}$ for all a' . This is a unique (up to scalar) basic vector in the intersection $V(a', \bar{\mu}_{a'} + 1) \cap \langle u_{b_{\bar{\mu}_{a'}+1}^R}, \dots, u_{b_k^R} \rangle$.*

Proof. By (5.10) we get

$$\langle u_{b_{\bar{\mu}_{a'}+1}^R}, \dots, u_{b_k^R} \rangle = \langle v_{b_{\bar{\mu}_{a'}+1}}, \dots, v_{b_k} \rangle = W_{k-i}.$$

Now the statement follows from Lemma 3.54. □

Next, we need to compute the minors in the right diagram.

Lemma 5.21. *Let $1 \leq a' \leq a-1$. Then*

$$\Delta_{I'_R(a', i)}(V^R) = \Delta_{I'(a', i)}(V) A_{a'+\bar{\mu}_{a'}} \cdots A_{a'+i-1},$$

where the scalars A_t are defined by (5.14) below.

Proof. By (5.11) and Lemma 5.20 the basis $\{u_t\}$ is triangular with respect to $\{v_t\}$, up to some relabeling. More precisely, for $a' + \bar{\mu}_{a'} \leq t \leq a' + i - 1$ we have

$$u_t \in A_t v_t + \text{span}(v_{b_j} \mid b_j < t) + \text{span}(v_s \mid \bar{\mu}_a < s < t). \quad (5.13)$$

where the leading coefficient A_t is given by

$$A_t = \begin{cases} \Delta_{I'(a, t-a)} / \Delta_{I'(a, t-a+1)} & \text{if } t \in \{a + \bar{\mu}_a, \dots, a + \bar{\lambda}_a - 1\}, \\ 1 & \text{otherwise.} \end{cases} \quad (5.14)$$

We claim that for all $t \in I'(a', i)$ all lower degree terms in the expansion (5.13) have indices which are also contained in $I'(a', i)$. This would ensure that the minors $\Delta_{I'_R(a', i)}(V^R)$ and $\Delta_{I'(a', i)}(V)$ are proportional. Indeed, we have

$$I'_R(a', i) = J(a', i) \cup \{b_{i+1}^R, \dots, b_k^R\} = \{b_1, \dots, b_{\bar{\mu}_{a'}}\} \cup \{a' + \bar{\mu}_{a'}, \dots, a' + i - 1\} \cup \{b_{i+1}^R, \dots, b_k^R\},$$

while

$$I'(a', i) = J(a', i) \cup \{b_{i+1}, \dots, b_k\} = \{b_1, \dots, b_{\bar{\mu}_{a'}}\} \cup \{a' + \bar{\mu}_{a'}, \dots, a' + i - 1\} \cup \{b_{i+1}, \dots, b_k\}.$$

Therefore

$$u_{b_1} \wedge \dots \wedge u_{b_{\bar{\mu}_{a'}}} \wedge u_{a' + \bar{\mu}_{a'}} \wedge \dots \wedge u_{a' + i - 1} = (1 \dots 1 \cdot A_{a' + \bar{\mu}_{a'}} \dots A_{a' + i - 1}) \times \\ v_{b_1} \wedge \dots \wedge v_{b_{\bar{\mu}_{a'}}} \wedge v_{a' + \bar{\mu}_{a'}} \wedge \dots \wedge v_{a' + i - 1},$$

Finally, by Lemma 5.19 we get

$$u_{b_{i+1}}^R \wedge \dots \wedge u_{b_k}^R = v_{b_{i+1}} \wedge \dots \wedge v_{b_k},$$

and the result follows. $\square$

Next, we consider the quiver Q for the cluster structure on $S_{\lambda/\mu}^\circ$ and the quiver Q^R for the cluster structure on $S_{\lambda_R/\mu_R}^\circ$. Clearly, Q^R is a subgraph of Q . We denote by $\hat{y}(a', i)$ and $\hat{y}^R(a', i)$ the exchange ratios at the vertices $I'(a', i)$ and $I'_R(a', i)$ respectively.

