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A Stability Problem Involving Approximate Identities, Discrete Convolution Operators, Singular Integral Operators, and Finite Sections

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A STABILITY PROBLEM INVOLVING APPROXIMATE IDENTITIES, DISCRETE CONVOLUTION OPERATORS, SINGULAR INTEGRAL OPERATORS, AND FINITE SECTIONS

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by

Ryan Pugh

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Abstract

A Stability Problem Involving Approximate Identities, Discrete Convolution Operators, Singular Integral Operators, and Finite Sections

by

Ryan Pugh

Let $n \in \mathbb{N}$ tend towards infinity and $r \in [0, 1)$ tend towards 1 with the condition that $n(1 - r) \to \lambda$ for some fixed $\lambda \in (0, \infty)$. A sequence $(F_{n,r})$ of bounded linear operators on a Hilbert space is called $\lambda-$stable if for all sufficiently large n and all r sufficiently close to 1 such that $n(1 - r)$ is sufficiently close to λ, each $F_{n,r}$ is invertible and these inverses are uniformly bounded. We consider the $\lambda-$stability problem for sequences arising from a C^*-algebra containing discrete convolution operators, singular integral operators, and their finite sections. Our main result is that a sequence in a certain C^*- algebra is $\lambda-$stable if and only if a certain collection of operators given by strong limits is invertible. As an application, we relate this result to approximate identities and discuss several concrete examples such as finite sections of Toeplitz operators $(T_n(k_\omega a))$ whose symbols are approximate identities applied to piecewise continuous functions and finite sections of singular integral operators.

Chapter 1

Introduction

Let X be a Hilbert space and let A be a bounded linear operator on X. In order to approximate the solution to the equation

$$Ax = y \tag{1.1}$$

for $x, y \in X$, we normally follow the following procedure: first, we choose a sequence of projections $R_n \in \mathcal{L}(X)$ which converges strongly to the identity operator I as $n \to \infty$. We also choose an approximating sequence of operators $A_n \in \mathcal{L}(\text{Im } R_n)$ that converges strongly to A. Rather than considering the equation in (1.1) directly, we consider the sequence of equations

$$A_n x_n = R_n y \tag{1.2}$$

where $x_n \in \text{Im } R_n$ and ask the question of whether or not solutions to this sequence can tell us information about the solution to (1.1). This relates directly to the notion of whether or not an approximation method is "applicable." Indeed, we say that the approximation method in (1.2) is *applicable* to the operator A (or that (A_n) is an *appropriate* approximating sequence for A) if there is an $n_0 \in \mathbb{N}$ such that A_n is invertible for all $n \geq n_0$ and if the sequence of (unique) solutions $(x_n)_{n \geq n_0}$ converges in norm to a solution x of (1.1). This is closely related to the notion of stability for the sequence (A_n). We call a sequence (A_n) *stable* if there exists an n_0 such that A_n is invertible for all $n \geq n_0$ and the inverses are uniformly bounded; i.e.,

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

The following proposition demonstrates the relationship between the notion of appropriate approximating sequences and stability. For a proof, see [8], Section 1.1.3 and Proposition 1.1.

Proposition 1.0.1. *Let (A_n) converge strongly to A. Then (A_n) is an appropriate approximating sequence for A if and only if A is invertible and the sequence (A_n) is stable.*

In this thesis we consider sequences of operators depending on two parameters, one parameter n going to infinity and the other parameter r going to 1 with the relationship $n(1-r) \to \lambda$ for some fixed $\lambda \in (0, \infty)$. An example of such a sequence is the sequence of finite sections of the Toeplitz operator with symbol a_r defined by $a_r(t) = (1 - \frac{r}{t})^{-\beta}(1 - tr)^{\beta}$, i.e., the sequence $(A_{n,r}) = (T_n(a_r))$; however, in this thesis we consider even more sequences arising from an algebra generated by a variety of different sequences of operators. In Chapter 3 we will extend the notion of stability of sequences depending on one parameter to what we call "λ-stability" for sequences of two parameters. Our ultimate goal is to find $\lambda-$stability criteria for a particular algebra of sequences of operators, showing that a sequence is $\lambda-$stable if and only if a certain collection of operators given by strong limits is invertible.

This thesis is structured as follows. In Chapter 2 we establish some preliminaries. In Chapter 3 we introduce the notion of λ-stability and introduce the algebra $\mathcal{F}_*$ for which we seek λ-stability criteria. The section ends by reducing λ-stability in the algebra $\mathcal{F}_*$ to a question of invertibility in another algebra. In Chapter 4 we introduce several new algebras and develop stability criteria for them. In Chapter 5 we use the theory developed in Chapter 4 to revisit the invertibility question posed at the end of Chapter 3, concluding the chapter by proving our main result. The thesis concludes by connecting this problem to approximate identities and applying our main results to concrete examples. In several places throughout this thesis we use Fredholm Theory for the algebra generated by Fourier convolutions and multiplication operators which is treated in detail in Appendix A. For convenience, Appendix B provides a list of notation used throughout the thesis.

Chapter 2

Preliminaries

2.1 Laurent, Toeplitz, and Hankel Operators

For a symbol $a \in L^\infty(\mathbb{S}^1)$, the Laurent operator $L(a)$ is the doubly-infinite matrix

$$L(a) = (a_{j-k}), \quad j, k \in \mathbb{Z}$$

acting on $\ell^2(\mathbb{Z})$ with a_{j-k} being the $(j-k)^{th}$ Fourier coefficient of a; i.e.,

$$a_{j-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta}) e^{-i(j-k)\theta} d\theta.$$

The Laurent operator $L(a)$ can be thought of as a *multiplication operator* $M(a) : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ defined by $M(a)f = af$. If we denote by Ξ the operator from $L^2(\mathbb{S}^1)$ into $\ell^2(\mathbb{Z})$ sending a function f to its sequence of Fourier coefficients, we have the relationship

$$L(a) = \Xi M(a) \Xi^{-1}$$

which yields the identities $L(ab) = L(a)L(b)$ and

$$\|L(a)\| = \|a\|_\infty \tag{2.1}$$

where the lefthandside is the operator norm and the righthandside is the L^∞ norm. One can also see that $L(a^*) = L(a)^*$, a property that we will make use of later. In our considerations of Laurent operators, we will often take the viewpoint of multiplication operators even if we do not explicitly say so.

Consider the following bounded linear operators on $\ell^2(\mathbb{Z})$ [1]:

$$P : (x_n)_{n\in\mathbb{Z}} \mapsto (y_n)_{n\in\mathbb{Z}} \quad \text{with} \quad y_n = \begin{cases} x_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{2.2}$$

$$J : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{-1-n})_{n\in\mathbb{Z}}. \tag{2.3}$$

Let I denote the identity on $\ell^2(\mathbb{Z})$ and set $Q := I - P$. For the operators P, Q, and J, the following properties hold: $P^* = P = P^2, J = J^*, J^2 = I$, and $JPJ = Q$. The operators P and Q are related to the *singular integral operator* on the unit circle $S_{\mathbb{S}^1}$ via the relationship $P = \dfrac{1 + S_{\mathbb{S}^1}}{2}$ and $Q = \dfrac{1 - S_{\mathbb{S}^1}}{2}$. Later we will consider the singular integral operator on the real and positive real line.

For $a \in L^\infty$, the *Toeplitz operator* $T(a)$ is defined by $T(a) = PL(a)P$ and the *Hankel operator* $H(a)$ is defined by $H(a) = PL(a)JP$. If we identify the image of P with $\ell^2(\mathbb{N})$, we can think of Toeplitz and Hankel operators as operators acting on $\ell^2(\mathbb{N})$. For $a \in L^\infty$, let $\tilde{a}$ denote the function defined by $\tilde{a}(t) = a(1/t)$ for $t \in \mathbb{S}^1$.

[1] We note that while these are considered as acting on $\ell^2(\mathbb{Z})$, there is the natural identification $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{S}^1)$ which we will freely go between in all of our considerations.

Then $JL(a)J = L(\tilde{a})$ and we have

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}).$$

A well-known fact that will be of use to us is that Hankel operators with continuous symbol are compact (see [2], Section 1.6 for example).

Finally, for $n \in \mathbb{N}$, define the following operators on $\ell^2(\mathbb{Z})$:

$$P_n : (x_k)_{k\in\mathbb{Z}} \mapsto (y_k)_{k\in\mathbb{Z}}, \qquad y_k = \begin{cases} x_k & \text{if } -n \le k < n \\ 0 & \text{if } k < -n \text{ or } k \ge n \end{cases} \tag{2.4}$$

$$U_n : (x_k)_{k\in\mathbb{Z}} \mapsto (x_{k-n})_{k\in\mathbb{Z}} \tag{2.5}$$

$$U_{-n} : (x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+n})_{k\in\mathbb{Z}} \tag{2.6}$$

$$P_n^+ := U_{-n}PU_n \tag{2.7}$$

$$Q_n^- := U_nQU_{-n} \tag{2.8}$$

The operators U_n and U_{-n} are shift operators that converge weakly to zero. We also have the relationship $P_n^+Q_n^- = Q_n^-P_n^+ = P_n$.

2.2 Composition Operators

Let $PC_{\pm 1}$ denote the set of all piecewise continuous functions that are continuous on $\mathbb{S}^1 \setminus \{-1, 1\}$, let $PC_{\pm 1}^0$ denote the set of all functions $f \in PC_{\pm 1}$ such that $f(-1 \pm 0) = 0$, and let PC_{-1}^0 denote the set of piecewise continuous functions f with $f(-1) = 0$. Given a continuous bijective function $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$, we define the composition operator generated by σ to be

$$C(\sigma) : f \mapsto f \circ \sigma \tag{2.9}$$

where $(f \circ \sigma)(t) = f(\sigma(t))$. In our work that follows, composition operators will be considered as operators defined on L^∞ or L^2. For $\tau \in \mathbb{S}^1$, we define the composition operator Y_τ by

$$Y_\tau = C(\sigma_\tau) \quad \text{with} \quad \sigma_\tau(t) = \tau t \tag{2.10}$$

and its inverse Y_τ^{-1} by

$$Y_\tau^{-1} = C(\widehat{\sigma}_\tau) \quad \text{with} \quad \widehat{\sigma}_\tau(t) = t/\tau. \tag{2.11}$$

This operator Y_τ can be thought of as an operator that takes a function and rotates it on the unit circle. For $r \in [0, 1)$, we define the operator C_r by

$$C_r = C(\sigma_r) \quad \text{with} \quad \sigma_r(t) = \frac{t + r}{1 + rt} \tag{2.12}$$

and its inverse by

$$C_r^{-1} = C(\widehat{\sigma}_r) \quad \text{with} \quad \widehat{\sigma}_r(t) = \frac{t - r}{1 - rt}. \tag{2.13}$$

Notice that the operator C_r is essentially stretching a function at 1 and squeezing it at -1. From this perspective it is not so hard to see that the characteristic functions χ_+ and χ_- on the upper and lower half plane of the unit circle respectively are invariant under this operator; that is,

$$C_r \chi_+ = \chi_+ \quad \text{and} \quad C_r \chi_- = \chi_-. \tag{2.14}$$

Finally, for ease of notation, define

$$G_{r,\tau} = C_r Y_\tau \tag{2.15}$$

and

$$G_{r,\tau}^{-1} = Y_\tau^{-1} C_r^{-1}. \tag{2.16}$$

Lemma 2.2.1. *For each $\tau \in \mathbb{S}^1$, the operator $Y_\tau : L^2 \to L^2$ is unitary. Moreover, $Y_\tau P Y_\tau^* = P, Y_\tau Q Y_\tau^* = Q$, and $Y_\tau L(a) Y_\tau^* = L(Y_\tau a)$ where $a \in L^\infty$.*

Proof. We refer the reader to [6], Lemma 5.1. □

The operators C_r are not isometries on L^2 in general. This fact motivates us to introduce the modified operators $R_r : L^2 \to L^2$ for $r \in [0, 1)$ defined by

$$(R_r f)(t) := \frac{\sqrt{1-r^2}}{1+rt} f\left(\frac{t+r}{1+rt}\right). \tag{2.17}$$

The following lemma will be useful in our considerations in this thesis. For a proof see [6], Lemma 5.2.

Lemma 2.2.2. *For each $r \in [0, 1)$, the operator $R_r : L^2 \to L^2$ is unitary. Moreover, $R_r P R_r^* = P, R_r Q R_r^* = Q$, and $R_r L(a) R_r^* = L(C_r a)$ when $a \in L^\infty$.*

Let (A_r) be a sequence of operators acting on a Hilbert space X. We say that (A_r) converges *strongly* to A as $r \to 1$ if $\|A_r x - Ax\| \to 0$ for all $x \in X$. We say that (A_r) converges *weakly* to A if $\langle x, A_r y\rangle \to \langle x, Ay\rangle$ as $r \to 1$ for all $x, y \in X$.

Lemma 2.2.3. *The sequences of operators (R_r) and (R_r^*) converge weakly to zero.*

Proof. Due to the uniform boundedness of (R_r) and (R_r^*), it is enough to prove weak convergence on a dense subset of $L^2(\mathbb{S}^1)$. To this end, let f be a function vanishing identically in a neighborhood of $1 \in \mathbb{S}^1$ and let g be a function vanishing identically in a neighborhood of $-1 \in \mathbb{S}^1$. For weak convergence of (R_r) we are tasked with showing $\langle g, R_r f\rangle$ converges to 0, but this is clear by definition - the inner product is identically 0 for r sufficiently close to 1. Similarly we have that $\langle f, R_r^* g\rangle$ converges to zero. □

We now recall the notion of convergence in measure. Let $\{f_r\}_{r\in[0,1)}$ be a sequence of uniformly bounded functions in L^∞. We say that f_r converges to f in measure as $r \to 1$ if for each fixed $\epsilon > 0$, the Lebesgue measure $\mu\left(K_{\epsilon,r}\right)$ of the set

$$K_{\epsilon,r} := \{t \in \mathbb{S}^1 : \left|f_r(t) - f(t)\right| \geq \epsilon\}$$

tends to 0 as r goes to 1. When this is the case, we write $f = \mu - \lim_{r\to 1} f_r$.

Lemma 2.2.4. *Let $\{f_r\}_{r\in[0,1)}$ be a sequence of functions in L^∞ and let $f \in L^\infty$. Then $L(f_r) \to L(f)$ strongly on L^2 as $r \to 1$ if and only if $\{f_r\}_{r\in[0,1)}$ is uniformly bounded and $f_r \to f$ in measure.*

Proof. Suppose $L(f_r) \to L(f)$ strongly. By the Uniform Boundedness Principle, we have that

$$\sup_{r\in[0,1)} \|L(f_r)\| < \infty.$$

Then since $\|L(f_r)\| = \|f_r\|_\infty$, it follows that $\{f_r\}_{r\in[0,1)}$ is uniformly bounded. Even further, it follows that f_r converges in L^2-norm to f. Then

$$\|f_r - f\|_{L^2}^2 \geq \frac{1}{2\pi}\int_{K_{\epsilon,r}} |f_r(t) - f(t)|^2 dt \geq \mu\left(K_{\epsilon,r}\right)\frac{\epsilon^2}{2\pi},$$

which implies that $\mu\left(K_{\epsilon,r}\right) \to 0$ as $r \to 1$ for each fixed $\epsilon > 0$. Thus f_r converges to f in measure.

Now suppose that $\{f_r\}_{r\in[0,1)}$ is uniformly bounded and $f_r \to f$ in measure. We aim to prove that $L(f_r)$ converges strongly to $L(f)$. Because of the uniform boundedness of $\{f_r\}_{r\in[0,1)}$ and hence $(L(f_r))_{r\in[0,1)}$, an approximation argument may be used. In particular, we will show that for each trigonometric polynomial p, $f_r p \to fp$ in L^2-norm. Set $M = \sup\limits_{r\in[0,1)} \|f_r - f\|_\infty < \infty$ and observe that

$$\begin{aligned}\|f_r - f\|_{L^2}^2 &= \frac{1}{2\pi}\int_{K_{\epsilon,r}} |f_r(t) - f(t)|^2 dt + \frac{1}{2\pi}\int_{\mathbb{S}^1\setminus K_{\epsilon,r}} |f_r(t) - f(t)|^2 dt \\ &\le \mu\left(K_{\epsilon,r}\right)\frac{M^2}{2\pi} + \epsilon^2.\end{aligned}$$

Since we may choose ϵ as small as desired, it follows that f_r converges to f in L^2 and so $f_r p$ converges to fp in L^2-norm for trigonometric polynomials p. Thus the claim is proven. □

2.3 The Fourier Transform

Let $\mathcal{F}$ denote the Fourier transform acting on the Schwartz space $S(\mathbb{R})$ of rapidly decaying C^∞ functions f via

$$(\mathcal{F}f)(x) = \int_{\mathbb{R}} e^{-2\pi i x z} f(z)dz, \quad x \in \mathbb{R}, \tag{2.18}$$

and let $\mathcal{F}^{-1}$ denote the inverse Fourier transform

$$(\mathcal{F}^{-1}f)(z) = \int_{\mathbb{R}} e^{2\pi i x z} f(x)dx, \quad z \in \mathbb{R}. \tag{2.19}$$

The Fourier transform extends by continuity to a unitary operator on $L^2(\mathbb{R})$;

we denote this extension also by $\mathcal{F}$. For $p > 1$ and b a bounded function, the operator $\mathcal{F}^{-1}b\mathcal{F}$ is well-defined on $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$. If this operator can be extended boundedly onto all of $L^p(\mathbb{R})$, then we call this extension a *Fourier convolution operator* and denote it by $W^0(b)$. In this case, the function b is called an L^p-Fourier multiplier. In this thesis we are concerned only with $p = 2$; in this situation the set of L^2-multipliers is exactly the algebra $L^\infty(\mathbb{R})$ of essentially bounded and measurable functions and we have $\|W^0(b)\|_{\mathcal{L}(L^2(\mathbb{R}))} = \|b\|_\infty$.

Chapter 3

The Stability Problem

In this section we introduce the notions of $\lambda-$convergence and $\lambda-$stability and introduce the algebra $\mathcal{F}_*$ for which we seek $\lambda-$stability criteria. Just as stability of sequences depending only on n is equivalent to invertibility modulo zero sequences, we show that this still holds true for $\lambda-$stability. By introducing certain strong limits, we prove a lifting theorem in order to say even more about when a sequence is $\lambda-$stable. We then apply a technique called localization, ending by reducing $\lambda-$stability in the algebra $\mathcal{F}_*$ to a question of invertibility in a smaller quotient algebra.

3.1 The Algebras $\mathcal{F}$ and $\mathcal{F}_*$

Throughout this thesis we let $n \in \mathbb{N}$ tend towards infinity and $r \in [0, 1)$ tend towards 1 with the condition that $n(1-r) \to \lambda$ for some fixed $\lambda \in (0, \infty)$. Let $\sigma_{n,r}$ be a sequence of real numbers. We define $\lambda-$convergence of $\sigma_{n,r}$ to σ as follows: we say

$$\lim_{(n,r)\overset{\lambda}{\to}(+\infty,1)} \sigma_{n,r} = \sigma$$

if for all open sets U containing σ there exist $n_0 \in \mathbb{N}, r_0 \in [0,1)$, and $\delta > 0$ such that if $n \geq n_0$, $r_0 \leq r < 1$, and $|(1-r)n - \lambda| < \delta$, then $\sigma_{n,r} \in U$. With this in mind, we define convergence in norm of a sequence of operators $(A_{n,r})$ to zero: we say $(A_{n,r})$ converges in norm to zero if for all $\epsilon > 0$ there exist $n_0 \in \mathbb{N}, r_0 \in [0,1)$, and $\delta > 0$ such that if $n \geq n_0$, $r_0 \leq r < 1$, and $|(1-r)n-\lambda| < \delta$, then $\|A_{n,r}\| < \epsilon$. The definition of strong convergence follows; we say a sequence of operators $(A_{n,r})$ converges strongly to an operator A if for all elements x of the domain we have

$$\|A_{n,r}x - Ax\| \to 0.$$

In this situation we write

$$\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} A_{n,r} = A.$$

In this notation the dependence on λ is not explicitly stated; however, we will always be assuming this.

Equivalently, when discussing $\lambda-$convergence of a sequence $\sigma_{n,r}$, we may pass to subsequences: we say

$$\lim_{(n,r)\overset{\lambda}{\to}(+\infty,1)} \sigma_{n,r} = \sigma$$

if for $n_i \to \infty, r_i \to 1$, and $n_i(1-r_i) \to \lambda$, $\lim_{i\to\infty} \sigma_{n_i,r_i} = \sigma$.

Let $n(1-r) \to \lambda$. We call a sequence $(F_{n,r})$ $\lambda-$*stable* if there exist $n_0 \in \mathbb{N}$, $r_0 \in [0,1)$, and $\delta > 0$ such that if $n \geq n_0$, $r \in [r_0, 1)$, and $|n(1-r) - \lambda| < \delta$, then $(F_{n,r})$ is invertible and the inverses are uniformly bounded; i.e,

$$\sup_{\substack{n\geq n_0 \\ r\in[r_0,1)}} \|F_{n,r}^{-1}\| < \infty.$$

Let $\mathcal{F}$ denote the space of sequences of bounded linear operators $\{A_{n,r}\}$ acting on $L^2(\mathbb{S}^1)$ for which

$$\|\{A_{n,r}\}\|_{\mathcal{F}} = \sup_{\substack{n\in\mathbb{N} \\ r\in[0,1)}} \|A_{n,r}\|_{\mathcal{L}(L^2(\mathbb{S}^1))} < \infty.$$

The space $\mathcal{F}$ is actually a C^*-algebra with norm given by the previous supremum and algebraic operations given by

$$\begin{aligned} \{A_{n,r}\} + \{B_{n,r}\} &:= \{A_{n,r} + B_{n,r}\}, \quad & z\{A_{n,r}\} &:= \{zA_{n,r}\}, \\ \{A_{n,r}\}\{B_{n,r}\} &:= \{A_{n,r}B_{n,r}\}, & \{A_{n,r}\}^* &:= \{A^*_{n,r}\}. \end{aligned}$$

We denote by $\mathcal{N}$ the *-ideal of $\mathcal{F}$ consisting of all sequences $\{C_{n,r}\} \in \mathcal{F}$ for which $\|\{C_{n,r}\}\|_{\mathcal{F}} \to 0$ as $n \to \infty$, $r \to 1$, and $n(1-r) \to \lambda$. In this thesis we are interested in stability of sequences belonging to a subalgebra of $\mathcal{F}$ generated by certain sequences and containing $\mathcal{N}$. Before introducing the subalgebra of interest, we start with the following useful theorem.

Theorem 3.1.1. *A sequence $(F_{n,r}) \in \mathcal{F}$ is λ-stable if and only if the coset $(F_{n,r})+\mathcal{N}$ is invertible in $\mathcal{F}/\mathcal{N}$.*

Proof. Suppose $(F_{n,r}) \in \mathcal{F}$ is λ-stable. Then there exist $n_0 \in \mathbb{N}$ and $r_0 \in [0,1)$ such that $F_{n,r}$ is invertible for all $n \geq n_0$ and $r \in [r_0, 1)$. Define $(A_{n,r})$ to be the sequence that is equal to $F^{-1}_{n,r}$ if $n \geq n_0$ and $r \in [r_0, 1)$ and 0 otherwise. This sequence is in $\mathcal{F}$ due to the uniform boundedness of the inverses and we have $(A_{n,r}F_{n,r}) - (I)$ and $(F_{n,r}A_{n,r}) - (I)$ belong to $\mathcal{N}$.

Now suppose $(F_{n,r}) + \mathcal{N}$ is invertible in $\mathcal{F}/\mathcal{N}$. Then there exists $(A_{n,r}) \in \mathcal{F}$ such that $A_{n,r}F_{n,r} = I + C_{n,r}$ with $\|C_{n,r}\| \to 0$. By definition, there exist $n_0 \in \mathbb{N}, r_0 \in [0,1)$, and $\delta > 0$ such that $\|C_{n,r}\| < 1/2$ for all $n \geq n_0, r \in [r_0, 1)$, and $|n(1-r) - \lambda| < \delta$. Now, if $\|C_{n,r}\| < 1/2$, we have $I + C_{n,r}$ is invertible and

$(I + C_{n,r})^{-1} A_{n,r}$ is the (left) inverse of $F_{n,r}$. Notice that

$$\|(I + C_{n,r})^{-1} A_{n,r}\| \leq \|(I + C_{n,r})^{-1}\| \cdot \|A_{n,r}\| \leq (1 - \|C_{n,r}\|)^{-1} \|A_{n,r}\| \leq 2\|A_{n,r}\|.$$

Then since $(A_{n,r}) \in \mathcal{F}$, it follows that this inverse is uniformly bounded. The argument for a right inverse is analagous; we just might have a different zero sequence $C'_{n,r}$, but this will be equal to $C_{n,r}$ modulo $\mathcal{N}$. Hence $(F_{n,r})$ is λ-stable.

□

In this thesis, we explore stability criteria for sequences of operators in the algebra

$$\mathcal{F}_* := \mathrm{alg}_{\mathcal{L}(L^2(\mathbb{S}^1))}\{(P), (P_n^+), (Q_n^-), (L(a)), (L(G_{r,\tau}^{-1} f)), (K), (Y_\tau^* R_r^* K_\tau R_r Y_\tau), \mathcal{N}\}$$

with $a \in PC, f \in PC, K, K_\tau \in \mathcal{K}(L^2(\mathbb{S}^1))$, and $\tau \in \mathbb{S}^1$. We remark that it would be equivalent to replace the condition $f \in PC$ with the condition that $f \in PC_{-1}^0$. Indeed, for a general $f \in PC$ we may use the representation

$$f = f(-1+0)\chi_+ + f(-1-0)\chi_- + d$$

where $d \in PC_{-1}^0$. Then $L(G_{r,\tau}^{-1} f)$ is

$$L(G_{r,\tau}^{-1} f) = f(-1+0) L(Y_\tau^* \chi_+) + f(-1-0) L(Y_\tau^* \chi_-) + L(G_{r,\tau}^{-1} d).$$

Then since $L(Y_\tau^* \chi_+)$ and $L(Y_\tau^* \chi_-)$ are already generated by $L(a)$, our remark follows. In some cases it will be convenient for us to use the condition $f \in PC_{-1}^0$; when this occurs we will explicitly say we are using this property.

For a sequence $(F_{n,r}) \in \mathcal{F}_*$, we define the following three strong limit operators as follows:

$$\Phi_0(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} F_{n,r} \tag{3.1}$$

$$\Phi_1(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} U_{-n}F_{n,r}U_n \tag{3.2}$$

$$\Phi_{-1}(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} U_nF_{n,r}U_{-n} \tag{3.3}$$

Notice that by the uniform boundedness of (U_n) and (U_{-n}), we have $\mathcal{N} \subseteq \ker \Phi_i$ for $i = 0, 1, -1$. The following three propositions show how each of the Φ_i acts on the generators of $\mathcal{F}_*$ for $i = 0, 1, -1$.

Proposition 3.1.2. *The strong limit $\Phi_0(F_{n,r})$ exists for all $(F_{n,r}) \in \mathcal{F}_*$. In particular, the generators are mapped as follows:*

$$\begin{array}{llll} (P) \mapsto P & (P_n^+) \mapsto I & (Q_n^-) \mapsto I & (Y_\tau^* R_r^* K_\tau R_r Y_\tau) \mapsto 0 \\ (L(a)) \mapsto L(a) & (L(G_{r,\tau}^{-1} f)) \mapsto 0 & (K) \mapsto K & \end{array}$$

where $a \in PC$ and $f \in PC_{-1}^0$.

Proof. This statement is clear for $(P), (P_n^+), (Q_n^-), (K)$, and $(L(a))$. For $(L(G_{r,\tau}^{-1}f))$, we note that

$$(G_{r,\tau}^{-1}f)(t) = f\left(\frac{t/\tau - r}{1 - rt/\tau}\right)$$

which converges locally uniformly to $f(-1)$ on $\mathbb{S}^1 \setminus \{\tau\}$ and hence $(L(G_{r,\tau}^{-1}f))$ converges strongly to $f(-1)I$. Then since we have $f \in PC_{-1}^0$, this is equal to 0.

To deal with $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$, we will use the fact that every compact operator K_τ on $\ell^p(\mathbb{Z})$ can be approximated as closely as desired by an operator whose matrix representation (a_{jk}) has only finitely many non-vanishing entries, allowing us to write

$$K_\tau = \sum_{j,k\in\mathbb{Z}} a_{jk} U_{-j}(PU_1 - U_1P)U_{k+1}$$

where $U_m = L(t^m)$ is the shift operator. Then to prove that the strong limit of $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$ is zero, it is enough to show that $(Y_\tau^* R_r^*(PL(t) - L(t)P)R_r Y_\tau)$ converges strongly to zero and that $(Y_\tau^* R_r^* L(t^j) R_r Y_\tau)$ converges strongly to something for any $j \in \mathbb{Z}$. For starters, we will show $(Y_\tau^* R_r^*(PL(t) - L(t)P)R_r Y_\tau)$ converges strongly to zero. To achieve this we will show that $(Y_\tau^* R_r^* L(t) R_r Y_\tau)$ converges strongly to a scalar multiple of the identity (this is enough since $P = Y_\tau^* R_r^* P R_r Y_\tau$). Notice that $Y_\tau^* R_r^* L(t) R_r Y_\tau = L\left(Y_\tau^* C_r^{-1} t\right)$. The symbol of this Laurent matrix is $\left(\dfrac{t/\tau - r}{1 - rt/\tau}\right)$ which converges to -1 locally uniformly on $\mathbb{S}^1 \setminus \{\tau\}$. Thus $L\left(\dfrac{t/\tau - r}{1 - rt/\tau}\right)$ converges strongly to $-I$.

To conclude our proof, we examine the strong limit of $(Y_\tau^* R_r^* L(t^j) R_r Y_\tau)$ for fixed $j \in \mathbb{Z}$. This sequence is equal to $L\left(Y_\tau^* C_r^{-1} t^j\right)$. Similar to what we just did, we see the symbol is $\left(\dfrac{t/\tau - r}{1 - rt/\tau}\right)^j$ which converges locally uniformly to $(-1)^j$ and hence this strong limit exists for any $j \in \mathbb{Z}$. Our proof is therefore complete.

□

Proposition 3.1.3. *The strong limit $\Phi_1(F_{n,r})$ exists for all $(F_{n,r}) \in \mathcal{F}_*$. In particular, the generators are mapped as follows:*

$$\begin{array}{cccc} (P) \mapsto I & (P_n^+) \mapsto I & (Q_n^-) \mapsto Q & (Y_\tau^* R_r^* K_\tau R_r Y_\tau) \mapsto 0 \\ (L(a)) \mapsto L(a) & (L(G_{r,\tau}^{-1} f)) \mapsto 0 & (K) \mapsto 0 & \end{array}$$

where $a \in PC$ and $f \in PC_{-1}^0$.

Proof. This statement is clear for $(P), (P_n^+)$, and (Q_n^-). Because we may view the shift operators $U_{\pm n}$ as Laurent operators $L(t^{\pm n})$ and Laurent operators commute with each other, we get the same strong limits for $L(a)$ and $(L(G_{r,\tau}^{-1} f))$ as we did in Proposition 3.1.2. The fact that (K) gets sent to zero is a consequence of the weak convergence of U_n and U_{-n} to zero. Thus the only generator left to check is $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$. We are aiming to compute the strong limit of

$U_{-n}Y_\tau^* R_r^* K_\tau R_r Y_\tau U_n$. Notice that

$$\begin{aligned}
U_{-n}Y_\tau^* R_r^* K_\tau R_r Y_\tau U_n &= L(t^{-n})Y_\tau^* R_r^* K_\tau R_r Y_\tau L(t^n) \\
&= Y_\tau^* R_r^* R_r Y_\tau L(t^{-n})Y_\tau^* R_r^* K_\tau R_r Y_\tau L(t^n) Y_\tau^* R_r^* R_r Y_\tau \\
&= Y_\tau^* R_r^* L(C_r Y_\tau t^{-n}) K_\tau L(C_r Y_\tau t^n) R_r Y_\tau \\
&= Y_\tau^* R_r^* L\left(\left(\tau \frac{t+r}{1+rt}\right)^{-n}\right) K_\tau L\left(\left(\tau \frac{t+r}{1+rt}\right)^{n}\right) R_r Y_\tau \\
&= Y_\tau^* R_r^* L\left(\left(\frac{t+r}{1+rt}\right)^{-n}\right) K_\tau L\left(\left(\frac{t+r}{1+rt}\right)^{n}\right) R_r Y_\tau \\
&= Y_\tau^* R_r^* L(C_r t^{-n}) K_\tau L(C_r t^n) R_r Y_\tau
\end{aligned}$$

To finish this proof, we will prove that $(L(C_r t^n))$ and $(L(C_r t^{-n}))$ converge strongly. The idea is that if we show that these two operators converge strongly, then $L(C_r t^{-n}) K_\tau L(C_r t^n)$ will converge in norm to some other compact operator K'. Then

$$\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} Y_\tau^* R_r^* L(C_r t^{-n}) K_\tau L(C_r t^n) R_r Y_\tau = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} Y_\tau^* R_r^* K' R_r Y_\tau$$

and so by Proposition 3.1.2 we get that this is equal to zero.

Let's start with strong convergence of $(L(C_r t^n))$. The strategy will be to prove local uniform convergence of the symbol on $\mathbb{S}^1 \setminus \{-1\}$. This implies convergence in measure and thus we may conclude that the Laurent operators converge strongly.

$$\begin{aligned}
C_r t^n &= \left(\frac{t+r}{1+rt}\right)^n \\
&= \left(1 + \frac{t+r-1-rt}{1+rt}\right)^n
\end{aligned}$$

$$= \Big(1 + \frac{r-1+t(1-r)}{1+rt}\Big)^n$$
$$= \Big(1 + \frac{(1-r)(t-1)}{1+rt}\Big)^n$$
$$= \Big(1 + \frac{\big[\frac{n(1-r)(t-1)}{1+rt}\big]}{n}\Big)^n \to \exp\Big(\lambda \frac{t-1}{1+t}\Big) \text{ on } \mathbb{S}^1 \setminus \{-1\}$$

Thus $L(C_r t^n)$ converges strongly to $L\Big(\exp\Big(\lambda \frac{t-1}{1+t}\Big)\Big)$. The argument for $(L(C_r t^{-n}))$ is analagous; following the same process as above we get that $(L(C_r t^{-n}))$ converges strongly to $L\Big(\exp\Big(\lambda \frac{1-t}{1+t}\Big)\Big)$. □

Proposition 3.1.4. *The strong limit* $\Phi_{-1}(F_{n,r})$ *exists for all* $(F_{n,r}) \in \mathcal{F}_*$. *In particular, the generators are mapped as follows:*

$$\begin{array}{cccc} (P) \mapsto 0 & (P_n^+) \mapsto P & (Q_n^-) \mapsto I & (Y_\tau^* R_r^* K_\tau R_r Y_\tau) \mapsto 0 \\ (L(a)) \mapsto L(a) & (L(G_{r,\tau}^{-1} f)) \mapsto 0 & (K) \mapsto 0 & \end{array}$$

where $a \in PC$ *and* $f \in PC_{-1}^0$.

Proof. As in the last proposition, the only generator for which this is not immediately clear is $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$. Following the same strategy as in the last proof, we have

$$U_n Y_\tau^* R_r^* K_\tau R_r Y_\tau U_{-n} = Y_\tau^* R_r^* L(C_r t^n) K_\tau L(C_r t^{-n}) R_r Y_\tau$$

which again converges strongly to zero for the exact same reasons as $Y_\tau^* R_r^* L(C_r t^{-n}) K_2 L(C_r t^n) R_r Y_\tau$ did in the last proof.

□

We will make use of the following lemma, a proof of which can be found in [8], Proposition 2.9:

Lemma 3.1.5. *The Toeplitz algebra generated by Laurent operators with continuous symbols and the projection* P *contains the compact operators.*

Remark 3.1.6. In light of Lemma 3.1.5, we can conclude that our algebra $\mathcal{F}_*$ automatically contains the compacts; i.e., we didn't need to explicitly include it as a generator. Nevertheless, it can be helpful to explicitly mention that these operators are in the algebra and to examine how they are mapped under all of our homomorphisms.

Define $\mathcal{J} := \{(C_{n,r}) + (K_1) + (U_{-n}K_2U_n) + (U_nK_3U_{-n}) : (C_{n,r}) \in \mathcal{N}, K_i \in \mathcal{K}(L^2(\mathbb{S}^1))\}$. This actually forms an ideal of $\mathcal{F}_*$:

Proposition 3.1.7. *The set* $\mathcal{J} = \{(C_{n,r}) + (K_1) + (U_{-n}K_2U_n) + (U_nK_3U_{-n}) : (C_{n,r}) \in \mathcal{N}, K_i \in \mathcal{K}(L^2(\mathbb{S}^1))\}$ *forms a closed, two-sided* $*-$*ideal of* $\mathcal{F}_*$.

Proof. We first show that $\mathcal{J}$ is contained in $\mathcal{F}_*$. We have $(C_{n,r})$ and (K_1) are in $\mathcal{F}_*$ by definition, so we just need to show that for any compact operator K the sequences $(U_{-n}KU_n)$ and (U_nKU_{-n}) belong to $\mathcal{F}_*$. For $a \in C(\mathbb{S}^1)$, our algebra $\mathcal{F}_*$ contains $U_{\mp n}L(a)U_{\pm n}$. It also contains $U_{\mp n}PU_{\pm n}$. These two facts together give us that for any operator A in the Toeplitz algebra, our algebra $\mathcal{F}_*$ contains $U_{\mp n}AU_{\pm n}$; in particular, we may take $A = K$ (Lemma 3.1.5).

The fact that $\mathcal{J}$ is self-adjoint and linear is clear by definition, so let's next prove that $\mathcal{J}$ is closed. Let

$$(A_{n,r}) = (C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n}) \in \mathcal{J}.$$

Since U_n and U_{-n} converge weakly to zero, it follows that for any compact operator L, $(U_{-n}LU_n)$ and (U_nLU_{-n}) converge strongly to zero. Consequently, $(A_{n,r})$ converges strongly to K_1, $(U_nA_{n,r}U_{-n})$ converges strongly to K_2, and $(U_{-n}A_{n,r}U_n)$ converges strongly to K_3. Then since $\|U_{\pm n}\| = 1$ we have by the Uniform Boundedness Principle

$$\|K_1\| \le \liminf_{\substack{n\to\infty\\ r\to 1}} \|A_{n,r}\|, \quad \|K_2\| \le \liminf_{\substack{n\to\infty\\ r\to 1}} \|A_{n,r}\|, \quad \|K_3\| \le \liminf_{\substack{n\to\infty\\ r\to 1}} \|A_{n,r}\|.$$

Thus, if $(A_{n,r}^{(j)}) \subset \mathcal{J}$ is a Cauchy sequence, then so are $(K_1^{(j)}) \subset \mathcal{K}, (K_2^{(j)}) \subset \mathcal{K}$, and $(K_3^{(j)}) \subset \mathcal{K}$. This means that there are compact operators K_1, K_2, and K_3 such that $\|K_1^{(j)} - K_1\| \to 0$, $\|K_2^{(j)} - K_2\| \to 0$, and $\|K_3^{(j)} - K_3\| \to 0$ as $j \to \infty$. This implies that there is a sequence $(A_{n,r}) \in \mathcal{J}$ such that $\|(A_{n,r}^{(j)}) - (A_{n,r})\| \to 0$ as $j \to \infty$. Thus $\mathcal{J}$ is closed.

We now prove that $\mathcal{J}$ has the absorption property of ideals. Again, let

$$(A_{n,r}) = (C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n}) \in \mathcal{J}$$

and now let $(B_{n,r})$ be any sequence in $\mathcal{F}_*$. Then

$$\begin{aligned} B_{n,r}A_{n,r} &= B_{n,r}(C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n}) \\ &= B_{n,r}C_{n,r} + (B_{n,r} - \Phi_0(B_{n,r}))K_1 + \Phi_0(B_{n,r})K_1 \\ &\quad + U_{-n}(U_nB_{n,r}U_{-n} - \Phi_{-1}(B_{n,r}))K_2U_n \\ &\quad + U_{-n}\Phi_{-1}(B_{n,r})K_2U_n + U_n(U_{-n}B_{n,r}U_n - \Phi_1(B_{n,r}))K_3U_{-n} \\ &\quad + U_n\Phi_1(B_{n,r})K_3U_{-n} \end{aligned}$$

Notice that $(B_{n,r}C_{n,r}) \in \mathcal{N}$ since $(B_{n,r})$ is uniformly bounded and $\Phi_0(B_{n,r})K_1 \in \mathcal{K}$ since the compacts form an ideal. We have also (by definition) that $(B_{n,r} - \Phi_0(B_{n,r})), (U_nB_{n,r}U_{-n} - \Phi_{-1}(B_{n,r}))$, and $(U_{-n}B_{n,r}U_n - \Phi_1(B_{n,r}))$ converge strongly to zero, and hence $(B_{n,r} - \Phi_0(B_{n,r}))K_1, (U_nB_{n,r}U_{-n} - \Phi_{-1}(B_{n,r}))K_2$, and $(U_{-n}B_{n,r}U_n - \Phi_1(B_{n,r}))K_3$ converge in norm to zero. Finally, since $\Phi_{-1}(B_{n,r})K_2$ and $\Phi_1(B_{n,r})K_3$ are compact the terms $U_{-n}\Phi_{-1}(B_{n,r})K_2U_n$ and $U_n\Phi_1(B_{n,r})K_3U_{-n}$ belong to $\mathcal{J}$. Thus $(B_{n,r}A_{n,r}) \in \mathcal{J}$. Passing to adjoints proves that $(A_{n,r}B_{n,r}) \in \mathcal{J}$.

□

The relevance of this ideal $\mathcal{J}$ is captured in the following Lifting Theorem:

Theorem 3.1.8. *(Lifting Theorem for $\mathcal{F}_*$) Let $(F_{n,r}) \in \mathcal{F}_*$. The following are equivalent:*

(a) *$(F_{n,r})$ is λ-stable*

(b) *$(F_{n,r}) + \mathcal{N}$ is invertible in $\mathcal{F}_*/\mathcal{N}$*

(c) *The operators $\Phi_0(F_{n,r}), \Phi_1(F_{n,r})$, and $\Phi_{-1}(F_{n,r})$ are invertible in $\mathcal{L}(L^2(\mathbb{S}^1))$ and the coset $(F_{n,r}) + \mathcal{J}$ is invertible in $\mathcal{F}_*/\mathcal{J}$.*

Proof. The equivalence of (a) and (b) follows from the fact that stability in $\mathcal{F}$ is equivalent to invertibility in $\mathcal{F}/\mathcal{N}$ (Theorem 3.1.1). This is enough since $\mathcal{F}_*/\mathcal{N}$ is a $*-$subalgebra of $\mathcal{F}/\mathcal{N}$ and C^*-algebras are inverse closed.

