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BY

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SOLUTION OF VISCOUSLY DAMPED LINEAR SYSTEMS USING A SET OF LOAD-DEPENDENT VECTORS

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SUMMARY

This paper considers a solution method for viscously damped linear structural systems which are subjected to transient loading. The equations of motion of such systems are written in a first-order form. A Krylov subspace is generated using the damped dynamic matrix and the static deflection from the first-order form of the equations of motion. Two convenient bases, Lanczos vectors and Ritz vectors, are constructed from the Krylov subspace. An approximate solution is then obtained by superposition of the Lanczos vectors or the Ritz vectors. In contrast to the traditional mode superposition method using complex eigenvectors, the Lanczos vectors or the Ritz vectors are less expensive to generate than the complex eigenvectors, yet yield comparable accuracy. In addition, there is no need for a static correction since the static deflection is already contained in the Krylov subspace. Numerical examples are presented to show the potential of using the Ritz vectors to compute responses of damped dynamic systems.

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INTRODUCTION

The analysis of viscously damped linear dynamic systems is useful in many aspects, e.g., flexible space structural systems, vibration control of dynamic systems, foundation-structure interaction, acoustic-structure interaction, etc. The differential equations of motion for a discretized model of such systems are expressed by

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{C}\,\dot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

where **M**, **C**, and **K** are, respectively, the $n \times n$ mass, damping, and stiffness matrices, and $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$, and $\mathbf{u}(t)$ are the $n \times 1$ acceleration, velocity, and displacement vectors. This second-order differential equation can be transformed into a first-order one[1] by doubling the size of the system as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}(t) \\ \ddot{\mathbf{u}}(t) \end{bmatrix} - \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{2}$$

which may be represented by

$$\mathbf{A}\,\dot{\mathbf{z}}(t) - \mathbf{B}\,\mathbf{z}(t) = \mathbf{y}(t) \tag{3}$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \tag{4}$$

and

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{5}$$

Equation (2) is not the only first-order form of equation (1); however, it is more suitable for our purpose. Note that the **A** and **B** in equation (2) are symmetric since the **M**, **C**, and **K** are symmetric under the usual finite element discretization process. This symmetry is useful in the construction of the present method.

To find the response of the damped dynamic systems, one can use the traditional mode superposition method. This method consists in finding the eigensolutions of the system expressed by equation (3) and using them to transform the coupled equations of motion into an equivalent set of decoupled equations. Then, the decoupled equations can be solved individually and their solutions are superposed to produce the response of the dynamic system. Computing the eigenvectors of a damped system is, however, much more expensive than computing the eigenvectors of an undamped system. Consequently, simplified methods are frequently used in which the eigenvectors of the associated undamped system (obtained by neglecting the damping matrix in the damped system) are used instead.[2] For some situations where the effect of damping is small compared to the effect of stiffness and inertia, the simplified methods may be satisfactory. For more general situations, the simplified methods may not be enough.[3–6] Therefore, it is desirable to have a method which can treat systems with arbitrary damping and is still efficient.

In many practical situations, the load applied to the system can be written into the form

$$\mathbf{f}(t) = \hat{\mathbf{f}}\,\epsilon(t) \tag{6}$$

in which the vector $\hat{\mathbf{f}}$ represents the spatial distribution and $\epsilon(t)$ represents the temporal variation of the applied load. In such situations, the information of $\hat{\mathbf{f}}$ can be utilized to enhance efficiency of the mode superposition method. Indeed, in the analysis of an undamped structural system using load-dependent vectors generated from the dynamic matrix $\mathbf{K}^{-1}\mathbf{M}$ and the static deflection shape $\mathbf{K}^{-1}\hat{\mathbf{f}}$ to compute an approximate response has been shown to be more efficient than using the eigenvectors of the undamped system.[7, 8] It is desirable that the idea of using load-dependent vectors can be extended to damped systems. In the formulation (2) for damped systems, we have $\mathbf{y}(t) = \hat{\mathbf{y}}\,\epsilon(t)$ and

$$\mathbf{B}^{-1}\hat{\mathbf{y}} = \begin{bmatrix} -\mathbf{K}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{K}^{-1}\hat{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix} \tag{7}$$

which is in fact the static deflection shape but in the $2n$-dimensional state vector form. Further, the *damped dynamic matrix* $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ can take the place of dynamic matrix in the undamped systems. Therefore, we are encouraged by this similarity to find a set of load-dependent vectors for a damped system in anticipation that they can produce approximate responses more efficiently than the eigenvectors of the damped system.

REDUCED SYSTEM

In practical analysis of complicated dynamic systems, the size of the equations of motion is usually very large. Therefore, how to efficiently obtain a satisfactory approximate solution of such systems has become an important topic for study. A commonly used approach in this regard is to reduce the original set of equations to a much smaller set and to find the approximate solution by solving the smaller set. In mathematical terms, this approach can be interpreted as a process of a *projection* of the original system onto a subspace. In the following we present the essential ingredients related to the current problem.

A set of n vectors $\mathbf{Q} = [\, \mathbf{q}_1, \ldots, \mathbf{q}_n \,]$ is used as a *basis* to express the solution $\mathbf{u}(t)$ since it has n components. That is, the exact solution to equation (3) can be expressed in the chosen basis $\mathbf{Q}$ as

$$\mathbf{z}(t) = \sum_{j=1}^{n} \mathbf{q}_j \, x_j(t) = \mathbf{Q} \, \mathbf{x}(t) \tag{8}$$

in which the basis $\mathbf{Q}$ serves to transform from the generalized coordinates $\mathbf{x}(t)$ to the geometric coordinates $\mathbf{z}(t)$. In principle, any set of linearly independent vectors can be used as a basis since the solution obtained will be the same no matter which basis is used. In practice, however, we hope to find a satisfactory approximate, $\mathbf{z}_m(t)$, by using only a small subset of $\mathbf{Q}$, i.e.,

$$\mathbf{z}_m(t) = \sum_{j=1}^{m} \mathbf{q}_j \, x_j(t) = \mathbf{Q}_m \, \mathbf{x}_m(t) \tag{9}$$

That is, we want to choose a set of basis vectors such that the first few of them, $\mathbf{Q}_m$ with m being much smaller than n, will produce a $\mathbf{z}_m(t)$ which is close to the exact solution $\mathbf{z}(t)$. This is anticipated because it is conjecture that only a small subspace of the whole solution space will be excited by the applied load under the condition of equation (6).

Geometrically, the $\mathbf{z}_m(t)$ is a projection of $\mathbf{z}(t)$ onto the subspace $span[\mathbf{Q}_m]$. Different $\mathbf{z}_m(t)$ can be obtained by introducing different $\mathbf{x}_m(t)$. In this study, we adopt an A-orthogonal projection method to find the approximate solution $\mathbf{z}_m(t)$. Here, $\mathbf{Q}$ is A-orthogonal if $\mathbf{q}_i^{\mathrm{T}} \mathbf{A} \, \mathbf{q}_j = 0$ for $i \neq j$. To proceed, we first pre-multiply equation (3) by $\mathbf{B}^{-1}$ to obtain

$$\mathbf{B}^{-1}\,\mathbf{A}\,\dot{\mathbf{z}}(t) - \mathbf{z}(t) = \mathbf{B}^{-1}\,\mathbf{y}(t) \tag{10}$$

where the damped dynamic matrix $\mathbf{B}^{-1}\,\mathbf{A}$ is unsymmetric but is symmetric with respect to the **A**-weighted inner product defined by

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = \mathbf{u}^{\mathrm{T}}\,\mathbf{A}\,\mathbf{v} \tag{11}$$

since for any two nonzero vectors **u** and **v**

$$(\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{u}, \mathbf{v})_{\mathbf{A}} = \mathbf{u}^{\mathrm{T}}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{v} = (\mathbf{u}, \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{v})_{\mathbf{A}} \tag{12}$$

