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STUDIES OF MULTIPERIPHERAL INTEGRAL EQUATIONS*†

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ABSTRACT

The CGL type of multiperipheral integral equation has been derived in terms of invariant variables, without assuming subenergies to be large compared with momentum transfers and particle masses. Both forward direction and nonforward direction have been worked out in detail, and the multiperipheral model with Toller-angle-dependent vertex functions has been discussed. We have furthermore demonstrated that all the qualitative physical properties of the ABFST model remain true in this generalized multiperipheral model.

1. INTRODUCTION

Since early 1963[1] the multi-Regge model, which is one kind of multiperipheral model, has been used with some success to describe production processes at high energies. Since 1968 two new developments have made this model more attractive; one is the "duality hypothesis" proposed by Dolen-Horn-Schmid[2] and extended by Chew-Pignotti,[3] another is the phenomenological model for production processes proposed by Chan-Łoskiewicz-Allison (CŁA).[4] The "duality hypothesis" asserts that if we extrapolate the smooth high-energy Regge representation down to low energy, the Regge representation gives a certain semi-local average over the resonance peaks. This duality simplifies multiperipheral calculations enormously. The phenomenological CŁA model supplements the multi-Regge model with the assumption that the structure of nonresonant low mass clusters is governed only by phase space. The CŁA model has been used to analyze data for $\pi^{\pm} p \to n\pi + p$, $K^{-} p \to n\pi + \Lambda$, $pp \to pp + n\pi$, $\bar{p}p \to n\pi$, and some other reactions of multiplicities n ranging from two to nine, and at p_{lab} ranging from 2 to 28 GeV/c.[4-6] The qualitative agreement with experiment is very good. Subsequently Białas, Michejda, and Turnau[6] used this model, incorporating the usual Regge phase factor, to calculate the absorptive part of the two-particle elastic scattering amplitude. Assuming the amplitude to be purely imaginary at high energy for small momentum transfer, their result gives a sharp momentum-transfer dependence of the elastic differential cross section. This result had not previously been achieved from a dynamical model.

Recently Chew, Goldberger, and Low (CGL)[7] and Halliday[8] have proposed a multiperipheral model which contains the multi-Regge or CLA models as a special case, and also leads to a unitarity integral equation for the elastic amplitude. The method of deriving this equation is very similar to the method used by Amati, Bertocchi, Fubini, Stanghellini and Tonin[9] in their 1962 papers (the ABFST model).

However, in Refs. 7 and 8 and also in later papers,[10] almost all[11,12] derivations have assumed subenergies to be large compared with momentum transfers and particle masses (i.e., the weak coupling limit), either right at the beginning of the calculation or at the time of writing down explicitly the kernel and limits of integration of the integral equation in terms of invariant variables. The assumption of high subenergies is unrealistic and hard to justify even if the duality concept is valid, because experimentally we know that most of the time in production processes at least one subenergy is not large.

In this paper we study the CGL type[7] of multiperipheral model in detail, using invariant variables. In both the forward and non-forward cases, all the limits of integration and the kernel of the integral equation have been worked out explicitly, without assuming that any subenergies are large. Furthermore, we have demonstrated that all qualitative physical properties--the logarithmic increase with energy of average multiplicity, and the energy independence of both the inelasticity and the transverse secondary momentum spectrum--remain as in the ABFST model.[9] The only ingredient necessary to show these properties is the existence of an integral equation with "short range" correlated kernel. Equally general is Regge asymptotic behavior for the solution of the integral equation. This point is crucial for bootstrap theory.

In Sec. 2, the CGL type of multiperipheral model is studied in detail for the forward-direction case, then the results are generalized to the nonforward direction in Sec. 3. In Sec. 4, we show that our results in Sec. 2 and Sec. 3 reduce to the previous weak coupling results when all subenergies become large. In Sec. 5, we demonstrate Regge asymptotic behavior and show that all the qualitative physical properties obtained from the ABFST model continue to hold in the CGL model. In Sec. 6 we discuss the possibility that the internal vertex functions have Toller-angle dependence. The last section presents some concluding remarks about this general multiperipheral model. The detailed mathematical evaluation of certain boundary functions, defined in previous sections, is contained in the Appendices.

2. THE INTEGRAL EQUATION AT FORWARD DIRECTION

Let us parameterize the production amplitude $T_{ab}^{\ n}$ for the process[13] (see Fig. 1)

$$a + b \rightarrow 0 + 1 + 2 + \cdots + (n+1) \qquad (2.1)$$

by the following multiperipheral model expression:

$$T_{ab}^{\ n}(p_a, p_b;\ p_0, p_1, \cdots, p_{n+1}) \equiv T_{ab}^{\ n}(p_a, p_b;\ Q_1, Q_2, \cdots, Q_{n+1})$$

$$= G_a(p_a, Q_1)\ f(-p_a, Q_1, Q_2)\ \beta(Q_1, Q_2)\ f(Q_1, Q_2, Q_3)\cdots$$

$$\cdot\ f(Q_n, Q_{n+1}, p_b)\ G_b(Q_{n+1}, p_b). \qquad (2.2)$$

In terms of invariant variables (see Fig. 1),

$$T_{ab}^{\ n}(p_a, p_b;\ Q_1, Q_2, \cdots, Q_{n+1}) = G_a(m_a^{\ 2}, t_1)\ f(\epsilon_1, m_a^{\ 2}, t_1, t_2)\ \beta(t_1, t_2)$$

$$\cdot\ f(\epsilon_2, t_1, t_2, t_3)\quad \beta(t_2, t_3)\cdots\beta(t_n, t_{n+1})$$

$$\cdot\ f(\epsilon_{n+1}, t_n, t_{n+1}, m_b^{\ 2})\ G_b(t_{n+1}, m_b^{\ 2}),$$

$$(2.3)$$

where $t_i = Q_i^{\ 2}$ is the ith momentum transfer and $\epsilon_i^{\ 2} = (p_{i-1} + p_i)^2$ is the ith subenergy squared, G_a and G_b are the "external" coupling constants, while $\beta(t_i, t_{i+1})$ is the "internal" coupling constant at ith vertex. The function f depends on the specific model.

The description "multiperipheral" means that either $|\beta(t_i, t_{i+1})|^2$ or $|f(\epsilon_i, t_{i-1}, t_i, t_{i+1})|^2$ decrease rapidly when any one of the t's becomes large and negative.

The unitarity relation tells us that $A_{ab}(p_a, p_b; 0)$, the imaginary part of the ab forward elastic scattering amplitude, will be equal to

$$A_{ab}(p_a, p_b; 0) \equiv \sum_{n=0}^{\infty} A_{ab}{}^{n}(p_a, p_b; 0) = \int |T_{ab}{}^{n}(p_a, p_b; Q_1, Q_2, \cdots, Q_{n+1})|^2 d\not{\phi}_n , \tag{2.4}$$

where $d\not{\phi}_n$ is the phase space element for the n+2 particle system,

$$d\not{\phi}_n = \frac{1}{(2\pi)^{3n-1}} \delta^+(p_0{}^2 - \mu_0{}^2) d^4 p_0 \; \delta^+(p_1{}^2 - \mu_1{}^2) d^4 p_1 \cdots \delta^+(p_{n+1}^2 - \mu_{n+1}^2) \cdot d^4 p_{n+1} \; \delta^4\Big(p_a + p_b - \sum_{i=0}^{n+1} p_i\Big)$$

$$= \frac{1}{(2\pi)^{3n+2}} \delta^+[(Q_1 + p_a)^2 - \mu_0{}^2] \, d^4 Q_1 \; \delta^+[(Q_2 - Q_1)^2 - \mu_1{}^2]$$

$$\cdot d^4 Q_2 \; \delta^+[(Q_3 - Q_2)^2 - \mu_2{}^2] \cdots \delta^+[(Q_{n+1} - Q_n)^2 - \mu_n{}^2]$$

$$\cdot d^4 Q_{n+1} \; \delta^+[(p_b - Q_{n+1})^2 - \mu_{n+1}^2] ,$$

and $A_{ab}{}^{n}$ is the contribution to A_{ab} from the n+2-particle intermediate state.

Just as in CGL,[7] let us introduce the modified absorptive function $B_a{}^{n}(p_a, p_b; Q_{n+1}; 0)$ by (see Fig. 2)

$$A_{ab}{}^{n}(p_a,p_b;0) = \frac{1}{(2\pi)^3}\int d^4Q_{n+1}\, B_a{}^{n}(p_a,p_b;Q_{n+1};0)\,|G_b(Q_{n+1},p_b)|^2 \cdot \delta^+[(Q_{n+1}-p_b)^2-\mu_{n+1}^2]; \tag{2.5}$$

then from Eqs. (2.1), (2.2), and (2.4), we can deduce the recursion relation between $B_a{}^{n}$ and B_a^{n-1} to be

$$B_a{}^{n}(p_a,p_b;Q_{n+1};0) = \frac{1}{(2\pi)^3}\int d^4Q_n\, B_a^{n-1}(p_a,Q_{n+1};Q_n;0)K(Q_n,Q_{n+1},p_b) \cdot \delta^+[(Q_n-Q_{n+1})^2-\mu_n{}^2], \tag{2.6}$$

with

$$K(Q_n,Q_{n+1},p_b) = |\beta(Q_n,Q_{n+1})\; f(q_n,Q_{n+1},p_b)|^2 . \tag{2.7}$$

If we define

$$B_a(p_a,p_b;Q'';0) \equiv \sum_{n=0}^{\infty} B_a{}^{n}(p_a,p_b;Q'';0), \tag{2.8}$$

we can derive the following integral equation for B_a from Eqs. (2.6) and (2.8) (see Fig. 3):

$$B_a(p_a,p_b;Q'';0) = B_a{}^{0}(p_a,p_b;Q'';0)+\frac{1}{(2\pi)^3}\int d^4Q'B_a(p_a,Q'';Q';0)K(Q',Q'',p_b) \cdot \delta^+[(Q'-Q'')^2-\mu'^2], \tag{2.9}$$

with

$$B_a{}^0(p_a,p_b;Q'';0) = |G_a(p_a,Q'')\, f(-p_a,Q'',p_b)|^2 \tag{2.10}$$

and from Eqs. (2.4) and (2.5), we get

$$A_{ab}(p_a,p_b;0) = \frac{1}{(2\pi)^3}\int d^4Q''\, B_a(p_a,p_b;Q'';0)\,|G_b(Q'',p_b)|^2\delta^+[(Q''-p_b)^2 - \mu''^2]. \tag{2.11}$$

For practical purposes, just as in ABFST[9] we wish to express these equations in terms of invariant variables. Let us then define the functions $A_{ab}(s''',0;t''',m_a{}^2)$, $B_a(s''',0;t''',t'',m_a{}^2,\mu''^2)$, and $K(\epsilon'',t''',t'',t',\mu''^2,\mu'^2)$ in terms of $A_{ab}(p_a,p_b;0)$, $B_a(p_a,p_b;Q'';0)$, and $K(Q',Q'',p_b)$ by (see Fig. 4)

$$A_{ab}(p_a,p_b;0) \equiv \int A_{ab}(s''',0;t''',m_a{}^2)\,\delta[(p_a+p_b)^2 - s''']\delta^+[p_b{}^2 - t''']$$
$$\cdot\,\delta[p_a{}^2 - m_a{}^2]\,ds'''\,dt'''\,dm_a{}^2\,, \tag{2.12}$$

$$B_a(p_a,p_b;Q'';0) \equiv \int B_a(s''',s'';0;t''',t'';m_a{}^2,\mu''^2)\,\delta[(p_a+p_b)^2 - s''']$$
$$\cdot\,\delta^+[p_a{}^2 - m_a{}^2]\delta[p_b{}^2 - t''']\delta[Q''^2 - t'']\delta[(p_2+Q'')^2 - s'']$$
$$\cdot\,\delta^+[(Q''-p_b)^2 - \mu''^2]ds'''\,ds''dt'''dt''dm_a{}^2\,d\mu''^2\,, \tag{2.13}$$

and

$$K(Q',Q',p_b) \equiv \int K(\epsilon'',t''',t'',t';\mu''^2,\mu'^2)\delta[(Q'-p_b)^2 - \epsilon'']$$
$$\cdot\,\delta[p_b{}^2 - t''']\delta[Q''^2 - t'']\delta[Q'^2 - t']\delta^+[(Q''-p_b)^2 - \mu''^2]$$
$$\cdot\,\delta^+[(Q'-Q'')^2 - \mu'^2]d\epsilon''\,dt'''dt''dt'\,d\mu''^2\,d\mu'^2\,. \tag{2.14}$$

For simplicity, we further wish to introduce three simpler notations $A_{ab}(s''', t''', 0)$, $B_a(s''', s''; t''', t''; 0)$, and $K(\epsilon'', t''', t'', t')$ by defining

$$A_{ab}(s''', 0; t''') \equiv A_{ab}(s''', 0; t''', m_a^2), \tag{2.15}$$

$$B(s''', s'', 0; t''', t'') \equiv B_a(s''', s'', 0; t''', t'', m_a^2, \mu''^2), \tag{2.16}$$

and

$$K(\epsilon'', t''', t'', t') \equiv K(\epsilon'', t''', t'', t', \mu''^2, \mu'^2) . \tag{2.17}$$

