Random Walks and Delocalization through Graph Eigenvector Structure

by

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Professor Olga Holtz
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Abstract

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In this thesis we prove the following results.

1. We show that the multiplicity of the second normalized adjacency matrix eigenvalue of any connected graph of maximum degree Δ is bounded by $O(n\Delta^{7/5}/\log^{1/5-o(1)} n)$ for any Δ, and by $O(n \log^{1/2} d/\log^{1/4-o(1)} n)$ for simple d-regular graphs when $d \geq \log^{1/4} n$.

2. Let G be a random d-regular graph. We prove that for every constant $\alpha > 0$, with high probability every eigenvector of the adjacency matrix of G with eigenvalue less than $-2\sqrt{d-2} - \alpha$ has $\Omega(n/\text{polylog}(n))$ nodal domains.

3. For every $d = p + 1$ for prime p and infinitely many n, we exhibit an n-vertex d-regular graph with girth $\Omega(\log_{d-1} n)$ and vertex expansion of sublinear sized sets upper bounded by $\frac{d+1}{2}$ whose nontrivial eigenvalues are bounded in magnitude by $2\sqrt{d-1} + O\left(\frac{1}{\log n}\right)$. This gives a high-girth version of Kahale's example showing Ramanujan graphs can have poor vertex expansion.

4. Anantharaman and Le Masson proved that any family of eigenbases of the adjacency operators of a family of graphs is quantum ergodic, assuming the graphs satisfy conditions of expansion and high girth. We show that neither of these two conditions is sufficient by itself to imply quantum ergodicity (which is a form of delocalization).

These results although different in nature, all exhibit the utility of the structure of eigenvectors. The main ingredient in the first result is a polynomial (in k) lower bound on the typical support of a closed random walk of length $2k$ in any connected graph, which in turn relies on new lower bounds for the entries of the Perron eigenvector of submatrices of the normalized adjacency matrix. The second result suggests Gaussian behavior of

eigenvectors of random regular graphs conjectured by Elon, a discrete analog of Berry's conjecture. The third result shows that properties that are sufficient to imply eigenvector delocalization are not strong enough to imply vertex expansion. The theorems and examples in the fourth result show why Anantharaman and Le Masson's quantum ergodicity result requires expansion both at a global scale (spectral expansion) and a local scale (high girth).

Contents

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Chapter 1

Introduction

1.1 Overview

Discrete graphs are ubiquitous models applicable to many areas, useful for any discrete system with some notion of connectivity. Some examples for which these are useful are populations, models of quantum systems, electrical networks, and Markov chains.

Often, analyzing these graphs is done through working with the adjacency matrix of the graph, and a surprising amount of information can be ascertained from the spectrum and eigenvectors of this matrix. This thesis will explore the following questions, each concerning the spectrum and eigenvectors of the adjacency matrix.

- **Eigenvalue multiplicity (Chapter 3).** Cheeger's Inequality and the multiway Cheeger inequality [AM85, LOGT14], prove that the connectivity of a graph is related to the largest eigenvalues of the *normalized* adjacency matrix. In a recent groundbreaking result, Jiang, Tidor, Yao, Zhang, and Zhao proved a new bound on the multiplicity of the second largest eigenvalue in the *unnormalized* adjacency matrix of connected, bounded degree graphs to solve the equiangular lines problem [JTY+21]. However, the upper bound they showed was far from the best known proven lower bound, and they asked whether a stronger upper bound on multiplicity could be shown. We do so for the normalized adjacency matrix, which answers their question in the case that the graph is regular.

- **Nodal Domains (Chapter 4).** Nodal domains have been used as a method to study the structure of eigenfunctions in both continuous and discrete spaces. See [Zel17] for a definition and history of nodal domains. It has been conjectured by Elon [Elo08] that eigenvectors of random regular graphs approximately have local statistics of Gaussian random variables. Elon, then Dekel, Lee, and Linial, observed that, according to computer simulations, eigenvectors of random regular graphs have a number of nodal domains that increases linearly in the eigenvalue as the eigenvalue becomes more negative [Elo08, DLL11], which aligns with Elon's conjecture. We prove that eigenvectors of the adjacency matrix with eigenvalue $-2\sqrt{d-2}$ have an almost linear number of nodal domains in n, the number of vertices of the graph.

- **Vertex Expansion (Chapter 5).** Random regular graphs are excellent *vertex expanders*, meaning that all small subsets of vertices neighbor an almost optimally large number of unique vertices, whereas no explicit family of deterministic graphs is proven to have close to this level of vertex expansion. Moreover, Kahale showed that spectral expansion by itself is not enough to imply vertex expansion better than $d/2$ [Kah95]. High-girth implies a graph expands optimally on a local scale, as opposed to global expansion given by optimal spectral expansion. The example given by Kahale has small cycles. Therefore, it was of interest to see whether high girth and expansion together were sufficient to imply vertex expansion. We give an example of a family of graphs that have high girth and almost optimal spectral expansion but have small subsets with poor vertex expansion, namely at most $(d+1)/2$.

- **Quantum Ergodicity (Chapter 6).** In [ALM15], Ananthanaraman and Le Masson introduce a *quantum ergodicity* result for eigenvectors of graphs with high girth and expansion analogous to Shnirelman's Theorem on Riemannian manifolds [Shn74]. In the paper, they state that the condition of spectral expansion acts in place of the condition of ergodicity of the geodesic flow. Therefore, we asked whether the condition of high girth can be dropped, or at least loosened. We show that neither expansion nor high girth are sufficient by themselves to imply quantum ergodicity. We also show that high girth cannot be relaxed to the weaker condition of having the Benjamini-Schramm limit have absolutely continuous spectrum, which is a condition used in later papers.

Connecting these various questions are a list of tools using the structure of eigenvectors of the adjacency matrix. Proving properties about the structure of these eigenvectors allows us to show a surprising amount about the graph and subgraphs, and vice versa. We now motivate and introduce the core ideas used in these proofs.

1.2 Core Ideas

Eigenvectors as a Distribution

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A. Moreover, let S denote a subset of vertices. Jiang et al. reduced the question on eigenvalue multiplicity to one on the change in spectra radius when passing to a principal submatrix A_S. Namely, it is useful to find conditions under which $\lambda_1(A_S) \ll \lambda_1(A)$ for $|S| = n-1$.

Jiang et al. use a combinatorial argument counting walks explictly to find a gap between $\lambda_1(A_S)$ and $\lambda_1(A)$. We find a gap by instead lower bounding entries of the eigenvector of $\lambda_1(A_S)$, then calculating the Rayleigh quotient of a test vector. We show that for a bounded

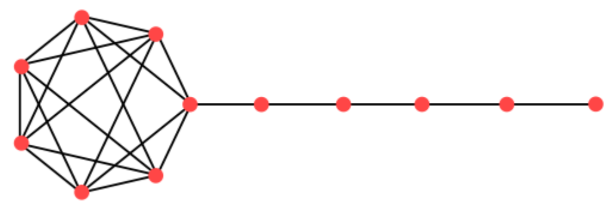

Figure 1.1: The top eigenvector of the adjacency matrix of the lollipop graph decays exponentially as we move down the path. However, if it is a subgraph of a regular graph, then we can extend it near the bulb of the lollipop.

degree, connected, regular graph, for any entry u on the boundary of S in G, we have that $\lambda_1(A) \geq \lambda_1(A_S) + \Omega(\psi_S(u)^2)$, where ψ_S is the eigenvector of $\lambda_1(A_S)$.

Therefore the problem becomes one about the structure of eigenvectors of A_S. To control the size of entries on the boundary of S, we interpret the eigenvector of $\lambda_1(A)$ as the limiting distribution of the endpoint of walks of length $k \to \infty$. We can similarly interpret the eigenvector of $\lambda_1(A_S)$ as a distribution of endpoints of walks that remain within S. Therefore, to show $\lambda_1(A_S) \ll \lambda_1(A)$, it is sufficient to show that a non-negligible portion of walks end at a vertex on the boundary of S.

If our graph is regular, we can do this by showing that some vertex in S that is not of maximum degree has many walks ending at it (see Section 3.1). If we drop the assumption of regularity, then there is not always a boundary vertex that has many walks ending at it. Consider the "lollipop" graph, which consists of a clique connected to a path (Figure 1.1). In the top eigenvector, the entries decay at an exponential rate down the path. If this is a subgraph of a regular graph, then we can extend the lollipop graph near the bulb. If it is not, then we only obtain an exponentially small increase. Intuitively, we sacrifice many choices by having our walk end at the bottom of the tail. Nevertheless, we give a generalization to irregular graphs for when we consider the *normalized* adjacency matrix.

There are at least two other ways that eigenvectors can be considered as a distribution, which have proved informative.

1. An eigenvector of a random graph can be considered as a joint distribution of n random variables with covariance that guarantees the entries satisfy the eigenvector equation. Recent work suggests that for randomly selected regular graphs, the statistics of eigenvectors will approximate the statistics of such a joint distribution of Gaussian random variables [BS19]. This result and interpretation is important to the result in Chapter 4. For sparse random regular graphs, there is also *probabilistic quantum unique ergodicity*, which says that for any fixed unit vector q orthogonal to the all ones vector, with high probability, $\max_i\{|\langle q, \psi_i\rangle|\} \leq \text{polylog}(n)/\sqrt{n}$ where the maximum is taken over $\{\psi_i\}$ a basis of eigenvectors [BHY19, HY21]. Up to the exponent of the polylog, this is the maximum if the ψ_i had Gaussian statistics.

2. The adjacency matrix can be seen as a specific case of a discrete Schrödinger operator

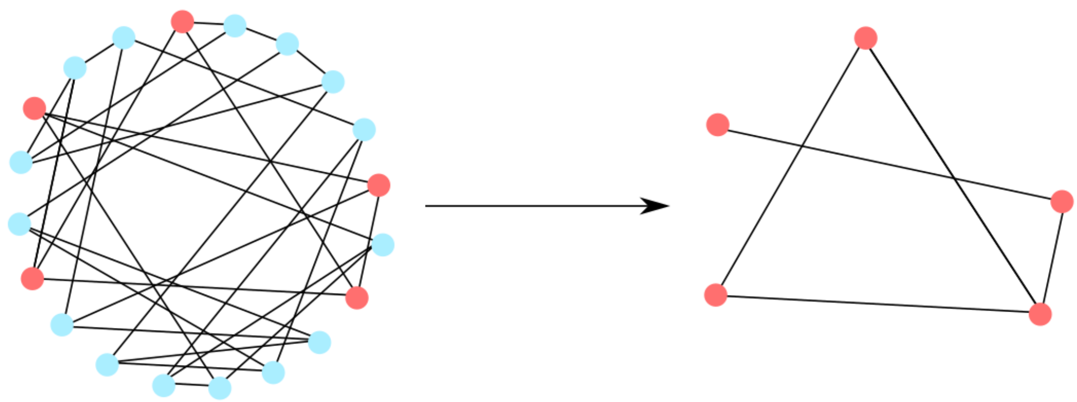

Figure 1.2: If the eigenvector ψ has negligible entries on the blue vertices, then it will be close to being an eigenvector on the induced subgraph on the red vertices.

$H := H_0 + V$, where H_0 is given by the discrete Laplacian, and V is a diagonal matrix of potentials. Therefore the eigenvector of the adjacency matrix, and more generally the eigenvector of a Schrodinger operator of eigenvalue λ, is a wave function with energy λ. Max Born gave an interpretation of the squared entries of the wave function as a probability distribution of the location of a quantum particle. Quantum ergodicity, studied in Chapter 6, is the notion that any reasonable test function (see Chapter 6) is equidistributed according to this distribution.

Eigenvectors and Graph Perturbation

Consider an eigenvector ψ of A with eigenvalue λ. If we delete the edge between vertices u and v and call the new adjacency matrix A', then $\|(A' - \lambda)\psi\| = \sqrt{\psi(u)^2 + \psi(v)^2}$. Similarly, $\psi^* A' \psi = \lambda - 2\psi(u)\psi(v)$. Therefore, the change in these two terms created by deleting an edge is given in terms of $\psi(u)$ and $\psi(v)$, and will be limited if these entries are small. The most extreme version of this statement is that if $\psi(u) = \psi(v) = 0$, then there is no error created by deleting this edge, and ψ is an eigenvector of A' with the same eigenvalue.

We use this type of error bound in multiple ways. First, if we add or remove edges in a graph, the change in quadratic form is given by the size of entries on the boundary of where we have cut or added edges. Specifically, if these entries are small, then ψ is still an eigenvector and has almost the same Rayleigh quotient. This idea is important in the proof in Chapter 4. If our eigenvector is delocalized, we can use its proximity to a Gaussian random variable to show that there are many nodal domains. If, however, it is localized, we show that any localized eigenvector with few nodal domains must have eigenvalue of modulus at most $2\sqrt{d-2} + \epsilon$ for any fixed ϵ. In order to do this, we use proximity of the localized eigenvector to an eigenvector on the induced graph on which it is localized. Given a subset of vertices S (now of arbitrary size), if we assume that the entries are negligible on all vertices not in S, then we can pass to the induced subgraph on S by deleting *all* edges that are not between two vertices in S (see Figure 1.2). Equivalently, the projection of ψ onto the subspace on entries of S will be close to an eigenvector of the same eigenvalue on the induced subgraph on A_S, this time in $|S|$-dimensional space.

This is also a key idea within Chapter 5, as we plant a subgraph that has poor vertex expansion in a Ramanujan graph and show that planting such a subgraph does not significantly change the spectral expansion. We do this by showing that the eigenvector mass on relevant vertices is sufficiently small.

Local Graph Structure Forcing Eigenvector Structure

A third general theme is that of certain structural elements of the graph inducing specific eigenvector structure. For example, eigenvectors of eigenvalue with large modulus are *forced* to be localized. A simple instance of this is the proof that there are many nodal domains for a random regular graph eigenvector with $\lambda < -d + 1 - \epsilon$ given in Section 1.3, where to avoid creating many nodal domains, an eigenvector must have exponentially increasing mass on many paths of vertices. Another example is used in a key part of Chapter 5 is a generalization of a result of Kahale that says that if a vertex v in a regular graph of degree d has a tree-like neighborhood of depth r, then for an eigenvector ψ of eigenvalue $|\lambda| \geq 2\sqrt{d-1}$, the sum of squared entries at distance k from v is at most the sum of squared entries at distance $k + 1$, assuming $k < r$ [Kah95].

We can also find this relation of local structure in the Cartesian product of a graph, where eigenvectors of the product graph can be explicitly given in terms of eigenvectors of the original graph. There are other, similar relations with other graph products, where multiple copies of the same graph are connected in a specific way. These graph products are used in Chapter 6 by giving examples of graphs that, despite having certain desirable combinatorial properties, have eigenvectors that are not equidistributed.

A final example is one given in [GS21] Section 1.1 of the connection between girth and eigenvector localization. For a d-regular graph, if an eigenvector is nonzero only on a small set of vertices S, then every vertex that neighbors S but is outside of S must neighbor S at least twice in order to satisfy the eigenvector equation. If we contract each of these vertices, we have divided the girth by at most two. However, the graph on S now has minimum degree d (see Figure 1.3). By expanding a ball at any vertex, if g is the girth of the new graph, then the size of the new graph is at least $(d-1)^{g/2}$, as this is the number of vertices in a neighborhood of size $g/2$ of any vertex, as the neighborhood is treelike. Therefore, as this new graph has at least half the girth of the original graph, we must have that the girth of the original graph is at most $4\log_{d-1}(|S|)$.

This result gives a shorter version of more involved results that show that eigenvector localization requires cancellation of entries, and therefore short cycles [GS21, AGS19].This can also be seen as a simpler version of an argument in Chapter 6 where the only way to create cancellations in the eigenvector equation, and thereby create localization, is by

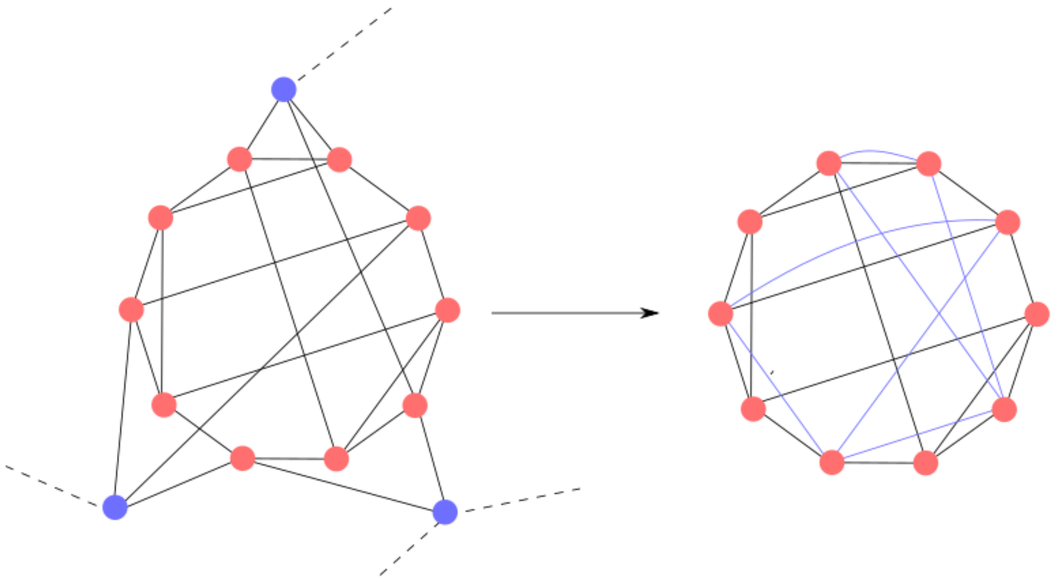

Figure 1.3: In the proof of [GS21], if S is the set of red vertices, and S form the only vertices on which the graph is nonzero, then every vertex neighboring S must have at least two edges going to S. If we then contract these vertices, the induced subgraph on S has minimum degree d.

having short cycles.

Having introduced some of the key ideas of the dissertation, we now introduce our full results in more detail.

1.3 Chapter Overviews

Eigenvalue Multiplicity

This section gives a more specific introduction to Chapter 3. In their beautiful work on the equiangular lines problem, Jiang, Tidor, Yao, Zhang, and Zhao [JTY+21] proved the following novel result constraining the distribution of the adjacency eigenvalues of *all* connected graphs of sufficiently low degree.

Theorem 1.3.1. *If G is a connected graph of maximum degree Δ on n vertices, then the multiplicity of the second largest eigenvalue of its adjacency matrix A_G is bounded by $O(n \log \Delta / \log\log(n))$.*

For their application to equiangular lines, [JTY+21] only needed to show that the multiplicity of the second eigenvalue is $o(n)$, but they asked whether the $O(n/\log\log(n))$ dependence in Theorem 1.3.1 could be improved, noting a huge gap between this and the best known lower bound of $\Omega(n^{1/3})$ achieved by certain Cayley graphs of $\mathrm{PSL}(2, p)$ (see

[JTY+21, Section 4]). Since then, this has been improved to $n^{2/5}-1$ for regular graphs and $\Omega(n/\sqrt{\log n})$ for irregular graphs [HSZZ21]. Apart from Theorem 1.3.1, there are as far as we are aware no known sublinear upper bounds on the second eigenvalue multiplicity for any general class of graphs, even if the question is restricted to Cayley graphs (unless one imposes a restriction on the spectral gap; see Section 1.3 for a discussion).

Meanwhile, in the theoretical computer science community, the largest eigenvalues of the *normalized* adjacency matrix $\tilde{A}_G := D_G^{-1/2} A_G D_G^{-1/2}$ (for D_G the diagonal matrix of degrees) have received much attention over the past decade due to their relation with graph partitioning problems and the unique games conjecture (see e.g. [Kol11, BRS11, LRTV12, OGT13, LOGT14, ABS15, BGH+15, LOG18]); in particular, many algorithmic tasks become easier on graphs with few large normalized adjacency eigenvalues. Thus, it is of interest to know how many of these eigenvalues there can be in the worst case.

In this work, we prove significantly stronger upper bounds than Theorem 1.3.1 on the second eigenvalue multiplicity for the normalized adjacency matrix. Graphs are undirected and allowed to have multiedges and self-loops, unless specified to be simple. Order the eigenvalues of $\tilde{A}_G$ as $\lambda_1(\tilde{A}_G) \geq \lambda_2(\tilde{A}_G) \geq \ldots \geq \lambda_n(\tilde{A}_G)$, and let $m_G(I)$ denote the number of eigenvalues of $\tilde{A}_G$ in an interval I.

Theorem 1.3.2. *If G is a connected graph of maximum degree Δ on n vertices with $\lambda_2(\tilde{A}_G) = \lambda_2$, then*[1]

$$m_G\left([(1 - \frac{\log\log_\Delta n}{\log_\Delta n})\lambda_2, \lambda_2]\right) = \widetilde{O}\left(n \cdot \frac{\Delta^{7/5}}{\log^{1/5} n}\right). \tag{1.1}$$

Because of the relationship $\tilde{A}_G = \frac{1}{d} A_G$ when G is regular, (1.1) gives a substantial improvement on Theorem 1.3.1 in the regular case (in the non-regular case, the results are incomparable as they concern different matrices). In addition to the stronger $O(n/\text{polylog}(n))$ bound, a notable difference between our result and Theorem 1.3.1 is that we control the number of eigenvalues in a small interval containing λ_2. Though we do not know whether the exponents in (1.1) are sharp, we show in Section 3.4 that constant degree bipartite Ramanujan graphs have at least $\Omega(n/\log^{3/2} n)$ eigenvalues in the interval appearing in (1.1), indicating that $O(n/\text{polylog}(n))$ is the correct regime for the maximum number of eigenvalues in such an interval when Δ is constant.

Theorem 1.3.2 is nontrivial for all $\Delta = \widetilde{o}(\log^{1/7} n)$; as remarked in [JTY+21], Paley graphs have degree $\Omega(n)$ and second eigenvalue multiplicity $\Omega(n)$, so some bound on the degree is required to obtain sublinear multiplicity. In Section 1.3, we present a variant of Theorem 1.3.2 (advertised in the abstract) which yields nontrivial bounds in the special case of simple d−regular graphs with degrees as large as $d = \exp(\log^{1/2-\delta} n)$, which is considerably larger than the regime $d = O(\text{polylog}(n))$ handled by [JTY+21].

The main new ingredient in the proof of Theorem 1.3.2 is a polynomial lower bound on the support of (i.e., number of distinct vertices traversed by) a simple random walk of

[1] All asymptotics are as $n \to \infty$ and the notation $\widetilde{O}(\cdot)$ suppresses polyloglog(n) terms.

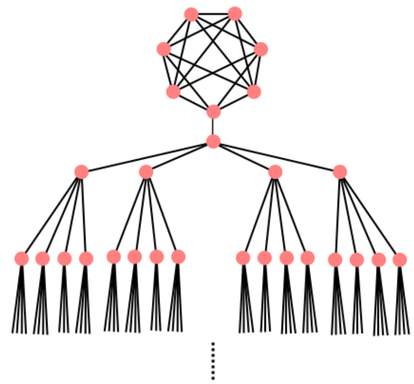

Figure 1.4: For a regular graph composed of a near-clique attached to an infinite tree, a closed walk of length $2k$ starting from within the near-clique does not typically go deeper than $O(\log k)$ down the tree. However, the support of such a closed walk is typically $k^{\Theta(1)}$. See Section 3.7 for a more detailed discussion.

fixed length conditioned to return to its starting point. The bound holds for any connected graph and any starting vertex and may be of independent interest.

Theorem 1.3.3. *Suppose G is connected and of maximum degree Δ on n vertices and x is any vertex in G. Let $\gamma_x^{2k} = (x = X_0, X_1, \ldots, X_{2k})$ denote a random walk of length $2k < n$ sampled according to the simple random walk on G starting at x. Then*

$$\mathbb{P}(\mathrm{support}(\gamma_x^{2k}) \le s | X_{2k} = X_0) \le \exp\left(-\frac{k}{65\Delta^7 s^4}\right) \qquad \text{for} \quad s \le \frac{1}{4}\left(\frac{k}{\Delta^7 \log \Delta}\right)^{1/5}. \tag{1.2}$$

In particular, this means that for constant Δ, the typical support of a closed random walk of length $2k$ is least $\Omega(k^{1/5})$. It may be tempting to compare Theorem 1.3.3 with the familiar fact that a random closed walk of length $2k$ on $\mathbb{Z}$ (or in continuous time, a standard Brownian bridge run for time $2k$) attains a maximum distance of $\Omega(\sqrt{k})$ from its origin. However, as seen in Figure 1.3, there are regular graphs for which a closed walk of length $2k$ from a particular vertex x travels a maximum distance of only polylog(k) with high probability. Theorem 1.3.3 reveals that nonetheless the number of *distinct* vertices traversed is always typically poly(k). We do not know if the specific exponent of $k^{1/5}$ supplied by Theorem 1.3.3 is sharp, but considering a cycle graph shows that it is not possible to do better than $k^{1/2}$.

Given Theorem 1.3.3, our proof of Theorem 1.3.2 follows the strategy of [JTY+21]: since most closed walks in G have large support, the number of such walks may be drastically reduced by deleting a small number of vertices from G. By a moment calculation relating the spectrum to self return probabilities and a Cauchy interlacing argument, this implies an upper bound on the multiplicity of $\lambda_2(\tilde{A}_G)$. The crucial difference is that we are able to delete only $n/\mathrm{polylog}(n)$ vertices whereas they delete $n/\mathrm{poly}\log\log(n)$.

The key ingredient in our proof of Theorem 1.3.3 is a result regarding the Perron eigenvector (i.e., the unique, strictly positive eigenvector with eigenvalue λ_1) of a submatrix of $\tilde{A}$.

Theorem 1.3.4. *For any graph $G = (V, E)$ of maximum degree Δ, take any set of vertices $S \subsetneq V$ such that the induced subgraph on S is connected, and let ψ_S be the ℓ_2-normalized Perron vector of $\tilde{A}_S$, the principal submatrix of $\tilde{A}$ corresponding to vertices in S. Then there is a vertex $u \in S$ which is adjacent to $V \setminus S$ such that*

$$\psi_S(u) \geq 1/(\Delta^{5/2}\lambda_1(\tilde{A}_S)|S|^{5/2}). \tag{1.3}$$

When we restrict this result to G being a d-regular graph and pass to the adjacency matrix, we achieve a result about the unnormalized adjacency matrix of irregular graphs that may be of independent interest.

Corollary 1.3.5. Let $H = (V, E)$ be an irregular connected graph of maximum degree Δ with at least two vertices, and let ϕ_H be the ℓ_2-normalized Perron vector of A_H. Then there is a vertex $u \in V$ with degree strictly less than Δ satisfying

$$\phi_H(u) \geq 1/(\Delta^2\lambda_1(A_H)|V|^{5/2}). \tag{1.4}$$

Corollary 1.3.5 may be compared with existing results in spectral graph theory on the "principal ratio" between the largest and smallest entries of the Perron vector of a connected graph. The known worst case lower bounds on this ratio are necessarily exponential in the diameter of the graph [CG07, TT15]. Corollary 1.3.5 articulates that there is always at least one vertex of non-maximal degree for which the ratio is only polynomial in the number of vertices.