The fact that the damped dynamic matrix is symmetric with respect to **A** is useful, as will be shown later. The $\mathbf{x}_m(t)$ is obtained by requiring that the residual vector resulting from the approximate solution $\mathbf{z}_m(t)$

$$\mathbf{r}_m(t) = \mathbf{B}^{-1}\,\mathbf{A}\,\dot{\mathbf{z}}_m(t) - \mathbf{z}_m(t) - \mathbf{B}^{-1}\,\mathbf{y}(t) \tag{13}$$

be **A**-orthogonal to the basis $\mathbf{Q}_m$, i.e.,

$$\mathbf{Q}_m^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_m(t) = \mathbf{0} \tag{14}$$

This requirement is known as the Galerkin condition in the literature. Substituting in turn equation (13), equation (9) and its time derivative into equation (14), we obtain an $m \times m$ reduced system:

$$\mathbf{Q}_m^{\mathrm{T}}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{Q}_m\,\dot{\mathbf{x}}_m(t) - \mathbf{Q}_m^{\mathrm{T}}\,\mathbf{A}\,\mathbf{Q}_m\,\mathbf{x}_m(t) = \mathbf{Q}_m^{\mathrm{T}}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{y}(t) \tag{15}$$

which actually is the **A**-orthogonal projection of the system given by equation (10) onto the subspace spanned by $\mathbf{Q}_m$. In essence, we solve the reduced system represented by equation (15) to find $\mathbf{x}_m(t)$ and then use equation (9) to construct an approximate solution $\mathbf{z}_m(t)$ to the original system.

The critical step in using the A-orthogonal projection method is the choice of the subspace *span*$[\mathbf{Q}_m]$ since the quality of the approximate solution is governed by the reduced system, which depends only on $\mathbf{Q}_m$. In this study, we choose the Krylov subspace spanned by $[\mathbf{b}, \mathbf{D}\mathbf{b}, \mathbf{D}^2\mathbf{b}, \ldots, \mathbf{D}^{m-1}\mathbf{b}]$ with $\mathbf{b} = \mathbf{B}^{-1}\,\hat{\mathbf{y}}$ and $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ as the subspace *span*$[\mathbf{Q}_m]$. It has been shown[9] that this Krylov subspace is very effective for extracting the least dominant eigenpairs of a damped dynamic system. Now we want to investigate whether this Krylov subspace is also effective for computing responses of damped dynamic systems.

TRIDIAGONAL FORM

An algorithm which generates an **A**-orthogonal set of vectors by applying the Gram-Schmidt orthogonalization procedure on the Krylov subspace spanned by $[\mathbf{q}, \mathbf{D}\mathbf{q}, \mathbf{D}^2\mathbf{q}, \ldots, \mathbf{D}^{m-1}\mathbf{q}]$ has been given by the authors.[9] This set can be looked upon as a variant of the standard Lanczos vectors. In the following, we summarize the formulations used to derive these Lanczos vectors for the purpose of notation and discussion. In practical computation, however, a reorthogonalization scheme[9] has to be appended to prevent the loss of orthogonality due to the round-off errors.

Given an arbitrary vector **b**, we normalize it to obtain the first Lanczos vector by

$$\delta_1 = \mathbf{b}^T \mathbf{A}\, \mathbf{b} \,/\, |\mathbf{b}^T \mathbf{A}\, \mathbf{b}| \tag{16a}$$

$$\gamma_1 = |\mathbf{b}^T \mathbf{A}\, \mathbf{b}|^{1/2} \tag{16b}$$

$$\mathbf{q}_1 = \mathbf{b} \,/\, \gamma_1 \tag{16c}$$

and compute the subsequent Lanczos vectors, $j = 1, \ldots, n-1$, from the three-term recurrence formula

$$\gamma_{j+1}\, \mathbf{q}_{j+1} = \mathbf{r}_{j+1} = \mathbf{B}^{-1} \mathbf{A}\, \mathbf{q}_j - \alpha_j\, \mathbf{q}_j - \beta_{j-1}\, \mathbf{q}_{j-1} \tag{16d}$$

where $\mathbf{q}_0 = \mathbf{0}$ and

$$\alpha_j = \delta_j\, \mathbf{q}_j^T \mathbf{A}\, \mathbf{B}^{-1} \mathbf{A}\, \mathbf{q}_j \tag{16e}$$

$$\beta_{j-1} = \delta_{j-1}\, \mathbf{q}_{j-1}^T \mathbf{A}\, \mathbf{B}^{-1} \mathbf{A}\, \mathbf{q}_j \tag{16f}$$

$$\delta_{j+1} = \mathbf{r}_{j+1}^T \mathbf{A}\, \mathbf{r}_{j+1} \,/\, |\mathbf{r}_{j+1}^T \mathbf{A}\, \mathbf{r}_{j+1}| \tag{16g}$$

$$\gamma_{j+1} = |\mathbf{r}_{j+1}^T \mathbf{A}\, \mathbf{r}_{j+1}|^{1/2} \tag{16h}$$

Note that the δ's are needed since **A** is not positive dfinite. This is different from the standard Lanczos algorithm[8] used in the analysis of undamped dynamic systems. The dimension of the Lanczos vectors here is $2n$ instead of n, as in the undamped case. However, the cost of computing these Lanczos vectors is not doubled because the structure of the matrices **A** and **B** can be exploited.

After m steps, we have m Lanczos vectors $\mathbf{Q}_m = [\,\mathbf{q}_1, \ldots, \mathbf{q}_m\,]$ satisfying the matrix form of the three-term recurrence formula

$$\gamma_{m+1}\, \mathbf{q}_{m+1}\, \mathbf{e}_m^{\mathrm{T}} = \mathbf{B}^{-1}\, \mathbf{A}\, \mathbf{Q}_m - \mathbf{Q}_m\, \mathbf{T}_m \tag{17}$$

where $\mathbf{e}_m^{\mathrm{T}}$ is the mth row of the $m \times m$ identity matrix $\mathbf{I}_m$, and $\mathbf{T}_m$ is a tri-diagonal matrix made up of the coefficients α_j, β_j, and γ_j :

$$\mathbf{T}_m = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_2 & \alpha_2 & \beta_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \gamma_{m-1} & \alpha_{m-1} & \beta_{m-1} \\ & & & & \gamma_m & \alpha_m \end{bmatrix} \tag{18}$$

The Lanczos vectors obtained in this way are $\mathbf{A}$-orthogonal, i.e., they satisfy

$$\mathbf{Q}_m^{\mathrm{T}}\, \mathbf{A}\, \mathbf{Q}_m = \mathbf{\Delta}_m \tag{19}$$

where $\mathbf{\Delta}_m$ is an $m \times m$ diagonal matrix with diagonal elements δ_i being 1 or -1. After pre-multiplying equation (17) by $\mathbf{Q}_m^{\mathrm{T}}\, \mathbf{A}$ and using equation (19) we can obtain

$$\mathbf{Q}_m^{\mathrm{T}}\, \mathbf{A}\, \mathbf{B}^{-1}\, \mathbf{A}\, \mathbf{Q}_m = \mathbf{\Delta}_m\, \mathbf{T}_m \tag{20}$$

This says that the $\mathbf{A}$-orthogonal projection of the damped dynamic matrix $\mathbf{B}^{-1}\,\mathbf{A}$ onto the Krylov subspace with the Lanczos vectors $\mathbf{Q}_m$ as its basis is a tri-diagonal matrix.