To show that (b) implies (c), suppose that $(F_{n,r}) + \mathcal{N}$ is invertible in $\mathcal{F}_*/\mathcal{N}$. Then there exists a sequence $(B_{n,r}) \in \mathcal{F}_*$ such that $(F_{n,r}B_{n,r}) = (I) + (C_{n,r})$ where $(C_{n,r}) \in \mathcal{N}$. Now, since $\mathcal{N} \subseteq \ker \Phi_j$ for $j = 0, 1, -1$, we have

$$\begin{aligned}\Phi_j(F_{n,r})\Phi_j(B_{n,r}) &= \Phi_j(F_{n,r}B_{n,r}) \\ &= \Phi_j(I + C_{n,r}) \\ &= \Phi_j(I) \\ &= I\end{aligned}$$

which shows that $\Phi_j(F_{n,r})$ has a right inverse. The existence of a left inverse can be shown similarly. To show that $(F_{n,r})$ is invertible in $\mathcal{F}_*/\mathcal{J}$ we can replicate the above argument, replacing the Φ_j with the canonical projection map $\pi_{\mathcal{J}} : \mathcal{F}_* \to \mathcal{F}_*/\mathcal{J}$ and using the fact $\mathcal{N} \subseteq \mathcal{J}$.

Finally, let's show that (c) implies (b). Suppose $(B_{n,r}) + \mathcal{J}$ is the left inverse of $(F_{n,r}) + \mathcal{J}$ in $\mathcal{F}_*/\mathcal{J}$. Then

$$B_{n,r}F_{n,r} = I + C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n}$$

where $C_{n,r} \in \mathcal{N}$ and $K_1, K_2, K_3 \in \mathcal{K}$. Define a new sequence $(B'_{n,r})$ by

$$B'_{n,r} := B_{n,r} - K_1\Phi_0(F_{n,r})^{-1} - U_{-n}K_2\Phi_{-1}(F_{n,r})^{-1}U_n - U_nK_3\Phi_1(F_{n,r})^{-1}U_{-n}.$$

We obtain that $B'_{n,r}F_{n,r}$ is equal to

$$\begin{aligned} B'_{n,r}F_{n,r} = I + C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n} - K_1\Phi_0(F_{n,r})^{-1}F_{n,r} \\ - U_{-n}K_2\Phi_{-1}(F_{n,r})^{-1}U_nF_{n,r} - U_nK_3\Phi_1(F_{n,r})^{-1}U_{-n}F_{n,r} \end{aligned}$$

which is equal to

$$I + C_{n,r} + C^{(1)}_{n,r} + U_{-n}C^{(2)}_{n,r}U_n + U_nC^{(3)}_{n,r}U_{-n}$$

where the three sequences

$$\begin{aligned} C^{(1)}_{n,r} &= K_1\Phi_0(F_{n,r})^{-1}(\Phi_0(F_{n,r}) - U_nF_{n,r}U_{-n}) \\ C^{(2)}_{n,r} &= K_2\Phi_{-1}(F_{n,r})^{-1}(\Phi_{-1}(F_{n,r}) - F_{n,r}) \\ C^{(3)}_{n,r} &= K_3\Phi_1(F_{n,r})^{-1}(\Phi_1(F_{n,r}) - U_{-n}F_{n,r}U_n) \end{aligned}$$

converge in norm to zero. Thus $(B'_{n,r}) + \mathcal{N}$ is the left inverse of $(F_{n,r}) + \mathcal{N}$ in $\mathcal{F}_*/\mathcal{N}$. The right invertibility can be shown analogously. □

3.2 Localization

Theorem 3.1.8 raises the question of when an element $(F_{n,r}) + \mathcal{J}$ is invertible. We will tackle this question via "Local Principle" by Allan and Douglas, a proof of which can be found in [4].

Theorem 3.2.1. *(Local Principle by Allan/Douglas) Let $\mathcal{A}$ be a C^*-algebra with identity element e and let Z be a closed subalgebra of the center of $\mathcal{A}$ which contains e (this means that every element of Z commutes with every element of $\mathcal{A}$). Denote the maximal ideal space of Z by Ω and for each maximal ideal $\omega \in \Omega$ let J_ω be the smallest closed two-sided ideal of $\mathcal{A}$ which contains the set ω. Then an element $a \in \mathcal{A}$ is invertible in A if and only if the coset $a + J_\omega$ is invertible in $\mathcal{A}/J_\omega$ for every $\omega \in \Omega$.*

To see how we can apply this to our situation, we need the following lemma.

Lemma 3.2.2. *The set $\mathcal{D}_1 = \{(L(f))+\mathcal{J} : f \in C(\mathbb{S}^1)\}$ is a $*$-subalgebra contained in the center of $\mathcal{F}_*/\mathcal{J}$. Moreover, $\mathcal{D}_1$ is $*$-isomorphic to $C(\mathbb{S}^1)$.*

Proof. We will show that $(L(f)) + \mathcal{J}$ commutes with each generator of $\mathcal{F}_*/\mathcal{J}$ when $f \in C(\mathbb{S}^1)$. We start by noting that since Laurent operators commute with eachother, we have that $(L(f)) + \mathcal{J}$ commutes with $(L(a)) + \mathcal{J}$ for $a \in PC$ and $(L(G_{r,\tau}^{-1}g)) + \mathcal{J}$ for $g \in PC$ automatically. We also have that $(L(f))$ commutes with compact operators modulo $\mathcal{J}$ for free, since the compacts form an ideal. Let's check the other generators now. Notice that

$$\begin{aligned} PL(f) - L(f)P &= PL(f)Q - QL(f)P \\ &= PL(f)JPJ - JPJL(f)P \\ &= PL(f)JPJ - JPL(\tilde{f})JP \end{aligned}$$

$$= H(f)J - JH(\tilde{f}) \in \mathcal{K}$$

where the final expression is compact since Hankel operators with continuous symbol are compact. Thus $(L(f))$ commutes with (P) modulo $\mathcal{J}$.

For $(P_n^+) + \mathcal{J}$, we have

$$\begin{aligned}
P_n^+ L(f) - L(f) P_n^+ &= U_{-n} P U_n L(f) - L(f) U_{-n} P U_n \\
&= U_{-n} P L(f) U_n - U_{-n} L(f) P U_n \\
&= U_{-n} (P L(f) - L(f) P) U_n \\
&= U_{-n} K U_n \in \mathcal{J}
\end{aligned}$$

where we used the identification $U_{\pm n} = L(t^{\pm n})$ to justify commuting in the second equality and our previous work with P for the final equality. An analogous argument can be used to show that

$$Q_n^- L(f) - L(f) Q_n^- = U_n K U_{-n} \in \mathcal{J}.$$

Let's turn our attention now to $(Y_\tau^* R_r^* K R_r Y_\tau) + \mathcal{J}$. We have that

$$\begin{aligned}
Y_\tau^* R_r^* K R_r Y_\tau L(f) - L(f) Y_\tau^* R_r^* K R_r Y_\tau &= Y_\tau^* R_r^* K R_r Y_\tau L(f) Y_\tau^* R_r^* R_r Y_\tau \\
&\quad - Y_\tau^* R_r^* R_r Y_\tau L(f) Y_\tau^* R_r^* K R_r Y_\tau \\
&= Y_\tau^* R_r^* K L(C_r Y_\tau f) R_r Y_\tau \\
&\quad - Y_\tau^* R_r^* L(C_r Y_\tau f) K R_r Y_\tau \\
&= Y_\tau^* R_r^* (K L(C_r Y_\tau f) - L(C_r Y_\tau f) K) R_r Y_\tau
\end{aligned}$$

We have that $(C_rY_\tau f)(t) = f(\tau\frac{t+r}{1+rt})$ converges locally uniformly to $f(\tau)$ and so $L(C_rY_\tau f)$ converges strongly to $f(\tau)I$. Thus $KL(C_rf)$ and $L(C_rf)K$ converge in norm to $f(\tau)K$. Combined with the fact that the $R_rY_\tau^*$ and $Y_\tau^*R_r^*$ are uniformly bounded, this implies that $Y_\tau^*R_r^*(KL(C_rf) - L(C_rf)K)R_rY_\tau = 0$ modulo $\mathcal{J}$, as desired. Since the generators of $\mathcal{F}_*/\mathcal{J}$ commute with the set $\mathcal{D}_1$, we have proven that $\mathcal{D}_1$ is a central subalgebra.

We now prove that $\mathcal{D}_1$ is $*$-isomorphic to $C(\mathbb{S}^1)$. We will do so by proving that the map $\Lambda : C(\mathbb{S}^1) \to \mathcal{D}_1$ defined by $\Lambda(f) = (L(f)) + \mathcal{J}$ is a $*$-isomorphism. Most of the requirements of a $*$-isomorphism are clear based on properties of Laurent operators, such as $L(ab) = L(a)L(b), L(a+b) = L(a)+L(b)$, and $L(a^*) = L(a)^*$. Surjectivity is also clear by definition. The only properties left to check are injectivity and continuity. For injectivity, let $f \in \ker\Lambda$. Then $(L(f)) \in \mathcal{J}$, meaning $(L(f)) = (C_{n,r})+(K_1)+(U_{-n}K_2U_n)+(U_nK_3U_{-n})$ where $(C_{n,r}) \in \mathcal{N}, K_i \in \mathcal{K}(L^2(\mathbb{S}^1))$. Then

$$L(f) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} L(f) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} (C_{n,r} + K_1 + U_{-n}K_2U_n + U_nK_3U_{-n}) = K_1$$

This implies $f = 0$ since a multiplication operator cannot be compact unless it vanishes. Thus Λ is injective. Now, since injective $*$-homomorphisms preserve spectra and are hence an isometry, we get also that Λ is continuous and we have proven it is a $*$-isomorphism. $\square$

One can show that for a compact set K, the maximal ideal space $\mathcal{M}(C(K))$ is homeomorphic to K itself (this is done showing the map $\Gamma : K \to \mathcal{M}(C(K))$ that sends a point $k \in K$ to the functional $\phi_k \in \mathcal{M}(C(K))$ defined by $\phi_k(f) = f(k)$ is a homeomorphism). In our case, we have that the maximal ideal space of $\mathcal{D}_1$ is homeomorphic to $\mathbb{S}^1$. For $t_0 \in \mathbb{S}^1$, we denote by $\mathcal{J}_{t_0}$ the smallest closed ideal of

$\mathcal{F}_*/\mathcal{J}$ containing $(L(c)) + \mathcal{J}$ where $c \in C(\mathbb{S}^1)$ vanishes at t_0; i.e.,

$$\mathcal{J}_{t_0} = \text{clos id}_{\mathcal{F}_*/\mathcal{J}}\{(L(c)) + \mathcal{J} : c \in C(\mathbb{S}^1), c(t_0) = 0\}.$$

Corollary 3.2.3. *Let $(F_{n,r}) \in \mathcal{F}_*$. Then $(F_{n,r}) + \mathcal{J}$ is invertible in $\mathcal{F}_*/\mathcal{J}$ if and only if $((F_{n,r}) + \mathcal{J}) + \mathcal{J}_{t_0}$ is invertible in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ for all $t_0 \in \mathbb{S}^1$.*

Proof. On account of Lemma 3.2.2, we can employ Theorem 3.2.1 in the setting $\mathcal{A} = \mathcal{F}_*/\mathcal{J}$ and $Z = \mathcal{D}_1$. As we have mentioned before, since $\mathcal{D}_1 \cong C(\mathbb{S}^1)$, their maximal ideal spaces are homeomorphic − that is, the maximal ideal space of $\mathcal{D}_1$ is homemorphic to $\mathbb{S}^1$. Putting all of this together gives the claim. □

The algebras $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ which arise from localization are called *local algebras* (at t_0). Our goal now is to understand invertibility in each of these local algebras. We start by stating the generators of each local algebra.

Proposition 3.2.4. *The local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ is generated by the following elements:*

$$\begin{array}{cc} (\{P\} + \mathcal{J}) + \mathcal{J}_{t_0}, & \left(\{P_n^+\} + \mathcal{J}\right) + \mathcal{J}_{t_0}, \\ \left(\{Q_n^-\} + \mathcal{J}\right) + \mathcal{J}_{t_0}, & \left(\{L(\chi_+)\} + \mathcal{J}\right) + \mathcal{J}_{t_0}, \\ \left(\left\{L(G_{r,t_0}^{-1} f)\right\} + \mathcal{J}\right) + \mathcal{J}_{t_0}, & \left(\{Y_{t_0}^* R_r^* K_{t_0} R_r Y_{t_0}\} + \mathcal{J}\right) + \mathcal{J}_{t_0} \end{array}$$

where $f \in PC$ and K_{t_0} is compact.

Proof. With the definition of $\mathcal{F}_*$ and $\mathcal{J}$, one can see that $\mathcal{F}_*/\mathcal{J}$ is generated by cosets with the following representatives:

(i) $\{P\}, \{P_n^+\}, \{Q_n^-\}$,

(ii) $\{L(a)\}$ with $a \in PC$ and $\left\{L(G_{r,\tau}^{-1} f)\right\}$ for each $\tau \in \mathbb{S}^1$, and

(iii) $\{Y_\tau^* R_r^* K_\tau R_r Y_\tau\}$ for each $\tau \in \mathbb{S}^1$ and K_τ compact.

We are left with showing that modulo $\mathcal{J}_{t_0}$ (and $\mathcal{N}$), the elements in (i)-(iii) reduce to the corresponding elements in the statement of the proposition. The items in (i) are clear, so we begin with (ii).

The generator $(L(G_{r,\tau}^{-1} f))$ for $f \in PC$ can be reduced to $(L(G_{r,t_0}^{-1} f))$ since we have $(L(G_{r,\tau}^{-1} f)) = 0$ modulo $\mathcal{J}_{t_0}$ when $\tau \neq t_0$. For the element $(L(a))$ with $a \in PC$ we may use the fact that any $a \in PC$ can be expressed as a linear combination of χ_+, χ_-, and a function continuous and vanishing at t_0 (and hence belonging to $\mathcal{J}_{t_0}$). Thus $(L(a))$ can be reduced to $(L(\chi_+))$ and $(L(\chi_-))$; however, since our algebra is unital (take $f = 1$ for $L(G_{r,\tau}^{-1} f)$) we may choose just $(L(\chi_+))$.

Finally, $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$ can be simplified to $(Y_{t_0}^* R_r^* K_\tau R_r Y_{t_0})$. To see this, for $\tau \neq t_0$ we let g be a continuous function with $g(t_0) = 1$ and vanishing in a neighborhood of τ. Then mod $\mathcal{J}_{t_0}$ we have

$$
\begin{aligned}
Y_\tau^* R_r^* K_\tau R_r Y_\tau &= Y_\tau^* R_r^* K_\tau R_r Y_\tau L(g) \\
&= Y_\tau^* R_r^* K_\tau R_r Y_\tau L(g) Y_\tau^* R_r^* R_r Y_\tau \\
&= Y_\tau^* R_r^* K_\tau L(C_r Y_\tau g) R_r Y_\tau.
\end{aligned}
$$

Now, $L(C_r Y_\tau g)$ converges strongly to $g(\tau)I = 0$. Thus $K_\tau L(C_r Y_\tau g)$ converges in norm to zero and hence for $\tau \neq t_0$ the element $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$ is zero mod $\mathcal{J}$ and $\mathcal{J}_{t_0}$. □

Recall that we aim to understand invertibility in each local algebra. Fortunately, our considerations can be reduced to a single local algebra:

Proposition 3.2.5. *For each $t_0 \in \mathbb{S}^1$, the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ is isomorphic to $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$.*

Proof. For $t_0 \in \mathbb{S}^1$, define the map $\Lambda_{t_0} : \mathcal{F}_* \to (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ by

$$(F_{n,r}) \mapsto (Y_{t_0}^{-1} F_{n,r} Y_{t_0} + \mathcal{J}) + \mathcal{J}_{t_0}$$

Under this map, the sequences $(P), (P_n^+)$, and (Q_n^-) are sent to the cosets with themselves as representatives. The sequence $(L(a))$ is sent to the coset with representative $(L(a_{t_0}))$ where $a_{t_0}(t) = a(t/t_0)$. Next, notice that $(L(G_{r,\tau}^{-1} f))$ is mapped to the coset with representative $(L(G_{r,t_0\tau}^{-1} f))$ and $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$ is mapped to the coset with representative $(Y_{t_0\tau}^* R_r^* K_\tau R_r Y_{t_0\tau})$.

Continuing, we see that under this map (K) is sent to $(Y_{t_0}^{-1} K Y_{t_0} + \mathcal{J}) + \mathcal{J}_{t_0} = 0$ since $Y_{t_0}^{-1} K Y_{t_0}$ is a (different) compact operator, zero sequences are sent to zero sequences, and $(U_{\pm n} K U_{\mp n})$ is mapped to $(U_{\pm n} K' U_{\mp n} + \mathcal{J}) + \mathcal{J}_{t_0}$ for some different compact operator K'. For continuous f vanishing at 1, we have $\Lambda_{t_0}(L(f)) = (L(f_{t_0}) + \mathcal{J}) + \mathcal{J}_{t_0}$ where $f_{t_0}(t) = f(t/t_0)$ vanishes at $t = t_0$ (i.e., $\mathcal{J}_1$ is mapped into $\mathcal{J}_{t_0}$). Thus the map Λ_{t_0} factors through the quotient $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ and we have a map $\hat{\Lambda}_{t_0} : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 \to (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$.

This map is clearly a $*-$homomorphism, and similar arguments that we have just made show that the map

$$\Gamma_{t_0} : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0} \to (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$$

which sends $(F_{n,r} + \mathcal{J}) + \mathcal{J}_1$ to $(Y_{t_0} F_{n,r} Y_{t_0}^{-1} + \mathcal{J}) + \mathcal{J}_{t_0}$ is well-defined and the inverse of $\hat{\Lambda}_{t_0}$. Thus the map $\hat{\Lambda}_{t_0}$ is a bijective $*-$homomorphism and so $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$ is isomorphic to $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$.

□

Thus, to fully understand stability in our algebra $\mathcal{F}_*$, the question of invertibility in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ remains. In order to develop invertibility criteria, we introduce and study two new algebras which serves as the starting point of the next chapter.

Chapter 4

Some New Algebras: Laying the Groundwork

Our goal now is to figure out when an element $(F_{n,r} + \mathcal{J}) + \mathcal{J}_1$ is invertible. To do so, we seek to "identify" the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$; that is, we want to find an algebra of operators that $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is isomorphic to so that invertibility in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ can be reduced to invertibility of an operator. In order to identify the algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$, we proceed as follows: first we define a new algebra $\mathcal{B}$ and develop stability criteria for this algebra, identifying $\mathcal{B}/\mathcal{N}$ as a direct sum of two different algebras. One of these direct summands will be directly related to $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ in the sense we will have a surjective map from $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ into it [1]. To understand the other direct summand we will introduce two new algebras: and algebra $\mathcal{C}$ which is a subalgebra of $\mathcal{F}_*$ and an associated larger algebra $\widehat{\mathcal{C}}$ containing $\mathcal{C}$. With the help of Fredholm Theory, we will use information about all of these algebras to identify $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ with an algebra of operators given by a certain strong limit.

[1] Actually we will see that $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is isomorphic to this direct summand!

4.1 Stability in the Algebra $\mathcal{B}$

Define the algebra $\mathcal{B}$ to be the algebra of sequences of operators acting on $L^2(\mathbb{S}^1)$ generated by the following elements:

$$\mathcal{B} := \text{alg}_{\mathcal{L}(L^2(\mathbb{S}^1))} \quad \big\{(P), (P_n^+), (Q_n^-), (L(\chi_+)), (L(C_r^{-1}f)), \\ \big(R_r^* K R_r), (Y_{-1}R_r^* K' R_r Y_{-1}), \mathcal{N}\big\}$$

where $f \in PC$, $K, K' \in \mathcal{K}(L^2(\mathbb{S}^1))$.

As we remarked in the definition of $\mathcal{F}_*$, we again could take $f \in PC^0_{-1}$ rather than general PC. This viewpoint will prove useful when we perform computations of strong limits.

By construction, there is a surjective map $\tau : \mathcal{B}/\mathcal{N} \to (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ defined by $\tau(B_{n,r} + \mathcal{N}) = (B_{n,r} + \mathcal{J}) + \mathcal{J}_1$ (to see this, compare the generators of $\mathcal{B}/\mathcal{N}$ to those of $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ which were established in Proposition 3.2.4 and note that in the proof there we show that $(Y_{-1}R_r^* K' R_r Y_{-1}) = 0$ modulo $\mathcal{J}$ and $\mathcal{J}_1$). In an effort to establish stability criteria for $\mathcal{B}$, we start by showing that a particular strong limit exists for all elements of $\mathcal{B}$. To help us, we start with the following lemma:

Lemma 4.1.1. *The sequence of operators $(R_r Y_{-1} R_r^*)$ converges weakly to zero.*

Proof. One can check that $Y_{-1}R_r^* = R_r Y_{-1}$ so that $R_r Y_{-1} R_r^* = R_r R_r Y_{-1}$. Now, by following the definition, we see that $R_r R_r = R_s$ for $s = \frac{2r}{1+r^2}$. Then $R_s Y_{-1}$ converges weakly to zero for the same reason that R_r does; the proof is nearly identical except when considering the inner product $\langle R_s Y_{-1} f, g\rangle$ we take f vanishing in a neighborhood of -1. $\square$

Proposition 4.1.2. *For each* $(B_{n,r}) \in \mathcal{B}$*, the strong limit* $\underset{\substack{n\to\infty \\ r\to 1}}{s\text{-}lim}\ R_r B_{n,r} R_r^*$ *exists. In particular, under this map the generators are mapped as follows:*

$$(P) \mapsto P \qquad\qquad (L(\chi_+)) \mapsto L(\chi_+)$$
$$(L(t^{-n})PL(t^n)) \mapsto L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big) \qquad (L(C_r^{-1}f)) \mapsto L(f)$$
$$(L(t^n)QL(t^{-n}) \mapsto L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big) \qquad (R_r^*KR_r) \mapsto K$$
$$(Y_{-1}R_r^*K'R_rY_{-1}) \mapsto 0$$

Proof. We prove this by checking that this strong limit exists for each generator. Most of the strong limits are clear; for $(P), (L(\chi_+)), (R_r^*KR_r)$, and $(L(C_r^{-1}f))$ the strong limits are $P, L(\chi_+), K$, and $L(f)$ respectively. The fact that the element $(Y_{-1}R_r^*K'R_rY_{-1})$ is sent to zero is a consequence of the weak convergence of $R_rY_{-1}R_r^*$ to zero (Lemma 4.1.1). The only ones that require a bit of work are $(L(t^{-n})PL(t^n))$ and $(L(t^n)QL(t^{-n}))$. Notice that

$$R_rL(t^{-n})PL(t^n)R_r^* = L(C_rt^{-n})PL(C_rt^n)$$

and

$$R_rL(t^n)QL(t^{-n})R_r^* = L(C_rt^n)QL(C_rt^{-n}).$$

We saw in the proof of Proposition 3.1.3 that $L(C_rt^n)$ converges strongly to $L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)$ and $(L(C_rt^{-n}))$ converges strongly to $L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)$. Thus, putting all of this together, we see that $(L(C_rt^{-n})PL(C_rt^n))$ converges strongly to

$$L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)$$

and $(L(C_r t^n) Q L(C_r t^{-n})$ converges strongly to

$$L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)$$

as desired. $\square$

We will denote the map that sends each element $(B_{n,r}) \in \mathcal{B}$ to the strong limit $\underset{\substack{n\to\infty\\ r\to 1}}{\text{s-lim}}\, R_r B_{n,r} R_r^*$ by ϵ and we will denote the algebra of operators $\epsilon(\mathcal{B})$ by $\mathcal{A}$. Because of the uniform boundedness of R_r and R_r^*, this map actually factors through $\mathcal{B}/\mathcal{N}$; we will denote this map also by ϵ. It is important to note that we can construct a map δ from our local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ into this algebra $\mathcal{A}$; in fact, this map will play a crucial role in the future.

Proposition 4.1.3. *The map $\delta : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 \to \mathcal{A}$ defined by $\delta((F_{n,r}+\mathcal{J})+\mathcal{J}_1) = \underset{\substack{n\to\infty\\ r\to 1}}{\text{s-lim}}\, R_r F_{n,r} R_r^*$ is well-defined and has the property that $\epsilon = \delta\circ\tau$.*

Proof. The fact that $\epsilon = \delta\circ\tau$ is true by construction. The only thing that needs to be done is to verify that δ is well-defined; that is, we must show that $\mathcal{J}$ and $\mathcal{J}_1$ are in the kernel of δ. In particular, we need to show that $\mathcal{N}, \mathcal{K}, U_{-n}K_1U_n, U_nK_2U_{-n}$ and $L(f)$ for continuous functions f on the unit circle vanishing at 1 get sent to zero under δ where $K_1, K_2 \in \mathcal{K}$. The fact that $\mathcal{N}$ and $\mathcal{K}$ are in the kernel of δ is a consequence of the fact that R_r converges weakly to zero (Lemma 2.2.3) and R_r and R_r^* are uniformly bounded. For $L(f)$, we have that $R_rL(f)R_r^* = L(C_rf)$ and $C_rf = f\big(\frac{t+r}{1+rt}\big)$ which converges to $f(1) = 0$ locally uniformly on $\mathbb{S}^1 \setminus \{-1\}$. Hence C_rf converges to 0 in measure and thus $L(C_rf)$ converges strongly to 0.

To complete this proof, we consider the strong limits $\underset{\substack{n\to\infty\\ r\to 1}}{\text{s-lim}}\, R_rU_{\pm n}KU_{\mp n}R_r^*$ for

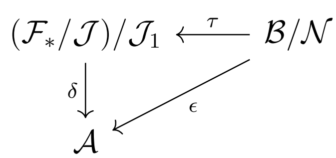
K compact. Since

$$R_r U_{\pm n} K U_{\mp n} R_r^* = R_r U_{\pm n} R_r^* R_r K R_r^* R_r U_{\mp n} R_r^*$$

it suffices to show that $R_r U_{\pm n} R_r^*$ and $R_r K R_r^*$ converge strongly. We have already seen in this proof that $R_r K R_r^*$ converges strongly to zero, and we have already shown in the proof of Proposition 4.1.2 that $R_r U_n R_r^* = (L(C_r t^n))$ and $R_r U_{-n} R_r^* = (L(C_r t^{-n}))$ converge strongly to some operators. Thus $\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r U_{\pm n} K U_{\mp n} R_r^* = 0$ and so $\mathcal{J} \subseteq \ker\, \delta$.

□

Altogether, we have have shown the existence of certain homomorphisms such that the following diagram commutes:



As we continue on our journey to identify $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$, we will expand on this diagram. Define

$$\mathcal{I} := \{(C_{n,r}) + (R_r^* K R_r) : (C_{n,r}) \in \mathcal{N}, K \in \mathcal{K}\}.$$

This forms an ideal of $\mathcal{B}$ and we have the following Lifting Theorem:

Theorem 4.1.4. *(Lifting Theorem for $\mathcal{B}$) Let $(B_{n,r}) \in \mathcal{B}$. The following are equivalent:*

(a) $(B_{n,r})$ is λ-stable

(b) $(B_{n,r}) + \mathcal{N}$ is invertible in $\mathcal{B}/\mathcal{N}$

(c) $\epsilon(B_{n,r}) = \underset{\substack{n\to\infty \\ r\to 1}}{\text{s-lim}}\, R_r B_{n,r} R_r^*$ *is invertible in* $\mathcal{L}(L^2(\mathbb{S}^1))$ *and* $(B_{n,r}) + \mathcal{I}$ *is invertible in* $\mathcal{B}/\mathcal{I}$

Proof. This can be proven in much the same way as Theorem 3.1.8. □

As we did with our algebra $\mathcal{F}_*/\mathcal{J}$, we will apply Theorem 3.2.1 to our algebra $\mathcal{B}/\mathcal{I}$ in order to deduce when an element is invertible in $\mathcal{B}/\mathcal{I}$. To this end, we need a central subalgebra of $\mathcal{B}/\mathcal{I}$.

Lemma 4.1.5. *The set* $\mathcal{D}_2 = \{(L(C_r^{-1}f)) + \mathcal{I} : f \in C(\mathbb{S}^1)\}$ *is a* $*$*-subalgebra contained in the center of* $\mathcal{B}/\mathcal{I}$*. Moreover,* $\mathcal{D}_2$ *is* $*$*-isomorphic to* $C(\mathbb{S}^1)$*.*

Proof. We first prove that $(L(C_r^{-1}f))$ commutes with each generator of $\mathcal{B}$ modulo $\mathcal{I}$ when $f \in C(\mathbb{S}^1)$. Because Laurent operators commute with each other, we only need to check this for $(P), (P_n^+), (Q_n^-), (R_r^* K R_r)$ and $(Y_{-1} R_r^* K R_r Y_{-1})$. The element $(R_r^* K R_r)$ is automatic since these sequences are already in the ideal $\mathcal{I}$ and ideals have the absorption property.

Let's handle $(Y_{-1} R_r^* K' R_r Y_{-1})$ next. We have

$$Y_{-1} R_r^* K' R_r Y_{-1} L(C_r^{-1} f) - L(C_r^{-1} f) Y_{-1} R_r^* K' R_r Y_{-1}$$

is equal to

$$Y_{-1} R_r^* K' R_r Y_{-1} L(C_r^{-1} f) Y_{-1} R_r^* R_r Y_{-1} - Y_{-1} R_r^* R_r Y_{-1} L(C_r^{-1} f) Y_{-1} R_r^* K' R_r Y_{-1}$$

which is equal to

$$Y_{-1} R_r^* K' L(C_r Y_{-1} C_r^{-1} f) R_r Y_{-1} - Y_{-1} R_r^* L(C_r Y_{-1} C_r^{-1} f) K' R_r Y_{-1}$$

which finally can be expressed as

$$Y_{-1}R_r^*(K'L(C_rY_{-1}C_r^{-1}f) - L(C_rY_{-1}C_r^{-1}f)K')R_rY_{-1}.$$

This term belongs to $\mathcal{N}$. Indeed, notice that since $(C_rY_{-1}C_r^{-1}f)(t) = f\left(-\frac{t+2r+r^2t}{1+2rt+r^2}\right)$ we have $C_rY_{-1}C_r^{-1}f$ converges locally uniformly to $f(-1)$ on $\mathbb{S}^1 \setminus \{-1\}$. Thus $L(C_rY_{-1}C_r^{-1}f)$ converges strongly to $f(-1)I$ and hence the term

$$KL(C_rY_{-1}C_r^{-1}f) - L(C_rY_{-1}C_r^{-1}f)K$$

converges in norm to 0. The uniform boundedness of $Y_{-1}R_r^*$ and R_rY_{-1} finishes the argument.

Let's now settle (P). Observe that

$$\begin{aligned} PL(C_r^{-1}f) - L(C_r^{-1}f)P &= PR_r^*L(f)R_r - R_r^*L(f)R_rP \\ &= R_r^*PL(f)R_r - R_r^*L(f)PR_r \\ &= R_r^*(PL(f) - L(f)P)R_r \end{aligned}$$

which belongs to $\mathcal{I}$ since $PL(f) - L(f)P$ is compact (see the proof of Lemma 3.2.2).

Let's now turn to (P_n^+) and (Q_n^-). For (P_n^+), we have

$$\begin{aligned} P_n^+L(C_r^{-1}f) - L(C_r^{-1}f)P_n^+ &= L(t^{-n})PL(t^n)L(C_r^{-1}f) - L(C_r^{-1}f)L(t^{-n})PL(t^n) \\ &= L(t^{-n})PL(C_r^{-1}f)L(t^n) - L(t^{-n})L(C_r^{-1}f)PL(t^n) \\ &= L(t^{-n})PR_r^*L(f)R_rL(t^n) - L(t^{-n})R_r^*L(f)R_rPL(t^n) \\ &= R_r^*L(C_rt^{-n})(PL(f) - L(f)P)L(C_rt^n)R_r \end{aligned}$$

From previous considerations, $PL(f) - L(f)P$ is compact and since $L(C_r t^{-n})$ and $L(C_r t^n)$ converge strongly we have $L(C_r t^{-n})(PL(f) - L(f)P)L(C_r t^n)$ converges in norm to a compact operator K. Thus, mod $\mathcal{N}$, we have that the above difference is of the form $R_r^* K R_r \in \mathcal{I}$. The argument for (Q_n^-) is analagous.

We now prove that $\mathcal{D}_2$ is $*$-isomorphic to $C(\mathbb{S}^1)$. We will show that the map

$$\Gamma : C(\mathbb{S}^1) \to \mathcal{D}_2$$

defined by $\Gamma(f) = (L(C_r^{-1} f)) + \mathcal{I}$ is a $*$-isomorphism. As we saw in the proof of Lemma 3.2.2, most of the properties of a $*$-isomorphism are clear; we just need to check injectivty and continuity. To this end, suppose $f \in \ker \Gamma$. Then $(L(C_r^{-1} f)) \in \mathcal{I}$, meaning $(L(C_r^{-1} f)) = (C_{n,r} + R_r^* K R_r)$ where $(C_{n,r}) \in \mathcal{N}$ and K is compact. Thus

$$L(f) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r L(C_r^{-1} f) R_r^* = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r (C_{n,r} + R_r^* K R_r) R_r^* = K.$$

Since the only compact multiplication operator is zero, this implies $f = 0$ and so Γ is injective. Because we are working with C^*-algebras, this also implies that Γ is an isometry and is thus continuous.

□

For $t_0 \in \mathbb{S}^1$, we denote by $\mathcal{I}_{t_0}$ the smallest closed ideal of $\mathcal{B}/\mathcal{I}$ containing $(L(C_r^{-1} g)) + \mathcal{I}$ where $g \in C(\mathbb{S}^1)$ vanishes at t_0; i.e.,

$$\mathcal{I}_{t_0} = \text{clos id}_{\mathcal{B}/\mathcal{I}} \{(L(C_r^{-1} g)) + \mathcal{I} : g \in C(\mathbb{S}^1), g(t_0) = 0\}.$$

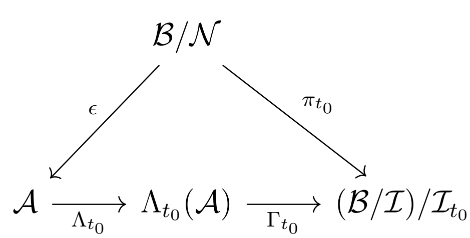
Corollary 4.1.6. *Let $(B_{n,r}) \in \mathcal{B}$. Then $(B_{n,r}) + \mathcal{I}$ is invertible in $\mathcal{B}/\mathcal{I}$ if and only* *if* $(B_{n,r} + \mathcal{I}) + \mathcal{I}_{t_0}$ *is invertible in* $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$ *for all* $t_0 \in \mathbb{S}^1$.

Proof. On account of Lemma 4.1.5, we can employ Theorem 3.2.1 in the setting $\mathcal{A} = \mathcal{B}/\mathcal{I}$ and $Z = \mathcal{D}_2$. As we have seen before, since $\mathcal{D}_2 \cong C(\mathbb{S}^1)$, their maximal ideal spaces are homeomorphic $-$ that is, the maximal ideal space of $\mathcal{D}_2$ is homemorphic to $\mathbb{S}^1$. Putting all of this together gives the claim. □

When combined with Corollary 4.1.6, Theorem 4.1.4 tells us that an element $(B_{n,r}) + \mathcal{N}$ is invertible in $\mathcal{B}/\mathcal{N}$ if and only if $\epsilon(B_{n,r} + \mathcal{N}) \in \mathcal{A}$ is invertible and $(B_{n,r} + \mathcal{I}) + \mathcal{I}_{t_0}$ is invertible in $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$ for all $t_0 \in \mathbb{S}^1$. Said differently, since we are working with C^*- algebras, we have that $\mathcal{B}/\mathcal{N}$ is isomorphic to a subalgebra of the direct sum of $\mathcal{A}$ and each of the $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$. We can actually reduce this to a direct sum of fewer algebras; this fact is a result of the following proposition.

Proposition 4.1.7. *Fix* $t_0 \in \mathbb{S}^1 \setminus \{-1\}$ *and let* $(B_{n,r}) \in \mathcal{B}$. *If* $\epsilon(B_{n,r} + \mathcal{N})$ *is invertible in* $\mathcal{A}$, *then* $(B_{n,r} + \mathcal{I}) + \mathcal{I}_{t_0}$ *is invertible in* $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$.

Proof. We will prove this statement by constructing for each $t_0 \in \mathbb{S}^1 \setminus \{-1\}$ maps Λ_{t_0} and Γ_{t_0} such that the following diagram commutes:



where π_{t_0} is the canonical projection map.

Let's start by noting where the generators of $\mathcal{B}/\mathcal{N}$ get sent to under π_{t_0}. The fact that $(P) + \mathcal{N}$ gets sent to $(\{P\} + \mathcal{I}) + \mathcal{I}_{t_0}$ and $(R_r^* K R_r) + \mathcal{N}$ gets sent to 0 is clear. The element $(Y_{-1} R_r^* K R_r Y_{-1})$ is also zero in the local algebra when $t_0 \neq -1$. To see this, let g be continuous and vanishing in a neighborhood of -1

and equal to 1 at 1. Then mod $\mathcal{I}$ and $\mathcal{I}_{t_0}$ we have

$$\begin{aligned} Y_{-1}R_r^*K'R_rY_{-1} &= Y_{-1}R_r^*K'R_rY_{-1}L(C_r^{-1}g) \\ &= Y_{-1}R_r^*K'R_rY_{-1}L(C_r^{-1}g)Y_{-1}R_r^*R_rY_{-1} \\ &= Y_{-1}R_r^*K'L(C_rY_{-1}C_r^{-1}g)R_rY_{-1}. \end{aligned}$$

We have already seen in the proof of Lemma 4.1.5 that $KL(C_rY_{-1}C_r^{-1}g)$ converges in norm to $g(-1)K = 0$ and hence $Y_{-1}R_r^*KR_rY_{-1}$ equals zero in the local algebra.

Let's analyze $(L(t^{-n})PL(t^n)) + \mathcal{N}$ and $(L(t^n)QL(t^{-n}) + \mathcal{N}$ next. Let g be a continuous function that is zero in a neighborhood of -1 and 1 in a neighborhood of t_0. Notice that since $L(C_r^{-1}g) = I$ modulo $\mathcal{I}_{t_0}$, we have (still mod $\mathcal{I}_{t_0}$)

$$L(t^n) = L(C_r^{-1}g)L(t^n) = R_r^*L(g)R_rL(t^n)R_r^*R_r = R_r^*L(gC_rt^n)R_r$$

and similarly

$$L(t^{-n}) = R_r^*L(gC_rt^{-n})R_r.$$

We have seen in the proof of Proposition 4.1.2 that C_rt^n and C_rt^{-n} converge locally uniformly on $\mathbb{S}^1 \setminus \{-1\}$; denote these limits by g_λ and g_λ^{-1} respectively. Even further, since g vanishes in a neighborhood of -1, we have uniform convergence of gC_rt^n to gg_λ and gC_rt^{-n} to gg_λ^{-1}. Hence, modulo $\mathcal{N}$, we have the equalities $L(gC_rt^n) = L(gg_\lambda)$ and $L(gC_rt^{-n}) = L(gg_\lambda^{-1})$. The functions gg_λ and gg_λ^{-1} are continuous, so modulo $\mathcal{I}_{t_0}$ we have $L(gg_\lambda) = g(t_0)g_\lambda(t_0)I = g_\lambda(t_0)I$ and similarly $L(gg_\lambda^{-1}) = g_\lambda^{-1}(t_0)I$. Direct computation shows that $g_\lambda(t_0)g_\lambda^{-1}(t_0) = 1$. Putting all of this together gives us that under π_{t_0} the generator $(L(t^{-n})PL(t^n))+\mathcal{N}$ gets sent to $(\{P\}+\mathcal{I})+\mathcal{I}_{t_0}$and the generator $(L(t^n)QL(t^{-n})+\mathcal{N}$ gets sent to $(\{Q\}+\mathcal{I})+\mathcal{I}_{t_0}$ for $t_0 \neq -1$.

Let's now turn to $(L(C_r^{-1}f)) + \mathcal{N}$ for $f \in PC$. Here we use the representation

$$f = f(t_0+0)Y_{t_0}^*\chi_+ + f(t_0-0)Y_{t_0}^*\chi_- + a$$

where a is continuous and vanishing at t_0. Then in the local algebra we have

$$L(C_r^{-1}f) = f(t_0+0)L(C_r^{-1}Y_{t_0}^*\chi_+) + f(t_0-0)L(C_r^{-1}Y_{t_0}^*\chi_-).$$

To summarize, we have that under π_{t_0} for $t_0 \neq -1$ the generators of $\mathcal{B}/\mathcal{N}$ are mapped as follows:

Generator in $\mathcal{B}/\mathcal{N}$	Image in $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}, t_0 \neq -1$
$(P) + \mathcal{N}$	$(\{P\} + \mathcal{I}) + \mathcal{I}_{t_0}$
$(R_r^*KR_r) + \mathcal{N}$	$(\{0\} + \mathcal{I}) + \mathcal{I}_{t_0}$
$(Y_{-1}R_r^*K'R_rY_{-1}) + \mathcal{N}$	$(\{0\} + \mathcal{I}) + \mathcal{I}_{t_0}$
$(L(t^{-n})PL(t^n)) + \mathcal{N}$	$\{(P\} + \mathcal{I}) + \mathcal{I}_{t_0}$
$(L(t^n)QL(t^{-n}) + \mathcal{N}$	$(\{Q\} + \mathcal{I}) + \mathcal{I}_{t_0}$
$(L(C_r^{-1}f)) + \mathcal{N}$, $f \in PC$	$(\{f(t_0+0)L(C_r^{-1}Y_{t_0}^*\chi_+) + f(t_0-0)L(C_r^{-1}Y_{t_0}^*\chi_-)\}$ $+\mathcal{I}) + \mathcal{I}_{t_0}$

Table 4.1: Images of Generators of $\mathcal{B}$ in the Local Algebras

We now define the maps Λ_{t_0} and Γ_{t_0} that make the diagram at the beginning of the proof commute. Define $\Lambda_{t_0} : \mathcal{A} \to \Lambda_{t_0}(\mathcal{A})$ by

$$\Lambda_{t_0}(A) = \operatorname*{s-lim}_{s\to 1} R_s Y_{t_0} A Y_{t_0}^* R_s^*$$

and define $\Gamma_{t_0} : \Lambda_{t_0}(\mathcal{A}) \to (\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$ by

$$\Gamma_{t_0}(A) = (R_r^* Y_{t_0}^* A Y_{t_0} R_r + \mathcal{I}) + \mathcal{I}_{t_0}$$

so that the composition sends an operator $A \in \mathcal{A}$ to the element

$$(R_r^* Y_{t_0}^* (\text{s-}\lim_{s\to 1} R_s Y_{t_0} A Y_{t_0}^* R_s^*) Y_{t_0} R_r + \mathcal{I}) + \mathcal{I}_{t_0} \in (\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}.$$

Recall from Proposition 4.1.2 the images of each element of $\mathcal{B}$ in $\mathcal{A}$ are as follows:

$$\begin{aligned}
(P) &\mapsto P & (L(\chi_+)) &\mapsto L(\chi_+)\\
(L(t^{-n})PL(t^n)) &\mapsto L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big) & (L(C_r^{-1}f)) &\mapsto L(f)\\
(L(t^n)QL(t^{-n}) &\mapsto L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big) & (R_r^*KR_r) &\mapsto K\\
(Y_{-1}R_r^*K'R_rY_{-1}) &\mapsto 0
\end{aligned}$$

We proceed from this starting point and show that under our new composition map the images agree with the images in π_{t_0}. Since $R_s Y_{t_0} P Y_{t_0}^* R_s^* = P$ and $R_r^* Y_{t_0}^* P Y_{t_0} R_r = P$, it follows that P does in fact get sent to $(P + \mathcal{I}) + \mathcal{I}_{t_0}$. Due to the weak convergence of R_s and R_s^* to zero, we have also that K is mapped to zero under the composition.