Lemma 5.22. Assume that $a' < a - 1$ and $I'_R(a', i)$ is a mutable box of Q^R . Then the exchange ratios at (a', i) agree: $\hat{y}(a', i) = \hat{y}^R(a', i)$.

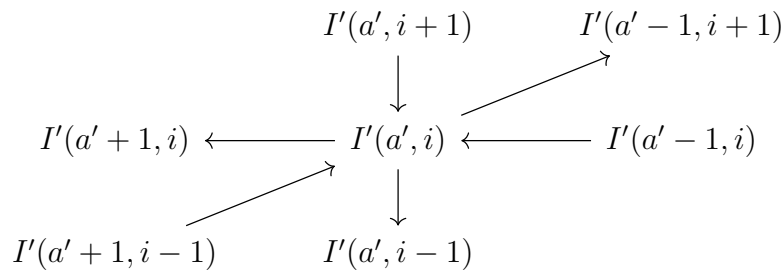
Proof. Let us denote the factor in Lemma 5.21 by

$$A(a', i) = A_{a' + \bar{\mu}_{a'}} \dots A_{a' + i - 1}.$$

Note that $A(a', i + 1) = A(a', i) \cdot A_{a' + i}$. Now we need to compute the exchange ratios for a mutable box $\square_{a', i}$. Since it is mutable, it is not in the boundary ribbon and $\square_{a' - 1, i}$, $\square_{a' - 1, i + 1}$, $\square_{a' + 1, i + 1}$ are all in λ_R/μ_R . For the other neighbors of $\square_{a', i}$, we make the following observations:

- (a) Either the box $\square_{a', i - 1}$ is in λ_R/μ_R , or it is in μ_R . In the latter case $i = \bar{\mu}_{a'} + 1$ and $\square_{a' + 1, i - 1}$ is also in μ .
- (b) If $a' < a - 1$, then either the box $\square_{a' + 1, i}$ is in λ_R/μ_R , or it is in μ_R . In the latter case $\square_{a' + 1, i - 1}$ is also in μ , and we have $b_i = a' + i$ (since this is the vertical step at the boundary of μ at height i).

Case 1 : All six neighbors of $\square_{a', i}$ are present:



The exchange ratio gets multiplied by

$$\frac{A(a' - 1, i)}{A(a' - 1, i + 1)} \cdot \frac{A(a', i + 1)}{A(a', i - 1)} \cdot \frac{A(a' + 1, i - 1)}{A(a' + 1, i)} = \frac{1}{A_{a' + i - 1}} \cdot (A_{a' + i} A_{a' + i - 1}) \cdot \frac{1}{A_{a' + i}} = 1.$$

Case 2 :

$$\begin{array}{ccccc}
 & I'(a', i+1) & & I'(a'-1, i+1) & \\
 & \downarrow & & \nearrow & \\
 I'(a'+1, i) & \longleftarrow & I'(a', i) & \longleftarrow & I'(a'-1, i) \\
 \mu & & \mu & &
 \end{array}$$

In this case the exchange ratio gets multiplied by

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot \frac{A(a', i+1)}{A(a'+1, i)} = \frac{1}{A_{a'+i-1}} \cdot \frac{A_{a'+i} A_{a'+i-1}}{A_{a'+i}} = 1.$$

Case 3:

$$\begin{array}{ccccc}
 & I'(a', i+1) & & I'(a'-1, i+1) & \\
 & \downarrow & & \nearrow & \\
 I'(a'+1, i) & \longleftarrow & I'(a', i) & \longleftarrow & I'(a'-1, i) \\
 & & \downarrow & & \\
 \mu & & I'(a', i-1) & &
 \end{array}$$

The exchange ratio gets multiplied by

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot \frac{A(a', i+1)}{A(a', i-1)} \cdot \frac{1}{A(a'+1, i)} = \frac{1}{A_{a'+i-1}} \cdot (A_{a'+i} A_{a'+i-1}) \cdot \frac{1}{A_{a'+i}} = 1.$$

Case 4:

$$\begin{array}{ccccc}
 & I'(a', i+1) & & I'(a'-1, i+1) & \\
 & \downarrow & & \nearrow & \\
 \mu & & I'(a', i) & \longleftarrow & I'(a'-1, i) \\
 & & \downarrow & & \\
 \mu & & I'(a', i-1) & &
 \end{array}$$

The exchange ratio gets multiplied by

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot \frac{A(a', i+1)}{A(a', i-1)} = \frac{1}{A_{a'+i-1}} \cdot (A_{a'+i} A_{a'+i-1}) = A_{a'+i}.$$

But by observation (b) above we have $a' + i = b_i$ and $A_{a'+i} = A_{b_i} = 1$.