By using the definition of $B_a(s''', s'', 0; t''', t'')$, $A_{ab}(s''' 0; t''')$, and $K(\epsilon'', t''', t'', t')$, Eqs. (2.9), (2.10), and (2.11) can be expressed in terms of invariant variables. These equations then assume the following forms (see Fig. 4):

$$B_a(s''', s'', 0; t''', t'') = B_a^0(s''', s'', 0; t''', t'') + \frac{1}{8\pi^3} \int B_a(s'', s', 0; t'', t') \cdot K(\epsilon'', t''', t'', t') \Psi_1(s''', s'', s'; t''', t'', t'; \epsilon'') ds' dt' d\epsilon'', \tag{2.18}$$

$$A_{ab}(s''', 0; t''') = \frac{1}{8\pi^3} \int B_a(s''', s'', 0; t''', t'') |G_b(t'', t''')|^2 \cdot \not{\Phi}_1(s''', s''; t''', t'') ds'' dt'' , \tag{2.19}$$

and if we assume that the production amplitude T_{ab}^n has the form given by Eq. (2.3), then $B_a^0(s''', s'', 0; t''', t'')$ and $K(\epsilon'', t''', t'', t')$ have the following corresponding forms:

$$B_a{}^0(s''',s'',0;t''',t'') = |G_a(m_a{}^2,t'')\ f(s''',m_a{}^2,t'',t''')|^2\ , \tag{2.20}$$

$$K(\epsilon'',t''',t'',t') = |\ \beta(t'',t')\ f(\epsilon'',t',t'',t''')|^2\ , \tag{2.21}$$

with

$$\begin{aligned} s''' &= (p_a + p_b)^2 & m_a{}^2 &= p_a{}^2 \\ s'' &= (p_a + Q'')^2 & t''' &= p_b{}^2 \\ s' &= (p_a + Q')^2 & t'' &= Q''^2 \\ \epsilon'' &= (Q' - p_b)^2 & t' &= Q'^2. \end{aligned}$$

The functions Ψ_1 and Φ_1 in Eqs. (2.18) and (2.19) are boundary functions defined as follows:

$$\Phi_1(s''',s'';t''',t'') = \int d^4Q''\ \delta[Q''^2 - t'']\delta[(p_a + Q'')^2 - s''] \cdot \delta^+[(Q'' - p_b)^2 - \mu''^2] \tag{2.22}$$

and

$$\Psi_1(s''',s'',s';t''',t'',t';\epsilon'') = \int d^4Q'\ \delta[Q'^2 - t']\delta[(p_a + Q')^2 - s'] \cdot \delta^+[(Q' - Q'')^2 - \mu'^2]\delta[(Q' - p_b)^2 - \epsilon'']. \tag{2.23}$$

We have worked out the functions Φ_1 and Ψ_1 in detail in Appendices A and B. If we assume $s''',s'',s' \gg t''',t'',t';m_a{}^2,\mu'^2,\mu''^2$ (see the justifiction of this approximation in Appendix B), we can derive the following asymptotic forms:

$$\not{\phi}_1(s''', s''; t''', t'') = \frac{\pi}{2s'''} \Theta[(s''')^{1/2} - (s'')^{1/2} - \mu'']$$

$$\times \ \Theta[(t''_{max} - t'')(t'' - t''_{min})] \tag{2.24}$$

and

$$\Psi_1(s''', s'', s'; t''', t'', t', \epsilon'') = \frac{1}{2s''} \Theta[(s'')^{1/2} - (s')^{1/2} - \mu']$$

$$\times \Theta[(t'_{max} - t')(t' - t'_{min})] \frac{\Theta[\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon_{min})]}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon_{min})]^{1/2}}$$

$$= \frac{1}{\pi} \not{\phi}_1(s'', s'; t'', t') \frac{\Theta[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]^{1/2}} . \tag{2.25}$$

An attractive form is

$$\Psi_1(s''', s'', s'; t''', t'', t'; \epsilon'') = \frac{1}{\pi} \not{\phi}_1(s'', s'; t'', t') \int_0^\pi d\theta \ \delta(-\epsilon'' + C + D \cos\theta), \tag{2.26}$$

with

$$t''_{\substack{max\\min}} \approx \begin{cases} -\frac{s''}{s'''}\left(-t''' + \frac{\mu''^2}{1 - s''/s'''}\right) \\ -\infty \end{cases} \tag{2.27}$$

$$t'_{\substack{max\\min}} \approx \begin{cases} -\frac{s'}{s''}\left(-t'' + \frac{\mu'^2}{1 - s'/s''}\right) \\ -\infty \end{cases} \tag{2.28}$$

$$\epsilon''_{\substack{max\\min}}\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right) = C \pm D \tag{2.29}$$

and

$$C = C\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t' \right)$$

$$\approx t' + t''' + (\mu'^2 - t'' - t')\left(\frac{s'''}{s''} \right) + (\mu''^2 - t''' - t'')\left(\frac{s'}{s''} \right)$$

$$+ 2t''\left(\frac{s'}{s''} \right)\left(\frac{s'''}{s''} \right), \qquad (2.30)$$

$$D = D\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t' \right)$$

$$\approx 2\left[t''' + (\mu''^2 - t''' - t'')\left(\frac{s'''}{s''} \right) + t''\left(\frac{s'''}{s''} \right)^2 \right]^{1/2}$$

$$\cdot \left[t' + (\mu'^2 - t'' - t')\left(\frac{s'}{s''} \right) + t''\left(\frac{s'}{s''} \right)^2 \right]^{1/2} . \qquad (2.31)$$

By using Eq. (2.24) to Eq. (2.31), we can rewrite Eqs. (2.18) and (2.19) as

$$B_a(s''', s'', 0; t''', t'') \approx B_a^0(s''', s'', 0; t''', t'') + \frac{1}{16\pi^3} \int_0^{s''} \frac{ds'}{s''}$$

$$\cdot \int_{-\infty}^{\frac{s'}{s''}\left(-t'' + \frac{\mu'^2}{1 - s'/s''} \right)} dt' \; B_a(s'', s', 0; t'', t')$$

$$\cdot \int_{\epsilon''_{min}}^{\epsilon''_{max}} d\epsilon'' \frac{K(\epsilon'', t''', t'', t')}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon_{min})]^{1/2}} \qquad (2.32)$$

or

$$B_a(s''', s'', 0; t''', t'') \approx B_a^{\,0}(s''', s'', 0; t''', t'') + \frac{1}{16\pi^3} \int_0^{s''} \frac{ds'}{s''} \cdot \int_{-\infty}^{-\frac{s'}{s''}\left(-t'' + \frac{\mu'^2}{1 - s'/s''}\right)} dt'\, B_a(s'', s', 0; t'', t') \cdot \int_0^{\pi} d\phi\, K(\epsilon''(\phi); t''', t'', t') , \tag{2.32'}$$

with $\epsilon''(\phi) = C + D \cos \phi$, and

$$A_{ab}(s''', 0; t''') \approx \frac{1}{16\pi^2} \int_0^{s'''} \frac{ds''}{s'''} \int_0^{-\frac{s''}{s'''}\left(-t''' + \frac{\mu''^2}{1 - s''/s'''}\right)} dt'' B_a(s''', s'', 0; t''', t'') \left|G^b(t'', t''')\right|^2 . \tag{2.33}$$

In high-energy collisions, if we are only studying the Regge asymptotic behavior of B_a and A_{ab}, we can neglect the $B_a^{\,0}(s''', s'', t''', t''; 0)$ term in (2.32) or (2.32') that corresponds to the total elastic cross section $\sigma^{e\ell}(s''')$. We expect, at least in Regge theory, that $\sigma^{e\ell}$ will decrease to zero at extremely high energy (at present accelerator accessible energies $\sigma^{e\ell}/\sigma^{tot} \lesssim 1/5$). Equation (2.32) or Eq. (2.32') then becomes

$$B_a(s''', s'', 0; t''', t'') \approx \frac{1}{16\pi^3} \int_0^{s''} \frac{ds'}{s''} \int_{-\infty}^{-\frac{s'}{s''}\left(-t + \frac{\mu'^2}{1 - s'/s''}\right)} dt'$$

$$\cdot\, B_a(s'', s', 0; t'', t') K\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right),$$

(2.34)

where

$$K\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right) = \int_{\epsilon_{min}\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right)}^{\epsilon_{max}\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right)} d\epsilon''$$

$$\cdot\, \frac{K(\epsilon''; t''', t'', t')}{[(\epsilon_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]^{1/2}}$$

(2.35)

or

$$K\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right) = \int_0^{\pi} d\phi\, K[\epsilon''(\phi); t''', t'', t'] \,, \qquad (2.35')$$

with

$$\epsilon''(\phi) = C\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right) + D\left(\frac{s''}{s'''}, \frac{s'}{s''}; t''', t'', t'\right)\cos\phi \,.$$

The kernel of Eq. (2.34), together with the limits of integration, has the important scaling property of being invariant under the group of multiplicative transformations

$$s''' \to c\, s''', \qquad s'' \to c\, s'', \qquad s' \to c\, s' \, .$$

As in ABFST,[9] this suggests that the solution of Eq. (2.34) can be expressed by the irreducible representation of the group. So we write

$$B_a(s''',s'',0;t''',t'') = \left(\frac{s'''}{s_0}\right)^{\alpha(0)} b_a^{\alpha}\left(\frac{s''}{s'''},0;t''',t''\right), \tag{2.36}$$

where s_0 is a constant. Furthermore if we set

$$y'' = s''/s''' \qquad \text{and} \qquad y' = s'/s'' ,$$

we find from Eqs. (2.34) and (2.33) the integral equation for b_a to be

$$b_a^{\alpha}(y'',0;t''',t'') = \frac{1}{16\pi^3}\,(y'')^{\alpha(0)} \int_0^1 dy' \int_{-\infty}^{-y'\left(-t''+\frac{\mu'^2}{1-y'}\right)} dt' \cdot b_a^{\alpha}(y',0;t'',t')K(y'',y';t''',t'',t'), \tag{2.37}$$

with

$$A_{ab}(s''',0;t''') = \frac{1}{16\pi^2}\left(\frac{s'''}{s_0}\right)^{\alpha(0)} \int_0^1 dy'' \int_{-\infty}^{-y''\left(-t'''+\frac{\mu''^2}{1-y''}\right)} dt'' \cdot b_a^{\alpha}(y'',0;t''',t'')\,|G_b(t'',t''')|^2 . \tag{2.38}$$

Equations (2.36), (2.37), and (2.38) are the most important results of this section. For a specified kernel $K(\epsilon'';t''',t'',t')$ and external coupling G^b, we can use these equations to solve for the leading output Regge pole $\alpha(0)$. For example if we let

$$f(\epsilon_i;t_{i-1},t_i,t_{i+1}) = \left(\frac{\epsilon_i + a}{s_0}\right)^{\alpha_{in}(t_i)} ,$$

then

$$K(\epsilon'',t''',t'',t') = |\beta(t'',t')|^2 \left(\frac{\epsilon'' + a}{s_0}\right)^{2\alpha_{in}(t'')} ,$$

and by using Eq. (2.33), we can easily get

$$K(y'',y';t''',t'',t') = \pi\ |\beta(t'',t')|^2 \left(\frac{\epsilon''_{max}}{s_0}\right)^{2\alpha_{in}(t'')} \cdot {}_2F_1\left[-2\alpha_{in}(t''), \frac{1}{2}, 1; 1 - \frac{\epsilon''_{min} + a}{\epsilon''_{max} + a}\right], \tag{2.39}$$

where ${}_2F_1$ is the usual hypergeometric function.

We are not attempting to solve (2.37) for any specified kernel in this paper; what we do in the next section is to generalize this result to the nonforward direction. In Sec. 5, we will use Eqs. (2.36), (2.37), and (2.38) to study production processes at extremely high energy to show that many of the properties of the ABFST model[9] are preserved in this more general model.

3. THE INTEGRAL EQUATION AT NONFORWARD DIRECTIONS

We wish to generalize this model to the nonforward direction. The unitarity relation tells us (see also Fig. 5)

$$A_{ab}(p_a, p_b; \Delta) = \sum_{n=0}^{\infty} A_{ab}^{\ n}(p_a, p_b; \Delta) , \qquad (3.1)$$

with

$$A_{ab}^{\ n}(p_a, p_b; \Delta) = \int T_{ab}^{*}(p_a - \Delta,\ p_b + \Delta;\ Q_1 + \Delta,\ Q_2 + \Delta, \cdots, Q_{n+1} + \Delta)$$

$$\cdot\ T_{ab}(p_a, p_b;\ Q_1, Q_2, \cdots, Q_{n+1})\ d\phi_n , \qquad (3.2)$$

where

$$T_{ab}^{*}(p_a - \Delta,\ p_b + \Delta;\ Q_1 + \Delta,\ Q_2 + \Delta, \cdots, Q_{n+1} + \Delta) = G_a(p_a - \Delta,\ Q_1 + \Delta)$$

$$\cdot\ f(-p_a + \Delta,\ Q_1 + \Delta,\ Q_2 + \Delta)\ \beta(Q_1 + \Delta,\ Q_2 + \Delta)$$

$$\cdot\ f(Q_1 + \Delta,\ Q_2 + \Delta,\ Q_3 + \Delta) \cdots \beta(Q_n + \Delta,\ Q_{n+1} + \Delta)$$

$$\cdot\ f(Q_n + \Delta,\ Q_{n+1} + \Delta,\ p_b + \Delta)\ \ G_b(Q_{n+1} + \Delta,\ p_b + \Delta). \qquad (3.3)$$

Just as in the forward case (Sec. 2), we introduce the modified absorptive part $B_a^{\ n}(p_a, p_b;\ Q_{n+1};\ \Delta)$ by

$$A_{ab}^{\ n}(p_a, p_b; \Delta) = \frac{1}{8\pi^3} \int B_a^{\ n}(p_a,\ p_b;\ Q_{n+1};\ \Delta)\ G_b^{\ *}(Q_{n+1} + \Delta,\ p_b + \Delta)$$

$$\cdot\ G_b(Q_{n+1},\ p_b)\ \delta^{+}[(Q_{n+1} - p_b)^2 - \mu_{n+1}^2] d^4 Q_{n+1} . \qquad (3.4)$$