Let's now consider the elements $L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)$ and $L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)$.

Under the map Λ_{t_0}, we have that $L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)$ is sent to

$$L\Big(\exp\Big(\lambda\frac{(t_0-1)}{(t_0+1)}\Big)\Big) =: L(g_\lambda)$$

and $L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)$ is sent to

$$L\Big(\exp\Big(-\lambda\frac{(t_0-1)}{(t_0+1)}\Big)\Big) =: L(g_{-\lambda}).$$

As before, if we take g to be a continuous function vanishing in a neighborhood of -1 and equal to 1 in a neighborhood of t_0, in $\mathcal{I}_{t_0}$ we have

$$L(g_\lambda) = L(C_r^{-1}g)L(g_\lambda) = R_r^* L(gC_r g_\lambda)R_r = C_r g_\lambda(t_0)I$$

and similarly

$$L(g_{-\lambda}) = C_r g_{-\lambda}(t_0)I.$$

Then since $C_r g_\lambda(t_0) C_r g_{-\lambda}(t_0) = 1$, we have that the generators $L\Big(\exp\Big(\lambda\dfrac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\dfrac{t-1}{1+t}\Big)\Big)$ and $L\Big(\exp\Big(\lambda\dfrac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\dfrac{1-t}{1+t}\Big)\Big)$ are mapped to $(P+\mathcal{I})+\mathcal{I}_{t_0}$ and $(Q+\mathcal{I})+\mathcal{I}_{t_0}$ respectively. Then since $R_r^* Y_{t_0}^* P Y_{t_0} R_r = P$ and $R_r^* Y_{t_0}^* Q Y_{t_0} R_r = Q$ we have that under Γ_{t_0} these elements get sent to exactly where we need them to be sent.

Finally, we look at $L(f)$ for $f \in PC$. We know that $\operatorname*{s-lim}_{s\to 1} R_s Y_{t_0} L(f) Y_{t_0}^* R_s^* = \operatorname*{s-lim}_{s\to 1} L(C_s Y_{t_0} f)$. We can write

$$Y_{t_0} f = f(t_0+0)\chi_+ + f(t_0-0)\chi_- + g$$

where $g(1 \pm 0) = 0$. We then obtain

$$C_s Y_{t_0} f = f(t_0+0)\chi_+ + f(t_0-0)\chi_- + C_s g.$$

Now, $C_s g$ converges to zero locally uniformly on $\mathbb{S}^1 \setminus \{-1\}$ (since $g(1\pm 0) = 0$) and hence $C_s Y_{t_0} f$ converges to $f(t_0+0)\chi_+ + f(t_0-0)\chi_-$ in measure. Thus $L(f)$ gets mapped to $f(t_0+0)L(\chi_+) + f(t_0-0)L(\chi_-)$ under Λ_{t_0}. Then under Γ_{t_0} we have that this is sent to $f(t_0+0)L(C_r^{-1}Y_{t_0}^*\chi_+) + f(t_0-0)L(C_r^{-1}Y_{t_0}^*\chi_-)$ as desired.

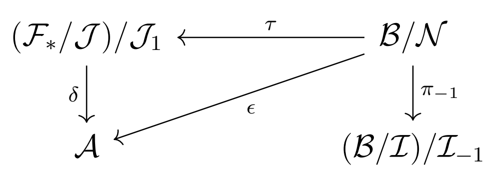
Having checked each generator, we have that the diagram at the beginning of our proof commutes and so our work is done.

□

Corollary 4.1.8. *The C^*-algebra $\mathcal{B}/\mathcal{N}$ is isomorphic to a $*-$subalgebra of the direct sum $\mathcal{A}\bigoplus(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$.*

Proof. Theorem 4.1.4 and Proposition 4.1.7 together tell us that an element $(B_{n,r})+\mathcal{N}\in\mathcal{B}/\mathcal{N}$ is invertible if and only if its images $\epsilon(B_{n,r})$ and $(\{B_{n,r}\}+\mathcal{I})+\mathcal{I}_{-1}$ are invertible in $\mathcal{A}$ and $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ respectively. In other words, the map that sends $(B_{n,r})+\mathcal{N}\in\mathcal{B}/\mathcal{N}$ to the element $\big(\epsilon(B_{n,r}),\pi_{-1}(B_{n,r})\big)\in\mathcal{A}\bigoplus(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ preserves spectra. Being a $*-$homomorphism, this means that this mapping is necessarily an isometry and hence injective. Thus we have an isomorphism onto the image. □

As of right now, our picture looks like this:



In what follows, we will examine the algebras $\mathcal{A}$ and $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$, further identifying them as different algebras of operators and expanding on this picture. We will use this more complete picture to help answer our question of invertibility in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$.

4.2 The Algebra $\mathcal{A}$

For a subset D of the real axis, we denote by χ_D the characteristic function of D. This can be regarded as a multiplication operator which we will sometimes write as $M(\chi_D)$ (other times context will clarify what is meant). For $n \geq 1$, we define the bounded linear operators E_n and E_{-n} by

$$E_n : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}), (x_i)_{i\in\mathbb{Z}} \mapsto \sqrt{n} \sum_{i=-\infty}^{\infty} x_i \chi_{[\frac{i}{n}, \frac{i+1}{n}]} \tag{4.1}$$

$$E_{-n} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}), f \mapsto \left(\sqrt{n} \int_{-\infty}^{\infty} f(x)\chi_{[\frac{i}{n}, \frac{i+1}{n}]}(x)dx\right)_{i=-\infty}^{\infty} \tag{4.2}$$

We have that $E_{-n}^* = E_n$ and $E_{-n}E_n = I$. The operator $L_n := E_n E_{-n}$ on $L^2(\mathbb{R})$ converges strongly on $L^2(\mathbb{R})$ to the identity operator I as $n \to \infty$ (for more information and proof, one can check [8], Sections 2.2.1 and 2.2.3 and Proposition 2.3).

Let $S_{\mathbb{R}}$ be the *singular integral operator on the real axis*; i.e., the operator defined by

$$(S_{\mathbb{R}}f)(x) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - x} dy. \tag{4.3}$$

The operator $S_{\mathbb{R}}$ is bounded on $L^2(\mathbb{R})$, is its own inverse, and is in fact a *Fourier convolution operator* with generating function $a(z) = \mathrm{sgn}(z)$; i.e.,

$$S_{\mathbb{R}} = \mathcal{F}^{-1} M\left(a\right) \mathcal{F} \tag{4.4}$$

(see [8] Section 2.1.1).

Keeping $a(z) = \mathrm{sgn}(z)$, we have that

$$P_{\mathbb{R}} = \frac{1 + S_{\mathbb{R}}}{2} = \mathcal{F}^{-1} M\left(\frac{1+a}{2}\right) \mathcal{F} \tag{4.5}$$

which will be a useful representation when it comes to computations.

Let σ denote the function

$$\sigma(e^{2\pi i\phi}) = -\frac{\sin^2(\pi\phi)}{\pi^2}\sum_{m\in\mathbb{Z}}\frac{\operatorname{sgn}(m+\frac{1}{2})}{(\phi+m)^2}, \quad \phi\in(0,1). \tag{4.6}$$

This function σ is continuous on $\mathbb{S}^1\setminus\{1\}$ and has a jump discontinuity at 1 with one-sided limits $\sigma(1+0) = -1$ and $\sigma(1-0) = 1$. Moreover, for all $n \geq 1$, we have

$$L(\sigma) = E_{-n}S_{\mathbb{R}}E_n. \tag{4.7}$$

We can now define the operator $\Psi_1 : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 \to \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ by

$$\Psi_1(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_nF_{n,r}E_{-n} \tag{4.8}$$

The algebra $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ is an algebra of operators on $L^2(\mathbb{R})$ and we will prove that it is in fact isomorphic to the algebra $\mathcal{A}$. Before we do this, we first must prove that this map is well-defined and compute where each generator is mapped to. To this end, we need some auxiliary results.

For $s,t\in\mathbb{R}$ and $\tau>0$, define the following kinds of shift operators on $L^2(\mathbb{R})$:

$$M_s : L^2(\mathbb{R})\to L^2(\mathbb{R}), (M_sf)(x) = e^{2\pi ixs}f(x) \tag{4.9}$$

$$U_t : L^2(\mathbb{R})\to L^2(\mathbb{R}), (U_tf)(x) = f(x-t) \tag{4.10}$$

$$Z_\tau : L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+), (Z_\tau f)(x) = \tau^{\frac{1}{2}}f(\tau x) \tag{4.11}$$

These operators will play an important role both in this section and in the

Fredholm theory established in Appendix A. There is a nice relationship between the first two operators and the Fourier transform:

Proposition 4.2.1. *$M_t\mathcal{F}^{-1} = \mathcal{F}^{-1}U_t$ and $\mathcal{F}M_{-t} = U_{-t}\mathcal{F}$.*

Proof. We will prove $M_t\mathcal{F}^{-1} = \mathcal{F}^{-1}U_t$ since the other relation will follow from the uniqueness of inverses. Let $f \in L^2(\mathbb{R})$. By definition,

$$\begin{aligned}(M_t\mathcal{F}^{-1}f)(x) &= \int_{-\infty}^{\infty} e^{2\pi ixt}e^{2\pi ixz}f(z)dz \\ &= \int_{-\infty}^{\infty} e^{2\pi ixt+2\pi ixz}f(z)dz \\ &= \int_{-\infty}^{\infty} e^{2\pi ix(t+z)}f(z)dz \\ &= \int_{-\infty}^{\infty} e^{2\pi ixy}f(y-t)dy \quad \text{(by setting } y = t+z\text{)} \\ &= (\mathcal{F}^{-1}U_tf)(x)\end{aligned}$$

as desired. □

Lemma 4.2.2. *For any $k \in \mathbb{Z}$, $E_nL(t^k)E_{-n} = L_n\mathcal{F}M(e^{2\pi ix\frac{k}{n}})\mathcal{F}^{-1}$.*

Proof. By Proposition 4.2.1, we have $M(e^{2\pi ix\frac{k}{n}})\mathcal{F}^{-1} = \mathcal{F}^{-1}U_{\frac{k}{n}}$, so we must show that $E_nL(t^k)E_{-n} = L_nU_{\frac{k}{n}}$. That is, for $f \in L^2(\mathbb{R})$), we aim to prove

$$E_nL(t^k)E_{-n}f = L_nU_{\frac{k}{n}}f.$$

Direct computation gives that the lefthand side is

$$E_nL(t^k)E_{-n}f = n\sum_{i\in\mathbb{Z}}\Big(\int_{-\infty}^{\infty} f(x)\chi_{[\frac{i-k}{n},\frac{i-k+1}{n}]}(x)dx\Big)\chi_{[\frac{i}{n},\frac{i+1}{n}]}$$

The righthand side is

$$L_n U_{\frac{k}{n}} f = n \sum_{i\in\mathbb{Z}} \Big(\int_{\frac{i}{n}}^{\frac{i+1}{n}} f\big(x - \frac{k}{n}\big) dx \Big) \chi_{[\frac{i}{n}, \frac{i+1}{n}]}$$
$$= n \sum_{i\in\mathbb{Z}} \Big(\int_{\frac{i-k}{n}}^{\frac{i-k+1}{n}} f(y) dy \Big) \chi_{[\frac{i}{n}, \frac{i+1}{n}]}$$

Since the lefthand side is equal to the righthand side, our work is done. □

Lemma 4.2.3. *Let K be a compact operator on $L^2(\mathbb{S}^1)$. Then*

(a) $(K) \in ker\ \Psi_1$

(b) $(U_{-n} K U_n) \in ker\ \Psi_1$

(c) $(U_n K U_{-n}) \in ker\ \Psi_1$

(d) $\Psi_1(L(f)) = f(1)I$ *if* f *is a continuous function. In particular,* $(L(f)) \in ker\ \Psi_1$ *if* f *is continuous and* $f(1) = 0$.

Proof. Assertion (a) follows from the weak convergence of E_n and uniform boundedness of E_{-n}. Assertions (b) and (c) will follow from (a) if we can prove that $\Psi_1(U_n)$ and $\Psi_1(U_{-n})$ exist. Writing $U_{\pm n} = L(t^{\pm n})$ and using Lemma 4.2.2, we have

$$E_n U_{\pm n} E_{-n} = L_n \mathcal{F} M(e^{\pm 2\pi i x}) \mathcal{F}^{-1}.$$

By Proposition 4.2.1,

$$L_n \mathcal{F} M(e^{\pm 2\pi i x}) \mathcal{F}^{-1} = L_n U_{\pm 1}$$

where we remark for the sake of clarity that $U_{\pm 1}$ is the operator acting on $L^2(\mathbb{R})$ defined in Equation 4.10. This converges strongly to $U_{\pm 1}$ and hence (b) and (c) hold.

To prove (d), it suffices to show that $\Psi_1(L(t)) = I$. Again using Lemma 4.2.2 and Proposition 4.2.1, we have $E_n L(t) E_{-n} = L_n U_{\frac{1}{n}}$ which converges strongly to the identity.

□

The next two theorems highlight a relationship between the operators E_n, E_{-n}, R_r, and R_r^*, namely that $E_n R_r^*$ and $R_r E_{-n}$ converge *strongly*. The fact that these converge strongly will be a useful tool for us in our considerations and proofs.

Theorem 4.2.4. *The sequence of operators $E_n R_r^* : L^2(\mathbb{S}^1) \to L^2(\mathbb{R})$ converges strongly to the operator A given by $(Af)(y) = \frac{\sqrt{2\lambda}}{2\pi} \int_{-\infty}^{\infty} \frac{f(\frac{1+ix}{1-ix})}{1-ix} e^{-i\lambda yx} dx$.*

Proof. Because the sequence $(E_n R_r^*)$ is uniformly bounded, we may prove strong convergence on a dense subset. In other words, we are seeking to find an operator A such that $\|E_n R_r^* f - Af\|_{L^2(\mathbb{R})}$ can be made as small as desired where f is taken from a dense subset of $L^2(\mathbb{S}^1)$. To this end, we take f to be a function vanishing identically in a neighborhood of -1 in $\mathbb{S}^1$. By definition, $(R_r^* f)(t) = \frac{\sqrt{1-r^2}}{1-rt} f(\frac{t-r}{1-rt})$. The m$^{\text{th}}$ Fourier coefficient is then given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sqrt{1-r^2}}{1-re^{i\theta}} f(\frac{e^{i\theta}-r}{1-re^{i\theta}}) e^{-im\theta} d\theta.$$

Thus we have

$$E_n R_r^* f = \sum_{m\in\mathbb{Z}} \frac{\sqrt{n(1-r^2)}}{2\pi} \left(\int_{-\pi}^{\pi} \frac{f(\frac{e^{i\theta}-r}{1-re^{i\theta}})}{1-re^{i\theta}} e^{-im\theta} d\theta \right) \chi_{[\frac{m}{n}, \frac{m+1}{n}]}$$

For convergence of this sequence in $L^2(\mathbb{R})$ we will employ the Dominated Convergence Theorem; we will find its pointwise limit and then find a dominating function. Let's first take a look at the pointwise limit; fix $y \in \mathbb{R}$. Then there exists an $M = \lfloor ny \rfloor$ such that $y \in [\frac{M}{n}, \frac{M+1}{n}]$. This means

$$(E_n R_r^* f)(y) = \frac{\sqrt{n(1-r^2)}}{2\pi} \int_{-\pi}^{\pi} \frac{f(\frac{e^{i\theta}-r}{1-re^{i\theta}})}{1-re^{i\theta}} e^{-i\lfloor ny \rfloor \theta} d\theta$$

It will be convenient to transform this to an integral of a function on the real line; to this end, we set $\frac{e^{i\theta}-r}{1-re^{i\theta}} = \frac{1+ix}{1-ix}$ and obtain that $\theta = 2\arctan(\epsilon x)$ with $\epsilon = \frac{1-r}{1+r}$. We can then rewrite our previous expression for $(E_n R_r^* f)(y)$ as

$$\frac{\sqrt{n(1-r^2)}}{2\pi} \int_{-\infty}^{\infty} \frac{f(\frac{1+ix}{1-ix})}{1-re^{2i\arctan(\epsilon x)}} e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)} \cdot \frac{2\epsilon}{1+(\epsilon x)^2} dx$$

Define $g(x) := f(\frac{1+ix}{1-ix})$. Then since f is vanishing identically in a neighborhood of -1 on the unit circle, we have that there exists some $L \in \mathbb{R}$ such that supp(g) $\subseteq [-L, L]$. We have

$$(E_n R_r^* f)(y) = \frac{\sqrt{n(1-r^2)}}{2\pi} \int_{-\infty}^{\infty} \frac{g(x)}{1-re^{2i\arctan(\epsilon x)}} e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)} \cdot \frac{2\epsilon}{1+(\epsilon x)^2} dx$$

The first thing to explore is the limit of the integrand. One can see via the squeeze theorem that the limit of $e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)}$ is the same as the limit of $e^{-2iny\arctan(\epsilon x)}$. By looking at Taylor series, we see that $-2iny\arctan(\epsilon x) = -2iny\epsilon x +$ terms that converge to zero. This then converges to $-i\lambda xy$ and so $e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)}$ converges to $e^{-i\lambda xy}$.

For the limit of $\frac{g(x)}{1-re^{2i\arctan(\epsilon x)}} \cdot \frac{2\epsilon}{1+(\epsilon x)^2}$ we will make use of the identity

$$e^{2i\theta} = \frac{1+i\tan\theta}{1-i\tan\theta}$$

with $\theta = \arctan(\epsilon x)$. In particular, we have $e^{2i\arctan(\epsilon x)} = \frac{1+i\epsilon x}{1-i\epsilon x}$ which when combined with the fact that $r = \frac{1-\epsilon}{1+\epsilon}$ yields that

$$\frac{1}{1 - re^{2i\arctan(\epsilon x)}} = \frac{(1+\epsilon)(1-i\epsilon x)}{2\epsilon(1-ix)}.$$

Thus

$$\begin{aligned}\frac{g(x)}{1 - re^{2i\arctan(\epsilon x)}} \cdot \frac{2\epsilon}{1+(\epsilon x)^2} &= \frac{g(x)(1+\epsilon)(1-i\epsilon x)}{2\epsilon(1-ix)} \cdot \frac{2\epsilon}{1+(\epsilon x)^2} \\ &= \frac{g(x)(1+\epsilon)(1-i\epsilon x)}{(1-ix)(1+(\epsilon x)^2)}\end{aligned}$$

which converges to $\dfrac{g(x)}{1-ix}$. Altogether, we have shown that

$$\int_{-\infty}^{\infty} \lim_{\substack{n\to\infty \\ r\to 1}} \frac{\sqrt{n(1-r^2)}}{2\pi} \frac{g(x)}{1 - re^{2i\arctan(\epsilon x)}} e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)} \frac{2\epsilon}{1+(\epsilon x)^2} dx$$

is equal to

$$\int_{-\infty}^{\infty} \frac{\sqrt{2\lambda}}{2\pi} \frac{g(x)}{1-ix} e^{-i\lambda yx} dx.$$

To complete the proof, we must find functions h_1 and h_2 in $L^2(\mathbb{R})$ such that

$$\left| \frac{g(x)}{1 - re^{2i\arctan(\epsilon x)}} e^{-2i\lfloor ny \rfloor \arctan(\epsilon x)} \cdot \frac{2\epsilon}{1+(\epsilon x)^2} \right| \leq h_1(x)$$

and

$$|(E_n R_r^* f)(y)| \leq h_2(y)$$

since the Dominated Convergence Theorem will then be in action. Finding h_1 is not too bad; from our previous work we have that

$$\left|\frac{g(x)}{1-re^{2i\arctan(\epsilon x)}}e^{-2i\lfloor ny\rfloor\arctan(\epsilon x)}\cdot\frac{2\epsilon}{1+(\epsilon x)^2}\right| = \left|\frac{g(x)(1+\epsilon)(1-i\epsilon x)}{(1-ix)(1+(\epsilon x)^2)}\right|$$

and since x will be coming from a bounded domain $\left|\frac{(1+\epsilon)(1-i\epsilon x)}{(1-ix)(1+(\epsilon x)^2)}\right|$ is bounded, say by M_1. Then we can take $h_1(x) = M_1|g(x)|$.

Finding h_2 will require a little more work, but is also not too bad. We will first integrate the expression for $(E_nR_r^*f)(y)$ by parts, taking $u = \frac{g(x)}{1-re^{2i\arctan(\epsilon x)}}$ and $dv = \frac{2\epsilon e^{-2i\lfloor ny\rfloor\arctan(\epsilon x)}}{1+(\epsilon x)^2}$. Then $(E_nR_r^*f)(y)$ is equal to

$$\left.\frac{ig(x)e^{-2i\lfloor ny\rfloor\arctan(\epsilon x)}}{1-re^{2i\arctan(\epsilon x)}\lfloor ny\rfloor}\right|_{-\infty}^{\infty}$$

minus the integral

$$\int_{-\infty}^{\infty}\frac{(1-re^{2i\arctan(\epsilon x)})g'(x)+\frac{2i\epsilon rg(x)e^{2i\arctan(\epsilon x)}}{1+(\epsilon x)^2}}{(1-re^{2i\arctan(\epsilon x)})^2}\frac{ie^{-2i\lfloor ny\rfloor\arctan(\epsilon x)}}{\lfloor ny\rfloor}dx.$$

This term

$$\left.\frac{ig(x)e^{-2i\lfloor ny\rfloor\arctan(\epsilon x)}}{1-re^{2i\arctan(\epsilon x)}\lfloor ny\rfloor}\right|_{-\infty}^{\infty}$$

is zero since $\mathrm{supp}(g)\subseteq[-L,L]$. We therefore focus on bounding the norm of the remaining integral. We start by noting that

$$|(E_nR_r^*f)(y)| \le \int_{-\infty}^{\infty}\left|\frac{(1-re^{2i\arctan(\epsilon x)})g'(x)+\frac{2i\epsilon rg(x)e^{2i\arctan(\epsilon x)}}{1+(\epsilon x)^2}}{(1-re^{2i\arctan(\epsilon x)})^2 n(y-1)}\right|dx$$

which is less than or equal to

$$\int_{-\infty}^{\infty}\left|\frac{\Big(\frac{2\epsilon(1-ix)g'(x)}{(1+\epsilon)(1-i\epsilon x)}+\frac{2i\epsilon rg(x)}{1+(\epsilon x)^2}\frac{2\epsilon(1-ix)}{(1+\epsilon)(1-i\epsilon x)}\Big)(1+\epsilon)^2(1-i\epsilon x)^2}{(2\epsilon(1-ix))^2 n(y-1)}\right|dx.$$

Finally, this is less than or equal to

$$\frac{1}{y-1}\int_{-\infty}^{\infty}\left|\frac{\Big(\frac{2(1-ix)g'(x)}{(1+\epsilon)(1-i\epsilon x)}+\frac{2irg(x)}{1+(\epsilon x)^2}\frac{2\epsilon(1-ix)}{(1+\epsilon)(1-i\epsilon x)}\Big)(1+\epsilon)^2(1-i\epsilon x)^2}{4\epsilon n(1-ix)^2}\right|dx.$$

This final integral is bounded by some M_2 since x is coming from a bounded domain, and hence we may choose the function $h_2(y) = \dfrac{M_2}{y-1} \in L^2(\mathbb{R})$ and our proof is complete. □

Theorem 4.2.5. *The sequence of operators* $R_r E_{-n} : L^2(\mathbb{R}) \to L^2(\mathbb{S}^1)$ *converges strongly to the operator* T *given by* $(T\chi_{[a,b]})(t) = \sqrt{\dfrac{2}{\lambda}}\dfrac{e^{a\lambda\frac{t-1}{1+t}} - e^{b\lambda\frac{t-1}{1+t}}}{1-t}$.

Proof. Due to the uniform boundedness of $(R_r E_{-n})$, to prove strong convergence it is enough to show that $\|R_r E_{-n} f - Tf\|_{L^2(\mathbb{S}^1)} \to 0$ as $n \to \infty, r \to 1$ for f from a dense subset of $L^2(\mathbb{R})$. We thus take f to be a characteristic function, $f = \chi_{[a,b]}$. By definition,

$$E_{-n}f = (\sqrt{n}\int_{-\infty}^{\infty}\chi_{[a,b]}(x)\chi_{[\frac{k}{n},\frac{k+1}{n}]}(x)dx)_{k\in\mathbb{Z}} = (y_k)_{k\in\mathbb{Z}}$$

where

$$y_k = \begin{cases} \dfrac{\sqrt{n}}{n} & \text{if } \lfloor na \rfloor + 1 \le k \le \lfloor bn \rfloor - 1 \\ \left(b - \dfrac{\lfloor bn \rfloor}{n}\right)\sqrt{n} & \text{if } k = \lfloor bn \rfloor \\ \left(\dfrac{\lfloor na \rfloor + 1}{n} - a\right)\sqrt{n} & \text{if } k = \lfloor na \rfloor \\ 0 & \text{otherwise} \end{cases}$$

Identifying $\ell^2(\mathbb{Z})$ with $L^2(\mathbb{S}^1)$, we have $E_{-n}f = \sum_{k\in\mathbb{Z}} y_k t^k$. Then

$$\begin{aligned} R_r E_{-n} f &= \frac{\sqrt{1-r^2}}{1+rt} \sum_{k\in\mathbb{Z}} y_k \left(\frac{t+r}{1+rt}\right)^k \\ &= \frac{\sqrt{1-r^2}}{1+rt}\left(b - \frac{\lfloor bn \rfloor}{n}\right)\sqrt{n}\left(\frac{t+r}{1+rt}\right)^{\lfloor bn \rfloor} \\ &\quad + \frac{\sqrt{1-r^2}}{1+rt}\left(\frac{\lfloor na \rfloor + 1}{n} - a\right)\sqrt{n}\left(\frac{t+r}{1+rt}\right)^{\lfloor na \rfloor} \\ &\quad + \frac{\sqrt{1-r^2}}{1+rt}\frac{1}{\sqrt{n}} \sum_{k=\lfloor na \rfloor + 1}^{\lfloor bn \rfloor - 1} \left(\frac{t+r}{1+rt}\right)^k \end{aligned}$$

To prove that $\|R_r E_{-n} f - Tf\|_{L^2(\mathbb{S}^1)} \to 0$, we will use the L^2- Dominated Convergence Theorem; that is, we will show that the sequence $\{R_r E_{-n} f\}$ converges to Tf pointwise almost everywhere and that $\exists g \in L^2(\mathbb{S}^1)$ such that $|(R_r E_{-n} f)(t)| \le g(t)$ for all $n \in \mathbb{N}, r \in [0, 1)$, and almost every $t \in \mathbb{S}^1$. Let's start the journey with pointwise almost everywhere convergence, specifically pointwise convergence on $\mathbb{S}^1 \setminus \{\pm 1\}$.

Pointwise convergence on $\mathbb{S}^1 \setminus \{\pm 1\}$. First notice that the first term in the sum above goes to zero. Indeed, for fixed $t \in \mathbb{S}^1 \setminus \{-1\}$ we have

$$\Big|\frac{\sqrt{1-r^2}}{1+rt}\left(b - \frac{\lfloor bn \rfloor}{n}\right)\sqrt{n}\left(\frac{t+r}{1+rt}\right)^{\lfloor bn \rfloor}\Big| = \Big|\frac{\sqrt{(1-r)n}\sqrt{1+r}}{1+rt}\Big|\Big|\frac{bn - \lfloor bn \rfloor}{n}\Big|$$

$$\leq \Big|\frac{\sqrt{(1-r)n}\sqrt{1+r}}{1+rt}\Big| \cdot \frac{1}{n}$$

The first term converges locally uniformly to $\dfrac{\sqrt{2\lambda}}{1+t}$ on $\mathbb{S}^1 \setminus \{-1\}$ and hence $\Big|\dfrac{\sqrt{(1-r)n}\sqrt{1+r}}{1+rt}\Big| \cdot \dfrac{1}{n}$ will converge to zero. Notice that even for t approaching -1 we do not have any blow up happening; here we have

$$\Big|\frac{\sqrt{1-r^2}}{1+rt}(b - \frac{\lfloor bn \rfloor}{n})\sqrt{n}(\frac{t+r}{1+rt})^{\lfloor bn \rfloor}\Big| \approx \Big|\frac{1}{n(1+rt)}\Big| \leq \frac{1}{n(1-r)} \to \frac{1}{\lambda}$$

A nearly identical argument will work to prove that the term

$$\frac{\sqrt{1-r^2}}{1+rt}(\frac{\lfloor na \rfloor + 1}{n} - a)\sqrt{n}(\frac{t+r}{1+rt})^{\lfloor na \rfloor}$$

converges pointwise to zero on $\mathbb{S}^1 \setminus \{-1\}$.

Let's now turn our attention to the final term in the sum expression for $E_{-n}R_r f$:

$$\begin{aligned}
\frac{\sqrt{1-r^2}}{1+rt}\frac{1}{\sqrt{n}}\sum_{\lfloor na \rfloor+1}^{\lfloor bn \rfloor -1}(\frac{t+r}{1+rt})^k &= \frac{\sqrt{1-r^2}}{(1+rt)\sqrt{n}}\frac{-(\frac{t+r}{1+rt})^{\lfloor bn \rfloor -1} + (\frac{t+r}{1+rt})^{\lfloor an \rfloor}}{1-(\frac{t+r}{1+rt})} \\
&= \sqrt{\frac{1-r^2}{n}}\frac{-(\frac{t+r}{1+rt})^{\lfloor bn \rfloor -1} + (\frac{t+r}{1+rt})^{\lfloor an \rfloor}}{1+rt-t-r} \\
&= \sqrt{\frac{1-r^2}{n}}\frac{-(\frac{t+r}{1+rt})^{\lfloor bn \rfloor -1} + (\frac{t+r}{1+rt})^{\lfloor an \rfloor}}{(1-t)(1-r)} \\
&= \sqrt{\frac{1-r^2}{n(1-r)^2}}\frac{-(\frac{t+r}{1+rt})^{\lfloor bn \rfloor -1} + (\frac{t+r}{1+rt})^{\lfloor an \rfloor}}{(1-t)} \\
&= \sqrt{\frac{1+r}{n(1-r)}}\frac{-(\frac{t+r}{1+rt})^{\lfloor bn \rfloor -1} + (\frac{t+r}{1+rt})^{\lfloor an \rfloor}}{(1-t)}
\end{aligned}$$

Notice that we must avoid $t = 1$. Now, $\sqrt{\frac{1+r}{n(1-r)}} \to \sqrt{\frac{2}{\lambda}}$. We also have that

$$\begin{aligned}\Big(\frac{t+r}{1+rt}\Big)^{an} &= \Big(1+\frac{t+r-1-rt}{1+rt}\Big)^{na}\\ &= \Big(1+\frac{(1-r)(t-1)}{1+rt}\Big)^{na}\\ &= \Big(1+n(1-r)\cdot\frac{(t-1)}{n(1+rt)}\Big)^{na} \to e^{\lambda a(\frac{t-1}{1+t})}\end{aligned}$$

where convergence is locally uniform on $\mathbb{S}^1 \setminus \{-1\}$. To relate this fact to the convergence of $(\frac{t+r}{1+rt})^{\lfloor an\rfloor}$, we note that

$$\Big(\frac{t+r}{1+rt}\Big)^{\lfloor an\rfloor} = \Big(\frac{t+r}{1+rt}\Big)^{an}\cdot\Big(\frac{t+r}{1+rt}\Big)^{\lfloor an\rfloor-an} = \Big(\frac{t+r}{1+rt}\Big)^{an}\cdot\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}$$

Observe that $\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor} \to 1$. Indeed, if we choose r sufficiently close to 1 we will have that $\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}$ stays away from -1 since $\frac{1+rt}{t+r} \to 1$ as $r \to 1$ and $na - \lfloor na\rfloor \in [0,1)$. We can then take the principle branch of log with branch cut $(-\infty, 0]$ and look at $\log\Big(\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}\Big)$. Because $na-\lfloor na\rfloor \in [0,1)$ we see that $\log\Big(\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}\Big) = (na-\lfloor na\rfloor)\log\Big(\frac{1+rt}{t+r}\Big)$. Thus

$$\begin{aligned}\Big|\log\Big(\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}\Big)\Big| &= \Big|na-\lfloor na\rfloor\Big|\cdot\Big|\log\Big(\frac{1+rt}{t+r}\Big)\Big|\\ &\le \Big|\log\Big(\frac{1+rt}{t+r}\Big)\Big| < \epsilon\end{aligned}$$

where the final estimate comes from the facts that $\frac{1+rt}{t+r} \to 1$ locally uniformly on $\mathbb{S}^1 \setminus \{-1\}$ and log is continuous on its principle branch. We have thus shown that $\log\Big(\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}\Big)$ converges to 0, i.e., $\Big(\frac{1+rt}{t+r}\Big)^{an-\lfloor an\rfloor}$ converges to 1. Hence $(\frac{t+r}{1+rt})^{\lfloor an\rfloor}$ and $(\frac{t+r}{1+rt})^{an}$ have the same limit. The same argument works for $(\frac{t+r}{1+rt})^{\lfloor bn\rfloor-1}$. We have thus shown that $\{R_rE_{-n}f\}$ converges to $\{Tf\}$ pointwise on $\mathbb{S}^1 \setminus \{\pm 1\}$, i.e., that we have pointwise convergence almost everywhere.

Finding a dominating function. The first 2 terms we dealt with converged to zero as long as we avoided $t = -1$ and hence are bounded by some M. We also showed that as we approach -1 there is no blowup and thus we still have boundedness. The last term that we dealt with is also bounded − this is clear for $t \neq 1$ since there we have uniform convergence. Thus to finish up the proof we need only show that there are no issues at $t = 1$ for this last term. Let's do it:

$$\begin{aligned}
|(R_r E_{-n} f)(1)| &= \left| \frac{\sqrt{1-r^2}}{1+r} \frac{1}{\sqrt{n}} \sum_{\lfloor na \rfloor + 1}^{\lfloor bn \rfloor - 1} \left(\frac{1+r}{1+r} \right)^k \right| \\
&= \left| \frac{\sqrt{1-r^2}}{1+r} \frac{1}{\sqrt{n}} (\lfloor bn \rfloor - 1 - \lfloor na \rfloor) \right| \\
&\leq \left| \frac{\sqrt{1-r^2}}{1+r} \frac{1}{\sqrt{n}} (bn - 1 - (na - 1)) \right| \\
&= \left| \frac{\sqrt{1-r^2}}{1+r} \frac{1}{\sqrt{n}} (bn - na) \right| \\
&= \left| \frac{\sqrt{1-r^2}}{1+r} \sqrt{n} (b-a) \right| \\
&= \left| \frac{\sqrt{1+r}}{1+r} \sqrt{(1-r)n} (b-a) \right| < c
\end{aligned}$$

We have now shown that each piece is bounded for every $t \in \mathbb{S}^1$ and thus we can bound the entire sum. The dominating function will then be a constant, which is integrable since $\mathbb{S}^1$ is a finite measure space and our proof is complete. □

Theorem 4.2.6. *The operator $\Psi_1 : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 \to \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ is a well-defined mapping and the generators of $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ are mapped as follows:*

$$\begin{array}{lll}
(P) \mapsto \chi_{[0,\infty)} & (P_n^+) \mapsto \chi_{[-1,\infty)} & (Q_n^-) \mapsto \chi_{(-\infty,1]} \\
(L(\chi_+)) \mapsto Q_{\mathbb{R}} & (L(C_r^{-1} f)) \mapsto W^0\left(f\left(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\right)\right) & (R_r^* K R_r) \mapsto K'
\end{array}$$

where $f \in PC^0_{-1}$ and $K, K' \in \mathcal{K}(L^2(\mathbb{S}^1))$.

Proof. The fact that this mapping is well-defined is a consequence of Lemma 4.2.3. We now show the images of each generator under this map. The fact that $(R_r^* K R_r)$ is mapped to another compact operator is a result of the strong convergence of $E_n R_r^*$ and $R_r E_{-n}$ (Theorems 4.2.4 and 4.2.5). The images of $(P), (P_n^+)$, and (Q_n^-) are a result of the facts that L_n converges strongly to the identity and $P = E_{-n}\chi_{[0,\infty)}E_n$, $P_n^+ = E_{-n}\chi_{[-1,\infty)}E_n$, and $Q_n^- = E_{-n}\chi_{(-\infty,1]}E_n$. To deal with $(L(\chi_+))$, we use the fact that any piecewise continuous function g has the representation

$$g = g(1+0)\frac{1-\sigma}{2} + g(1-0)\frac{1+\sigma}{2} + d \tag{4.12}$$

where σ is the function defined in (4.6) and d is a function on $\mathbb{S}^1$ which is continuous at 1 and vanishes there. Taking $g = \chi_+$, we get that

$$\begin{aligned} L(\chi_+) &= L\Big(\frac{1-\sigma}{2}\Big) + L(d) \\ &= E_{-n}\Big(\frac{1-S_{\mathbb{R}}}{2}\Big)E_n + L(d) \end{aligned}$$

and so $E_n L(\chi_+) E_{-n} = L_n\Big(\frac{1-S_{\mathbb{R}}}{2}\Big)L_n + E_n L(d) E_{-n}$ which converges strongly to $\frac{1-S_{\mathbb{R}}}{2}$.

Finally we approach $L(C_r^{-1}f)$ for $f \in PC^0_{\pm 1}$. Recall that $(C_r^{-1}f)(t) = f(\frac{t-r}{1-rt})$. Using a geometric series, we can write

$$\frac{t-r}{1-rt} = (t-r)\sum_{k=0}^{\infty}(rt)^k$$

and thus we can use Lemma 4.2.2 to write

$$E_n L(C_r^{-1} f) E_{-n} = L_n \mathcal{F} M\Big(f\big(\frac{e^{2\pi i \frac{x}{n}} - r}{1 - r e^{2\pi i \frac{x}{n}}} \big) \Big) \mathcal{F}^{-1}.$$

Focusing on the argument of f and using a Taylor series expansion, we have

$$\begin{aligned} \frac{e^{2\pi i \frac{x}{n}} - r}{1 - r e^{2\pi i \frac{x}{n}}} &= \frac{-r + 1 + \frac{2\pi i x}{n} + \cdots}{1 - r(1 + \frac{2\pi i x}{n} + \cdots)} \\ &= \frac{n(1-r) + 2\pi i x + \cdots}{n(1-r) - r 2\pi i x - \cdots} \\ &\to \frac{\lambda + 2\pi i x}{\lambda - 2\pi i x} \end{aligned}$$

Thus $E_n L(C_r^{-1} f) E_{-n} \to \mathcal{F} M\Big(f\Big(\frac{\lambda + 2\pi i x}{\lambda - 2\pi i x} \Big) \Big) \mathcal{F}^{-1} = W^0\Big(f\Big(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x} \Big) \Big)$, as desired. □

We are still aiming to prove that $\mathcal{A}$ is $*$-isomorphic to $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$. To do so, we will first prove a few more things regarding the Fourier transform that will aid us in computations when we define the isomorphism.

Lemma 4.2.7. *Let b be a Fourier-multiplier and let J be the operator defined by*

$$(Jb)(x) = b(-x) =: \tilde{b}(x).$$

Then

1. $\mathcal{F} M(b) = \mathcal{F}^{-1} J M(b)$ *and*

2. $M(b) \mathcal{F}^{-1} = J M(\tilde{b}) \mathcal{F}$.

Proof. For (1), let $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned}(\mathcal{F}^{-1}JM(b)f)(x) &= \int_{\mathbb{R}} e^{2\pi ixz} b(-z)f(-z)dz \\ &= -\int_{\infty}^{-\infty} e^{-2\pi ixy} b(y)f(y)dy \\ &= \int_{\mathbb{R}} e^{-2\pi ixy} b(y)f(y)dy \\ &= (\mathcal{F}M(b)f)(x)\end{aligned}$$

Assertion (2) can be proven analagously. □

Corollary 4.2.8. *Let b be a Fourier-multiplier. Then $\mathcal{F}M(b)\mathcal{F}^{-1} = W^0(\tilde{b})$.*

Proof. For a Fourier-multiplier b, we have

$$\begin{aligned}\mathcal{F}M(b)\mathcal{F}^{-1} &= \mathcal{F}^{-1}JM(b)\mathcal{F}^{-1} \\ &= \mathcal{F}^{-1}JJM(\tilde{b})\mathcal{F} \\ &= \mathcal{F}^{-1}M(\tilde{b})\mathcal{F} = W^0(\tilde{b})\end{aligned}$$

where the first equality is using (1) from Lemma 4.2.7 and the second equality is using (2) from this lemma. □

Lemma 4.2.9. *Let b be a Fourier multiplier. Then*

$$Z_t\mathcal{F}^{-1}M(b)\mathcal{F}Z_t^{-1} = \mathcal{F}^{-1}M(\widehat{b})\mathcal{F}$$

where $\widehat{b}(x) := b\left(\frac{x}{t}\right)$.