Case 5:

$$\begin{array}{ccccc}
& & I'(a', i+1) & & I'(a'-1, i+1) \\
& & \downarrow & \nearrow & \\
\mu & & I'(a', i) & \longleftarrow & I'(a'-1, i) \\
& & \mu & & \mu
\end{array}$$

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot A(a', i+1) = \frac{1}{A_{a'+i-1}} \cdot (A_{a'+i} A_{a'+i-1}) = A_{a'+i}.$$

Similarly to the previous case, we have $A_{a'+i} = 1$. $\square$

Now we deal with the left boundary where $a' = a - 1$.

Lemma 5.23. *Suppose that $a' = a - 1$ and $I'_R(a', i)$ is a mutable box of Q^R . Then the exchange ratios at (a', i) agree: $\hat{y}(a', i) = \hat{y}^R(a', i)$.*

Proof. We follow the cases in the proof of Lemma 5.22. In each case, we describe the contribution from $I'(a' + 1, i)$ and $I'(a' + 1, i - 1)$ which are present in λ/μ but missing in λ_R/μ_R and call these “missing factors”.

Case 1: The missing factor is $\Delta_{I'(a, i-1)}/\Delta_{I'(a, i)}$ while the rest of the exchange ratio is multiplied by

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot \frac{A(a', i+1)}{A(a', i-1)} = \frac{1}{A_{a'+i-1}} \cdot (A_{a'+i} A_{a'+i-1}) = A_{a'+i}.$$

Note that $a' + i = a + i - 1 \in \{a + \bar{\mu}_a, \dots, a + \bar{\lambda}_a - 1\}$, so by (5.14) we get

$$A_{a'+i} = \Delta_{I'(a, i-1)}/\Delta_{I'(a, i)},$$

so the missing factor is restored.

Case 2: The missing factor is $1/\Delta_{I'(a'+1, i)}$ while the rest of the exchange ratio is multiplied by

$$\frac{A(a'-1, i)}{A(a'-1, i+1)} \cdot A(a', i+1) = \frac{1}{A_{a'+i-1}} \cdot (A_{a'+i} A_{a'+i-1}) = A_{a'+i}.$$

Note that $a' + i = a + \bar{\mu}_a$, so $A_{a'+i} = \Delta_{I_\mu}/\Delta_{I'(a, i)} = 1/\Delta_{I'(a, i)}$,

Case 3 is similar to **Case 2**, and **Case 4** and **Case 5** follow Lemma 5.22 verbatim since there are no missing factors. $\square$

Here we discuss Example 1.3 in detail. Recall that $\lambda = (7, 7, 5, 3, 1)$, $\mu = (3, 1)$ and $a = 6$. We can label the boxes of λ/μ by $I'(a, i)$ and draw the quiver as follows:

$$u_7 = \frac{\Delta_{5,8,10,11,12}}{\Delta_{5,7,10,11,12}} v_7 = \frac{1}{\Delta_{5,7,10,11,12}} v_7 \quad \text{since } \Delta_{I_\mu} = 1. \quad (5.15)$$