Then from Eqs. (3.1) to (3.4), we can easily see that the following recursion relation holds for $B_a^{\ n}(p_a, p_b;\ Q_{n+1};\ \triangle)$:

$$B_a^{\ n}(p_a, p_b;\ Q_{n+1};\ \triangle) = \frac{1}{8\pi^3}\int B_a^{n-1}(p_a, Q_{n+1};\ Q_n;\ \triangle)\ K(Q_n, Q_{n+1}, p_b;\ \triangle)$$

$$\cdot\ \delta^+[(Q_n - Q_{n+1})^2 - \mu_n^{\ 2}]d^4Q_n\ , \qquad (3.5)$$

with

$$K(Q_n, Q_{n+1}, p_b;\ \triangle) = \beta^*(Q_n+\triangle,\ Q_{n+1}+\triangle)\ \beta(Q_n,\ Q_{n+1})\ f^*(Q_n+\triangle,\ Q_{n+1}+\triangle,\ p_b+\triangle)$$

$$\cdot\ f(Q_n,\ Q_{n+1},\ p_b)\ . \qquad (3.6)$$

By defining

$$B_a(p_a, p_b;\ Q'';\ \triangle) = \sum_{n=0}^{\infty} B_a^{\ n}(p_a, p_b;\ Q'';\ \triangle),$$

we get the following equations (see Fig. 6):

$$A_{ab}(p_a, p_b;\ \triangle) = \frac{1}{8\pi^3}\int d^4Q''\ B_a(p_a, p_b;\ Q'';\ \triangle)\ G_b^{\ *}(Q''+\triangle,\ p_b+\triangle)$$

$$\cdot\ G_b(Q'', p_b)\ \delta^+[(Q'' - p_b)^2 - \mu''^2]\ , \qquad (3.7)$$

$$B_a(p_a, p_b;\ Q'';\ \triangle) = B_a^{\ 0}(p_a, p_b;\ Q'';\ \triangle) + \frac{1}{8\pi^3}\int d^4Q'\ B_a(p_a, Q'';\ Q';\ \triangle)$$

$$\cdot\ K(Q', Q'', p_b;\ \triangle)\ \delta^+[(Q' - Q'')^2 - \mu'^2], \qquad (3.8)$$

with

$$B_a^{\ 0}(p_a, p_b; Q''; \triangle) = G_a^{\ *}(p_a-\triangle,\ Q''+\triangle)\ G_a(p_a, Q'')\ f^*(-p_a+\triangle,\ Q''+\triangle,\ p_b+\triangle)$$

$$\cdot\ f(-p_a,\ Q'',\ p_b) \qquad (3.9)$$

and

$$K(Q',Q'',p_b;\ \Delta) = \beta^*(Q'+\Delta,\ Q''+\Delta)\ \beta(Q',Q'')\ f^*(Q'+\Delta,\ Q''+\Delta,\ p_b+\Delta)f(Q',Q'',p_b). \tag{3.10}$$

Just as in Sec. 2, we wish to translate these formulas into invariant variables. If we introduce the functions $A_{ab}(s''',T;t''',t_+''')$, $B_a(s''',s'',T;\ t''',t'';\ t_+''',t_+'')$, and $K(\epsilon'',T;\ t''',t'',t';\ t_+''',t_+'',t_+')$ in a way similar to that of Sec. 2, we can get the following results:

$$A_{ab}(s''',T;\ t''',t_+''') = \frac{1}{8\pi^3}\int B_a(s''',s'',T;\ t''',t'';\ t_+''',t_+'')\ G_b^*(t_+''',t_+'')$$
$$\cdot\ G_b(t'',t''')\ \Phi_2(s''',s'',T;\ t''',t'';\ t_+''',t_+'')$$
$$\cdot\ ds''\ dt''\ dt_+'' \tag{3.11}$$

$$B_a(s''',s'',T;\ t''',t'';\ t_+''',t_+'') = B_a^0(s''',s'',T;\ t''',t'';\ t_+''',t_+'')$$
$$+\ \frac{1}{8\pi^3}\int ds'\ dt'\ dt_+'\ d\epsilon''\ B_a(s'',s',T;t'',t';t_+'',t_+')$$
$$\cdot\ K(\epsilon'',T;\ t''',t'',t';\ t_+''',t_+'',t_+')$$
$$\cdot\ \Psi_2(s''',s'',s',T;\ t''',t'',t';\ t_+''',t_+'',t_+'), \tag{3.12}$$

with

$$B_a^{\,0}(s''', s'', T;\ t''', t'';\ t_+''', t_+'') = G_a^{\,*}(m_a^{\,2}, t_+'')\ G_a(m_a^{\,2}, t'')\ f^*(s''', t_+'', t_+''')$$

$$\cdot\ f(s''', t'', t''') , \qquad (3.13)$$

$$K(\epsilon'', T;\ t''', t'', t';\ t_+''', t_+'', t_+') = \beta^*(t_+', t_+'')\ \beta(t', t'')\ f^*(\epsilon'', t_+', t_+'', t_+''')$$

$$\cdot\ f(\epsilon'', t', t'', t'''), \qquad (3.14)$$

and

$$t_+''' = (p_b + \triangle)^2, \qquad t_+'' = (Q'' + \triangle)^2, \qquad t_+' = (Q' + \triangle)^2 .$$

The functions Φ_2 and Ψ_2 are boundary functions defined by

$$\Phi_2(s''', s'', T;\ t'', t'';\ t_+''', t_+'') \equiv \int d^4Q\ \delta(Q^2 - t'')\ \delta^+[(Q'' + \triangle)^2 - t_+'']$$

$$\cdot\ \delta[(p_a + Q'')^2 - s'']\ \delta^+[(Q''-p_b)^2 - \mu''^2]$$

$$(3.15)$$

and

$$\Psi_2(s''', s'', s', T;\ t''', t'', t';\ t_+''', t_+'', t_+';\ \epsilon'')$$

$$\equiv \int d^4Q'\ \delta(Q'^2 - t')\ \delta[(Q' + \triangle)^2 - t_+']\ \delta[(p_a + Q')^2 - s']$$

$$\cdot\ \delta[(Q' - p_b)^2 - \epsilon'']\ \delta^+[(Q' - Q'')^2 - \mu'^2] .$$

$$(3.16)$$

As shown in detail in Appendices C and D, if

$s''', s'', s' \gg t''', t'', t';\ t_+''', t_+'', t_+';\ m_a^{\,2}, \mu'^2, \mu''^2$, we can get the following expressions for Φ_2 and Ψ_2 :

$$\Phi_2(s''', s'', T;\ t''', t'';\ t_+''', t_+'') = \frac{1}{2s'''}\, \Theta[(s''')^{1/2} - (s'')^{1/2} - \mu'']$$

$$\cdot\ \Theta[(t_{max}''' - t'')(t'' - t_{min}'')]$$

$$\cdot\ \frac{\Theta[(t_{+max}'' - t_+'')(t_+'' - t_{+min}'')]}{[(t_{+max}'' - t_+'')(t_+'' - t_{+min}'')]^{1/2}}\ , \tag{3.17}$$

and

$$\Psi_2(s''', s'', s', T;\ t''', t'', t';\ t_+''', t_+'', t_+';\ \epsilon'')$$

$$=\ \Phi_2(s'', s', T;\ t'', t';\ t_+'', t_+')\ \delta[\epsilon'' - C - D\cos(\phi' - \phi_b)], \tag{3.18}$$

with

$$\cos\phi' = \frac{t' + T - \left(\frac{s'}{s''}\right)(T + t'' - t_+'')}{2\left[Tt' + u'T\left(\frac{s'}{s''}\right) + t''T\left(\frac{s'}{s''}\right)^2\right]^{1/2}}\ ,$$

$$\cos\phi_b = \frac{t''' + T - t_+''' - \left(\frac{s'''}{s''}\right)(T + t'' - t_+'')}{2\left[t''T + u''T\left(\frac{s'''}{s''}\right) + t''T\left(\frac{s'''}{s''}\right)^2\right]^{1/2}}\ ,$$

or

$$\Psi_2(s''', s'', s', T;\ t''', t'', t';\ t_+''', t_+'', t_+';\ \epsilon'')$$

$$= \Psi_1(s''', s'', s', t''', t'', t';\ \epsilon'')\ \delta\Big\{t' - t_+' + T - \left(\frac{s'}{s''}\right)(T + t''' - t_+''')$$

$$- 2\left[Tt' + u'T\left(\frac{s'}{s''}\right) + t''T\left(\frac{s'}{s''}\right)^2\right]^{1/2}\cos\left[\cos^{-1}\left(\frac{\epsilon'' - C}{D}\right) + \phi_b\right]\Big\}, \tag{3.18'}$$

with Ψ_1, t''_{max}, t''_{min}, t'_{max}, t'_{min}, C, and D defined as in Sec. 2, and

$$u' = \mu'^2 - t' - t'', \qquad u'' = \mu''^2 - t'' - t''',$$

$$t''_{+\,\substack{max\\min}} = t'' + T - \left(\frac{s''}{s'''}\right)(T + t''' - t'''_+) \pm 2\left[t''T + u''T\left(\frac{s''}{s'''}\right) + t'''T\left(\frac{s''}{s'''}\right)^2\right]^{1/2} \tag{3.19}$$

and

$$t'_{+\,\substack{max\\min}} = t'' + T - \left(\frac{s'}{s''}\right)(T + t'' - t''_+) \pm 2\left[t'T + u'T\left(\frac{s'}{s''}\right) + t''T\left(\frac{s'}{s''}\right)^2\right]^{1/2} . \tag{3.20}$$

Using Eqs. (3.17) to (3.20), and assuming that we can neglect the inhomogeneous term $B_a^{\,0}$ at high energy, we can rewrite Eq. (3.11) and Eq. (3.12) in the following approximated forms:

$$A_{ab}(s''',T;\, t''',t'''_+) \approx \frac{1}{16\pi^3}\int_0^{s'''}\frac{ds''}{s'''}\int_{-\infty}^{-(s''/s''')\left(-t''' + \frac{\mu''^2}{1 - s''/s'''}\right)} dt''$$

$$\cdot \int_{t''_{+min}\left(\frac{s''}{s'''},T;t''',t'''_+,t''\right)}^{t''_{+max}\left(\frac{s''}{s'''},T;t''',t'''_+,t''\right)} dt''_+$$

$$\cdot\, B_a(s''',s'',T;\, t''',t'';\, t'''_+,t''_+)$$

$$\cdot\, \frac{G_b^{\,*}(t''_+,t'''_+)\; G_b(t'',t''')}{[(t''_{+max} - t'')(t''_+ - t''_{+min})]^{1/2}} \tag{3.21}$$

or

$$A_{ab}(s''',T;\ t''',t_+''') \approx \frac{1}{16\pi^3} \int_0^{s'''} \frac{ds''}{s'''} \int_{-\infty}^{-(s''/s''')(-t''' + \frac{\mu''^2}{1 - s''/s'''})} dt''$$

$$\cdot \int_0^{\pi} d\phi\ B_a(s''',s'',T;\ t''',t'';\ t_+''',t_+'')$$

$$\cdot\ G_b^{\ *}(t_+'',t_+''')\ G_b(t''',t'')\ , \qquad (3.21')$$

with

$$t_+'' = t'' + T - (\frac{s''}{s'''})(T + t''' - t_+''') + 2[t''T + u''T(\frac{s''}{s'''}) + t'''T(\frac{s''}{s'''})^2\ ]^{1/2}\ \cos\phi$$

and

$$B_a(s''',s'',T;\ t''',t'';\ t_+''',t_+'') \approx \frac{1}{16\pi^3} \int_0^{s'} \frac{ds'}{s''} \int_{-\infty}^{-(s'/s'')(-t'' + \frac{\mu'^2}{1-s'/s''})} dt'$$

$$\cdot \int_{t'_{min}(\frac{s'}{s''},T;t'',t_+'',t')}^{t'_{max}(\frac{s'}{s''},T;t'',t_+'',t')} dt'_+$$

$$\cdot\ B_a(s'',s',T;\ t'',t';\ t_+'',t_+')\ \frac{K(\epsilon'',T;\ t''',t'',t';\ t_+''',t_+'',t_+')}{[(t'_{+max} - t'_+)(t'_+ - t'_{+min})]^{1/2}}\ , \qquad (3.22)$$

with

$$\epsilon'' = \epsilon''(t''',t'',t';\ t_+''',t_+'',t_+';\ T) = C + D\ \cos\ (\phi' - \phi_b)$$

or

$$B_a(s''', s'', T;\ t''', t'';\ t_+''', t_+'') \approx \frac{1}{16\pi^3} \int_0^{s''} \frac{ds'}{s''} \int_{-\infty}^{-(s'/s'')(-t'' + \frac{\mu'^2}{1 - s'/s''})} dt'$$

$$\cdot \int_{\epsilon''_{min}(\frac{s''}{s'''},\ \frac{s'}{s''};\ t''', t'', t')}^{\epsilon''_{max}(\frac{s''}{s'''},\ \frac{s'}{s''};\ t''', t'', t')} d\epsilon''$$

$$\cdot B_a(s'', s', T;\ t'', t';\ t_+'', t_+') \frac{K(\epsilon'', T;\ t''', t'', t';\ t_+''', t_+'', t_+')}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]^{1/2}}, \qquad (3.22')$$

with

$$t_+' \equiv t_+'(\frac{s''}{s'''}, \frac{s'}{s''}, T;\ t''', t'', t';\ t_+''', t_+'')$$

$$= t' + T - (s'/s'')(T + t'' - t_+'') - 2[t'T + u'T(s'/s'') + t''T(s'/s'')]^{1/2} \cdot \cos[\cos^{-1}\left(\frac{\epsilon'' - C}{D}\right) + \phi_b] .$$

Let us investigate the situation when Δ approaches zero, i.e., in the forward direction; then

$$T \to 0, \qquad t_+''' \to t''', \qquad t_+'' \to t'', \qquad t_+' \to t',$$

and Eq. (3.21') and Eq. (3.22') will be reduced to Eq. (2.31) and Eq. (2.30) respectively, i.e.,reduced to the forward direction case.