The proof of Theorem 1.3.4 is based on an analysis of hitting times in the simple random walk on G via electrical flows, and appears in Section 3.1. Combined with a perturbation-theoretic argument, it enables us to show that any small connected induced subgraph S of G can be extended to a slightly larger induced subgraph with significantly larger Perron value $\lambda_1(\tilde{A}_S)$. With some further combinatorial arguments, this implies that closed walks cannot concentrate on small sets, yielding Theorem 1.3.3 in Section 3.2, which is finally used to deduce Theorem 1.3.2 in Section 3.3.

We show in Section 3.4 via an explicit example (Figure 1.5) that the exponent of 5/2 appearing in Corollary 1.3.5 is sharp up to polylogarithmic factors. We conclude with a discussion of open problems in Section 3.5.

Remark 1.3.6 (Higher Eigenvalues)**.** An update of the preprint of [JTY+21] generalizes Theorem 1.3.1 to the multiplicity of the jth eigenvalue. Our results can also be generalized in this manner by some nominal changes to the arguments in Section 3.3, but for simplicity we focus on λ_2 in this paper.

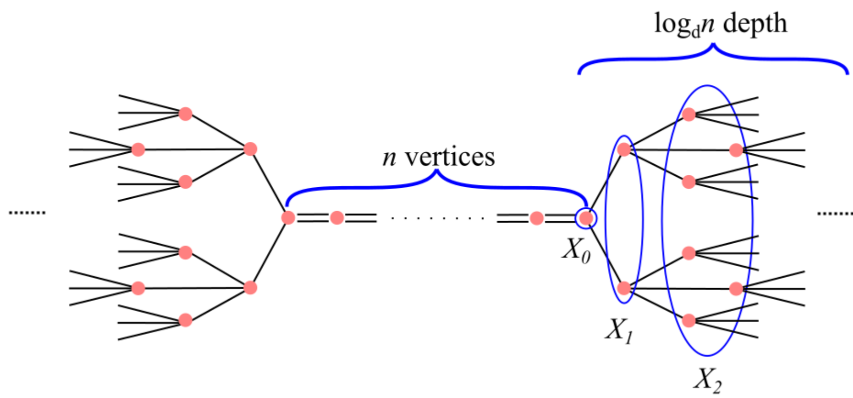

Figure 1.5: An example of a graph where all vertices u that are not of maximum degree have $\psi(u) = \tilde{O}(n^{-5/2})$. The circled sets X_0, X_1 and X_2 will be used in the analysis of the graph in Section 3.4.

Higher degree regular graphs

If $G = (V, E)$ is a simple, d-regular graph, and $S \subsetneq V$ such that $|S| \leq d$, then necessarily all vertices of S are adjacent to vertices in $V \setminus S$. Therefore we can improve the bound from Theorem 1.3.4 by assuming the vertex on the boundary is the maximizer of the Perron vector, which has value $\psi_S(u) \geq 1/\sqrt{|S|}$. This leads to the following variants of our main results for simple, regular graphs of sufficiently high degree.

Theorem 1.3.7. *G is simple, $d-$regular, and connected with $\lambda_2 = \lambda_2(A_G)$, then*

$$m_G\left([(1 - \frac{\log\log_d n}{\log_d n})\lambda_2, \lambda_2]\right) = \begin{cases} \widetilde{O}\left(\frac{n}{d}\right) & \text{when } d = o(\log^{1/4} n) \\ \widetilde{O}\left(\frac{n \log^{1/2} d}{\log^{1/4} n}\right) & \text{when } d = \Omega(\log^{1/4} n). \end{cases} \tag{1.5}$$

The above theorem is based on the following corresponding result for closed walks.

Theorem 1.3.8. *If G is simple, $d-$regular, and connected on n vertices and γ is a random closed walk of length $2k < n$ started at any vertex in G, then:*

$$\Pr(\text{support}(\gamma) \leq s) \leq \exp\left(-\frac{k}{100 s^3}\right) \qquad \text{for} \quad s \leq \min\left\{\frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4}, \frac{d}{2}\right\}. \tag{1.6}$$

The proofs of both theorems appear in Section 3.6

Related Work

Eigenvalue Multiplicity. Despite the straightforward nature of the question, relatively little is known about eigenvalue multiplicity of general graphs. As discussed in [JTY+21], if one assumes that G is a bounded degree expander graph, then the bound of Theorem

1.3.1 can be improved to $O(n/\log n)$. On the other hand, if G is assumed to be a Cayley graph of bounded doubling constant K (indicating non-expansion), then [LM08] show that the multiplicity of the second eigenvalue is at most $\exp(\log^2 K)$. In the context of Cayley graphs, one interesting new implication of Theorem 1.3.7 is that all Cayley graphs of degree $O(\exp(\log^{1/2-\delta} n))$ have second eigenvalue multiplicity $O(n/\log^{\delta/2} n)$.

Distance regular graphs of diameter D have exactly $D+1$ distinct eigenvalues (see [God93] 11.4.1 for a proof). However, besides the top eigenvalue (which must have multiplicity 1), generic bounds on the multiplicity of the other eigenvalues are not known. As expanding graphs have diameter $\Theta(\log_d n)$, the average multiplicity of eigenvalues besides λ_1 for expanding distance regular graphs is $\Theta(n/\log_d n)$. It is tempting to see this as a hint that the multiplicity of the second eigenvalue could be $\Omega(n/\log_d n)$.

Sublinear multiplicity does not necessarily hold for eigenvalues in the interior of the spectrum even assuming bounded degree. In particular, Rowlinson has constructed connected $d-$regular graphs with an eigenvalue of multiplicity at least $n(d-2)/(d+2)$ [Row19] for constant d.

Higher Order Cheeger Inequalities. The results of [LRTV12, LOGT14] imply that if a $d-$regular graph G has a second eigenvalue multiplicity of m, then its vertices can be partitioned into $\Omega(m)$ disjoint sets each having edge expansion $O(\sqrt{d(1-\lambda_2)\log m})$. Combining this with the observation that a set cannot have expansion less than the reciprocal of its size shows that $m = O(n/\text{polylog}(n))$ whenever $1-\lambda_2(\tilde{A}_G) \leq 1/\log^c n$ for any $c > 1$, i.e., the graph is sufficiently non-expanding. Our main theorem may be interpreted as saying that this phenomenon persists for all graphs.

Support of Walks. There are as far as we are aware no known lower bounds for the support of a random closed walk of fixed length in a general graph (or even Cayley graph). It is relatively easy to derive such bounds for bounded degree graphs if the length of the walk is sufficiently larger than the mixing time of the simple random walk on the graph; the key feature of Theorem 1.3.3, which is needed for our application, is that the length of the walk can be taken to be much smaller.

For Cayley graphs, closed walks, and the vertices encountered, have been studied. Erschler studied the typical distance from the starting vertex of the midpoint of a closed walk [Ers06]. Benjamini, Izkovsky and Kesten studied the support of closed walks on Cayley graphs of polynomial growth [BIK07]. Their result implies that for graphs such that the random walk is recurrent, the support is $o(k)$. Our result can be interpreted as attempting to find the correct order of this $o(k)$.

The support of open walks (namely removing the condition that the walk ends at the starting point) is better understood. There are Chernoff-type bounds on the size of the support of a random walk based on the spectral gap [Gil98, Kah97]. Such bounds and their variants are an important tool in derandomization.

Entries of the Perron Vector. There is a large literature concerning the magnitude of the entries of the Perron eigenvector of a graph — see [Ste14, Chapter 2] for a detailed discussion of results up to 2014. Rowlinson showed sufficient conditions on the Perron eigenvector for which changing the neighborhood of a vertex increases the spectral radius [Row90]. Cvetković, Rowlinson, and Simić give a condition which, if satisfied, means a given edge swap increases the spectral radius [CRS93]. Cioabă showed that for a graph of maximum degree Δ and diameter D, $\Delta - \lambda_1 > 1/nD$ [Cio07]. Cioabă, van Dam, Koolen, and Lee then showed that $\lambda_1 \geq (n-1)^{1/D}$ [CVDKL10]. The results of [VMSK+11] prove a lemma similar to Lemma 3.2.2, giving upper and lower bounds on the change in spectral radius from the deletion of edges. However, their result does not quite imply Lemma 3.2.2, and we prove a slightly different statement.

Nodal Domains

Introduction

Here we give a deeper introduction to Chapter 4. Courant's nodal domain theorem states that the zero set of the kth smallest Dirichlet eigenfunction of the Laplacian on a smooth bounded domain in $\mathbb{R}^d$ partitions it into at most k connected components [CH53]. These components, known as the *nodal domains* of the eigenfunction, have garnered significant interest over time in spectral geometry and mathematical physics (see e.g. [Zel17]). The analogous definition for a finite discrete graph $G = (V, E)$ is the following.

Definition 1.3.9 (Nodal domains)**.** A (*weak*) *nodal domain* of a function $f : V \to \mathbb{R}$ on G is a maximal connected subgraph S of G such that $f(u) \geq 0$ for all $u \in S$ or $f(u) \leq 0$ for all $u \in S$. A *strong nodal domain* of $f : V \to \mathbb{R}$ on G is a maximal connected subgraph S of G such that $f(u) > 0$ for all $u \in S$ or $f(u) < 0$ for all $u \in S$.

Fiedler [Fie75] showed that for a tree, the eigenvector of the kth smallest eigenvalue of the discrete Laplacian (defined as $L_G = D_G - A_G$ where D_G is the diagonal matrix of vertex degrees and A_G is the adjacency matrix) has exactly k nodal domains. Davies, Gladwell, Leydold, and Stadler [DGLS00] showed that for an arbitrary graph that the kth Laplacian eigenvector has at most k nodal domains and at most $k + m - 1$ strong nodal domains, where m is the multiplicity of the kth eigenvalue. Berkolaiko [Ber08] showed that for a connected graph with n vertices and $n + \ell - 1$ edges (such that removing ℓ edges would produce a tree) the kth eigenvector of a Schrödinger operator with arbitrary potential has between $k - \ell$ and k nodal domains. Beyond these results, we are not aware of any lower bounds on the number of nodal domains of eigenvectors of any large class of graphs.

Our main result is the following lower bound on the number of nodal domains of a random regular graph[2]. We refer to a nodal domain with a single vertex as a *singleton* nodal domain.

[2]We restrict our attention to weak nodal domains as there are at least as many strong domains as weak domains.

Theorem 1.3.10. *Fix $d \geq 3$ and $\alpha > 0$ and let G be a random d-regular graph on n vertices. Then with probability $1 - o(1)$ as $n \to \infty$, every eigenvector of A_G with eigenvalue $\lambda \leq -2\sqrt{d-2} - \alpha$ has $\Omega\left(\frac{n}{\log^{C_{1.3.10}}(n)}\right)$ singleton nodal domains, where $C_{1.3.10} \leq 301$ is an absolute constant.*

Note that for large enough n, almost every $d-$regular graph has at least $\Omega(d^{-3/2}n)$ eigenvalues with $\lambda \leq -2\sqrt{d-2}$, as the spectrum of A_G converges weakly to the Kesten-McKay measure [McK81]. Since the Laplacian of a $d-$regular graph is equal to $dI - A_G$, the conclusion of the theorem also holds for the "high energy" eigenvectors of the Laplacian with eigenvalues $\lambda \geq d + 2\sqrt{d-2} + \alpha$; we will accordingly also refer to highly negative eigenvalues of the adjacency matrix as high energy.

The proof of Theorem 1.3.10 appears in Section 4.2 to Section 4.5 and employs tools from random matrix theory (ℓ_∞ delocalization of eigenvectors of random regular graphs [HY21]), graph limits (weak convergence of eigenvectors of random regular graphs [BS19]), and combinatorics (expansion and short cycle counts of random regular graphs), and is outlined in Section 1.3. The conceptual phenomenon articulated by the proof is that (under certain conditions) high energy eigenvectors of graphs cannot simultaneously have few nodal domains and be delocalized. A simple demonstration of this tension for the easier case of $d = 3, 4$ is presented in Section 1.3. Due to the use of a weak convergence argument, there is no effective bound on the $o(1)$ probability in the statement of Theorem 1.3.10, and the proof requires d to be constant.

We complement Theorem 1.3.10 by observing in Section 4.6 (Theorem 4.6.2) that by an application of the expander mixing lemma, *every* non-leading eigenvector f of a d-regular expander graph G with sufficiently large spectral gap has two nodal domains which together contain a constant fraction of the vertices of G.

History and Related Work

Random Graphs. Dekel, Lee, and Linial [DLL11] initiated the study of nodal domains of eigenvectors of Erdös-Rényi $G(n, p)$ random graphs. They showed that for constant p, with high probability all but $O(1)$ of the vertices are contained in two large nodal domains for every non-leading adjacency eigenvector. Arora and Bhaskara [AB11] improved this by establishing that when $p \geq n^{-1/19+o(1)}$ there are typically exactly 2 nodal domains in each non-leading eigenvector. H. Huang and Rudelson [HR20] proved that these two domains are approximately the same size for eigenvectors of eigenvalues macroscopically away from the edge when $p \in [n^{-c}, 1/2]$ for some fixed c and also for the first and last $e^{c(\log\log n)^2}$ eigenvectors when $p \in (0, 1)$ is constant. Linial suggested studying the shape of these nodal domains; for example, how many vertices are on the boundary of a domain, what is the distribution of distances to the boundary, etc. For sufficiently dense graphs sampled from $G(n, p)$, this geometry turned out to be trivial — in particular, Rudelson [Rud17, Section 5.2] showed that with high probability, for $G(n, p)$ with fixed $p \in (n^{-c}, 1)$, every vertex is adjacent to $\Omega(n/\text{polylog}n)$ vertices that have the opposite sign in each eigenvector f. This left open the question of nontrivial structure of the nodal domains for sparse graphs[3]. Theorem 1.3.10 and Theorem 4.6.2 show that both the number and the geometry of nodal domains is nontrivial for high energy eigenvalues of sparse random regular graphs.

In contrast to the situation for dense graphs, Dekel, Lee, and Linial observed that in simulations, a randomly selected d-regular graph with d constant has a number of nodal domains that increases as the eigenvalue becomes more negative. Our results confirm their observation that the most negative eigenvalues have many nodal domains.

Random Matrix Theory and Graph Limits. The results for $G(n, p)$ described above rely crucialy on delocalization estimates in random matrix theory.

The proof of Theorem 1.3.10 relies on both ℓ_∞ and ℓ_2 delocalization and combines them in a new way. We first consider no-gaps delocalization at scale $t = 1 - \delta$ for a small constant δ; if this property holds for an eigenvector, we employ a weak convergence result of Backhausz and Szegedy [BS19] to argue that the local distribution of eigenvector entries around a randomly chosen vertex behaves like a Gaussian wave (defined in Section 4.1), implying that a random vertex is a singleton nodal domain with constant probability. Otherwise, we apply the ℓ_∞ delocalization estimate of [BHY19, HY21] to the subset of δn vertices on which the eigenvector is ℓ_2-localized; the ℓ_∞ bound allows us to simplify and exploit the locally almost-treelike structure of the graph on this subset and deduce many singleton nodal domains via a different argument which hinges on the negativity of the eigenvalue λ. Thus, we sidestep the current lack of no-gaps estimates for random regular graphs, as well as the difficulty of examining individual eigenvector entries solely using the Green's function method[4].

Mathematical Physics. The field of quantum chaos aims to relate the classical dynamics of the geodesic flow on a manifold to the behavior of its high energy Laplacian eigenfunctions [Rud08], and the number of nodal domains has also been studied in this context [BGS02]. A guiding question in this area is Berry's random wave conjecture [Ber77], which asserts that the high energy eigenfunctions of quantum chaotic billiards behave like "Gaussian random waves" in the limit. Random d-regular graphs have studied as a discrete model of quantum chaos [KS97, BOS07, Smi13]; in particular, a discrete analogue of Berry's conjecture considered in [Elo08] asserts that the bulk eigenvectors of random d-regular graphs have a (locally) jointly Gaussian distribution with a specific *nonzero* covariance matrix depending on the degree d. This conjecture implies the existence of many nodal domains in random regular graphs. Theorem 1.3.10 proves the implication of the conjecture for sufficiently negative λ, and one branch of its proof (Section 4.2) is directly inspired by

[3] As a starting point, Eldan, H. Huang, and Rudelson asked in 2020 [Rud20] whether the most negative eigenvector of a sparse $G(n, p)$ graph has more than two nodal domains.

[4] The Green's function $(A - zI)^{-1}$ of a random regular graph can only approximate that of the infinite tree when $\Im(z) \geq \text{polylog} n/n$, meaning that it inherently reflects the aggregate behavior of polylog$n$ eigenvectors.

the "Gaussian wave" heuristic, which we make rigorous via the weak convergence result of [BS19].

Low degree case

As a warm-up, we prove a weaker version of Theorem 1.3.10 which applies to any eigenvector of a regular graph with sufficiently negative eigenvalue and an ℓ_∞ bound.

Proposition 1.3.11. *For $\alpha, \eta > 0$, $d \geq 3$, assume f is an eigenvector of a d-regular graph $G = (V, E)$ with eigenvalue $\lambda \leq -(d-1) - \alpha$ and*

$$\|f\|_\infty \leq \frac{\eta}{\sqrt{n}}. \tag{1.7}$$

Then f has at least

$$\frac{n}{(2\eta)^{2+\frac{\log(d-1)}{\log(1+\alpha/(d-1))}}}$$

nodal domains.

Proof. Assume that $u \in V$ is not a singleton nodal domain and $|f(u)| \geq \frac{1}{2\sqrt{n}}$. Then u has at most $d-1$ neighbors v such that $f(u)f(v) \leq 0$, so as $\sum_{v \sim u} f(v) = \lambda f(u)$, we must have that for some neighbor v of u, $|f(v)| \geq (1 + \alpha/(d-1))|f(u)|$. Repeating this argument, if there are no singleton nodal domains at distance at most k from u, then there is a path $(u = x_0, \ldots, x_k)$ such that $|f(x_i)| \geq (1 + \alpha/(d-1))|f(x_{i-1})|$ for each i. By (1.7), we must have $k \leq \tilde{k}$ for

$$\tilde{k} := \frac{\log(2\eta)}{\log(1 + \frac{\alpha}{d-1})}.$$

Every u with $|f(u)| \geq \frac{1}{2\sqrt{n}}$ must have a vertex w that is a singleton nodal domain and $d(u, w) \leq \tilde{k}$. By (1.7), there are at least $\frac{3}{4}n/\eta^2$ vertices u with $|f(u)| \geq 1/2\sqrt{n}$.

Any vertex w has at most $d(d-1)^{\tilde{k}-1}$ vertices at distance at most $\tilde{k}$. Therefore there are at least

$$\frac{\frac{3}{4} \cdot \frac{n}{\eta^2}}{d(d-1)^{\tilde{k}-1}} \geq \frac{n}{(2\eta)^{2+\frac{\log(d-1)}{\log(1+\alpha/(d-1))}}}$$

singleton nodal domains. ◻

The ℓ_∞ delocalization bound of [HY21] corresponds to $\eta = \text{polylog} n$. Thus if $d \leq 4, \alpha > 0$ are fixed and $\lambda \leq -(d-1) - \alpha$, Proposition 1.3.11 yields $\Omega(n/\text{polylog} n)$ nodal domains for an eigenvector of a random d-regular graph, recovering the conclusion of Theorem 1.3.10 up to polylogarithmic factors in the spectral window $[-d, -(d-1) - \alpha]$. We recall that every nontrivial eigenvalue λ of a random $d-$regular graph satisfies $|\lambda| \leq 2\sqrt{d-1} + o(1)$ with high probability [Fri03], so for $d > 5$ there are typically no eigenvectors with $\lambda \leq -(d-1)$ and Proposition 1.3.11 is vacuous. To improve the required bound on λ from $-(d-1)$ to $-2\sqrt{d-2}$, we shift from a local analysis of the entries of f to a more global one.

Proof outline and organization

In Section 4.2, we use the weak convergence result of Backhausz and Szegedy [BS19] to show that, with high probability, if the ℓ_2 mass of an eigenvector f is not concentrated on a set of size δn for a small constant δ, then it has many singleton nodal domains. The remainder of the proof focuses on the case where the eigenvector f is ℓ_2-localized on a small set $S \subset G$. In Section 4.3 we give a deterministic upper bound of the spectral radius of "almost treelike" graphs in terms of their maximum degree, average degree, and girth; in particular, the bound implies that certain small subgraphs of G have small spectral radius, with high probability. In Section 4.4 we show that if f has few singleton nodal domains in S, then we may pass to an edge subgraph $H \subset G[S]$ (of the induced subgraph $G[S]$) of maximum degree at most $d-1$ such that the restriction of f to S, denoted by f_S, satisfies

$$f_S^T A_H f_S \approx f^T A_G f = \lambda. \tag{1.8}$$

This is the step in which both the ℓ_2-localization assumption and the ℓ_∞ bound of [HY21] are crucially used. If λ is sufficiently negative, (1.8) violates the spectral radius bound of Section 4.3 applied to H, so we conclude that there must be many singleton nodal domains of f in S. We combine the above cases to prove Theorem 1.3.10 in Section 4.5. We conclude by showing that any sparse expander graph contains two nodal domains whose total size is large in Section 4.6.

Vertex Expansion

Chapter 5 is concerned with expander graphs, which are ubiquitous in theoretical computer science. A natural and highly well-studied quantity associated with a d-regular graph is its *edge expansion* defined as

$$\min_{|S| \le \epsilon n} E(S, \overline{S})/|S|,$$

for some constant ϵ. Namely it is the minimum ratio of edges leaving a set S to the size of S for all S of appropriately bounded size. While edge expansion is known to be intractable to compute, there are explicit constructions of good edge expanders, and it is closely related to the second largest magnitude eigenvalue of its adjacency matrix, also known as *spectral expansion* of a graph, via the expander mixing lemma and Cheeger's inequality [Alo86]. Spectral expansion is easily computable. In particular, an application of the expander mixing lemma proves that small enough sets in graphs with spectral expansion $o(d)$ have near-optimal edge expansion of $(1 - o_d(1))d$.

A natural analog to edge expansion is *vertex expansion*, defined as

$$\min_{|S| \le \epsilon n} |\Gamma(S)|/|S|$$

for some constant ϵ, where $\Gamma(S)$ is the neighborhood of the set S (potentially containing vertices of S). However, as difficult as edge expansion is to ascertain, vertex expansion has proven far more challenging.

As witnessed by the neighborhood of a vertex, we cannot hope for vertex expansion greater than $d-1$. Therefore we call a graph a *lossless vertex expander* if for every δ, there exists an ϵ such that there is vertex expansion $d-1-\delta$ for sets of size ϵn. Lossless vertex expanders exist since a random d-regular graph is one with high probability (see [HLW18, Theorem 4.16] for a proof). However no deterministic efficient construction of such graphs is known. In an effort to understand lossless vertex expansion better and give explicit constructions, a natural question to ask is: *what properties of random graphs leads to lossless vertex expansion*?

Since a random d-regular graph is near-Ramanujan with high probability [Fri03], and since near-Ramanujan graphs have near-optimal edge expansion, it is natural to inquire if spectral expansion has any implications for vertex expansion as well. Kahale [Kah95] showed that the spectral expansion gives a bound on the vertex expansion. Specifically, Ramanujan graphs (namely graphs with optimal spectral expansion) have vertex expansion at least $d/2$. While this is a nontrivial implication, it falls short of achieving the coveted *losslessness* property. Kahale also proved that the bound of $d/2$ is tight. In particular, he exhibited an infinite family of near-Ramanujan graphs with vertex expansion $d/2$, which means spectral expansion alone is not sufficient for lossless vertex expansion.

The occurrence of a copy of $K_{2,d}$[5] as a subgraph is the obstruction to lossless vertex expansion in Kahale's example. Kahale's example deviates from a random graph in that it is highly unlikely for a random graph to contain a copy of $K_{2,d}$ as a subgraph. More generally, random graphs have the property that with high probability any two "short" cycles are far apart, which Kahale's example doesn't satisfy. Thus, it is natural to ask if the "near-Ramanujan" property in conjunction with the "separatedness of cycles" property of random graphs break past the $d/2$ barrier of Kahale. The "separatedness of cycles" property is especially interesting to consider since it is a key property of random graphs exploited in proofs of Alon's conjecture [Fri03, Bor19]. A concrete question we can ask is: *Do Ramanujan graphs with $\Omega(\log_{d-1} n)$ girth have lossless vertex expansion?*

An affirmative answer to the above question would prove that the Ramanujan graphs of Lubotzky, Phillips, and Sarnak [LPS88] are lossless vertex expanders. Towards answering the above question, we prove the following negative result:

Theorem 1.3.12. *For every $d = p+1$ for prime $p \geq 3$, there is an infinite family of d-regular graphs G on n vertices of girth $\geq \left(\frac{2}{3} - o_n(1)\right)\log_{d-1} n$ where there is a set of vertices U such that $|\Gamma(U)| \leq (d+1)|U|/2$, $|U| \leq n^{1/3}$, and $\max\{\lambda_2(G), -\lambda_n(G)\} \leq 2\sqrt{d-1} + O(1/\log_{d-1} n)$.*

We also complement the above with a positive result which can be summarized as "small enough sets in Ramanujan graphs expand nearly losslessly":

Theorem 1.3.13. *Let G be a d-regular Ramanujan graph with girth $C\log_{d-1} n$, then every set of S of size $\leq n^{\kappa}$ for $\kappa < \frac{C}{4}$ has vertex expansion $(1-o_d(1))d$.*

After posting our preprint, Amitay Kamber informed us that a theorem in an alternative version of [Kah95] gives the same bound by a different argument. Moreover, his theorem

[5] complete bipartite graph with 2 vertices on one side and d vertices on the other

does not depend on the spectral expansion of the graph. However, our proof may be of interest as an alternate method.

Overview of Proof of 1.3.12

Our proof is inspired by that of Kahale's. At a high level, Kahale embeds a copy of $K_{2,d}$ within a Ramanujan graph. We proceed similarly to Kahale, but instead of embedding a $K_{2,d}$, we embed a single subgraph H that is high girth but a lossy vertex expander and show that if H has size n^{α} for some $0 < \alpha \leq 1/3$, the overall graph is still near-Ramanujan.

Our proof involves two steps: the first step is in proving that the subgraph H being embedded has spectral radius bounded by $2\sqrt{d-1}$, and the second step is in proving that planting H within a Ramanujan graph results in a near-Ramanujan graph. For the first step, we describe an infinite graph containing H and bound its spectral radius via a trace moment method. The trace moment method involves bounding the number of closed walks satisfying certain properties within a graph, and is inspired by an encoding argument from Bordenave's proof of Friedman's theorem [Bor19].