To find u_8 and u_9 , we can expand v_9 and v_{10} in the basis $(v_5, v_7, v_8, v_{11}, v_{12})$ using Cramer's Rule:

$$v_9 = \frac{\Delta_{9,7,8,11,12}}{\Delta_{5,7,8,11,12}}v_5 + \frac{\Delta_{5,9,8,11,12}}{\Delta_{5,7,8,11,12}}v_7 + \frac{\Delta_{5,7,9,11,12}}{\Delta_{5,7,8,11,12}}v_8 + \frac{\Delta_{5,7,8,9,12}}{\Delta_{5,7,8,11,12}}v_{11} + \frac{\Delta_{5,7,8,11,9}}{\Delta_{5,7,8,11,12}}v_{12}, \quad (5.16)$$

$$v_{10} = \frac{\Delta_{10,7,8,11,12}}{\Delta_{5,7,8,11,12}}v_5 + \frac{\Delta_{5,10,8,11,12}}{\Delta_{5,7,8,11,12}}v_7 + \frac{\Delta_{5,7,10,11,12}}{\Delta_{5,7,8,11,12}}v_8 + \frac{\Delta_{5,7,8,10,12}}{\Delta_{5,7,8,11,12}}v_{11} + \frac{\Delta_{5,7,8,11,10}}{\Delta_{5,7,8,11,12}}v_{12}, \quad (5.17)$$

Given that $u_8 \in \langle v_5, v_7, v_8 \rangle \cap \langle v_{10}, v_{11}, v_{12} \rangle$ and $u_9 \in \langle v_5, v_7, v_8, v_9 \rangle \cap \langle v_{11}, v_{12} \rangle$, we rewrite (5.16) and (5.17) as

$$v_{10} - \frac{\Delta_{5,7,8,10,12}}{\Delta_{5,7,8,11,12}}v_{11} - \frac{\Delta_{5,7,8,11,10}}{\Delta_{5,7,8,11,12}}v_{12} = \frac{\Delta_{10,7,8,11,12}}{\Delta_{5,7,8,11,12}}v_5 + \frac{\Delta_{5,10,8,11,12}}{\Delta_{5,7,8,11,12}}v_7 + \frac{\Delta_{5,7,10,11,12}}{\Delta_{5,7,8,11,12}}v_8, \quad (5.18)$$

$$v_9 - \frac{\Delta_{9,7,8,11,12}}{\Delta_{5,7,8,11,12}}v_5 - \frac{\Delta_{5,9,8,11,12}}{\Delta_{5,7,8,11,12}}v_7 - \frac{\Delta_{5,7,9,11,12}}{\Delta_{5,7,8,11,12}}v_8 = \frac{\Delta_{5,7,8,9,12}}{\Delta_{5,7,8,11,12}}v_{11} + \frac{\Delta_{5,7,8,11,9}}{\Delta_{5,7,8,11,12}}v_{12}. \quad (5.19)$$

We find that (5.18) defines u_8 under the normalization of (5.10), whereas (5.19) requires rescaling by $\Delta_{5,7,8,11,12}/\Delta_{5,7,8,10,12}$ to obtain the following:

$$u_8 = v_{10} - \frac{\Delta_{5,7,8,10,12}}{\Delta_{5,7,8,11,12}}v_{11} - \frac{\Delta_{5,7,8,11,10}}{\Delta_{5,7,8,11,12}}v_{12} = \frac{\Delta_{10,7,8,11,12}}{\Delta_{5,7,8,11,12}}v_5 + \frac{\Delta_{5,10,8,11,12}}{\Delta_{5,7,8,11,12}}v_7 + \frac{\Delta_{5,7,10,11,12}}{\Delta_{5,7,8,11,12}}v_8, \quad (5.20)$$

$$u_9 = v_{11} + \frac{\Delta_{5,7,8,11,9}}{\Delta_{5,7,8,9,12}}v_{12} = \frac{\Delta_{5,7,8,11,12}}{\Delta_{5,7,8,9,12}}v_9 - \frac{\Delta_{9,7,8,11,12}}{\Delta_{5,7,8,9,12}}v_5 - \frac{\Delta_{5,9,8,11,12}}{\Delta_{5,7,8,9,12}}v_7 - \frac{\Delta_{5,7,9,11,12}}{\Delta_{5,7,8,9,12}}v_8. \quad (5.21)$$