Using equations (19) and (20), one may simplify the reduced system expressed by equation (15) to :

$$\mathbf{\Delta}_m\, \mathbf{T}_m\, \dot{\mathbf{x}}_m(t) - \mathbf{\Delta}_m\, \mathbf{x}_m(t) = \mathbf{Q}_m^{\mathrm{T}}\, \mathbf{A}\, \mathbf{B}^{-1}\, \mathbf{y}(t) \tag{21}$$

or

$$\mathbf{T}_m\, \dot{\mathbf{x}}_m(t) - \mathbf{x}_m(t) = \mathbf{\Delta}_m\, \mathbf{Q}_m^{\mathrm{T}}\, \mathbf{A}\, \mathbf{B}^{-1}\, \mathbf{y}(t) \tag{22}$$

since $\mathbf{\Delta}_m\, \mathbf{\Delta}_m = \mathbf{I}_m$. This first-order system of differential equations is only slightly coupled. Its solution can be obtained either by a step-by-step integration method or by finding its eigensolutions and using mode superposition. It should be noted that the $\mathbf{T}_m$ is not positive-definite since its eigenvalues are complex in general. Therefore, one may have to choose an equation solver with an appropriate pivoting strategy when applying a step-by-step integration method to solve equation (22).

Since the starting vector is chosen to be the static deflection shape, we have $\mathbf{B}^{-1}\, \hat{\mathbf{y}} = \mathbf{b} = \gamma_1\, \mathbf{q}_1$ according to equation (16c) and

$$\mathbf{\Delta}_m \, \mathbf{Q}_m^{\mathrm{T}} \, \mathbf{A} \, \mathbf{B}^{-1} \, \hat{\mathbf{y}} = \gamma_1 \, \mathbf{\Delta}_m \, \mathbf{Q}_m^{\mathrm{T}} \, \mathbf{A} \, \mathbf{q}_1 = (\gamma_1, 0, \ldots, 0)^{\mathrm{T}} \tag{23}$$

after using $\mathbf{q}_i^{\mathrm{T}} \mathbf{A} \, \mathbf{q}_j = 0$ for $i \neq j$. This indicates that only the first element of the right-hand side vector in equation (22) is nonzero. This special right-hand side vector tends to make $x_j(t)$ dwindle as j increases. Thus, choosing the static deflection as the starting vector can help the sequence $\mathbf{z}_j(t)$, $j = 1, \ldots,$ converge fast so that only a small number of the Lanczos vectors is needed to obtain a desired accuracy.

In order to determine how many Lanczos vectors are required to obtain a desired accuracy we need to assess the errors resulting from a given approximation. That is, we need to estimate the error $\mathbf{z}(t) - \mathbf{z}_m(t)$. Since this quantity is not computable unless the exact solution $\mathbf{z}(t)$ is known, the following criterion can be adopted to determine when to stop in the algorithm. For each generated Lanczos vector we compute the participation factor

$$h_j = \mathbf{q}_j^{\mathrm{T}} \hat{\mathbf{y}} = \mathbf{q}_j^{\mathrm{T}} \begin{Bmatrix} \hat{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix} \tag{24}$$

Here, the h_j provides a measure of the extent to which the $\mathbf{q}_j$ participates in synthesizing the load $\hat{\mathbf{f}}$ on the structure. Thus, we can terminate the Lanczos algorithm at the jth step when the h_j is small enough. The participation factor needs not be computed explicitly. Pre-multiplying the three term recurrence formula (15.d) by $\hat{\mathbf{y}}^{\mathrm{T}}$, we obtain

$$\gamma_{j+1} \hat{\mathbf{y}}^{\mathrm{T}} \mathbf{q}_{j+1} = \hat{\mathbf{y}}^{\mathrm{T}} \, \mathbf{B}^{-1} \, \mathbf{A} \, \mathbf{q}_j - \alpha_j \, \hat{\mathbf{y}}^{\mathrm{T}} \mathbf{q}_j - \beta_{j-1} \, \hat{\mathbf{y}}^{\mathrm{T}} \mathbf{q}_{j-1} \tag{25}$$

Since $\hat{\mathbf{y}}^{\mathrm{T}} \, \mathbf{B}^{-1} = \mathbf{b}^{\mathrm{T}} = \gamma_1 \, \mathbf{q}_1^{\mathrm{T}}$, the first term on the right-hand side of equation (25) becomes $\gamma_1 \, \mathbf{q}_1^{\mathrm{T}} \, \mathbf{A} \, \mathbf{q}_j = 0$ when $j \neq 1$. Thus, equation (25) can be simplified to

$$\gamma_{j+1} \, \hat{\mathbf{y}}^{\mathrm{T}} \, \mathbf{q}_{j+1} = - \alpha_j \, \hat{\mathbf{y}}^{\mathrm{T}} \, \mathbf{q}_j - \beta_{j-1} \, \hat{\mathbf{y}}^{\mathrm{T}} \, \mathbf{q}_{j-1} \tag{26}$$

After making use of the definition of the participation factors, we have

$$h_{j+1} = - \frac{\alpha_j h_j + \beta_{j-1} h_{j-1}}{\gamma_{j+1}} \tag{27}$$

which indicates that the h_{j+1} can be obtained simply from the previous h's and the Lanczos coefficients α's, β's, and γ's.

DIAGONAL FORM

To decouple the system represented by equation (21) we solve the eigenproblem

$$\mathbf{\Delta}_m \mathbf{T}_m \mathbf{S}_m = \mathbf{\Delta}_m \mathbf{S}_m \mathbf{\Theta}_m^{-1} \tag{28}$$

to find the set of vectors $\mathbf{S}_m = [\, \mathbf{s}_1, \ldots, \mathbf{s}_m \,]$ which satisfies

$$\mathbf{S}_m^{\mathrm{T}} \mathbf{\Delta}_m \mathbf{S}_m = \mathbf{I}_m \qquad \mathbf{S}_m^{\mathrm{T}} \mathbf{\Delta}_m \mathbf{T}_m \mathbf{S}_m = \Theta_m^{-1} \tag{29}$$

Introducing the transformation $\mathbf{x}_m(t) = \mathbf{S}_m \mathbf{w}_m(t)$ and pre-multiplying both sides of equation (21) by $\mathbf{S}_m^{\mathrm{T}}$, we obtain the decoupled set of equations :

$$\mathbf{\Theta}_m^{-1} \dot{\mathbf{w}}_m(t) - \mathbf{w}_m(t) = \mathbf{Y}_m^{\mathrm{T}} \mathbf{A} \mathbf{B}^{-1} \hat{\mathbf{y}} \, \epsilon(t) \tag{30}$$

where $\mathbf{Y}_m = \mathbf{Q}_m \mathbf{S}_m$ contains the Ritz vectors and has the following properties :

$$\mathbf{Y}_m^{\mathrm{T}} \mathbf{A} \mathbf{Y}_m = \mathbf{S}_m^{\mathrm{T}} \mathbf{Q}_m^{\mathrm{T}} \mathbf{A} \mathbf{Q}_m \mathbf{S}_m = \mathbf{S}_m^{\mathrm{T}} \mathbf{\Delta}_m \mathbf{S}_m = \mathbf{I}_m \tag{31a}$$

$$\mathbf{Y}_m^{\mathrm{T}} \mathbf{A} \mathbf{B}^{-1} \mathbf{A} \mathbf{Y}_m = \mathbf{S}_m^{\mathrm{T}} \mathbf{Q}_m^{\mathrm{T}} \mathbf{A} \mathbf{B}^{-1} \mathbf{A} \mathbf{Q}_m \mathbf{S}_m = \mathbf{S}_m^{\mathrm{T}} \mathbf{\Delta}_m \mathbf{T}_m \mathbf{S}_m = \Theta_m^{-1} \tag{31b}$$

Using these properties, one can easily verify that equation (30) actually is the projected system obtained from using the Ritz vectors as a basis to express the Krylov subspace. This projected system is equivalent to the one expressed by equation (21) since $\mathbf{Y}_m$ and $\mathbf{Q}_m$ span the same Krylov subspace and Therefore $\mathbf{Y}_m \mathbf{w}_m(t)$ is equal to $\mathbf{Q}_m \mathbf{x}_m(t)$. Accordingly we can use the decoupled set in equation (30) to study the solution $\mathbf{z}_m(t)$ to equation (21) or (22).