The second step is in proving that our method of embedding a copy of H within a Ramanujan graph does not perturb the eigenvalues by a large amount. Towards doing so, we use the fact that the spectral radius of H is bounded by $2\sqrt{d-1}$ in conjunction with Kahale's argument about dispersion of eigenvalues in high-girth graphs.

Overview of Proof of 1.3.13

We first prove that if a set S in a Ramanujan graph has "lossy" vertex expansion, then we can construct a graph H on vertex set S such that (i) the girth of H is at least half the girth of G, and (ii) the average degree of H is "high" (in particular, the worse the vertex expansion of S, the higher the average degree of H). We then employ the irregular Moore bound, which gives a quantitative tradeoff between the average degree of a graph and its girth. In particular, this would imply that a Ramanujan graph with "lossy" vertex expansion necessarily must have "low" girth.

Related Work

Applications of Vertex Expanders There are many applications of expander graphs where having vertex expansion is particularly useful. For example, lossless expanders are particularly of interest in the field of error correcting codes [LMSS01, SS96, Spi96]. Lossless vertex expanders give linear error correcting codes that are decodable in linear time [SS96]. Guruswami, Lee and Razborov [GLR08] use bipartite vertex expanders to construct large subspaces of $\mathbb{R}^n$ where all vectors x in the subspace satisfy $(\log n)^{-O(\log\log\log n)}\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$.

Explicit Constructions Constructions of Ramanujan graphs of [LPS88, Mar88, Mor94] of all degrees that are of the form $p^r + 1$ for p prime, as well as the construction of near-

Ramanujan graphs of every degree of [MOP20] have vertex expansion $\sim \frac{d}{2}$ just by virtue of being Ramanujan via Kahale's result. In fact no deterministic construction has improved upon the $d/2$ bound obtained from solely spectral information. In a remarkable work, Capalbo et. al. [CRVW02] exhibited an explicit construction of a bipartite graph where subsets of one side of the bipartite graph expand losslessly to the other, using a zig-zag product so the the losslessness of a small, random-like graph boosts the expansion from a large, potentially lossy vertex expanding graph.

Quantum Ergodicity Quantum ergodicity is another area where both local and global properties of random-like graphs are used. In particular, Anantharaman and Le Masson [ALM15] proved that graphs that have few short cycles (and are therefore close to high girth) and spectral expansion are quantum ergodic, which in this context means the eigenvectors are equidistributed across vertices. Anantharaman, as well as Brooks, Le Masson, and Lindenstrauss exhibited alternative proofs [Ana17, BLML16]. The proof from [BLML16] shows that quantum ergodicity is equivalent to the mixing of a certain graphical operator. They then use high girth to show that this is equivalent to showing mixing on the infinite tree, then expansion to show the nonbacktracking operator mixes on the tree.

Eigenvector Delocalization Ganguly and Srivastava, and later Alon, Ganguly and Srivastava [GS21, AGS19] give a perturbation of the LPS graph similar to Kahale's argument, but instead of individual vertices, two trees are added and connected to the graph. By assuming the tree is sufficiently deep and carefully connecting the tree to the rest of the graph, the authors create a graph that is high girth but contains eigenvectors that are localized. These graphs are also lossy vertex expanders. However, they show that these graphs cannot be Ramanujan, but rather have spectral radius at least $(2+c)\sqrt{d-1}$ where $c > 0$ is a constant. Alon [Alo21] used eigenvector delocalization to create near-Ramanujan expanders of every degree by perturbing known constructions of Ramanujan or near-Ramanujan graphs. Paredes [Par21] used similar techniques to remove short cycles in a graph while preserving expansion and uses this to algorithmically create graphs that are near-Ramanujan and also have girth at least $\Omega(\sqrt{\log n})$.

Complexity of Constraint Satisfaction Problems Proofs that it is hard for even linear degree Sum-of-Squares to refute random 3XOR and 3SAT instances on n variables [Gri01, Sch08] rely on lossless vertex expansion of some sets in a graph underlying a random instance, which suggests a connection between deterministic algorithms for constructing lossless vertex expanders and algorithms for explicit hard instances for Sum-of-Squares.

Quantum Ergodicity

Chapter 6 concerns the topic of quantum ergodicity. While classical integral systems often have periodic orbits in phase space, eigenstates of quantized chaotic systems tend to be uniformly distributed (see [Zel05]). This phenomenon is expressed in the high energy limit through the concept of quantum ergodicity. Consider a compact Riemannian manifold (M, g) and a basis of eigenfunctions $\{\psi_j\}$ of the Laplace-Beltrami operator Δ on M with eigenvalues $\{\lambda_j\}$. We say $\{\psi_j\}$ is *quantum ergodic* if for every continuous test function $a : M \to \mathbb{R}$,

$$\lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \psi_j, a\psi_j \rangle - \int_M a \, d\mathrm{Vol} \right|^2 = 0.$$

Here $\langle \psi_j, a\psi_j \rangle := \int_M a(x)|\psi(x)|^2 \, d\mathrm{Vol}(x)$ and $N(\lambda) := |\{\lambda_j \leq \lambda\}|$. Shnirelman's Theorem [Shn74, DV85, Zel87] states that if the geodesic flow of M is ergodic with respect to the Liouville measure, then $\{\psi_j\}$ is quantum ergodic.

Discrete graphs have provided a fruitful model for quantum chaos [KS97, KS99], and Brooks and Lindenstrauss initiated the study of conditions of localization and delocalization of eigenvectors on large regular discrete graphs [BL13]. They proved that if small sets in a graph expand well (for example the graph has high girth), all eigenvectors are delocalized in a quantifiable way depending on this expansion.

It is in this context that Anantharaman and Le Masson proved a result on discrete graphs analogous to Shnirelman's Theorem [ALM15]. To introduce this result, we consider an infinite family of d-regular graphs $(G_n) = (V_n, E_n)$ with d constant and $|V_n| = n$. We write $\mathcal{A}_n$ to denote the adjacency operator of G_n. $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathcal{A}_n$. We also require the following definitions.

Definition 1.3.14. The family of graphs (G_n) is said to satisfy *EXP* if there is a constant $\epsilon > 0$ such that for each $\mathcal{A}_n$, $\max\{\lambda_2, |\lambda_n|\} \leq (1 - \epsilon)d$.

Definition 1.3.15 ([BS01])**.** Take μ to be a measure over isomorphism classes of rooted, potentially infinite graphs. The family of graphs (G_n) is said to have Benjamini-Schramm limit μ if for each fixed $R > 0$, as $n \to \infty$, the distribution of isomorphism classes of rooted balls of radius R in G_n around a root selected from V_n uniformly at random converges weakly to the distribution of isomorphism classes of balls of radius R around the roots of graphs according to μ. For a graph H, we say (G_n) has unrooted Benjamini-Schramm limit H if the limiting measure μ is H with a root of H selected uniformly at random.

Definition 1.3.16. The family of graphs (G_n) is said to satisfy *BST* if it has unrooted Benjamini-Schramm limit T_d, where T_d is the infinite d-regular tree. Namely, for all fixed $R > 0$,

$$\lim_{n \to \infty} \frac{|\{x \in V_n, \rho(x) < R\}|}{n} \to 0,$$

where $\rho(x)$ is the injectivity radius of x (the largest r such that the ball of radius r around x is a tree).

These properties together are enough to guarantee quantum ergodicity.

Theorem 1.3.17 ([ALM15] Theorem 1)**.** *Assume that* $(G_n) = (V_n, E_n)$ *is a family of graphs that satisfies EXP and BST. Let* $a_n : V_n \to \mathbb{R}$ *be series of functions such that* $\sum_{v \in V_n} a_n(v) = 0$ *and* $\|a_n\|_\infty \leq 1$*. Then for any series of orthonormal eigenbases* $(\psi_1^{(n)}, \dots, \psi_n^{(n)})$ *of* $(\mathcal{A}_n)$*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\langle \psi_i^{(n)}, a_n \psi_i^{(n)} \rangle|^2 = 0, \tag{1.9}$$

where

$$\langle \psi_i^{(n)}, a_n \psi_i^{(n)} \rangle = \sum_{v \in V_n} a_n(v) |\psi_i^{(n)}(v)|^2.$$

A series of eigenvectors that satisfies (1.9) is called *quantum ergodic*. In fact, the theorem can be generalized to more general operators a_n than given, but the above formulation is sufficient for our purposes.

Anantharaman and Le Masson suggest that the EXP condition is analogous to the requirement of ergodicity in Shnirelman's Theorem. Therefore, it is natural to wonder whether EXP alone is sufficient to necessitate quantum ergodicity, as no other assumptions are made in Shnirelman's Theorem. However, we show that this is not the case.

Theorem 1.3.18. *There is an infinite family of graphs* (G'_n) *satisfying EXP that have a family of orthonormal eigenbases of the adjacency operators* $(\psi_1^{(n)}, \dots, \psi_n^{(n)})$ *that violates quantum ergodicity. Specifically, there is a series of functions* $a_n : V_n \to \mathbb{R}_n$*,* $\|a_n\|_\infty \leq 1$*,* $\sum_{v \in V_n} a_n(v) = 0$ *such that for each* n*,*

$$\frac{1}{n} \sum_{i=1}^{n} |\langle \psi_i^{(n)}, a_n \psi_i^{(n)} \rangle|^2 = 1/2.$$

The family of graphs in Theorem 1.3.18 is $(G_n \square C_4)$ for any family of graphs (G_n) that satisfies EXP. Here $G_1 \square G_2$ denotes the Cartesian product of graphs G_1 and G_2, and C_4 is the cycle graph of length 4. Intuitively, EXP measures expansion at global scales, whereas the Cartesian product creates a pattern on a local scale that causes localization of eigenvectors. The Cartesian product is particularly useful because of the explicit formula of its eigenvectors based on the eigenvectors of the two original graphs. Therefore, because C_4 has an eigenbasis with localized eigenvectors, the series of graphs $(G_n \square C_4)$ all have many localized eigenvectors. In fact, C_4 can be replaced with any graph with an adjacency operator with localized eigenvectors. Moreover, $(G_n \square C_4)$ satisfies EXP because of the relationship between eigenvalues of the adjacency operator of the Cartesian product with those of the adjacency operators of the original graphs. For the various properties of the Cartesian product, see Section 2.3 of Cvetković, Rowlinson, and Simic [CRS97].

Considering we cannot fully remove the requirement of BST, we then try to relax it. In order to necessitate quantum ergodicity in Schrödinger operators [AS19a] and quantum graphs [AMS21]), along with requirements similar to EXP and BST, an extra requirement is added that the imaginary part of the entries of the Green's function of the Benjamini Schramm limit is bounded for all $z \in \mathbb{C}^+$, where $\mathbb{C}^+$ is the upper half of the complex plane.

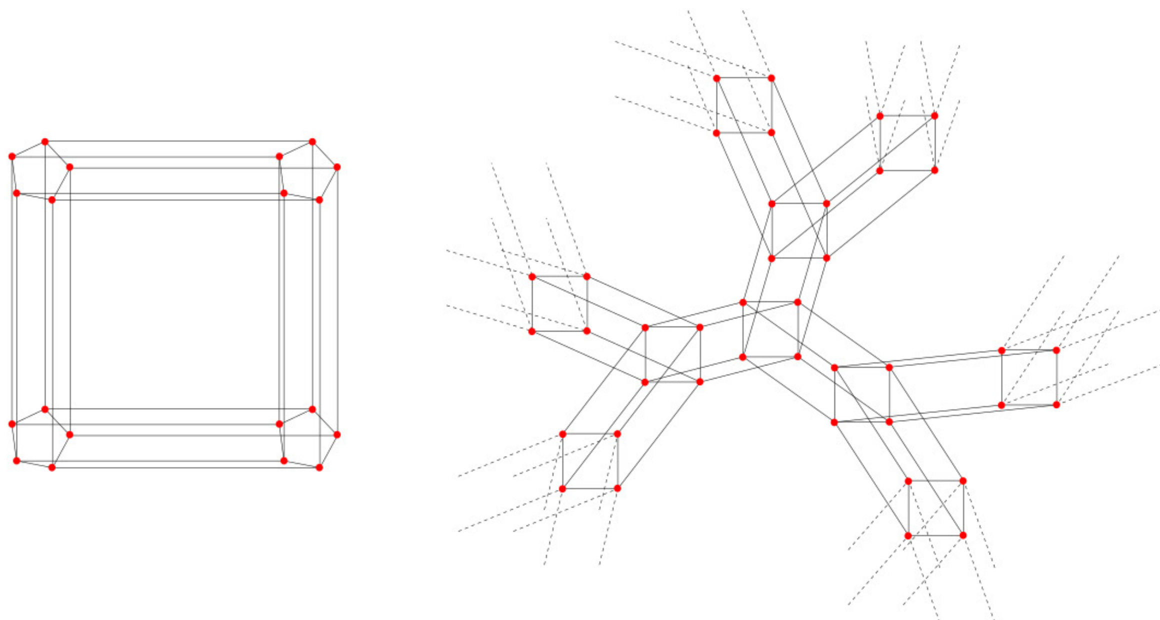

Figure 1.6: Denote by C_k the cycle graph of length k, and T_d the infinite d regular tree. The figure gives $C_4 \square C_5$ and a portion of $T_3 \square C_4$. The Cartesian product $G_1 \square G_2$ can be thought of as replacing each vertex of G_1 with a copy of G_2. Note that $G_1 \square G_2 \cong G_2 \square G_1$.

For the adjacency operator, this property is a generalization of BST, as the Green's function of the infinite tree is known to be have bounded imaginary part for all $z \in \mathbb{C}^+$ (see [AW13] for a proof). Therefore we asked whether we could relax BST to a condition bounding the imaginary part of entries of the Green's function of the Benjamini-Schramm limit.

The first step is to calculate the Benjamini Schramm limit of $(G_n \square X)$. Of course, by Theorem 1.3.17, $(G_n \square C_4)$ cannot satisfy BST. In fact, the Cartesian product creates many cycles at every vertex. We show that the Benjamini-Schramm limit commutes with the Cartesian product, as for any graph X, the sequence of graphs $(G_n \square X)$ converges to the Cartesian product of the Benjamini-Schramm limit of (G_n) with X. Therefore if our family of d-reguler graphs (G_n) satisfies BST, then the Benjamini-Schramm limit of $(G_n \square C_4)$ is $T_d \square C_4$.

Examining the entries of the Green's function in our example, we show that that for an infinite graph G_1 and finite G_2, the Green's function of $G_1 \square G_2$ follows the pattern of the spectrum of the Cartesian product of finite graphs. Namely, we prove the following, which could be of independent interest. Here $\mathcal{G}_G^z$ denotes the Green's function of G at z.

Theorem 1.3.19. *Consider a (potentially infinite) graph G_1 and a finite graph G_2 with adjacency operators $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Let $\psi_1, \dots \psi_k$ be an orthonormal eigenbasis of $\mathcal{A}_2$ with eigenvalues $\lambda_1, \dots, \lambda_k$.*

We have

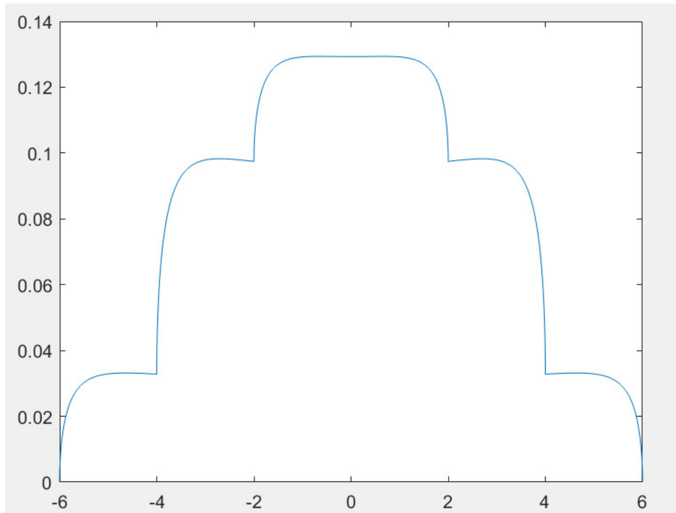

Figure 1.7: A plot of the spectral density of $T_5 \square C_4$. It is the sum of the Kesten-McKay measure shifted by the different eigenvalues of $\mathcal{A}_{C_4}$.

$$\mathcal{G}^z_{G_1 \square G_2} = \sum_{i=1}^{k} \mathcal{G}^{z-\lambda_i}_{G_1} \otimes \psi_i \psi_i^T.$$

Therefore the entries of the Green's function of $T_d \square C_4$ can be written as a linear combination of entries of Green's functions on T_d. As this latter quantity has bounded imaginary part everywhere, $\mathcal{G}^z_{T_d \square C_4}$ also has bounded imaginary part. This means that by taking (G_n) to satisfy both EXP and BST, the family of graphs $(G_n \square C_4)$ satisfies EXP and has bounded imaginary part of the Green's function in the Benjamini-Schramm limit, but nevertheless by Theorem 1.3.18 it violates quantum ergodicity. Therefore, in general, BST cannot be generalized to the requirement of having bounded imaginary part of the Green's function.

We end the chapter with Section 6.3, which shows that BST by itself is not sufficient to necessitate quantum ergodicity.

Theorem 1.3.20. *There is an infinite family of graphs (H_n) satisfying BST that have a family of orthonormal eigenbases of the adjacency operators $(\psi_1^{(n)}, \dots, \psi_n^{(n)})$ that violates quantum ergodicity. Specifically, there is a series of functions $a_n : V_n \to \mathbb{R}_n$, $\|a_n\|_\infty \leq 1$, $\sum_{v \in V_n} a_n(v) = 0$ such that for each n,*

$$\frac{1}{n} \sum_{i=1}^{n} |\langle \psi_i^{(n)}, a_n \psi_i^{(n)} \rangle|^2 \geq 1/d.$$

As EXP measures global expansion, BST measures local expansion, so our example creates localization by creating patterns on a global scale. Our construction is similar to those of [GS21, AGS19, MM21] in that we take a set of high girth graphs and connect them in such a way that creates a geometric phenomenon without destroying girth.

Related work

Other results in graph quantum ergodicity. For an overview of results before 2019, see [AS19b]. Since the original proof of quantum ergodicity, Anantharaman [Ana17] and Brooks, Le Masson, and Lindenstrauss [BLML16] have given alternate proofs of Theorem 1.3.17. Quantum ergodicity statements have since then been found for a variety of graphical models, including quantum graphs [ISW20, AMS21] and the Anderson model on the Bethe lattice [AS17].

Green's function of the Cartesian product. Chung and Yau, then Ellis [CY00, Ell03] proved that the entries of the Green's function of a Cartesian product can be expressed as a contour integral of a function of Green's functions on the two original graphs, assuming that both graphs are finite.

1.4 Bibliographic Note

The principal results of this thesis have appeared previously as published or submitted papers. Chapter 3 was joint work with Peter Rasmussen and Nikhil Srivastava and appeared in [MRS21]. Chapter 4 was joint work with Shirshendu Ganguly, Nikhil Srivastava, and Sidhanth Mohanty, and appeared in [GMMS21]. Chapter 5 was joint work with Sidhanth Mohanty and appeared in [MM21]. Chapter 6 appeared in [McK21].

Chapter 2

Preliminaries

A graph $G = (V, E)$ is a set of vertices V that are connected by edges E. The adjacency matrix A of a graph G is a $|V| \times |V|$ matrix with rows and columns corresponding to vertices. We have

$$A(x, y) = \begin{cases} 1 & x \text{ and } y \text{ are connected by an edge} \\ 0 & \text{otherwise.} \end{cases}$$

For a subset of vertices $S \subseteq V(G)$ we use $G[S]$ to denote the induced subgraph of G on S. We use $N(S)$ to denote the set of vertices that have a neighbor in S. We use $E(S, T)$ to denote the collection of edges with one endpoint in S and one endpoint in T. We use $\overline{S}$ to denote the the set of vertices $V(G) \setminus S$. We use $B_G(S, \ell)$ to denote the induced subgraph on the set of all vertices of distance at most ℓ from S, and we write $B_G(v, \ell) := B_G(\{v\}, \ell)$.

Definition 2.0.1. The *girth* $g(G)$ of a graph G is the length of the smallest cycle in G.

Definition 2.0.2. For $G = (V, E)$, the *valency* of $a \in V$ to $B \subset V$ is $|\Gamma(a) \cap B|$, where $\Gamma(S)$ for $S \subset V$ is the set of neighbors of S in G.

Definition 2.0.3. The *ball* of radius h around a set $U \subset V$, denoted $\mathrm{Ball}_h(U)$, is the set of vertices of distance at most h from U.

Definition 2.0.4. The *vertex expansion* of a set $U \subset V$ is

$$\Psi(U) := \frac{|\Gamma(U)|}{|U|}.$$

Similarly, the ϵ-vertex expansion of a graph G is:

$$\Psi_\epsilon(G) = \min_{|U| \leq \epsilon |V|} \Psi(U)$$

where U ranges over subsets of V, and ϵ is an arbitrary constant.

Definition 2.0.5. The *ε-edge expansion* of a graph G, denoted $\Psi_\varepsilon(G)$, is defined as:

$$\Phi_\varepsilon(G) := \max_{\substack{S \subseteq V(G) \\ |S| \leq \varepsilon n}} \frac{|E(S, \overline{S})|}{|S|}.$$

Definition 2.0.6 (Bicycle-freeness)**.** We say G is ℓ-bicycle-free if for every vertex v, $B_G(v, \ell)$ contains at most 1 cycle.

The eigenvalues of the adjacency matrix of G are denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \lambda_n(G)$. When G is clear from the context, it is dropped.

Definition 2.0.7. The *spectral expansion* of a finite graph G, denoted $\lambda(G)$ is defined as $\max\{\lambda_2(G), -\lambda_n(G)\}$, which can equivalently be described as the "second largest absolute eigenvalue".

We now state the following standard fact known as the *expander mixing lemma* (see [HLW18, Lemma 2.5]).

Lemma 2.0.8 (Expander Mixing Lemma)**.** *Let G be a d-regular graph on n vertices. For any two subsets of vertices, $S, T \subseteq V(G)$, let $e(S,T)$ be the number of pairs of vertices (x, y) such that $x \in S, y \in T$ and $\{x, y\}$ is an edge in G. Then:*

$$\left| E(S,T) - \frac{d}{n}|S| \cdot |T| \right| \leq \lambda(G) \sqrt{|S| \cdot |T|}.$$

Definition 2.0.9. Given a graph G, we use A_G to denote its adjacency matrix. When G is a finite graph on n vertices, the eigenvalues of A_G can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$.

Definition 2.0.10. We use B_G to denote the *nonbacktracking matrix* of a graph G which is a matrix with rows and columns indexed by directed edges of G defined as follows:

$$B[(u,v),(w,x)] = \begin{cases} 1 & \text{if } v = w \text{ and } u \neq x \\ 0 & \text{otherwise.} \end{cases}$$

We state the "irregular Moore bound" of [AHL02] which articulates a tradeoff between the average degree of a graph and its girth.

Lemma 2.0.11. *Let G be a n-vertex graph with average degree-d. Then*

$$g(G) \leq 2 \log_{d-1} n + 2.$$

Given a vector $f \in \mathbb{R}^{V(G)}$, we use f_S to denote the vector in $\mathbb{R}^S$ obtained by restricting f to coordinates in S. We also will write $\|f\| := \|f\|_2$. For a matrix A, we use $\|A\|$ to denote the spectral norm of A.

We write $\Pr_{x\sim\mu}(E)$ to denote the probability that a random variable x sampled from the distribution μ satisfies E.

All logarithms are base e unless noted otherwise.

Electrical Flows. We use $\mathrm{Reff}_H(\cdot,\cdot)$ to denote the effective resistance between two vertices in H, viewing each edge of the graph as a unit resistor. See e.g. [DS84] or [Bol13, Chapter IX] for an introduction to electrical flows and random walks on graphs.

Graphs. For a matrix M, we use M_S to denote the principal submatrix of M corresponding to the indices in S. Consider a graph $G = (V, E)$ and a subset $H \subset V$. Let $P := AD^{-1}$ be the transition matrix of the simple random walk matrix on G, where A is the adjacency matrix and D is the diagonal matrix of degrees. We will also use the normalized adjacency matrix $\tilde{A} := D^{-1/2}AD^{-1/2}$. Note that P and $\tilde{A}$ are similar, and that $\tilde{A}$ is symmetric. P_S and $\tilde{A}_S$ are submatrices of P and $\tilde{A}$; they are not the transition matrices and normalized adjacency matrices of the induced subgraph on S. Note P_S and $\tilde{A}_S$ are also similar.

Perron Eigenvector. We use ψ_S to denote the ℓ_2-normalized eigenvector corresponding to $\lambda_1(\tilde{A}_S)$, which is a simple eigenvalue if S is connected. Note that for connected S, ψ_S is strictly positive by the Perron-Frobenius theorem.

A simple graph refers to a graph without multiedges or self-loops. We assume the maximum degree $\Delta \geq 2$ for all connected regular graphs, since otherwise the graph is just an edge, so $\log \Delta > 0$.

Chapter 3

Eigenvalue Multiplicity

3.1 Lower Bounds on the Perron Eigenvector

In this section we prove Theorem 1.3.4, which is a direct consequence of the following slightly more refined result. In a graph $G = (V, E)$, define the boundary of S as the set of vertices in S adjacent to $V \backslash S$ in G.

Theorem 3.1.1 (Large Perron Entry)**.** *Let $G = (V, E)$ be a connected graph of maximum degree Δ and $S \subsetneq V$ such that the induced subgraph on S is connected. Then there is a vertex $u \in S$ on the boundary of S such that*

$$\psi_S(u)/\psi_S(t) \geq 1/(\Delta^{5/2}\lambda_1(\tilde{A}_S)|S|^2) \tag{3.1}$$

where $t = \arg\max_{w \in S} \psi_S(w)$.

At a high level, the proof proceeds as follows. First, we show that there exists a vertex $x \in S$ adjacent to the boundary of S such that a random walk started at x is somewhat likely to hit t before it hits the boundary of S. Second, we express the ratio of $D_S^{1/2}\psi_S(x)$ and $D_S^{1/2}\psi_S(t)$ as a limit as $k \to \infty$ of the ratio $\mathbb{P}Y_x^k/\mathbb{P}Y_t^k$, where Y_v^k is the event that the simple random walk started at v remains in S for k steps; we bound this ratio from below using the hitting time estimate from the first step. Third, by the eigenvector equation the ratio of the entries of an eigenvector at two neighboring vertices is bounded. Hence, x is adjacent to some vertex u on the boundary of S satisfying the theorem.