Also, we can easily verify that the kernel and integration limits of Eqs. (3.22) and (3.22') still have the important scaling property of invariance under the transformation

$$s''' \to c\, s''', \qquad s'' \to c\, s'', \qquad s' \to c\, s'.$$

This suggests that B_a can be written as

$$B_a(s''', s'', T;\ t''', t'';\ t_+''', t_+'') = (s'''/s_0)^{\alpha(T)}\ b_a^{\alpha}\left(\frac{s''}{s'''}, T; t''', t''; t_+''', t_+''\right). \tag{3.23}$$

If we set

$$y'' = s''/s''' , \qquad y' = s'/s''' ,$$

we will get the following integral equation for b_a^{α} :

$$b_a^{\alpha}(y'', T;\ t''', t'';\ t_+''', t_+'') = \frac{1}{16\pi^3}\,(y'')^{\alpha(T)} \int_0^1 dy' \int_{-\infty}^{-y'\left(-t'' + \frac{\mu'^2}{1-y'}\right)} dt'$$

$$\cdot \int_{\epsilon''_{min}(y'', y'; t''', t'', t')}^{\epsilon''_{max}(y'', y'; t''', t'', t')} d\epsilon''$$

$$\cdot\ b_a^{\alpha}(y', T;\ t'', t';\ t_+'', t_+')\ \frac{K(\epsilon'', T;\ t''', t'', t';\ t_+''', t_+'', t_+')}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]^{1/2}} \tag{3.24}$$

or

$$b_a^{\alpha}(y'',T;\ t''',t'';\ t_+''',t_+'') = \frac{1}{16\pi^3}(y'')^{\alpha(T)} \int_0^1 dy' \int_{-\infty}^{-y'(-t''+\frac{\mu'^2}{1-y'})} dt'$$

$$\cdot \int_{t'_{+min}(y',T;t'',t';t_+'')}^{t'_{max}(y',T;t'',t';t_+'')} dt_+'$$

$$\cdot\ b_a^{\alpha}(y',T;t'',t';t_+'',t_+')\ \frac{K(\epsilon'',T;\ t''',t'',t';\ t_+''',t_+'',t_+')}{[(t''_{+max}-t'')(t'_+-t'_{+min})]^{1/2}} \qquad (3.24')$$

and

$$A_{ab}(s''',T;\ t''',t_+''') = \frac{1}{16\pi^3}(s'''/s_0)^{\alpha(T)} \int_0^1 dy'' \int_{-\infty}^{-y''(-t'''+\frac{\mu''^2}{1-y''})} dt''$$

$$\cdot \int_0^{\pi} d\phi\ \ b_a^{\alpha}(y'',T;\ t''',t'';\ t_+''',t_+'')\ G_b^*(t_+'',t_+''')\ G_b(t'',t''') \qquad (3.25)$$

or

$$A_{ab}(s''',T;\ t''',t_+''') = \frac{1}{16\pi^3}(s'''/s_0)^{\alpha(T)} \int_0^1 dy'' \int_{-\infty}^{-y''(-t''' + \frac{\mu''^2}{1-y''})} dt''$$

$$\cdot \int_{t''_{+min}(y'',T;t''',t'';t_+''')}^{t''_{+max}(y'',T;t''',t'';t_+''')} dt''_+$$

$$\cdot\ b_a{}^{\alpha}(y'',T;\ t''',t'';\ t_+''',t_+'')\ \frac{G_b{}^*(t_+'',t_+''')\ G_b(t'',t''')}{[(t''_{+max} - t'')(t'' - t''_{+min})]^{1/2}}\ . \qquad (3.25')$$

Equations (3.23) through (3.25') are the essential results of this section, and we have shown that both A_{ab} and B_a have Regge asymptotic behavior.

4. THE WEAK COUPLING LIMIT (HIGH-SUBENERGY APPROXIMATION)

Now let us assume that all the subenergies ϵ_i are also large compared with $s_0 \approx 1(\mathrm{GeV}/c)^2$ and with all the t's and the particle masses. We will call this condition the weak coupling limit or the high-subenergy approximation. Even though at present we know this approximation to be unrealistic (i.e., experimentally most production events have at least one subenergy smaller or comparable to the t's and μ^2's), nevertheless the weak coupling limit was the original motivation for inventing the multi-Regge model,[7,8,10,14] which is one kind of multiperipheral model. Because of this history and also because many papers have investigated dynamical properties under this approximation, it is worthwhile to show that our equations in Sec. 2 and Sec. 3 can be reduced to those simplified equations which have been derived in Ref. (7) and Ref. (10) under the weak coupling approximation.

When $\epsilon_i \gg s_0, t$'s, μ^2's, one can easily show that $s''/s' \gg 1$, $s'''/s'' \gg 1$, and if we assume the multi-Regge model, then the function $f(\epsilon_i, t_{i-1}, t_i, t_{i+1})$ takes the simple form $(\epsilon_i/s_0)^{\alpha_{in}(t_i)}$, where we have thrown the Regge phase factor into the coupling constants G_a, G_b, and the β's, and α_{in} is the input exchange trajectory. From Eqs. (2.27) and (2.28), we know that both t''_{max} and t'_{max} will approach zero, so from Eq. (2.29) to (2.31) we will get

$$\begin{aligned}\epsilon''_{max} &\approx (\mu'^2 - t'' - t)(s'''/s'') + 2(t't'')^{1/2}(s'''/s'') \\ &= \left\{\mu'^2 + \left[(-t)^{1/2} + (-t'')^{1/2}\right]^2\right\}(s'''/s'') \qquad (4.1)\end{aligned}$$

and

$$\epsilon''_{min} \approx (\mu'^2 - t'' - t')(s'''/s'') - 4(t't'')^{1/2} (s'''/s'')$$

$$= \left\{ \mu'^2 + [(-t)^{1/2} - (-t'')^{1/2}]^2 \right\} (s'''/s'') . \qquad (4.2)$$

(i) The Forward Case $\Delta^2 = 0$

By using Eqs. (4.1), (4.2), (2.37), and (2.39), we obtain the following results:

$$b_a^{\alpha}(y'',0;\ t''',t'') = \frac{1}{16\pi^2} \int_0^1 dy' \int_{-\infty}^0 dt'\ (y'')^{\alpha(0)-2\alpha_{in}(t'')} \cdot \gamma(t'',t')\ b_a^{\alpha}(y',0;\ t'',t'), \qquad (4.3)$$

where

$$\gamma(t'',t') = \frac{1}{2} \left\{ \frac{\mu'^2 + [(-t')^{1/2} + (-t'')^{1/2}]^2}{s_0} \right\}^{2\alpha_{in}(t'')} \beta^2(t',t'') \cdot {}_2F_1 \left\{ -2\,\alpha_{in}(t''),\ -\frac{1}{2};1;\ \frac{4(t't'')^{1/2}}{\mu'^2 + \left[(-t')^{1/2} + (-t'')^{1/2}\right]^2} \right\}. \qquad (4.4)$$

If we make the ansatz

$$b_a^{\alpha}(y'',0;\ t''',t'') = (y'')^{\alpha(0)-2\alpha_{in}(t'')}\ b_a^{\alpha}(t''), \qquad (4.5)$$

then from Eq. (4.3) we get

$$b_a^{\alpha}(t'') = \int_{\infty}^{0} dt'\ \frac{b_a^{\alpha}(t')\ \gamma(t'',t')}{\alpha(0) - 2\,\alpha_{in}(t') + 1} \qquad (4.6)$$

for

$$\alpha(0) \quad > \quad 2\,\alpha_{in}(t') - 1 \,.$$

Now if we further assume that $\gamma(t'',t')$ can be factorized as

$$\gamma(t'',t') \quad = \quad h_1(t')\, h_2(t''), \qquad (4.7)$$

where h_1 and h_2 are some arbitrary functions, then we can let

$$b_a^{\alpha}(t'') \; = \; c h_2(t''), \qquad (4.8)$$

where c is an arbitrary constant; and Eq. (4.6) will be reduced to

$$1 \; = \int_{-\infty}^{0} dt' \; \frac{h_1(t')\, h_2(t')}{\alpha(0) - 2\,\alpha_{in}(t') + 1} \,. \qquad (4.9)$$

If we define

$$g^2(0) \quad = \quad [\alpha(0) - 2\,\bar{\alpha}_{in} + 1] \int_{-\infty}^{0} dt' \; \frac{h_1(t')\, h_2(t')}{\alpha(0) - 2\,\alpha_{in}(t') + 1} \,, \qquad (4.10)$$

we will get

$$\alpha(0) \quad = \quad 2\,\bar{\alpha}_{in} - 1 + g^2(0) \,. \qquad (4.11)$$

Furthermore, if we assume $\alpha_{in}(t)$ to be a flat trajectory α_{in}, then

$$\alpha(0) \quad = \quad 2\,\alpha_{in} - 1 + g^2, \qquad (4.12)$$

with

$$g^2 \quad = \int_{-\infty}^{0} dt' \; h_1(t')\, h_2(t'),$$

which is just the Chew-Pignotti[14] result.

$$\alpha(T) = 2\,\bar{\alpha}_{in} - 1 + g^2(T)\,, \tag{4.23}$$

with

$$g^2(T) = \left(\alpha(T) - 2\,\bar{\alpha}_{in} + 1\right) \int_{-\infty}^{0} dt' \int_{t_{+min}}^{t'_{+max}} dt'_+$$

$$\cdot \frac{h_1(t', t'_+;\, T)\; h_2(t', t'_+;\, T)}{[(t'_{+max} - t'_+)(t'_+ - t'_{+min})]^{1/2}\; [\alpha(T) - \alpha_{in}(t') - \alpha_{in}(t'_+)]}\,. \tag{4.24}$$

This is just the CGL result.[7]

5. THE AVERAGE MULTIPLICITY, THE INELASTICITY, AND THE SPECTRA OF SECONDARIES IN ULTRAHIGH-ENERGY COLLISIONS

As in ABFST,[9] we are going to use the model described in Sec. 2 to study the quantities which can be measured in ultrahigh-energy production processes. What we want to demonstrate in this section is that all the qualitative properties of the ABFST model[9] remain true in this generalized multiperipheral model.

(i) Average Multiplicity of Secondaries $\langle n \rangle$

We know the reaction

$$a + b \rightarrow 0 + 1 + 2 + \cdots + (n+1) \tag{2.1}$$

will contribute n secondary particles;[15] this means that the average number of secondary particles $\langle n \rangle$ in the final state at an energy s'' will be given by

$$\langle n \rangle = \frac{\sum_{n=0}^{\infty} n \int |T_{ab}{}^{n}|^{2} \, d\not{\phi}_{n}}{\sum_{n=0}^{\infty} \int |T_{ab}{}^{n}|^{2} \, d\not{\phi}_{n}} = \frac{\sum_{n=0}^{\infty} n \, A_{ab}{}^{n}(s'', 0; m_b{}^2)}{\sum_{n=0}^{\infty} A_{ab}{}^{n}(s'', 0; m_b{}^2)} . \tag{5.1}$$

We call $\langle n \rangle$ the average multiplicity.

Now let us replace $\beta(t_i, t_{i+1})$ by $g\,\beta'(t_i, t_{i+1})$; then Eq. (2.3) becomes

$$T_{ab}{}^{n} = g^{n}\, G_a(m_a{}^2, t_1)\, f(\epsilon_1, m_a{}^2, t_1, t_2)\, \beta'(t_1, t_2)\, f(\epsilon_2, t_1, t_2, t_3) \cdot \beta'(t_2, t_3) \cdots \beta'(t_n, t_{n+1})\, f(\epsilon_{n+1}, t_n, t_{n+1}, m_b{}^2)\, G_b(t_{n+1}, m_b{}^2) , \tag{5.2}$$

so

$$n \int |T_{ab}{}^{n}|^2 \, d\not{\phi}_n = g^2 \cdot \frac{\partial}{\partial g^2} \left(\int |T_{ab}{}^{n}|^2 \, d\not{\phi}_n \right),$$

and

$$\langle n \rangle = g^2 \frac{\partial}{\partial g^2} \left[\ell n \, A_{ab}(s''', 0; m_b{}^2) \right] . \qquad (5.3)$$

From Eq. (2.36), we can write

$$A_{ab}(s''', 0; m_b{}^2) = C_{\alpha(0)}(g^2, m_b{}^2) \, (s'''/s_0)^{\alpha(0)}, \qquad (5.4)$$

where

$$C_{\alpha(0)}(g^2, m_b{}^2) = \frac{1}{16\pi^2} \int_0^1 dy'' \int_{-\infty}^{-y''\left(-m_b{}^2 + \frac{\mu''^2}{1-y''}\right)} dt'' \cdot b_a{}^{\alpha}(y'', 0; m_b{}^2, t'') \, |G_b(t'', m_b{}^2)|^2 , \qquad (5.5)$$

so

$$\langle n \rangle = g^2 \frac{\partial}{\partial g^2} \left[\ell n \, C_{\alpha(0)}(g^2, m_b{}^2) \right] + g^2 \frac{\partial \alpha(0)}{\partial g^2} \, \ell n (s'''/s'') . \qquad (5.6)$$

At very high energy we can neglect the first term to get

$$\langle n \rangle \approx g^2 \frac{\partial \alpha(0)}{\partial g^2} \, \ell n (s'''/s_0) , \qquad (5.7)$$

so we have shown that at very high energy the average multiplicity $\langle n \rangle$ will increase as $\ell n(s'''/s_0)$ in our model.