Since $\mathbf{B}^{-1} \hat{\mathbf{y}} = \gamma_1 \mathbf{q}_1$, we have

$$\mathbf{Y}_m^{\mathrm{T}} \mathbf{A} \mathbf{B}^{-1} \hat{\mathbf{y}} = \mathbf{S}_m^{\mathrm{T}} \mathbf{Q}_m^{\mathrm{T}} \mathbf{A} \gamma_1 \mathbf{q}_1 = \gamma_1 \delta_1 (s_{11}, s_{12}, \ldots, s_{1m})^{\mathrm{T}} \tag{32}$$

where the s_{1j} is the first element of the vector $\mathbf{s}_j$ for $j = 1, \ldots, m$. With this, the jth equation of the projected system given by equation (30) can be expressed as

$$\dot{w}_j(t) - \theta_j w_j(t) = \gamma_1 \delta_1 \theta_j s_{1j} \epsilon(t) = g_j(t) \tag{33}$$

where both the θ_j and the $g_j(t)$ are complex-valued for underdamped modes.[10] Equation (33) is a first-order differential equation and its solution can be obtained by a closed-form formulation for simple $\epsilon(t)$ or by a numerical method for complicated $\epsilon(t)$.

NUMERICAL SOLUTION BY A PIECEWISE EXACT METHOD

In practical applications, the excitation function $\epsilon(t)$ usually is not known in the form of an analytical expression but rather is supplied at a set of discrete points, such as most earthquake ground acceleration data. A direct and accurate method for obtaining the response to this type of excitation is to interpolate the excitation function and to solve the resulting equation exactly. In this method, only the unknown excitation function is interpolated and no other numerical approximations in the integration process is made. Accordingly, it is called a *piecewise exact method.* In the following, we use a piecewise linear interpolation of the excitation function to derive a formula for evaluating the response.

Consider a typical uncoupled equation, whose subscript is dropped for simplicity,

$$\dot{w}(t) - \theta\, w(t) = g(t) \tag{34}$$

Under the assumption that $g(t)$ is a piecewise linear function, equation (34) may be written as

$$\dot{w}(t) - \theta\, w(t) = g_i + \frac{\Delta g}{\Delta t}\, t \tag{35}$$

where

$$\Delta t = t_{i+1} - t_i \qquad \Delta g = g_{i+1} - g_i \tag{36}$$

Equation (35) is valid between two consecutive discrete points i and $i+1$ and its solution, for $0 \leq t \leq \Delta t$, is

$$w(t) = a + b\,t + c\,e^{\theta t} \tag{37}$$

where a, b and c are integration constants. Substituting this solution into equation (35) and using the initial condition $w(t_i) = w_i$, one can obtain the following expressions for the integration constants

$$a = -\frac{1}{\theta} g_i - \frac{1}{\theta^2} \frac{\Delta g}{\Delta t} \tag{38}$$

$$b = -\frac{1}{\theta} \frac{\Delta g}{\Delta t} \tag{39}$$

$$c = w_i - a \tag{40}$$

These constants are substituted in the equation (37) to give the solution $w(t)$ for $0 \le t \le \Delta t$ and this can be used to provide a recursive solution to the equation (34) in increments Δt.

The integration constants can also be expressed in terms of real-valued quantities as

$$\begin{Bmatrix} a_R \\ a_I \end{Bmatrix} = -\frac{1}{(\theta_R^2+\theta_I^2)^2} \begin{bmatrix} \theta_R^2-\theta_I^2 & 2\theta_R\theta_I \\ -2\theta_R\theta_I & \theta_R^2-\theta_I^2 \end{bmatrix} \begin{Bmatrix} \dfrac{\Delta g_R}{\Delta t} \\ \dfrac{\Delta g_I}{\Delta t} \end{Bmatrix} - \frac{1}{\theta_R^2+\theta_I^2} \begin{bmatrix} \theta_R & \theta_I \\ -\theta_I & \theta_R \end{bmatrix} \begin{Bmatrix} g_{Ri} \\ g_{Ii} \end{Bmatrix} \tag{41}$$

$$\begin{Bmatrix} b_R \\ b_I \end{Bmatrix} = -\frac{1}{\theta_R^2+\theta_I^2} \begin{bmatrix} \theta_R & \theta_I \\ -\theta_I & \theta_R \end{bmatrix} \begin{Bmatrix} \dfrac{\Delta g_R}{\Delta t} \\ \dfrac{\Delta g_I}{\Delta t} \end{Bmatrix} \tag{42}$$

and the solution $w(t)$, for $0 \le t \le \Delta t$, accordingly as

$$\begin{Bmatrix} w_R(t) \\ w_I(t) \end{Bmatrix} = \begin{Bmatrix} a_R + b_R t \\ a_I + b_I t \end{Bmatrix} + e^{\theta_R t} \begin{bmatrix} \cos(\theta_I t) & -\sin(\theta_I t) \\ \sin(\theta_I t) & \cos(\theta_I t) \end{bmatrix} \begin{Bmatrix} w_{Ri} - a_R \\ w_{Ii} - a_I \end{Bmatrix} \tag{43}$$

where the subscripts R and I represent, respectively, the real and the imaginary parts of a complex quantity.

NUMERICAL EXAMPLES

We solve two simple problems using the proposed method. To make the comparison precise, we use the Ritz vectors as the basis and solve the decoupled reduced system by the piecewise exact method since there will be some numerical errors involved in employing a step-by-step integration method to solve the slightly coupled system obtained by using the Lanczos vectors as the basis. The first example shows the results obtained from superposition of different numbers of Ritz vectors. The second example compares the result from superposition of the Ritz vectors and the result from superposition of the same number of complex eigenvectors.

Example 1 : The system shown in Figure 1 is a cantilever beam with a lumped translational viscous-damper attached at the tip. The length, Young's modulus, density, inertia, and area of the beam are, respectively, 5 m, 500 N/m^2, 1 Kg/m^3, 1 m^4, and 1 m^2. The damping coefficient of the tangential damper is 1 N-sec/m. The load is suddenly applied to the beam, so the $\epsilon(t)$ is an unit step function. The cantilever beam is modeled by 5 equal finite elements deduced from elementary beam theory. The consistent mass is used to define **M**. The damping matrix **C** has only one nonzero element representing the magnitude of the lumped damper. The system has 10 degrees of freedom and the order of the associated (**A**, **B**) is 20. We compute the time history of the vertical displacement at the location where the load is applied. Figure 2 shows the results obtained for various j, where j is the number of pairs of Ritz vectors used in the summation. The result of the $j=10$ case is the exact solution since there are only 10 degrees of freedom in the system. It is seen from Figure 2 that the approximate solution in each case is very close to the exact solution except for the $j=2$ and $j=6$ two cases, where the solution actually diverges as time increases. This is because there exists at least one pair of Ritz values whose real parts are positive in the $j=2$ and $j=6$ cases. That is, the reduced system obtained in these two cases is *unstable* even though the original system is stable. Note that this undesirable phenomenon does not exist when dealing with undamped systems. An easy way to avoid obtaining a divergent solution in such situation is to simply neglect the unstable Ritz modes during the process of superposition. This has been done for the above $j=2$ and $j=6$ two cases and the results thus obtained are stable and as good as those in the other cases.