Proof. Write $S = M \sqcup B$, where B is the boundary of S and $M = S \setminus B$. If $t \in B$ then we are done, so assume not. Let $\mathbb{P}_x^G(\cdot)$ denote the law of the simple random walk (SRW) $(X_i)_{i=0}^{\infty}$ on G started at $X_0 = x$, and for any subset $T \subset V$, let $\tau_T := \{\min i : X_i \in T\}$ denote the hitting time of the SRW to that subset; if $T = \{u\}$ is a singleton we will simply write τ_u.

Step 1. We begin by showing that there is a vertex $x \in M$ adjacent to B for which the random walk started at x is reasonably likely to hit t before B. To do so, we use the well-known connection between hitting probabilities in random walks and electrical flows. Define a new graph $K = (V' = V \setminus B \cup \{s\}, E')$ by contracting all vertices in B to a single vertex s. Let

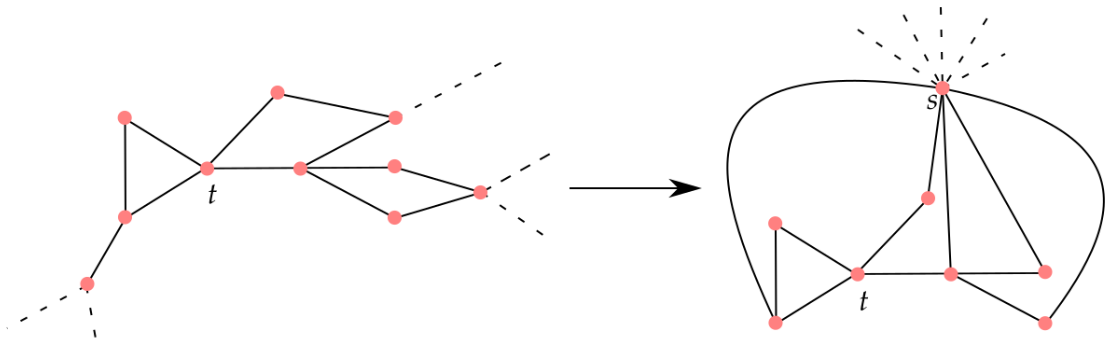

Figure 3.1: In Step 1 of the proof of Theorem 3.1.1, we lower bound the probability that a random walk started at a certain vertex x adjacent to B reaches t before reaching B. We do this by contracting B to a vertex s, then lower bounding the current from s to t, which establishes the existence of the desired x. The left graph in the figure is G and the right graph is the contracted graph K, with the dotted lines indicating edges leaving the set of interest $S = M \sqcup B$.

$f : V' \to [0, 1]$ be the vector of voltages in the electrical flow in K with boundary conditions $f(s) = 0, f(t) = 1$, regarding every edge as a unit resistor. By Ohm's law, the current flow from s to t is equal to $1/\mathrm{Reff}_K(s, t)$. We have the crude upper bound

$$\mathrm{Reff}_K(s, t) \le \mathrm{distance}_K(s, t) \le |S|,$$

since S is connected, so the outflow of current from s is at least $1/|S|$. By Kirchhoff's current law, there must be a flow of at least $1/(|S| \deg_K(s))$ on at least one edge $(s, x) \in E'$. By Ohm's law again, for this particular $x \in V'$ we must have

$$f(x) \ge \frac{1}{|S| \deg_K(s)} = \frac{1}{|S||\partial_G B|} \ge \frac{1}{\Delta |S|^2}, \tag{3.2}$$

where $\partial_G B$ denotes the edge boundary of B in G. Appealing to e.g. [Bol13, Chapter IX, Theorem 8], this translates to the probabilistic bound

$$\mathbb{P}_x^G(\tau_t < \tau_B) = \mathbb{P}_x^K(\tau_t < \tau_s) = f(x) \ge \frac{1}{\Delta |S|^2}. \tag{3.3}$$

Finally, since $f(s) = f(y) = 0$ for every $y \in V \setminus S$, we must in fact have $x \in M$.

Step 2. We now use (3.3) to show that $\psi_S(x)$ is large. Because $\tilde{A}_S = D_S^{-1/2} P_S D_S^{1/2}$, the top eigenvector of P_S is $D_S^{1/2}\psi_S / \|D_S^{1/2}\psi_S\|$. Let $P' : (P + I)/2$ denote the lazy random walk[1] on G,

[1]This modification is only to ensure non-bipartiteness; if S is not bipartite we may take the simple random walk

and to ease notation let $\mathbb{P}'_x(\cdot) := \mathbb{P}'^{G}_x(\cdot)$ denote the law of the lazy random walk on G started at x. Note that the eigenvectors of P_S, as well as $\mathbb{P}_x(\tau_t < \tau_B)$, do not change when passing to P'_S.

For the lazy random walk, the Perron-Frobenius theorem implies that

$$\frac{(D_S^{1/2}\psi_S)(w)}{\|D_S^{1/2}\psi_S\|} = \lim_{k\to\infty} \frac{\mathbf{1}_S^T {P'_S}^k e_w}{\|\mathbf{1}_S^T {P'_S}^k\|},$$

for every $w \in S$, where $\mathbf{1}_S \in \mathbb{R}^S$ is the all ones vector. We further have

$$\mathbf{1}_S^T {P'_S}^k e_w = \mathbb{P}'_w(\tau_{V\setminus S} > k),$$

namely the probability a random walk of length k starting at w stays in S.

We are interested in the ratio

$$\frac{(D_S^{1/2}\psi_S)(x)}{(D_S^{1/2}\psi_S)(t)} = \lim_{k\to\infty} \frac{\mathbb{P}'_x(\tau_{V\setminus S} > k)}{\mathbb{P}'_t(\tau_{V\setminus S} > k)}. \tag{3.4}$$

Fix an integer $k > 0$. The numerator of (3.4) is bounded as

$$\begin{aligned}
\mathbb{P}'_x(\tau_{V\setminus S} > k) &\geq \mathbb{P}'_x(\tau_{V\setminus S} > k | \tau_t < \tau_B)\mathbb{P}'_x(\tau_t < \tau_B) \\
&\geq \frac{1}{\Delta|S|^2}\mathbb{P}'_x(\tau_{V\setminus S} > k | \tau_t < \tau_B) \qquad \text{by (3.3)} \\
&\geq \frac{1}{\Delta|S|^2}\sum_{\theta=0}^{k-1} \mathbb{P}'_x(\tau_{V\setminus S} > k | \tau_t = \theta, \tau_t < \tau_B)\mathbb{P}'_x(\tau_t = \theta | \tau_t < \tau_B) \qquad (3.5) \\
&= \frac{1}{\Delta|S|^2}\sum_{\theta=0}^{k-1} \mathbb{P}'_t(\tau_{V\setminus S} > k - \theta)\mathbb{P}'_x(\tau_t = \theta | \tau_t < \tau_B) \\
&\geq \frac{1}{\Delta|S|^2}\sum_{\theta=0}^{k-1} \mathbb{P}'_t(\tau_{V\setminus S} > k)\mathbb{P}'_x(\tau_t = \theta | \tau_t < \tau_B). \qquad (3.6)
\end{aligned}$$

Observe that $\mathbb{E}'_x\tau_B < \infty$ since G is connected. Thus,

$$\begin{aligned}
\sum_{\theta=0}^{k-1} \mathbb{P}'_x(\tau_t = \theta | \tau_t < \tau_B) &= 1 - \mathbb{P}'_x(\tau_t \geq k | \tau_t < \tau_B) \\
&\geq 1 - \frac{\mathbb{P}'_x(\tau_B \geq k)}{\mathbb{P}'_x(\tau_t < \tau_B)} \\
&\geq 1 - \frac{\mathbb{E}'_x\tau_B}{k}\cdot\Delta|S|^2 \quad \text{by Markov and (3.3).}
\end{aligned}$$

Combining this bound with (3.6), we have

$$\mathbb{P}'_x(\tau_{V\setminus S} > k) \geq \frac{1}{\Delta|S|^2}\left(1 - \frac{\mathbb{E}'_x\tau_B}{k}\cdot\Delta|S|^2\right)\mathbb{P}'_t(\tau_{V\setminus S} > k)$$

Taking the limit as $k \to \infty$ in (3.4) yields

$$\frac{(D_S^{1/2}\psi_S)(x)}{(D_S^{1/2}\psi_S)(t)} \geq \frac{1}{\Delta |S|^2}.$$

Step 3. Since x is adjacent to B, we can choose a $u \in B$ adjacent to x. The eigenvector equation and nonnegativity of the Perron vector now imply $\Delta\lambda_1(A_S)\psi_S(u) \geq \psi_S(x)$, whence

$$(D_S^{1/2}\psi_S)(u) \geq \frac{1}{\lambda_1(\tilde{A}_S)\Delta^2|S|^2}(D_S^{1/2}\psi_S)(t). \tag{3.7}$$

Therefore, as D is a diagonal matrix, and the entries of D range from 1 to Δ, it must be the case that

$$\psi_S(u) \geq \frac{1}{\lambda_1(\tilde{A}_S)\Delta^{5/2}|S|^2}\psi_S(t).$$

□

Remark 3.1.2. As the proof shows, the right-hand side of (3.1) may be replaced with $1/\Delta^{3/2}\lambda_1(\tilde{A}_S)|\partial_G B|R$ where B is the boundary of S in G and R is the maximum effective resistance between two vertices in S.

Proof of Corollary 1.3.5. Given an irregular graph H, construct a Δ−regular graph G containing H as an induced subgraph (it is trivial to do this if we allow G to be a multigraph). Repeating the above proof on G with $S = H$ and observing that $D_S^{1/2}$ is a multiple of the identity since G is regular, (3.7) yields the desired conclusion. □

3.2 Support of Closed Walks

In this section we prove Theorem 1.3.3, which is an immediate consequence of the following slightly stronger result. Let $W^{2k,s}$ denote the event a simple random walk of length $2k$ has support at most s and ends at its starting point.

Theorem 3.2.1 (Implies Theorem 1.3.3)**.** *If G is connected and of maximum degree Δ on n vertices, then for every vertex $x \in G$ and $k < n/2$,*

$$\mathbb{P}_x(W^{2k,s}) \leq \exp\left(-\frac{k}{65\Delta^7 s^4}\right)\mathbb{P}_x(W^{2k,2s}) \qquad \text{for} \quad s \leq \frac{1}{4}\left(\frac{k}{\Delta^7 \log\Delta}\right)^{1/5}. \tag{3.8}$$

The proof requires a simple lemma lower bounding the increase in the Perron value of a subgraph upon adding a vertex in terms of the Perron vector.

Lemma 3.2.2 (Perturbation of λ_1)**.** *Take the normalized adjacency matrix $\tilde{A} := D^{-1/2}AD^{-1/2}$ of a graph $G = (V, E)$ of maximum degree Δ. For any $S \subsetneq V$ and vertex $u \in S$, the submatrix which includes the subset $S' = (S \cup \{v\}, E(S) \cup \{(u, v)\})$, which adds a vertex v and the edge (u, v) to S, satisfies*

$$\lambda_1(\tilde{A}_{S'}) \geq \frac{1}{2}\left(\lambda_1(\tilde{A}_S) + \sqrt{\lambda_1(\tilde{A}_S)^2 + \Delta^{-2}\psi_S(u)^2}\right).$$

Proof. The largest eigenvalue of $\tilde{A}$ is at least the quadratic form associated with the unit vectors

$$g_\alpha(x) = \begin{cases} \sqrt{1-\alpha^2}\psi_S(x) & x \in V \\ \alpha & x = v \end{cases}$$

for $0 \leq \alpha \leq 1$. We have $g_\alpha^T \tilde{A} g_\alpha = (1-\alpha^2)\lambda_1(\tilde{A}_S) + d_u^{-1/2} d_v^{-1/2} \alpha \sqrt{1-\alpha^2}\psi_S(u)$, where d_u is the degree of u in G. This quantity is maximized when

$$\alpha = \sqrt{\frac{1}{2} - \frac{\lambda_1(\tilde{A}_S)}{2\sqrt{\lambda_1(\tilde{A}_S)^2 + d_u^{-1} d_v^{-1}\psi_S(u)^2}}},$$

at which

$$g_\alpha^T \tilde{A} g_\alpha = \frac{1}{2}\left(\lambda_1(\tilde{A}_S) + \sqrt{\lambda_1(\tilde{A}_S)^2 + d_u^{-1} d_v^{-1}\psi_S(u)^2}\right).$$

□

Combining Lemma 3.2.2 and Theorem 3.1.1 yields a bound on the increase of the top eigenvalue of the submatrix corresponding to an induced subgraph that may be achieved by adding vertices to it.

Lemma 3.2.3 (Support Extension)**.** *For any connected graph $G = (V, E)$ of maximum degree Δ, consider its normalized adjacency matrix $\tilde{A}$. For any connected subset $S \subsetneq V$ such that $2 \leq |S| = s < |V|/2$, there is a connected subset $T \subset V$ containing S such that $|T| = 2s$ and*

$$\lambda_1(\tilde{A}_T) \geq \lambda_1(\tilde{A}_S)\left(1 + \frac{5}{128\Delta^7 s^4}\right).$$

Proof. Define $\lambda_1 := \lambda_1(\tilde{A}_S)$ and note that $\lambda_1 \geq 1/\Delta$ since S contains at least one edge. As ψ_S is a normalized vector with s entries, $\psi_S(t) \geq 1/\sqrt{s}$. Therefore $\psi_S(u) \geq 1/(\Delta^{5/2}\lambda_1 s^{5/2})$. Take v to be any vertex in $V \setminus S$ that neighbors u in G. By Lemma 3.2.2,

$$\begin{aligned}\lambda_1(\tilde{A}_{S\cup\{v\}}) &\geq \frac{1}{2}\left(\lambda_1 + \sqrt{\lambda_1^2 + \Delta^{-2}\psi_S(u)^2}\right)\\ &\geq \lambda_1 + \frac{\psi_S(u)^2}{4\lambda_1\Delta^2} - \frac{\psi_S(u)^4}{16\lambda_1^3\Delta^4}\\ &\geq \lambda_1 + \frac{1}{6\lambda_1^3\Delta^7 s^5} \qquad \text{as } \psi_S(u)^2/\lambda_1^2 \leq \Delta^2\\ &\geq \lambda_1 + \frac{1}{6\Delta^7 s^5} \qquad \text{since } \lambda_1 \leq 1. \end{aligned} \tag{3.9}$$

Assuming that $s < |V|/2$, we can iterate this process s times, adding the vertices $\{v_1, \ldots v_s\}$. At each step we add the vertex v_i and increase the Perron eigenvalue of $\tilde{A}_{S\cup\{v_1,\ldots,v_{i-1}\}}$ by at least $1/(6\Delta^7(s+i-1)^5)$. Therefore, defining $T = S \cup \{v_1, \ldots v_s\}$, we have

$$\lambda_1(\tilde{A}_T) \geq \lambda_1 + \frac{1}{6\Delta^7}\sum_{i=1}^{s}\frac{1}{(s+i-1)^5} \geq \lambda_1 + \frac{5}{128\Delta^7 s^4},$$

where the last inequality follows from approximating the sum with the integral. As $\lambda_1 \leq 1$, this translates to the desired multiplicative bound. □

Proof of Theorem 3.2.1. We begin by showing (3.8). Let Γ_x^s be the set of connected subgraphs of G with s vertices containing x. Choose S to be the maximizer of $e_x^T \tilde{A}_S^{2k} e_x$ among $S \in \Gamma_x^s$, and let $T \in \Gamma_x^{2s}$ be the extension of S guaranteed by Lemma 3.2.3 to satisfy

$$\lambda_1(\tilde{A}_T) \geq \left(1 + \frac{5}{128\Delta^7 s^4}\right)\lambda_1(\tilde{A}_S).$$

P_S^{2k} has the same diagonal entries as $\tilde{A}_S^{2k}$, so

$$\mathbb{P}_x(W^{2k,s}) \leq \sum_{S'\in\Gamma_x^s} e_x^T \tilde{A}_{S'}^{2k} e_x,$$

since each walk of length $2k$ satisfying $W^{2k,s}$ is contained in at least one $S' \in \Gamma_x^s$. Furthermore, $|\Gamma_x^s| \leq \Delta^{2s}$ since each subgraph of Γ_x^s may be encoded by one of its spanning trees, which may in turn be encoded by a closed walk rooted at x traversing the edges of the tree. We then have

$$\begin{aligned}\mathbb{P}_x(W^{2k,s}) &\leq |\Gamma_x^s| e_x^T \tilde{A}_S^{2k} e_x\\ &\leq \Delta^{2s}\lambda_1(\tilde{A}_S)^{2k}\\ &\leq \Delta^{2s}\left(1 + \frac{5}{128\Delta^7 s^4}\right)^{-2k}\lambda_1(\tilde{A}_T)^{2k}.\end{aligned} \tag{3.10}$$

We will bound the right hand side in terms of $\mathbb{P}_x(W^{2k,2s})$.

We claim that for every $z \in T$,

$$e_x^T \tilde{A}_T^{2k} e_x \geq \Delta^{-4s} e_z^T \tilde{A}_T^{2k-4s} e_z. \tag{3.11}$$

To see this, let π be a path in T of length $\ell \leq 2s$ between x and z, which must exist since T is connected and has size $2s$. Then every closed walk of length $2k - 2\ell$ in T rooted at z may be extended to a walk of length $2k$ in T rooted at x by attaching π and its reverse. Performing the walk of π twice occurs with probability at least $\Delta^{-2\ell}$. Since all of the walks produced this way are distinct, we have

$$e_x^T \tilde{A}_T^{2k} e_x \geq \Delta^{-2\ell} e_z^T \tilde{A}_T^{2k-2\ell} e_z.$$

By the same argument $e_z^T \tilde{A}_T^{2k-2\ell} e_z \geq \Delta^{-4s+2\ell} e_z^T \tilde{A}_T^{2k-4s} e_z$, and inequality (3.11) follows.

Choose $z \in T$ to be the maximizer of $e_z^T \tilde{A}_T^{2k-4s} e_z$, for which we have:

$$e_z^T \tilde{A}_T^{2k-4s} e_z \geq \frac{1}{2s} \operatorname{Tr}(P_T^{2k-4s}) \geq \frac{\lambda_1(\tilde{A}_T)^{2k-4s}}{2s}.$$

Combining this with (3.11) and substituting in (3.10), we obtain

$$\begin{aligned} \mathbb{P}_x(W^{2k,s}) &\leq \Delta^{6s} \cdot 2s \left(1 + \frac{5}{128\Delta^7 s^4}\right)^{-2k} \lambda_1(\tilde{A}_T)^{4s} e_x^T \tilde{A}_T^{2k} e_x \\ &\leq \Delta^{6s} \cdot 2s \left(1 + \frac{5}{128\Delta^7 s^4}\right)^{-2k} \lambda_1(\tilde{A}_T)^{4s} \mathbb{P}_x(W^{2k,2s}). \end{aligned}$$

Applying the inequality $e^{x/2} \leq 1 + x$ for $0 < x < 1$ and the bound $\lambda_1(\tilde{A}_T) < 1$, we obtain

$$\mathbb{P}_x(W^{2k,s}) \leq \exp\left(6s \log \Delta + \log(2s) - \frac{5k}{128\Delta^7 s^4}\right) \mathbb{P}_x(W^{2k,2s}), \tag{3.12}$$

which implies

$$\mathbb{P}_x(W^{2k,s}) \leq \exp\left(-\frac{k}{65\Delta^7 s^4}\right) \mathbb{P}_x(W^{2k,2s})$$

whenever

$$s \leq \frac{1}{4} \left(\frac{k}{\Delta^7 \log(\Delta)}\right)^{1/5},$$

establishing (3.8).

□

3.3 Bound on Eigenvalue Multiplicity

In this section we prove Theorem 1.3.2, restated below in slightly more detail.

Theorem 3.3.1 (Detailed Theorem 1.3.2)**.** *Let G be a maximum degree Δ connected graph on n vertices. If*[2] $\Delta \le \log^{1/7} n/\log\log n$ *then the spectrum of the normalized adjacency matrix* $\tilde{A}$ *satisfies*

$$m_G\left([(1-\frac{\log\log_\Delta n}{\log_\Delta n})\lambda_2, \lambda_2]\right) = O\left(n\cdot\frac{\Delta^{7/5}(\log^{2/5}\Delta)\log\log n}{\log^{1/5} n}\right). \tag{3.13}$$

Proof. For now, assume that $|\lambda_n(P)| \le |\lambda_2(P)|$. Let $\mathbb{P}(\cdot)$ denote the law of a simple random walk (SRW) γ of length $2k$ on G, started at a vertex chosen uniformly at random (i.e., *not* from the stationary measure of the SRW). Let $W^{2k} := W^{2k,n}$ denote the event that γ returns to its starting vertex after $2k$ steps. In an abuse of notation, let $W^{2k,\ge s+1} := W^{2k}\backslash W^{2k,s}$ be the event that a walk of length $2k$ is closed and has support at least $s+1$.

Set $k := \frac{1}{3}\log_\Delta n$ and $c := 2\log k$ and let s be a parameter satisfying

$$\mathbb{P}(W^{2k,s}) \le e^{-c}\mathbb{P}(W^{2k}) \tag{3.14}$$

to be chosen later. Delete cn/s vertices from G uniformly at random and call the resulting graph H.

If γ has support at least $s+1$, then the probability that none of the vertices of γ are deleted is at most

$$\left(1-\frac{s}{n}\right)^{\frac{cn}{s}} \le e^{-c}.$$

Thus,

$$\mathbb{E}_H\mathbb{P}(\gamma\subset H|\gamma\in W^{2k,\ge s+1}) \le e^{-c},$$

where $\mathbb{E}_H$ is the expectation over H. It then follows by the probabilistic method that there exists a deletion such that the resulting subgraph H of G satisfies

$$\mathbb{P}(W^{2k,\ge s+1}\cap\{\gamma\subset H\}) \le e^{-c}\mathbb{P}(W^{2k,\ge s+1}).$$

Write $\lambda_2 := \lambda_2(\tilde{A}_G)$ and let m' be the number of eigenvalues of H in the interval $[(1-\epsilon)\lambda_2, \lambda_2]$ for $\epsilon := c/2\log_\Delta(n)$. Since $2k$ is even,

$$\begin{aligned}
m'(1-\epsilon)^{2k}\lambda_2^{2k} &\le \mathrm{Tr}(\tilde{A}_H^{2k}) \\
&= n\mathbb{P}(W^{2k}\cap\{\gamma\subset H\}) \\
&= n(\mathbb{P}(W^{2k,s}\cap\{\gamma\subset H\}) + \mathbb{P}(W^{2k,\ge s+1}\cap\{\gamma\subset H\})) \\
&\le n(\mathbb{P}(W^{2k,s}) + e^{-c}\mathbb{P}(W^{2k,\ge s+1})) \quad \text{by our choice of } H \\
&\le n(e^{-c}\mathbb{P}(W^{2k}) + e^{-c}\mathbb{P}(W^{2k,\ge s+1})) \quad \text{by (3.14)} \\
&\le 2e^{-c}\,\mathrm{Tr}(\tilde{A}_G^{2k}) \\
&\le 2e^{-c}(n\lambda_2^{2k}+1).
\end{aligned}$$

We may assume that the diameter of G is at least 10 as otherwise $\Delta \ge n^{1/10}$, making the theorem statement vacuous. Because of the diameter, we can take four edges

[2] If $\Delta \ge \log^{1/7} n/\log\log n$ then (1.1) is vacuously true.

$(u_1, v_1), (a_1, b_2), (u_2, v_2), (a_2, b_2)$ such that the distance between each pair of edges is at least 2. Then consider the vectors ϕ_1, ϕ_2 such that for $i \in \{1, 2\}$

$$\phi_i(x) = \begin{cases} 1 & x \in \{u_i, v_i\} \\ -1 & x \in \{a_i, b_i\} \\ 0 & \text{otherwise} \end{cases}$$

Choose real numbers α and β such that at least one is nonzero. We have

$$\frac{(\alpha\phi_1 + \beta\phi_2)^T D^{-1/2} A D^{-1/2} (\alpha\phi_1 + \beta\phi_2)}{(\alpha\phi_1 + \beta\phi_2)^T (\alpha\phi_1 + \beta\phi_2)} \geq \frac{\frac{4}{\Delta}(\alpha^2 + \beta^2)}{4(\alpha^2 + \beta^2)} \geq \frac{1}{\Delta}.$$

Therefore by Courant Fischer

$$\lambda_2 \geq \min_{\alpha,\beta} \frac{(\alpha\phi_1 + \beta\phi_2)^T D^{-1/2} A D^{-1/2} (\alpha\phi_1 + \beta\phi_2)}{(\alpha\phi_1 + \beta\phi_2)^T (\alpha\phi_1 + \beta\phi_2)} \geq \frac{1}{\Delta}.$$

By our choice of k, this means $n\lambda_2^{2k} \geq 1$. Moreover,

$$\epsilon \leq \frac{2 \log\log n}{2 \log_\Delta n} \leq \frac{\log \Delta \log\log n}{\log n} < 1/2,$$

based on our assumptions on Δ. Thus, $1 - \epsilon \geq e^{-1.5\epsilon}$. Combining these facts,

$$m' \lambda_2^{2k} \leq 4 e^{3k\epsilon - c} n \lambda_2^{2k},$$

yielding

$$m' \leq 4ne^{3k\epsilon - c} \leq 4ne^{-c/2} = O\left(\frac{n}{\log_\Delta n}\right).$$

As we created H by deleting cn/s vertices, it follows by Cauchy interlacing that the number of eigenvalues of $\tilde{A}$ in $[(1-\epsilon)\lambda_2, \lambda_2]$ is at most

$$\frac{cn}{s} + O\left(\frac{n}{\log_\Delta n}\right).$$

We now show that taking

$$s := \frac{1}{4}\left(\frac{k}{\Delta^7 \log \Delta}\right)^{1/5}$$

satisfies (3.14). Applying Theorem 3.2.1 equation (3.8) to each $x \in G$ and summing, we have

$$\begin{aligned} \frac{\mathbb{P}(W^{2k,s})}{\mathbb{P}(W^{2k})} &\leq \exp\left(-\frac{k}{65\Delta^7 s^4}\right) \\ &\leq \exp\left(-\Omega\left(\frac{\log n \log^{2/5} \Delta}{\Delta^{7/5}}\right)\right) \\ &\ll \exp(-c) = \exp(-\Theta(\log\log_\Delta n)), \end{aligned}$$

satisfying (3.14) for sufficiently large n, and we conclude that

$$m_G\left([(1-\frac{\log\log_\Delta n}{\log_\Delta n})\lambda_2,\lambda_2]\right)=O\left(n\cdot\frac{\Delta^{7/5}\log^{2/5}\Delta\log\log n}{\log^{1/5}n}\right),$$

as desired.