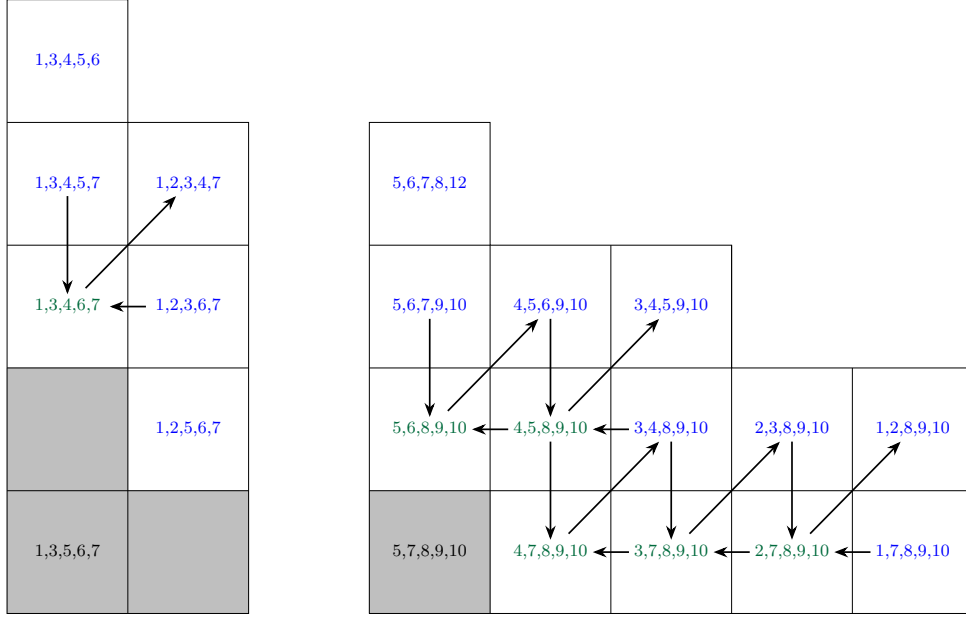
After the cut, we get the left matrix $V^L = (w_1, \dots, w_7)$ where

$$w_1 = v_5, w_2 = v_7, w_3 = v_8, w_4 = v_9, w_5 = v_{10}, w_6 = v_{11}, w_7 = v_{12},$$

and the right matrix $V^R = (u_1, \dots, u_{10})$ where u_7, u_8, u_9 are given by (5.15), (5.20), (5.21) and

$$u_i = v_i \quad \text{for } 1 \leq i \leq 6 \quad \text{and} \quad u_{10} = v_{12}.$$

We can fill in the diagrams $\lambda^{6,L}/\mu^{6,L}$ and $\lambda^{6,R}/\mu^{6,R}$ with $I'_L(a', i)$ and $I'_R(a', i)$ respectively:



The reader is invited to check that the corresponding minors are rescaled as in Lemma 5.21.

5.6. Proof of Theorem 1.2

In this section we prove Theorem 1.2 which we repeat for the reader's convenience. Recall that U_a has a cluster structure defined by Proposition 5.7.

Theorem 5.24. *Let $n > 0$ and $0 < k \leq n$. Let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n - k)$ -rectangle. Choose $1 \leq a \leq n - k$, and let U_a be the principal open set in $S_{\lambda/\mu}^\circ$ defined by the non-vanishing of the cluster variables in the a -th column of λ/μ (from the right). Then, we have an isomorphism*

$$\Phi_a : U_a \xrightarrow{\sim} S_{\lambda^a, L/\mu^a, L}^\circ \times S_{\lambda^a, R/\mu^a, R}^\circ.$$

Furthermore, Φ_a is a quasi-cluster isomorphism.

Proof. We use the commutative diagram (5.6) to summarize the results of this section. By (5.5), the configurations of flags Ω^L and Ω^R yield well-defined points in the respective braid varieties $X(w_0\beta^L)$ and $X(w_0\beta^R)$. We can formally restate the constructions in Sections 5.3 and 5.4 by writing

$$V^L = \Xi(\Omega^L), \quad V^R = \Xi(\Omega^R)$$

where $\Xi = \tilde{\Omega}^{-1}$ is defined in the proof of Theorem 3.56. In particular, the proof of Theorem 3.56 implies that

$$V^L \in S_{\lambda^a, L/\mu^a, L}^\circ, \quad V^R \in S_{\lambda^a, R/\mu^a, R}^\circ$$

and $\Omega(V^L) = \Omega^L$, $\Omega(V^R) = \Omega^R$. Therefore Φ_a is well defined. Note that applying row operations on V corresponds to simultaneous row operations on V^L and V^R .