Example 2 : The frame structure shown in Figure 3 is considered. The structure is modeled by 10 beam elements. The length of each element is equal to 1 m. Material properties are the same as in Example 1. The damping coefficient of the concentrated damper is 10 N-sec/m. The load is such that the $\epsilon(t)$ is an unit step function with a magnitude 1000 N. The coefficient matrices of the system are obtained in the same way as in Example 1. We compute the time history of the horizontal displacement at the node 8. Figure 4(a) compares the solution obtained from superposition of 4 pairs of Ritz vectors and the solution from superposition of 4 pairs of eigenvectors. Similarly, Figure 4(b) compares the solution from 10 pairs of Ritz vectors and from 10 pairs of eigenvectors. It is seen that the overall responses obtained from Ritz vectors are no worse than those obtained from the same number of eigenvectors. In fact, the Ritz vectors give more accurate descriptions of the peak values of the responses than the eigenvectors.

Example 3 : The dam-foundation system shown in Figure 5 is studied. The dam is similar to the Pine Flat Dam, considered previously in reference 11. The material properties for the dam and foundation are : Young's modulus 2.16 MN/m^2, mass density 2.15 $kN\text{-}sec^2/m^4$, and Poisson ratio 0.25. A 9-node plane strain element is used for the dam and foundation. The system has totally 1154 degrees of freedom. Rayleigh damping ratio of 5% of critical is assumed to account for the energy dissipation due to the material damping in dam and foundation. Concentrated viscous dampers are attached along the two ends and the bottom of the foundation to simulate radiation damping, resulting in the non-proportionality of the damping matrix. The dam is subjected to the S69E components of the Taft earthquake of July 21, 1952. We compute the time history of the horizontal displacement at the top of the dam. Figure 6 shows the solutions obtained from superposition of 5 pairs of Ritz vectors, 10 pairs of Ritz vectors, 5 pairs of eigenvectors, and 10 pairs of eigenvectors. It is seen from the figure that there is practically no difference among the four solutions. The CPU used to compute 5 Ritz vectors is 360 seconds and 10 Ritz vectors is 770 seconds; while the CPU used to extract 5 eigenvectors is 1700 seconds and 10 eigenvectors is 2800 seconds. For this large system, using the Ritz vectors instead of eigenvectors, therefore, can result in significant saving in the computational effort with no loss of accuracy.

FINAL REMARKS

From the examples shown here, it is apparent that the approximation obtained using the load-dependent Ritz vectors is no worse than that from the same number of eigenvectors. On the other hand, the Ritz vectors (or equivalently the Lanczos vectors) can be generated with only a fraction of the computational effort required for extracting the eigenvectors. In addition, even when an unstable reduced system occurs, one can still obtain a good approximation simply by neglecting the unstable Ritz modes during the process of superposition. It is, therefore, advantageous to use the Ritz vectors for computing dynamic responses of large damped systems.

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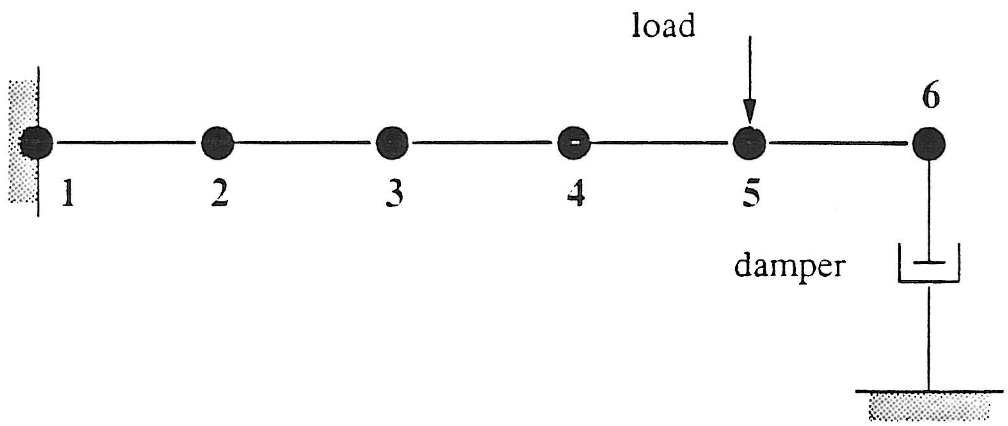