If $|\lambda_n|>|\lambda_2|$, we can do a lazy walk with probability of moving $p=\frac{1}{2}$, therefore making all eigenvalues nonnegative. This is equivalent to doubling the degree of every vertex by adding loops. This is the equivalent of taking the simple random walk on a graph with maximum degree 2Δ, requiring $s\le\frac{1}{11}\left(\frac{k}{\Delta^7\log\Delta}\right)^{1/5}$, yielding the same asymptotics. □

3.4 Examples

In this section, we consider examples demonstrating some of the points raised in the introduction regarding the tightness of our results. As most of our results in this section are combinatorial rather than probabilistic, we will consider multiplicity in the non-normalized adjacency matrix A. For regular graphs, this is equivalent.

Bipartite Ramanujan Graphs

We show that bipartite Ramanujan graphs (see [LPS88]; known to exist for every $d\ge 3$ by [MSS15]) have high multiplicity near λ_2. This means that the bound of $n/\log^{\Theta(1)}n$ of Theorem 1.3.2 is tight.

Theorem 3.4.1 (Friedman [Fri91] Corollary 3.6)**.** *Let G be a d-regular graph on n vertices. Then*

$$\lambda_2(A_G)\ge 2\sqrt{d-1}(1-O(1/\log^2 n)).$$

Lemma 3.4.2 (McKay [McK81] Lemma 3)**.** *The number of closed walks on the infinite d-regular tree of length $2k$ starting at a fixed vertex is $\Theta\left(\frac{4^k(d-1)^k}{k^{3/2}}\right)$.*

Proposition 3.4.3. *There exists a constant $\alpha>0$ such that for fixed d, every bipartite d-regular bipartite Ramanujan graph G on n vertices satisfies*

$$m_G\left([\lambda_2(1-\alpha\frac{\log\log(n)}{\log(n)}),\lambda_2]\right)=\Omega(n/\log^{3/2}(n)).$$

Proof. By Theorem 3.4.1,

$$\lambda_2\left(1-\alpha\frac{\log\log(n)}{\log(n)}\right)\le 2\sqrt{d-1}\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right),$$

for sufficiently large n. Let $k=\beta\log n$ for some constant β to be set later and suppose that there are m eigenvalues of A_G in the interval $[2\sqrt{d-1}\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right),\lambda_2]$. Recall that the spectrum of a bipartite graph is symmetric around 0. From Lemma 3.4.2 it follows that for some constant C,

$$\begin{aligned} Cn\left(\frac{4^k(d-1)^k}{k^{3/2}}\right) &\leq \sum_{i=1}^{n} \lambda_i(A_G)^{2k} \\ &\leq 2d^{2k} + (n-2m)\left(2\sqrt{d-1}\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right)\right)^{2k} + 2m(2\sqrt{d-1})^{2k}. \end{aligned}$$

If we let β be sufficiently small and $\alpha > \frac{3}{2\beta}$, rearranging yields

$$\begin{aligned} \frac{m}{n} &\geq \frac{C\frac{4^k(d-1)^k}{k^{3/2}} - \frac{2d^{2k}}{n} - \left(2\sqrt{d-1}\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right)\right)^{2k}}{2(2\sqrt{d-1})^{2k}\cdot\left(1-\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right)^{2k}\right)} \\ &= \Omega\left(\frac{1-2n^{2\beta-1}}{k^{3/2}} - \frac{\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right)^{2k}}{1-\left(1-\frac{1}{2}\alpha\frac{\log\log(n)}{\log(n)}\right)^{2k}}\right) \\ &= \Omega\left(\frac{1}{k^{3/2}} - \frac{1}{e^{\alpha\beta\log\log(n)}}\right) \\ &= \Omega\left(\frac{1}{k^{3/2}}\right). \end{aligned}$$

□

Mangrove Tree

This section shows that the dependence on $|V|$ in Corollary 1.3.5 is tight up to polylogarithmic factors. Our example begins with a path of multiedges containing n vertices, where each multiedge of the path is composed of $d/2$ edges for some even d. At both ends of the path, we attach a tree of depth $\log_{d-1} n$. The roots have degree $d/2$ and all other vertices (besides the leaves) have degree d. Therefore the only vertices in the graph that are not degree d are the leaves of the two trees. Call this graph Q. An example of this graph is shown in Figure 1.5.

Proposition 3.4.4. *For every vertex u of degree less than d, $\psi_Q(u) = \tilde{O}(n^{-5/2})$, where $\tilde{O}$ suppresses dependence on logarithmic factors and d.*

Therefore, we cannot hope to do significantly better than our analysis in Lemma 3.2.3, in which we find a vertex u of non-maximal degree with $\psi(u) \geq 1/(d\lambda_1 n^{5/2})$.

Proof. For simplicity, call $\lambda_1(A_Q) = \lambda_1$ and $\psi_Q = \psi$. By the symmetry of the graph, the value of ψ at vertices in the tree is determined by the distance from the root. Call the entries of ψ corresponding to the tree $r_0, r_1, \ldots, r_\ell$, where the index indicates the distance from the root.

By the discussion in the proof of Kahale [Kah95] Lemma 3.3, if we define

$$\theta := \log\left(\frac{\lambda_1}{2\sqrt{d-1}} + \sqrt{\frac{\lambda_1^2}{4(d-1)} - 1}\right),$$

then for $0 \le i \le \ell$, entries of the eigenvector must satisfy

$$\frac{r_i}{r_0} = \frac{\sinh((\ell+1-i)\theta)(d-1)^{-i/2}}{\sinh((\ell+1)\theta)}$$

where ℓ is the depth of the tree.

Therefore, $r_\ell/r_0 = \frac{\sinh(\theta)(d-1)^{-\ell/2}}{\sinh((\ell+1)\theta)}$. Examining the various terms, $\sinh(\theta) \le d$ and $(d-1)^{-\ell/2} = \frac{1}{\sqrt{n}}$. To bound the third term, we use the definition $\sinh(x) = (e^x - e^{-x})/2$, which yields

$$\sinh((\ell+1)\theta) \ge \frac{1-o_n(1)}{2}\left(\frac{\lambda_1}{2\sqrt{d-1}} + \sqrt{\frac{\lambda_1^2}{4(d-1)} - 1}\right)^{\log_{d-1} n+1}.$$

λ_1 is at least the spectral radius of the path of length n with $d/2$ multiedges between vertices. This spectral radius is $d\cos(\pi/(n+1))$. This gives

$$\begin{aligned}
\sinh((\ell+1)\theta) &\ge \frac{1-o_n(1)}{2(2\sqrt{d-1})^{\log_{d-1} n+1}}\left(d(1-\frac{\pi^2}{2n^2}) + \sqrt{d^2(1-\frac{\pi^2}{2n^2})^2 - 4d + 4}\right)^{\log_{d-1} n+1} \\
&\ge \frac{1-o_n(1)}{2(2\sqrt{d-1})^{\log_{d-1} n+1}}(d+d-2)^{\log_{d-1} n+1}\left(1 - O\left(\frac{d}{n^2}\right)\right)^{\log_{d-1} n+1} \\
&\ge \frac{1-o_n(1)}{2}e^{-O(d\log_{d-1} n/n^2)}\sqrt{n} \ge \frac{\sqrt{n}}{3}
\end{aligned}$$

for large enough n. Therefore

$$\frac{r_\ell}{r_0} = \frac{\sinh(\theta)(d-1)^{-\ell/2}}{\sinh((\ell+1)\theta)} \le \frac{3d}{n}. \tag{3.15}$$

At this point, we know the ratio between r_ℓ and r_0, but still need to bound the overall mass of the eigenvector on the tree. A "regular partition" is a partition of vertices $V = \bigsqcup_{i=0}^k X_j$ where the number of neighbors a vertex $u \in X_i$ has in X_j does not depend on u. We can create a *quotient matrix*, where entry i, j corresponds to the number of neighbors a vertex $u \in X_i$ has in X_j. For an overview of quotient matrices and their utility, see Godsil, [God93, Chapter 5]. In our partition, every vertex in the path is placed in a set by itself. The vertices of each of the two trees are partitioned into sets according to their distance from the two roots. Call the matrix according to this partition B_Q. We denote by $B_Q(X_i, X_j)$ the entry in B_Q corresponding to edges from a vertex in X_i to X_j.

Define $X_0, \ldots X_\ell$ as the sets corresponding to vertices in the first tree of distance $0, \ldots, \ell$ from the root. For $1 \le j \le \ell - 1$, $B_Q(X_0, X_1) = d/2$. $B_Q(X_j, X_{j+1}) = d - 1$. Moreover, for $0 \le j \le \ell - 1$, $B_Q(X_{j+1}, X_j) = 1$. All values between vertices in the path are unchanged at $d/2$.

Consider the diagonal matrix D with $D_{i,i} := |X_i|^{-1/2}$. $D^{-1}B_QD$ is a symmetric matrix. Define $C := D^{-1}B_QD$ We now have $C(X_{j+1}, X_j) = C(X_j, X_{j+1}) = \sqrt{d-1}$ for $1 \le j \le \ell - 1$, and $C(X_0, X_1) = C(X_1, X_0) = \sqrt{d/2}$.

If a vector ϕ is an eigenvector of C, then $D\phi$ is an eigenvector of B_Q with the same eigenvalue. By the definition of D this means

$$\psi_C(X_i)^2 = \sum_{u \in X_i} \psi_{A_Q}(u)^2. \tag{3.16}$$

Define $C_{X_{0:\ell}}$ as the submatrix of C corresponding the the sets $\{X_0, \ldots, X_\ell\}$, then extended with zeros to have the same size as C. Every entry of $C + \frac{d/2-\sqrt{d-1}}{\sqrt{d-1}} C_{X_{0:\ell}}$ is less than or equal to the corresponding entry of the adjacency matrix of a path of length $n + 2\log_{d-1} n$ with $d/2$ edges between pairs of vertices. Also, ψ_C is a nonnegative vector. Therefore the quadratic form $\psi_C^T(C + \frac{d/2-\sqrt{d-1}}{\sqrt{d-1}} C_{X_{0:\ell}})\psi_C$ is at most the spectral radius of this path. Namely

$$\psi_C^T C \psi_C + \frac{d/2 - \sqrt{d-1}}{\sqrt{d-1}} \psi_C^T C_{X_{0:\ell}} \psi_C \le d\cos(\pi/(n + 2\log_{d-1} n + 1)).$$

Because C contains the path of length n, $\psi_C^T C \psi_C \ge d\cos(\pi/(n+1))$. Putting these together yields

$$\psi_C^T C_{X_{0:\ell}} \psi_C \le \frac{\sqrt{d-1}}{d/2 - \sqrt{d-1}} \cdot d(\cos(\pi/(n+2\log_{d-1} n+1)) - \cos(\pi/(n+1))) \le \frac{d\sqrt{d-1}}{d/2 - \sqrt{d-1}} \frac{3\pi^2 \log_d n}{n^3}. \tag{3.17}$$

Define $\psi_C(X_{1:\ell})$ as the projection of ψ_C on $\{X_1, \ldots X_\ell\}$. $C\psi_C(X_{1:\ell}) = C_{X_{0:\ell}}\psi_C(X_{1:\ell})$, so

$$\psi_C^T C_{X_{0:\ell}} \psi_C \ge \lambda_1 \|\psi_C(X_{1:\ell})\|^2 \ge d(\cos(\pi/(n+1)))\|\psi_C(X_{1:\ell})\|^2 \tag{3.18}$$

Combining (3.17) and (3.18) yields

$$\|\psi_C(X_{1:\ell})\|^2 \le \frac{\sqrt{d-1}}{d/2 - \sqrt{d-1}} \left(\frac{3\pi^2 \log_d n}{n^3}\right) / \cos(\pi/n + 1) \le \left(\frac{21\pi^2 \log_d n}{n^3}\right)$$

assuming $d \ge 4$ and n is sufficiently large.

Using (3.16) and the eigenvalue equation, we obtain

$$\psi_Q(r_0) = \psi_C(X_0) \le \lambda_1(A_C)\|\psi_C(X_{1:\ell})\| \le d \cdot \frac{5\pi \log_d^{1/2} n}{n^{3/2}}.$$

Therefore, according to (3.15)

$$r_\ell \le \frac{15d^2\pi(\log_d^{1/2} n)}{n^{5/2}}.$$

□

3.5 Open Problems

We conclude with some promising directions for further research.

Beyond the Trace Method: Polynomial Multiplicity Bounds

There is a large gap between our upper bound of $O(n/\log^{1/5} n)$ on the multiplicity of the second eigenvalue and the lower bound of $n^{2/5}$ mentioned after Theorem 1.3.1. It is very natural to ask, whether the bound of this paper may be improved. To improve the bound beyond $O(n/\text{polylog}(n))$, however, it appears that a very different approach is needed.

Open Problem 1 (Similar to Question 6.3 of [JTY$^+$21]). Let $d > 1$ be fixed integer. Does there exist an $\varepsilon > 0$ such that for every connected d-regular graph G on n vertices, the multiplicity of the second largest eigenvalue of A_G is $O(n^{1-\varepsilon})$?

In the present paper, we rely on the trace method to bound eigenvalue multiplicity through closed walks. There are three drawbacks to this approach that stops a bound on the second eigenvalue multiplicity below $n/\text{polylog}(n)$. First, considering walks of length $\omega(\log(n))$ makes the top eigenvalue dominate the trace, leaving no information behind. Second, considering the trace $\operatorname{Tr} A_G^k$ for $k = O(\log(n))$ it is impossible to distinguish eigenvalues that differ by $O(1/\log(n))$. Third, as covered in Section 3.4, there exist graphs such that there are $\Omega(n/\text{polylog}(n))$ eigenvalues in a range of that size around the second eigenvalue. Thus, the trace method reaches a natural barrier at $n/\text{polylog}(n)$).

Sharper Bounds for Closed Random Walks

We have no reason to believe that the exponent of $1/5$ appearing in Theorem 1.3.3 is sharp. In fact, we know of no example where where the answer is $o(k^{1/2})$. An improvement over Theorem 1.3.3 would immediately yield an improvement of Theorem 1.3.2.

Open Problem 2. Let $d > 1$ be a fixed integer. Does there exist an $\alpha > 1/5$ such that for every connected d-regular graph G on n vertices and every vertex x of G, a random closed walk of length $2k < n$ rooted at x has support $\Omega(k^\alpha)$ in expectation? Is $\alpha = 1/2$ true? Does such a bound hold for SRW in general?

3.6 Proofs for high degree regular graphs

Theorem 3.6.1 (Detailed Theorem 1.3.8). *If G is d-regular, has exactly h self-loops at every vertex, and no multi-edges*[3], *then*

$$\mathbb{P}_x(W^{2k,s}) \leq \exp\left(-\frac{k}{100s^3}\right)\mathbb{P}_x(W^{2k,2s}) \qquad \textit{for} \quad s \leq \min\left\{\frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4}, \frac{d-h}{2}\right\}. \tag{3.19}$$

[3]This technical assumption is used to handle the case when $|\lambda_n(A_G)| > \lambda_2(A_G)$ in Theorem 3.6.2. Here we take $h = 0$.

Proof. We show this via a small modification of the proof of Theorem 3.2.1. Assume $s \le (d-h)/2$. The key observation is that each vertex has at least $d-h$ edges in G to other vertices, so in a subgraph of size at most $2s-1$ every vertex has at least one edge in G leaving the subgraph. In this case, we can simply choose $u \in S$ as $u := \arg\max_{w \in S} \psi_S(w)$ in Lemma 3.2.3. Therefore, considering the adjacency matrix, (3.9) can be improved to

$$\lambda_1(A_{S\cup\{v\}}) \ge \frac{1}{2}\left(\lambda_1 + \sqrt{\lambda_1^2 + \psi_S(u)^2}\right) \ge \lambda_1 + \frac{\psi_S(u)^2}{6\lambda_1^2} \ge \lambda_1 + \frac{1}{6\lambda_1^2 s}.$$

Therefore, after adding s vertices to S according to the process of Lemma 3.2.3, we find a set $T \in \Gamma_x^{2s}$ satisfying

$$\lambda_1(A_T) \ge \lambda_1 + \frac{1}{6\lambda_1^2}\sum_{i=1}^{s}\frac{1}{s+i-1} \ge \lambda_1 + \frac{\log 2}{6\lambda_1^2} \ge \lambda_1\left(1 + \frac{1}{10\lambda_1^3}\right).$$

Using this improved bound, and keeping in mind that $\lambda_1(A_T) \le 2s$, we can replicate the argument above to get to the following improvement over (3.12):

$$\mathbb{P}_x(W^{2k,s}) \le \exp\left(2s\log d + 4s\log(2s) + \log(2s) - \frac{k}{80s^3}\right)\mathbb{P}_x(W^{2k,2s}).$$

This implies

$$\mathbb{P}_x(W^{2k,s}) \le \exp\left(7s\log d - \frac{k}{80s^3}\right)\mathbb{P}_x(W^{2k,2s}) \le \exp\left(-\frac{k}{100s^3}\right)\mathbb{P}_x(W_x^{2k,2s})$$

whenever

$$s \le \frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4},$$

establishing (3.19). □

Theorem 3.6.2 (Detailed Theorem 1.3.7)**.** *If G is simple and d-regular, then*

$$m_G\left([(1 - \frac{\log\log_d n}{\log_d n})\lambda_2, \lambda_2]\right) = \begin{cases} O\left(n \cdot \frac{\log d \log\log n}{d}\right) & \text{when } d\log^{1/2} d \le \alpha \log^{1/4} n \\ O\left(n \cdot \frac{\log^{1/2} d \log\log n}{\log^{1/4} n}\right) & \text{when } d\log^{1/2} d \ge \alpha\log^{1/4} n \end{cases} \tag{3.20}$$

for all[4] $d \le \exp(\sqrt{\log n})$*, where* $\alpha := \sqrt[4]{3}/4$*.*

Proof. The proof is the same as the proof of Theorem 1.3.2 in Section 3.3, except we choose different s.

[4]If $d \ge \exp(\sqrt{\log n})$ then (3.20) is vacuously true.

1. If $d \log^{1/2} d < \alpha \log^{1/4} n$ set

$$s := \min\left\{\frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4}, \frac{d-h}{2}\right\} = \frac{d}{2}$$

with $h = 0$. Applying Theorem 3.6.1 it is easily checked that (3.14) is satisfied for large enough n, yielding a bound of

$$m_G\left([(1 - \frac{\log\log_d n}{\log_d n})\lambda_2, \lambda_2]\right) = O\left(n \cdot \frac{\log d \log\log n}{d}\right).$$

2. If G is simple, d-regular and $d \log^{1/2} d \geq \alpha \log^{1/4} n$, set

$$s := \min\left\{\frac{1}{8}\left(\frac{k}{\log d}\right)^{1/4}, \frac{d-h}{2}\right\} = \frac{1}{8}\left(\frac{\log n}{\log^2 d}\right)^{1/4}$$

with $h = 0$. Then (3.14) is again satisfied by applying Theorem 3.2.1 equation (3.19), and we conclude that

$$m_G\left([(1 - \frac{\log\log_d n}{\log_d n})\lambda_2, \lambda_2]\right) = O\left(n \cdot \frac{\log^{1/2} d \log\log n}{\log^{1/4} n}\right).$$

□

3.7 Lollipop

Here, we show that if we do not assume that our graph is regular, the average support of a uniformly chosen (from the set of all such walks) closed walk of length k from a fixed vertex is no longer necessarily $k^{\Theta(1)}$ (as opposed to the average support of a random walk) . We take the lollipop graph, which consists of a clique of $(d+1)$ vertices for fixed $d \geq 3$ and a path of length n $\{u_1, \ldots, u_n\}$ attached to a vertex v of the clique, where $n \gg k$. Here $\psi := \psi(A)$ and $\lambda_1 := \lambda_1(A)$ are the Perron eigenvector and eigenvalue of the adjacency matrix of the graph.

Lemma 3.7.1. $\psi(v) \geq 1/\sqrt{d+2}$.

Proof. By symmetry, the value on all entries of the clique besides v are the same. Call this value $\psi(b)$. Then by the eigenvalue equation we have $\lambda_1\psi(b) = \psi(v) + (d-1)\psi(b)$, so as $\lambda_1 \geq d$, it must be that $\psi(v) \geq \psi(b)$.

Similarly, to satisfy the eigenvalue equation, vertices on the path must satisfy the recursive relation

$$\lambda_1\psi(u_i) = \psi(u_{i-1}) + \psi(u_{i+1}) \quad 1 \leq i \leq n-1$$
$$\lambda_1\psi(u_n) = \psi(u_{n-1})$$

where we define $v = u_0$. To satisfy this equation, we must have $\psi(u_i) \geq (\lambda_1 - 1)\psi(u_{i+1})$ for each i, so as $\lambda_1 \geq d \geq 3$, $\psi(v) \geq \sum_{i=1}^{n} \psi(u_k)$. As the Perron vector is nonnegative, $\psi(v)^2 \geq \sum_{i=1}^{n} \psi(u_k)^2$, and

$$(d+2)\psi(v)^2 \geq \psi(v)^2 + d\psi(b)^2 + \sum_{i=1}^{n} \psi(u_k)^2 = 1,$$

so $\psi(v) \geq 1/\sqrt{d+2}$. □

Call γ_v^{2k} the number of closed walks of length $2k$, and $\gamma_v^{2k,\geq \ell+d+1}$ as the subset of these walks with support at least $\ell + d + 1$.

Proposition 3.7.2. *For $\ell \geq 2\log(k)/\log(\lambda_1/2)$,*

$$|\gamma_v^{2k,\geq \ell+d+1}| = O(k^{-2})|\gamma_v^{2k}|.$$

Proof. For a closed walk to have support $\ell + d + 1$, it must contain u_ℓ. For such walks, once the path is entered, at least 2ℓ steps must be spent in the path, as the walk must reach u_ℓ and return. Therefore, closed walks starting at v that reach u_ℓ can be categorized as follows. First, there is a closed walk from v to v. Then there is a closed walk from v to v going down the path containing u_ℓ. On this excursion, the walk can only go forward or backwards, and it spends at least 2ℓ steps within the path. For each of these steps, there are 2 options. If we remain in the path after 2ℓ steps, upper bound the number of choices until returning to v by λ_1 at each step. After returning to v, the remaining steps form another closed walk. The number of closed walks from v of length i is at most λ_1^i. Therefore the number of closed walks with an excursion to u_ℓ is at most

$$\sum_{i=0}^{2k} \lambda_1^i 2^{2\ell} \lambda_1^{2k-2\ell-i} = (2k+1)\lambda_1^{2k-2\ell} 2^{2\ell}.$$

The total number of closed walks starting at v is at least $\psi(v)^2\lambda_1^n$. Therefore the fraction of closed walks that have support at least ℓ is at most

$$\frac{(2k+1)2^{2\ell}\lambda_1^{2k-2\ell}}{\lambda_1^{2k}/(d+2)} = \frac{(d+2)(2k+1)2^{2\ell}}{\lambda_1^{2\ell}}$$

so for $\ell \geq 2(\log k)/\log(\lambda_1/2)$, this is $O(k^{-2})$.

□

Remark 3.7.3. Instead of adding a path, we can add a tree (as exhibited in Figure 1.3). According to the same analysis, the probability a walk reaches depth further than $\Theta(\log k)$ is small. Therefore, in Theorem 1.3.3 we can not get a sufficient bound on support from passing to depth, but must deal with support itself.

Chapter 4

Nodal Domains

4.1 Preliminaries

Lemma 4.1.1 (Edge expansion in random graphs [HLW18, Theorem 4.16])**.** *Let G be a random d-regular graph. For every $\delta > 0$, there is an $\varepsilon > 0$ such that:*

$$\Phi_\varepsilon(G) \geq d - 2 - \delta.$$

Lemma 4.1.2 (Bicycle-freeness in random regular graphs [Bor19, Lemma 9])**.** *Let G be a random d-regular graph. There exists an absolute constant $c_{4.1.2} \in (0, 1)$ such that with probability $1 - o(1)$, G is ℓ-bicycle-free for any $\ell \leq c_{4.1.2} \log_{d-1} n$.*

We write $G \backslash F$ to signify $(V, E \backslash F)$. We use Lemma 4.1.2 to derive the following:

Lemma 4.1.3. *Let G be a random d-regular graph. Then with probability $1 - o_n(1)$ there exists a collection of edges F with cardinality bounded by $(d-1)n^{1-c_{4.1.2}/2}$ such that $G \setminus F$ has girth $\ell := \frac{c_{4.1.2}}{2} \log_{d-1} n$.*

Proof. Let $\mathcal{C}$ be the collection of all cycles in G of length at most ℓ. By Lemma 4.1.2, G is 2ℓ-bicycle-free. Consequently, the collection of graphs given by $\mathcal{C}' := \{B_G(C, \ell) : C \in \mathcal{C}\}$ must be pairwise vertex-disjoint. Indeed, if there are distinct $C, C' \in \mathcal{C}$ for which $B_G(C, \ell)$ and $B_G(C', \ell)$ share a vertex v, then $B_G(v, 2\ell)$ contains both C and C' contradicting bicycle-freeness.

By bicycle-freeness, for any $C \in \mathcal{C}$, the number of vertices in $B_G(C, \ell)$ is at least $(d-1)^{\ell-1} = \frac{n^{c_{4.1.2}/2}}{d-1}$, and by vertex-disjointness of the balls around cycles, $|\mathcal{C}'| \leq (d-1)n^{1-c_{4.1.2}/2}$. However, since $|\mathcal{C}| = |\mathcal{C}'|$, we have the same bound on $|\mathcal{C}|$. We can then construct F by choosing one edge per $C \in \mathcal{C}$, which completes the proof. □

Delocalization of eigenvectors of random regular graphs

A key ingredient in our proof is the following result about ℓ_∞-delocalization of eigenvectors in random regular graphs, as stated in [HY21, Theorem 1.4], which built on the previous result [BHY19].

Theorem 4.1.4. *Let $d \geq 3$ be a constant, and let G be a random d-regular graph. With probability $1 - O(n^{-1+o(1)})$ for all eigenvectors v:*

$$||v||_\infty \leq \frac{\log^{C_{\text{HY}}} n}{\sqrt{n}} ||v||,$$

where $C_{\text{HY}} \leq 150$ is an absolute constant independent of d.

Gaussian wave

Our results also use results concerning the before-mentioned Gaussian wave.

Definition 4.1.5. Consider the infinite d-regular tree T_d with vertex set V_d and origin o. An *eigenvector process* with eigenvalue λ is a joint distribution $\{X_v\}_{v \in V_d}$, such that it is invariant under all automorphisms of the tree, $\mathbb{E}(X_o^2) = 1$, and satisfies the eigenvector equation

$$\lambda X_o = \sum_{v \sim o} X_v \tag{4.1}$$

with probability 1.

Observe that the eigenvector process must satisfy the eigenvector equation at every vertex by automorphism invariance, and that by taking the expectation of (4.1) and automorphism invariance, if $\mathbb{E}(X_o) \neq 0$, then $\lambda = d$.

Definition 4.1.6. A *Gaussian wave* is an eigenvector process that is also a Gaussian process.