Next, we prove that Φ_a is an isomorphism. The matrices V^L and V^R are defined up to row operations on both (independent from each other), but we can fix a unique representative for V^L

by requiring that $v_{b_1}, \dots, v_{b_k}$ is a standard basis. This determines the (framed) flag $\mathcal{F}^a$, and we can apply unique row operations on V^R which match both framed flags $\mathcal{F}^W$ and $\mathcal{F}^a$. Now all vectors v_i are contained in the union of columns of V^L and V^R .

Lemma 5.21 implies that all cluster variables in Q^R are multiplied by monomials in frozen, and by Lemmas 5.22 and 5.23 the exchange ratios are preserved. So Φ_a is a quasi-cluster isomorphism. $\square$

5.7. Splicing on n -strands

As mentioned in Section 3.6, the skew shaped positroid $S_{\lambda/\mu}^\circ$ is isomorphic to the braid variety $X\left(\beta(w_{\lambda/\mu})\beta(w_0^{(n)})\right)$, where $w_{\lambda/\mu} \in S_n$, cf. (3.21). Selecting $1 \leq a \leq n - k$, we have

$$w_{\lambda^{a,L}/\mu^{a,L}} \in S_{n-a}, \quad w_{\lambda^{a,R}/\mu^{a,R}} \in S_{a+k}.$$

We choose an embedding $S_{a+k} \hookrightarrow S_n$ as the subgroup of permutations that leave fixed the elements $a + k + 1, \dots, n$, and we embed $S_{n-a} \hookrightarrow S_n$ as the subgroup of elements that leave fixed the elements $1, \dots, a$. Upon these embeddings, we have a length-additive decomposition

$$w_{\lambda/\mu} = w_{\lambda^{a,L}/\mu^{a,L}} w_{\lambda^{a,R}/\mu^{a,R}}, \quad (5.22)$$

see Figure 5.1. By results of [GKSS26], the decomposition (5.22) implies that we have an open embedding

$$X\left(\beta(w_{\lambda^{a,L}/\mu^{a,L}})\beta(w_0^{(n)})\right) \times X\left(\beta(w_{\lambda^{a,R}/\mu^{a,R}})\beta(w_0^{(n)})\right) \hookrightarrow X\left(\beta(w_{\lambda/\mu})\beta(w_0^{(n)})\right)$$

that induces a quasi-cluster equivalence between the left-hand side and an open subset of the right-hand side given by the freezing of some cluster variables. We do not know whether, upon the corresponding identifications, this coincides with the embedding $S_{\lambda^{a,L}/\mu^{a,L}}^\circ \times S_{\lambda^{a,R}/\mu^{a,R}}^\circ \hookrightarrow S_{\lambda/\mu}^\circ$.

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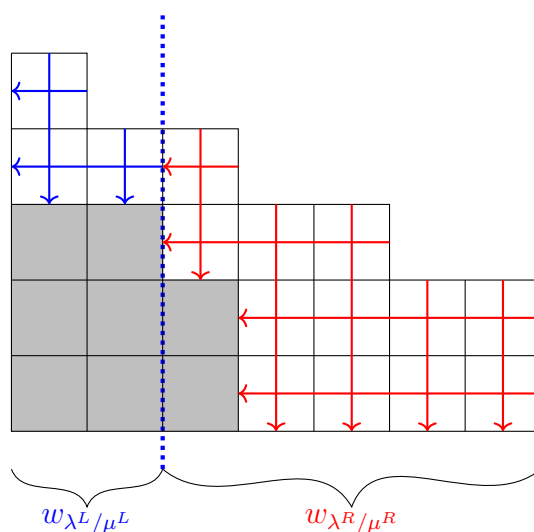


Figure 5.1: Length-additive decomposition $w_{\lambda/\mu} = w_{\lambda^a,L}/\mu^{a,L} w_{\lambda^a,R}/\mu^{a,R}$.

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