(ii) The Inelasticity η

Now let us consider the spectrum of the "primary" particle which is directly connected with the incident particle. If we choose the "a" particle to be the target, and the "b" particle to be the incident particle, then the "primary" particle will be the (n+1)th particle in the $T_{ab}^{\ n}$ amplitude.

Let the momentum of the "primary" particle be p" (see Fig. 7). Then the spectrum of the "primary" particle will simply be

$$dn(p'') = B_a(p_a, p_b; p_b - p''; 0) \, |G_b(p_b - p'', p_b)|^2 \, \delta^+(p''^2 - \mu''^2) \, d^4p'' \tag{5.8}$$

or

$$dn(p'') = B_a(s''', s'', 0; t''', t'') \, |G_b(t'', t''')|^2 \, \not{\Phi}_1(s''', s''; t''', t'') ds'' dt''. \tag{5.8'}$$

Now let us use this spectrum to calculate the average energy carried away by the "primary" particle. If E" is the energy of the "primary" particle, then in the rest frame of m_a,

$$E'' = \frac{p'' \cdot p_a}{m_a}, \tag{5.9}$$

but

$$s'' = (p_a + Q'')^2 = (p_a + p_b - p'')^2 = s''' + \mu''^2 - 2p''\cdot p_a - 2p''\cdot p_b$$

and

$$t'' = (p_b - p'')^2 = m_b^{\ 2} + \mu''^2 - 2p''\cdot p_b ,$$

so

$$E'' = \frac{s''' - s'' + t'' - m_b^{\ 2}}{2m_a} \approx \frac{s''' - s''}{2m_a} . \tag{5.10}$$

Then the inelasticity η can be defined as

$$\eta = 1 - \frac{\langle E'' \rangle}{E_b}$$

$$= 1 - \frac{2m_a}{s'''}$$

$$\cdot \frac{\int \frac{(s'''-s'')}{2m_a} B_a^{\alpha}(s''',s'',0;m_b^2,t)\,|G_b(t'',m_b^2)|^2 \not{\phi}_1(s''',s'';m_b^2,t'')ds''dt''}{\int B_a^{\alpha}(s''',s'',0;m_b^2,t'')\,|G_b(t'',m_b^2|^2 \not{\phi}_1(s''',s'';m_b^2,t'')ds''\,dt''}$$

$$\approx 1 - \frac{\int_0^1 (1-y'')dy'' \int_{-\infty}^{-y''(-m_b^2+\frac{\mu''}{1-y''})} dt''\, b_a^{\alpha}(y'',0;m_b^2,t'')\,|G_b(t'',m_b^2)|^2}{\int_0^1 dy'' \int_{-\infty}^{-y'(-m_b^2+\frac{\mu''}{1-y''})} dt''\, b_a^{\alpha}(y'',0;m_b^2,t'')\,|G_b(t'',m_b^2)|^2}\,; \qquad (5.11)$$

so we have proved that the inelasticity η is independent of the incident energy in the high-energy limit.

(iii) <u>Spectra of Secondaries</u>

Let $N(k)d^4k$ be the number of final states, such that one of the secondary particles has its four momentum between k and $k + dk$, then

$$N(k) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \int (T_{ab}{}^n)^* (T_{ab}{}^n)\ \delta^4(Q_{i+1} - Q_i - k)d\not{\phi}_n$$

$$= \sum_{n=i}^{\infty} \sum_{i=1}^{\infty} \int (T_{ab}{}^n)^* (T_{ab}{}^n)\ \delta^4(Q_{i+1} - Q_i - k)d\not{\phi}_n\ . \qquad (5.12)$$

From the definitions of $T_{ab}{}^{n}$ and $d\not{\Phi}_n$, we can write $N(k)$ as (see Fig. 8)

$$N(k) = \sum_{n-i=0}^{\infty} \sum_{i-1=0}^{\infty} \frac{1}{(2\pi)^{3n+1}} \int \Bigg\{ |G_a(p_a,Q_1)f(-p_a,Q_1,Q_2)\beta(Q_1,Q_2)$$

$$\cdot f(Q_1,Q_2,Q_3)\cdots\beta(Q_{i-1},Q_i)f(Q_{i-1},Q_i,Q_{i+1})|^2 \delta^+[(Q_1+p_a)^2 - \mu_0{}^2]$$

$$\cdot d^4Q_1\ \delta^+[(Q_2-Q_1) - \mu_1{}^2]d^4Q_2\cdots d^4Q_{i-1}\ \delta^+[(Q_i-Q_{i-1})^2 - \mu_{i-1}^2]\Bigg\}$$

$$\cdot\ |\beta(Q_i,Q_{i+1})|^2\ \delta^+[(Q_{i+1}-Q_i)^2 - \mu_i{}^2]d^4Q_i\ d^4Q_{i+1}\ \delta^4(Q_{i+1}-Q_i-k)$$

$$\cdot \Bigg\{ |G_b(Q_{n+1},p_b)f(Q_n,Q_{n+1},p_b)\beta(Q_n,Q_{n+1})f(Q_{n-1},Q_n,Q_{n+1})\beta(Q_{n-1},Q_n)$$

$$\cdot \cdots\beta(Q_{i+1},Q_{i+2})f(Q_i,Q_{i+1},Q_{i+2})|^2\ \delta^+[(p_b-Q_{n+1})^2 - \mu_{n+1}^2]d^4Q_{n+1}$$

$$\cdot\ \delta^+[(Q_{n+1}-Q_n)^2 - \mu_n{}^2]d^4Q_n\cdots d^4Q_{i+2}\ \delta[(Q_{i+2}-Q_{i+1})^2 - \mu_{i+1}^2]\Bigg\}. \tag{5.13}$$

By using the definition of the function $B_a{}^n$, $B_b{}^m$,

$B_a = \sum_{n=0}^{\infty} B_a{}^n$, and $B_b = \sum_{m=0}^{\infty} B_b{}^m$, just as in the ABFST model[9] we can easily show that (see Fig. 9)

$$N(k) = \frac{1}{(2\pi)^6} \int B_a(p_a, Q'; Q; 0) \delta^+[(Q'-Q)^2 - \mu_k^2] \delta^4(Q'-Q-\mathbf{k}) d^4Q \, d^4Q' \cdot B_b(p_b, -Q; -Q'; 0) . \tag{5.14}$$

If we translate this equation in terms of invariant variables, Eq. (5.14) will become (see Fig. 9)

$$N(k) = \int B_a(w', w, 0'; t', t) dt \, dw \, |\beta(t,t')|^2 \, dt' \, dv \cdot B_b(v', v, 0; t, t') \, I(w, v'; t', t; k) \, \delta^+(k^2 - \mu_k^2) , \tag{5.15}$$

where

$$w = (p_a + Q)^2 , \qquad v = (p_b - Q')^2 , \qquad t = Q^2 ,$$

$$w' = (p_a + Q')^2, \qquad v' = (p_b - Q)^2 , \qquad t' = Q'^2 ,$$

and

$$I(w, v; t', t; k) = \int d^4Q \, d^4Q' \, \delta(Q^2 - t) \, \delta(Q'^2 - t') \, \delta[(p_a + Q)^2 - w] \cdot \delta[(p_b - Q')^2 - v] \, \delta^4(Q' - Q - k) . \tag{5.16}$$

This boundary function I has been evaluated in Ref. 9 and in Appendix E . We are able to show the following result (for details see Appendix E):

$$N(k) = (s'''/s_0)^{\alpha(0)} F, \tag{5.17}$$

$$F = \left(\frac{\mu_k^2 + \underset{\sim}{k}_T^2}{s_0}\right)^{\alpha(0)+1} \iint |\beta(t,t')|^2 \, dt \, dt' \iint dx \, dz [(1+x)(1+z)]^{\alpha(0)+1}$$

$$\cdot \, b_a^{\alpha}\left(\frac{x}{1+x}, 0; t', t\right) b_b^{\alpha}\left(\frac{z}{1+z}, 0; t, t'\right)$$

$$\cdot \, T[-t - (\mu_k^2 + \underset{\sim}{k}_T^2)\, x(1+z); \; -t' - (\mu_k^2 + \underset{\sim}{k}_T^2) z(1+x); \underset{\sim}{k}_T^2], \tag{5.18}$$

where $T(a; b; c)$ is the usual triangle function given by

$$T(a;b;c) = \frac{\Theta(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac)}{(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac)^{1/2}},$$

where $\underset{\sim}{k}_T$ and k_L are the transverse and longitudinal components of the four-momentum k with respect to the incident direction, both defined in c.m. frame of a and b. And k_0 is the time component of k.

The limits of integration of Eq. (5.18) have been analyzed in Ref. 9, and the important fact demonstrated that, when the energy of the secondary (k_0) is small compared with the total energy $(s'')^{1/2}$, the function F is independent both of the incident and the secondary energy, i.e.,

$$F = F(\mu_k^2, \; \underset{\sim}{k}_T^2). \tag{5.19}$$

The reason is simply that the function T in Eq. (5.18) is different from zero only in a region of the xy plane which is independent of s''' and k_0. But

$$\delta^+(k^2 - \mu_k^2)d^4k = \frac{1}{2k_0} \delta^+(k^2 - \mu_k^2)dk^2 \, d^2\underset{\sim}{k}_T \, dk_L$$

$$= \frac{1}{2} \frac{d^2\underset{\sim}{k}_T \, dk_L}{(\mu_k^2 + \underset{\sim}{k}_T^2 + k_L^2)^{1/2}} , \qquad (5.20)$$

so

$$dN(k) = N(k)\,\delta^+(\mu_k^2 - k^2)d^4k$$

$$= \frac{1}{2k_0} \delta^+(\mu_k^2 - k^2) \, F(k^2, \underset{\sim}{k}_T^2)dk^2 \, d^2\underset{\sim}{k}_T \, dk_L$$

$$= \frac{1}{2} F(\mu_k^2, \underset{\sim}{k}_T^2) \frac{d^2\underset{\sim}{k}_T \, dk_L}{(\mu_k^2 + \underset{\sim}{k}_T^2 + k_L^2)^{1/2}} . \qquad (5.21)$$

This means the spectra of the transverse momenta and longitudinal momenta are independent of the initial energy $(s''')^{1/2}$, so the average transverse momentum of the secondary is independent of s''' also. All the properties which we have shown in this section (and also shown by ABFST[9]), will be preserved by any multiperipheral model with "finite range" correlated kernel (which may include Toller-angle dependence or even more complicated kernels).

6. THE INTEGRAL EQUATION WHEN TOLLER-ANGLE DEPENDENCE IS INCLUDED

For the same process as described by Eq. (2.1), if we want to include the Toller-angle dependence in the internal vertex function β, we need to allow a dependence of β on four different Q's. In this situation the production amplitude $T_{ab}{}^{n}$ can be written as (see Fig. 1)

$$\begin{aligned} T_{ab}{}^{n} &= G_a(p_a,Q_1)\, f(-p_a,Q_1,Q_2)\, \beta(-p_a,Q_1,Q_2,Q_3) \\ &\quad \cdot f(Q_1,Q_2,Q_3)\, \beta(Q_1,Q_2,Q_3,Q_4)\, f(Q_2,Q_3,Q_4) \\ &\quad \cdot \cdots \beta(Q_{n-1},Q_n,Q_{n+1},p_b)\, f(Q_n,Q_{n+1},p_b)\, G_b(Q_{n+1},p_b) \\ &= F_a(p_a,Q_1,Q_2)\, F(-p_a,Q_1,Q_2,Q_3)\, F(Q_1,Q_2,Q_3,Q_4) \\ &\quad \cdot \cdots F(Q_{n-1},Q_n,Q_{n+1},p_b)\, G_b(Q_{n+1},p_b) , \end{aligned} \tag{6.1}$$

where

$$F_a(p_a,Q_1,Q_2) = G_a(p_a,Q_1)\, f(-p_a,Q_1,Q_2) , \tag{6.2}$$

$$F(Q_{i-1},Q_i,Q_{i+1},Q_{i+2}) = \beta(Q_{i-1},Q_i,Q_{i+1},Q_{i+2})\, f(Q_i,Q_{i+1},Q_{i+2}) , \tag{6.3}$$

or

$$T_{ab}{}^{n}(p_a,p_b;\, Q_1,Q_2,\cdots Q_{n+1})$$

$$= G_a(m_a{}^2,t_1)\; f(\epsilon_1,m_a{}^2,t_1,t_2)\; \beta(t_1,\phi_1,t_2)\; f(\epsilon_2,t_1,t_2,t_3)$$

$$\cdot\; \beta(t_2,\phi_2,t_3)\cdots\beta(t_n,\phi_n,t_{n+1})\; f(\epsilon_{n+1},t_n,t_{n+1},m_b{}^2)$$

$$\cdot\; G_b(t_{n+1},m_b{}^2) \quad , \qquad (6.4)$$

where ϕ_i is the ith Toller angle defined in the rest frame of particle "i" by

$$\cos\phi_i = \frac{(\underset{\sim}{Q}_{i+2} \times \underset{\sim}{p}_{i+1})\cdot(\underset{\sim}{Q}_{i-1} \times \underset{\sim}{p}_{i-1})}{|\underset{\sim}{Q}_{i+2} \times \underset{\sim}{p}_{i+1}|\cdot|\underset{\sim}{Q}_{i+1} \times \underset{\sim}{p}_{i-1}|} \qquad i=1,2,3,\cdots,n \;, (6.5)$$

with

$$p_a = -Q_0 \;, \qquad p_b = Q_{n+2} \;,$$

$\underset{\sim}{Q}_i$ and $\underset{\sim}{p}_i$ being the three momenta of Q_i and p_i in the rest frame of particle "i." The Toller angle ϕ_i here is in one-to-one correspondence with the "Toller angle" ω_i defined in a different way through an O(2,1) group variable by Bali, Chew, and Pignotti,[17] when all the particles are spinless.