Theorem 4.1.7 (Theorem 1.1 of [Elo09])**.** *For any $-d \leq \lambda \leq d$, there exists a unique Gaussian wave with parameter λ.*

We call this Gaussian wave Λ_λ.

Definition 4.1.8. The *Lévy Prokhorov distance* between two Borel probability measures μ_1 and μ_2 on $\mathbb{R}^k$ is given by

$$\tilde{d}(\mu_1, \mu_2) := \inf\{\epsilon > 0 | \forall A \in \mathcal{B}_k, \mu_1(A) \leq \mu_2(A_\epsilon) + \epsilon \text{ and } \mu_2(A) \leq \mu_1(A_\epsilon) + \epsilon\},$$

where $\mathcal{B}_k$ is the set of Borel measurable sets in $\mathbb{R}^k$ and A_ϵ is the neighborhood of radius ϵ around A.

Define C_ℓ to be the number of vertices in $B_{T^d}(v, \ell)$, where T_d is the infinite d-regular tree, and v is an arbitrary vertex. Namely

$$C_\ell := 1 + \frac{d((d-1)^\ell - 1)}{d-2}.$$

A vector $f \in \mathbb{R}^{V(G)}$ on the vertices of a graph G on n vertices defines the following distribution $\nu_{G,f,\ell}$ on $\mathbb{R}^{C_\ell}$. Select a vertex $u \in V$ uniformly at random. Order the vertices in $B(u, \ell)$ by starting a breadth first search at u, breaking ties in the order of the search uniformly at random. Create the vector $x := (x_1, \ldots x_{C_\ell})$ such that $x_k := \sqrt{n} f(u_k)$, where u_k is the kth vertex in this breadth first search. If $B(u, \ell)$ has fewer than C_ℓ vertices, then have $x_k = 0$ for $1 \leq k \leq C_\ell$. Finally, let $\nu_{G,f,\ell}$ be the distribution of $\overline{x}(u)$.

Theorem 4.1.9 (Theorem 2 of [BS19]). *For every $d \geq 3$, $\epsilon > 0$ and $R \in \mathbb{N}$, there exists N such that for $n > N$, with probability at least $1 - \epsilon$, a random regular graph of degree d on n vertices has the following property. Any eigenvector f of G is such that $\nu_{G,f,R}$ is at most ϵ in Lévy-Prokhorov distance from the distribution of $\sigma \cdot \Lambda_\lambda$ restricted to the vertices of $B_{T^d}(o, R)$ for some $\sigma \in [0, 1]$, where λ is the eigenvalue of f.*

In fact, [BS19] proves that there is an N and a $\delta > 0$ such that a $G(n, d)$ graph has this property for all normalized vectors f such that there exists a constant λ such that $\|(A - \lambda I)f\| \leq \delta$. Namely, this statement is true for all "pseudo-eigenvectors".

4.2 Either ℓ_2-localization or many nodal domains

In this section, we show (Lemma 4.2.2) that if an eigenvector of a random regular graph is appropriately delocalized in ℓ_2, then its proximity to the Gaussian wave implies it has many nodal domains. We begin by showing that the root vertex in a Gaussian wave with negative parameter λ has a constant probability of being a singleton domain.

Lemma 4.2.1. *For $d \geq 3$ and $0 < \alpha \leq d$, let*

$$c_{4.2.1} := \frac{\alpha^d}{3^{d+2} d^{d+1}}.$$

Assume that $\lambda \leq -\alpha$. With probability at least $c_{4.2.1}$, $\{o\}$ is a singleton nodal domain in Λ_λ with all entries in $B(o, 1)$ of modulus at least $\alpha/5d$.

Proof. The proof proceeds by using the covariance of the Gaussian wave to pass to a Gaussian vector with i.i.d. entries, then showing that with probability at least $c_{4.2.1}$, this vector has a direction and norm that imply Lemma 4.2.1.

The distribution of Λ_λ restricted to $B(o, 1)$ is given by the multivariate normal distribution $N(\mathbf{0}, \Sigma)$ for a $(d + 1) \times (d + 1)$ covariance matrix Σ. The distribution according to $N(\mathbf{0}, \Sigma)$ is the same as the distribution of $\Sigma^{1/2} g$, where g is a length $(d + 1)$ vector with i.i.d. Gaussian $N(0, 1)$ entries. Denote by $\{v_1, \ldots, v_d\}$ the neighbors of o and denote by e_v the elementary vector on v. Notice that $\langle \Sigma^{1/2} e_o, \Sigma^{1/2} e_o \rangle = \mathbb{E}(X_o^2) = 1$, and by the eigenvector equation and automorphism invariance $\langle \Sigma^{1/2} e_o, \Sigma^{1/2} e_{v_i} \rangle = \mathbb{E}(X_o X_{v_i}) = \lambda/d \leq -\alpha/d$.

Let $\tilde{g} := g/\|g\|$. Next, we show that if $\tilde{g}$ is sufficiently close to $\Sigma^{1/2} e_o$, then it must have negative inner product with $\Sigma^{1/2} e_{v_i}$ for each $1 \leq i \leq d$.

If $\langle \tilde{g}, \Sigma^{1/2} e_o \rangle \geq 1 - \frac{\alpha^2}{16 d^2}$,

$$
\begin{aligned}
\langle \tilde{g}, \Sigma^{1/2} e_{v_i} \rangle &= 1 - \frac{1}{2} \|\tilde{g} - \Sigma^{1/2} e_{v_i}\|^2 \\
&\leq 1 - \frac{1}{2} \left(\|\Sigma^{1/2} e_o - \Sigma^{1/2} e_{v_i}\| - \|\tilde{g} - \Sigma^{1/2} e_o\| \right)^2 \\
&\leq 1 - \left(\sqrt{1 - \langle \Sigma^{1/2} e_o, \Sigma^{1/2} e_{v_i} \rangle} - \sqrt{1 - \langle \Sigma^{1/2} e_o, \tilde{g} \rangle} \right)^2 \\
&\leq 1 - \left(\sqrt{1 + \frac{\alpha}{d}} - \sqrt{\frac{\alpha^2}{16d^2}} \right)^2 \\
&\leq -\frac{\alpha}{d} - \frac{\alpha^2}{16d^2} + \frac{\alpha}{2d} \sqrt{1 + \frac{\alpha}{d}} \\
&\leq -\frac{\alpha}{5d}
\end{aligned}
$$

for each i. The first inequality is the triangle inequality. The second is the parallelogram law. The last inequality is true as $\alpha/d \leq 1$.

The probability that $\|g\| \geq 1$ is at least the probability that the first coordinate of g has modulus at least 1. As this coordinate is standard normal, this probability is at least 0.3. The probability that $\langle \tilde{g}, e_o \rangle \geq 1 - \frac{\alpha^2}{16d^2}$ is the surface area of the spherical cap where this inequality is true divided by the surface area of the sphere. The surface area of the spherical cap is at least the volume of the d dimensional sphere base of the spherical cap. The radius of the d-dimensional sphere is

$$
\sqrt{1 - \left(1 - \frac{\alpha^2}{16d^2}\right)^2} = \sqrt{\frac{\alpha^2}{8d^2} - \frac{\alpha^4}{256d^4}} \geq \frac{\alpha}{3d},
$$

meaning that the probability that $\langle \tilde{g}, e_o \rangle \geq 1 - \frac{\alpha^2}{16d^2}$ is at least

$$
\left(\left(\frac{\alpha}{3d} \right)^d \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \right) \Big/ \left(\frac{2\pi^{(d+1)/2}}{\Gamma(\frac{d}{2} + \frac{1}{2})} \right) \geq \frac{\alpha^d}{3^d d^{d+1} \sqrt{\pi}}.
$$

The probability that both $\langle g, e_o \rangle \geq 1 - \frac{\alpha^2}{16d^2}$ and $\langle g, \Sigma^{1/2} e_{v_i} \rangle \leq -\alpha/2d$ for each i is at least the probability that $\|g\| \geq 1$ and $\langle \tilde{g}, e_o \rangle \geq 1 - \frac{\alpha^2}{16d^2}$. By rotational invariance of g these are independent, so this probability is at least

$$
0.3 \cdot \frac{\alpha^d}{3^d d^{d+1} \sqrt{\pi}} \geq \frac{\alpha^d}{3^{d+2} d^{d+1}}.
$$

□

Lemma 4.2.2. *For any $d \geq 3$ $\delta > 0$ and $0 < \alpha \leq d$, there exists $N = N(d, \delta, \alpha)$ such that if $n > N$, then with probability at least $1 - \delta$ with respect to $G(n, d)$, for any eigenvector f with eigenvalue less than $-\alpha$ either*

1. *f has at least $c_{4.2.1}n/2$ singleton nodal domains, or*

2. *There is a set of vertices $S \subset V$, $|S| \leq \delta n$ such that $\sum_{v \in S} f(v)^2 \geq 1 - \delta$.*

Proof. Define $\mu = \mu(d, \lambda, \sigma)$ to be the distribution of the Gaussian wave $\sigma \cdot \Lambda_\lambda$ restricted to $B(o, 1)$. Assume that $\tilde{d}(\mu, \nu_{G,f,1}) \leq \epsilon$, for $\epsilon \leq c_{4.2.1}/2$ to be fixed later. We consider two cases depending on the relationship between σ and ϵ. If σ is much larger than ϵ, then the eigenvector is delocalized, and we can use Lemma 4.2.1. Otherwise, the eigenvector is localized.

First, assume $\sigma \geq 10\epsilon d\alpha^{-1}$. Define A to be the set of vectors $\overline{x} := (x_o, x_{v_1}, \ldots, x_{v_d}) \in \mathbb{R}^{d+1}$ such that

1. $\min\{|x_o|, |x_{v_1}|, \ldots, |x_{v_d}|\} \geq \frac{\sigma\alpha}{5d}$ and

2. $x_o \cdot x_{v_i} < 0$ for each $1 \leq i \leq d$.

By Lemma 4.2.1, $\mu(A) \geq c_{4.2.1}$. By the definition of A, a given vector $\overline{x} \in A$ is such that all entries are of modulus at least $\frac{\sigma\alpha}{5d}$. Moreover, by the assumption on σ, we have $\epsilon \leq \frac{\sigma\alpha}{10d}$. Therefore, for a vector $\overline{y} := (y_o, y_{v_1}, \ldots, y_{v_d})$ such that $\|\overline{x} - \overline{y}\| \leq \epsilon$, the entries of $\overline{y}$ are of the same sign as the entries of $\overline{x}$. Therefore, if $x_o \cdot x_{v_i} < 0$ for each $1 \leq i \leq d$, then $y_o \cdot y_{v_i} < 0$ for each $1 \leq i \leq d$, meaning that if $B(o, 1)$ is colored as per $\overline{y}$, then $\{o\}$ is a singleton nodal domain.

As $\tilde{d}(\mu, \nu_{G,f,1}) \leq \epsilon$, we have $\nu_{G,f,1}(A_\epsilon) \geq \mu(A) - \epsilon \geq c_{4.2.1}/2$. By the previous paragraph, all vectors in A_ϵ correspond to singleton nodal domains, so there are at least $c_{4.2.1}n/2$ singleton domains of f in G.

Now assume $\sigma < 10\epsilon d\alpha^{-1}$. In this case, we will show that because $\nu_{G,f,1}$ is close to a Gaussian with low variance, the distribution of entries of f must be concentrated around 0.

Denote by μ_0 the distribution of the value on o in μ, and $\nu_0 := \nu_{G,f,0}$. Note that μ_0 is the distribution $N(0, \sigma^2)$. The Euclidean distance between two points can only decrease when projecting onto a single coordinate, therefore the Lévy Prokhorov distance can only decrease as well. This means that as $\tilde{d}(\mu, \nu_{G,f,1}) \leq \epsilon$, then $\tilde{d}(\mu_0, \nu_0) \leq \epsilon$. Therefore for each $z \geq 0$,

$$\Pr_{x \sim \nu_0}(x \in [-z - \epsilon, z + \epsilon]) \geq \Pr_{x \sim \mu_0}(x \in [-z, z]) - \epsilon.$$

Fix $z := \sigma\sqrt{2\log\frac{1}{\epsilon}}$ and observe that by Gaussian tail bounds

$$\Pr_{x \sim \mu_0}(x \notin [-z, z]) \leq 2\epsilon. \tag{4.2}$$

Also, by examining the endpoints of the interval, we have

$$\mathbb{E}_{x \sim \nu_0}\left(\mathbf{1}\Big[x \in [-z - \epsilon, z + \epsilon]\Big] \cdot x^2\right) \leq \left(\sigma\sqrt{2\log\frac{1}{\epsilon}} + \epsilon\right)^2.$$

By assumption $\sigma < 10\epsilon d\alpha^{-1}$. Therefore

$$\left(\sigma\sqrt{2\log\frac{1}{\epsilon}}+\epsilon\right)^2 \leq \left(10\epsilon d\alpha^{-1}\sqrt{2\log\frac{1}{\epsilon}}+\epsilon\right)^2 = \left(1+10d\alpha^{-1}\sqrt{2\log\frac{1}{\epsilon}}\right)^2\epsilon^2 \leq 250d^2\alpha^{-2}\epsilon^2\log\frac{1}{\epsilon}.$$

As $\frac{1}{\epsilon} > \log\frac{1}{\epsilon}$ and $\mathbb{E}_{x\sim\nu_0}(x^2) = 1$, this means that

$$\mathbb{E}_{x\sim\nu_0}\left(\mathbf{1}\Big[x \notin [-z-\epsilon, z+\epsilon]\Big]\cdot x^2\right) \geq 1 - 250d^2\alpha^{-2}\epsilon.$$

Combining this with (4.2) and the definition of ν_0, this means that if $S = \{u \in V | f(u)^2 \geq 2\sigma^2\log\frac{1}{\epsilon}\}$, then $|S| \leq 2\epsilon n$, and

$$\sum_{u\in S} f(u)^2 = \frac{1}{n}\sum_{u\in S} nf(u)^2 = \mathbb{E}_{x\sim\nu_0}\left(\mathbf{1}\Big[x \notin [-z-\epsilon, z+\epsilon]\Big]\cdot x^2\right) \geq 1 - 250d^2\alpha^{-2}\epsilon.$$

It is therefore sufficient to choose N as per Theorem 4.1.9 for

$$\epsilon < \min\left\{\frac{c_{4.2.1}}{2}, \frac{\alpha^2}{250d^2}\delta\right\}.$$

□

4.3 Spectral radius bounds

The main result of this section is Lemma 4.3.6, where we prove bounds on the spectral radius of high-girth graphs with bounded maximum degree and hereditary degree (defined below) approximately equal to 2.

Definition 4.3.1. The *hereditary degree* of a graph H is defined as:

$$\max_{H'\subseteq H} \mathrm{AvgDegree}(H')$$

where $\mathrm{AvgDegree}(H') = 2|E(H')|/|V(H')|$.

Definition 4.3.2. Given a collection of edges F, we will use $v(F)$ to denote the number of vertices adjacent to F, and $c(F)$ to denote the number of connected components formed by F.

Definition 4.3.3. Given a graph H and a collection of edges $F \subseteq E(H)$, we use 1_F to denote its indicator vector in $\mathbb{R}^{E(H)}$. The *spanning forest polytope* of H is defined to be the convex hull of $\{1_F : F \text{ forest}\}$.

We will also need the following two ingredients.

Lemma 4.3.4. *[Kes59] If T is a forest with maximum degree bounded by Δ, then $\lambda_{\max}(A_T) \leq 2\sqrt{\Delta - 1}$.*

The following fact about the spanning forest polytope is a consequence of [KV12, Theorem 13.21].

Lemma 4.3.5. *The spanning forest polytope of a graph H is equal to the feasible region of the following linear program:*

$$\begin{aligned} y &\in \mathbb{R}^{E(H)} \\ y &\geq 0 \\ \sum_{e \in F} y_e &\leq v(F) - c(F) \qquad\qquad \forall F \subseteq E(H). \end{aligned}$$

Lemma 4.3.6. *Let H be a graph with hereditary degree $2(1+\delta)$, maximum degree Δ, and girth g. Then:*

$$\lambda_{\max}(A_H) \leq 2\frac{1+\delta}{1-\frac{1}{g}}\sqrt{\Delta - 1}.$$

Proof. Since A_H is a symmetric matrix with nonnegative entries,

$$\lambda_{\max}(A_H) = \max_{f \in \mathbb{R}^{V(H)} \setminus \{0\}} \frac{f^\top A_H f}{\|f\|^2}.$$

We will bound $f^\top A_H f$ for any f. Observe that:

$$f^\top A_H f = \sum_{\{u,v\} \in E(H)} f_u f_v.$$

We will prove that there is a spanning forest T for which:

$$\frac{1-\frac{1}{g}}{1+\delta} f^\top A_H f \leq f^\top A_T f \tag{4.3}$$

which by Lemma 4.3.4 is bounded by $2\sqrt{\Delta - 1}$ hence implying

$$f^\top A_H f \leq 2\frac{1+\delta}{1-\frac{1}{g}}\sqrt{\Delta - 1}.$$

To prove (4.3) we exhibit a distribution $\mathcal{D}$ on spanning forests such that:

$$\mathbb{E}_{\boldsymbol{T} \sim \mathcal{D}}\left[f^\top A_{\boldsymbol{T}} f\right] = \frac{1-\frac{1}{g}}{1+\delta} f^\top A_H f.$$

Let $y \in \mathbb{R}^{E(H)}$ be the vector with $\frac{1-\frac{1}{g}}{1+\delta}$ in every entry. We claim that y is inside the spanning forest polytope of H. To verify this, it suffices to check if y satisfies the linear constraints given by the linear program description of the polytope from Lemma 4.3.5. By construction, each $y_e \geq 0$.

For any $F \subseteq E(H)$, write it as $F_1 \cup \cdots \cup F_{c(F)}$ where each F_i is a connected component given by F. Since the girth of H is at least g, for any $|F_i| < g$ we know F_i forms a tree and hence $|F_i| = v(F_i) - 1$. For the remaining components, we know $|F_i| \leq v(F_i)(1+\delta)$ by our bound on the hereditary average degree. Now:

$$\begin{aligned}
\sum_{e \in F} y_e &= \sum_{i=1}^{c(F)} \sum_{e \in F_i} y_e \\
&= \sum_{i=1}^{c(F)} \frac{1-\frac{1}{g}}{1+\delta} |F_i| \\
&= \sum_{i \in [c(F)] : |F_i| < g} \frac{1-\frac{1}{g}}{1+\delta} |F_i| + \sum_{i \in [c(F)] : |F_i| \geq g} \frac{1-\frac{1}{g}}{1+\delta} |F_i| \\
&\leq \sum_{i \in [c(F)] : |F_i| < g} (v(F_i) - 1) + \sum_{i \in [c(F)] : |F_i| \geq g} \left(1 - \frac{1}{g}\right) v(F_i) \\
&\leq \sum_{i=1}^{c(F)} (v(F_i) - 1) \\
&= v(F) - c(F).
\end{aligned}$$

The second to last inequality follows from the fact that for a graph of girth g, a subgraph with at least g edges has at least g vertices. Since y is in the spanning forest polytope of H it must be expressible as a convex combination $p_1 T_1 + \cdots + p_s T_s$ of indicator vectors of spanning forests in H. Let $\mathcal{D}$ be the distribution given by choosing spanning forest T_i with probability p_i. Notice that for $\boldsymbol{T} \sim \mathcal{D}$ the probability that any given edge e is chosen is $\frac{1-\frac{1}{g}}{1+\delta}$. Now:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{T} \sim \mathcal{D}} \left[f^\top A_{\boldsymbol{T}} f \right] &= \mathbb{E}_{\boldsymbol{T} \sim \mathcal{D}} \left[\sum_{\{u,v\} \in E(H)} \mathbf{1}[e \in \boldsymbol{T}] f_u f_v \right] \\
&= \sum_{\{u,v\} \in E(H)} f_u f_v \Pr[e \in \boldsymbol{T}] \\
&= \frac{1-\frac{1}{g}}{1+\delta} \sum_{\{u,v\} \in E(H)} f_u f_v \\
&= \frac{1-\frac{1}{g}}{1+\delta} f^\top A_H f,
\end{aligned}$$

which completes the proof. □

4.4 ℓ_2-localization implies many nodal domains

In this section G is a d-regular graph and f is a vector in $\mathbb{R}^{V(G)}$. We prove that under some suitable assumptions on G and f, it is not possible for f to simultaneously be localized and have few nodal domains. Next, we verify that all of these conditions are simultaneously satisfied by random graphs and eigenvectors corresponding to sufficiently negative eigenvalues with high probability.

The conditions we impose on G are:

Almost high-girth: There is $F \subseteq E(G)$ such that $|F| \leq O(n^{1-c})$ and the girth of $G \setminus F$ is at least $c \log_{d-1} n$ for some absolute constant $c > 0$.

Lossless edge expansion: $\Phi_\varepsilon(G) \geq d - 2 - \delta$ for some constants $\varepsilon > 0$ and $0 < \delta < d - 2$ (see Definition 2.0.5).

The conditions we impose on f are:

ℓ_2-localization: There is a set $S \subseteq V(G)$ of size εn such that $\|f_S\|^2 \geq (1 - \eta)\|f\|^2$ for some small constant $\eta > 0$ such that $4d\sqrt{\eta} < \delta\sqrt{d-2}$.

ℓ_∞-delocalization: $\|f\|_\infty \leq \dfrac{\log^C n}{\sqrt{n}}\|f\|$ for some constant C.

High energy: $f^\top A_G f = \lambda\|f\|^2$ for $\lambda < -2(1 + 2\delta)\sqrt{d-2}$.

We note that the labels for the conditions on G and f are not definitions of those properties, but rather for readability in back-referencing.

The key result of this section is the following. We emphasize that all nodal domains considered are weak nodal domains of f defined with respect to the graph G, and not its subgraphs.

Lemma 4.4.1. *If G and f satisfy the above conditions then f must have $\Omega\left(\frac{n}{\log^{2C+1} n}\right)$ singleton nodal domains.*

A key lemma in service of proving Lemma 4.4.1 is:

Lemma 4.4.2. *Let G, f and S satisfy the above conditions, and let c, d, δ and η be the parameters from above. If f has fewer than $\frac{n}{\log^{2C+1} n}$ singleton nodal domains in S, then there is a subgraph H of G on vertex set S such that:*

1. *The girth of H is at least $c \log_{d-1} n$.*
2. *The maximum degree of H is at most $d - 1$.*

3. *The hereditary degree of H is at most* $2 + \delta$.

4. $f_S^\top A_H f_S \le (\lambda + 4d\sqrt{\eta})\|f_S\|^2$.

Proof. Let H be the graph obtained by starting with $G[S]$, and then deleting the edge subgraph

$$L := L_+ \cup L_\circ \cup (F \cap E(G[S]))$$

where L_+ is the subgraph obtained by choosing every edge $\{u, v\}$ in $G[S]$ such that $f_S(u)f_S(v) \ge 0$, and $L_\circ$ is obtained by choosing one arbitrary incident edge in $G[S]$ to each singleton nodal domain $v \in S$ with degree d in $G[S]$.

Proof of 1. H is a subgraph of $G \setminus F$ and hence has girth at least $c\log_{d-1} n$.

Proof of 2. Every vertex v with degree d in $G[S]$ has an incident edge in L: indeed, if v is a singleton nodal domain with degree d in $G[S]$ then one of its incident edges is added to $L_\circ$; otherwise v has a neighbor $u \in S$ such that $f_S(u)f_S(v) \ge 0$, which means $\{u, v\} \in L_+$. Consequently, every vertex in H has degree bounded by $d - 1$.

Proof of 3. For any $T \subseteq S$, since $|T| \le \varepsilon n$, it must be the case that $|E(T, \overline{T})| \ge d - 2 - \delta$ by "lossless edge expansion". Since G is a d-regular graph, the average degree of $G[T]$ must be at most $2 + \delta$. Consequently since $H[T]$ is a subgraph of $G[T]$, the average degree of $H[T]$ is also bounded by $2 + \delta$.

Proof of 4. First observe that:

$$\begin{aligned}
\lambda\|f\|^2 &= f^\top A_G f \\
&= f_S^\top A_G f_S + 2 f_{\overline{S}}^\top A_G f_S + f_{\overline{S}}^\top A_G f_{\overline{S}} \\
&\ge f_S^\top A_{G[S]} f_S - 2d\sqrt{\eta}\|f\|^2 - d\eta\|f\|^2 \\
&\ge f_S^\top A_{G[S]} f_S - 3d\sqrt{\eta}\|f\|^2
\end{aligned}$$

where the third line follows from "ℓ_2-localization", Cauchy-Schwarz inequality, and $\lambda_{\max}(A_G) \le d$. Consequently $f_S^\top A_{G[S]} f_S \le (\lambda + 3d\sqrt{\eta})\|f\|^2$. Next, observe that:

$$\begin{aligned}
f_S^\top A_{G[S]} f_S &= f_S^\top A_H f_S + f_S^\top A_L f_S \\
&= f_S^\top A_H f_S + f_S^\top A_{L_+} f_S + f_S^\top A_{L_\circ} f_S + f_S^\top A_{F\cap E(G[S])} f_S \\
&\ge f_S^\top A_H f_S + f_S^\top A_{L_\circ} f_S + f_S^\top A_{F\cap E(G[S])} f_S && \text{(since } f_S^\top A_{L_+} f_S \ge 0\text{)} \\
&\ge f_S^\top A_H f_S - 2|L_\circ| \cdot \|f_S\|_\infty^2 - 2|F \cap E(G[S])| \cdot \|f_S\|_\infty^2 \\
&\ge f_S^\top A_H f_S - \left(\frac{2}{\log n} + O(n^{-c}\log^{2C} n)\right)\|f\|^2,
\end{aligned}$$

where the last inequality is because $|L_\circ| \le \frac{n}{\log^{2C+1} n}$ by assumption, $\|f_S\|_\infty^2 \le \frac{\log^{2C} n}{n}\|f\|^2$ by "ℓ_∞-delocalization", and $|F \cap E(G[S])| = O\left(n^{1-c}\right)$ by "almost-high girth".

Chaining the above two inequalities together gives us:

$$f_S^\top A_H f_S \le \left(\lambda + 3d\sqrt{\eta} + O\left(\frac{1}{\log n}\right)\right)\|f\|^2 \le (\lambda + 4d\sqrt{\eta})\|f\|^2.$$

Since $\lambda + 4d\sqrt{\eta} < 0$ and $\|f\|^2 \ge \|f_S\|^2$, the above is bounded by $(\lambda + 4d\sqrt{\eta})\|f_S\|^2$, completing the proof of 4. □

We are now ready to prove 4.4.1.