Proof. We will prove this by showing the following:

1. $Z_t \mathcal{F}^{-1} = \mathcal{F}^{-1} Z_t^{-1}$

2. $\mathcal{F} Z_t^{-1} = Z_t \mathcal{F}$

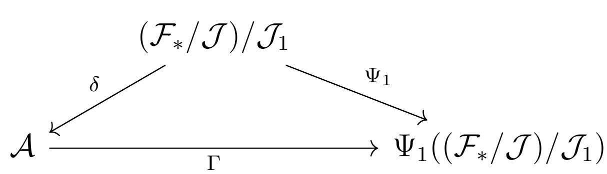
This will prove the lemma since $Z_t^{-1} M(b) Z_t = M(\widehat{b})$ with $\widehat{b}$ defined in the lemma statement. We will only prove (1) since the second is analagous. For a function f, we have

$$\begin{aligned}
(Z_t \mathcal{F}^{-1} f)(x) &= \sqrt{t}(\mathcal{F}^{-1} f)(tx) \\
&= \sqrt{t} \int_{\mathbb{R}} e^{2\pi i t x z} f(z) dz \\
&= \sqrt{t} \int_{\mathbb{R}} e^{2\pi i y x} f\left(\frac{y}{t}\right) \frac{dy}{t} \\
&= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{2\pi i y x} f\left(\frac{y}{t}\right) dy \\
&= (\mathcal{F}^{-1} Z_t^{-1} f)(x)
\end{aligned}$$

□

Theorem 4.2.10. *The algebra $\mathcal{A}$ is $*$-isomorphic to $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$.*

Proof. We will prove this by mapping the generators of $\mathcal{A}$ to the generators of $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ via a series of unitary transformations (which we will denote by Γ) and such that the following diagram commutes:



Recall that under δ, the generators of $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ are mapped as follows:

$$\begin{aligned}
(P) &\mapsto P & (L(C_r^{-1}f)) &\mapsto L(f) \\
(Q_n^-) &\mapsto L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)QL\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big) & (L(\chi_+)) &\mapsto L(\chi_+) \\
(P_n^+) &\mapsto L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)PL\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big) & (R_r^*KR_r) &\mapsto K
\end{aligned}$$

Define $F : L^2(\mathbb{S}^1) \to L^2(\mathbb{R})$ to be the operator taking $a \in L^2(\mathbb{S}^1)$ and sending it to the function $b \in L^2(\mathbb{R})$ defined by $b(x) = \frac{1}{1-ix}a(\frac{1+ix}{1-ix})$. We then have a mapping

$$\begin{aligned}
\Delta : \mathcal{L}(L^2(\mathbb{S}^1)) &\to \mathcal{L}(L^2(\mathbb{R})) \\
A &\to FAF^{-1}
\end{aligned} \tag{4.13}$$

This acts on the algebra $\mathcal{A}$ as follows: $P_{\mathbb{S}^1}$ gets sent to $P_{\mathbb{R}}$, $L(\chi_+)$ gets sent to $\chi_{[0,\infty)}$, $L(f)$ gets sent to $M(g)$ for $g(x) = f\big(\frac{1+ix}{1-ix}\big)$, $L\Big(\exp\Big(\lambda\frac{t-1}{1+t}\Big)\Big)$ gets sent to $M(e^{-\lambda ix})$, and $L\Big(\exp\Big(\lambda\frac{1-t}{1+t}\Big)\Big)$ gets sent to $M(e^{\lambda ix})$.

Next for each of the resulting operators A, we transform it via $\mathcal{F}A\mathcal{F}^{-1}$. Here we have

$$\begin{aligned}
\mathcal{F}P_{\mathbb{R}}\mathcal{F}^{-1} &= \mathcal{F}\frac{1+S_{\mathbb{R}}}{2}\mathcal{F}^{-1} \\
&= \frac{1+\operatorname{sgn}(x)}{2} \\
&= \chi_{[0,\infty)}
\end{aligned}$$

where we are using the representation in (4.5) in these equalities. For $M(\chi_{[0,\infty)})$ we get

$$\mathcal{F}M(\chi_{[0,\infty)})\mathcal{F}^{-1} = \mathcal{F}^{-1}M(\chi_{(-\infty,0)})\mathcal{F}$$

$$= Q_{\mathbb{R}}$$

where the first equality is making use of Corollary 4.2.8. This corollary also gives us that $\mathcal{F}M(g)\mathcal{F}^{-1} = W^0(\tilde{g})$ where $\tilde{g}(x) = g(-x) = f(\frac{1-ix}{1+ix})$.

Now, $\mathcal{F}M(e^{-\lambda ix})\mathcal{F}^{-1}$ and $\mathcal{F}M(e^{\lambda ix})\mathcal{F}^{-1}$ are actually shift operators; using Lemma 4.2.1 gives that they are equal to $U_{-\frac{\lambda}{2\pi}}$ and $U_{\frac{\lambda}{2\pi}}$ respectively. Thus $M(e^{-\lambda ix})PM(e^{\lambda ix})$ gets sent to $U_{-\frac{\lambda}{2\pi}}\chi_{[0,\infty)}U_{\frac{\lambda}{2\pi}} = \chi_{[-\frac{\lambda}{2\pi},\infty)}$ and similarly the term $M(e^{\lambda ix})QM(e^{-\lambda ix})$ gets sent to $\chi_{(-\infty,\frac{\lambda}{2\pi}]}$.

Finally, for the resulting operator A we transform it into the operator $Z_{\frac{\lambda}{2\pi}}AZ_{\frac{\lambda}{2\pi}}^{-1}$ where $Z_{\frac{\lambda}{2\pi}}$ refers to the shift operator Z_t defined in Equation (4.11) for $t = \frac{\lambda}{2\pi}$. Under this map, we have that $\chi_{[0,\infty)}$ is left unchanged, $\chi_{[-\frac{\lambda}{2\pi},\infty)}$ is sent to $\chi_{[-1,\infty)}$, and $\chi_{(-\infty,\frac{\lambda}{2\pi}]}$ is sent to $\chi_{(-\infty,1]}$.

The operator $Q_{\mathbb{R}}$ is left unchanged under this map as well: indeed, using its representation as a Fourier convolution and Lemma 4.2.9 we have

$$\begin{aligned} Z_{\frac{\lambda}{2\pi}}Q_{\mathbb{R}}Z_{\frac{\lambda}{2\pi}}^{-1} &= Z_{\frac{\lambda}{2\pi}}\mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x)}{2}\mathcal{F}Z_{\frac{\lambda}{2\pi}}^{-1} \\ &= \mathcal{F}^{-1}\frac{1-\operatorname{sgn}(\frac{2\pi}{\lambda}x)}{2}\mathcal{F} \\ &= \mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x)}{2}\mathcal{F} \\ &= Q_{\mathbb{R}}. \end{aligned}$$

Next, again using Lemma 4.2.9, we get that the Fourier convolution $W^0(\tilde{g})$ is mapped to $W^0(\hat{g})$ where $\hat{g}(x) = \tilde{g}(\frac{2\pi}{\lambda}x) = g(-\frac{2\pi}{\lambda}x) = f(\frac{1-i\frac{2\pi}{\lambda}x}{1+i\frac{2\pi}{\lambda}x}) = f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)$.

Lastly, we will consider the image of compacts under this sequence of transformations. To summarize, we have defined Γ from the start of the proof to be the map that sends A to $Z_{\frac{\lambda}{2\pi}}\mathcal{F}FAF^{-1}\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1}$. We are left with showing that for a compact operator K,

$$Z_{\frac{\lambda}{2\pi}}\mathcal{F}FKF^{-1}\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1} = \operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} E_n R_r^* K R_r E_{-n}.$$

We do this by proving

$$Z_{\frac{\lambda}{2\pi}}\mathcal{F}F = \sqrt{\pi}\operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} E_n R_r^*$$

and

$$F^{-1}\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1} = \frac{1}{\sqrt{\pi}}\operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} R_r E_{-n}.$$

To start, let $f \in L^2(\mathbb{S}^1)$. Then

$$\begin{aligned}
(Z_{\frac{\lambda}{2\pi}}\mathcal{F}Ff)(x) &= \sqrt{\frac{\lambda}{2\pi}}(\mathcal{F}Ff)\left(\frac{\lambda}{2\pi}x\right) \\
&= \sqrt{\frac{\lambda}{2\pi}}\int_{-\infty}^{\infty} e^{-2\pi i \frac{\lambda}{2\pi}xz}(Ff)(z)dz \\
&= \frac{\sqrt{2\lambda}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-i\lambda xz} f(\frac{1+iz}{1-iz})}{1-iz}dz \\
&= \sqrt{\pi}((\operatorname*{s-lim}_{\substack{n\to\infty\\ r\to 1}} E_n R_r^*)f)(x)
\end{aligned}$$

(to see the final equality, compare the expression to that in the statement of Theorem 4.2.4). To finish up, consider an arbitrary characteristic function $\chi_{[a,b]}$. We have

$$\begin{aligned}
(F^{-1}\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1}\chi_{[a,b]})(t) &= \frac{2}{1+t}(\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1}\chi_{[a,b]})\left(\frac{t-1}{i(t+1)}\right) \\
&= \frac{2}{1+t}\int_{-\infty}^{\infty} e^{2\pi i \frac{t-1}{i(t+1)}z}(Z_{\frac{\lambda}{2\pi}}^{-1}\chi_{[a,b]})(z)dz \\
&= \frac{2}{1+t}\int_{-\infty}^{\infty} e^{2\pi \frac{t-1}{t+1}z}\frac{\sqrt{2\pi}}{\sqrt{\lambda}}\chi_{[a,b]}\left(\frac{2\pi z}{\lambda}\right)dz
\end{aligned}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\lambda}} \frac{2}{1+t} \int_{\frac{\lambda a}{2\pi}}^{\frac{\lambda b}{2\pi}} e^{2\pi \frac{t-1}{t+1} z} dz$$

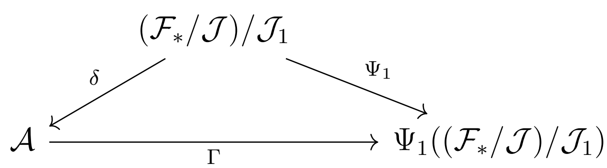
$$= \frac{1}{\sqrt{\pi}} \frac{e^{\lambda a \frac{t-1}{t+1}} - e^{\lambda b \frac{t-1}{t+1}}}{1-t}$$

$$= \frac{1}{\sqrt{\pi}} ((\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r E_{-n})\chi_{[a,b]})(t)$$

and thus our proof is complete.

□

Remark 4.2.11. In the preceding proof, for each $A \in \mathcal{A}$ we computed by hand the image $Z_{\frac{\lambda}{2\pi}} \mathcal{F} F A F^{-1} \mathcal{F}^{-1} Z_{\frac{\lambda}{2\pi}}^{-1}$. This allowed us to directly see that $\mathcal{A}$ is isomorphic to $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$. There is another way for us to see that the diagram



commutes using the strong convergence of $E_n R_r^*$ and $R_r E_{-n}$. Indeed, for $(F_{n,r}) \in (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ we have

$$\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n F_{n,r} E_{-n} = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n R_r^* R_r F_{n,r} R_r^* R_r E_{-n}$$

which can be written as

$$\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n R_r^* \left(\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r F_{n,r} R_r^* \right) \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r E_{-n}.$$

In the previous proof we showed that

$$Z_{\frac{\lambda}{2\pi}} \mathcal{F} F = \sqrt{\pi} \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n R_r^*$$

and

$$F^{-1}\mathcal{F}^{-1}Z_{\frac{\lambda}{2\pi}}^{-1} = \frac{1}{\sqrt{\pi}} \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r E_{-n}.$$

Thus

$$\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n F_{n,r} E_{-n} = Z_{\frac{\lambda}{2\pi}} \mathcal{F} F \left(\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r F_{n,r} R_r^* \right) F^{-1}\mathcal{F}^{-1} Z_{\frac{\lambda}{2\pi}}^{-1}$$

which is precisely what we saw. Even further, this tells us that the existence of the strong limit $\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_r F_{n,r} R_r^*$ automatically implies the existence of the strong limit $\operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n F_{n,r} E_{-n}$.

Remark 4.2.12. Recall the diagram

$$\begin{array}{ccc} (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\quad\tau\quad} & \mathcal{B}/\mathcal{N} \\ {\scriptstyle\delta}\big\downarrow & \swarrow_{\epsilon} & \\ \mathcal{A} & & \end{array}$$

from before. The work from this section and Theorem 4.2.10 permit us to draw

$$\begin{array}{ccc} (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\quad\tau\quad} & \mathcal{B}/\mathcal{N} \\ {\scriptstyle\delta}\big\downarrow & \swarrow_{\epsilon} & \\ \mathcal{A} & & \\ {\scriptstyle\cong}\big\updownarrow & & \\ \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) & & \end{array}$$

Since we have commutativity in the above diagram, we can define the map $\widehat{\Psi}_1 : \mathcal{B}/\mathcal{N} \to \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ so that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\quad\tau\quad} & \mathcal{B}/\mathcal{N} \\ {\scriptstyle\Psi_1}\big\downarrow & \swarrow_{\widehat{\Psi}_1} & \\ \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) & & \end{array}$$

4.3 Two New Algebras $\mathcal{C}$ and $\widehat{\mathcal{C}}$

As we did with our algebra $\mathcal{A}$, we will show that the local algebra $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ is also isomorphic to an algebra of operators on $L^2(\mathbb{R})$. In order to do this, we will need to introduce a new algebra $\mathcal{C}$ and a larger algebra $\widehat{\mathcal{C}}$ containing $\mathcal{C}$. Let $\mathcal{C}$ denote the algebra generated by the following elements:

$$\begin{aligned}\mathcal{C} := \operatorname{alg}_{\mathcal{L}(L^2(\mathbb{S}^1))} \quad & \big\{(P), (P_n^+), (Q_n^-), (L(\chi_+)), (L(\chi_-)), \\ & \big(E_{-n}K_1E_n), (Y_{-1}E_{-n}K_2E_nY_{-1}), \mathcal{N}\big\}\end{aligned}$$

where $K_1, K_2 \in \mathcal{K}(L^2(\mathbb{R}))$.

The algebra $\mathcal{C}/\mathcal{N}$ is a subalgebra of $\mathcal{F}_*/\mathcal{N}$ and hence the map Ψ_1 is also defined on this algebra. We have by construction a surjective map ι from $\mathcal{C}/\mathcal{N}$ into $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ that sends a sequence $(C_n)+\mathcal{N}$ to the sequence $(\{C_n\}+\mathcal{I})+\mathcal{I}_{-1}$. We remark that here we are using the generator $(L(C_r^{-1}f))$ with $f \in PC_{-1}^0$ to get that in $((\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1})/\mathcal{I}_{t_0}$ this term goes to zero.

Define the operator $\Psi_{-1} : \mathcal{C} \to \Psi_{-1}(\mathcal{C})$ by

$$\Psi_{-1}(C_n) := \operatorname*{s-lim}_{n\to\infty} E_nY_{-1}C_nY_{-1}E_{-n} \tag{4.14}$$

This map can also be defined on our algebra $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$; we will denote this by $\widehat{\Psi}_{-1}$. Explicitly,

$$\widehat{\Psi}_{-1}(B_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_nY_{-1}B_{n,r}Y_{-1}E_{-n} \tag{4.15}$$

In order to prove that this map is well-defined, we will make use of the following lemma.

Lemma 4.3.1. *The sequence of operators* $E_nY_{-1}E_{-n} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *converges weakly to zero.*

Proof. Since the sequence $E_nY_{-1}E_{-n}$ is uniformly bounded, it is enough to prove weak convergence on a dense subset. To this end, we prove $\langle E_nY_{-1}E_{-n}f, g\rangle_{L^2(\mathbb{R})} \to 0$ for characteristic functions $f = \chi_{[a,b]}$ and $g = \chi_{[c,d]}$. By definition,

$$E_nY_{-1}E_{-n}f = n\sum_{k\in\mathbb{Z}}[(-1)^k\int_{\mathbb{R}} f(x)\chi_{[\frac{k}{n},\frac{k+1}{n}]}(x)dx]\chi_{[\frac{k}{n},\frac{k+1}{n}]}.$$

Thus $\langle E_nY_{-1}E_{-n}\chi_{[a,b]}, \chi_{[c,d]}\rangle$ is equal to

$$\int_{\mathbb{R}} n\sum_{k\in\mathbb{Z}}[(-1)^k\int_{\mathbb{R}} \chi_{[a,b]}(x)\chi_{[\frac{k}{n},\frac{k+1}{n}]}(x)dx]\chi_{[\frac{k}{n},\frac{k+1}{n}]}(y)\chi_{[c,d]}(y)dy$$

which is equal to

$$\int_c^d n\sum_{k\in\mathbb{Z}}[(-1)^k\int_a^b \chi_{[\frac{k}{n},\frac{k+1}{n}]}(x)dx]\chi_{[\frac{k}{n},\frac{k+1}{n}]}(y)dy.$$

This is precisely

$$\begin{aligned}
&n\int_c^d \sum_{k=\lfloor na\rfloor+1}^{\lfloor bn\rfloor-1}[(-1)^k\frac{1}{n}]\chi_{[\frac{k}{n},\frac{k+1}{n}]}(y)dy \\
&+ n\int_c^d \left(\frac{\lfloor na\rfloor+1}{n} - a\right)(-1)^{\lfloor na\rfloor}\chi_{[\frac{\lfloor na\rfloor}{n},\frac{\lfloor na\rfloor+1}{n}]}(y)dy \\
&+ n\int_c^d \left(b - \frac{\lfloor nb\rfloor}{n}\right)(-1)^{\lfloor nb\rfloor}\chi_{[\frac{\lfloor nb\rfloor}{n},\frac{\lfloor nb\rfloor+1}{n}]}(y)dy.
\end{aligned}$$

For the second term, we have

$$\left| n\int_c^d (\frac{\lfloor na\rfloor+1}{n}-a)(-1)^{\lfloor na\rfloor}\chi_{[\frac{\lfloor na\rfloor}{n},\frac{\lfloor na\rfloor+1}{n}]}(y)dy\right| \le n\left(\frac{\lfloor na\rfloor+1}{n}-a\right)\frac{1}{n}$$
$$= \frac{\lfloor na\rfloor+1}{n}-a$$
$$= \frac{\lfloor na\rfloor-na+1}{n}$$
$$\le \frac{2}{n}$$

which goes to zero as n goes towards infinity. For the third term, we have

$$\left| n\int_c^d \left(b-\frac{\lfloor nb\rfloor}{n}\right)(-1)^{\lfloor nb\rfloor}\chi_{[\frac{\lfloor nb\rfloor}{n},\frac{\lfloor nb\rfloor+1}{n}]}(y)dy\right| \le n\left(b-\frac{\lfloor nb\rfloor}{n}\right)\frac{1}{n}$$
$$= \frac{bn-\lfloor nb\rfloor}{n}$$
$$\le \frac{1}{n}$$

which also tends to zero as n goes to infinity.

Finally we consider

$$n\int_c^d \sum_{k=\lfloor na\rfloor+1}^{\lfloor bn\rfloor-1}[(-1)^k\frac{1}{n}]\chi_{[\frac{k}{n},\frac{k+1}{n}]}(y)dy.$$

In this case, we are essentially left with considering the convergence of a series of the form

$$\sum_{k=\lfloor na\rfloor+1}^{\lfloor bn\rfloor-1}\frac{(-1)^k}{n}.$$

Actually, to be precise, one may consider separate cases based on how the interval $[c,d]$ intersects $\left[\frac{\lfloor na\rfloor+1}{n},\frac{\lfloor nb\rfloor}{n}\right]$ similar to as we have done before. We can then treat the edge cases as we did before and then the main sum will be like that written above with possibly different starting and ending values for k. But this sum converges to zero as n goes to infinity; indeed, we have the bound

$$-\frac{2}{n} \leq \sum_{k=\lfloor na \rfloor+1}^{\lfloor bn \rfloor -1} \frac{(-1)^k}{n} \leq \frac{2}{n}$$

and so by the Squeeze Theorem this goes to zero as n goes to infinity.

□

Theorem 4.3.2. *The map $\widehat{\Psi}_{-1}$ is a well-defined map on $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} (\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1} & \xrightarrow{\widehat{\Psi}_{-1}} & \\ \uparrow \iota & & \searrow \\ \mathcal{C}/\mathcal{N} & \xrightarrow[\Psi_{-1}]{} & \Psi_{-1}(\mathcal{C}) \end{array}$$

and the generators of $\mathcal{C}$ are mapped under Ψ_{-1} as follows:

$$\begin{array}{ccc} (P) \mapsto \chi_{[0,\infty)} & (P_n^+) \mapsto \chi_{[-1,\infty)} & (Q_n^-) \mapsto \chi_{(-\infty,1]} \\ (L(\chi_+)) \mapsto P_{\mathbb{R}} & (E_{-n}K_1E_n) \mapsto 0 & (Y_{-1}E_{-n}K_2E_nY_{-1}) \mapsto K_2 \end{array}$$

Proof. We start by showing that this map is well-defined. The fact that $\mathcal{N}$ is in the kernel follows from the uniform boundedness of E_nY_{-1} and $Y_{-1}E_{-n}$. For $(L(f))$ with f continuous on $\mathbb{S}^1$ and vanishing at -1, we can use the representation in Equation (4.12) with $g = Y_{-1}f$ to write

$$Y_{-1}f = f(-1)\frac{1-\sigma}{2} + f(-1)\frac{1+\sigma}{2} + d = d$$

where d is a function on $\mathbb{S}^1$ that is continuous and vanishing at 1. Thus

$$E_nY_{-1}L(f)Y_{-1}E_{-n} = E_nL(d)E_{-n}$$

which converges strongly to zero. The last thing to check is that the term $E_nY_{-1}R_r^*KR_rY_{-1}E_{-n}$ converges strongly to zero for K compact. Due to the strong convergence of $E_nR_r^*$ and R_rE_{-n}, it follows that $E_nR_r^*KR_rE_{-n}$ converges in norm to some other compact operator K'. Writing

$$E_nY_{-1}R_r^*KR_rY_{-1}E_{-n} = E_nY_{-1}E_{-n}E_nR_r^*KR_rE_{-n}E_nY_{-1}E_{-n}$$

we get that mod $\mathcal{N}$

$$E_nY_{-1}R_r^*KR_rY_{-1}E_{-n} = E_nY_{-1}E_{-n}K'E_nY_{-1}E_{-n}.$$

Now since $\mathcal{N}$ is in the kernel of $\widehat{\Psi}_{-1}$ and $E_nY_{-1}E_{-n}$ converges weakly to zero (Lemma 4.3.1), it follows that this element converges strongly to zero.

The fact that the diagram commutes is clear; it follows directly from the fact that the map ι is essentially an inclusion and that $\mathcal{I}$ and $\mathcal{I}_{-1}$ are in the kernel of $\widehat{\Psi}_{-1}$.

To round out the proof, we will compute the images of Ψ_{-1} when acting on C. Notice that since $Y_{-1}PY_{-1} = P$, the image of (P) under Ψ_{-1} is the same as its image under Ψ_1. For (P_n^+), we have

$$Y_{-1}P_n^+Y_{-1} = Y_{-1}L(t^{-n})PL(t^n)Y_{-1} = L((-t)^{-n})PL((-t)^n) = L(t^{-n})PL(t^n) = P_n^+$$

and so the image of (P_n^+) under Ψ_{-1} is also the same as its image under Ψ_1. Similarly, (Q_n^-) has the same image under Ψ_1 and Ψ_{-1}.

For $(L(\chi_+))$, we have $Y_{-1}L(\chi_+)Y_{-1} = L(Y_{-1}\chi_+)$. We can use the representation in Equation (4.12) with $g = Y_{-1}\chi_+$ to write

$$Y_{-1}\chi_+ = \chi_+(-1+0)\frac{1-\sigma}{2} + \chi_+(-1-0)\frac{1+\sigma}{2} + d$$

where σ is the function defined in (4.6) and d is a function on $\mathbb{S}^1$ that is continuous and vanishing at 1. Thus

$$Y_{-1}L(\chi_+)Y_{-1} = L\left(\frac{1+\sigma}{2}\right) + L(d)$$

and so $E_n Y_{-1} L(\chi_+) Y_{-1} E_{-n} = L_n \left(\frac{1+S_{\mathbb{R}}}{2}\right) L_n + E_n L(d) E_{-n}$ which converges strongly to $\frac{1+S_{\mathbb{R}}}{2} = P_{\mathbb{R}}$.

Next we consider the image of $(E_{-n}K_1E_n)$ under Ψ_{-1}. For this, we aim to consider the strong limit $E_nY_{-1}E_{-n}K_1E_nY_{-1}E_{-n}$ but as we have already seen in this proof, this converges strongly to zero by Lemma 4.3.1.

Finally, for $(Y_{-1}E_{-n}K_2E_nY_{-1}) = A$, we have $E_nY_{-1}AY_{-1}E_{-n} = L_nK_2L_n$ which converges strongly to K_2.

$\square$

We will see that $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ is actually $*$-isomorphic to $\Psi_{-1}(\mathcal{C})$ via the mapping $\widehat{\Psi}_{-1}$. Surjectivity is clear, but to prove injectivity we will first need to identify the kernel of Ψ_{-1} when acting on $\mathcal{C}/\mathcal{N}$. In order to do this, we will need to develop stability criteria for $\mathcal{C}$.

We define $C(\dot{\mathbb{R}})$ to be the Banach algebra of all continuous functions f on the real line $\mathbb{R}$ possessing finite limits $f(+\infty)$ and $f(-\infty)$ such that $f(+\infty) = f(-\infty)$. Let $\widehat{\mathcal{C}}$ denote the algebra generated by the following elements:

$$\widehat{\mathcal{C}} := \mathrm{alg}_{\mathcal{L}(L^2(\mathbb{S}^1))} \quad \big\{(P), (P_n^+), (Q_n^-), (L(\chi_+)), (L(f)), (E_{-n}K_1E_n), \\ \big(Y_{-1}E_{-n}K_2E_nY_{-1}), (E_{-n}M(g)E_n), (I)\big\}$$

where K_1, K_2 are compact, $f \in PC^0_{\pm 1}$, and $g \in C(\dot{\mathbb{R}})$. It can be straightforwardly checked that our operators Ψ_1 and Ψ_{-1} can be extended to all of $\widehat{\mathcal{C}}$; one need only verify that they are defined for the element $(E_{-n}M(g)E_n)$ for $g \in C(\dot{\mathbb{R}})$. Notice that our algebra $\mathcal{C}$ is a subalgebra of $\widehat{\mathcal{C}}$, meaning that we can specialize the stability criteria for $\widehat{\mathcal{C}}$ to $\mathcal{C}$. Define the set $\mathcal{J}'$ by

$$\mathcal{J}' = \{(C_n + E_{-n}K_1E_n + Y_{-1}E_{-n}K_2E_nY_{-1}) : C_n \in \mathcal{N}, K_1, K_2 \in \mathcal{K}\}$$

This forms an ideal of $\widehat{\mathcal{C}}$ and we have the following lifting theorem:

Theorem 4.3.3. *(Lifting Theorem for $\widehat{\mathcal{C}}$) Let $(C_n) \in \widehat{\mathcal{C}}$. The following are equivalent:*

(a) (C_n) is stable

(b) $(C_n) + \mathcal{N}$ is invertible in $\widehat{\mathcal{C}}/\mathcal{N}$

(c) $\Psi_1(C_n)$ and $\Psi_{-1}(C_n)$ are invertible in $\mathcal{L}(L^2(\mathbb{R}))$ and $(C_n) + \mathcal{J}'$ is invertible in $\widehat{\mathcal{C}}/\mathcal{J}'$.

Proof. This can be proven in the same way as Theorem 3.1.8. □

As we have done many times before, we will now localize over a central subalgebra of $\widehat{\mathcal{C}}/\mathcal{J}'$. To help us prove a particular set is a central subalgebra, we start with the following lemma. An alternative proof can be found in [9], Lemma 3.2.

Lemma 4.3.4. *Let $g \in C(\dot{\mathbb{R}})$. Then $L_n M(g) L_n - M(g) L_n \in \mathcal{N}$.*

Proof. By the uniform boundedness of L_n, we can use an approximation argument. Let g be a smooth, compactly supported function and let $f \in L^2(\mathbb{R})$. By definition,

$$L_n M(g) L_n f = n^2 \sum_{k=-\infty}^{\infty} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} g(y)dy \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right] \chi_{[\frac{k}{n}, \frac{k+1}{n}]}.$$

By the Mean Value Theorem for integrals, for each $k \in \mathbb{Z}$ there exists a point $x_k \in [\frac{k}{n}, \frac{k+1}{n}]$ such that $g(x_k) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(y)dy$. Thus

$$L_n M(g) L_n f = n^2 \sum_{k=-\infty}^{\infty} \left[\frac{1}{n} g(x_k) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right] \chi_{[\frac{k}{n}, \frac{k+1}{n}]}.$$

Also by definition, we have

$$M(g) L_n f = gn \sum_{k=-\infty}^{\infty} \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right) \chi_{[\frac{k}{n}, \frac{k+1}{n}]}.$$

Then

$$L_n M(g) L_n f - M(g) L_n f = n \sum_{k=-\infty}^{\infty} \left[(g(x_k) - g) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right] \chi_{[\frac{k}{n}, \frac{k+1}{n}]}.$$

Thus, setting $(\star) = L_n M(g) L_n f - M(g) L_n f$, we have

$$\begin{aligned} \|(\star)\|^2 &= \int_{\mathbb{R}} \left(n \sum_{k=-\infty}^{\infty} \left[(g(x_k) - g) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right] \chi_{[\frac{k}{n}, \frac{k+1}{n}]}(y) \right)^2 dy \\ &= \int_{\mathbb{R}} n^2 \sum_{k \in \mathbb{Z}} |g - g(x_k)|^2 \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right|^2 \chi_{[\frac{k}{n}, \frac{k+1}{n}]}(y)dy \end{aligned}$$

$$= n^2 \sum_{k\in\mathbb{Z}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |g - g(x_k)|^2 \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right|^2 dy$$
$$= n \sum_{k\in\mathbb{Z}} |g - g(x_k)|^2 \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right|^2 .$$

Notice that

$$\begin{aligned}
\left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right|^2 &\leq \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left| f(x) \right| dx \right)^2 \\
&= \left(\int_{\mathbb{R}} \left| f(x)\chi_{[\frac{k}{n},\frac{k+1}{n}]}(x) \right| dx \right)^2 \\
&\leq \|f\|_2^2 \|\chi_{[\frac{k}{n},\frac{k+1}{n}]}\|_2^2 \\
&= \frac{1}{n^2}\|f\|_2^2
\end{aligned}$$

where we have made use of Hölder's inequality for the second inequality. Thus

$$\begin{aligned}
\|L_n M(g) L_n f - M(g) L_n f\|^2 &= n \sum_{k\in\mathbb{Z}} |g - g(x_k)|^2 \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \right|^2 \\
&\leq \frac{1}{n}\|f\|_2^2 \sum_{k\in\mathbb{Z}} |g - g(x_k)|^2 \\
&\leq \frac{1}{n}\|f\|_2^2 \sum_{k\in\mathbb{Z}} \left(\frac{1}{n}\right)^2 \\
&\leq \frac{1}{n^2}\|f\|_2^2
\end{aligned}$$

where the second to last inequality is making use of the smoothness of g to get a uniform bound on $|g - g(x_k)|^2$ and the final inequality is making use of the fact g is compactly supported. Since this final expression goes to zero as n goes to infinity, our proof is complete. □

Lemma 4.3.5. *The set* $\mathcal{D}_3 := \{(E_{-n}M(g)E_n) + \mathcal{J}' : g \in C(\dot{\mathbb{R}})\}$ *is a central subalgebra of* $\widehat{\mathcal{C}}/\mathcal{J}'$*. Moreover,* $\mathcal{D}_3$ *is isomorphic to* $C(\dot{\mathbb{R}})$*.*

Proof. We start by proving that this is a central subalgebra. We first show that $(E_{-n}M(g)E_n)$ commutes with $(P), (P_n^+)$, and (Q_n^-) mod $\mathcal{J}'$. We will make use of the equalities $E_nPE_{-n} = L_n\chi_{[0,\infty)}$, $E_nP_n^+E_{-n} = L_n\chi_{[-1,\infty)}$, and $E_nQ_n^-E_{-n} = L_n\chi_{(-\infty,1]}$. Let's start with (P). Observe that

$$\begin{aligned} PE_{-n}M(g)E_n - E_{-n}M(g)E_nP &= E_{-n}E_nPE_{-n}M(g)E_n - E_{-n}M(g)E_nPE_{-n}E_n \\ &= E_{-n}L_n\chi_{[0,\infty)}M(g)E_n - E_{-n}M(g)L_n\chi_{[0,\infty)}E_n \\ &= E_{-n}L_nM(g)\chi_{[0,\infty)}E_n - E_{-n}M(g)\chi_{[0,\infty)}L_nE_n \\ &= E_{-n}M(g)\chi_{[0,\infty)}E_n - E_{-n}M(g)\chi_{[0,\infty)}E_n \\ &= 0. \end{aligned}$$

This same argument holds for (P_n^+) and (Q_n^-); we just change $\chi_{[0,\infty)}$ to $\chi_{[-1,\infty)}$ and $\chi_{(-\infty,1]}$ respectively.

Next we prove that $(E_{-n}M(g)E_n)$ commutes with $L(t)$, which will imply that it commutes with Laurent operators with continuous symbols. We will then use this to show that it commutes with $L(f)$ for $f \in PC_{\pm 1}$. Consider

$$L(t)E_{-n}M(g)E_nL(t^{-1}) - E_{-n}M(g)E_n.$$

To show that this is in $\mathcal{N}$, it will be convenient to think of the operator $E_{-n}M(g)E_n$

as a diagonal matrix (g_{ij}) whose entries are

$$g_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ n\int_{\frac{j}{n}}^{\frac{j+1}{n}} g(x)dx & \text{if } i = j \end{cases}$$

(to see this, we let $E_{-n}M(g)E_n$ act on the basis element e_j whose j^{th} entry is 1 and has zeros everywhere else). Then as a matrix, $L(t)E_{-n}M(g)E_nL(t^{-1})$ is also diagonal but with the entries shifted; i.e., if $L(t)E_{-n}M(g)E_nL(t^{-1}) = (a_{ij})$ then $a_{ij} = g_{(i-1)(j-1)}$. The difference $L(t)E_{-n}M(g)E_nL(t^{-1}) - E_{-n}M(g)E_n$ is then a diagonal matrix with diagonal entries given by

$$n\int_{\frac{j-1}{n}}^{\frac{j}{n}} g(x)dx - n\int_{\frac{j}{n}}^{\frac{j+1}{n}} g(x)dx = n\int_{\frac{j}{n}}^{\frac{j+1}{n}} g\left(x - \frac{1}{n}\right) - g(x)dx.$$

But this goes uniformly to zero due to the uniform continuity of g and so

$$L(t)E_{-n}M(g)E_nL(t^{-1}) - E_{-n}M(g)E_n \in \mathcal{N}.$$

Next we show that $(E_{-n}M(g)E_n)$ commutes with $L(f)$ for $f \in PC_{\pm 1}$. For such f we have the representation

$$f = \alpha\sigma + \beta Y_{-1}\sigma + d$$

where σ is given in Equation 4.6 and α and β are chosen so that the function d is continuous on all of $\mathbb{S}^1$ (they will depend on the values of $f(1 \pm 0)$ and $f(-1 \pm 0)$). Thus, since we have already handled the continuous symbol case, our consideration of

$$L(f)E_{-n}M(g)E_n - E_{-n}M(g)E_nL(f)$$

can be reduced to examining

$$L(\sigma)E_{-n}M(g)E_n - E_{-n}M(g)E_nL(\sigma).$$

Notice that since $L(\sigma) = E_{-n}S_{\mathbb{R}}E_n$, we have the term

$$L(\sigma)E_{-n}M(g)E_n - E_{-n}M(g)E_nL(\sigma)$$

is equal to

$$E_{-n}L_nS_{\mathbb{R}}L_nM(g)E_n - E_{-n}M(g)L_nS_{\mathbb{R}}L_nE_n.$$

We have also, mod $\mathcal{N}$, the equality

$$E_{-n}L_nS_{\mathbb{R}}L_nM(g)E_n - E_{-n}M(g)L_nS_{\mathbb{R}}L_nE_n = E_{-n}S_{\mathbb{R}}M(g)E_n - E_{-n}M(g)S_{\mathbb{R}}E_n$$

due to Lemma 4.3.4. This in turn is equal to $E_{-n}(S_{\mathbb{R}}M(g) - M(g)S_{\mathbb{R}})E_n$. Thus if we can show $S_{\mathbb{R}}M(g) - M(g)S_{\mathbb{R}}$ is compact our work will be done since $E_{-n}KE_n \in \mathcal{J}'$. That $S_{\mathbb{R}}M(g) - M(g)S_{\mathbb{R}}$ is compact can be seen in several ways; one way is to note that under each of the homomorphisms given by the Fredholm Theory in Appendix A.1, Equations A.1, A.2, and A.3, $M(g)$ is sent to a scalar multiple of the identity since $g \in C(\dot{\mathbb{R}})$ and so $S_{\mathbb{R}}M(g) - M(g)S_{\mathbb{R}}$ is sent to zero under all of these homomorphisms and hence is compact.

The final thing to check is that for $f, g \in C(\dot{\mathbb{R}})$, the term $E_{-n}M(g)E_n$ commutes mod $\mathcal{J}'$ with $E_{-n}M(f)E_n$. For this we have

$$\begin{aligned} E_{-n}M(g)E_nE_{-n}M(f)E_n &= E_{-n}M(g)L_nM(f)E_n \\ &= E_{-n}M(g)L_nM(f)L_nE_n \end{aligned}$$

$$
\begin{aligned}
&= E_{-n}M(g)M(f)L_nE_n \\
&= E_{-n}M(f)M(g)L_nE_n \\
&= E_{-n}M(f)L_nM(g)L_nE_n \\
&= E_{-n}M(f)E_nE_{-n}M(g)E_n
\end{aligned}
$$

where these equalities are holding mod $\mathcal{N}$ by Lemma 4.3.4. Thus $E_{-n}M(g)E_n$ commutes mod $\mathcal{J}'$ with $E_{-n}M(f)E_n$.

We now prove that $\mathcal{D}_3$ is $*$-isomorphic to $C(\dot{\mathbb{R}})$. We will show that the map

$$\Gamma : C(\dot{\mathbb{R}}) \to \mathcal{D}_3$$

defined by $\Gamma(f) = (E_{-n}M(f)E_n) + \mathcal{J}'$ is a $*$-isomorphism.

We first check the properties of a $*$-isomorphism. For additivity, we have

$$E_{-n}M(f+g)E_n = E_{-n}(M(f)+M(g))E_n = E_{-n}M(f)E_n + E_{-n}M(g)E_n$$

as needed. We also have $M(f^*) = M(f)^*$, which when paired with the fact $(E_{-n})^* = E_n$ and $(E_n)^* = E_{-n}$ gives $\Gamma(f^*) = \Gamma(f)^*$. For multiplicativity, we want to show that $\Gamma(fg) - \Gamma(f)\Gamma(g) \in \mathcal{J}'$. But we have already seen this when we were showing $E_{-n}M(g)E_n$ commutes mod $\mathcal{J}'$ with $E_{-n}M(f)E_n$.

All that remains is checking injectivty and continuity. To this end, suppose $f \in \ker \Gamma$. Then $(M(f)) \in \mathcal{J}'$, meaning

$$(E_{-n}M(f)E_n) = (C_n + E_{-n}K_1E_n + Y_{-1}E_{-n}K_2E_nY_{-1})$$

where $C_n \in \mathcal{N}, K_1, K_2 \in \mathcal{K}$. Then

$$\begin{aligned} M(f) &= \operatorname*{s-lim}_{n\to\infty} E_n E_{-n} M(f) E_n E_{-n} \\ &= \operatorname*{s-lim}_{n\to\infty} E_n (C_n + E_{-n} K_1 E_n + Y_{-1} E_{-n} K_2 E_n Y_{-1}) E_{-n} \\ &= K_1. \end{aligned}$$

Since the only compact multiplication operator is zero, this implies $f = 0$ and so Γ is injective. Because we are working with C^*-algebras, this also implies that Γ is an isometry and is thus continuous.

□

For $x \in \dot{\mathbb{R}}$, we denote by $\mathcal{J}'_x$ the smallest closed ideal of $\widehat{\mathcal{C}}/\mathcal{J}'$ containing $(E_{-n}M(g)E_n) + \mathcal{J}'$ where $g \in C(\dot{\mathbb{R}})$ vanishes at x; i.e.,

$$\mathcal{J}'_x = \text{clos id}_{\widehat{\mathcal{C}}/\mathcal{J}'}\{(E_{-n}M(g)E_n) + \mathcal{J}' : g \in C(\dot{\mathbb{R}}), g(x) = 0\}.$$

Corollary 4.3.6. *Let $(C_n) \in \widehat{\mathcal{C}}$. Then $(C_n) + \mathcal{J}'$ is invertible in $\widehat{\mathcal{C}}/\mathcal{J}'$ if and only if $(C_n + \mathcal{J}') + \mathcal{J}'_x$ is invertible in $(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$ for all $x \in \dot{\mathbb{R}}$.*

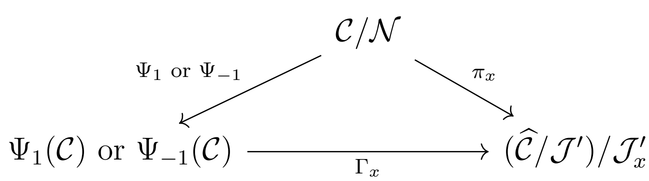
Proof. On account of Lemma 4.3.5, we can employ Theorem 3.2.1 in the setting $\mathcal{A} = \widehat{\mathcal{C}}/\mathcal{J}'$ and $Z = \mathcal{D}_3$. As we have seen before, since $\mathcal{D}_3 \cong C(\dot{\mathbb{R}})$, their maximal ideal spaces are homeomorphic − that is, the maximal ideal space of $\mathcal{D}_3$ is homemorphic to $\dot{\mathbb{R}}$. Putting all of this together gives the claim. □

Altogether, we have shown that an element $(C_n) \in \widehat{\mathcal{C}}$ is stable if and only if its images under Ψ_1 and Ψ_{-1} are invertible and its cosets $(C_n + \mathcal{J}') + \mathcal{J}'_x$ are invertible in $(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$ for all $x \in \dot{\mathbb{R}}$. But we are really only interested in stability for the algebra $\mathcal{C}$, so we proceed as follows: since $\mathcal{C}$ is a $*$-subalgebra of $\widehat{\mathcal{C}}$ and

C^*-algebras are inverse closed, we can apply the stability criteria for $\widehat{\mathcal{C}}$ to $\mathcal{C}$. We will see that for elements in $\mathcal{C}$, invertibility of their images in each of the local algebras $(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$ is implied by invertibility of their image under Ψ_1 and Ψ_{-1}, thus reducing stability in $\mathcal{C}$ to invertibility of two operators.