For the forward case, just as in Sec. 2, let us introduce a function $B_a{}^n(p_a,p_b;\, Q_{n+1},Q_n;\, 0)$ by

$$A_{ab}{}^{n}(p_a,p_b;\,0) = \frac{1}{(2\pi)^6}\int d^4Q_{n+1}\, d^4Q_n\; B_a{}^n(p_a,p_b;\, Q_{n+1},Q_n;\,0)$$

$$\times\; |G_b(Q_{n+1},p_b|^2\; \delta^+[(Q_{n+1}-Q_n)^2-\mu_n{}^2]\; \delta^+[(p_b-Q_{n+1})^2-\mu_{n+1}^2],$$

$$(6.6)$$

then the function $B_a^{\,n}(p_a,p_b;\, Q_{n+1},Q_n;\, 0)$ will satisfy a recursion relation

$$B_a^{\,n}(p_a,p_b;\, Q_n,Q_{n+1};\, 0) = \frac{1}{8\pi^3}\int d^4Q_{n-1}\, B_a^{\,n-1}(p_a,Q_{n+1};\, Q_n,Q_{n-1};\, 0)$$

$$\cdot\ \delta^+[(Q_n - Q_{n-1})^2 - \mu_{n-1}^{\;2}]\ K(Q_{n-1},Q_n,Q_{n+1},p_b), \qquad (6.7)$$

where

$$K(Q_{n-1},Q_n,Q_{n+1},p_b) = |F(Q_{n-1},Q_n,Q_{n+1},p_b)|^2 \ . \qquad (6.8)$$

By defining

$$B_a(p_a,p_b;\, Q'',Q';\, 0) \equiv \sum_{n=0}^{\infty} B_a^{\,n}(p_a,p_b;\, Q'',Q';\, 0) \qquad (6.9)$$

with

$$B_a^{\,0}(p_a,p_b;\, Q'',Q';\, 0) = |F_a(p_a,Q'',p_b)|^2\ \delta^4(p_a + Q')\ , \qquad (6.10)$$

we can get the following integral equation for $B_a(p_a,p_b;\, Q'',Q';\, 0)$ (see also Fig. 10):

$$B_a(p_a,p_b;\, Q'',Q';\, 0) = B_a^{\,0}(p_a,p_b;\, Q'',Q';\, 0) + \frac{1}{8\pi^3}\int d^4Q$$

$$\cdot\ B_a(p_a,Q'';\, Q',Q;\, 0)\ \delta^+[(Q - Q')^2 - \mu^2]\ K(Q,Q',Q'',p_b) \quad (6.11)$$

and

$$A_{ab}(p_a,p_b;\, 0) = \frac{1}{(2\pi)^6}\int d^4Q'\ d^4Q''\ B_a(p_a,p_b;\, Q'',Q';\, 0)\,|G_b(Q'',p_b)|^2$$

$$\cdot\ \delta^+[(Q'' - Q')^2 - \mu'^2]\ \delta^+[(Q'' - p_b)^2 - \mu''^2] \ . \qquad (6.12)$$

As in Sec. 2, we want to translate these equations into invariant variables. Let us introduce $B_a(s''', s'', s'; t''', t'', t'; \epsilon''; m_a^2, \mu'^2, \mu''^2)$ and $K(\phi'; t''', t'', t', t; \epsilon'', \epsilon'; \epsilon_{12}; \mu^2, \mu'^2, \mu''^2)$ by (see also Fig. 11)

$$
\begin{aligned}
B_a(p_a, p_b; Q'', Q'; 0) \equiv \int & B_a(s''', s'', s'; t''', t'', t'; \epsilon''; m_a^2, \mu'^2, \mu''^2) \\
& \cdot \delta[(p_a + p_b)^2 - s'''] \cdot \delta[(p_a + Q'')^2 - s''] \, \delta(p_b^2 - t''') \, \delta(Q''^2 - t'') \\
& \cdot \delta(Q'^2 - t') \, \delta^+(p_a^2 - m_a^2) \cdot \delta[(Q' - p_b)^2 - \epsilon''] \, \delta^+[(Q' - Q'')^2 - \mu'^2] \\
& \cdot \delta^+[(Q'' - p_b)^2 - \mu''^2] \, ds''' \, ds'' \, ds' \, dt''' \, dt'' \, dt' \\
& \cdot d\epsilon'' \, dm_a^2 \, d\mu'^2 \, d\mu''^2
\end{aligned}
\tag{6.13}
$$

and

$$
\begin{aligned}
K(Q, Q', Q'', p_b) \equiv \int & K(\phi'; t''', t'', t', t; \epsilon'', \epsilon'; \epsilon_{12}; \mu^2, \mu'^2, \mu''^2) \\
& \cdot \delta(p_b^2 - t''') \, \delta(Q''^2 - t'') \cdot \delta(Q'^2 - t') \, \delta(Q^2 - t) \\
& \cdot \delta[(Q' - p_b)^2 - \epsilon''] \, \delta[(Q - Q'')^2 - \epsilon'] \, \delta^+[(Q - Q')^2 - \mu^2] \\
& \cdot \delta^+[(Q' - Q'')^2 - \mu'^2] \, \delta^+[(Q'' - p_b)^2 - \mu''^2] \, \delta[(Q - p_b)^2 - \epsilon_{12}] \\
& \cdot \delta \left\{ \cos \phi' - \frac{[\underset{\sim}{p}_b \times (\underset{\sim}{p}_b - \underset{\sim}{Q}'')] \cdot [\underset{\sim}{Q} \times (\underset{\sim}{Q} - \underset{\sim}{Q}')]}{|\underset{\sim}{p}_b \times (\underset{\sim}{p}_b - \underset{\sim}{Q}'')| \, |\underset{\sim}{Q} \times (\underset{\sim}{Q} - \underset{\sim}{Q}')|} \right\} \\
& \cdot d\phi \, dt''' \, dt'' \, dt' \, dt \cdot d\epsilon'' \, d\epsilon' \, d\epsilon_{12} \, d\mu^2 \, d\mu'^2 \, d\mu''^2 \ .
\end{aligned}
\tag{6.14}
$$

However, for simplicity of writing, we define

$$B_a(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$\equiv\ B_a(s''',s'',s';\ t''',t'',t';\ \epsilon'';\ m_a{}^2,\mu'^2,\mu''^2), \tag{6.15}$$

$$K(\emptyset';\ t''',t'',t',t;\ \epsilon'',\epsilon',\epsilon_{12})$$

$$\equiv\ K(\emptyset';\ t''',t'',t',t;\ \epsilon'',\epsilon';\ \epsilon_{12};\ \mu^2,\mu'^2,\mu''^2)\ . \tag{6.16}$$

By using Eq. (2.12), Eq. (2.15), and the definitions of $B_a(s''',s'',s';\ t''',t'',t';\ \epsilon'')$ and $K(\emptyset';\ t''',t'',t';\ \epsilon'',\epsilon',\epsilon_{12})$, Eq. (6.11) and Eq. (6.12) will become the following forms (see also Fig. 11):

$$B_a(s''',s'',s';\ t''',t'',t';\ \epsilon'') = B_a{}^0(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$+\ \frac{1}{8\pi^3}\int ds\ dt\ d\epsilon_{12}\ d\epsilon'\cdot B_a(s'',s',s;\ t'',t',t;\ \epsilon')$$

$$\cdot\ K(\emptyset';\ t''',t'',t',t;\ \epsilon'',\epsilon')\cdot \Psi_3(s''',s'',s',s;\ t''',t'',t',t;\ \epsilon',\epsilon'',\epsilon_{12}), \tag{6.17}$$

$$A_{ab}(s''',0;\ t''') = \frac{1}{(2\pi)^6}\int ds''\ ds'\ d\epsilon''\ dt''\ dt'$$

$$B_a(s''',s'',s';\ t''',t'',t';\ \epsilon)\,|G_b(t'',t''')|^2$$

$$\cdot\ \Phi_1(s''',s'';\ t''',t'')\,\Psi_1(s''',s'',s';\ t''',t'',t';\ \epsilon'')\ , \tag{6.18}$$

with

$$\Psi_3(s''',s'',s',s;\ t''',t'',t',t;\ \epsilon',\epsilon'',\epsilon_{12})$$

$$= \int d^4Q\ \delta[(p_a + Q)^2 - s]\ \delta(Q^2 - t)$$

$$\cdot\ \delta[(Q - Q'')^2 - \epsilon']\ \delta[(Q - p_b)^2 - \epsilon_{12}]\ \delta^+[(Q - Q')^2 - \mu^2]\ , \tag{6.19}$$

$$B_a^{\,0}(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$= |G_a(m_a^{\,2},t'')\ f(\epsilon'',m_a^{\,2},t'',t''')|^2\ \delta(t' - m_a^{\,2}) \cdot \delta(\epsilon'' - s''')\ \delta(s'')\ , \tag{6.20}$$

and $s = (p_a + Q)^2$, $\epsilon' = (Q - Q'')^2$, $\epsilon_{12} = (Q - p_b)^2$, $t = Q^2$,

$$K(\phi';\ t''',t'',t',t;\ \epsilon'',\epsilon') = |\beta(t',\phi',t'')\ f(\epsilon'',t',t'',t''')|^2, \tag{6.21}$$

where ϕ' is a function of $\epsilon_{12},\epsilon'',\epsilon',t''',t'',t'$, and t.

The third kind of boundary function Ψ_3 has been worked out in detail in Appendix F; the result is

$$\Psi_3(s''',s'',s',s;\ t''',t'',t',t;\ \epsilon',\epsilon'',\epsilon_{12})$$

$$= \delta(\epsilon_{12} - C'' + D'')\ \Psi_1(s'',s',s;\ t'',t',t;\ \epsilon'), \tag{6.22}$$

with

$$C'' \approx t + t''' + u\left(\frac{s'''}{s'}\right) + \bar{\epsilon}''\left(\frac{s}{s'}\right) + 2t'\left(\frac{s}{s'}\right)\left(\frac{s'''}{s'}\right),$$

$$D'' \approx 2[t + u\left(\frac{s}{s'}\right) + t'\left(\frac{s}{s'}\right)^2]^{\frac{1}{2}}[t''' + \bar{\epsilon}''\left(\frac{s'''}{s'}\right) + t'\left(\frac{s'''}{s'}\right)^2]^{\frac{1}{2}}$$

$$\bullet\ \cos[\cos^{-1}\left(\frac{\epsilon' - C'}{D'}\right) - \phi_b]\ ,$$

$$C' \approx t + t'' + u\left(\frac{s''}{s'}\right) + u'\left(\frac{s}{s'}\right) + 2t'\left(\frac{s}{s'}\right)\left(\frac{s''}{s'}\right) ,$$

$$D' \approx 2[t + u\left(\frac{s}{s'}\right) + t'\left(\frac{s}{s'}\right)^2]^{\frac{1}{2}} [t'' + u'\left(\frac{s''}{s'}\right) + t'\left(\frac{s''}{s'}\right)^2]^{\frac{1}{2}} ,$$

$$u = \mu^2 - t - t' , \qquad \bar{\epsilon}'' = \epsilon'' - t' - t''' ,$$

at high energy, and assuming $s \gg t''', t'', t', t'; m_a^2, \mu^2, \mu'^2, \mu''^2, \epsilon''$.

So if we neglect the inhomogeneous term B_a^0 in Eq. (6.17) at extremely high energy, then the kernel and limits of integration of the homogeneous integral equation of B_a still have the scaling property

$$s''' \to cs''' , \quad s'' \to cs'' , \quad s' \to cs' , \quad s \to cs ,$$

so we still can prove that A_{ab} and B_a have Regge asymptotic behavior, i.e., $\sim (s''')^{\alpha(0)}$. In this section we have demonstrated that we can easily include the Toller-angle dependence in our scheme at forward direction, but there is in principle no difficulty in generalizing to the nonforward case. Furthermore, this kind of scheme will also work for any correlation kernel of finite length. That is, we can always write down an integral equation for the modified absorptive function B_a, and we can also prove that B_a and A_{ab} have Regge asymptotic behavior for any multiperipheral model whose kernel only involves a finite correlation interval.

7. CONCLUSION

In this paper we have explicitly derived the CGL type[7] of multiperipheral integral equation in terms of invariant variables, without making the high-energy approximation, and we have demonstrated that the absorptive part A_{ab} and the modified absorptive part B_a will have Regge asymptotic behavior as a result of the high-energy scaling invariance of the kernel. This property will always be obtained so long as the kernel only involves a finite-link correlation, and this is also the only requirement to prove those qualitative properties that have been shown by ABFST[9] to be true in high-energy production collisions. The form of our general integral equations is more complicated than that resulting from the kinematic approximation that all subenergies are large compared with the momentum transfers and the masses involved. However, it is hard to justify the latter kinematic approximation, and the general equation derived in this paper is still not hopeless to solve numerically with presently existing computers.

For example, one specific model, that of CZA,[4] has been adopted to describe the $N\bar{N}$ annihilation process,[16] and by the arguments of Ting[18] we may hope to generate an output ω trajectory from pure nucleon-trajectory exchange. Because in the CZA model the subenergies are not all large compared with the t's, the integral equations described in Sec. 2 and Sec. 3 of this paper have to be used in order to get a realistic result. This calculation is in progress.