Proof of 4.4.1. We prove the desired statement by contradiction. If f has less than $\frac{n}{\log^{2C+1} n}$ singleton nodal domains then consider the subgraph H that is promised by Lemma 4.4.2. On one hand by 4 of 4.4.2:

$$f_S^\top A_H f_S \le (\lambda + 4d\sqrt{\eta})\|f_S\|^2 \le (-2(1+2\delta)\sqrt{d-2} + \delta\sqrt{d-2})\|f_S\|^2 = -2\left(1 + \frac{3}{2}\delta\right)\sqrt{d-2}\|f_S\|^2.$$

which implies that the spectral radius of A_H is lower bounded by $2\left(1+\frac{3}{2}\delta\right)\sqrt{d-2}$. On the other hand, by 1, 2 and 3 of 4.4.2 in conjunction with 4.3.6 the spectral radius of A_H is upper bounded by $\frac{(2+\delta)\sqrt{d-2}}{1-\frac{1}{c\log_{d-1} n}}$, which is at most $2(1+\delta)\sqrt{d-2}$, which is a contradiction. □

Remark 4.4.3 (Sharpness of Lemma 4.4.2)**.** We remark that 2 in Lemma 4.4.2 is the source of the $\lambda \le -2\sqrt{d-2} - \alpha$ hypothesis in Theorem 1.3.10; reducing the degree of H below $d-1$ would yield a larger spectral window in Theorem 1.3.10. The entirely local argument of the Lemma is seen to be sharp by taking $G[S] = \cup_{i=1}^k T_i$ to be a disjoint union of finite d−ary trees T_i of depth $O(\log\log n)$ such that the graph distance between any two trees is at least 2, and f to be an eigenfunction of a $(d-1)$-ary tree $T_i' \subset T_i$ with eigenvalue $\lambda \approx -2\sqrt{d-2}$ in each copy, and zero everywhere else. Then f and G satisfy the hypotheses of Lemma 4.4.2 locally, f has no singleton nodal domains in $G[S]$ (since each vertex has a path on which $f = 0$ to the leaves of the tree), and there is no subgraph of $G[S]$ of maximum degree strictly less than $d-1$ satisfying 4. Thus, improving Lemma 4.4.2 will require either additional hypotheses or a more global examination of the structure of G and f.

4.5 Many nodal domains in random regular graphs

We are now ready to prove our main result.

Proof of Theorem 1.3.10. By Lemma 4.2.2, with probability $1 - o(1)$, every eigenvector f either has $\Omega(n)$ singleton nodal domains or satisfies "ℓ_2-localization".

We define the following events:

- $\mathcal{E}_1$: G satisfies "almost-high girth" with constant $c_{4.1.2}$ and "lossless edge expansion"; f satisfies "ℓ_∞-delocalization" with constant C_{HY} and "high energy",
- $\mathcal{E}_2$: f has at least $c_{4.2.1}n/2$ singleton nodal domains,
- $\mathcal{E}_3$: f satisfies "ℓ_2-localization" and has fewer than $c_{4.2.1}n/2$ singleton nodal domains.

Clearly, when $\mathcal{E}_2$ occurs there are $\Omega(n/\log^{2C+1} n)$ nodal domains. Next, observe that when both $\mathcal{E}_1$ and $\mathcal{E}_3$ occur, the conditions of Lemma 4.4.1 are satisfied and, f has $\Omega(n/\log^{2C+1} n)$ singleton nodal domains.

Thus, it suffices to lower bound $\Pr[\mathcal{E}_2 \cup (\mathcal{E}_1 \cap \mathcal{E}_3)]$. Since $\mathcal{E}_2$ and $\mathcal{E}_3$ are mutually exclusive, $\mathcal{E}_2$ and $\mathcal{E}_3 \cap \mathcal{E}_1$ are also mutually exclusive, and hence:

$$\Pr[\mathcal{E}_2 \cup (\mathcal{E}_1 \cap \mathcal{E}_3)] = \Pr[\mathcal{E}_2] + \Pr[\mathcal{E}_1 \cap \mathcal{E}_3] \geq \Pr[\mathcal{E}_2] + \Pr[\mathcal{E}_3] - \Pr\left[\overline{\mathcal{E}_1}\right] \tag{4.4}$$

Lemma 4.2.2 implies that $\Pr[\mathcal{E}_2] + \Pr[\mathcal{E}_3] = 1 - o(1)$. We further have $\Pr\left[\overline{\mathcal{E}_1}\right] = o(1)$ by a combination of Lemma 4.1.3, Lemma 4.1.1 and Theorem 4.1.4. Thus,

$$\Pr[\mathcal{E}_2 \cup (\mathcal{E}_1 \cap \mathcal{E}_3)] = 1 - o(1),$$

which completes the proof.

□

4.6 Large Nodal Domains in Expanders

In this section we prove that as a consequence of expansion in random graphs, for any eigenvector of a random d-regular graph, most vertices are part of a macroscopic nodal domain. Key to our result in this section is the following lemma, which proves that by the expander mixing lemma, the only way to have the "correct" number of internal edges in a large subgraph is to have a large connected component.

Lemma 4.6.1. *Let G be a n-vertex d-regular graph and let $S \subseteq V(G)$ of size cn, where c is arbitrary. Also assume $\lambda(G) < d$. Then $G[S]$ has a connected component of size at least:*

$$\left(c - \frac{2(1-c)\lambda(G)}{d - \lambda(G)}\right) \cdot n.$$

Proof. By the expander mixing lemma, we know that the average degree of $G[S]$ is:

$$\begin{aligned}\mathrm{AvgDegree}(G[S]) &= \frac{|E(S,S)|}{|S|} \\ &\geq cd - \lambda(G)(1-c).\end{aligned}$$

Let the size of the connected component C^* in $G[S]$ with maximum average degree be $c'n$. We know that AvgDegree($G[C^*]$) is at least AvgDegree($G[S]$), and by the expander mixing lemma:

$$\begin{aligned}\text{AvgDegree}(G[C^*]) &= \frac{e(C^*, C^*)}{|C^*|} \\ &\leq c'd + \lambda(G)(1 - c').\end{aligned}$$

Consequently, we have:

$$\begin{aligned} c'd + \lambda(G)(1 - c') &\geq cd - \lambda(G)(1 - c) \\ c'(d - \lambda(G)) &\geq c(d + \lambda(G)) - 2\lambda(G) \\ c' &\geq c \cdot \frac{d + \lambda(G)}{d - \lambda(G)} - \frac{2\lambda(G)}{d - \lambda(G)} \\ &= c - \frac{2(1 - c)\lambda(G)}{d - \lambda(G)}. \end{aligned}$$

which proves the claim. □

The result about nodal domains (which actually really applies to any signing of the vertices independent of being an eigenvector) in expanders is:

Theorem 4.6.2. *Let G be a d-regular graph and let f be any eigenvector of A_G. Suppose C_1 and C_2 be the two largest nodal domains in f, then $|C_1| + |C_2| \geq \left(1 - \frac{2\lambda(G)}{d - \lambda(G)}\right) n$.*

Proof. Let $S_+ := \{v \in V(G) : f(v) \geq 0\}$ and $S_- := \{v \in V(G) : f(v) < 0\}$. Let's denote $|S_+|$ as cn and $|S_-|$ as $(1 - c)n$. By Lemma 4.6.1 we know that the largest component C_+ in S_+ has size at least $\left(c - \frac{2(1-c)\lambda(G)}{d-\lambda(G)}\right) \cdot n$ and the largest component C_- in S_- (which is distinct from C_+) has size at least $\left(1 - c - \frac{2c\lambda(G)}{d-\lambda(G)}\right) \cdot n$. It then follows:

$$\begin{aligned} |C_1| + |C_2| &\geq |C_+| + |C_-| \\ &\geq \left(1 - \frac{2\lambda(G)}{d - \lambda(G)}\right) \cdot n. \end{aligned}$$

□

Remark 4.6.3. When G is a random d-regular graph, then by Friedman's Theorem $\frac{2\lambda(G)}{d-\lambda(G)} = O\left(\frac{1}{\sqrt{d}}\right)$ [Fri03], and so for large enough d, the statement implies that a large constant fraction of the vertices are part of the two largest nodal domains. For instance, when $d \geq 99$, at least half the vertices are part of the two largest nodal domains.

Chapter 5

Lossless Expansion

5.1 Operator Theory

In this section, let V be a countable set and $T : \ell_2(V) \to \ell_2(V)$ be a bounded linear operator.

Definition 5.1.1. The *spectrum* of T, which we denote $\mathsf{spec}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not invertible.

Definition 5.1.2. The *spectral radius* of T, which we denote $\rho(T)$ is defined as $\sup\{|\lambda| : \lambda \in \mathsf{spec}(T)\}$.

Remark 5.1.3. The *operator norm* of T, which we write as $\|T\|$ is equal to $\sqrt{\rho(TT^*)}$ where T^* is the adjoint of T.[1]

Remark 5.1.4. $\rho(T) = \lim_{\ell\to\infty} \|T^\ell\|^{1/\ell}$.

Remark 5.1.5 (Consequence of [Que96, Theorem 6])**.** Suppose T is a self-adjoint operator, and Φ is a basis of $\ell_2(V)$. Then:

$$\rho(T) = \sup_{\phi\in\Phi} \limsup_{k\to\infty} |\langle \phi, T^k\phi\rangle|^{1/k}.$$

Remark 5.1.6. Let A be any principal submatrix of T. Then $\rho(A) \le \rho(T)$.

Corollary 5.1.7. If H is a subgraph of (possibly infinite) graph G, then $\rho(A_H) \le \rho(A_G)$.

5.2 Infinite Trees Hanging from a Biregular Graph

Let H be any $(2, d-1)$-biregular graph where the partition with degree-$(d-1)$ vertices is called U and the partition with degree-2 vertices is called V. Let X be the infinite graph constructed from H as follows:

[1]Since $\ell_2(V)$ comes equipped with the inner product $\langle f, g\rangle := \sum_{v\in V} f(v)g(v)$, T^* is simple the "transpose" of T.

At every vertex in U, the $(d-1)$-regular partition, glue an infinite tree where the root has degree-1 and the remaining vertices have degree-d. At every vertex in V, the 2-regular partition, glue an infinite tree where the root has degree-$(d-2)$ and every other vertex has degree-d.

Note that X is a d-regular infinite graph. The main result of this section is:

Lemma 5.2.1. $\rho(A_X) \le 2\sqrt{d-1}$.

To prove 5.2.1, we instead turn our attention to the nonbacktracking matrix of X, called B_X. In particular, we bound $\rho(B_X)$ and then employ the Ihara–Bass formula of [AFH15] for infinite graphs to translate the bound on $\rho(B_X)$ into a bound on $\rho(A_X)$.

Thus, we first prove:

Lemma 5.2.2. $\rho(B_X) \le \sqrt{d-1}$.

We use the following version of the Ihara–Bass formula of [AFH15] for infinite graphs.

Theorem 5.2.3. *Let G be a (possibly infinite) graph. Then*

$$\mathrm{spec}(B_G) = \{\pm 1\} \cup \{\lambda : (D_G - I) - \lambda A_G + \lambda^2 I \text{ is not invertible}\}.$$

An immediate corollary that we will use is:

Corollary 5.2.4. Let G be a d-regular graph. Then $\rho(B_G) \le \sqrt{d-1}$ implies that $\rho(A_G) \le 2\sqrt{d-1}$.

Proof. If there is μ in $\mathrm{spec}(A_G)$ such that $|\mu| > 2\sqrt{d-1}$, then $\mu I - A_G$ is not invertible. Consequently, by 5.2.3 $\lambda = \frac{\mu + \sqrt{\mu^2 - 4(d-1)}}{2}$, which is greater than $\sqrt{d-1}$, is in $\mathrm{spec}(B_G)$. □

In light of 5.2.4, we see that 5.2.2 implies 5.2.1.

Towards proving 5.2.2, we first make a definition.

Definition 5.2.5. We call a walk W a $(a \times b)$-linkage if it can be split into a segments, each of which is a length-b nonbacktracking walk.

Proof of 5.2.2. By 5.1.4

$$\rho(B_X) = \limsup_{\ell \to \infty} \|B_X^\ell\|^{1/\ell}.$$

Since $\|B_X^\ell\| = \sqrt{\rho(B_X^\ell (B_X^*)^\ell)}$ it suffices to bound $\rho(T)$ where $T := B_X^\ell (B_X^*)^\ell$ is a bounded self-adjoint operator, and hence by 5.1.5:

$$\rho(T) = \max_{uv \in \vec{E}(X)} \limsup_{k \to \infty} |\langle 1_{uv}, T^k 1_{uv} \rangle|^{1/k}.$$

The quantity $\langle 1_{uv}, T^k 1_{uv} \rangle$ is bounded by the number of $(2k \times (\ell+1))$-linkages that start and end at vertex u, which we can bound via an encoding argument. In particular, we will give an algorithm to uniquely encode such linkages and bound the total number of possible encodings.

5.3 Encoding Linkages

Each length-$(\ell+1)$ nonbacktracking segment can be broken into 3 consecutive phases (of which some can possibly be empty): the phase where distance to H decreases on each step (Phase 1), the second phase where distance to H does not change on each step (Phase 2), and the third phase where distance to H increases on each step (Phase 3). We further break the third phase into two (possibly empty) subphases — the first subphase where the distance to u decreases on each step (Phase 3a), and the second subphase where the distance to u increases on each step (Phase 3b).

To encode the linkage, for each length-$(\ell+1)$ nonbacktracking we specify four numbers denoting the lengths of Phases 1, 2, 3a, and 3b. Note that Phase 2 is nonempty only if it is contained in H. For each step ab in Phase 2 that goes from U (the $(d-1)$-regular partition) to V (the 2-regular partition) we specify a number i in $[d-1]$ such that b is the ith neighbor of a within H. If the first step ab in Phase 2 is from V to U we specify a number in $[2]$ denoting if b is the first or second neighbor of a. For each step ab in Phase 3b we specify a number i in $[d-1]$ such that b is the ith neighbor of a that does not lie in the path between between u and H.

5.4 Recovering Linkages from Encodings

We recover a linkage from its encoding "segment-by-segment". Suppose the first t segments have been recovered, we show how to recover the $(t+1)$-th segment. Let x be the vertex the walk is at after it has traversed the first t segments. The steps taken in Phase 1 can be recovered from the length of the Phase since there is a unique path from any vertex to H. The steps in Phase 2 alternate between stepping from V to U and from U to V. It is easy to recover the first step of Phase 2 as well as any step from U to V; a step ab from V to U that is not the first step of Phase 2 is uniquely determined by the previous step, since a has 2 neighbors in U and by the nonbacktracking nature of the walk there is only one choice for b. Note that Phase 3a is nonempty only if u is not in H and all the steps are contained in the same branch as u. Since there is a unique shortest path between the start vertex of Phase 3a and u, the steps taken in Phase 3a can be recovered from its length. Finally, it is easy to recover the steps taken in Phase 3b since they are explicitly given in the encoding.

5.5 Counting Encodings

Now we turn our attention to bounding the total number of encodings. For given $\alpha, \beta \geq 0$ such that $\alpha + \beta = 2k(\ell+1)$ we first bound the number of walks such that α steps occur in Phase 2 (i.e. are within H) and β steps occur outside Phase 2 (i.e. are outside H). Let $v_1, v_2, \ldots, v_{2k(\ell+1)}$ be the sequence of vertices visited by the walk in order. By $d(x,y)$ we denote the graphical distance between vertices x and y, and for a set of vertices A, we write $d(x,A) := \min_{y\in A} d(x,y)$. Since $d(v_1,H) = d(v_{2k(\ell+1)},H)$, $|d(v_i,H) - d(v_{i+1},H)| \leq 1$ always and $|d(v_i,H) - d(v_{i+1},H)| = 0$ for every step in Phase 2, the number of steps of the walk that occur in Phase 3 of their respective segments is at most $\frac{\beta}{2}$. In particular, the number of steps that occur in Phase 3b of their respective segments is bounded by $\frac{\beta}{2}$. The following bounds hold:

- The number of possible encodings of the lengths of phases is at most $(\ell+1)^{8k}$.
- The number of possible encodings of the first step of Phase 2 of each segment is at most 2^{2k}.
- The number of possible encodings of the list of U-to-V steps in Phase 2 is at most $(d-1)^{\frac{\alpha+1}{2}}$ because the steps taken in Phase 2 alternate between going from V to U and from U to V.
- The number of possible encodings of the list of steps in Phase 3b is at most $(d-1)^{\frac{\beta}{2}}$.

The above bounds combine to give a bound of

$$(\ell+1)^{8k}2^{2k}(d-1)^{\frac{\alpha+1}{2}}(d-1)^{\frac{\beta}{2}} \leq (\ell+1)^{8k}2^{2k}\sqrt{d-1}^{2k(\ell+1)+1}.$$

As there are at most $2k\ell$ choices for (α,β) pairs, the number of $(2k\times(\ell+1))$-linkages is at most

$$2k\ell(\ell+1)^{8k}2^{2k}\sqrt{d-1}^{2k(\ell+1)+1}.$$

Thus,

$$\rho(T) \leq \limsup_{k\to\infty}\left(2k\ell(\ell+1)^{8k}2^{2k}\sqrt{d-1}^{2k(\ell+1)+1}\right)^{1/k} = 4(\ell+1)^8\sqrt{d-1}^{2(\ell+1)}.$$

Consequently,

$$\rho(B_X) \leq \limsup_{\ell\to\infty}\rho(T)^{1/2\ell} \leq \limsup_{\ell\to\infty}\left(4(\ell+1)^8\sqrt{d-1}^{2(\ell+1)}\right)^{1/2\ell} = \sqrt{d-1}.$$

□

5.6 High-Girth Near-Ramanujan graphs with Lossy Vertex Expansion

We will plant a high girth graph with low spectral radius within a d-regular Ramanujan graph. We will show that such a construction is a spectral expander, but has low vertex expansion. By $u \sim_G v$, we mean that u and v are adjacent in the graph G. We will write $u \sim v$ when the graph is clear from context.

Consider a $(2, d-1)$ biregular bipartite graph $H = (U, V, E)$, with vertex components U and V. U is the degree-$(d-1)$ component and V the degree-2 component. Therefore if we define $\gamma := |U|$, requiring γ to be even, then $|V| = (d-1)\gamma/2$. Call the vertices of U and V $\{u_1, \dots, u_\gamma\}$ and $\{v_1, \dots, v_{\gamma(d-1)/2}\}$, respectively. We connect U and V in such a way to maximize the girth of H.

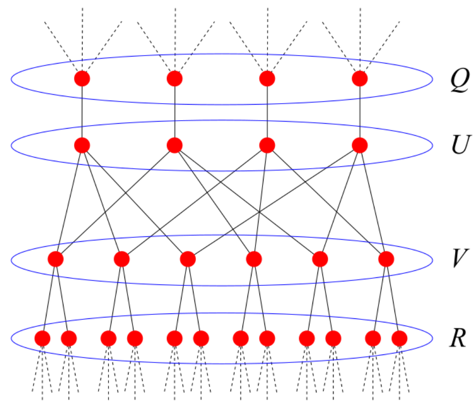

Figure 5.1: H', with labeled components for $d = 4, \gamma = 4$. Note that $\Psi(U) = (d+1)/2$. To create G', we connect Q and R to a well spaced matching in G.

Lemma 5.6.1.

$$g(H) \geq 2\log_{d-1}\gamma.$$

Proof. Because of the valency conditions on H, there is a graph $\tilde{H}$ on γ vertices $\{\tilde{u}_1, \dots, \tilde{u}_\gamma\}$, where $\tilde{u}_i \sim_{\tilde{H}} \tilde{u}_j$ if and only if $\exists v_k \in H$ such that $u_i \sim_H v_k$ and $u_j \sim_H v_k$. Namely, U corresponds to the vertex set of $\tilde{H}$, and V corresponds to the edge set. $\tilde{H}$ is $d-1$ regular, and, as paths in $\tilde{H}$ of length r correspond to paths of length $2r$ in H, $g(H) = 2g(\tilde{H})$.

By a result of Linial and Simkin [LS21], there exists a graph $\tilde{H}$ that has girth at least $c\log_{d-2}\gamma$, for any $c \in (0,1)$, assuming γ is even. Therefore by setting $c = \log(d-2)/\log(d-1)$, we have that $g(\tilde{H}) \geq \log_{d-1}\gamma$ and $g(H) \geq 2\log_{d-1}\gamma$. □

We add a new set of vertices $Q = \{q_1, \dots, q_\gamma\}$ and add a matching between Q and U, adding the edge $q_i u_i$ for $1 \leq i \leq \gamma$. Similarly, we add another set of vertices $R = \{r_{i,j}\}, 1 \leq i \leq \gamma(d-1)/2, 1 \leq j \leq d-2$. For each $1 \leq i \leq \gamma(d-1)/2$, we then add an edge from v_i to each of $r_{i,j}$ for $1 \leq j \leq (d-2)$.

We call H' the graph on $U \cup V \cup Q \cup R$. At this point vertices of U and V have degree-d, and vertices of Q and R have degree-1. Also, note $\Psi(U) = (d+1)/2$. We wish to embed H' into a larger, high girth expander, and show that this new graph maintains high girth and expansion, even though the set U is a lossy vertex expander. Our argument follows that of [Kah95, Section 5], but instead of embedding individual vertices, we will embed H'.

Theorem 5.6.2 (Theorem 1.3.12 in more detail)**.** *For every $d = p + 1$ for prime $p \geq 3$, there is an infinite family of d-regular graphs $G_m = (V_m, E_m)$ on m vertices, such that $\exists U_m \subset V_m$ with $\Psi(U_m) = (d+1)/2$ for $|U_m| \leq m^{1/3}$, $g(G_m) = (\frac{2}{3} - o_m(1))\log_{d-1} m$, and such that $\lambda(G_m) \leq 2\sqrt{d-1} + O(1/\log_{d-1} m)$.*

Proof. By the result of Lubotzky, Phillips and Sarnak [LPS88], for such d, there exists an infinite family of d-regular graphs, where graphs of n vertices have girth $(\frac{4}{3} - o_n(1)) \log_{d-1} n$ and have spectral expansion $\leq 2\sqrt{d-1}$.

For a given graph $G = (V, E)$ of this type of size n, we attach H' by removing a matching $M \subset E$, $M = \{(a_{1,1}, a_{1,2}), \ldots, (a_{k,1}, a_{k,2})\}$ for

$$k := \gamma(d-1)(2 + (d-1)(d-2))/4. \tag{5.1}$$

We take a matching such that the pairwise distance between edges in the matching is maximized in G.

Lemma 5.6.3. *In a d-regular graph on n vertices, there exists a matching M of size k such that for every pair of edges $(a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2}) \in M$, $i_1 \neq i_2$,*

$$d((a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2})) \geq \log_{d-1} n - \log_{d-1} \gamma - O_n(1).$$

Proof. For a given pair of adjacent vertices $(a_{i,1}, a_{i,2})$, as our graph is d regular, there are at most $1 + d\frac{(d-1)^r - 1}{d-2}$ vertices at distance at most r from $a_{i,1}$, and at most $(d-1)^r$ vertices at distance r from $a_{i,2}$ and distance $r+1$ from $a_{i,1}$. Therefore for any $d \geq 4$, the number of edges at distance at most r from a given edge is less than $4(d-1)^r$. We then greedily add edges by choosing an arbitrary edge with vertices at distance at least r away from all already chosen edges. A kth such edge will exist as long as $4k(d-1)^r \leq n$. For our k given in (5.1) we can set $r = \log_{d-1} n - \log_{d-1} \gamma - O_n(1)$. □

To connect H' to G, we first delete the matching M. Then for every vertex of Q and R, we add $d-1$ edges to the set of vertices of M, connecting to each vertex of M exactly once. Namely, the induced subgraph on $(Q \cup R) \cup M$ is a $(d-1, 1)$ biregular bipartite graph. Call $G' = (V', E')$ the new graph formed from G and H'.

We wish to show that G' remains high girth and a good spectral expander. For the girth of G', cycles are either completely contained in H', completely contained in G, or a mix between the two. Cycles in H' have length at least $2\log_{d-1} \gamma$ by 5.6.1. Cycles in G have length at least $(\frac{4}{3} - o_n(1)) \log_{d-1} n$ by the construction of [LPS88]. For cycles that are a mix of H' and G, we must go from one vertex of H' to another vertex of H' through G. Therefore by 5.6.3, the length of such a cycle is at least $\log_{d-1} n - \log_{d-1} \gamma - O_n(1)$, giving

$$g(G') \geq \min\{2\log_{d-1} \gamma, \log_{d-1} n - \log_{d-1} \gamma - O_n(1)\}.$$

To show that the spectrum is not adversely affected, we follow the argument of [Kah95, Theorem 5.2], with some adjustments. For our new graph, assume that there is an eigenvector $g \perp \mathbf{1}$ corresponding to an eigenvalue $|\mu| > 2\sqrt{d-1}$.

Call A the adjacency matrix of G', and A_G the adjacency matrix of G padded with zeros so it is of the same size as A. Then we have

$$g^* A g = g_G^* A_G g_G + g_{H'}^* A g_{H'} - 2\sum_{i=1}^{k} g(a_{i,1}) g(a_{i,2}) + \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u) g(a_{i,j})$$

where g_G and $g_{H'}$ are the projections of g onto G and H', respectively.

We know that

$$|g_G^* A_G g_G| \leq 2\sqrt{d-1}\|g_G\|^2 + \frac{d}{n}\left(\sum_{u\in G} g(u)\right)^2$$

by decomposing g into parts parallel and perpendicular to the all ones vector.

By a combination of 5.2.1 and 5.1.7, the spectral radius of H' is at most $2\sqrt{d-1}$, and therefore we have

$$|g_G^* A_G g_G| + |g_{H'}^* A g_{H'}| \leq 2\sqrt{d-1}\|g\|^2 + \frac{d}{n}\left(\sum_{u\in H'} g(u)\right)^2$$

as $\sum_G g(u) = -\sum_{H'} g(u)$, considering $g \perp \mathbf{1}$.

To show that $|\mu| = 2\sqrt{d-1} + O(1/\log n)$, we then need to show

$$\frac{1}{\|g\|^2}\left(\frac{d}{n}\left(\sum_{H'} g_{H'}(u)\right)^2 - 2\sum_{i=1}^{k} g(a_{i,1})g(a_{i,2}) + \sum_{\substack{u\in Q\cup R\\ a_{i,j}\in M\\ u\sim a_{i,j}}} g(u)g(a_{i,j})\right) = O\left(\frac{1}{\log n}\right). \tag{5.2}$$

The first term of (5.2) can be bounded as

$$\frac{d}{n}\left(\sum_{H'} g_{H'}(u)\right)^2 \leq \frac{d}{n}|H'|\,\|g_{H'}\|^2 \leq \frac{\gamma(2+(d-1)(d-2))d}{2n}\|g_{H'}\|^2. \tag{5.3}$$

The second term we can bound as

$$\left|2\sum_{i=1}^{k} g(a_{i,1})g(a_{i,2})\right| \leq \sum_{a_{i,j}\in M} g(a_{i,j})^2. \tag{5.4}$$

Now we will bound the last term of (5.2) using the Cauchy-Schwarz inequality.

$$\left|\sum_{\substack{u\in Q\cup R\\ a_{i,j}\in M\\ u\sim a_{i,j}}} g(u)g(a_{i,j})\right| \leq \sqrt{(d-1)\sum_{u\in Q\cup R} g(u)^2}\sqrt{\sum_{a_{i,j}\in M} g(a_{i,j})^2}. \tag{5.5}$$

We use the following lemma to bound the right hand sides of (5.4) and (5.5). The lemma is a generalized version of [Kah95, Lemma 5.1]. The result follows from the same proof, which we reproduce for completeness. Here, for two vectors $a, b \in \mathbb{R}^n$, $a \leq b$ if $\forall i \in [n], a(i) \leq b(i)$.