Theorem 4.3.7. *Let $(C_n) \in \mathcal{C}$. Then invertibility of $\Psi_1(C_n)$ and invertibility of $\Psi_{-1}(C_n)$ imply invertibility of the coset $(C_n + \mathcal{J}') + \mathcal{J}'_x$ for all $x \in \dot{\mathbb{R}}$.*

Proof. To prove this statement, for each $x \in \dot{\mathbb{R}}$ we will construct a homomorphism Γ_x from either $\Psi_1(\mathcal{C})$ or $\Psi_{-1}(\mathcal{C})$ into $(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$ such that the following diagram commutes:



So that it is clear what we need these homomorphisms to do, the image of each generator of $\mathcal{C}/\mathcal{N}$ in $\Psi_{-1}(\mathcal{C})$ and each of the $(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$ are summarized in the following table.

		$(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$			
$\mathcal{C}/\mathcal{N}$	Ψ_{-1}	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$
(P)	$\chi_{[0,\infty)}$	(0)	(0)	(0)	(P)
(P_n^+)	$\chi_{[-1,\infty)}$	(0)	(P_n^+)	(I)	(I)
(Q_n^-)	$\chi_{(-\infty,1]}$	(I)	(I)	(I)	(I)
$(L(\chi_+))$	$P_{\mathbb{R}}$	$(L(\chi_+))$	$(L(\chi_+))$	$(L(\chi_+))$	$(L(\chi_+))$
$(E_{-n}K_1E_n)$	0	(0)	(0)	(0)	(0)
$(Y_{-1}E_{-n}K_2E_nY_{-1})$	K_2	(0)	(0)	(0)	(0)

Table 4.2: Images of Generators of $\mathcal{C}$ in the Local Algebras for $x \leq 0$

		$(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$			
$\mathcal{C}/\mathcal{N}$	Ψ_{-1}	$0 < x < 1$	$x = 1$	$x > 1$	$x = \infty$
(P)	$\chi_{[0,\infty)}$	(I)	(I)	(I)	(P)
(P_n^+)	$\chi_{[-1,\infty)}$	(I)	(I)	(I)	(P)
(Q_n^-)	$\chi_{(-\infty,1]}$	(I)	(Q_n^-)	(0)	(Q)
$(L(\chi_+))$	$P_{\mathbb{R}}$	$(L(\chi_+))$	$(L(\chi_+))$	$(L(\chi_+))$	$(L(\chi_+))$
$(E_{-n}K_1E_n)$	0	(0)	(0)	(0)	(0)
$(Y_{-1}E_{-n}K_2E_nY_{-1})$	K_2	(0)	(0)	(0)	(0)

Table 4.3: Images of Generators of $\mathcal{C}$ in the Local Algebras for $x > 0$

Note that the only difference for Ψ_1 is $(L(\chi_+))$ is mapped to $Q_{\mathbb{R}}$, $(E_{-n}K_1E_n)$ is mapped to K_1, and $(Y_{-1}E_{-n}K_2E_nY_{-1})$ is mapped to zero. Thus when we construct the maps Γ_x, all requirements will be the same except we have the choice of whether we wish to work with $P_{\mathbb{R}}$ or $Q_{\mathbb{R}}$. It will be made explicit which we are using in each case. Throughout our constructions, we will make use of the shift operators U_t and Z_τ defined in equations (4.10) and (4.11) for particular values of t.

Case 1: $x < -1$.

In this case we work with Ψ_{-1} and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (0), \quad \chi_{[-1,\infty)} \mapsto (0), \quad \chi_{(-\infty,1]} \mapsto (I), \quad P_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0).$$

This will be easier if we transform the sequences on $\mathcal{L}(L^2(\mathbb{S}^1))$ to be sequences on $\mathcal{L}(L^2(\mathbb{R}))$ via the transformation Δ defined in (4.13). From this point of view we now need to map $P_{\mathbb{R}}$ to $M(\chi_{[0,\infty)})$. We start by taking an operator A and sending it to the sequence $(\operatorname{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_2AU_{-2}Z_\tau)$. Under this, we have

$$\chi_{[0,\infty)} \mapsto (0), \quad \chi_{[-1,\infty)} \mapsto (0), \quad \chi_{(-\infty,1]} \mapsto (I), \quad P_{\mathbb{R}} \mapsto (P_{\mathbb{R}}), \quad K \mapsto (0)$$

and so now we just need a map that send $P_{\mathbb{R}}$ to $M(\chi_{[0,\infty)})$. But we can do this using the Fourier transform; indeed, recall the representation

$$P_{\mathbb{R}} = \mathcal{F}^{-1}\frac{1+\operatorname{sgn}(x)}{2}\mathcal{F}$$

given in (4.5). By sending a sequence (A) to $(\mathcal{F}A\mathcal{F}^{-1})$, this sends $(P_{\mathbb{R}})$ to $M\left(\left(\frac{1+\operatorname{sgn}(x)}{2}\right)\right) = (M(\chi_{[0,\infty)}))$ and so our job is done. All in all, the map Γ_x is the map defined by

$$\Gamma_x : A \mapsto \left(\left(\Delta^{-1}\mathcal{F}\left(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_2AU_{-2}Z_\tau\right)\mathcal{F}^{-1}\right) + \mathcal{J}'\right) + \mathcal{J}'_x.$$

<u>Case 2:</u> $x = -1$.

In this case we work with Ψ_1 and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (0), \quad \chi_{[-1,\infty)} \mapsto (P_n^+), \quad \chi_{(-\infty,1]} \mapsto (I), \quad Q_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

We start by sending an operator A to the sequence $\left(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_1AU_{-1}Z_\tau\right)$. Under this, we have

$$\chi_{[0,\infty)} \mapsto (0), \quad \chi_{[-1,\infty)} \mapsto (\chi_{[0,\infty)}), \quad \chi_{(-\infty,1]} \mapsto (I), \quad Q_{\mathbb{R}} \mapsto (Q_{\mathbb{R}}), \quad K \mapsto (0)$$

Next we take the resulting sequence (A) and map it to $\left(\mathcal{F}^{-1}A\mathcal{F}\right)$. Here we automatically have $(\chi_{[0,\infty)})$ being mapped to $(P_{\mathbb{R}})$ since

$$P_{\mathbb{R}} = \mathcal{F}^{-1}\frac{1+\operatorname{sgn}(x)}{2}\mathcal{F}.$$

We also have

$$\mathcal{F}^{-1}Q_{\mathbb{R}}\mathcal{F} = \mathcal{F}^{-1}\mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x)}{2}\mathcal{F}\mathcal{F}$$

$$= \mathcal{F}^{-1}\mathcal{F}^{-1}\chi_{(-\infty,0]}\mathcal{F}\mathcal{F}$$

$$= \mathcal{F}^{-1}\mathcal{F}\chi_{[0,\infty)}\mathcal{F}^{-1}\mathcal{F}$$

$$= \chi_{[0,\infty)}$$

where we made use of Corolllary 4.2.8 in the second to last equality. Finally, we take these sequences of operators in $\mathcal{L}(L^2(\mathbb{R}))$ and send them to sequences of operators in $\mathcal{L}(L^2(\mathbb{S}^1))$ via the map Δ^{-1} as we did in Case 1. This sends $(P_\mathbb{R})$ to $(P_{\mathbb{S}^1})$ and $(M(\chi_{[0,\infty)}))$ to $(L(\chi_+))$. To finish up, we take the sequence of operators (A) and send them to $(U_{-n}AU_n)$ which sends (P) to (P_n^+) and leaves $(L(\chi_+))$ invariant.

<u>Case 3:</u> $-1 < x < 0$.

In this case we work with Ψ_{-1} and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (0), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (I), \quad P_\mathbb{R} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

For this we can define the map

$$\Gamma_x : A \mapsto \left(\left(\Delta^{-1}\mathcal{F}\left(\underset{\tau\to\infty}{\text{s-lim}}\ Z_\tau^{-1}U_{\frac{1}{2}}AU_{-\frac{1}{2}}Z_\tau\right)\mathcal{F}^{-1}\right) + \mathcal{J}'\right) + \mathcal{J}'_x$$

(the reasoning is nearly identical to that of Case 1).

<u>Case 4:</u> $x = 0$.

In this case we work with Ψ_1 and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (P), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (I), \quad Q_\mathbb{R} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

We start by sending an operator A to the sequence $\left(\underset{\tau\to\infty}{\text{s-lim}}\ Z_\tau^{-1}AZ_\tau\right)$. Under this, we have

$$\chi_{[0,\infty)} \mapsto (\chi_{[0,\infty)}), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (I), \quad Q_\mathbb{R} \mapsto (Q_\mathbb{R}), \quad K \mapsto (0)$$

Next we take the resulting sequence (A) and map it to $\big(\mathcal{F}^{-1}A\mathcal{F}\big)$. As we saw in Case 2, this sends $(\chi_{[0,\infty)})$ to $(P_{\mathbb{R}})$ and $(Q_{\mathbb{R}})$ to $\big(\chi_{[0,\infty)}\big)$. Taking these sequences of operators in $\mathcal{L}(L^2(\mathbb{R}))$ and sending them to sequences of operators in $\mathcal{L}(L^2(\mathbb{S}^1))$ via the map Δ^{-1} as we did in Case 1 finishes the job.

Case 5: $0 < x < 1$.

In this case we work with Ψ_{-1} and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (I), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (I), \quad P_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

For this we can define the map

$$\Gamma_x : A \mapsto \left(\left(\Delta^{-1}\mathcal{F}\left(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_{-\frac{1}{2}}AU_{+\frac{1}{2}}Z_\tau\right)\mathcal{F}^{-1}\right) + \mathcal{J}'\right) + \mathcal{J}'_x$$

(the reasoning is nearly identical to that of Case 1).

Case 6: $x = 1$.

In this case we work with Ψ_1 and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (I), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (Q_n^-), \quad Q_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

We start by sending an operator A to the sequence $\left(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_{-1}AU_1Z_\tau\right)$. Under this, we have

$$\chi_{[0,\infty)} \mapsto (I), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (\chi_{(-\infty,0]}), \quad Q_{\mathbb{R}} \mapsto (Q_{\mathbb{R}}), \quad K \mapsto (0)$$

Next we take the resulting sequence (A) and map it to $\big(\mathcal{F}^{-1}A\mathcal{F}\big)$. This sends $(\chi_{(-\infty,0]})$ to $(Q_{\mathbb{R}})$ and $(Q_{\mathbb{R}})$ to $\big(\chi_{[0,\infty)}\big)$. Taking these sequences of operators in $\mathcal{L}(L^2(\mathbb{R}))$ and sending them to sequences of operators in $\mathcal{L}(L^2(\mathbb{S}^1))$ via the map Δ^{-1} as we did in Case 1 sends $(Q_{\mathbb{R}})$ to $(Q_{\mathbb{S}^1})$ and $\big(\chi_{[0,\infty)}\big)$ to $(L(\chi_+))$. Finally we take the sequence (A) and map it to (U_nAU_{-n}) to complete this case.

Case 7: $x > 1$.

In this case we work with Ψ_{-1} and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (I), \quad \chi_{[-1,\infty)} \mapsto (I), \quad \chi_{(-\infty,1]} \mapsto (0), \quad P_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

For this we can define the map

$$\Gamma_x : A \mapsto \left(\left(\Delta^{-1} \mathcal{F} \left(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1} U_{-2} A U_2 Z_\tau \right) \mathcal{F}^{-1} \right) + \mathcal{J}' \right) + \mathcal{J}'_x$$

(the reasoning is nearly identical to that of Case 1).

<u>Case 8:</u> $x = \infty$.

In this case we work with Ψ_1 and are looking for a map that does the following:

$$\chi_{[0,\infty)} \mapsto (P), \quad \chi_{[-1,\infty)} \mapsto (P), \quad \chi_{(-\infty,1]} \mapsto (Q), \quad Q_{\mathbb{R}} \mapsto (L(\chi_+)), \quad K \mapsto (0)$$

We start by sending an operator A to the sequence $(\operatorname*{s-lim}_{\tau\to\infty} Z_\tau A Z_\tau^{-1})$. Due to the weak convergence of Z_τ, compact operators are sent to zero here. The other operators are mapped as follows:

$$\chi_{[0,\infty)} \mapsto (\chi_{[0,\infty)}), \quad \chi_{[-1,\infty)} \mapsto (\chi_{[0,\infty)}), \quad \chi_{(-\infty,1]} \mapsto (\chi_{(-\infty,0]}), \quad Q_{\mathbb{R}} \mapsto (Q_{\mathbb{R}})$$

Next we take the resulting sequence (A) and map it to $\big(\mathcal{F}^{-1} A \mathcal{F}\big)$. This sends $(\chi_{[0,\infty)})$ to $(P_{\mathbb{R}})$, $(\chi_{(-\infty,0]})$ to $(Q_{\mathbb{R}})$, and $(Q_{\mathbb{R}})$ to $\big(\chi_{[0,\infty)}\big)$. Taking these sequences of operators in $\mathcal{L}(L^2(\mathbb{R}))$ and sending them to sequences of operators in $\mathcal{L}(L^2(\mathbb{S}^1))$ via the map Δ^{-1} as we did in Case 1 sends $(P_{\mathbb{R}})$ to $(P_{\mathbb{S}^1})$, $(Q_{\mathbb{R}})$ to $(Q_{\mathbb{S}^1})$, and $\big(\chi_{[0,\infty)}\big)$ to $(L(\chi_+))$, as required.

$\square$

The previous theorem, when combined with the Lifting Theorem for $\widehat{\mathcal{C}}$, tells us that invertibility of an element $(C_n) + \mathcal{N} \in \mathcal{C}/\mathcal{N}$ is dependent only upon invertibility of $\Psi_1(C_n)$ and $\Psi_{-1}(C_n)$. Thus, since we are in a C^*-algebra situation, we actually have that $\mathcal{C}/\mathcal{N}$ is isomorphic to a subalgebra of the direct sum $\Psi_1(\mathcal{C}) \bigoplus \Psi_{-1}(\mathcal{C})$:

Corollary 4.3.8. *The C^*-algebra $\mathcal{C}/\mathcal{N}$ is isomorphic to a $*-$subalgebra of the direct sum $\Psi_1(\mathcal{C}) \bigoplus \Psi_{-1}(\mathcal{C})$.*

Proof. Theorems 4.3.3 and 4.3.7 yield that an element $(C_n) + \mathcal{N} \in \mathcal{C}/\mathcal{N}$ is invertible if and only if $\Psi_1(C_n)$ and $\Psi_{-1}(C_n)$ are invertible. Said differently, the mapping from $\mathcal{C}/\mathcal{N}$ into $\Psi_1(\mathcal{C}) \bigoplus \Psi_{-1}(\mathcal{C})$ preserves spectra. Since we are in the C^*-algebra setting, this mapping is an isometry and hence injective. We thus have an isomorphism onto its image in the direct sum. $\square$

4.4 Identifying $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$

In this section we will finally identify $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ with the algebra of operators $\Psi_{-1}(\mathcal{C})$; Corollary 4.3.8 will prove to be a key ingredient on our way to proving this. Recall the following commutative diagram given in the statement of Theorem 4.3.2:

$$\begin{array}{ccc} (\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1} & \xrightarrow{\widehat{\Psi}_{-1}} & \\ \uparrow \iota & & \searrow \\ \mathcal{C}/\mathcal{N} & \xrightarrow[\Psi_{-1}]{} & \Psi_{-1}(\mathcal{C}) \end{array}$$

From this point of view, in order to prove that $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ is isomorphic to $\Psi_{-1}(\mathcal{C})$ we need only show that $\widehat{\Psi}_{-1}$ is injective. The strategy is as follows: using Corollary 4.3.8 and Fredholm Theory, we will identify the kernel of Ψ_{-1}. We will then use the surjectivity of ι in order make a conclusion about the kernel of $\widehat{\Psi}_{-1}$. In our efforts to identify the kernel of Ψ_{-1}, the algebra $\text{alg}_{\mathcal{L}(L^2(\mathbb{R}))}\{\chi_{[0,\infty)}, S_{\mathbb{R}}\}$ will make an appearance. We will first study this algebra, realizing it as a matrix algebra. This viewpoint will be advantageous in our analysis of the kernel of Ψ_{-1}.

In what follows we will be using some notation and following the work done in [5], Section 8. We denote by η the isometry from $L^2(\mathbb{R}) \to L^2(\mathbb{R}^+) \bigoplus L^2(\mathbb{R}^+)$ that sends f to $(f_1, f_2)^T$ with $f_1(x) = f(x)$ and $f_2(x) = f(-x)\, \forall x \in \mathbb{R}^+$. We can now define the $*-$isomorphism Φ_η defined by

$$\Phi_\eta : A \mapsto \eta A \eta^{-1} \tag{4.16}$$

that maps $\mathcal{L}(L^2((\mathbb{R}))$ onto $\mathcal{L}(L^2((\mathbb{R}^+))^{2\times 2}$. We now introduce two operators on $\mathcal{L}(L^2((\mathbb{R}))$ onto $\mathcal{L}(L^2((\mathbb{R}^+))$: let $S = S_{\mathbb{R}^+}$ be the *singular integral operator on the positive real line* and let N be the *Hankel operator*:

$$(Sf)(x) = \frac{1}{\pi i}\int_0^\infty \frac{f(y)}{y-x}dy, \qquad x \in \mathbb{R}^+ \tag{4.17}$$

$$(Nf)(x) = \frac{1}{\pi i}\int_0^\infty \frac{f(y)}{y+x}dy, \qquad x \in \mathbb{R}^+ \tag{4.18}$$

Then we have

$$\Phi_\eta(\chi_{[0,\infty)}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Phi_\eta(S_{\mathbb{R}}) = \begin{pmatrix} S & -N \\ N & -S \end{pmatrix} \tag{4.19}$$

where I refers to the identity operator on $L^2(\mathbb{R}^+)$.

We define the C^*-algebra Σ_2^0 by

$$\Sigma_2^0 := \text{alg}_{\mathcal{L}(L^2((\mathbb{R}^+))^{2\times 2}} \left\{ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} S & -N \\ N & -S \end{pmatrix} \right\}. \tag{4.20}$$

From our discussion, we have the following corollary:

Corollary 4.4.1. *The $*-$isomorphism Φ_η maps $alg_{\mathcal{L}(L^2(\mathbb{R}))}\{\chi_{[0,\infty)}, S_\mathbb{R}\}$ onto Σ_2^0.*

Thus, our study of $\text{alg}_{\mathcal{L}(L^2(\mathbb{R}))}\{\chi_{[0,\infty)}, S_\mathbb{R}\}$ can be transformed into the study of Σ_2^0. Recall the Mellin transform $M : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ is given by

$$(Mf)(z) = \int_0^\infty x^{-iz-\frac{1}{2}} f(x)dx, \quad z \in \mathbb{R} \tag{4.21}$$

and its inverse $M^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}^+)$ is given by

$$(M^{-1}f)(x) = \frac{1}{2\pi}\int_{-\infty}^\infty x^{iz-\frac{1}{2}} f(z)dz, \quad x \in \mathbb{R}^+ \tag{4.22}$$

For a multiplication operator $b \in L^\infty(\mathbb{R})$, we denote by $M^0(b)$ the *Mellin convolution operator*

$$M^0(b) := M^{-1}bM : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+). \tag{4.23}$$

One can show various properties, such as $\|M^0(b)\| = \|b\|, M^0(b)^* = M^0(b^*)$, and $M^0(b_1b_2) = M^0(b_1)M^0(b_2)$. These properties together yield that the mapping $b \mapsto M^0(b)$ is a $*$-isomorphism. In [8], Section 2.1.2, Equations 4 and 5, they show that our operators S and N are actually Mellin convolution operators:

$$S = M^0(s), \qquad s(z) = \coth\left(\pi z + \frac{\pi i}{2}\right) \qquad (z \in \mathbb{R}), \tag{4.24}$$

$$N = M^0(n), \qquad n(z) = -i(\cosh(\pi z))^{-1} \qquad (z \in \mathbb{R}). \tag{4.25}$$

From this point of view, we have (since $s^2 - n^2 = 1$)

$$SN = NS \quad \text{and} \quad S^2 - N^2 = 1. \tag{4.26}$$

Let $PC_\infty(\mathbb{R})$ denote the set of all continuous functions f on $\mathbb{R}$ for which the limits at infinity and negative infinity exist and are finite, and let $C^0_\infty(\mathbb{R})$ denote the set of all continuous functions f on $\mathbb{R}$ for which $\lim_{x\to\pm\infty} f(x) = 0$. The set $PC_\infty(\mathbb{R})$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ which contains the function s and $C^0_\infty(\mathbb{R})$ is the smallest closed ideal of $PC_\infty(\mathbb{R})$ which contains the function n.

In order to describe Σ^0_2, we introduce the following sets:

$$\Sigma_1 := \{\alpha I + \beta S + M^0(b) : \alpha, \beta \in \mathbb{C}, b \in C^0_\infty(\mathbb{R})\} \tag{4.27}$$

$$\Sigma^0_1 := \{M^0(b) : b \in C^0_\infty(\mathbb{R})\} \tag{4.28}$$

Proposition 4.4.2.

(a) Σ_1 *is a* C^**-algebra and* Σ^0_1 *is a* $*$*-ideal of* Σ_1.

(b) $\Sigma^0_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, D \in \Sigma_1, B, C \in \Sigma^0_1 \right\}.$

Proof. A proof can be found in [5], Proposition 8.2. □

Proposition 4.4.3. *The kernel of* Ψ_{-1} *is equal to* $\{C_n + E_{-n} K E_n : C_n \in \mathcal{N}, K \in \mathcal{K}\}$.

Proof. We will use the following scheme for our proof:

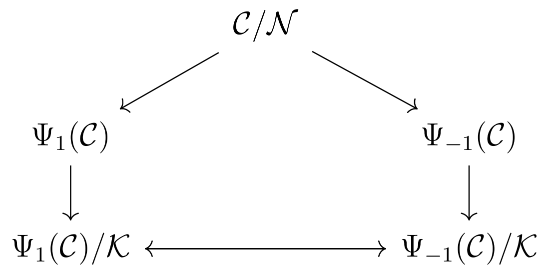
$$\begin{array}{ccc} & \mathcal{C}/\mathcal{N} & \\ \swarrow & & \searrow \\ \Psi_1(\mathcal{C}) & & \Psi_{-1}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \Psi_1(\mathcal{C})/\mathcal{K} & & \Psi_{-1}(\mathcal{C})/\mathcal{K} \end{array}$$

From the Fredholm Theory outlined in Appendix A.1 and A.2, the algebras $\Psi_1(\mathcal{C})/\mathcal{K}$ and $\Psi_{-1}(\mathcal{C})/\mathcal{K}$ can be further decomposed into a direct sum of algebras given by their images under various homomorphisms. Now, we can construct maps from each component of the direct sum that $\Psi_1(\mathcal{C})/\mathcal{K}$ is isomorphic to into the components of the direct sum for $\Psi_{-1}(\mathcal{C})/\mathcal{K}$. Indeed, the identity mapping sends $H^{+-}(\Psi_1(\mathcal{C}))$ into $H^{++}(\Psi_{-1}(\mathcal{C}))$, $H^{++}(\Psi_1(\mathcal{C}))$ into $H^{+-}(\Psi_{-1}(\mathcal{C}))$, $H^{-+}(\Psi_1(\mathcal{C}))$ into $H^{--}(\Psi_{-1}(\mathcal{C}))$, and $H^{--}(\Psi_1(\mathcal{C}))$ into $H^{-+}(\Psi_{-1}(\mathcal{C}))$. For mapping $H_{s,\infty}(\Psi_1(\mathcal{C}))$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C}))$ and $H_{\infty,0}(\Psi_1(\mathcal{C}))$ into $H_{\infty,0}(\Psi_{-1}(\mathcal{C}))$, it suffices to find a multiplicative map that keeps the characteristic functions invariant and that sends $Q_{\mathbb{R}}$ to $P_{\mathbb{R}}$. For this, it is more convenient to view the algebras $H_{s,\infty}(\Psi_1(\mathcal{C}))$, $H_{s,\infty}(\Psi_{-1}(\mathcal{C}))$, $H_{\infty,t}(\Psi_1(\mathcal{C}))$, and $H_{\infty,t}(\Psi_{-1}(\mathcal{C}))$ (which are all $\text{alg}_{\mathcal{L}(L^2(\mathbb{R}))}\{\chi_{[0,\infty)}, S_{\mathbb{R}}\}$) as the matrix algebra Σ_2^0. From this perspective, the map that leaves the characteristic functions invariant and sends $Q_{\mathbb{R}}$ to $P_{\mathbb{R}}$ is not so hard to see: we simply send a matrix $\begin{pmatrix} M^0(a_1) & M^0(a_2) \\ M^0(a_3) & M^0(a_4) \end{pmatrix}$ to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} M^0(\widetilde{a_1}) & M^0(\widetilde{a_2}) \\ M^0(\widetilde{a_3}) & M^0(\widetilde{a_4}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\widetilde{a}(x) = a(-x)$. This map works since $s(-x) = -s(x)$ and $n(-x) = n(x)$.

Thus we actually have the following scheme:

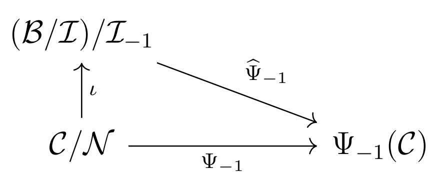


Now let $(C_n) \in \ker \Psi_{-1}$. Then tracking (C_n) down the righthandside of the scheme and into $\Psi_1(\mathcal{C})/\mathcal{K}$, we have that (C_n) is 0 in $\Psi_1(\mathcal{C})/\mathcal{K}$; i.e., $\Psi_1(C_n) := K$ is compact. Consider the element $(C_n - E_{-n}KE_n) \in \mathcal{C}/\mathcal{N}$. Under Ψ_1 this gets sent to zero and under Ψ_{-1} this is also sent to 0. In other words, since $\mathcal{C}/\mathcal{N}$ is isomorphic to a subalgebra of the direct sum $\Psi_1(\mathcal{C}) \bigoplus \Psi_{-1}(\mathcal{C})$, this element is identically 0. Thus $(C_n) = (E_{-n}KE_n)$ and our proof is complete.

□

Theorem 4.4.4. *The local algebra* $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ *is* $*$*-isomorphic to* $\Psi_{-1}(\mathcal{C})$.

Proof. Recall the following diagram:



We need to prove that $\widehat{\Psi}_{-1}$ is injective, so let $(B_{n,r} + \mathcal{I}) + \mathcal{I}_{-1} \in \ker \widehat{\Psi}_{-1}$. By surjectivity of ι, there exists a sequence (C_n) such that $\iota(C_n) = (B_{n,r} + \mathcal{I}) + \mathcal{I}_{-1}$. But since $\Psi_{-1} = \widehat{\Psi}_{-1} \circ \iota$, this means that $(C_n) \in \ker \Psi_{-1}$. Using Proposition 4.4.3, we can conclude that $(C_n) = (D_n + E_{-n}KE_n)$ where $D_n \in \mathcal{N}$ and K is compact. Thus $(B_{n,r}) = (D_n + E_{-n}KE_n)$. Since $\mathcal{N} \subseteq \mathcal{I}$, $D_n \in \mathcal{I}$. It remains to show that $(E_{-n}KE_n)$ is zero in the local algebra. But this is not so bad, since

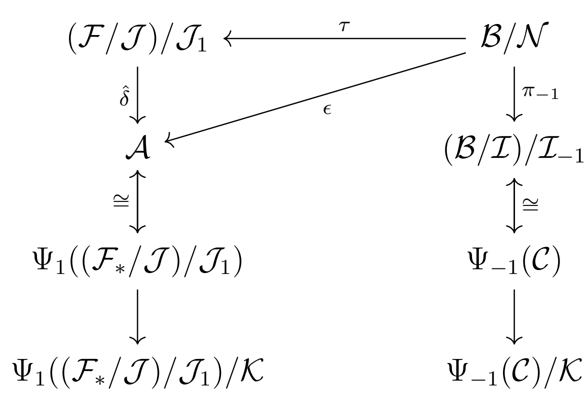
$$E_{-n}KE_n = R_r^*R_rE_{-n}KE_nR_r^*R_r = R_r^*K'R_r \in \mathcal{I}$$

where the last equality only holds mod $\mathcal{N}$ due to the strong convergence of R_rE_{-n} and $E_nR_r^*$. Therefore $\widehat{\Psi}_{-1}$ is injective and thus is actually a $*$-isomorphism and our proof is complete. □

Corollary 4.4.5. *The C^*-algebra $\mathcal{B}/\mathcal{N}$ is isomorphic to a $*-$subalgebra of the direct sum $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) \bigoplus \Psi_{-1}(\mathcal{C})$.*

Proof. This is a direct result of Corollary 4.1.8 and Theorems 4.2.10 and 4.4.4. □

Let's take a moment to reflect on what we have accomplished thus far. We started this section with the goal of identifying $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ with an algebra of operators. In order to work towards this goal, we introduced several new algebras and identified these new algebras with algebras of operators. The work that we have done can be summarized in the following scheme:



We will make full use of the isomorphisms we demonstrated and actually work with the following diagram:

$$
\begin{array}{ccc}
(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\quad\tau\quad} & \mathcal{B}/\mathcal{N} \\
\Big\downarrow{\scriptstyle \Psi_1} & \swarrow{\scriptstyle \widehat{\Psi}_1} & \Big\downarrow{\scriptstyle \widehat{\Psi}_{-1}\circ\pi_{-1}} \\
\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) & & \Psi_{-1}(\mathcal{C}) \\
\Big\downarrow & & \Big\downarrow \\
\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)/\mathcal{K} & & \Psi_{-1}(\mathcal{C})/\mathcal{K}
\end{array}
$$

In the next chapter we will use this scheme in order to identify the kernel of the map τ and to ultimately achieve our goal of identifying $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ with an algebra of operators.

Chapter 5

The Algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$

The goal of this chapter is to finally identify the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ with an algebra of operators. In the first section we find and prove that a concrete element belongs to the kernel of the map τ. In the second section we use this element to prove that the kernel of $\widehat{\Psi}_1$ is equal to the kernel of τ. In the final section we summarize our findings and present our main result. Throughout this chapter we make use of the Fredholm theory presented in Appendix A.

5.1 Exploring the Kernel of τ

In this section, we seek to find an element that belongs to the kernel of τ. We proceed as follows: first we find an element $(B_{n,r}) \in \mathcal{B}/\mathcal{N}$ that gets sent to a compact operator K under $\Psi_1 \circ \tau$ but that does not get sent to a compact operator in $\Psi_{-1}(\mathcal{C})$. We then consider the difference of the image of this element under τ with the preimage of K under Ψ_1 in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$. We denote this new element by $(F_{n,r})$. We first concretely identify what the compact operator K looks like, and then we fully describe what $(F_{n,r})$ looks like by expressing it as a matrix and providing a formula for its entries. Finally, by choosing a specific function with desired properties, we exhibit an element belonging to the kernel of τ.

For this first step, we will use the following diagram:

$$\begin{array}{ccc} (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\quad\tau\quad} & \mathcal{B}/\mathcal{N} \\ \Big\downarrow{\scriptstyle \Psi_1} & \swarrow{\scriptstyle \widehat{\Psi}_1} & \Big\downarrow{\scriptstyle \widehat{\Psi}_{-1}\circ\pi_{-1}} \\ \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) & & \Psi_{-1}(\mathcal{C}) \end{array}$$

Define $\chi := \chi_+ - \chi_-$ and let f be a continuous function on the unit circle with $f(1) = 1$ and $f(-1) = 0$. Consider the element

$$(B_{n,r}) := (I - P_n^+)(I - L(C_r^{-1}f))(L(\chi))(I - Q_n^-) + \mathcal{N} \in \mathcal{B}/\mathcal{N}.$$

Direct computations show that

$$(\Psi_1 \circ \tau)(B_{n,r}) = \chi_{(-\infty,-1)} W^0\Big(1 - f\Big(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\Big)\Big)(Q_{\mathbb{R}} - P_{\mathbb{R}})\chi_{(1,\infty)}$$

gets sent to zero by each of the homomorphisms given from the Fredholm theory discussed in Appendix A. We can thus conclude that this element is in fact compact in $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$; let's call this compact K. In $\mathcal{B}/\mathcal{N}$ we then consider the element the element

$$((I - P_n^+)(I - L(C_r^{-1}f))(L(\chi))(I - Q_n^-)) - (E_{-n}KE_n) + \mathcal{N}.$$

Notice that this element is nonzero; one way to notice this is by observing that it is sent to $(((I - P_n^+)(L(\chi))(I - Q_n^-)) - (E_{-n}KE_n) + \mathcal{I}) + \mathcal{I}_{-1}$ in $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ which, when identified with $\Psi_{-1}(\mathcal{C})$, is not zero. This is seen, for example, by observing that it gets mapped to $-\chi_{(-\infty,0]}S_{\mathbb{R}}\chi_{[0,\infty)} \neq 0$ under $H_{\infty,0}$. Its image in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$

is the element

$$\begin{aligned}(F_{n,r}) &:= ((\tau(B_{n,r}) - E_nKE_n) + \mathcal{J}) + \mathcal{J}_1 \\ &= ((I - P_n^+)(I - L(C_r^{-1}f))(L(\chi))(I - Q_n^-) - (E_{-n}KE_n) + \mathcal{J}) + \mathcal{J}_1.\end{aligned}$$

In order to better understand $(F_{n,r})$ we will first seek to describe K more concretely. From what we have already seen, K is the operator

$$\chi_{(-\infty,-1)}W^0\Big(1 - \hat{g}\Big)(Q_{\mathbb{R}} - P_{\mathbb{R}})\chi_{(1,\infty)}$$

where $\hat{g}(x) = f(\frac{\lambda-2\pi ix}{\lambda+2\pi ix})$. This can be rewritten as

$$\chi_{(-\infty,-1)}(Q_{\mathbb{R}} - P_{\mathbb{R}})\chi_{(1,\infty)} - \chi_{(-\infty,-1)}W^0(\hat{g})(Q_{\mathbb{R}} - P_{\mathbb{R}})\chi_{(1,\infty)}.$$

Now, since $Q_{\mathbb{R}} - P_{\mathbb{R}} = -S_{\mathbb{R}}$, we can conclude that

$$K = -\chi_{(-\infty,-1)}S_{\mathbb{R}}\chi_{(1,\infty)} + \chi_{(-\infty,-1)}W^0(\hat{g})S_{\mathbb{R}}\chi_{(1,\infty)}.$$

Let's now use our newfound knowledge to give our element $(F_{n,r})$ a bit of a makeover:

$$\begin{aligned}(F_{n,r}) &= (I - P_n^+)(I - L(C_r^{-1}f))(L(\chi))(I - Q_n^-) - (E_{-n}KE_n) \\ &= (I - P_n^+)(I - L(C_r^{-1}f))(L(\chi))(I - Q_n^-) \\ &\quad - (E_{-n}(-\chi_{(-\infty,-1)}S_{\mathbb{R}}\chi_{(1,\infty)} + \chi_{(-\infty,-1)}W^0(\hat{g})S_{\mathbb{R}}\chi_{(1,\infty)})E_n) \\ &= (I - P_n^+)(L(\chi))(I - Q_n^-) + (E_{-n}\chi_{(-\infty,-1)}S_{\mathbb{R}}\chi_{(1,\infty)}E_n) \qquad (5.1)\end{aligned}$$

$$- (I - P_n^+)(L(C_r^{-1} f\chi))(I - Q_n^-) + (E_{-n}\chi_{(-\infty,-1)} W^0(\hat{g}) W^0(\mathrm{sgn}(x))\chi_{(1,\infty)} E_n) \tag{5.2}$$

Let's tackle (5.1) first. Recall the function σ which is continuous on $\mathbb{S}^1 \setminus \{1\}$ with one sided limits $\sigma(1+0) = -1$ and $\sigma(1-0) = 1$ and for which we have the relation $L(\sigma) = E_{-n} S_{\mathbb{R}} E_n$ for all $n \geq 1$. With this in mind, we can write the second term of (5.1) as

$$\begin{aligned} E_{-n}\chi_{(-\infty,-1)} S_{\mathbb{R}} \chi_{(1,\infty)} E_n &= E_{-n}\chi_{(-\infty,-1)} E_n E_{-n} S_{\mathbb{R}} E_n E_{-n} \chi_{(1,\infty)} E_n \\ &= (I - P_n^+)(L(\sigma))(I - Q_n^-) \end{aligned}$$

where the first equality holds because of the fact that L_n commutes with characteristic functions with integer endpoints. We then have that (5.1) is equal to $(I - P_n^+)(L(\chi + \sigma))(I - Q_n^-)$ and since $\chi(1+0) = 1$ and $\chi(1-0) = -1$ we have that this belongs to $\mathcal{J}_1$. Thus the whole expression (5.1) is in $\mathcal{J}_1$ and hence zero in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$. We can thus conclude that in our local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$, we have

$$(F_{n,r}) = -(I - P_n^+)(L(C_r^{-1} f\chi))(I - Q_n^-) + (E_{-n}\chi_{(-\infty,-1)} W^0(\hat{g}) W^0(\mathrm{sgn}(x))\chi_{(1,\infty)} E_n). \tag{5.3}$$

This remaining expression will require a bit more work. Define $h := f\chi$. Then

$$W^0(\hat{g}\mathrm{sgn}(x)) = -W^0\Big(h\Big(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\Big)\Big)$$

Then for $\hat{h}(x) = h\Big(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\Big)$, we rewrite (5.3) as

$$(F_{n,r}) = -(I - P_n^+)(L(C_r^{-1} h) + E_{-n} W^0(\hat{h}) E_n)(I - Q_n^-).$$

Proposition 5.1.1. *Let f be a continuous function on the unit circle with $f(1) = 1$ and $f(-1) = 0$, set $h := f\chi$, and define $\hat{h}(x) = h\Big(\dfrac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\Big)$. Let $(F_{n,r})$ be the element defined by*

$$(F_{n,r}) = -(I - P_n^+)(L(C_r^{-1}h) + E_{-n}W^0(\hat{h})E_n)(I - Q_n^-).$$

Then the $m^{th} = (l-j)^{th}$ entry of its matrix representation is given by

$$\frac{1}{n}\int_0^1\int_0^1\int_{\mathbb{R}} e^{2\pi i(\frac{s-t+m}{n})x}\hat{h}(x)dxdtds - \frac{2\epsilon_r}{\lambda}\int_{\mathbb{R}}\hat{h}(x)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx$$

when $m < -2n$ and 0 otherwise.

Proof. We will tackle this expression by first exploring what $E_{-n}W^0(\hat{h})E_n$ looks like as a matrix and then looking at what $L(C_r^{-1}h)$ looks like as a matrix. $W^0(\hat{h})$ is an integral operator with kernel $k(x-y) = (\mathcal{F}^{-1}\hat{h})(x-y)$. From this point of view, we can see that $E_{-n}W^0(\hat{h})E_n$ is in fact a Laurent operator. Indeed, let e_j be the vector with 1 in the j^{th} position and zeros everywhere else. Then

$$\begin{aligned}(E_{-n}W^0(\hat{h})E_n)(e_j) &= (E_{-n}W^0(\hat{h}))(\sqrt{n}\chi_{[\frac{j}{n},\frac{j+1}{n}]})\\ &= E_{-n}(\sqrt{n}\int_{\mathbb{R}} k(x-y)\chi_{[\frac{j}{n},\frac{j+1}{n}]}(y)dy)\\ &= \left(n\int_{\mathbb{R}}\left[\int_{\mathbb{R}} k(x-y)\chi_{[\frac{j}{n},\frac{j+1}{n}]}(y)dy\right]\chi_{[\frac{l}{n},\frac{l+1}{n}]}(x)dx\right)_{l\in\mathbb{Z}}\end{aligned}$$

So the lj^{th} entry of the matrix is

$$n\int_{\mathbb{R}}\left[\int_{\mathbb{R}} k(x-y)\chi_{[\frac{j}{n},\frac{j+1}{n}]}(y)dy\right]\chi_{[\frac{l}{n},\frac{l+1}{n}]}(x)dx = n\int_{\frac{l}{n}}^{\frac{l+1}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}} k(x-y)dydx$$

Letting $x = \dfrac{s+l}{n}$ and $y = \dfrac{t+j}{n}$, we have that this equals

$$\frac{1}{n}\int_0^1\int_0^1 k\Big(\frac{s-t+l-j}{n}\Big)dtds$$

which depends only on $l-j$ and is hence a Laurent operator. Recalling that $k(x-y) = (\mathcal{F}^{-1}\hat{h})(x-y)$, we can actually write this double integral as a triple integral:

$$\frac{1}{n}\int_0^1\int_0^1 k\Big(\frac{s-t+l-j}{n}\Big)dtds = \frac{1}{n}\int_0^1\int_0^1\int_{\mathbb{R}} e^{2\pi i(\frac{s-t+l-j}{n})x}\hat{h}(x)dxdtds.$$

Now let's think about $L(C_r^{-1}h)$. The m^{th} Fourier coefficient of $C_r^{-1}h$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h\Big(\frac{e^{i\theta}-r}{1-re^{i\theta}}\Big)e^{-im\theta}d\theta.$$

Let's rewrite this so that we get it in terms of $\hat{h}$. To do this, we set

$$\frac{e^{i\theta}-r}{1-re^{i\theta}} = \frac{\lambda-2\pi ix}{\lambda+2\pi ix} = t$$

and seek to write θ in terms of x. By performing algebra, we end up with $x = \dfrac{-\lambda(1+r)}{2\pi(1-r)}\tan(\frac{\theta}{2})$ and so $\theta = 2\arctan\Big(\dfrac{-2\pi x(1-r)}{\lambda(1+r)}\Big)$. Letting $\epsilon_r = \dfrac{1-r}{1+r}$, we have that $\theta = 2\arctan(\dfrac{-2\pi\epsilon_r}{\lambda}x)$ and thus have that the m^{th} Fourier coefficient of $C_r^{-1}h$ is

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} h\Big(\frac{e^{i\theta}-r}{1-re^{i\theta}}\Big)e^{-im\theta}d\theta &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{h}(x)e^{-im2\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}(2\arctan(\frac{-2\pi\epsilon_r}{\lambda}x))'dx\\ &= \frac{-2\epsilon_r}{\lambda}\int_{\mathbb{R}}\hat{h}(x)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx.\end{aligned}$$

Recall that we are trying to figure out what

$$-(I-P_n^+)(L(C_r^{-1}h)+E_{-n}W^0(\hat{h})E_n)(I-Q_n^-)$$

looks like. From the work we just did, we know what the entries of the matrix representation for the middle piece is, namely the $m^{th}=(l-j)^{th}$ entry is given by

$$\frac{1}{n}\int_0^1\int_0^1\int_{\mathbb{R}} e^{2\pi i(\frac{s-t+m}{n})x}\hat{h}(x)dxdtds-\frac{2\epsilon_r}{\lambda}\int_{\mathbb{R}}\hat{h}(x)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx.$$

Multiplying on the left by $(I-P_n^+)$ and on the right by $(I-Q_n^-)$ makes the entries below the $-n^{th}$ row and to the left of the n^{th} column zero, and hence we are left with a Hankel matrix. In terms of the entries of the matrix expression we found, this corresponds to the requirement $m<-2n$. $\square$

We now turn to exhibiting a specific element in the kernel of τ by picking a particular function f. Take the function f defined by $f(e^{i\theta})=1-\dfrac{|\theta|}{\pi}$. Define

$$(\widehat{F}_{n,r}):=(I-P_n^+)(I-L(C_r^{-1}f))(L(\chi))(I-Q_n^-)-(E_{-n}KE_n)\in(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$$

for this specific f and let $(\widehat{B}_{n,r})$ be the element in $\mathcal{B}/\mathcal{N}$ that gets mapped to $(\widehat{F}_{n,r})$ under τ. Explicitly,

$$(\widehat{B}_{n,r}):=(I-P_n^+)(I-L(C_r^{-1}f))(L(\chi))(I-Q_n^-)-(E_{-n}KE_n)+\mathcal{N} \tag{5.4}$$

with $f(e^{i\theta})=1-\dfrac{|\theta|}{\pi}$. We have the following:

Theorem 5.1.2. *Let* $f(e^{i\theta}) = 1 - \dfrac{|\theta|}{\pi}$ *and let* K *be the compact operator*

$$K = -\chi_{(-\infty,-1)} S_{\mathbb{R}} \chi_{(1,\infty)} + \chi_{(-\infty,-1)} W^0(\hat{g}) S_{\mathbb{R}} \chi_{(1,\infty)}.$$

Then the element $(\widehat{B}_{n,r})$ *defined by*

$$(\widehat{B}_{n,r}) := (I - P_n^+)(I - L(C_r^{-1} f))(L(\chi))(I - Q_n^-) - (E_{-n} K E_n) + \mathcal{N}$$

belongs to the kernel of τ.