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APPENDIX A. THE BOUNDARY FUNCTION $\Phi_1(s''', s''; t''', t'')$

In this appendix we discuss the boundary function Φ_1 which is defined by Eq. (2.22), that is

$$\Phi_1(s''', s''; t''', t'')$$

$$= \int d^4Q'' \, \delta[Q''^2 - t''] \, \delta[(p_a + Q'')^2 - s''] \, \delta^+[(p_b - Q'')^2 - \mu''^2] \; . \quad (A.1)$$

We know

$$\delta^+[(p_b - Q'')^2 - \mu''] = \delta[(p_b - Q'')^2 - \mu''^2] \, \Theta[(p_b - Q'')_0 - \mu'']$$

$$= \delta[(p_b - Q'')^2 - \mu''^2] \, \Theta[E_b - E_{Q''} - \mu''] \quad , \quad (A.2)$$

but

$$\Theta(E_b - E_{Q''} - \mu'') = \Theta[(E_a + E_b) - (E_a + E_{Q''}) - \mu''] \; , \quad (A.3)$$

and because $\Theta(E_b - E_{Q''} - \mu'')$ is an invariant, we can evaluate in any frame. Let us calculate in the c.m. frame of particle "a" and "b"; then

$$s''' = (p_a + p_b)^2 = (E_a + E_b)^2$$

and

$$s'' = (p_a + Q'')^2 = (E_a + E_{Q''})^2 - (\underset{\sim}{k}_a + \underset{\sim}{k}_{Q''})^2 ,$$

so

$$E_a + E_b = (s''')^{\frac{1}{2}} \qquad \text{and} \qquad E_a + E_{Q''} > (s'')^{\frac{1}{2}} \; .$$

Equation (A.2) can be rewritten as

$$\delta^+[(p_b - Q'')^2 - \mu''^2] = \delta[(p_b - Q'')^2 - \mu''^2] \, \Theta[(s''')^{\frac{1}{2}} - (s'')^{\frac{1}{2}} - \mu''^2] , \quad (A.4)$$

so from (A.1) and (A.4), we get

$$\Phi_1(s''',s''; t''',t'') = \Theta[(s''')^{\frac{1}{2}} - (s'')^{\frac{1}{2}} - \mu''] \, \Phi_1'(s''',s''; t''',t''), \tag{A.5}$$

where

$$\Phi_1'(s''',s''; t''',t'')$$
$$= \int d^4Q'' \, \delta(Q''^2 - t'') \, \delta[(p_a + Q'')^2 - s''] \, \delta[(p_b - Q'')^2 - \mu''^2] \, . \tag{A.6}$$

Now let us concentrate on the function Φ_1'. Because it is also an invariant, we can evaluate in any frame. Let us go to the rest frame of particle "a" so that

$$p_a = (m_a,0,0,0) \, ,$$

$$p_b = (E_b,0,0,k_b) \quad \text{and} \quad k_b > 0 \, ,$$

$$Q'' = (E'',k'' \sin\Theta'' \cos\phi'', k'' \sin\Theta'' \sin\phi'', k'' \cos\Theta'')$$
$$\text{and} \quad k'' > 0 \, ,$$

with $p_a^{\,2} = m_a^{\,2}$, $p_b^{\,2} = t'''$. Because

$$s''' = (p_a + p_b)^2 = m_a^{\,2} + t''' + 2m_aE_b$$

and

$$t''' = E_b^{\,2} - k_b^{\,2} \, ,$$

we can get

$$E_b = \frac{s''' - m_a^{\,2} - t'''}{2m_a} = \frac{\bar{s}'''}{2m_a}$$

and

$$k_b^{\,2} = \frac{(s''' - m_a^{\,2} - t''')^2}{4m_a^{\,2}} - m_b^{\,2} = \left[\frac{\lambda(s''',m_a^{\,2},t''')}{2m_a}\right]^2 ,$$

where

$$\bar{s}''' = s''' - m_a^{\,2} - t'''$$

and

$$\lambda(a,b,c) = (a^2 + b^2 - c^2 - 2ab - 2bc - 2ca)^{\frac{1}{2}}.$$

Now we can rewrite Eq. (A.6):

$$\Phi_1' \equiv \Phi_1'(s''',s''; t''',t'')$$

$$= \int_0^\infty k''^2 dk'' \int_{-1}^{+1} d\cos\theta'' \int_{-\infty}^{+\infty} dE'' \int_0^{2\pi} d\phi''\, \delta(E''^2 - k''^2 - t'')$$

$$\times\, \delta(m_a^{\,2} + t'' - s'' + 2m_aE'')\, \delta(t''' + t'' - \mu''^2 - 2E''E_b + 2k_bk'' \cos\theta''). \tag{A.7}$$

If we define

$$\bar{s}'' = s'' - m_a^{\,2} - t'' \qquad \text{and} \qquad u'' = \mu''^2 - t'' - t''',$$

then Eq. (A.7) will become the following expression after we integrate over $d\phi''$:

$$\Phi_1' = 2\pi \int_0^\infty k''^2\, dk'' \int_{-1}^{+1} d\cos\theta'' \int_{-\infty}^{+\infty} dE''\, \delta[E''^2 - (k''^2 + t'')]$$

$$\cdot\, \delta(-\bar{s}'' + 2m_aE'')\, \delta\left[-u'' - \frac{\bar{s}'''}{m_a} E_{Q''} + \frac{\lambda(s''',m_a^{\,2},t''')}{m_a} k'' \cos\theta''\right]$$

Equation (A.8) Continued

Equation (A.8) Continued.

$$= 2\pi\int_0^\infty \frac{k''^2\, dk''}{2(k''^2 + t'')^{\frac{1}{2}}}\int_{-1}^{+1} d\cos\Theta \left\{\delta[-\bar{s}'' + 2m_a(k''^2 + t'')^{\frac{1}{2}}]\right.$$

$$\cdot\ \delta\left[-u'' - \frac{\bar{s}'''}{m_a}(k''^2 + t'')^{\frac{1}{2}} + \frac{\lambda(s''',m_a^2,t''')}{m_a} k''\cos\Theta''\right]$$

$$+\ \delta\left[-\bar{s}'' - 2m_a(k''^2 + t'')^{\frac{1}{2}}\right]$$

$$\left.\cdot\ \delta\left[-u'' + \frac{\bar{s}'''}{m_a}(k''^2 + t'')^{\frac{1}{2}} + \frac{\lambda(s''',m_a^2,t''')}{m_a} k''\cos\Theta''\right]\right\}$$

$$= \frac{\pi m_a}{2\lambda(s''',m_a^2,m_b^2)}\int_0^\infty \frac{dk^2}{(k''^2 + t'')^{\frac{1}{2}}}\left(\delta[-\bar{s}'' + 2m_a(k''^2 + t'')^{\frac{1}{2}}]\right.$$

$$\cdot\ \Theta\left\{\frac{\lambda^2(s''',m_a^2,t''')}{m_a^2}k''^2 - \left[u'' + \frac{\bar{s}'''}{m_a}(k''^2 + t'')^{\frac{1}{2}}\right]^2\right\}$$

$$+\ \delta[-\bar{s}'' - 2m_a(k''^2 + t'')^{\frac{1}{2}}]$$

$$\left.\cdot\ \Theta\left\{\frac{\lambda^2(s''',m_a^2,t''')}{m_a^2}k''^2 - \left[u'' - \frac{\bar{s}'''}{m_a}(k''^2 + t'')^{\frac{1}{2}}\right]^2\right\}\right)$$

$$= \frac{\pi}{2\lambda(s''',m_a^2,m_b^2)}\,\Theta\left\{\left[\left(\frac{\bar{s}''}{2m_a}\right)^2 - t''\right]\frac{\lambda^2(s''',m_a^2,t''')}{m_a^2} - \left(u'' + \frac{\bar{s}'''\bar{s}''}{2m_a^2}\right)^2\right\}$$

$$= \frac{\pi}{2\lambda(s''',m_a^2,m_b^2)}\,\Theta(-m_a^2u''^2 - t''\bar{s}'''^2 - t'''\bar{s}''^2 - \bar{s}''\bar{s}'''u'' + 4m_a^2t'''t''), \tag{A.8}$$

for $\bar{s}'' > 0$ and $(\bar{s}''/2m_a)^2 > t''$. If we assume $s''' \gg m_a^{\,2}, \mu''^2, t'''$, we get

$$\Phi_1(s''',s''; t''',t'')$$

$$= \frac{\pi}{2s'''} \; \Theta[(s''')^{\frac{1}{2}} - (s'')^{\frac{1}{2}} - \mu''] \; \Theta[(t''_{max} - t'')(t'' - t''_{min})], \qquad \text{(A.9)}$$

with

$$t''_{\substack{max\\min}} = -\frac{(s''' - s'')}{2} \pm \left[\left(\frac{s''' - s''}{2}\right)^2 - \mu''^2 s'' + \frac{s''}{s'''} t'''(s''' - s'')\right]^{\frac{1}{2}}. \qquad \text{(A.10)}$$

Because $(s''')^{\frac{1}{2}} > (s'')^{\frac{1}{2}} + \mu''$, we know

$$s''' - s'' > 2\mu''(s''')^{\frac{1}{2}} + \mu'' \gg m_a^{\,2}, t''', \mu''^2$$

at high energy. It follows that

$$t''_{\substack{max\\min}} \approx \begin{cases} -\left(\dfrac{s''}{s'''}\right)\left(-t''' + \dfrac{\mu''^2}{1 - \frac{s''}{s'''}}\right) \\[2ex] -(s''' - s'') \sim -\infty \end{cases} . \qquad \text{(A.11)}$$

APPENDIX B. THE BOUNDARY FUNCTION $\Psi_1(s''', s'', s'; t''', t'', t'; \epsilon'')$

In this Appendix we discuss the boundary function ψ_1 which is defined by Eq. (2.23), that is

$$\Psi_1(s''', s'', s'; t''', t'', t'; \epsilon'')$$

$$= \int d^4Q' \; \delta(Q'^2 - t') \; \delta[(p_a + Q')^2 - s'] \; \delta[(Q' - p_b)^2 - \epsilon'']$$

$$\cdot \; \delta^+[(Q' - Q'')^2 - \mu'^2] \quad , \tag{B.1}$$

so

$$\Psi_1(s''', s'', s'; t''', t'', t'; \epsilon'')$$

$$= \Theta[(s'')^{\frac{1}{2}} - (s')^{\frac{1}{2}} - \mu'] \; \psi_1'(s''', s'', s'; t''', t'', t'; \epsilon'') \quad , \tag{B.2}$$

with

$$\Psi_1'(s''', s'', s'; t''', t'', t'; \epsilon'')$$

$$= \int d^4Q' \; \delta(Q'^2 - t') \; \delta[(p_a + Q')^2 - s'] \; \delta[(Q' - p_a)^2 - \epsilon'']$$

$$\cdot \; \delta[(Q' - Q'')^2 - \mu'^2] \quad . \tag{B.3}$$

Let us concentrate on the function Ψ_1', evaluated in the rest frame of particle a, such that

$$p_a = (m_a, 0, 0, 0) \quad ,$$

$$Q'' = (E'', 0, 0, k''), \qquad k'' > 0 \quad ,$$

$$p_b = (E_b, k_b \sin\Theta_b, 0, k_b \cos\Theta_b), \qquad k_b > 0 \quad ,$$

$$Q' = (E', k' \sin\Theta' \cos\phi', k' \sin\Theta' \sin\Theta', k' \cos\Theta')$$

and $k' > 0$.

Just as in Appendix A, we can easily show the following:

$$E'' = \frac{\bar{s}''}{2m_a} , \qquad k'' = \frac{\lambda(s'',m_a^2,t'')}{2m_a} ,$$

$$E_b = \frac{\bar{s}'''}{2m_a} , \qquad k_b = \frac{\lambda(s''',m_a^2,t''')}{2m_a} . \tag{B.4}$$

Furthermore

$$\mu''^2 = (Q'' - p_b)^2 = t'' + m_b^2 - 2E_bE'' + 2k''k_b \cos\Theta_b ,$$

so

$$\cos\Theta_b = \frac{(U'' + 2E''E_b)}{2k''k_b} = \frac{(2m_a^2 u'' - \bar{s}'''\bar{s}'')}{\lambda(s'',m_a^2,t'')\ \lambda(s''',m_a^2,t''')} . \tag{B.5}$$

Now from Eq. (B.3), we can rewrite Ψ_1' as

$$\Psi_1' \equiv \Psi_1'(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$= \int_0^{\infty} k'^2 dk' \int_{-1}^{+1} d\cos\Theta' \int_{-\infty}^{+\infty} dE'$$

$$\cdot\ \delta[E'^2 - (k'^2 + t')]\ \delta(m_a^2 + t' - s' + 2m_aE')$$

$$\cdot\ \delta(t' + t'' - \mu'^2 - 2E'E'' + 2k'k'' \cos\Theta') \int_0^{2\pi} d\phi'$$

$$\cdot\ \delta[t' + m_b^2 - \epsilon'' - 2E'E_b$$

$$+ 2k'k_1[\cos\Theta' \cos\Theta_b + \sin\Theta' \sin\Theta_b \cos\phi')] . \tag{B.6}$$