Lemma 5.6.4 (Lemma 5.1 of [Kah95]). *Consider a graph on a vertex set W, a subset X of W, a positive integer h, and $s \in L^2(W)$. Let X_i be the set of nodes at distance i from X. Assume the following conditions hold:*

(1) For $h-1 \leq i, j \leq h$, all nodes in X_i have the same number of neighbors in X_j.

(2) If $u \in X_{h-1}$ and $v \in X_h$ and $u \sim v$, then $s(u)/s(v)$ does not depend on the choices of u and v.

(3) s is nonnegative and $As \leq \mu s$ on $\mathrm{Ball}_{h-1}(X)$, where μ is a positive real number.

Proof. Let A be the adjacency matrix of W. Let P_{h-1} and $P_h(X)$ be the orthogonal projections onto X_{h-1} and onto X_h, respectively. Let $P_{\leq h-1}$ and $P_{\leq h}(X)$ be the orthogonal projections onto $\mathrm{Ball}_{h-1}(X)$ and $\mathrm{Ball}_h(X)$, respectively. We need to show that

$$\frac{\|P_h g\|^2}{\|P_h s\|^2} \geq \frac{\|P_{h-1} g\|^2}{\|P_{h-1} s\|^2}.$$

Call $A_h = P_{\leq h} A P_{\leq h}$ (so A_h performs the adjacency operator on $\mathrm{Ball}_h(X)$). By the conditions of the lemma, we know that there are constants α, β and γ such that

$$P_h A_h s = \gamma P_h s \tag{5.6}$$

and

$$A_h P_h s = \alpha P_h s + \beta P_{h-1} s. \tag{5.7}$$

By assumption,

$$A_h s \leq \mu P_{\leq h-1} s + \gamma P_h s. \tag{5.8}$$

Therefore by applying $P_{\leq h-1}$ to both sides of (5.8),

$$\begin{aligned} P_{\leq h-1} A_h s &\leq \mu P_{\leq h-1} s \\ &\leq \mu P_{\leq h} s - \mu P_h s. \end{aligned}$$

Now we apply A_h to both sides:

$$\begin{aligned} A_h P_{\leq h-1} A_h s &\leq \mu A_h s - \mu A_h P_h s \\ &\leq \mu A_h s - \mu(\alpha P_h s + \beta P_{h-1} s) && \text{by (5.7)} \\ &\leq \left(\mu^2 P_{\leq h-1} + \mu(\gamma - \alpha) P_h - \mu\beta P_{h-1}\right) s. && \text{by (5.8)} \end{aligned}$$

Define the matrix $B := \mu^2 P_{\leq h-1} + \mu(\gamma-\alpha) P_h - \mu\beta P_{h-1} - A_h P_{h-1} A_h$. B has no positive entries on the off-diagonal. Take any eigenvector ψ of B. Without loss of generality assume that ψ has a positive entry. Then take $i = \mathrm{argmax}_u \psi(u)/s(u)$. As $\psi \leq (\psi(i)/s(i))s$, $(B\psi)(i) \geq (B(\psi(i)/s(i))s)(i)$. The quantity on the right is nonnegative, meaning that the eigenvalue with eigenvector ψ is nonnegative. As ψ was arbitrary, B is positive semidefinite.

Because B is positive semidefinite,

$$g^* A_h P_{\leq h-1} A_h g \leq g^* \left(\mu^2 P_{\leq h-1} + \mu(\gamma - \alpha) P_h - \mu\beta P_{h-1}\right) g. \tag{5.9}$$

For any orthogonal projection P, $P^2 = P$. Therefore $g^* A_h P_{\leq h-1} A_h g = ||P_{\leq h-1} A_h g||^2$. Moreover (5.9) becomes

$$||P_{\leq h-1} A_h g||^2 \leq \mu^2 ||P_{\leq h-1} g||^2 + \mu(\gamma - \alpha)||P_h g||^2 - \mu\beta ||P_{h-1} g||^2.$$

By assumption, $||P_{\leq h} A_h g|| = \mu ||P_{\leq h} g||$. Therefore

$$(\gamma - \alpha)||P_h g||^2 \geq \beta ||P_{h-1} g||^2. \tag{5.10}$$

As A_h and P_h are self adjoint, $s^* A_h P_h s = s^* P_h A_h s$, so $\alpha ||P_h s||^2 + \beta ||P_{h-1}||^2 = \gamma ||P_h s||^2$. Combining this with (5.10), we obtain (5.11). □

For any $g \in L^2(W)$ *such that* $|Ag(u)| = \mu |g(u)|$ *for* $u \in \mathrm{Ball}_{h-1}(X)$*, we have*

$$\frac{\sum_{v \in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \geq \frac{\sum_{v \in X_{h-1}} g(v)^2}{\sum_{v \in X_{h-1}} s(v)^2}. \tag{5.11}$$

To use the lemma, we set $X_0 = U \cup V$, and h will vary from $2 \leq h \leq \lfloor r/2 \rfloor$. Assuming that the girth of G' is at least r, the $\lfloor r/2 \rfloor$ neighborhoods of each vertex do not overlap.

Our test vector decays exponentially, with a small adjustment.

$$s(y) = \begin{cases} \frac{1}{(d-1)^{h/2}} & y \in X_{h,U} \\ \frac{2}{\sqrt{d-1}} - \frac{1}{(d-1)^{3/2}} & y \in X_{0,V} \\ \left(\frac{2}{d-2} - \frac{2}{(d-1)(d-2)}\right) \frac{1}{(d-1)^{(h-1)/2}} & y \in X_{h,V}, h \geq 1. \end{cases}$$

For this assignment of values we have $As \leq (2\sqrt{d-1})s$. In fact, this inequality is sharp at all coordinates except for $y \in X_{1,V}$.

For this s, we have that $\sum_{y \in X_h} s(y)^2$ is constant for $h = 1, \ldots, \lfloor r/2 \rfloor$. Also, recall $Q \cup R = X_1$ and $M = X_2$. By 5.6.4, as g corresponds to an eigenvalue $|\mu| > 2\sqrt{d-1}$, the mass on each of first 2 layers of X can only be at most $2/(r-2)$ of the total mass.

Combining (5.3), (5.4), and (5.5), we can bound (5.2) as

$$\begin{aligned} (5.2) &\leq \frac{\gamma(2 + (d-1)(d-2))d}{2n} ||g_{H'}||^2 + \sum_{a \in X_2} g(a)^2 + \sqrt{(d-1)\sum_{u \in X_1} g(u)^2} \sqrt{\sum_{a \in X_2} g(a)^2} \\ &\leq \left(\frac{\gamma(2 + (d-1)(d-2))d}{2n} + (1 + \sqrt{d-1}) \frac{2}{r-2} \right) ||g||^2. \end{aligned}$$

If we set $\gamma = n^{1/3}$ and $r = \frac{2}{3} \log_{d-1} n - O_n(1)$, for fixed d this becomes

$$O\left(\frac{1}{\log n} \right) ||g||^2,$$

meaning that $\mu \leq 2\sqrt{d-1} + O(1/\log n)$. This also gives the desired bounds on vertex expansion and girth, by setting $U = U_m$. Because $|V'| = (1 + o_n(1))n$, the bounds on $\Psi(U_m)$, $g(G')$ and $\lambda(G')$ given in terms of n do not change when they are given in terms of m. □

5.7 Lossless Expansion of Small Sets

In this section, we prove that sufficiently small sets in a high-girth spectral expander expand losslessly.

Theorem 5.7.1 (Theorem 1.3.13 in detail)**.** *Let G be a d-regular graph on n vertices with girth at least $2\alpha \log_{d-1} n + 4$. Then for any set S with n^{κ} vertices,*

$$\frac{|\Gamma(S)|}{|S|} \geq d - \lambda(G) - \frac{d^{2\kappa/\alpha}}{2} - \frac{d}{n^{1-\kappa}}.$$

Proof. Let S be a set of vertices of size n^{κ} in G. Let e_S denote the number of internal edges within S. Let n_i denote the number of vertices in $\Gamma(S)$ that have i edges from S incident to it. Then: $|\Gamma(S)| = n_1 + n_2 + \cdots + n_d$ and $|E(S, \Gamma(S))| = n_1 + 2n_2 + \cdots + dn_d$. Note that $|E(S, \Gamma(S))|$ is also equal to $d|S| - 2e_S$. Now consider the graph H_S on vertex set S and edge set given by induced edges on S along with new edges introduced by adding an arbitrary spanning tree for every set of i vertices that are neighbors of a vertex in $\Gamma(S)$ with exactly i neighbors in S. The number of edges in H_S is equal to

$$e_S + n_2 + 2n_3 + \cdots + (d-1)n_d = e_S + |E(S, \partial S)| - |\Gamma(S)| = d|S| - e_S - |\Gamma(S)|.$$

The edges in H_S that are not in G correspond to paths of length at most 2 in G. Therefore $g(H_S) \geq \frac{1}{2} g(G) \geq \alpha \log_{d-1} n + 2$. As a consequence of the expander mixing lemma (2.0.8), $e_S \leq \left(\lambda(G) + \frac{d|S|}{n}\right)|S|$. Consequently,

$$|E(H_S)| \geq \left(d - \lambda(G) - \frac{d|S|}{n}\right)|S| - |\Gamma(S)|,$$

which means the average degree is lower bounded by

$$2\left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|}\right).$$

Thus by the irregular Moore bound (2.0.11),

$$g(H_S) \leq \frac{2 \log n^{\kappa}}{\log\left(2\left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|}\right) - 1\right)} + 2$$

and hence

$$\frac{\alpha}{\log(d-1)} \leq \frac{2\kappa}{\log\left(2\left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|}\right) - 1\right)}.$$

This implies

$$d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|} - \frac{1}{2} \leq \frac{d^{2\kappa/\alpha}}{2},$$

and finally by rearranging the above and plugging in $|S| = n^\kappa$

$$\frac{|\Gamma(S)|}{|S|} \geq d - \lambda(G) - \frac{d^{2\kappa/\alpha} - 1}{2} - \frac{d}{n^{1-\kappa}}.$$

□

Remark 5.7.2. If G is a n-vertex d-regular Ramanujan graph with girth $\frac{4}{3}\log_{d-1} n$ (which is a condition satisfied by the Ramanujan graphs of [LPS88]) then for every set S of size n^κ for $\kappa < 1/3$,

$$\frac{|\Gamma(S)|}{|S|} \geq d(1 - o_d(1)).$$

Chapter 6

Quantum Ergodicty

6.1 EXP is not sufficient for quantum ergodicity

For a graph $G = (V, E)$, we will use $|G|$ to denote the number of vertices $|V|$. I_n refers to the identity operator of dimension n. Unless otherwise specified, for a graph G, we will use the notation $G = (V_G, E_G)$ with adjacency operator $\mathcal{A}_G$. For $u, v \in V$, we will write $u \sim v$ to signify $(u, v) \in E$.

Definition 6.1.1. Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *Cartesian product* $G_1 \square G_2$ of the graphs G_1 and G_2 is defined as follows. $G_1 \square G_2$ has vertex set $V_1 \times V_2$, and for $(u_1, u_2), (v_1, v_2) \in V_1 \times V_2$, $(u_1, u_2) \sim (v_1, v_2)$ if and only if either

1. $u_1 \sim v_1$ in G_1 and $u_2 = v_2$ or

2. $u_1 = v_1$ and $u_2 \sim v_2$ in G_2.

An equivalent characterization is that if G_1 and G_2 have adjacency operations $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, then $G_1 \square G_2$ is the graph with adjacency operator $\mathcal{A}_1 \otimes I_{|G_2|} + I_{|G_1|} \otimes \mathcal{A}_2$.

Note that the Cartesian product is well defined for locally finite graphs (graphs where each vertex has finite degree), even if the graphs have an infinite number of vertices. However, in this section, we assume that all graphs are finite.

Given $\phi_i : V_i \to \mathbb{R}$ for $i \in \{1, 2\}$, we define $\psi_{\phi_1,\phi_2} : V_1 \times V_2 \to \mathbb{R}$ to be

$$\psi_{\phi_1,\phi_2}(u_1, u_2) := \phi_1(u_1) \cdot \phi_2(u_2). \tag{6.1}$$

The key property of Cartesian products we use is the following:

Fact 6.1.2. If ϕ_1 is an eigenvector of $\mathcal{A}_1$ of eigenvalue λ_1 and ϕ_2 is an eigenvector of $\mathcal{A}_2$ with eigenvalue λ_2, then ψ_{ϕ_1,ϕ_2} is an eigenvector of $\mathcal{A}_{G_1 \square G_2}$ of eigenvalue $\lambda_1 + \lambda_2$.

Take (G_n) as a family of d-regular graphs (V_n, E_n) that satisfies EXP with a fixed parameter $\epsilon > 0$ with adjacency operators $(\mathcal{A}_n)$. Let C_4 denote the cycle graph of length 4. $G_n \square C_4$ is a $d + 2$ regular graph on $4n$ vertices.

Proposition 6.1.3. *The family of graphs* $(G_n \Box C_4)$ *satisfies EXP but does not satisfy BST.*

Proof. As the spectrum of $\mathcal{A}_{C_4}$ is $\{2, 0, 0, -2\}$, by Fact 6.1.2, the adjacency operator of $G_n \Box C_4$ satisfies EXP with parameter $\frac{\min\{d\epsilon, 2\}}{d+2}$, which is constant for constant d.

Given $(u_1, u_2) \in V_n \times V_{C_4}$, take (v_1, v_2) such that $u_1 \sim v_1$ in G_n and $u_2 \sim v_2$ in C_4. This defines a 4−cycle in $G_n \Box C_4$, given by

$$(u_1, u_2) \sim (u_1, v_2) \sim (v_1, v_2) \sim (v_1, u_2) \sim (u_1, u_2).$$

By the regularity of G_n and C_4, such a vertex pair (v_1, v_2) will exist for any (u_1, u_2) (in fact, a total of $2d$ such pairs will exist). Therefore, there is a cycle of length 4 starting at any given vertex, so if $R \geq 2$, then $\frac{|\{(u_1,u_2) \in V_n \times V_{C_4}, \rho(x) < R\}|}{4n} = 1$. This means $G_n \Box C_4$ does not satisfy BST. □

Theorem 6.1.4 (Implies Theorem 1.3.18)**.** *Each graph in the family* $(G_n \Box C_4)$ *admits an eigendecomposition that violates quantum ergodicity.*

Proof. We order and label the four vertices of V_{C_4} $\{1, 2, 3, 4\}$. The localized eigenbasis of $\mathcal{A}_{C_4}$ we will use is given by the following table:

eigenvector	eigenvalue
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	2
$(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0)$	0
$(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$	0
$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	-2

.

We define $a_n : (V_n \times V_{C_4}) \to \mathbb{R}$,

$$a_n(u_1, u_2) = \begin{cases} 1 & u_2 \in \{1, 3\} \\ -1 & u_2 \in \{2, 4\}. \end{cases}$$

Because $\sum_{V_n \times V_{C_4}} a_n(u_1, u_2) = 0$ and $\|a_n\|_\infty = 1$, a_n satisfies the conditions of Theorem 1.3.17. Define $\psi_{\phi,i}$ as the eigenvector from (6.1) corresponding to the normalized eigenvector ϕ of $\mathcal{A}_n$ and the ith eigenvector of C_4 given in the table. By Fact 6.1.2, for any $u \in V_n$,

$$\psi_{\phi,2}(u, 2) = \psi_{\phi,2}(u, 4) = \psi_{\phi,3}(u, 1) = \psi_{\phi,3}(u, 3) = 0.$$

Therefore

$$|\langle \psi_{\phi,2}, a_{4n} \psi_{\phi,2} \rangle| = |\langle \psi_{\phi,3}, a_{4n} \psi_{\phi,3} \rangle| = 1$$

and

$$|\langle \psi_{\phi,1}, a_{4n} \psi_{\phi,1} \rangle| = |\langle \psi_{\phi,4}, a_{4n} \psi_{\phi,4} \rangle| = 0.$$

For any n,

$$\frac{1}{4n} \sum_{\phi,i} |\langle \psi_{\phi,i}, a_{4n} \psi_{\phi,i} \rangle|^2 = \frac{1}{2},$$

meaning this family of eigenbases is not quantum ergodic. □

6.2 Green's function on the infinite Cartesian product

Our goal in this section is to show the Benjamini-Schramm limit of $(G_n \square C_4)$ from Section 6.1 has bounded imaginary part of the Green's function under the added assumption that (G_n) satisfies BST. Therefore the requirement of BST cannot be generalized to this looser requirement on the Benjamini-Schramm limit. This relies on the Cartesian product commuting with both the Benjamini Schramm limit and, in a sense, with the Green's function itself.

Take μ to be a measure over isomorphism classes of rooted graphs, and X to be a finite graph. The measure $\mu_{\square X}$ is defined over the same space as μ, such that for any set Γ of isomorphism classes,

$$\mu_{\square X}(\Gamma) = \frac{1}{|X|} \sum_{v \in V_X} \mu(\{(G, o) : \exists (H, o') \in \Gamma \text{ s.t. } (G \square X, (o, v)) \cong (H, o')\}).$$

Proposition 6.2.1. *If the Benjamini-Schramm limit of (G_n) is μ, then the Benjamini-Schramm limit of $(G_n \square X)$ is $\mu_{\square X}$.*

Proof. Because (G_n) converges to μ, $\forall \epsilon > 0$, $\exists N$ such that for $n > N$, the distribution of $1/\epsilon$ rooted balls in G_n is within ϵ in any metrization of the weak topology of that of μ. We denote by $B_r(G, o)$ the ball of radius r around the root o in G. We have that

$$B_{1/\epsilon}(G_n \square X, (o, v)) \cong B_{1/\epsilon}(B_{1/\epsilon}(G_n, o) \square X, (o, v)).$$

This is to say that the distribution over $1/\epsilon$ neighborhoods in $G_n \square X$ only depends on G_n up to vertices of distance $1/\epsilon$. Therefore the distribution of $1/\epsilon$ balls with root (u, v) in $G_n \square X$ obtained by sampling u at random is within ϵ of the measure $\mu_{\square X}$ conditioned on the root being of the form $(\cdot, v)$ for specific $v \in V_X$. Sampling uniformly over $v \in V_X$ and sending $\epsilon \to 0$ gives the result. □

Consider a graph G with adjacency operator $\mathcal{A}$ and $z \in \mathbb{C}_+$. The Green's function $\mathcal{G}_G^z$ is the unique operator such that $(\mathcal{A} - z)\mathcal{G}_G^z = I_{|G|}$.

Theorem 6.2.2 (Restatement of Theorem 1.3.19)**.** *Take any (potentially infinite) graph G_1 and a finite graph G_2. Let $\psi_1, \ldots \psi_k$ be an orthonormal eigenbasis of $\mathcal{A}_2$ with eigenvalues $\lambda_1, \ldots, \lambda_k$.*
We have

$$\mathcal{G}^z_{G_1 \square G_2} = \sum_{i=1}^{k} \mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i \psi_i^T.$$

Originally I wrote a proof for when $G_1 = T_d$ which calculated entries of the Green's function recursively, similar to the proof of the Kesten-McKay measure using recursion (see for example Section 3 of [AW13]). The proof below was then sent to me by Mostafa Sabri, which generalizes to any G_1 and is less computationally intensive.

Proof. The adjacency operator of $G_1 \square G_2$ is $\mathcal{A}_1 \otimes I_{|G_2|} + I_{|G_1|} \otimes \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the adjacency operators of G_1 and G_2 respectively.

$$\begin{aligned}
(\mathcal{A}_1 \otimes I_{|G_2|} + I_{|G_1|} \otimes \mathcal{A}_2 - z)(\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T) &= \mathcal{A}_1\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T + \mathcal{G}_{G_1}^{z-\lambda_i} \otimes \mathcal{A}_2\psi_i\psi_i^T - z(\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T) \\
&= \mathcal{A}_1\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T + \lambda_i(\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T) - z(\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T) \\
&= (\mathcal{A}_1 + \lambda_i - z)\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T \\
&= I_{|G_1|} \otimes \psi_i\psi_i^T.
\end{aligned}$$

Therefore

$$\sum_{i=1}^{k}(\mathcal{A}_1 \otimes I_{|G_2|} + I_{|G_1|} \otimes \mathcal{A}_2 - z)(\mathcal{G}_{G_1}^{z-\lambda_i} \otimes \psi_i\psi_i^T) = \sum_{i=1}^{k} I_{|G_1|} \otimes \psi_i\psi_i^T = I_{|G_1 \square G_2|}$$

as desired. □

Theorem 6.2.3. *EXP and having bounded imaginary part in entries of the Green's function of the Benjamini-Schramm limit do not necessarily imply quantum ergodicity.*

Proof. Take the family (G_n) to satisfy EXP and BST. $(G_n \square C_4)$ satisfies EXP and, by Proposition 6.2.1, has Benjamini-Schramm limit $T_d \square C_4$. The entries of $\mathcal{G}_{T_d}^z$ have bounded imaginary part for $z \in \mathbb{C}^+$, so by Theorem 1.3.19 so does $\mathcal{G}_{T_d \square C}^z$, as $\psi_i\psi_i^T$ has norm 1. However, by Theorem 6.1.4, $(G_n \square C_4)$ has a family of eigenbases that violates quantum ergodicity. □

6.3 BST is not sufficient for quantum ergodicity

Consider any family of d-regular graphs (F_n) for d even and $d \geq 8$ such that (F_n) satisfies BST and $|F_n| = n$. We construct a family of graphs (H_n) as follows. Delete an arbitrary edge of F_n, and call this new graph F_n'. Create $d/2$ copies of F_n'. Then add a vertex v_n, and add an edge from v_n to each of the d vertices of degree $d-1$, two for each copy of F_n'. Call this graph $H_n = (V_{H_n}, E_{H_n})$.

Proposition 6.3.1. (H_n) *satisfies BST but not EXP.*

Proof. Take a vertex $x \in V_{H_n} \backslash v_n$. To differentiate between the injectivity radii of H_n and F_n, we will write $\rho_{H_n}(x)$ and $\rho_{F_n}(x)$. The former refers to the injectivity radius of x in H_n, whereas the latter is the injectivity radius of the vertex corresponding to x in F_n. We claim that $\rho_{H_n}(x) \geq \rho_{F_n}(x)$. To see this, take a cycle through x in H_n. If it intersects v_n, then it must intersect both neighbors of v_n in the copy of F_n that contains x. Therefore the length of such a cycle is at least $2\rho_{F_n}(x) + 2$. If a cycle does not intersect v_n, it remains in F_n and has length at least $2\rho_{F_n}(x) + 1$. Putting these together, we have $\rho_{H_n}(x) \geq \rho_{F_n}(x)$. Therefore, (H_n) satisfies BST.

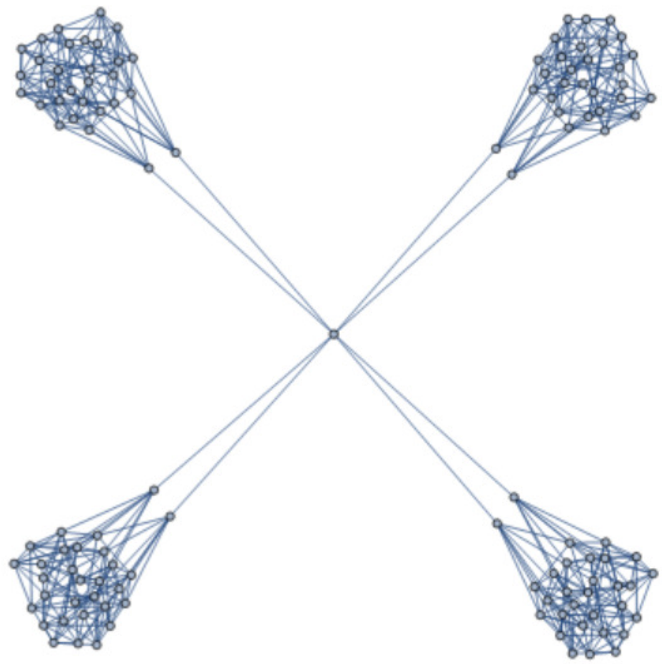

Figure 6.1: An example of a graph H_n from Section 6.3.

By Cheeger's inequality (see for example [Chu96]), because there are only 2 edges from one copy of F'_n to the rest of the graph, $\lambda_2 \geq (1 - 4/n)d$, meaning (H_n) does not satisfy EXP. □

Theorem 6.3.2 (Implies Theorem 1.3.20)**.** *The family of graphs* (H_n) *has an orthonormal eigenbasis which violates quantum ergodicity.*

Proof. Enumerate the copies of F'_n in H_n $F'_{n,1}, \dots, F'_{n,d/2}$. For any eigenvector ϕ of $\mathcal{A}_{F'_n}$, a normalized eigenvector χ of $\mathcal{A}_{H_n}$ of the same eigenvalue is given by

$$\chi(u) = \begin{cases} \phi(u)/\sqrt{2} & u \in V_{F'_{n,1}} \\ -\phi(u)/\sqrt{2} & u \in V_{F'_{n,2}} \\ 0 & \text{otherwise.} \end{cases}$$

Call X the set of eigenvectors of this type. We then set

$$a_n(u) = \begin{cases} 1 & u \in V_{F'_{n,1}}, V_{F'_{n,2}} \\ -1 & u \in V_{F'_{n,3}}, V_{F'_{n,4}} \\ 0 & \text{otherwise.} \end{cases}$$

a_n satisfies the conditions of a test function for quantum ergodicity. If we take Λ to be an eigenbasis of $\mathcal{A}_{H_n}$ that contains X,

$$\frac{1}{\frac{d}{2}n+1}\sum_{\psi\in\Lambda}|\langle\psi, a_n\psi\rangle|^2 \geq \frac{1}{\frac{d}{2}n+1}\sum_{\chi\in X}|\langle\chi, a_n\chi\rangle|^2 = \frac{1}{\frac{d}{2}n+1}\sum_{\chi\in X}1 = \frac{n}{\frac{d}{2}n+1} \geq \frac{1}{d}$$

violating quantum ergodicity. □

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