Proof. For the function $f(e^{i\theta}) = 1 - \dfrac{|\theta|}{\pi}$, we have

$$\hat{h}(x) = \begin{cases} -1 + \dfrac{2}{\pi} \arctan(\frac{2\pi}{\lambda} x) & \text{if } x > 0 \\ 1 + \dfrac{2}{\pi} \arctan(\frac{2\pi}{\lambda} x) & \text{if } x < 0 \end{cases}$$

Based on our previous proposition and discussion, to complete this proof we need only show that the Laurent matrix whose $m^{th} = (l-j)^{th}$ entry is 0 when $m \geq -2n$ and given by

$$\frac{1}{n} \int_0^1 \int_0^1 \int_{\mathbb{R}} e^{2\pi i (\frac{s-t+m}{n}) x} \hat{h}(x) dx dt ds - \frac{2\epsilon_r}{\lambda} \int_{\mathbb{R}} \hat{h}(x) \frac{e^{-2im \arctan(\frac{-2\pi\epsilon_r}{\lambda} x)}}{1 + (\frac{-2\pi\epsilon_r}{\lambda} x)^2} dx$$

when $m < -2n$ and $\epsilon_r = \frac{1-r}{1+r}$ belongs to $\mathcal{N}$. Let's start with the second integral:

$$\begin{aligned} &-\frac{2\epsilon_r}{\lambda} \int_{\mathbb{R}} \hat{h}(x) \frac{e^{-2im \arctan(\frac{-2\pi\epsilon_r}{\lambda} x)}}{1 + (\frac{-2\pi\epsilon_r}{\lambda} x)^2} dx \\ &\qquad = -\frac{2\epsilon_r}{\lambda} \int_0^\infty \Big(-1 + \frac{2}{\pi} \arctan(\frac{2\pi}{\lambda} x) \Big) \frac{e^{-2im \arctan(\frac{-2\pi\epsilon_r}{\lambda} x)}}{1 + (\frac{-2\pi\epsilon_r}{\lambda} x)^2} dx \end{aligned}$$

$$-\frac{2\epsilon_r}{\lambda}\int_{-\infty}^{0}\Big(1+\frac{2}{\pi}\arctan(\frac{2\pi}{\lambda}x)\Big)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx$$

$$=\frac{2\epsilon_r}{\lambda}\int_{0}^{\infty}\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx-\frac{2\epsilon_r}{\lambda}\int_{-\infty}^{0}\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx \quad (5.5)$$

$$-\frac{2\epsilon_r}{\lambda}\int_{\mathbb{R}}\frac{2}{\pi}\arctan(\frac{2\pi}{\lambda}x)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx \quad (5.6)$$

The integrals in (5.5) yield (by setting $x\mapsto -x$ into the second one)

$$\frac{2\epsilon_r}{\lambda}\int_{0}^{\infty}\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx-\frac{2\epsilon_r}{\lambda}\int_{0}^{\infty}\frac{e^{-2im\arctan(\frac{2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{2\pi\epsilon_r}{\lambda}x)^2}dx$$

$$=\frac{2\epsilon_r}{\lambda}\int_{0}^{\infty}\frac{2i\sin\Big(2m\arctan(\frac{2\pi\epsilon_r}{\lambda}x)\Big)}{1+(\frac{2\pi\epsilon_r}{\lambda}x)^2}dx.$$

Let $u=\arctan(\frac{2\pi\epsilon_r}{\lambda}x)$. We then have

$$\begin{aligned}\frac{2\epsilon_r}{\lambda}\int_{0}^{\infty}\frac{2i\sin\Big(2m\arctan(\frac{2\pi\epsilon_r}{\lambda}x)\Big)}{1+(\frac{2\pi\epsilon_r}{\lambda}x)^2}dx &= \frac{2\epsilon_r}{\lambda}\frac{\lambda}{2\pi\epsilon_r}\int_{0}^{\pi/2}2i\sin(2mu)du\\ &=\frac{2i}{\pi}\int_{0}^{\pi/2}\sin(2mu)du\\ &=\frac{-i\cos(m\pi)}{m\pi}+\frac{i}{m\pi}.\end{aligned}$$

Now let's return to (5.6). By again splitting this integral into two integrals (one over the positive real axis and one over the negative real axis) and applying $x\mapsto -x$ so that we have both integrals over the positive real line, we obtain

$$\begin{aligned}&-\frac{2\epsilon_r}{\lambda}\int_{\mathbb{R}}\frac{2}{\pi}\arctan\left(\frac{2\pi}{\lambda}x\right)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx\\ &\qquad=-\frac{8i\epsilon_r}{\lambda\pi}\int_{0}^{\infty}\arctan\left(\frac{2\pi}{\lambda}x\right)\frac{\sin\Big(2m\arctan(\frac{2\pi\epsilon_r}{\lambda}x)\Big)}{1+(\frac{2\pi\epsilon_r}{\lambda}x)^2}dx.\end{aligned}$$

Let $u = \arctan(\frac{2\pi\epsilon_r}{\lambda}x)$. Our integral then becomes

$$-\frac{4i}{\pi^2}\int_0^{\pi/2} \arctan\left(\frac{\tan(u)}{\epsilon_r}\right)\sin(2mu)du.$$

By performing integration by parts with $\mu = \arctan\left(\frac{\tan(u)}{\epsilon_r}\right)$ and $dv = \sin(2mu)$, we see that this integral becomes

$$\frac{4i}{\pi^2}\frac{\arctan\left(\frac{\tan(u)}{\epsilon_r}\right)\cos(2mu)}{2m}\Bigg|_0^{\pi/2} - \frac{4i}{\pi^2}\int_0^{\pi/2}\frac{\cos(2mu)\sec^2(u)}{1+\left(\frac{\tan(u)}{\epsilon_r}\right)^2}\frac{1}{2m\epsilon_r}du$$

which is equal to

$$\frac{4i}{\pi^2}\frac{\pi}{2}\frac{\cos(m\pi)}{2m} - \frac{2i}{m\epsilon_r\pi^2}\int_0^{\pi/2}\frac{\cos(2mu)\sec^2(u)}{1+\left(\frac{\tan(u)}{\epsilon_r}\right)^2}du$$
$$= \frac{i\cos(m\pi)}{m\pi} - \frac{2i}{m\epsilon_r\pi^2}\int_0^{\pi/2}\frac{\cos(2mu)\sec^2(u)}{1+\left(\frac{\tan(u)}{\epsilon_r}\right)^2}du$$

and so we really just need to consider the integral that remains in the expression. By first rewriting $\sec^2(u) = \dfrac{1}{\cos^2(u)}$ and multiplying the top and bottom by ϵ_r^2, we get the integral is equal to

$$\frac{-2i}{m\epsilon_r\pi^2}\int_0^{\pi/2}\frac{\epsilon_r^2\cos(2mu)}{\epsilon_r^2\cos^2 u + \sin^2 u}du.$$

Notice that the integrand is an even function, and so our integral can be expressed as

$$\frac{-i}{m\epsilon_r\pi^2}\int_{-\pi/2}^{\pi/2}\frac{\epsilon_r^2\cos(2mu)}{\epsilon_r^2\cos^2 u + \sin^2 u}du.$$

Let γ denote the unit circle. We do a substitution $z = e^{2iu}$ to transform our integral to one over the unit circle:

$$\begin{aligned}
&\frac{-i}{m\epsilon_r\pi^2}\int_\gamma \frac{\epsilon_r^2\frac{1}{2}(z^m+z^{-m})}{\epsilon_r^2(\sqrt{z}+\frac{1}{\sqrt{z}})^2(\frac{1}{2})^2+(\frac{1}{2i})^2(\sqrt{z}-\frac{1}{\sqrt{z}})^2}\frac{1}{2iz}dz\\
&\quad=\frac{-1}{4m\epsilon_r\pi^2}\int_\gamma \frac{\epsilon_r^2(z^m+z^{-m})}{\epsilon_r^2z(\sqrt{z}+\frac{1}{\sqrt{z}})^2\frac{1}{4}-\frac{1}{4}z(\sqrt{z}-\frac{1}{\sqrt{z}})^2}dz\\
&\quad=\frac{-\epsilon_r^2}{m\epsilon_r\pi^2}\int_\gamma \frac{(z^m+z^{-m})}{(\epsilon_r z+\epsilon_r+z-1)(\epsilon_r z+\epsilon_r-z+1)}dz\\
&\quad=\frac{-\epsilon_r}{m\pi^2}\int_\gamma \frac{(z^m+z^{-m})}{(z+\frac{\epsilon_r-1}{\epsilon_r+1})(z+\frac{\epsilon_r+1}{\epsilon_r-1})(\epsilon_r+1)(\epsilon_r-1)}dz.
\end{aligned}$$

Recall that we have $\epsilon_r = \dfrac{1-r}{1+r}$. This means that $\dfrac{\epsilon_r-1}{\epsilon_r+1} = -r$ and similarly $\dfrac{\epsilon_r+1}{\epsilon_r-1} = -\dfrac{1}{r}$ and so our integral becomes

$$\begin{aligned}
&\frac{-\epsilon_r}{m\pi^2(\epsilon_r^2-1)}\int_\gamma \frac{(z^m+z^{-m})}{(z-r)(z-\frac{1}{r})}dz\\
&\quad=\frac{-\epsilon_r}{m\pi^2(\epsilon_r^2-1)}\Big[\int_\gamma \frac{z^m}{(z-r)(z-\frac{1}{r})}dz+\int_\gamma \frac{z^{-m}}{(z-r)(z-\frac{1}{r})}dz\Big].
\end{aligned}$$

Now, since $m<0$ and $r<1$, we have (via residue theory)

$$\int_\gamma \frac{z^{-m}}{(z-r)(z-\frac{1}{r})}dz = 2\pi i\frac{r^{-m}}{r-\frac{1}{r}}.$$

For $\int_\gamma \frac{z^m}{(z-r)(z-\frac{1}{r})}dz$ we will employ partial fractions. We write

$$\frac{1}{z^{-m}(z-r)(z-\frac{1}{r})} = \frac{A_1}{z}+\frac{A_2}{z^2}+\cdots+\frac{A_{-m}}{z^{-m}}+\frac{B}{z-r}+\frac{C}{z-\frac{1}{r}}.$$

Solving this, we get that $B = \dfrac{r^m}{r-\frac{1}{r}}$ and $A_1 = -\dfrac{r^m-r^{-m}}{r-\frac{1}{r}}$. We need not worry about A_i for $i>1$ since in these cases $\dfrac{A_i}{z^i}$ will integrate to zero over the unit circle. Similarly, we don't need to worry about C since $\dfrac{1}{r}$ lives outside of the unit circle (although it was necessary to compute in order to find A_1). Thus, we have

$$\int_\gamma \frac{z^m}{(z-r)(z-\frac{1}{r})}dz = 2\pi i\Big(\frac{r^{-m}-r^m}{r-\frac{1}{r}} + \frac{r^m}{r-\frac{1}{r}}\Big).$$

Altogether,

$$\begin{aligned}\frac{-\epsilon_r}{m\pi^2(\epsilon_r^2-1)}\Big[\int_\gamma \frac{z^m}{(z-r)(z-\frac{1}{r})}dz + \int_\gamma \frac{z^{-m}}{(z-r)(z-\frac{1}{r})}dz\Big]& \\ = \frac{-2\pi i\epsilon_r}{m\pi^2(\epsilon_r^2-1)}\Big(\frac{r^{-m}-r^m}{r-\frac{1}{r}} + \frac{r^m}{r-\frac{1}{r}} + \frac{r^{-m}}{r-\frac{1}{r}}\Big)& \\ = \frac{-2i\epsilon_r}{m\pi(\epsilon_r^2-1)}\Big(\frac{2r^{-m}}{r-\frac{1}{r}}\Big)& \\ = \frac{-i}{m\pi r^m}&\end{aligned}$$

where this final equality is coming again from our definition of ϵ_r.

We've made some good progress: so far we have shown that for our particular choice of function f (and thus particular choice of $\hat{h}$), the resulting integral

$$-\frac{2\epsilon_r}{\lambda}\int_{\mathbb{R}} \hat{h}(x)\frac{e^{-2im\arctan(\frac{-2\pi\epsilon_r}{\lambda}x)}}{1+(\frac{-2\pi\epsilon_r}{\lambda}x)^2}dx$$

is equal to

$$\frac{-i\cos(m\pi)}{m\pi} + \frac{i}{m\pi} + \frac{i\cos(m\pi)}{m\pi} - \frac{i}{m\pi r^m} = \frac{i}{m\pi} - \frac{i}{m\pi r^m}.$$

Next, we examine

$$I := \frac{1}{n}\int_0^1\int_0^1\int_{\mathbb{R}} e^{2\pi i(\frac{s-t+m}{n})x}\hat{h}(x)dxdtds.$$

Once we have this, we will be able to explicitly describe the entries of the matrix that we are seeking to prove belongs to $\mathcal{N}$. For our choice of $\hat{h}$, we have that the above triple integral is equal to

$$
\begin{aligned}
I = & \frac{1}{n}\int_0^1\int_0^1\int_0^\infty e^{2\pi i(\frac{s-t+m}{n})x}\Big(-1+\frac{2}{\pi}\arctan(\frac{2\pi}{\lambda}x)\Big)dxdtds \\
& +\frac{1}{n}\int_0^1\int_0^1\int_{-\infty}^0 e^{2\pi i(\frac{s-t+m}{n})x}\Big(1+\frac{2}{\pi}\arctan(\frac{2\pi}{\lambda}x)\Big)dxdtds \\
= & -\frac{1}{n}\int_0^1\int_0^1\int_0^\infty e^{2\pi i(\frac{s-t+m}{n})x}dxdtds \\
& +\frac{2}{n\pi}\int_0^1\int_0^1\int_0^\infty e^{2\pi i(\frac{s-t+m}{n})x}\arctan(\frac{2\pi}{\lambda}x)dxdtds \\
& +\frac{1}{n}\int_0^1\int_0^1\int_{-\infty}^0 e^{2\pi i(\frac{s-t+m}{n})x}dxdtds \\
& +\frac{2}{n\pi}\int_0^1\int_0^1\int_{-\infty}^0 e^{2\pi i(\frac{s-t+m}{n})x}\arctan(\frac{2\pi}{\lambda}x)dxdtds
\end{aligned}
$$

Similar to before, we can write

$$
-\frac{1}{n}\int_0^1\int_0^1\int_0^\infty e^{2\pi i(\frac{s-t+m}{n})x}dxdtds+\frac{1}{n}\int_0^1\int_0^1\int_{-\infty}^0 e^{2\pi i(\frac{s-t+m}{n})x}dxdtds
$$

as

$$
-\frac{2i}{n}\int_0^1\int_0^1\int_0^\infty \sin\Big(2\pi(\frac{s-t+m}{n})x\Big)dxdtds
$$

and we can also write

$$
\begin{aligned}
& \frac{2}{n\pi}\int_0^1\int_0^1\int_0^\infty e^{2\pi i(\frac{s-t+m}{n})x}\arctan(\frac{2\pi}{\lambda}x)dxdtds \\
& +\frac{2}{n\pi}\int_0^1\int_0^1\int_{-\infty}^0 e^{2\pi i(\frac{s-t+m}{n})x}\arctan(\frac{2\pi}{\lambda}x)dxdtds
\end{aligned}
$$

as

$$\frac{4i}{n\pi}\int_0^1\int_0^1\int_0^\infty \arctan(\frac{2\pi}{\lambda}x)\sin\Big(2\pi(\frac{s-t+m}{n})x\Big)dxdtds.$$

For convenience, let $\zeta := \dfrac{s-t+m}{n}$. Then we have

$$I = 2i\int_0^1\int_0^1\int_0^\infty \sin(2\pi\zeta x)\Big(\frac{2}{n\pi}\arctan(\frac{2\pi}{\lambda}x)-\frac{1}{n}\Big)dxdtds.$$

Let's focus on the innermost integral first and do integration by parts on it with $u = \dfrac{2}{n\pi}\arctan(\frac{2\pi}{\lambda}x) - \dfrac{1}{n}$ and $dv = \sin(2\pi\zeta x)$. We then have

$$\frac{-\cos(2\pi\zeta x)\Big(\frac{2}{n\pi}\arctan(\frac{2\pi}{\lambda}x)-\frac{1}{n}\Big)}{2\pi\zeta}\Bigg|_0^\infty + \frac{2}{n\pi\zeta\lambda}\int_0^\infty \frac{\cos(2\pi\zeta x)}{1+(\frac{2\pi}{\lambda}x)^2}dx$$
$$= -\frac{1}{2\pi\zeta\ n} + \frac{2}{n\pi\zeta\lambda}\int_0^\infty \frac{\cos(2\pi\zeta x)}{1+(\frac{2\pi}{\lambda}x)^2}dx.$$

The integrand of the integral that remains is even, so we have

$$\frac{2}{n\pi\zeta\lambda}\int_0^\infty \frac{\cos(2\pi\zeta x)}{1+(\frac{2\pi}{\lambda}x)^2}dx = \frac{1}{n\pi\zeta\lambda}\int_{-\infty}^\infty \frac{\cos(2\pi\zeta x)}{1+(\frac{2\pi}{\lambda}x)^2}dx.$$

We will use complex analysis for this integral too. Define

$$f(z) = \frac{e^{2\pi\zeta iz}}{1+(\frac{2\pi}{\lambda}z)^2} = \frac{\lambda^2 e^{2\pi\zeta iz}}{4\pi^2(z-\frac{\lambda i}{2\pi})(z+\frac{\lambda i}{2\pi})}$$

and denote by Γ the closed lower semicircle with radius R traversed counterclockwise. Then

$$\int_\Gamma f(z)dz = 2\pi i\cdot \mathrm{Res}(f,-\frac{\lambda i}{2\pi}) = -\frac{\lambda e^{\zeta\lambda}}{2}.$$

One can show that the integral over the lower semicircle (excluding the real axis) tends to zero as R approaches infinity (which is due to the fact $\zeta < 0$), and so we have

$$\int_{\infty}^{-\infty} \frac{\cos(2\pi\zeta x)}{1+(\frac{2\pi}{\lambda}x)^2}dx = -\frac{\lambda e^{\zeta\lambda}}{2}.$$

Thus, the original integral I that we were analyzing is equal to

$$\begin{aligned} I &= 2i\int_0^1\int_0^1\int_0^{\infty} \sin(2\pi\zeta x)\Big(\frac{2}{n\pi}\arctan(\frac{2\pi}{\lambda}x) - \frac{1}{n}\Big)dxdtds \\ &= 2i\int_0^1\int_0^1 \Big(-\frac{1}{2\pi\zeta n} + \frac{1}{n\pi\zeta\lambda}\frac{\lambda e^{\zeta\lambda}}{2}\Big)dtds \\ &= -\frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{\zeta n}dtds + \frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{\zeta\lambda}}{n\zeta}dtds \\ &= -\frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{s-t+m}dtds + \frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m}dtds. \end{aligned}$$

We just went through a lot of computation, so let's take a moment to regain focus on what we are doing. We have an element $(\widehat{B}_{n,r})$ that we are aiming to prove belongs to the kernel of τ. We have seen that to prove this, it is enough to show that the matrix whose $m^{th} = (i-j)^{th}$ entry is given by certain integrals when $m < -2n$ and 0 everywhere else goes to zero in norm. We've now computed precisely what the entries of this matrix are: they are

$$\frac{i}{m\pi} - \frac{i}{m\pi r^m} - \frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{s-t+m}dtds + \frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m}dtds.$$

To finish up the proof, we need only show that the Hilbert-Schmidt norm of the matrix goes to zero.

Recall that the Hilbert-Schmidt norm of a matrix A is given by

$$\|A\|_{HS}^2 = \sum_{i,j} |a_{ij}|^2.$$

The matrix we are considering is of the form

$$\begin{bmatrix} \vdots & \vdots & \vdots & \iddots \\ a_{-2n-3} & a_{-2n-4} & a_{-2n-5} & \cdots \\ a_{-2n-2} & a_{-2n-3} & a_{-2n-4} & \cdots \\ a_{-2n-1} & a_{-2n-2} & a_{-2n-3} & \cdots \end{bmatrix}$$

The Hilbert-Schmidt norm of this is

$$\sum_{k=1}^{\infty} k(a_{-2n-k})^2$$

and so we proceed by proving this sum will converge to zero for the known values of a_{-2n-k} we have computed. We will start by dealing with $\frac{i}{m\pi} - \frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{s-t+m}dtds$.

Notice that

$$\begin{aligned} \Big|\frac{i}{m\pi} - \frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{s-t+m}dtds\Big| &= \Big|\frac{i}{\pi}\int_0^1\int_0^1 \frac{1}{m} - \frac{1}{s-t+m}dtds\Big| \\ &= \Big|\frac{i}{\pi}\int_0^1\int_0^1 \frac{s-t}{m(s-t+m)}dtds\Big| \\ &\leq \frac{1}{\pi}\int_0^1\int_0^1 \frac{|s-t|}{|m||s-t+m|}dtds \\ &\leq \int_0^1\int_0^1 \frac{1}{|m||1+m|}dtds \\ &= \frac{1}{m(m+1)} \end{aligned}$$

Going back to the Hilbert-Schmidt norm for these 2 pieces, we thus have

$$\Big|\sum_{k=1}^{\infty} k(c_{-2n-k})^2\Big| = \Big|\sum_{k=1}^{\infty} k\Big(\frac{i}{\pi}\int_0^1\int_0^1 \frac{s-t}{(-2n-k)(s-t+(-2n-k))}dtds\Big)^2\Big|$$

$$\leq \sum_{k=1}^{\infty} \frac{k}{(-2n-k)^2(-2n-k+1)^2}$$

and this final sum converges to zero as n approaches ∞. All that remains now is to consider the Hilbert-Schmidt norm for the entries $-\frac{i}{m\pi r^m} + \frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m}dtds$.

We have

$$\frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m}dtds - \frac{i}{m\pi r^m} = \frac{i}{\pi}\int_0^1\int_0^1 \frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m} - \frac{e^{\frac{m}{n}\lambda}}{m}dtds + \frac{i}{\pi}\frac{e^{\frac{m}{n}\lambda}}{m} - \frac{i}{m\pi r^m}$$

Now,

$$\begin{aligned}
\left|\frac{e^{(\frac{s-t+m}{n})\lambda}}{s-t+m} - \frac{e^{\frac{m}{n}\lambda}}{m}\right| &= \left|e^{\frac{m}{n}\lambda}\right|\left|\frac{e^{(\frac{s-t}{n})\lambda}}{s-t+m} - \frac{1}{m}\right| \\
&= \left|e^{\frac{m}{n}\lambda}\right|\left|\frac{me^{(\frac{s-t}{n})\lambda} - s + t - m}{m(s-t+m)}\right| \\
&\leq \left|e^{\frac{m}{n}\lambda}\right|\frac{|m||e^{(\frac{s-t}{n})\lambda} - 1| + |s-t|}{|m||s-t+m|} \\
&\leq \left|e^{\frac{m}{n}\lambda}\right|\frac{|m||e^{(\frac{s-t}{n})\lambda} - 1| + 1}{|m||s-t+m|} \\
&= \frac{\left|e^{\frac{m}{n}\lambda}\right||e^{(\frac{s-t}{n})\lambda} - 1|}{|s-t+m|} + \frac{\left|e^{\frac{m}{n}\lambda}\right|}{|m||s-t+m|} \\
&\leq \frac{2\frac{\lambda}{n}e^{\frac{m}{n}\lambda}}{|m+1|} + \frac{\left|e^{\frac{m}{n}\lambda}\right|}{|m+1|^2}
\end{aligned}$$

where in the final inequality we are using the fact that $|e^x - 1| \leq 2|x|$ when $|x| \leq 1$.

All is well when considering the Hilbert-Schmidt norm with $\frac{\left|e^{\frac{m}{n}\lambda}\right|}{|m+1|^2}$ since here we will have an infinite sum with a power of 3 in the denominator, so let's focus our attention on $\frac{2\frac{\lambda}{n}e^{\frac{m}{n}\lambda}}{|m+1|}$. Here we will use the fact that we can bound

$\dfrac{k}{(-2n-k+1)^2}$ by some M. The sum of interest is

$$\begin{aligned}\sum_{k=1}^{\infty} k(c_{-2n-k})^2 &= \sum_{k=1}^{\infty} k\frac{(\frac{2\lambda}{n}e^{-\frac{2n+k}{n}\lambda})^2}{(-2n-k+1)^2} \\ &\leq \frac{4\lambda^2 M}{n^2}\sum_{k=1}^{\infty} e^{-\frac{2\lambda k}{n}}e^{-2\lambda} \\ &= \frac{4\lambda^2 M e^{-2\lambda}}{n^2}\frac{e^{-\frac{2\lambda}{n}}}{1-e^{-\frac{2\lambda}{n}}}.\end{aligned}$$

This final expression converges to zero; one can see this by noting that the numerator is bounded and that in the denominator one n from the n^2 can be used to deal with the $1-e^{-\frac{2\lambda}{n}}$ term (maybe more easily seen if we expand using a Taylor series) and the other n will give the convergence to zero.

The last thing to do is to deal with the terms $\dfrac{i}{\pi}\dfrac{e^{\frac{m}{n}\lambda}}{m} - \dfrac{i}{m\pi r^m}$. Here we are aiming to show

$$-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{k}{(2n+k)^2}\Big((e^{-\frac{\lambda}{n}})^{2n+k} - r^{2n+k}\Big)^2$$

converges to zero. We will consider the sum without the $-\dfrac{1}{\pi^2}$.

First we rewrite it as

$$\sum_{k=1}^{\infty}\frac{1}{n}\frac{k}{n}\frac{1}{(2+\frac{k}{n})^2}\Big(e^{-\lambda(2+\frac{k}{n})} - e^{n(2+\frac{k}{n})ln(r)}\Big)^2$$

Define $\lambda_{n,r} := -nln(r)$. Then $\lambda_{n,r} \to \lambda$ and so for sufficiently large n and r sufficiently close to 1 we have $\dfrac{\lambda}{2} < \lambda_{n,r} < \dfrac{3\lambda}{2}$. With this notation, the sum we are looking at is

$$\sum_{k=1}^{\infty}\frac{1}{n}\frac{k}{n}\frac{1}{(2+\frac{k}{n})^2}\Big(e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\Big)^2$$

For starters, we know that $\dfrac{k/n}{(2+\frac{k}{n})} \leq 1$ and so our sum can be bounded above by

$$\sum_{k=1}^{\infty} \frac{1}{n} \frac{1}{(2+\frac{k}{n})} \left(e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right)^2.$$

We have also that for sufficiently large n and r sufficiently close to 1 the bound $\dfrac{\lambda}{2} < \lambda_{n,r} < \dfrac{3\lambda}{2}$ which will imply $\left|e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right| \leq 1$, giving us

$$\sum_{k=1}^{\infty} \frac{1}{n} \frac{1}{(2+\frac{k}{n})} \left(e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right)^2 \leq \sum_{k=1}^{\infty} \frac{1}{n} \frac{1}{(2+\frac{k}{n})} \left|e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right|.$$

Observe that

$$\begin{aligned}
\left|e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right| &= \left|\int_{\lambda}^{\lambda_{n,r}} e^{-x(2+\frac{k}{n})}\left(2+\frac{k}{n}\right)dx\right| \\
&\leq |\lambda - \lambda_{n,r}|\left(2+\frac{k}{n}\right)e^{-\frac{\lambda}{2}(2+\frac{k}{n})}
\end{aligned}$$

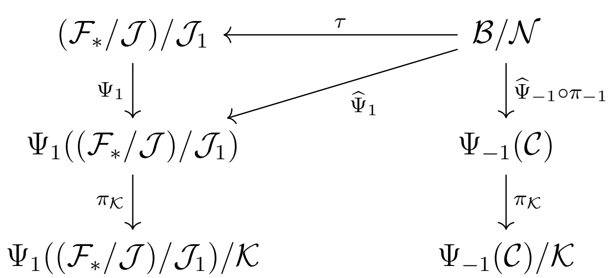
where the last inequality is also making use of the fact that for the x in the integral, $\dfrac{\lambda}{2} < x < \dfrac{3\lambda}{2}$. We thus have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{n} \frac{1}{(2+\frac{k}{n})} \left|e^{-\lambda(2+\frac{k}{n})} - e^{-\lambda_{n,r}(2+\frac{k}{n})}\right| &\leq \sum_{k=1}^{\infty} \frac{1}{n} \frac{1}{(2+\frac{k}{n})} |\lambda - \lambda_{n,r}|\left(2+\frac{k}{n}\right)e^{-\frac{\lambda}{2}(2+\frac{k}{n})} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{1}{n} |\lambda - \lambda_{n,r}| e^{-\frac{\lambda}{2}\frac{k}{n}} \\
&= \frac{e^{-\lambda}}{n} |\lambda - \lambda_{n,r}| \sum_{k=1}^{\infty} (e^{-\frac{\lambda}{2n}})^k \\
&\leq \frac{e^{-\lambda}}{n} |\lambda - \lambda_{n,r}| \sum_{k=0}^{\infty} (e^{-\frac{\lambda}{2n}})^k \\
&= \frac{e^{-\lambda}}{n} |\lambda - \lambda_{n,r}| \frac{1}{1 - e^{-\frac{\lambda}{2n}}}.
\end{aligned}$$

We know that $\dfrac{1}{n(1-e^{-\frac{\lambda}{2n}})}$ converges to $\dfrac{2}{\lambda}$ and since $\lambda_{n,r}$ converges to λ we have $\dfrac{e^{-\lambda}}{n}|\lambda - \lambda_{n,r}|\dfrac{1}{1-e^{-\frac{\lambda}{2n}}}$ converges to zero, as desired. Thus the Hilbert-Schmidt norm of the matrix we are considering converges to zero and so our element $(\widehat{B}_{n,r})$ is in the kernel of τ. $\square$

5.2 Identifying $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$

Recall the following scheme:



We just proved that $(\widehat{B}_{n,r}) \in \ker\tau$ and so clos $\mathrm{id}_{\mathcal{B}/\mathcal{N}}\{(\widehat{B}_{n,r})\} \subseteq \ker\tau$. Our next goal will be to show that $\ker\widehat{\Psi}_1 = \ker\tau$. Since $\widehat{\Psi}_1 = \Psi_1\circ\tau$, we automatically have $\ker\tau \subseteq \ker\widehat{\Psi}_1$. Our goal, then, is to show that $\ker\widehat{\Psi}_1 \subseteq \ker\tau$. We start with the following proposition. Recall that the maps $H^{\pm\pm}$, $H_{s,\infty}$, and $H_{\infty,0}$ are defined in Appendix A.

Proposition 5.2.1. *Let* $(B_{n,r}) \in \ker\widehat{\Psi}_1$. *Then*

(a) $\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r})))$ *is mapped to zero under* $H^{\pm\pm}$

(b) $\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r})))$ *is mapped to zero under* $H_{s,\infty}$ *for all* $s\in\mathbb{R}$

(c) The element $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))))$ *is contained in the closed ideal generated by* $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r}))))$.

In order to prove this proposition, it will be beneficial to view the algebra $H_{\infty,0}(\Psi_{-1}(\mathcal{C}))$ as the matrix algebra Σ_2^0 defined in (4.20). Recall that we have

$$H_{\infty,0}(\Psi_{-1}(\mathcal{C})) = \mathrm{alg}_{\mathcal{L}(L^2(\mathbb{R}))}\{\chi_{[0,\infty)}, S_{\mathbb{R}}\} \cong \Sigma_2^0$$

which has a very nice representation given in Proposition 4.4.2.

We will also need to make use of the following fact, which is stated and proven in [8], Proposition 2.2.1 (c)(iii).

Proposition 5.2.2. Σ_0^1 *is the smallest closed subalgebra of* Σ_1 *which contains* N^2 *and* $S_{\mathbb{R}^+}N^2$.

We are now fully equipped to prove Proposition 5.2.1.

Proof of Proposition 5.2.1. Let $(B_{n,r}) \in \ker \widehat{\Psi}_1$.

(a) and (b): To prove these two claims, we start by observing that the algebras $H^{\pm\pm}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1))$ are exactly the same as $H^{\pm\pm}(\Psi_1(\mathcal{C}))$ and similarly the algebras $H_{s,\infty}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1))$ are the same as $H_{s,\infty}(\Psi_1(\mathcal{C}))$. Then since an element in the kernel of $\widehat{\Psi}_1$ is zero in $H^{\pm\pm}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1))$ and $H_{s,\infty}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1))$, it suffices to show that there are injective maps from $H^{\pm\pm}(\Psi_1(\mathcal{C}))$ into $H^{\pm\pm}(\Psi_{-1}(\mathcal{C}))$ and $H_{s,\infty}(\Psi_1(\mathcal{C}))$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C}))$. But we have already constructed such maps in the proof of Proposition 4.4.3 and so our work is done.

(c): To prove $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))))$ is contained in the closed ideal generated by $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r}))))$, it is equivalent to show this claim for their images under Φ_η. The image of $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r}))))$ under Φ_η is $\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$.

We know by Proposition 4.4.2 (b) that

$$\Phi_\eta(H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))))) = \begin{pmatrix} \alpha_A I + \beta_A S + M^0(b_1) & M^0(b_2) \\ M^0(b_3) & \alpha_D I + \beta_D S + M^0(b_4) \end{pmatrix}$$
$$= M^0(B) + M^0(C)$$

where $B = \begin{pmatrix} \alpha_A 1 + \beta_A s & 0 \\ 0 & \alpha_D 1 + \beta_D s \end{pmatrix}$ and $C = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in (C^0_\infty(\mathbb{R}))^{2\times 2}$.

Define the map $M_{\pm\infty}$ by

$$M_{\pm\infty} : \Sigma^0_2 \to \mathbb{C}^4$$
$$M^0(B) + M^0(C) \mapsto (B(+\infty), B(-\infty))$$

Here $B(\pm\infty) = (B_{11}(\pm\infty), B_{22}(\pm\infty))$ where B_{ij} refers to the function in the ij^{th} position in the matrix B. Because $b(\pm\infty) = 0$ for $b \in C^0_\infty(\mathbb{R})$, this map factors through $(C^0_\infty(\mathbb{R}))^{2\times 2}$. Furthermore, since $s(+\infty) = 1$ and $s(-\infty) = -1$, we have that this map sends $M^0(B) + M^0(C)$ to $(\alpha_A + \beta_A, \alpha_A - \beta_A, \alpha_D + \beta_D, \alpha_D - \beta_D)$.

Following the elements in $\Psi_{-1}(\mathcal{C})$ into $H_{\infty,0}(\Psi_{-1}(\mathcal{C}))$, then into Σ^0_2 and finally into $\mathbb{C}^4$ via the map we have just defined, we have

$$\chi_{[0,\infty)} \mapsto (1,0,1,0) \qquad \chi_{[-1,\infty)} \mapsto (1,0,1,0)$$
$$\chi_{(-\infty,1]} \mapsto (0,1,0,1) \qquad P_{\mathbb{R}} \mapsto (1,0,0,1)$$

Recall the homomorphisms $H^{\pm\pm}$ defined in Appendix A.1, Equation (A.1). From Proposition A.1.2, these can be viewed as maps into $\mathbb{C}$. We now define the map σ on $\Psi_{-1}(\mathcal{C})/\mathcal{K}$ that sends an operator A to the 4-tuple

$$(H^{++}(A), H^{--}(A), H^{+-}(A), H^{-+}(A)) \in \mathbb{C}^4.$$

Amazingly, we have that the composition

$$\Psi_{-1}(\mathcal{C})/\mathcal{K} \xrightarrow{H_{\infty,0}} H_{\infty,0}(\Psi_{-1}(\mathcal{C})/\mathcal{K}) \xrightarrow{\Phi_\eta} \Phi_\eta(H_{\infty,0}(\Psi_{-1}(\mathcal{C})/\mathcal{K})) \xrightarrow{M_{\pm\infty}} \mathbb{C}^4$$

is exactly the same as doing the map

$$\Psi_{-1}(\mathcal{C}) \xrightarrow{\quad\sigma\quad} \mathbb{C}^4$$

But we have proven that for an element $(B_{n,r})$ in the kernel of $\widehat{\Psi}_1$,

$$H^{\pm\pm}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))) = 0.$$

Recalling that the image $(B_{n,r})$ in Σ_2^0 will be of the form

$$\begin{pmatrix} \alpha_A I + \beta_A S + M^0(b_1) & M^0(b_2) \\ M^0(b_3) & \alpha_D I + \beta_D S + M^0(b_4) \end{pmatrix},$$

we can conclude that if $(B_{n,r})$ in the kernel of $\widehat{\Psi}_1$, then

$$(\alpha_A + \beta_A, \alpha_A - \beta_A, \alpha_D + \beta_D, \alpha_D - \beta_D) = (0, 0, 0, 0).$$

In other words, $\alpha_A = \beta_A = \alpha_D = \beta_D = 0$. Thus, if $(B_{n,r}) \in \ker \widehat{\Psi}_1$, we have that

$$\Phi_\eta(H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))))) = \begin{pmatrix} M^0(b_1) & M^0(b_2) \\ M^0(b_3) & M^0(b_4) \end{pmatrix}$$

with $b_i \in C^0_\infty$. This means that to show $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))))$ is contained in the closed ideal generated by $H_{\infty,0}(\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r}))))$, it is equivalent to show that

$$\left\{ \begin{pmatrix} M^0(b_1) & M^0(b_2) \\ M^0(b_3) & M^0(b_4) \end{pmatrix} : b_i \in C^0_\infty \right\} \subseteq \operatorname{clos\,id}_{\Sigma^0_2} \left\{ \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \right\} =: \Sigma_3.$$

By Proposition 5.2.2, our work will be done if we prove that the matrices with N^2 in exactly one position and zero everywhere else and the matrices with SN^2 in exactly one position and zero everywhere else belong to Σ_3, along with the ability to multiply the N^2 and SN^2 on the left and right by arbitrary elements of Σ_1. This will imply that we can generate $M^0(b)$ with $b \in C^0_\infty$ in any position of the matrix and so our claim will be proven.

Let's start with the top left component. Notice that $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ are in Σ^0_2. Thus

$$\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} = \begin{pmatrix} N^2 & 0 \\ 0 & 0 \end{pmatrix} \in \Sigma_3$$

and so

$$\begin{pmatrix} N^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} SN^2 & 0 \\ 0 & 0 \end{pmatrix} \in \Sigma_3.$$

For $A, B \in \Sigma_1$ arbitrary, we can left multiply by the matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and right multiply by the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ to get the matrices with AN^2B and ASN^2B in the top left entry and zeros everywhere else. Therefore the matrix $\begin{pmatrix} M^0(b) & 0 \\ 0 & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3.

Next we show $\begin{pmatrix} 0 & 0 \\ M^0(b) & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3. Notice that

$$\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N^2 & 0 \end{pmatrix} \in \Sigma_3$$

and so

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 0 & 0 \\ N^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ SN^2 & 0 \end{pmatrix} \in \Sigma_3.$$

For arbitrary $A, B \in \Sigma_1$, we can left multiply by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ and right multiply by the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ to get the matrices with AN^2B and ASN^2B in the bottom left entry and zeros everywhere else. Therefore $\begin{pmatrix} 0 & 0 \\ M^0(b) & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3.

Let's now settle the bottom right component. Observe that

$$\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N^2 \end{pmatrix} \in \Sigma_3$$

and so

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & N^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & SN^2 \end{pmatrix} \in \Sigma_3.$$

For arbitrary $A, B \in \Sigma_1$, we can left multiply by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ and right multiply by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ to get the matrices with AN^2B and ASN^2B in the bottom right entry and zeros everywhere else. Therefore $\begin{pmatrix} 0 & 0 \\ M^0(b) & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3.

Finally, let's do the top right component. We have already shown that the matrix $\begin{pmatrix} M^0(b) & 0 \\ 0 & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3, so in particular $\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} AN & 0 \\ 0 & 0 \end{pmatrix}$ are in Σ_3. Thus

$$\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & N^2 \\ 0 & 0 \end{pmatrix} \in \Sigma_3$$

and

$$\begin{pmatrix} SN & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & SN^2 \\ 0 & 0 \end{pmatrix} \in \Sigma_3.$$

For arbitrary $A, B \in \Sigma_1$, we can left multiply by the matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and right multiply by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ to get the matrices with AN^2B and ASN^2B in the top right entry and zeros everywhere else. Therefore $\begin{pmatrix} 0 & M^0(b) \\ 0 & 0 \end{pmatrix}$ with $b \in C^0_\infty$ belongs to Σ_3 and our proof is complete.