Figure 1 A cantilever beam with a damper

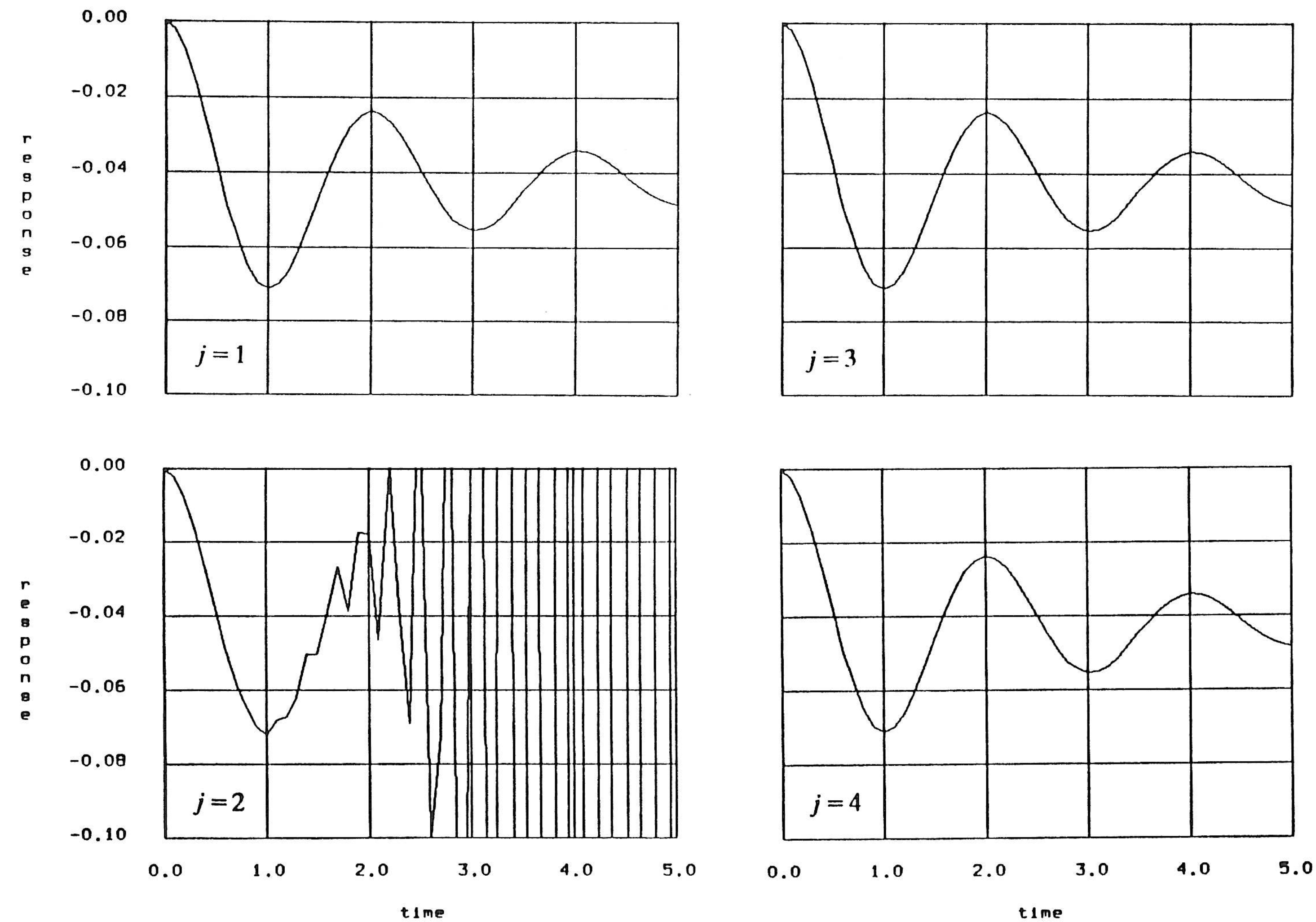


Figure 2 Vertical displacement at node 5 of Fig. 1

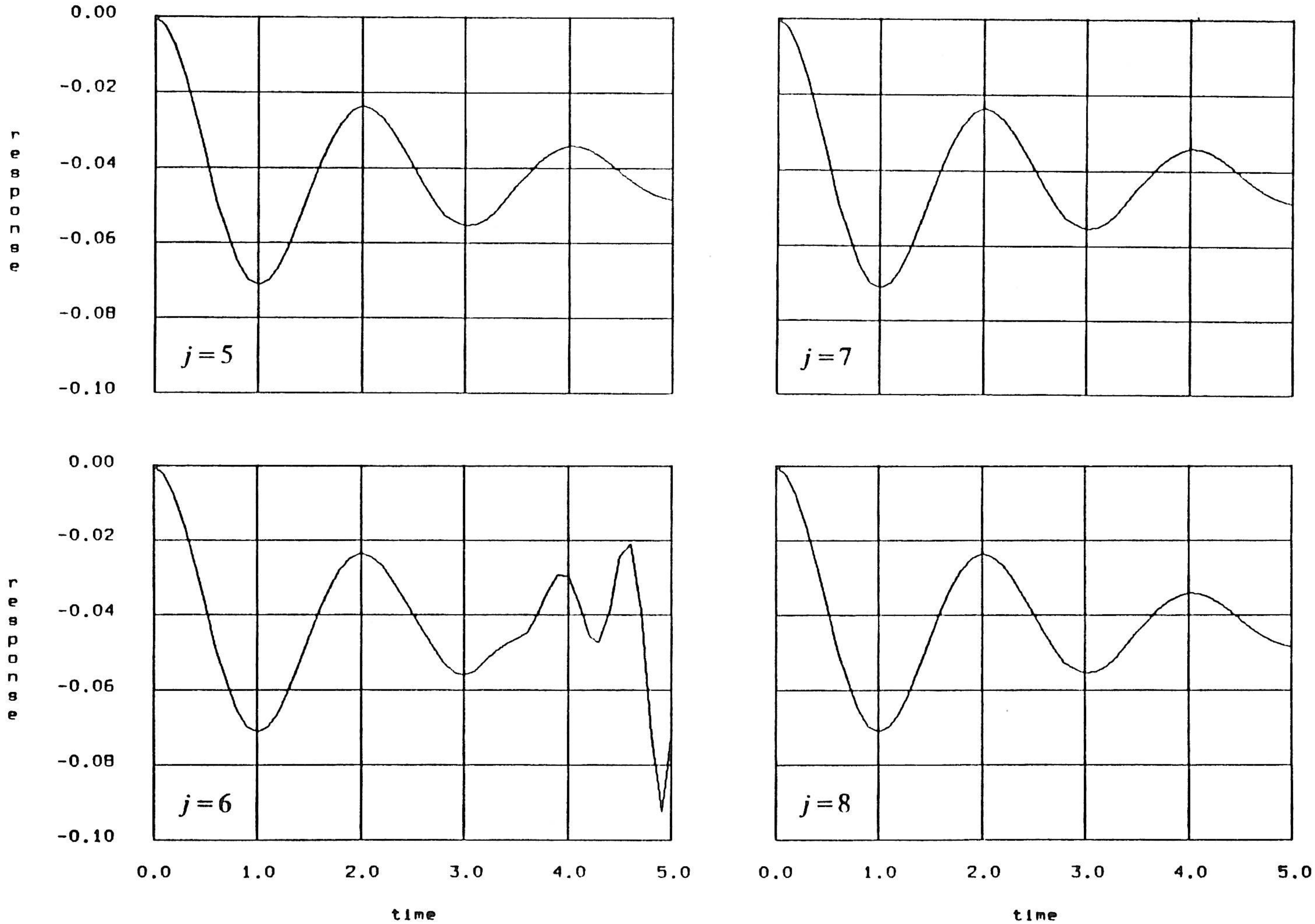
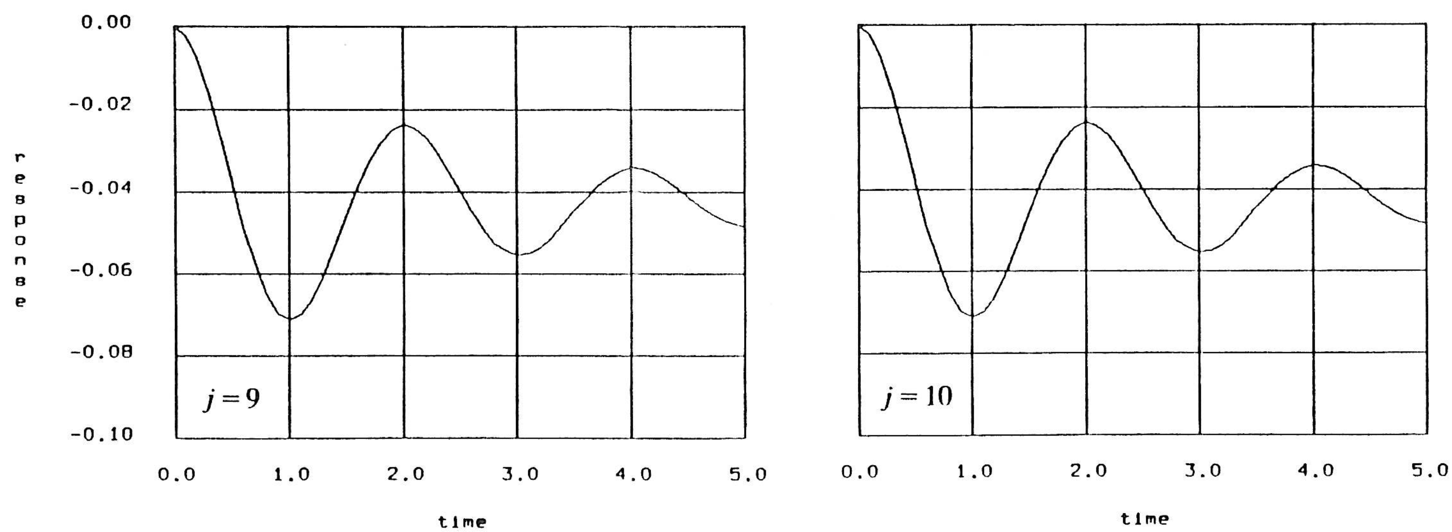