□

The following lemma will be of great use.

Lemma 5.2.3. *Let $f : X \to Y$ be an open and continuous map between topological spaces. Then for any subset $V \subseteq Y$, $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$. Here $\overline{V}$ refers to the closure of V in Y and $\overline{f^{-1}(V)}$ refers to the closure of $f^{-1}(V)$ in X.*

Proof. Since f is continuous, $f^{-1}(\overline{V})$ is closed. By definition, $\overline{f^{-1}(V)}$ is the smallest closed subset of X containing $f^{-1}(V)$. Since $V \subseteq \overline{V}$, it follows that $f^{-1}(V) \subseteq f^{-1}(\overline{V})$ and so $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$.

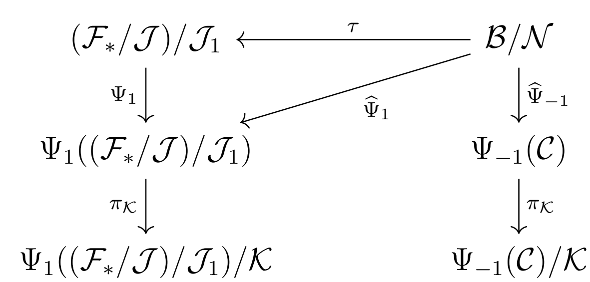
For the reverse containment, take $x \in f^{-1}(\overline{V})$ and let U be an open neighborhood of x. We want to show that $(U \setminus \{x\}) \cap f^{-1}(V) \neq \emptyset$ (i.e., that x is a limit point of $f^{-1}(V)$). Since $x \in f^{-1}(\overline{V})$, we know $f(x) \in \overline{V}$ and since $x \in U$ we have also that $f(x) \in f(U)$. This means $f(x) \in \overline{V} \cap f(U)$. The fact that $f(x) \in \overline{V}$ means that any open neighborhood of $f(x)$ minus the point $f(x)$ when intersected with V is nonempty. Since f is an open map by assumption, we have that $f(U)$ is an open neighborhood of $f(x)$. Thus $V \cap (f(U) \setminus \{f(x)\}) \neq \emptyset$ and hence $(U \setminus \{x\}) \cap f^{-1}(V) \neq \emptyset$, as desired. $\square$

Corollary 5.2.4. *Let* $(B_{n,r}) \in$ *ker* $\widehat{\Psi}_1$. *Then*

$$\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))) \in \textit{clos id}_{\ \Psi_{-1}(\mathcal{C})/\mathcal{K}}\{\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})))\}.$$

Proof. Recall from the Fredholm theory that the algebra $\Psi_{-1}(\mathcal{C})$ is $*$-isomorphic to a subalgebra of the direct sum of its images under a collection of homomorphisms. From Proposition 5.2.1, we may actually conclude that for $(B_{n,r}) \in \ker \widehat{\Psi}_1$, its image in this subalgebra of the direct sum is contained in the closed ideal generated by the image of $(\widehat{B}_{n,r})$ in this subalgebra of the direct sum. Now, the preimage of a principal ideal under an isomorphism is again a principal ideal and so using Lemma 5.2.3 we arrive at the claim. $\square$

Recall the following scheme that we worked hard to build:



Using Lemma 5.2.3 and Corollary 5.2.4 as the starting point, we will work our way up from the bottom of the scheme back to the top and make a statement about the kernel of $\widehat{\Psi}_1$.

Proposition 5.2.5. *Suppose* $(B_{n,r}) \in \mathcal{B}/\mathcal{N}$ *has the property that*

$$\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r}))) \in \textit{clos id}_{\ \Psi_{-1}(\mathcal{C})/\mathcal{K}}\{\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})))\}.$$

Then

$$\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r})) \in \textit{clos id}_{\ \Psi_{-1}(\mathcal{C})}\{\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})), \mathcal{K}\}.$$

In particular, $\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r})) \in \textit{clos id}_{\ \Psi_{-1}(\mathcal{C})}\{\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})), \mathcal{K}\}$ *if* $(B_{n,r}) \in \textit{ker}\ \widehat{\Psi}_1$.

Proof. Let $\mathfrak{J} = \text{id}_{\ \Psi_{-1}(\mathcal{C})/\mathcal{K}}\{\pi_{\mathcal{K}}(\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})))\}$ and $\mathfrak{L} = \text{id}_{\ \Psi_{-1}(\mathcal{C})}\{\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})), \mathcal{K}\}$. We will first show that $\pi_{\mathcal{K}}^{-1}(\mathfrak{J}) = \mathfrak{L}$. For ease of notation, set $y := \widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r}))$. We know from algebra that $\pi_{\mathcal{K}}^{-1}(\mathfrak{J})$ is an ideal of $\Psi_{-1}(\mathcal{C})$ containing y and $\mathcal{K}$. Then since $\mathfrak{L}$ is the smallest ideal containing y and $\mathcal{K}$, we have $\mathfrak{L} \subseteq \pi_{\mathcal{K}}^{-1}(\mathfrak{J})$.

For the reverse containment, take $x \in \pi_{\mathcal{K}}^{-1}(\mathfrak{J})$. Then $\pi_{\mathcal{K}}(x) \in \mathfrak{J}$. This means

$$\pi_{\mathcal{K}}(x) = \sum_i a_i \pi_{\mathcal{K}}(y) b_i$$

with $a_i, b_i \in \Psi_{-1}(\mathcal{C})/\mathcal{K}$. By surjectivity of $\pi_{\mathcal{K}}$, we have that there exist $c_i, d_i \in \Psi_{-1}(\mathcal{C})$ such that $a_i = \pi_{\mathcal{K}}(c_i)$ and $b_i = \pi_{\mathcal{K}}(d_i)$. Thus

$$\begin{aligned}
\pi_{\mathcal{K}}(x) &= \sum_i a_i \pi_{\mathcal{K}}(y) b_i \\
&= \sum_i \pi_{\mathcal{K}}(c_i)\pi_{\mathcal{K}}(y)\pi_{\mathcal{K}}(d_i) \\
&= \sum_i \pi_{\mathcal{K}}(c_i y d_i)
\end{aligned}$$

This gives

$$x + K = \sum_i (c_i y d_i + K_i)$$

for K, K_i compact and so

$$x = -K + \sum_i (c_i y d_i + K_i) \in \mathfrak{L}.$$

We have thus proven that $\pi_{\mathcal{K}}^{-1}(\mathfrak{J}) = \mathfrak{L}$. Notice that $\pi_{\mathcal{K}}$ is an open and continuous map (continuity is clear, and the fact that it is open can be shown directly or by invoking the Open Mapping Theorem since this is a continuous surjection between Banach spaces). We can thus invoke Lemma 5.2.3 to complete the proof. □

Proposition 5.2.6. *Suppose $(B_{n,r}) \in \mathcal{B}/\mathcal{N}$ has the property that*

$$\widehat{\Psi}_{-1}(\pi_{-1}(B_{n,r})) \in clos\ id\ _{\Psi_{-1}(\mathcal{C})}\{\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})), \mathcal{K}\}.$$

Then

$$\pi_{-1}(B_{n,r}) \in clos\ id\ _{(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}}\{(\pi_{-1}(\widehat{B}_{n,r})), (Y_{-1}E_{-n}\mathcal{K}E_nY_{-1})\}.$$

In particular, $\pi_{-1}(B_{n,r}) \in clos\ id\ _{(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}}\{(\pi_{-1}(\widehat{B}_{n,r})), (Y_{-1}E_{-n}\mathcal{K}E_nY_{-1})\}$ if $(B_{n,r}) \in ker\ \widehat{\Psi}_1$.

Proof. Because $\widehat{\Psi}_{-1}$ is an isomorphism, we have

$$\widehat{\Psi}_{-1}^{-1}(\text{id}\ _{\Psi_{-1}(\mathcal{C})}\{\widehat{\Psi}_{-1}(\pi_{-1}(\widehat{B}_{n,r})), \mathcal{K}\}) = \text{id}\ _{(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}}\{(\pi_{-1}(\widehat{B}_{n,r})), (Y_{-1}E_{-n}\mathcal{K}E_nY_{-1})\}$$

and since $\widehat{\Psi}_{-1}$ is an open map we again can use Lemma 5.2.3 to get the statement about closures. □

Theorem 5.2.7. *The kernel of $\widehat{\Psi}_1$ is equal to the kernel of τ.*

Proof. Because $\widehat{\Psi}_1 = \Psi_1 \circ \tau$, we have $\ker \tau \subseteq \ker \widehat{\Psi}_1$. We therefore only must show that $\ker \widehat{\Psi}_1 \subseteq \ker \tau$, so let $(B_{n,r}) \in \ker \widehat{\Psi}_1$ be arbitrary.

Recall from Corollary 4.4.5 that $\mathcal{B}/\mathcal{N}$ is $*$-isomorphic to a subalgebra of the direct sum $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) \bigoplus \Psi_{-1}(\mathcal{C})$. For the sake of this proof, we will actually consider it as a subalgebra of the direct sum $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) \bigoplus (\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$, which we are permitted to do because of Theorem 4.4.4. Being in the kernel of $\widehat{\Psi}_1$ means that the image in $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ is 0 and from the work we have done in the previous propositions, we know that if $(B_{n,r}) \in \ker \widehat{\Psi}_1$, then

$$\pi_{-1}(B_{n,r}) \in \text{clos id}_{\,(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}}\{(\pi_{-1}(\widehat{B}_{n,r})), (Y_{-1}E_{-n}\mathcal{K}E_nY_{-1})\}.$$

Recalling the definition of $(\widehat{B}_{n,r})$ in Equation 5.4, we can see directly that this element is sent to zero in $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ – in fact, it was constructed this way. We have also that, for compact K, $(Y_{-1}E_{-n}KE_nY_{-1})$ is mapped to 0 in $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ (this is a direct result of Lemma 4.3.1).

Putting all of this together, we have the following: when an element $(B_{n,r}) \in \ker \widehat{\Psi}_1$ is identified with its image in the subalgebra of the direct sum $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) \bigoplus (\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$, it is seen to belong to the closed ideal generated by $(0, \pi_{-1}(\widehat{B}_{n,r}))$ and $(0, Y_{-1}E_{-n}\mathcal{K}E_nY_{-1})$. Thus if we show both of these elements are in the kernel of τ our proof will be complete.

Because of Theorem 5.1.2 we have that $(\widehat{B}_{n,r}) \in \ker \tau$ and so $(0, \pi_{-1}(\widehat{B}_{n,r}))$ is sent to zero under τ. All that remains is proving $Y_{-1}E_{-n}KE_nY_{-1}$ is in the kernel of τ for $K \in \mathcal{K}$. To this end, let f be a continuous function vanishing in a neighborhood of -1 and equal to 1 at 1. Then mod $\mathcal{J}$ and $\mathcal{J}_1$ we have

$$\begin{aligned} Y_{-1}E_{-n}KE_nY_{-1} &= Y_{-1}R_r^*K'R_rY_{-1} \\ &= Y_{-1}R_r^*K'R_rY_{-1}L(f) \\ &= Y_{-1}R_r^*K'R_rY_{-1}L(f)Y_{-1}R_r^*R_rY_{-1} \\ &= Y_{-1}R_r^*K'L(C_rY_{-1}f)R_rY_{-1}. \end{aligned}$$

Notice that $(C_rY_{-1}f)(t) = f\left(-\dfrac{t+r}{1+rt}\right)$ and so $C_rY_{-1}f$ converges locally uniformly to $f(-1)I = 0$ on $\mathbb{S}^1 \setminus \{-1\}$. Thus $L(C_rY_{-1}f)$ converges strongly to 0 and hence $K'L(C_rY_{-1}f)$ converges in norm to 0. Then since $Y_{-1}R_r^*$ and R_rY_{-1} are uniformly bounded, we get that $Y_{-1}R_r^*K'L(C_rY_{-1}f)R_rY_{-1} \in \mathcal{N}$ and so the sequence $Y_{-1}E_{-n}KE_nY_{-1}$ is equal to 0 in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$. Thus $Y_{-1}E_{-n}KE_nY_{-1}$ is in the kernel of τ and we may conclude $\ker\widehat{\Psi}_1 \subseteq \ker\tau$. Therefore $\ker\widehat{\Psi}_1 = \ker\tau$, as desired. □

Corollary 5.2.8. *The local algebra* $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ *is* $*$*-isomorphic to the algebra of operators* $\mathcal{A}$.

Proof. Recall the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1 & \xleftarrow{\ \tau\ } & \mathcal{B}/\mathcal{N} \\
{\scriptstyle\Psi_1}\downarrow & \swarrow{\scriptstyle\widehat{\Psi}_1} & \\
\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1) & &
\end{array}$$

The only property of a $*-$isomorphism that is left for us to show for Ψ_1 is injectivity, so let $(F_{n,r})$ be in the kernel of Ψ_1. By surjectivity of τ, there exists a $(B_{n,r}) \in \mathcal{B}/\mathcal{N}$ such that $\tau(B_{n,r}) = (F_{n,r})$. Then

$$\widehat{\Psi}_1(B_{n,r}) = \Psi_1(\tau(B_{n,r})) = 0;$$

i.e., $(B_{n,r})$ is in the kernel of $\widehat{\Psi}_1$. Thus by Theorem 5.2.7 we have $(B_{n,r}) \in \ker\tau$. This means $(F_{n,r}) = 0$ and hence δ is injective. This shows that $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is $*-$isomorphic to $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$, and since we have already proven that is $*-$isomorphic to $\mathcal{A}$ in Theorem 4.2.10 the proof is complete. □

5.3 Summary and Main Results

We have just shown that the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is $*$-isomorphic to the algebra of operators $\mathcal{A}$. This tells us that invertibility in $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is equivalent to invertibility in $\mathcal{A}$. Recall from Proposition 3.2.5 that $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ is $*$-isomorphic to $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau$ for each $\tau \in \mathbb{S}^1$ via the rotation map which sends $(F_{n,r} + \mathcal{J}) + \mathcal{J}_1$ to $(Y_\tau F_{n,r} Y_\tau^{-1} + \mathcal{J}) + \mathcal{J}_\tau$. For each $\tau \in \mathbb{S}^1$, we define the map Ψ_τ on $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau$ by

$$\Psi_\tau(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n Y_\tau F_{n,r} Y_\tau^* E_{-n}. \tag{5.7}$$

Recall that this strong limit has dependence on λ given by the relationship $n(1-r) \to \lambda$ even though our notation does not reflect it. When $\tau = 1$ this is precisely the map Ψ_1 we have already examined when acting on $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$. From what we have already discussed, the fact that for each $\tau \in \mathbb{S}^1$ the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau$ is $*-$isomorphic to the algebra of operators $\Psi_\tau\left((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau\right)$ will follow once we prove that Ψ_τ is well-defined.

Lemma 5.3.1. *Let $\tau \in \mathbb{S}^1$ be fixed. Then the map*

$$\Psi_\tau : (\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau \to \Psi_\tau\left((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau\right)$$

defined by

$$\Psi_\tau(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n Y_\tau F_{n,r} Y_\tau^* E_{-n}$$

is well-defined. Moreover, it acts on each of the generators as follows:

$$
\begin{aligned}
(P) &\mapsto \chi_{[0,\infty)} & (Y_\tau^* R_r^* K_\tau R_r Y_\tau) &\mapsto Z_{\frac{\lambda}{2\pi}} \mathcal{F} F K_\tau F^{-1} \mathcal{F}^{-1} Z_{\frac{\lambda}{2\pi}}^{-1} \\
(P_n^+) &\mapsto \chi_{[-1,\infty)} & (L(G_{r,\tau}^{-1} f)) &\mapsto W^0\Big(f\Big(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x}\Big)\Big) \\
(Q_n^-) &\mapsto \chi_{(-\infty,1]} & (L(\chi_+)) &\mapsto \begin{cases} I & \textit{if } \tau \textit{ is on upper half plane} \\ 0 & \textit{if } \tau \textit{ is on lower half plane} \\ Q_{\mathbb{R}} & \textit{if } \tau = 1 \\ P_{\mathbb{R}} & \textit{if } \tau = -1 \end{cases}
\end{aligned}
$$

Proof. We first prove well-definedness. We start by observing that $\Psi_\tau(L(t)) = \tau I$; indeed, we have

$$
E_n Y_\tau L(t) Y_\tau^* E_{-n} = E_n L(\tau t) E_{-n} = \tau E_n L(t) E_{-n}
$$

which converges strongly to τI by Theorem 4.2.6. This fact automatically yields that $(L(f))$ is in the kernel of Ψ_τ when f is continuous and vanishing at τ (an approximation argument can be used). To show that (K) is in the kernel for K compact, we will use the representation

$$
K = \sum_{j,k \in \mathbb{Z}} a_{jk} U_{-j} (P U_1 - U_1 P) U_{k+1}
$$

where $U_m = L(t^m)$ is the shift operator as we have done before. Because of the fact $E_n Y_\tau P Y_\tau^* E_{-n} = E_n P E_{-n}$ converges strongly to $\chi_{[0,\infty)}$ (which was shown in Theorem 4.2.6) and $U_1 = L(t)$ converges strongly to a scalar multiple of the identity, if we can show that $E_n Y_\tau L(t^j) Y_\tau^* E_{-n}$ converges strongly to some limit for any fixed $j \in \mathbb{Z}$ the job will be done. By Lemma 4.2.2, we have

$$E_n Y_\tau L(t^j) Y_\tau^* E_{-n} = \tau^j L_n \mathcal{F} M(e^{2\pi i x \frac{j}{n}}) \mathcal{F}^{-1}$$

which converges strongly to $\tau^j I$. Thus constant compact sequences are mapped to zero under Ψ_τ. Finally, we handle sequences of the form $(U_n K U_{-n})$ and $(U_{-n} K U_n)$ where K is compact. We will show that $(U_n K U_{-n})$ is in the kernel of Ψ_τ; the argument for the other is analogous . Identifying $U_{\pm n}$ as $L(t^{\pm n})$, we have that

$$\begin{aligned} E_n Y_\tau U_n K U_{-n} Y_\tau^* E_{-n} &= E_n Y_\tau U_n Y_\tau^* E_{-n} E_n Y_\tau K Y_\tau^* E_{-n} E_n Y_\tau U_{-n} Y_\tau^* E_{-n} \\ &= L_n \mathcal{F} M(\tau^n e^{2\pi i x}) \mathcal{F}^{-1} (E_n Y_\tau K Y_\tau^* E_{-n}) L_n \mathcal{F} M(\tau^{-n} e^{-2\pi i x}) \mathcal{F}^{-1} \\ &= L_n \mathcal{F} M(e^{2\pi i x}) \mathcal{F}^{-1} (E_n Y_\tau K Y_\tau^* E_{-n}) L_n \mathcal{F} M(e^{-2\pi i x}) \mathcal{F}^{-1} \end{aligned}$$

where in the second equality we are using Lemma 4.2.2. Now using Proposition 4.2.1 we have $L_n \mathcal{F} M(e^{\pm 2\pi i x}) \mathcal{F}^{-1} = L_n U_{\pm 1}$ where $U_{\pm 1}$ is the shift operator defined in Equation 4.10. Since $L_n U_{\pm 1}$ converges strongly and we have already seen that $(E_n Y_\tau K Y_\tau^* E_{-n})$ converges strongly to zero, it follows that $(U_n K U_{-n})$ is in the kernel of Ψ_τ.

To finish the proof, we compute where each generator is sent. The images of the sequences $(P), (P_n^+)$, and (Q_n^-) are clear. For $L(\chi_+)$, observe that we have the equality $E_n Y_\tau L(\chi_+) Y_\tau^* E_{-n} = E_n L(Y_\tau \chi_+) E_{-n}$. Recall for $f \in PC$ we have the representation in (4.12). Using this representation with $f = Y_\tau \chi_+$ we have

$$Y_\tau \chi_+ = \chi_+(\tau + 0) \frac{1 - \sigma}{2} + \chi_+(\tau - 0) \frac{1 + \sigma}{2} + d$$

where d is continuous and vanishing at 1. Thus

$$\Psi_\tau(L(\chi_+)) = \begin{cases} I & \text{if } \tau \text{ is on upper half plane} \\ 0 & \text{if } \tau \text{ is on lower half plane} \\ Q_{\mathbb{R}} & \text{if } \tau = 1 \\ P_{\mathbb{R}} & \text{if } \tau = -1 \end{cases}$$

Next let's handle the element $(L(G_{r,\tau}^{-1}f))$. We have

$$E_n Y_\tau(L(G_{r,\tau}^{-1}f)Y_\tau^* E_{-n} = E_n L(C_r^{-1}f)E_{-n}$$

which converges strongly to $W^0\Big(f\Big(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\Big)\Big)$ as was proven in Theorem 4.2.6.

Finally we deal with $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$. Notice that

$$E_n Y_\tau Y_\tau^* R_r^* K_\tau R_r Y_\tau Y_\tau^* E_{-n} = E_n R_r^* K_\tau R_r E_{-n}$$

which converges strongly to $Z_{\frac{\lambda}{2\pi}} \mathcal{F} F K_\tau F^{-1} \mathcal{F}^{-1} Z_{\frac{\lambda}{2\pi}}^{-1}$, a fact we proved in Theorem 4.2.10.

□

Corollary 5.3.2. *For each $\tau \in \mathbb{S}^1$, the local algebra $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau$ is $*-$isomorphic to the algebra of operators $\Psi_\tau\big((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau\big)$.*

Theorem 5.3.3. *(Main Result) An element $(F_{n,r}) \in \mathcal{F}_*$ is $\lambda-$stable if and only if its images under $\Phi_0, \Phi_1, \Phi_{-1},$ and Ψ_τ are invertible for each $\tau \in \mathbb{S}^1$. Moreover, generators of $\mathcal{F}_*$ are mapped as follows by $\Phi_0, \Phi_1, \Phi_{-1},$ and Ψ_τ:*

Generator in $\mathcal{F}_*$	Φ_0	Φ_1	Φ_{-1}
(P)	P	I	0
(P_n^+)	I	I	P
(Q_n^-)	I	Q	I
$(L(a)), a \in PC$	$L(a)$	$L(a)$	$L(a)$
$(L(G_{r,t_0}^{-1} f)), f \in PC_{-1}^0$	0	0	0
(K)	K	0	0
$(Y_{t_0}^* R_r^* K_{t_0} R_r Y_{t_0})$	0	0	0

Table 5.1: Images of Generators of $\mathcal{F}_*$ under Φ_i for $i = 0, 1, -1$

Generator in $\mathcal{F}_*$	Ψ_τ
(P)	$\chi_{[0,\infty)}$
(P_n^+)	$\chi_{[-1,\infty)}$
(Q_n^-)	$\chi_{(-\infty,1]}$
$(L(a)), a \in PC$	$a(\tau+0)Q_{\mathbb{R}} + a(\tau-0)P_{\mathbb{R}}$
$(L(G_{r,t_0}^{-1} f)), f \in PC_{-1}^0$	$\begin{cases} W^0\left(f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right) & \text{if } t_0 = \tau \\ 0 & \text{if } t_0 \neq \tau \end{cases}$
(K)	0
$(Y_{t_0}^* R_r^* K_{t_0} R_r Y_{t_0})$	$\begin{cases} Z_{\frac{\lambda}{2\pi}} \mathcal{F} F K_\tau F^{-1} \mathcal{F}^{-1} Z_{\frac{\lambda}{2\pi}}^{-1} & \text{if } t_0 = \tau \\ 0 & \text{if } t_0 \neq \tau \end{cases}$

Table 5.2: Images of Generators of $\mathcal{F}_*$ under Ψ_τ

Proof. Theorem 3.1.8 tells us that an element $(F_{n,r}) \in \mathcal{F}_*$ is $\lambda-$stable if and only if its images under Φ_0, Φ_1, and Φ_{-1} are invertible and the coset $(F_{n,r}) + \mathcal{J}$ is invertible in $\mathcal{F}_*/\mathcal{J}$. Corollary 3.2.3 states that $(F_{n,r}) + \mathcal{J}$ is invertible in $\mathcal{F}_*/\mathcal{J}$ if and only if $(F_{n,r} + \mathcal{J}) + \mathcal{J}_\tau$ is invertible for each $\tau \in \mathbb{S}^1$, and finally Corollary 5.3.2 tells us that $(F_{n,r} + \mathcal{J}) + \mathcal{J}_\tau$ is invertible for each $\tau \in \mathbb{S}^1$ if and only if $\Psi_\tau(F_{n,r})$ is invertible for each $\tau \in \mathbb{S}^1$.

The images given in the table were computed and proven in Propositions 3.1.2, 3.1.3, and 3.1.4 and Lemma 5.3.1. □

Chapter 6

Relating to Approximate Identities

In this chapter we introduce an algebra $\mathcal{S}_\Omega(k_\omega, PC)$ of sequences of operators that are involving approximate identities. In the first section define approximate identities, provide examples, and introduce $\mathcal{S}_\Omega(k_\omega, PC)$. In the second section we establish a relationship between $\mathcal{S}_\Omega(k_\omega, PC)$ and our algebra $\mathcal{F}_*$, concluding by translating our stability criteria for $\mathcal{F}_*$ into stability criteria for $\mathcal{S}_\Omega(k_\omega, PC)$. Finally, we apply the result to some concrete examples.

6.1 Approximate Identities and the Algebra $\mathcal{S}_\Omega(k_\omega, PC)$

In order to define approximate identities we start with a function $K \in L^1(\mathbb{R})$ with the property

$$\int_{-\infty}^{\infty} K(x)dx = 1. \tag{6.1}$$

For $\omega \in [0, \infty)$, the bounded linear operator $k_\omega : L^\infty \to L^\infty$ defined by

$$(k_\omega f)(e^{ix}) = \int_{-\infty}^{\infty} f(e^{i(x-y)})\omega K(\omega y)dy$$

is called the *approximate identity* with kernel K. It can be shown that the functions $k_\omega f$ are continuous for each $\omega \in [0, \infty)$. Notice that (6.1) is the only requirement we impose on K; sometimes more things like non-negativity and decay conditions are required (for example, in [3], Section 3.14).

One example of an approximate identity is the Fejér-Cesaro means σ_n, $n \in \mathbb{N}$, defined in terms of Fourier coefficients by

$$(\sigma_n f)(e^{ix}) = \sum_{m=-n}^{n} \left(1 - \frac{|m|}{n+1}\right) f_m e^{imx}.$$

We also have

$$(\sigma_n f)(e^{ix}) = \int_0^{2\pi} k_n(x-t) f(e^{it})dt$$

where the Fejér kernel k_n is given by

$$k_n(x) = \frac{1}{2\pi(n+1)} \frac{\sin^2((n+1)x/2)}{\sin^2(x/2)}$$

for $x \in \mathbb{R}$.

If we let F denote the periodic extension of f, i.e., $F(x) = f(e^{ix})$ for $x \in \mathbb{R}$, then we have

$$(\sigma_n f)(e^{ix}) = \int_{-\infty}^{\infty} (n+1)K((n+1)(x-t))F(t)dt$$

with $K(x) = \dfrac{2}{\pi}\dfrac{\sin^2(x/2)}{x^2}$ (if interested, more information can be found in [1], Section 62).

Another example is the harmonic extension

$$h_\mu : \sum_{m=-\infty}^{\infty} e^{imx} a_m \mapsto \sum_{m=-\infty}^{\infty} \mu^{|m|} e^{imx} a_m$$

for $0 \le \mu < 1$. In this example we have $h_\mu a = k_\omega a$ with $K(x) = 1/(\pi(1+x^2))$ and $\omega = -1/\log \mu$.

Throughout what follows we let $\Omega \subseteq [0, \infty)$ be an unbounded index set and k_ω be an approximate identity. Much of the setup will parallel what we did for the algebra $\mathcal{F}$; this is not a coincidence. Let $\mathcal{S}_\Omega$ denote the set of all sequences $\{A_{n,\omega}\}_{n\in\mathbb{N},\omega\in\Omega}$ of bounded linear operators on L^2 for which

$$\|\{A_{n,\omega}\}\|_{\mathcal{S}_\Omega} := \sup_{\substack{n\in\mathbb{N}\\ \omega\in\Omega}} \|A_{n,\omega}\|_{\mathcal{L}(L^2(\mathbb{S}^1))} < \infty. \tag{6.2}$$

This space becomes a C^*-algebra when equipped with the above supremum norm and the following algebraic operations:

$$\begin{aligned} \{A_{n,\omega}\} + \{B_{n,\omega}\} &:= \{A_{n,\omega} + B_{n,\omega}\}, & z\{A_{n,\omega}\} &:= \{zA_{n,\omega}\}, \\ \{A_{n,\omega}\}\{B_{n,\omega}\} &:= \{A_{n,\omega}B_{n,\omega}\}, & \{A_{n,\omega}\}^* &:= \{A_{n,\omega}^*\}. \end{aligned}$$

We let $\mathcal{N}_\Omega$ denote the $*$-ideal of $\mathcal{S}_\Omega$ consisting of all sequence $\{A_{n,\omega}\}$ for which $\|\{A_{n,\omega}\}\| \to 0$ as $\omega \to \infty$ and let $\mathcal{S}_\Omega(k_\omega, PC)$ denote the smallest closed $*$-subalgebra of $\mathcal{S}_\Omega$ with $\mathcal{N}_\Omega \subseteq \mathcal{S}_\Omega(k_\omega, PC)$ containing the following elements:

$$(P), (P_n^+), (Q_n^-), (M(k_\omega a))$$

with $a \in PC$. In the next section we study stability criteria for this algebra.

6.2 Stability Criteria for $\mathcal{S}_\Omega(k_\omega, PC)$

We now proceed to relate the stability problem we solved in this thesis to a related problem involving approximate identities.

Proposition 6.2.1. *Let k_ω be an approximate identity and $a \in PC$. Then the sequence $\{a_r\}_{r\in[0,1)}$ defined by $a_r = k_\omega a$ with $\omega = (1+r)/(2(1-r))$ is contained in the smallest closed subalgebra of $\mathcal{F}$ containing constant sequences of piecewise continuous functions and sequences $\{G_{r,\tau}^{-1} f\}$ for $f \in PC_{\pm 1}^0$ and $\tau \in \mathbb{S}^1$.*

Proof. For a proof we refer the reader to [6], Proposition 4.6. There it is shown that

$$k_\omega a = a + \beta G_{r,\tau}^{-1}(f - \chi_+) + Y_\tau^{-1} n_r \tag{6.3}$$

where $\beta = a(\tau+0) - a(\tau-0)$, f is the function defined by

$$f\left(\frac{1+ix}{1-ix}\right) = \int_{-\infty}^{x} K(y)dy,$$

and $Y_\tau^{-1} n_r \in \mathcal{N}_\Omega$. □

We remark that the relationship $\omega = (1+r)/(2(1-r))$ also implicitly gives us a relationship between ω and n; that is, since $n(1-r) \to \lambda$ we have $\dfrac{n}{\omega} \to \lambda$.

Theorem 6.2.2. *Let $R = \{r \in [0,1) : \frac{1+r}{2(1-r)} \in \Omega\}$ and define the map*

$$\Xi : \mathcal{S}_\Omega(k_\omega, PC) \to \mathcal{F}_*$$
$$\{A_{n,\omega}\}_{n\in\mathbb{N},\omega\in\Omega} \mapsto \{A_{n,r}\}_{n\in\mathbb{N},r\in R}$$

Then a sequence $\{A_{n,\omega}\}_{n\in\mathbb{N},\omega\in\Omega}$ *is* $\lambda-$*stable if and only if the corresponding sequence* $\{A_{n,r}\}_{n\in\mathbb{N},r\in R}$ *is. In particular, a sequence* $\{A_{n,\omega}\}_{n\in\mathbb{N},\omega\in\Omega}$ *is* $\lambda-$*stable if and only if its images under* $\Phi_0\circ\Xi, \Phi_1\circ\Xi, \Phi_{-1}\circ\Xi,$ *and* $\Psi_\tau\circ\Xi$ *are all invertible. Moreover, these maps act on the generators of* $\mathcal{S}_\Omega(k_\omega, PC)$ *as follows:*

Generator in $\mathcal{S}_\Omega(k_\omega, PC)$	$\Phi_0\circ\Xi$	$\Phi_1\circ\Xi$	$\Phi_{-1}\circ\Xi$
(P)	P	I	0
(P_n^+)	I	I	P
(Q_n^-)	I	Q	I
$(M(k_\omega a)), a\in PC$	$L(a)$	$L(a)$	$L(a)$

Table 6.1: Images of Generators of $\mathcal{S}_\Omega(k_\omega, PC)$ under $\Phi_i\circ\Xi$ for $i=0,1,-1$

Generator in $\mathcal{S}_\Omega(k_\omega, PC)$	$\Psi_\tau\circ\Xi$
(P)	$\chi_{[0,\infty)}$
(P_n^+)	$\chi_{[-1,\infty)}$
(Q_n^-)	$\chi_{(-\infty,1]}$
$(M(k_\omega a)), a\in PC$	$a(\tau+0)W^0\left(f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)$ $+a(\tau-0)W^0\left(1-f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)$

Table 6.2: Images of Generators of $\mathcal{S}_\Omega(k_\omega, PC)$ under $\Psi_\tau\circ\Xi$

Proof. It is readily seen that Ξ is a $*$-homomorphism into $\mathcal{F}_*$; the generators are mapped to generators. The only one for which this may not be obvious is the multiplication operators, but this is established from Proposition 6.2.1. We may compose this map Ξ with each of our maps that give stability criteria for $\mathcal{F}_*$, resulting in a collection of $*$-homomorphisms from $\mathcal{S}_\Omega(k_\omega, PC)$ into algebras of operators. The statement about stability follows.

Regarding the table, much of it is the same as our table in the main results section. For the generator $(M(k_\omega a))$ with $a\in PC$ we are using the representation given in line (6.3) in the proof of Proposition 6.2.1 in order to arrive at our image under Ψ_τ. The function f that appears in the table is also defined there. $\square$

Remark 6.2.3. We remark that the map Ξ is actually a map into a subalgebra $\mathcal{F}_R$ of $\mathcal{F}_*$ due to the fact we are using the index set R instead of $[0,1)$; however, this is not an issue. In fact, the work done in this section is true for *any* unbounded index set $\Omega \subseteq [0,\infty)$. This can be seen through the viewpoint of something called *fractality.* The notion of fractality for usual sequences $(A_n)_{n\in\mathbb{N}}$ is described in [12], Section 4.2. The idea is as follows: let T be a homomorphism on an algebra of sequences, let $\nu : \mathbb{N} \to \mathbb{N}$ be a monotonically increasing map, and define Φ_ν as the map that sends a sequence (A_n) to the subsequence $(A_{\nu(n)})$. We call a homomorphism T a *fractal homomorphism* if for each ν there exists a homomorphism T_ν such that

$$T_\nu(\Phi_\nu(A_n)) = T(A_n).$$

This property is really saying that every subsequence of (A_n) contains the full information about the image of the entire sequence under T (this is the motivation for calling it fractal).

We can adapt this definition to work for our generalized sequences $(A_{n,r})$. We call a set $S \subseteq \mathbb{N} \times [0,1)$ $\lambda-$*suitable* if it has the property that there exist $(n_i, r_i) \in S$ such that $n_i(1-r_i) \to \lambda$ as $r_i \to 1$ and $n_i \to \infty$ as $i \to \infty$. With this, we are equipped to define fractality for generalized sequences. Similar to before, define the map Φ_S to be the map that ends a sequence $(A_{n,r})_{n\in\mathbb{N}, r\in[0,1)}$ to the subsequence $(A_{n,r})_{(n,r)\in S}$. We call T fractal if for all $\lambda-$suitable sets S, there exists a homomorphism T_S such that $T_S \circ \Phi_S = T$.

Our set R defined in Theorem 6.2.2 has the property that $\mathbb{N} \times R$ is $\lambda-$suitable by definition for any unbounded index set Ω. Even further, since the operators $\Phi_{-1}, \Phi_0, \Phi_1$, and Ψ_τ are all strong limit operators, they are defined on the subalgebra $\mathcal{F}_R$. Moreover, if we let $\mathcal{G}$ denote the direct sum of algebras that we have shown $\mathcal{F}_*/\mathcal{N}$ to be isomorphic to, the following diagram commutes (again, we are using the fact our operators are given by strong limits):

$$\begin{array}{ccc} \mathcal{F}_*/\mathcal{N} & \xrightarrow{\Phi_\nu} & \mathcal{F}_R/\mathcal{N} \\ & \nwarrow\!\!\searrow \quad \swarrow & \\ & \mathcal{G} & \end{array}$$

which is precisely what we need.

6.3 Concrete Examples

Let's now apply Theorem 6.2.2 to concrete examples, starting with sequences of finite sections of Toeplitz operators whose symbols are given by approximate identities applied to piecewise continuous functions. Recall that the function f appearing in the corollaries is defined by

$$f\left(\frac{1+ix}{1-ix}\right) = \int_{-\infty}^{x} K(y)dy$$

where K is the function coming from the approximate identities.

Corollary 6.3.1. *Let $a \in PC$. Then the sequence $(T_n(k_\omega a)) := (P_n T(k_\omega a) P_n)$ is stable if and only if the following operators are invertible:*

(i) $T(a)$

(ii) $T(\tilde{a})$

(iii) $a(\tau+0)\chi_{[0,1]} W^0\left(f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\chi_{[0,1]} + a(\tau-0)\chi_{[0,1]} W^0\left(1 - f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\chi_{[0,1]}$ *for each* $\tau \in \mathbb{S}^1$

Proof. This is a direct result of Theorem 6.2.2. The only simplification we have done is rewriting $QL(a)Q = JT(\tilde{a})J$ in order to see that invertibility of $QL(a)Q$ is equivalent to $T(\tilde{a})$ being invertible. $\square$

Remark 6.3.2. If we set $A := \chi_{[0,1]} W^0 \left(f\left(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x} \right) \right) \chi_{[0,1]}$, condition (iii) can be expressed as requiring the operator

$$A + \frac{a(\tau - 0)}{a(\tau + 0) - a(\tau - 0)} I$$

be invertible where $I = \chi_{[0,1]}$ (we may assume $a(\tau + 0) - a(\tau - 0) \neq 0$ since otherwise condition (iii) is a scalar multiple of $\chi_{[0,1]}$). This is now a question of the spectrum of A. By utlizing Fredholm Theory, one can realize that the essential spectrum of this operator A is

$$\sigma_{\text{ess}}(A) = [0, 1].$$

In the case where $\overline{\text{im } f} \subseteq [0, 1]$, then we may conclude even further that the spectrum of A is also equal to $[0, 1]$ (this has to do with positivity in a Hilbert space). To summarize, we have that condition (iii) is requiring the invertibility of the operator

$$A + \frac{a(\tau - 0)}{a(\tau + 0) - a(\tau - 0)} I$$

with $A := \chi_{[0,1]} W^0 \left(f\left(\frac{\lambda - 2\pi i x}{\lambda + 2\pi i x} \right) \right) \chi_{[0,1]}$ which is equivalent to requiring $\frac{a(\tau-0)}{a(\tau+0)-a(\tau-0)}$ to not be in the spectrum of A. When $\overline{\text{im } f} \subseteq [0, 1]$, this boils down to the condition $\frac{a(\tau-0)}{a(\tau+0)-a(\tau-0)} \notin [0, 1]$ which can further be expressed as requiring the line segment connecting $a(\tau - 0)$ and $a(\tau + 0)$ not crossing 0.

Next we apply our stability criteria to sequences of finite sections of Toeplitz operators with piecewise continuous symbols.

Corollary 6.3.3. *Let $a \in PC$. Then the sequence $(T_n(a)) := (P_n T(a) P_n)$ is stable if and only if the following operators are invertible:*

(i) $T(a)$

(ii) $T(\tilde{a})$

(iii) $\chi_{[0,1]}(a(\tau+0)Q_{\mathbb{R}} + a(\tau-0)P_{\mathbb{R}})\chi_{[0,1]}$ *for each* $\tau \in \mathbb{S}^1$

We now apply our stability criteria to sequences of Toeplitz operators whose symbols are given by approximate identities applied to piecewise continuous functions.

Corollary 6.3.4. *Let $a \in PC$. Then the sequence $(T(k_\omega a))$ is stable if and only if the following operators are invertible:*

(i) $T(a)$

(ii) $L(a)$

(iii) $a(\tau+0)\chi_{[0,\infty)}W^0\left(f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\chi_{[0,\infty)}+a(\tau-0)\chi_{[0,\infty)}W^0\left(1-f\left(\frac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\chi_{[0,\infty)}$ *for each* $\tau \in \mathbb{S}^1$

Remark 6.3.5. We remark that the operator in (iii) is a Wiener-Hopf operator with piecewise continuous symbol. By rewriting it in a similar way to what we did in Remark 6.3.2, its invertibility can be reduced to a question of spectrum and one can find it if needed.

Finally, we apply the result to singular integral operators.

Corollary 6.3.6. *Let $a, b \in PC$. Then the sequence $(M(k_\omega a)P + M(k_\omega b)Q)$ is stable if and only if the following operators are invertible:*

(i) $L(a)P + L(b)Q$

(ii) $L(a)$

(iii) $L(b)$

(iv) $$\left(a(\tau+0)W^0\left(f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)+a(\tau-0)W^0\left(1-f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\right)\chi_{[0,\infty)} + \left(b(\tau+0)W^0\left(f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)+b(\tau-0)W^0\left(1-f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\right)\chi_{(-\infty,0)}$$ *for each* $\tau \in \mathbb{S}^1$

Corollary 6.3.7. *Let* $a, b \in PC$*. Then the sequence* $(P_n(M(k_\omega a)P + M(k_\omega b)Q)P_n)$ *is stable if and only if the following operators are invertible:*

(i) $L(a)P + L(b)Q$

(ii) $T(\tilde{a})$

(iii) $T(b)$

(iv) $$\chi_{[-1,1]}\left(a(\tau+0)W^0\left(f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)+a(\tau-0)W^0\left(1-f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\right)\chi_{[0,1]} + \chi_{[-1,1]}\left(b(\tau+0)W^0\left(f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)+b(\tau-0)W^0\left(1-f\left(\tfrac{\lambda-2\pi ix}{\lambda+2\pi ix}\right)\right)\right)\chi_{[-1,0)}$$ *for each* $\tau \in \mathbb{S}^1$

Corollary 6.3.8. *Let* $a, b \in PC$*. Then the sequence* $(P_n(M(a)P + M(b)Q)P_n)$ *is stable if and only if the following operators are invertible:*

(i) $L(a)P + L(b)Q$

(ii) $T(\tilde{a})$

(iii) $T(b)$

(iv) $\chi_{[-1,1]}(a(\tau+0)Q_{\mathbb{R}}+a(\tau-0)P_{\mathbb{R}})\chi_{(0,1]}+\chi_{[-1,1]}(b(\tau+0)Q_{\mathbb{R}}+b(\tau-0)P_{\mathbb{R}})\chi_{(-1,0)}$ *for each* $\tau\in\mathbb{S}^1$

Appendix A

Fredholm Theory for Algebra of Fourier Convolutions and Multiplication Operators

In this appendix we treat the Fredholm Theory for the algebra generated by Fourier convolution operators with piecewise continuous symbols and multiplication operators. We start with the general Fredholm Theory for this algebra and since many of the algebras we consider in this thesis are subalgebras of this, we discuss its specialization to them in the subsequent subsection.

A.1 The General Theory

For the general Fredholm Theory presented here we follow the work of [10]. In this discussion, we will specialize to the unweighted L^2 spaces. We let $\mathcal{A}(PC(\dot{\mathbb{R}}), PC)$ denote the smallest closed subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ containing all operators aI of multiplication by a function $a \in PC(\dot{\mathbb{R}})$ and all Fourier convolutions $W^0(b)$ for

$b \in PC$. We write $\mathcal{A}^{\mathcal{K}}(PC(\dot{\mathbb{R}}), PC)$ to denote the image of this algebra in the Calkin algebra (i.e., $\mathcal{A}^{\mathcal{K}}(PC(\dot{\mathbb{R}}), PC)$ is $\mathcal{A}(PC(\dot{\mathbb{R}}), PC)$ modulo the space of compact operators). Recall for $s, t \in \mathbb{R}$ and $\tau > 0$, we have defined the following kinds of shift operators[1]:

$$M_s : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (M_s f)(x) = e^{2\pi i x s} f(x)$$

$$U_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (U_t f)(x) = f(x - t)$$

$$Z_\tau : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+), (Z_\tau f)(x) = \tau^{\frac{1}{2}} f(\tau x)$$

With these definitions, we are equipped to define some strong limit operators. For $A \in \mathcal{L}(L^2(\mathbb{R}))$, let

$$H^{\pm\pm}(A) = \operatorname*{s-lim}_{t\to\pm\infty} \operatorname*{s-lim}_{s\to\pm\infty} M_{-t} U_{-s} A U_s M_t \tag{A.1}$$

Here the first superscript in $H^{\pm\pm}$ refers to the strong limit with respect to $s \to \pm\infty$ and the second one with respect to $t \to \pm\infty$. We also define the following strong limits for $s, t \in \mathbb{R}$:

$$H_{s,\infty}(A) = \operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1} U_{-s} A U_s Z_\tau \tag{A.2}$$

$$H_{\infty,t}(A) = \operatorname*{s-lim}_{\tau\to\infty} Z_\tau M_{-t} A M_t Z_\tau^{-1} \tag{A.3}$$

[1] We remark that if the reader refers to [10], there the notation for these operators is slightly different: our M_s is the same as their U_{-s}; our U_t is the same as their V_t; and our Z_τ is the same as their Z_τ^{-1}.

Interestingly, the invertibility of the images of an operator $A \in \mathcal{A}(PC(\dot{\mathbb{R}}), PC)$ under each of these homomorphisms is enough to tell us if A is Fredholm; indeed, we have the following:

Theorem A.1.1. *The algebra $\mathcal{A}^{\mathcal{K}}(PC(\dot{\mathbb{R}}), PC)$ is inverse closed in the Calkin algebra $\mathcal{L}(L^2(\mathbb{R}))/\mathcal{K}(L^2(\mathbb{R}))$ and an operator $A \in \mathcal{A}(PC(\dot{\mathbb{R}}), PC)$ is Fredholm if and only if $H_{s,\infty}(A), H_{\infty,t}(A)$ and $H^{\pm\pm}(A)$ are invertible $\forall s, t \in \mathbb{R}$.*

Proof. For a proof of this statement, we refer the reader to [10], Theorem 5.6.2 and Corollary 5.6.3. □

Notice that since being Fredholm is equivalent to being invertible modulo compact operators, we may actually view the Calkin algebra as isomorphic to a subalgebra of the direct sum of $H^{\pm\pm}(\mathcal{A}(PC(\dot{\mathbb{R}}), PC))$ with $H_{s,\infty}(\mathcal{A}(PC(\dot{\mathbb{R}}), PC))$ and $H_{\infty,t}(\mathcal{A}(PC(\dot{\mathbb{R}}), PC))$ for all $s, t \in \mathbb{R}$ (here we are making use of the C^*-algebra properties to make this conclusion). Because these homomorphisms give us the information we need to understand when an operator is Fredholm in our algebra, it is worthwhile to see how they act on the elements of the algebra $\mathcal{A}(PC(\dot{\mathbb{R}}), PC)$. The following three propositions achieve this.

Proposition A.1.2. *The strong limits $H^{\pm\pm}$ exist for $A \in \mathcal{A}(PC(\dot{\mathbb{R}}), PC)$ and these mappings are algebra homomorphisms onto the algebra $\mathbb{C}I$. In particular, for $a \in PC(\dot{\mathbb{R}})$ and $b \in PC$,*

$$\begin{array}{ll} H^{+\pm}(aI) = a(+\infty)I, & H^{-\pm}(aI) = a(-\infty)I, \\ H^{\pm+}(W^0(b)) = b(+\infty)I, & H^{\pm-}(W^0(b)) = b(-\infty)I, \end{array}$$

and $H^{\pm\pm}(K) = 0$ for compact operators K.

Proof. Let's first deal with aI for $a \in PC(\dot{\mathbb{R}})$. We will think of this as a multiplication operator and write it as $M(a)$. Here we have

$$M_{-t}U_{-s}M(a)U_sM_t = M_{-t}M(\hat{a})M_t = M(\hat{a})$$

where $\hat{a}(x) = a(x+s)$. Then as $s \to \infty$ $M(\hat{a})$, converges strongly to $a(+\infty)I$. Similarly, as $s \to -\infty$ we have $M(\hat{a})$, converges strongly to $a(-\infty)I$. Thus we have proven the claim for aI.

Let's now discuss $W^0(b)$. We will use Proposition 4.2.1 often in our computation. Here we have

$$\begin{aligned} M_{-t}U_{-s}W^0(b)U_sM_t &= M_{-t}U_{-s}\mathcal{F}^{-1}M(b)\mathcal{F}U_sM_t \\ &= M_{-t}\mathcal{F}^{-1}M_{-s}M(b)M_s\mathcal{F}M_t \\ &= \mathcal{F}^{-1}U_{-t}M_{-s}M(b)M_sU_t\mathcal{F} \\ &= \mathcal{F}^{-1}U_{-t}M(b)U_t\mathcal{F} \\ &= \mathcal{F}^{-1}M(\hat{b})\mathcal{F}. \end{aligned}$$

where $\hat{b}(x) = b(x+t)$. Thus as t approaches infinity, $\mathcal{F}^{-1}M(\hat{b})\mathcal{F}$ converges strongly to $b(+\infty)I$. Similarly, as $t \to -\infty$ we will have strong convergence to $b(-\infty)I$ as desired. □

Proposition A.1.3. *Let $a \in L^\infty(\mathbb{R})$, $b \in PC$, and let χ_+ (resp. χ_-) denote the characteristic function for the positive (resp. negative) real axis. Then for $t \in \mathbb{R}$,*

(i) $H_{\infty,t}(aI) = a(-\infty)\chi_-I + a(+\infty)\chi_+I$

(ii) $H_{\infty,t}(W^0(b)) = b(t^-)Q_{\mathbb{R}} + b(t^+)P_{\mathbb{R}}$

Proof. As before, for $a \in L^\infty(\mathbb{R})$ we think of aI as a multiplication operator $M(a)$.

We have

$$Z_\tau M_{-t} M(a) M_t Z_\tau^{-1} = Z_\tau M(a) Z_\tau^{-1} = M(\hat{a})$$

where $\hat{a}(x) = a(\tau x)$. Then if $x < 0$, we have strong convergence of $M(\hat{a})$ to $a(-\infty)I$ as $\tau \to \infty$ and similarly if $x > 0$ we have strong convergence to $a(+\infty)I$ as $\tau \to \infty$. Thus, aI is mapped to $a(-\infty)\chi_- I + a(+\infty)\chi_+ I$.

Let's now consider $W^0(b)$ for $b \in PC$. It is sufficient to prove the claim for $t = 0$ (the other cases will follow from this since they correspond to shifting the symbol b). We may write b as

$$b(0^-)\chi_- + b(0^+)\chi_+ + c$$

where $c \in PC$ is continuous and vanishing at 0. Using the representation of $P_\mathbb{R}$ in (4.5), we see that $W^0(b(0^-)\chi_- + b(0^+)\chi_+) = b(0^-)Q_\mathbb{R} + b(0^+)P_\mathbb{R}$, so our job will be done if we show $Z_\tau W^0(c) Z_\tau^{-1}$ converges strongly to 0 as $\tau \to \infty$. But this is clear, since $Z_\tau W^0(c) Z_\tau^{-1} = W^0(\hat{c})$ where $\hat{c}(x) = c(\frac{x}{\tau})$ and $c(0) = 0$.

□

Proposition A.1.4. *Let $s \in \mathbb{R}, a \in PC$, and b be a Fourier multiplier. Then*

(i) $H_{s,\infty}(aI) = a(s^-)\chi_- I + a(s^+)\chi_+ I$

(ii) $H_{s,\infty}(W^0(b)) = b(-\infty)Q_\mathbb{R} + b(+\infty)P_\mathbb{R}$

Proof. This can be proven in a similar way to the previous proposition by following the definitions. □

The following tables show the images of the elements belonging to $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ and $\Psi_{-1}(\mathcal{C})$ under each of these homomorphisms. In the following sections we will see that in these cases, only some of these homomorphisms play a role.

$(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$	Image under Ψ_1	H^{++}	H^{+-}	H^{--}	H^{-+}
(P)	$\chi_{[0,\infty)}$	I	I	0	0
$(L(\chi_+))$	$Q_{\mathbb{R}}$	0	I	I	0
$(L(t^{-n})PL(t^n))$	$\chi_{[-1,\infty)}$	I	I	0	0
$(L(t^n)QL(t^{-n}))$	$\chi_{(-\infty,1]}$	0	0	I	I
$(L(C_r^{-1}f))$ $f \in PC_{-1}^0$	$W^0(\hat{g})$ $\hat{g}(x) = f(\frac{\lambda-2\pi ix}{\lambda+2\pi ix})$	0	0	0	0
$(R_r^* K R_r)$	K'	0	0	0	0

Table A.1: Images of the Generators of $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ under $H^{\pm\pm}$

$(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$	Image under Ψ_1	$H_{s,\infty}$	$H_{\infty,t}$
(P)	$\chi_{[0,\infty)}$	$\begin{cases} 0 & \text{if } s<0 \\ \chi_{[0,\infty)} & \text{if } s=0 \\ I & \text{if } s>0 \end{cases}$	$\chi_{[0,\infty)}$
$(L(\chi_+))$	$Q_{\mathbb{R}}$	$Q_{\mathbb{R}}$	$\begin{cases} I & \text{if } t<0 \\ Q_{\mathbb{R}} & \text{if } t=0 \\ 0 & \text{if } t>0 \end{cases}$
$(L(t^{-n})PL(t^n))$	$\chi_{[-1,\infty)}$	$\begin{cases} 0 & \text{if } s<-1 \\ \chi_{[0,\infty)} & \text{if } s=-1 \\ I & \text{if } s>-1 \end{cases}$	$\chi_{[0,\infty)}$
$(L(t^n)QL(t^{-n}))$	$\chi_{(-\infty,1]}$	$\begin{cases} I & \text{if } s<1 \\ \chi_{(-\infty,0)} & \text{if } s=1 \\ 0 & \text{if } s>1 \end{cases}$	$\chi_{(-\infty,0)}$
$(L(C_r^{-1}f))$ $f \in PC_{-1}^0$	$W^0(\hat{g})$ $\hat{g}(x) = f(\frac{\lambda-2\pi ix}{\lambda+2\pi ix})$	0	$\hat{g}(t^-)Q_{\mathbb{R}} + \hat{g}(t^+)P_{\mathbb{R}}$
$(R_r^* K R_r)$	K'	0	0

Table A.2: Images of the Generators of $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ under $H_{s,\infty}$ and $H_{\infty,t}$

Generator in $\mathcal{B}/\mathcal{N}$	Image in $\Psi_{-1}(\mathcal{C})$	H^{++}	H^{+-}	H^{--}	H^{-+}
(P)	$\chi_{[0,\infty)}$	I	I	0	0
$(L(\chi_+))$	$P_{\mathbb{R}}$	I	0	0	I
$(L(t^{-n})PL(t^n))$	$\chi_{[-1,\infty)}$	I	I	0	0
$(L(t^n)QL(t^{-n}))$	$\chi_{(-\infty,1]}$	0	0	I	I
$(L(C_r^{-1}f))$ $f \in PC_{-1}^0$	0	0	0	0	0
$(R_r^* K R_r)$	0	0	0	0	0
$(Y_{-1}R_r^* K' R_r Y_{-1})$	K'	0	0	0	0

Table A.3: Images of the Generators of $\mathcal{B}/\mathcal{N}$ under $H^{\pm\pm}$

Generator in $\mathcal{B}/\mathcal{N}$	Image in $\Psi_{-1}(\mathcal{C})$	$H_{s,\infty}$	$H_{\infty,t}$
(P)	$\chi_{[0,\infty)}$	$\begin{cases} 0 & \text{if } s < 0 \\ \chi_{[0,\infty)} & \text{if } s = 0 \\ I & \text{if } s > 0 \end{cases}$	$\chi_{[0,\infty)}$
$(L(\chi_+))$	$P_{\mathbb{R}}$	$P_{\mathbb{R}}$	$\begin{cases} 0 & \text{if } t < 0 \\ P_{\mathbb{R}} & \text{if } t = 0 \\ I & \text{if } t > 0 \end{cases}$
$(L(t^{-n})PL(t^n))$	$\chi_{[-1,\infty)}$	$\begin{cases} 0 & \text{if } s < -1 \\ \chi_{[0,\infty)} & \text{if } s = -1 \\ I & \text{if } s > -1 \end{cases}$	$\chi_{[0,\infty)}$
$(L(t^n)QL(t^{-n}))$	$\chi_{(-\infty,1]}$	$\begin{cases} I & \text{if } s < 1 \\ \chi_{(-\infty,0)} & \text{if } s = 1 \\ 0 & \text{if } s > 1 \end{cases}$	$\chi_{(-\infty,0)}$
$(L(C_r^{-1}f))$ $f \in PC_{-1}^0$	0	0	0
$(R_r^* K R_r)$	0	0	0
$(Y_{-1}R_r^* K' R_r Y_{-1})$	K'	0	0

Table A.4: Images of the Generators of $\mathcal{B}/\mathcal{N}$ under $H_{s,\infty}$ and $H_{\infty,t}$

A.2 The Fredholm Theory Applied to $\Psi_1(\mathcal{C})$, $\Psi_{-1}(\mathcal{C})$, and $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$

For the larger algebra $\mathcal{A}(PC(\dot{\mathbb{R}}), PC)$, each of the homomorphisms $H^{\pm\pm}$, $H_{s,\infty}$, and $H_{\infty,t}$ are important $\forall s, t \in \mathbb{R}$; however, for our subalgebras some of these maps are redundant. We start by examining $H_{s,\infty}$.

Proposition A.2.1. *Let A be an element in $\Psi_1(\mathcal{C})/\mathcal{K}$ or $\Psi_{-1}(\mathcal{C})/\mathcal{K}$. Then if $H_{s,\infty}(A)$ is invertible for $s = 1, -1$, and 0, then $H_{s,\infty}(A)$ is invertible for all $s \in \mathbb{R}$. In other words, the algebras $H_{s,\infty}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)/\mathcal{K})$ and $H_{s,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ are redundant in the Fredholm Theory for $s \neq \pm 1, 0$.*

Proof. To prove that invertibility of $H_{-1,\infty}(A)$ implies invertibility of $H_{s,\infty}(A)$ for $s < -1$ for $A \in \Psi_1(\mathcal{C})/\mathcal{K}$, we construct a map Λ_{-1} from $H_{-1,\infty}(\Psi_1(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_1(\mathcal{C})/\mathcal{K})$ for $s < -1$ so that the following diagram commutes:

$$\begin{array}{ccccc} & & \Psi_1(\mathcal{C})/\mathcal{K} & & \\ & \swarrow & & \searrow & \\ H_{-1,\infty}(\Psi_1(\mathcal{C})/\mathcal{K}) & & \xrightarrow[\Lambda_{-1}]{\hspace{3cm}} & & H_{s,\infty}(\Psi_1(\mathcal{C})/\mathcal{K}) \end{array}$$

This boils down to finding a multiplicative map that sends $Q_{\mathbb{R}}$ to itself and $\chi_{[0,\infty)}$ to 0 (one can see this by tracking the generators in Tables A.1 and A.2). The map Λ_{-1} defined by $\Lambda_{-1}(A) = \operatorname{s-lim}_{t\to\infty} Z_t^{-1} U_1 A U_{-1} Z_t$ does the job. Indeed, for $\chi_{[0,\infty)}$ we've got

$$Z_t^{-1} U_1 \chi_{[0,\infty)} U_{-1} Z_t = \chi_{[t,\infty)}$$

which converges strongly to 0 as t goes to infinity. For $Q_{\mathbb{R}}$, we have

$$
\begin{aligned}
Z_t^{-1}U_1Q_{\mathbb{R}}U_{-1}Z_t &= Z_t^{-1}U_1\mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x)}{2}\mathcal{F}U_{-1}Z_t \\
&= \mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x/t)}{2}\mathcal{F} \\
&= \mathcal{F}^{-1}\frac{1-\operatorname{sgn}(x)}{2}\mathcal{F} = Q_{\mathbb{R}}
\end{aligned}
$$

where in the second equality we are using Proposition 4.2.1 and Lemma 4.2.9. This same map works for mapping $H_{-1,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ for $s < -1$, since the only difference in requirement is leaving $P_{\mathbb{R}}$ invariant (which this map Λ_{-1} also does).

Next we show that invertibility of $H_{0,\infty}(A)$ implies invertibility of $H_{s,\infty}(A)$ for $-1 < s < 0$ for $A \in \Psi_1(\mathcal{C})/\mathcal{K}$. But this reduces to finding a map that sends $Q_{\mathbb{R}}$ to itself and $\chi_{[0,\infty)}$ to 0, which we have already demonstrated. Again, the same map still works for mapping $H_{0,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ for $-1 < s < 0$.

To show that invertibility of $H_{1,\infty}(A)$ implies invertibility of $H_{s,\infty}(A)$ for $0 < s < 1$ for $A \in \Psi_1(\mathcal{C})/\mathcal{K}$, we again construct a map Λ such that the following diagram commutes for $0 < s < 1$:

$$
\begin{array}{ccc}
 & \Psi_1(\mathcal{C})/\mathcal{K} & \\
\swarrow & & \searrow \\
H_{1,\infty}(\Psi_1(\mathcal{C})/\mathcal{K}) & \xrightarrow[\Lambda]{} & H_{s,\infty}(\Psi_1(\mathcal{C})/\mathcal{K})
\end{array}
$$

But the map Λ_{-1} that we have just defined does this job; the computations are nearly identical to what we have already done. This same mapping works for sending $H_{1,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ for $0 < s < 1$.

Finally we show that invertibility of $H_{1,\infty}(A)$ implies invertibility of $H_{s,\infty}(A)$ for $s > 1$ for $A \in \Psi_1(\mathcal{C})/\mathcal{K}$. Similar to before, we construct a map Λ_1 from

$H_{1,\infty}(\Psi_1(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_1(\mathcal{C})/\mathcal{K})$ for $s > 1$ so that the following diagram commutes:

$$\begin{array}{ccc} & \Psi_1(\mathcal{C})/\mathcal{K} & \\ \swarrow & & \searrow \\ H_{1,\infty}(\Psi_1(\mathcal{C})/\mathcal{K}) & \xrightarrow[\Lambda_1]{} & H_{s,\infty}(\Psi_1(\mathcal{C})/\mathcal{K}) \end{array}$$

The map Λ_1 defined by $\Lambda_1(A) = \text{s-}\lim\limits_{t\to\infty} Z_t^{-1}U_{-1}AU_1Z_t$ achieves this; the computations to verify this are analagous to what we have already done. Again, this same map will work for mapping $H_{1,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ into $H_{s,\infty}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ for $s > 1$.

□

Corollary A.2.2. *Let A be an element in $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)/\mathcal{K}$. Then if $H_{s,\infty}(A)$ is invertible for $s = 1, -1$, and 0, then $H_{s,\infty}(A)$ is invertible for all $s \in \mathbb{R}$. In other words, the algebras $H_{s,\infty}(\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)/\mathcal{K})$ are redundant in the Fredholm Theory for $s \neq \pm 1, 0$.*

Proof. The algebra $\Psi_1(\mathcal{C})$ is a subalgebra of $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$, but their images under each of the $H_{s,\infty}$ for $s \in \mathbb{R}$ are the same. The proof of Proposition A.2.1 can thus carry over to this situation. □

Proposition A.2.3. *Let A be an element in $\Psi_1(\mathcal{C})/\mathcal{K}$ or $\Psi_{-1}(\mathcal{C})/\mathcal{K}$. Then if $H_{\infty,t}(A)$ is invertible for $t = 0$, then $H_{\infty,t}(A)$ is invertible for all $t \in \mathbb{R}$. In other words, the algebras $H_{\infty,t}(\Psi_{-1}(\mathcal{C})/\mathcal{K})$ are redundant in the Fredholm Theory for $t \neq 0$.*

Proof. Let's first settle this for $\Psi_{-1}(\mathcal{C})/\mathcal{K}$. To prove this statement, we will construct homomorphisms Λ and Λ' such that the following diagrams commute:

$$\begin{array}{ccc} & \Psi_{-1}(\mathcal{C})/\mathcal{K} & \\ \swarrow & & \searrow \\ H_{\infty,0}(\Psi_{-1}(\mathcal{C})/\mathcal{K}) & \xrightarrow[\Lambda]{} & H_{\infty,t}(\Psi_{-1}(\mathcal{C})/\mathcal{K}) \end{array}$$

$$\begin{array}{ccc} & \Psi_{1}(\mathcal{C})/\mathcal{K} & \\ \swarrow & & \searrow \\ H_{\infty,0}(\Psi_{1}(\mathcal{C})/\mathcal{K}) & \xrightarrow[\Lambda']{} & H_{\infty,t}(\Psi_{1}(\mathcal{C})/\mathcal{K}) \end{array}$$

first for $t > 0$ and then for $t < 0$. Notice, however, that if we construct one of the maps the other will be exactly the same. Indeed, the only difference in requirements between Λ and Λ' are when one sends $P_{\mathbb{R}}$ to I the other must send $Q_{\mathbb{R}}$ to 0 and vice versa; however, a homomorphism that sends $P_{\mathbb{R}}$ to I must necessarily send $Q_{\mathbb{R}}$ to 0. We thus only construct Λ in this proof.

We first construct the map Λ for $t > 0$. This map must leave characteristic functions unchanged but send $P_{\mathbb{R}}$ to I. The map Λ that sends an operator A to $\operatorname*{s-lim}_{t\to\infty} M_{-t}AM_t$ achieves this. Indeed, since the characteristic functions can formally be viewed as multiplication operators they commute with $M_{\pm t}$ and so are invariant under this map. For $P_{\mathbb{R}}$ we have

$$\begin{aligned} M_{-t}P_{\mathbb{R}}M_t &= M_{-t}\mathcal{F}^{-1}\frac{1+\operatorname{sgn}(x)}{2}\mathcal{F}M_t \\ &= \mathcal{F}^{-1}U_{-t}\frac{1+\operatorname{sgn}(x)}{2}U_t\mathcal{F} \\ &= \mathcal{F}^{-1}\frac{1+\operatorname{sgn}(x+t)}{2}\mathcal{F} \end{aligned}$$

which converges strongly to I as $t \to \infty$.

For $t < 0$, we require a map that sends $P_{\mathbb{R}}$ to 0 and leaves the characteristic functions unchanged. Here we define Λ to be the map that sends A to $\operatorname{s-lim}_{t\to\infty} M_t A M_{-t}$. $\square$

Appendix B

List of Notation

B.1 Operators

$L(a)$	Laurent operator with symbol a
$M(a)$	Multiplication operator with symbol a
$T(a)$	Toeplitz operator with symbol a
$H(a)$	Hankel operator with symbol a
P	The operator that sends a sequence $(x_n)_{n\in\mathbb{Z}}$ to $(y_n)_{n\in\mathbb{Z}}$ with $y_n = x_n$ if $n \geq 0$ and $y_n = 0$ if $n < 0$
Q	$I - P$
J	The operator that sends a sequence $(x_n)_{n\in\mathbb{Z}}$ to the sequence $(x_{-1-n})_{n\in\mathbb{Z}}$
S_Γ	The singular integral operator on the space Γ
P_n	The operator that sends a sequence $(x_k)_{k\in\mathbb{Z}}$ to the sequence $(y_k)_{k\in\mathbb{Z}}$ where $y_k = x_k$ if $-n \leq k < n$ and $y_k = 0$ if $k < -n$ or $k \geq n$
U_n	The operator that sends $(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k-n})_{k\in\mathbb{Z}}$
U_{-n}	The operator that sends $(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+n})_{k\in\mathbb{Z}}$

P_n^+	The operator $U_{-n}PU_n$
Q_n^-	The operator U_nQU_{-n}
Y_τ	The operator on $L^2(\mathbb{S}^1)$ defined by $(Y_\tau f)(t) = f(\tau t)$
Y_τ^*	The operator on $L^2(\mathbb{S}^1)$ defined by $(Y_\tau^* f)(t) = f(t/\tau)$
C_r	The operator on $L^2(\mathbb{S}^1)$ defined by $(C_r f)(t) = f(\frac{t+r}{1+rt})$
C_r^{-1}	The operator on $L^2(\mathbb{S}^1)$ defined by $(C_r f)(t) = f(\frac{t-r}{1-rt})$
$G_{r,\tau}$	The operator C_rY_τ
R_r	The operator on $L^2(\mathbb{S}^1)$ defined by $(R_r f)(t) = \frac{\sqrt{1-r^2}}{1+rt} f\left(\frac{t+r}{1+rt}\right)$
R_r^*	The operator on $L^2(\mathbb{S}^1)$ defined by $(R_r^* f)(t) = \frac{\sqrt{1-r^2}}{1-rt} f\left(\frac{t-r}{1-rt}\right)$
$\mathcal{F}$	The Fourier Transform
$\mathcal{F}^{-1}$	The inverse Fourier Transform
$W^0(b)$	The Fourier convolution of a symbol b, i.e., $\mathcal{F}^{-1}b\mathcal{F}$
Φ_0	The strong limit map defined in Equation 3.1
Φ_1	The strong limit map defined in Equation 3.2
Φ_{-1}	The strong limit map defined in Equation 3.3
τ	The surjective map from $\mathcal{B}/\mathcal{N}$ into $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ defined by $\tau(B_{n,r} + \mathcal{N}) = (B_{n,r} + \mathcal{J}) + \mathcal{J}_1$
ϵ	The strong limit map defined on $\mathcal{B}/\mathcal{N}$ defined by $\epsilon(B_{n,r}) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_rB_{n,r}R_r^*$
δ	The surjective map from $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ into $\mathcal{A}$ defined by $\delta((F_{n,r} + \mathcal{J}) + \mathcal{J}_1) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} R_rF_{n,r}R_r^*$
E_n	The operator from $\ell^2(\mathbb{Z})$ into $L^2(\mathbb{R})$ that maps $(x_i)_{i\in\mathbb{Z}}$ to $\sqrt{n}\sum_{i=-\infty}^{\infty} x_i\chi_{[\frac{i}{n},\frac{i+1}{n}]}$

E_{-n} The operator from $L^2(\mathbb{R})$ into $\ell^2(\mathbb{Z})$ that sends a function f to $\left(\sqrt{n}\int_{-\infty}^{\infty} f(x)\chi_{[\frac{i}{n},\frac{i+1}{n}]}(x)dx\right)_{i=-\infty}^{\infty}$

L_n The operator E_nE_{-n}

Ψ_1 The operator from $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1$ onto $\Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ defined by

$\Psi_1(F_{n,r}) = \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_nF_{n,r}E_{-n}$

$\widehat{\Psi}_1$ The map from $\mathcal{B}/\mathcal{N} \to \Psi_1((\mathcal{F}_*/\mathcal{J})/\mathcal{J}_1)$ given by $\widehat{\Psi}_1 = \Psi_1 \circ \tau$

M_s The operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ defined by $(M_sf)(x) = e^{2\pi ixs}f(x)$ for $s \in \mathbb{R}$

U_t The operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ defined by $(U_tf)(x) = f(x-t)$ for $t \in \mathbb{R}$

Z_τ The operator from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ defined by $(Z_\tau f)(x) = \tau^{\frac{1}{2}}f(\tau x)$ for $\tau > 0$

ι The surjective map from $\mathcal{C}/\mathcal{N}$ into $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ that sends a sequence $(C_n) + \mathcal{N}$ to the sequence $(\{C_n\} + \mathcal{I}) + \mathcal{I}_{-1}$

Ψ_{-1} The operator from $\mathcal{C}$ onto $\Psi_{-1}(\mathcal{C})$ defined by

$\Psi_{-1}(C_n) = \operatorname*{s-lim}_{n\to\infty} E_nY_{-1}C_nY_{-1}E_{-n}$

$\widehat{\Psi}_{-1}$ The operator from $(\mathcal{B}/\mathcal{I})/\mathcal{I}_{-1}$ onto $\Psi_{-1}(\mathcal{C})$ defined by

$\widehat{\Psi}_{-1}(B_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_nY_{-1}B_{n,r}Y_{-1}E_{-n}$

η The isometry from $L^2(\mathbb{R}) \to L^2(\mathbb{R}^+) \bigoplus L^2(\mathbb{R}^+)$ that sends f to $(f_1, f_2)^T$ with $f_1(x) = f(x)$ and $f_2(x) = f(-x)\,\forall x \in \mathbb{R}^+$

Φ_η The $*-$isomorphism defined by $\Phi_\eta : A \mapsto \eta A\eta^{-1}$ that maps $\mathcal{L}(L^2((\mathbb{R}))$ onto $\mathcal{L}(L^2((\mathbb{R}^+))^{2\times 2}$

N The Hankel operator on $\mathcal{L}(L^2((\mathbb{R}))$ onto $\mathcal{L}(L^2((\mathbb{R}^+))$ defined by $(Nf)(x) = \dfrac{1}{\pi i}\int_0^\infty \dfrac{f(y)}{y+x}dy, \quad x \in \mathbb{R}^+$

M	The Mellin Transform, i.e., the operator from $L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ defined by $(Mf)(z) = \int_0^\infty x^{-iz-\frac{1}{2}} f(x)dx, \quad z \in \mathbb{R}$
M^{-1}	The inverse Mellin Transform, i.e., the operator from $L^2(\mathbb{R}) \to L^2(\mathbb{R}^+)$ defined by $(M^{-1}f)(x) = \dfrac{1}{2\pi} \int_{-\infty}^\infty x^{iz-\frac{1}{2}} f(z)dz, \quad x \in \mathbb{R}^+$
$M^0(b)$	The Mellin convolution operator for a multiplication operator $b \in L^\infty(\mathbb{R})$ defined by $M^{-1}bM : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$
Ψ_τ	The map defined on $(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_\tau$ for each $\tau \in \mathbb{S}^1$ defined by $\Psi_\tau(F_{n,r}) := \operatorname*{s-lim}_{\substack{n\to\infty \\ r\to 1}} E_n Y_\tau F_{n,r} Y_\tau^* E_{-n}$
$H^{\pm\pm}$	The strong limit map $\operatorname*{s-lim}_{t\to\pm\infty} \operatorname*{s-lim}_{s\to\pm\infty} M_{-t}U_{-s}AU_sM_t$ where the first superscript in $H^{\pm\pm}$ refers to the strong limit with respect to $s \to \pm\infty$ and the second one with respect to $t \to \pm\infty$
$H_{s,\infty}$	The strong limit map $\operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1}U_{-s}AU_sZ_\tau$ for $s \in \mathbb{R}$
$H_{\infty,t}$	The strong limit map $\operatorname*{s-lim}_{\tau\to\infty} Z_\tau M_{-t}AM_tZ_\tau^{-1}$ for $t \in \mathbb{R}$

B.2 Algebras

$C(\mathbb{S}^1)$	The space of continuous functions on the unit circle
PC	The space of piecewise continuous functions on the unit circle
$PC_{\pm 1}$	The set of all piecewise continuous functions that are continuous on $\mathbb{S}^1 \setminus \{-1, 1\}$
$PC^0_{\pm 1}$	The set of all functions $f \in PC_{\pm 1}$ such that $f(-1 \pm 0) = 0$
PC^0_{-1}	The set of piecewise continuous functions f with $f(-1) = 0$
$\mathcal{F}$	The space of sequences of uniformly bounded linear operators $(A_{n,r})$ acting on a Hilbert space
$\mathcal{K}$	The ideal of compact operators

$\mathcal{N}$	The ideal of sequences converging in norm to 0
$\mathcal{F}_*$	The algebra generated in $\mathcal{L}(L^2(\mathbb{S}^1))$ by $\mathcal{N}$ and the elements (P), (P_n^+), (Q_n^-), $(L(a))$, $(L(G_{r,\tau}^{-1}f))$, (K), and $(Y_\tau^* R_r^* K_\tau R_r Y_\tau)$ with $a \in PC, f \in PC, K, K_\tau \in \mathcal{K}(L^2(\mathbb{S}^1))$, and $\tau \in \mathbb{S}^1$.
$\mathcal{J}$	The ideal $\{(C_{n,r}) + (K_1) + (U_{-n}K_2U_n) + (U_nK_3U_{-n}) : (C_{n,r}) \in \mathcal{N}, K_i \in \mathcal{K}(L^2(\mathbb{S}^1))\}$
$\mathcal{J}_{t_0}$	clos $\text{id}_{\mathcal{F}_*/\mathcal{J}}\{(L(c)) + \mathcal{J} : c \in C(\mathbb{S}^1), c(t_0) = 0\}$ for $t_0 \in \mathbb{S}^1$
$(\mathcal{F}_*/\mathcal{J})/\mathcal{J}_{t_0}$	The local algebra at the point $t_0 \in \mathbb{S}^1$ with generators given in Proposition 3.2.4
$\mathcal{B}$	The algebra generated in $\mathcal{L}(L^2(\mathbb{S}^1))$ by $\mathcal{N}$ and the elements (P), (P_n^+), (Q_n^-), $(L(\chi_+))$, $(L(C_r^{-1}f))$, $(R_r^*KR_r)$, and $(Y_{-1}R_r^*K'R_rY_{-1})$ where $f \in PC$, $K, K' \in \mathcal{K}(L^2(\mathbb{S}^1))$
$\mathcal{A}$	The algebra of operators $\epsilon(\mathcal{B}/\mathcal{N})$
$\mathcal{I}$	The ideal $\{(C_{n,r}) + (R_r^*KR_r) : (C_{n,r}) \in \mathcal{N}, K \in \mathcal{K}\}$
$\mathcal{I}_{t_0}$	clos $\text{id}_{\mathcal{B}/\mathcal{I}}\{(L(C_r^{-1}g)) + \mathcal{I} : g \in C(\mathbb{S}^1), g(t_0) = 0\}$ for $t_0 \in \mathbb{S}^1$
$(\mathcal{B}/\mathcal{I})/\mathcal{I}_{t_0}$	The local algebra at the point $t_0 \in \mathbb{S}^1$
$\mathcal{C}$	The algebra generated in $\mathcal{L}(L^2(\mathbb{S}^1))$ by $\mathcal{N}$ and the elements (P), (P_n^+), (Q_n^-), $(L(\chi_+))$, $(L(\chi_-))$ $(E_{-n}K_1E_n)$, and $(Y_{-1}E_{-n}K_2E_nY_{-1})$ where $K_1, K_2 \in \mathcal{K}(L^2(\mathbb{R}))$
$C(\dot{\mathbb{R}})$	The Banach algebra of all continuous functions f on the real line $\mathbb{R}$ possessing finite limits $f(+\infty)$ and $f(-\infty)$ such that $f(+\infty) = f(-\infty)$
$\widehat{\mathcal{C}}$	The algebra generated in $\mathcal{L}(L^2(\mathbb{S}^1))$ by $\mathcal{N}$ and the elements (P), (P_n^+), (Q_n^-), $(L(\chi_+))$, $(L(\chi_-))$, $(L(f))$, $(E_{-n}K_1E_n)$, $(Y_{-1}E_{-n}K_2E_nY_{-1})$, and $(E_{-n}M(g)E_n)$ where K_1, K_2 are compact, $f \in PC_{\pm 1}^0$, and $g \in C(\dot{\mathbb{R}})$
$\mathcal{J}'$	The ideal $\{(C_n + E_{-n}K_1E_n + Y_{-1}E_{-n}K_2E_nY_{-1}) : C_n \in \mathcal{N}, K_1, K_2 \in \mathcal{K}\}$

$\mathcal{J}'_x$	$\mathcal{J}'_x = \text{clos id}_{\widehat{\mathcal{C}}/\mathcal{J}'}\{(E_{-n}M(g)E_n) + \mathcal{J}' : g \in C(\dot{\mathbb{R}}), g(x) = 0\}$
$(\widehat{\mathcal{C}}/\mathcal{J}')/\mathcal{J}'_x$	The local algebra at a point $x \in \dot{\mathbb{R}}$
$PC_\infty(\mathbb{R})$	The set of all continuous functions f on $\mathbb{R}$ for which the limits at infinity and negative infinity exist and are finite
$C^0_\infty(\mathbb{R})$	The set of all continuous functions f on $\mathbb{R}$ for which $\lim_{x\to\pm\infty} f(x) = 0$
Σ^0_2	$\text{alg}_{\mathcal{L}(L^2((\mathbb{R}^+))^{2\times 2}}\left\{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} S & -N \\ N & -S \end{pmatrix}\right\}$
Σ_1	$\{\alpha I + \beta S + M^0(b) : \alpha, \beta \in \mathbb{C}, b \in C^0_\infty(\mathbb{R})\}$
Σ^0_1	$\{M^0(b) : b \in C^0_\infty(\mathbb{R})\}$
k_ω	The bounded linear operator defined from $L^\infty \to L^\infty$ for defined by $$(k_\omega f)(e^{ix}) = \int_{-\infty}^{\infty} f(e^{i(x-y)})\omega K(\omega y)dy$$ for $\omega \in [0, \infty)$; called the approximate identity with kernel K
$\mathcal{S}_\Omega$	The set of all uniformly bounded sequences $\{A_{n,\omega}\}_{n\in\mathbb{N},\omega\in\Omega}$ of bounded linear operators on $L^2(\mathbb{S}^1)$
$\mathcal{N}_\Omega$	The $*$-ideal of $\mathcal{S}_\Omega$ consisting of all sequence $\{A_{n,\omega}\}$ for which $\\|\{A_{n,\omega}\}\\| \to 0$ as $\omega \to \infty$
$\mathcal{S}_\Omega(k_\omega, PC)$	The smallest closed $*$-subalgebra of $\mathcal{S}_\Omega$ with $\mathcal{N}_\Omega \subseteq \mathcal{S}_\Omega(k_\omega, PC)$ containing the elements (P), (P_n^+), (Q_n^-), and $(M(k_\omega a))$ with $a \in PC$

B.3 Miscellaneous

$\mathbb{S}^1$	The unit circle
χ_+	The characteristic function on the upper half plane of the unit circle

χ_- The characteristic function on the lower half plane of the unit circle

χ_D The characteristic function of a subset D of the real axis

χ The function $\chi_+ - \chi_-$

σ The function defined by $\sigma(e^{2\pi i\phi}) = -\frac{\sin^2(\pi\phi)}{\pi^2}\sum_{m\in\mathbb{Z}}\frac{\operatorname{sgn}(m+\frac{1}{2})}{(\phi+m)^2}$ for $\phi \in (0,1)$ with the property that $L(\sigma) = E_{-n}S_{\mathbb{R}}E_n$ for all $n \in \mathbb{N}$

Bibliography

[1] N. I. Akhiezer. *Theory of Approximation*. Frederick Ungar Publishing Co., 1956.

[2] A. Böttcher and B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices*. Universitext. Springer New York, 1999.

[3] A. Böttcher and B. Silbermann. *Analysis of Toeplitz Operators*. Springer Monographs in Mathematics. Springer Berlin, Heidelberg, 2 edition, 2006.

[4] Ronald G Douglas et al. *Banach algebra techniques in the theory of Toeplitz operators*, volume 15. American Mathematical Soc., 1980.

[5] T. Ehrhardt, S. Roch, and B. Silbermann. Finite section method for singular integrals with operator-valued pqc-coefficients. In A. Böttcher and I. Gohberg, editors, *Singular Integral Operators and Related Topics*, pages 204–243, Basel, 1996. Birkhäuser Basel.

[6] T. Ehrhardt and B. Silbermann. Approximate identities and stability of discrete convolution operators with flip. In Jürgen Rossmann, Peter Takáč, and Günther Wildenhain, editors, *The Maz'ya Anniversary Collection*, pages 103–132, Basel, 1999. Birkhäuser Basel.

[7] Israel Gohberg and Izrail Aronovich Fel_dman. *Convolution equations and projection methods for their solution*, volume 41. American Mathematical Soc., 2005.

[8] R. Hagen, S. Roch, and B. Silbermann. *Spectral theory of approximation methods for convolution equations*, volume 74. Birkhäuser, 1995.

[9] Andreas Rathsfeld. Reduktionsverfahren für singuläre integralgleichungen mit stückweise stetigen koeffizienten. *Mathematische Nachrichten*, 127(1):125–143, 1986.

[10] S. Roch, P. Santos, and B. Silbermann. *Non-commutative Gelfand theories. A tool-kit for operator theorists and numerical analysts.* Springer Science & Business Media, 2011.

[11] S. Roch, P. A. Santos, and B. Silbermann. Finite section method in some algebras of multiplication and convolution operators and a flip. *Z. Anal. Anwendungen*, 16(3):575–606, 1997.

[12] Steffen Roch and B. Silbermann. C*-algebra techniques in numerical analysis. *Journal of Operator Theory*, 35, 01 1996.

[13] B. Silbermann, R. Hagen, and S. Roch. C^*-algebra techniques—a powerful tool in numerical analysis. In *Analysis, numerics and applications of differential and integral equations (Stuttgart, 1996)*, volume 379 of *Pitman Res. Notes Math. Ser.*, pages 214–218. Longman, Harlow, 1998.