Figure 2 Vertical displacement at node 5 of Fig. 1



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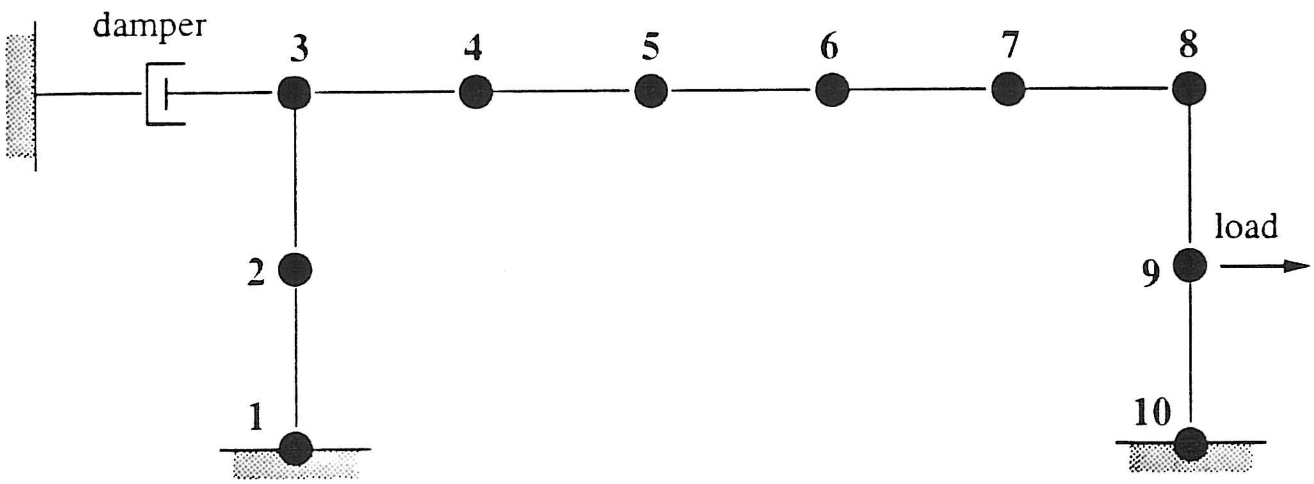


Figure 3 A damped dynamic system

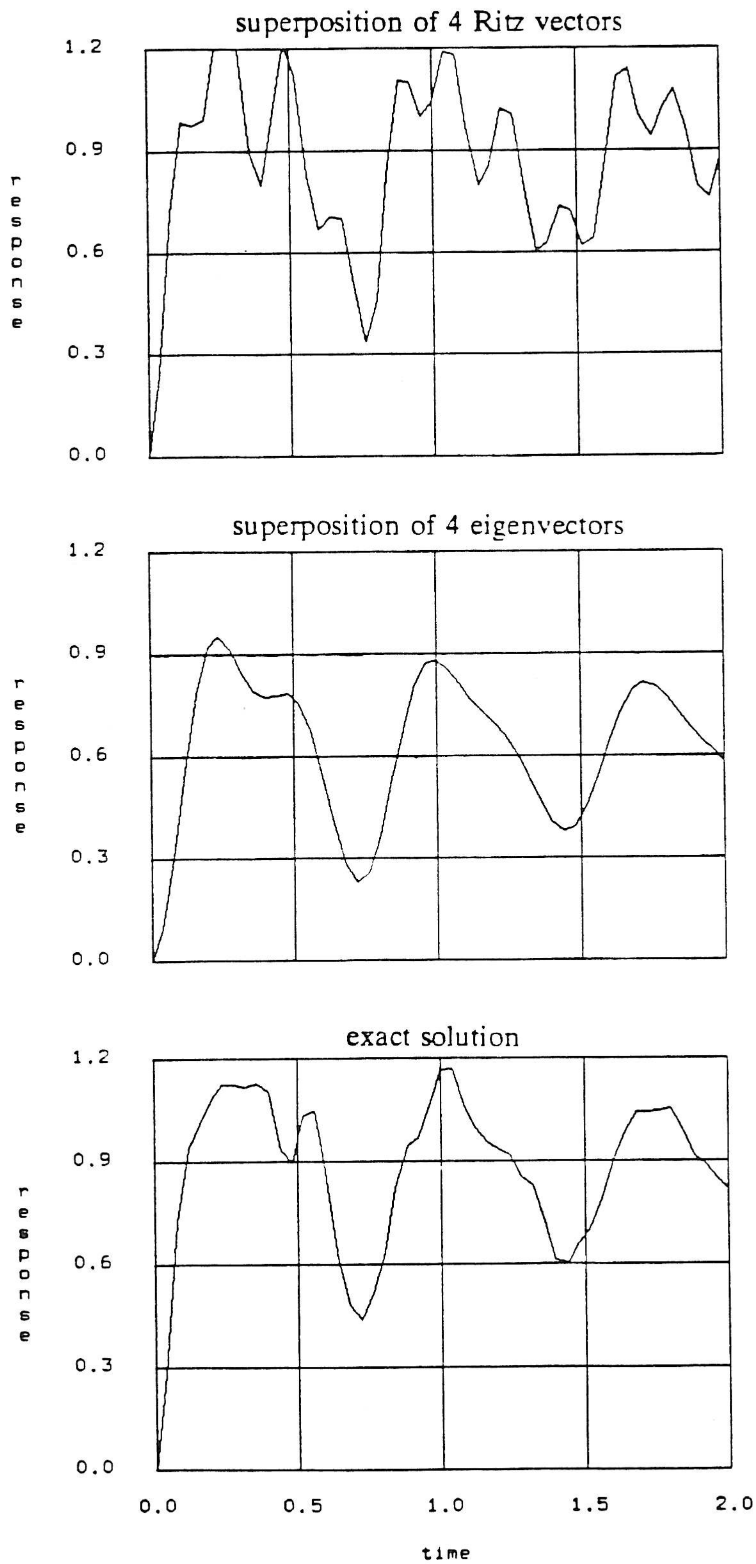


Figure 4a Horizontal displacement at node 8 of Fig. 3

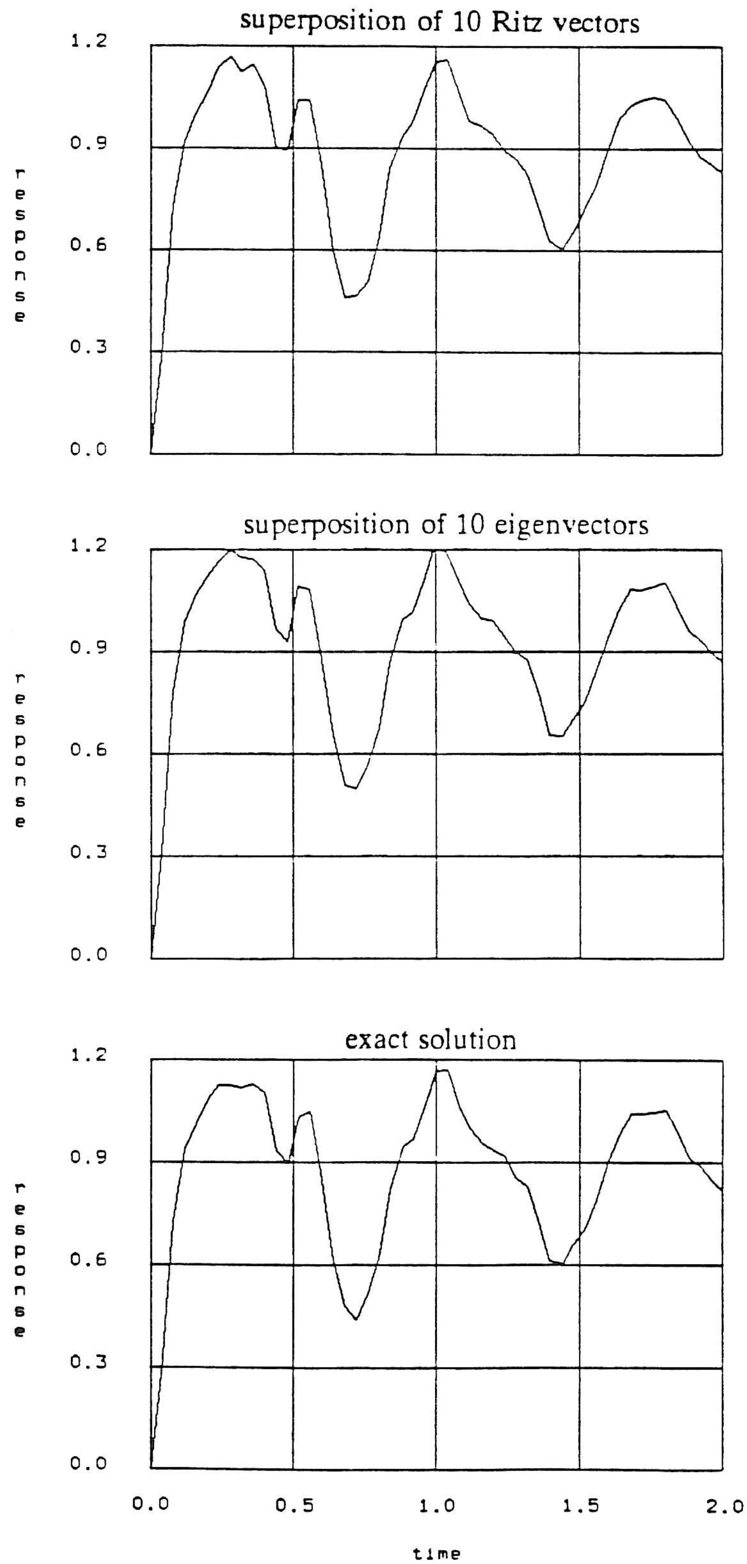
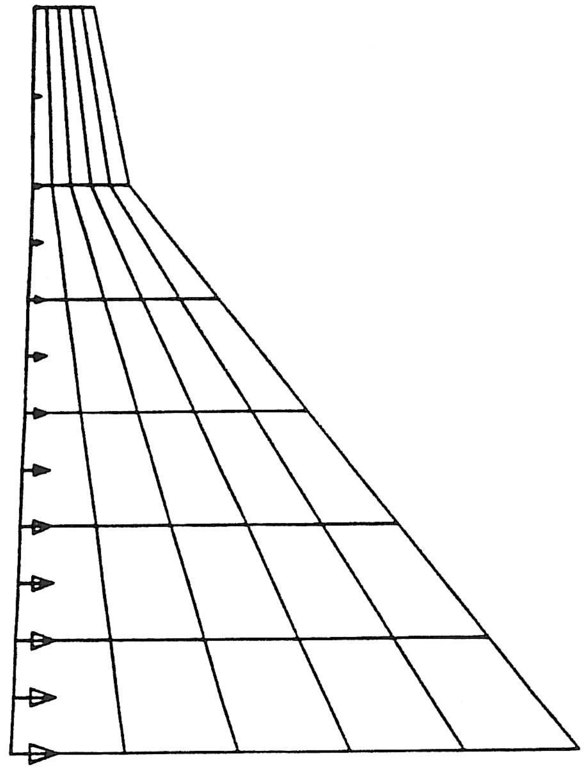


Figure 4b Horizontal displacement at node 8 of Fig. 3



detail for dam and
spatial distribution of loading

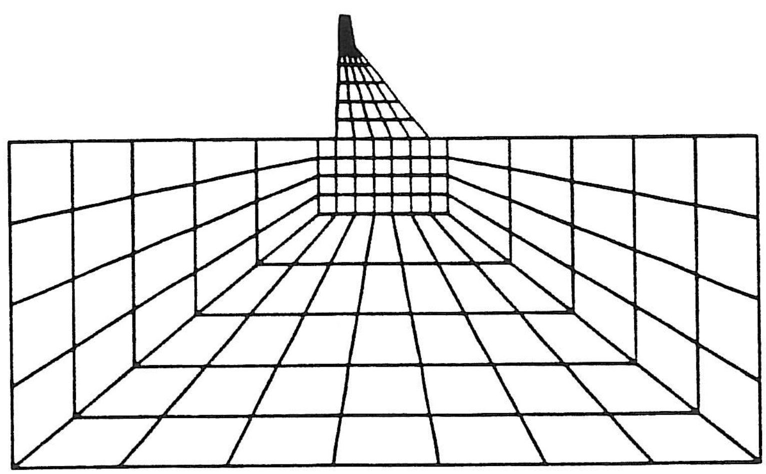


Figure 5 - Dam on foundation

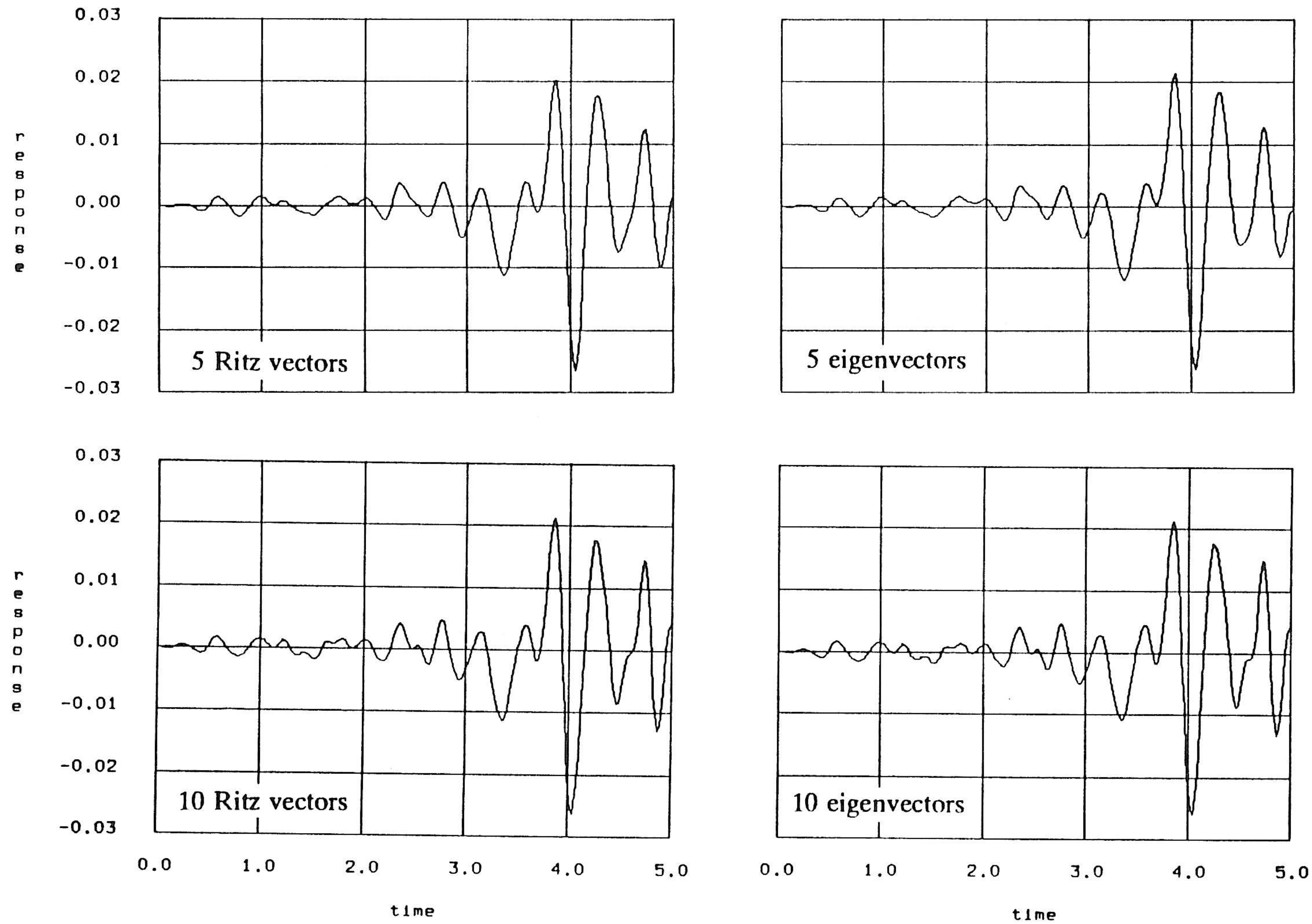


Figure 6 - Horizontal displacement at top Fig. 5