According to the definition of Φ_1', the first three integrals and delta functions will give us $(1/2\pi)\Phi_1'(s'',S';\ t'',t')$ so Eq. (B.4) can be rewritten as

$$\Psi_1' = \frac{\Phi_1'(s'',s';\ t'',t')}{2\pi} \int_0^{2\pi} d\phi'\ \delta(H) \tag{B.7}$$

or

$$\Psi_1' = \frac{\Theta(-m_a{}^2 u'^2 - t'\bar{s}''^2 - t''\bar{s}'^2 - \bar{s}'\bar{s}''u' + 4m_a{}^2 t't'')}{4\lambda(s'',m_a{}^2,t'')}$$

$$\cdot \int_0^{2\pi} d\phi'\ \delta(H) \ , \tag{B.7'}$$

where

$$H \equiv -\bar{\epsilon}'' - 2E'E_b + 2k'k_b \cos\Theta' \cos\Theta_b + 2k'k_b \sin\Theta' \sin\Theta_b \cos\phi, \tag{B.8}$$

with

$$\bar{s}' = s' - m_a{}^2 - t' \ , \quad u' = \mu'^2 - t' - t'', \quad \bar{\epsilon}'' = \epsilon'' - t' - t'' \ ,$$

$$E' = \frac{\bar{s}'}{2m_a} \ , \qquad k' = \frac{\lambda(s',t',t'')}{2m_a} \ , \text{ and} \tag{B.9}$$

$$\cos\Theta' = \frac{(u'' + 2E''E_b)}{2k'k''} \ .$$

We rewrite H in the form

$$H = -\bar{\epsilon}'' + \bar{C} + D\cos\phi = -\epsilon'' + C + D\cos\phi', \tag{B.10}$$

with

$$\bar{C} = -2E'E_b + 2k'k_b \cos\theta' \cos\theta_b \ , \qquad \text{(B.11)}$$

$$C = t' + t''' + \bar{C} \ , \qquad \text{(B.11')}$$

$$D = 2k'k_b \sin\theta' \sin\theta_b \ . \qquad \text{(B.12)}$$

From Eq. (B.4), Eq. (B.9), and Eq. (B.10), we can get

$$\bar{C} = -\frac{\bar{s}'\bar{s}'''}{2m_a^2} + \frac{k_b(u' + 2E'E'')}{k''} \cdot \frac{(u'' + 2E''E_b)}{2k''k_b}$$

$$= -\frac{\bar{s}'\bar{s}'''}{2m_a^2} + \frac{2m_a^2}{\bar{s}''^2 - 4m_a^2 t''}\left(u' + \frac{\bar{s}'\bar{s}''}{2m_a^2}\right)\left(u'' + \frac{\bar{s}''\bar{s}'''}{2m_a^2}\right) \ . \qquad \text{(B.13)}$$

We will further assume that $s''', s'', s' \gg t', t'', t''', m_a^2, \mu'^2, \mu''^2$; then $\bar{s}' \approx s'$, $\bar{s}'' \approx s''$, $\bar{s}''' \approx s'''$. The first part of this approximation $s''', s'', s' \gg m_a^2, \mu'^2, \mu''^2$, can be easily justified because we are only interested in high-energy collisions. Even though there exists a part of the phase space where s'' and s' are comparable to m_a^2, μ'^2, μ''^2, the percentage is very small if s''' is very large, and the percentage will decrease as s''' increases. The second part of the approximation--$s''', s'', s' \gg t''', t'', t'$--can be justified only if we invoke the dynamical assumption of the multiperipheral model that the general production amplitude T_{ab}^n falls off very fast as any of the t's become large. This condition has been confirmed experimentally in many 2-to-2 and 2-to-3 amplitudes. If such is generally true then in that part of the phase space where s's and t's are comparable and large, the kernel and $G_b(t'',t''')$ will be very small and will not contribute significantly to B_a and A_{ab}. Therefore the approximation $s''', s'', s' \gg t''', t'', t'$ is reasonable.

Under this approximation, Eq. (B.13) becomes

$$\overline{C} \approx -\frac{s's'''}{2m_a^2} + \frac{2m_a^2}{s''^2}\left(1 + \frac{4m_a^2 t''}{s''^2}\right)\left(u' + \frac{s's''}{2m_a^2}\right)\left(u'' + \frac{s''s'''}{2m_a^2}\right) \tag{B.14}$$

$$\approx u'\left(\frac{s'''}{s''}\right) + u''\left(\frac{s'}{s''}\right) + 2t''\left(\frac{s'}{s''}\right)\left(\frac{s'''}{s''}\right)$$

and

$$D = 2k'k_b \sin\Theta' \sin\Theta_b$$

$$= \frac{1}{2k''}\left[4k'^2k''^2 - (u' + 2E'E'')^2\right]^{\frac{1}{2}} \cdot \frac{1}{k''}\left[4k''^2k_b^2 - (u'' + 2E''E_b)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\,\frac{\left[\lambda^2(s',m_a^2,t')\,\lambda^2(s'',m_a^2,t'') - (2m_a^2u' + \bar{s}'\bar{s}'')^2\right]^{\frac{1}{2}}}{m_a\,\lambda(s'',m_a^2,t'')}$$

$$\cdot\,\frac{\left[\lambda^2(s'',m_a^2,t'')\,\lambda^2(s''',m_a^2,t''') - (2m_a^2u' + \bar{s}''\bar{s}''')^2\right]^{\frac{1}{2}}}{m_a\,\lambda(s'',m_a^2,t'')}$$

$$\approx \frac{1}{2}\,\frac{\left[(s'^2 - 4m_a^2t')(s''^2 - 4m_a^2t'') - 4m_a^2u's's'' - s'^2s''^2\right]^{\frac{1}{2}}}{m_a\,s''}$$

$$\cdot\,\frac{\left[(s'' - 4m_a^2t'')(s'' - 4m_a^2t''') - 4m_a^2u''s''s''' - s''^2s'''^2\right]^{\frac{1}{2}}}{m_a\,s''}$$

$$\approx 2\left\{\left[t' + u'\left(\frac{s'}{s''}\right) + t''\left(\frac{s'}{s''}\right)^2 \quad t''' + u''\left(\frac{s'''}{s''}\right) + t''\left(\frac{s'''}{s''}\right)^2\right]\right\}^{\frac{1}{2}}. \tag{B.15}$$

Then

$$\Psi_1 \equiv \Psi_1(s''', s'', s'; t''', t'', t', \epsilon'')$$

$$= \frac{\Phi_1(s'', s'; t'', t')}{2\pi} \int_0^{2\pi} d\phi' \; \delta(-\bar{\epsilon}'' + \bar{C} + D \cos \phi')$$

$$= \frac{\Phi_1}{2\pi} \int_0^{2\pi} d\phi' \; \delta(-\epsilon'' + C + D \cos \phi')$$

$$= \frac{\Phi_1(s'', s'; t'', t')}{\pi} \frac{\Theta[D^2 - (-\bar{\epsilon}'' + \bar{C})^2]}{[D^2 - (-\bar{\epsilon}'' + \bar{C})^2]^{\frac{1}{2}}}$$

$$= \frac{\Phi_1}{\pi} \frac{\Theta[D^2 - (-\epsilon'' + C)^2]}{[D^2 - (-\epsilon'' + C)^2]^{\frac{1}{2}}}$$

$$= \frac{\Phi_1(s'', s'; t'', t')}{\pi} \frac{\Theta[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]}{[(\epsilon''_{max} - \epsilon'')(\epsilon'' - \epsilon''_{min})]^{\frac{1}{2}}} , \qquad (B.16)$$

where

$$\epsilon''_{\substack{max \\ min}} = C \pm D , \qquad (B.17)$$

$$C = t' + t'' + u'\left(\frac{s'''}{s''}\right) + u''\left(\frac{s'}{s''}\right) + 2t''\left(\frac{s'}{s''}\right)\left(\frac{s'''}{s''}\right) . \qquad (B.18)$$

APPENDIX C. THE BOUNDARY FUNCTION $\Phi_2(s''', s'', T;\ t''', t'';\ t_+''', t_+'')$

The boundary function Φ_2 is defined by Eq. (3.21) as

$$\Phi_2(s''', s'', T;\ t''', t'';\ t_+''', t_+'')$$

$$= \int d^4Q''\ \delta(Q''^2 - t'')\ \delta[(Q'' + \triangle)^2 - t_+'']\ \delta[(p_a + Q'')^2 - s'']$$

$$\cdot\ \delta^+[(Q'' - p_b)^2 - \mu''^2]\ , \quad (C.1)$$

so

$$\Phi_2 \equiv \Phi_2(s''', s'', T;\ t''', t'';\ t_+''', t_+'')$$

$$= \Theta[(s''')^{\frac{1}{2}} - (s'')^{\frac{1}{2}} - \mu'']\ \Phi_2'(s''', s'', T;\ t''', t'';\ t_+''', t'')\ , \quad (C.2)$$

with

$$\Phi_2'(s''', s'', T;\ t''', t'';\ t_+''', t'') \equiv \int d^4Q''\ \delta(Q''^2 - t'')\ \delta[(Q'' + \triangle)^2 - t_+'']$$

$$\cdot\ \delta[(p_a + Q'') - s'']\ \delta[(Q'' - p_b)^2 - \mu''^2]\ . \quad (C.3)$$

Just as in Appendix A, let us evaluate Φ_2' in a special frame such that

$$\begin{aligned}
p_a &= (m_a, 0, 0, 0)\ , \\
p_b &= (E_b, 0, 0, k_b)\ , \qquad k_b > 0\ , \\
\triangle &= (\triangle_0, \triangle_x, 0, \triangle_z)\ , \qquad \triangle_x > 0\ , \qquad\qquad (C.4) \\
Q'' &= (E'', k''\sin\Theta''\cos\phi'', k''\sin\Theta''\sin\phi'', k''\cos\Theta'')\ , \\
& \qquad\qquad k'' > 0\ , \\
\triangle^2 &\equiv T\ .
\end{aligned}$$

Because (see Fig. 5)

$$p_a^+ = p_a - \triangle\ , \qquad p_b^+ = p_b + \triangle\ ,$$

it follows that

$$m_a^2 = (p_a^+)^2 = m_a^2 + T - 2m_a\Delta_0 , \qquad \Delta_0 = \frac{T}{2m_a} , \tag{C.5}$$

$$t_+^{\prime\prime\prime} = (p_b^+)^2 = t^{\prime\prime\prime} + T + 2E_b\Delta_0 - 2k_b\Delta_z ,$$

$$\Delta_z = \frac{1}{2k_b}(T + 2E_b\Delta_0 + t^{\prime\prime\prime} - t_+^{\prime\prime\prime}) \tag{C.6}$$

$$\approx \frac{E_b}{k_b}\Delta_0 \approx \Delta_0 ,$$

and from

$$T = \Delta^2 = \Delta_0^2 - \Delta_x^2 - \Delta_z^2 , \text{ we get } \quad \Delta_x = (\Delta_0^2 - \Delta_z^2 - T)^{\frac{1}{2}} \approx (-T)^{\frac{1}{2}} . \tag{C.7}$$

Now we can rewrite Eq. (C.1) as

$$\Phi_2 = \Theta[(s^{\prime\prime\prime})^{\frac{1}{2}} - (s^{\prime\prime})^{\frac{1}{2}} - \mu^{\prime\prime}]\int_0^{\infty} k^{\prime\prime 2}dk^{\prime\prime}\int_{-1}^{+1} d\cos\Theta^{\prime\prime}\int_{-\infty}^{+\infty} dE^{\prime\prime}$$

$$\cdot\ \delta(E^{\prime\prime 2} - k^{\prime\prime 2} - t^{\prime\prime})\ \delta(m_a^2 + t^{\prime\prime} - s^{\prime\prime} + 2m_aE^{\prime\prime})$$

$$\cdot\ \delta(t^{\prime\prime\prime} + t^{\prime\prime} - \mu^{\prime\prime 2} - 2E^{\prime\prime}E_b + 2k_bk^{\prime\prime}\cos\Theta^{\prime\prime})\int_0^{2\pi} d\phi$$

$$\cdot\ \delta(t^{\prime\prime} + T - t_+^{\prime\prime} + 2E^{\prime\prime}\Delta_0 - 2k^{\prime\prime}\cos\Theta^{\prime\prime}\,\Delta_z - 2k^{\prime\prime}\sin\Theta^{\prime\prime}\,\Delta_x\cos\phi^{\prime\prime})$$

Equation (C.8) continued.

Equation (C.8) continued.

$$= \frac{\Phi_1(s''',s''; t''',t'')}{\pi} \int_0^{\pi} d\phi$$

$$\cdot\, \delta(-t''_+ + t'' + T + 2E''\Delta_0 - 2k'' \cos\Theta'' \Delta_z - 2k'' \sin\Theta'' \Delta_x \cos\phi)$$

$$= \frac{\Phi_1}{\pi} \int_0^{\pi} d\phi\ \delta(-t''_+ + C' - D' \cos\phi) \,, \tag{C.8}$$

or

$$\Phi_2 = \frac{\Phi_1}{\pi} \frac{\Theta[D'^2 - (C' - t''_+)^2]}{[D'^2 - (C' - t''_+)^2]^{\frac{1}{2}}} = \frac{\Phi_1}{\pi} \frac{\Theta[(t''_{+max} - t''_+)(t''_+ - t''_{+min})]}{[(t''_{+max} - t''_+)(t''_+ - t''_{+min})]^{\frac{1}{2}}} \,, \tag{C.9}$$

with

$$C' = t'' + T + 2E''\Delta_0 - 2k'' \cos\Theta'' \Delta_z \,, \tag{C.10}$$

$$D' = 2k'' \sin\Theta'' \Delta_x \,, \tag{C.11}$$

and

$$t''_{+\substack{max \\ min}} = C' \pm D' \,. \tag{C.12}$$

From Appendix A, Eq. (C.5), and Eq. (C.6), we can rewrite C' as

$$C' = t'' + T + \frac{\bar{s}''T}{2m_a^2} - \frac{2(u'' + 2E_b E'')}{2k_b} \cdot \frac{(T + 2E_b\Delta_0 + t''' - t''_+)}{2k_b}$$

Equation (C.13) continued

Equation (C.13) continued

$$\approx \; t'' + T + \frac{s''T}{2m_a^2} - \frac{2m_a^2}{s'''^2}\left(1 + \frac{4m_a^2 m_b^2}{s'''^2}\right)$$

$$\bullet \left[\frac{s'''^2 s''T}{4m_a^2} + \frac{s''' s''}{2m_a^2}\,(T + t''' - t_+''')\right]$$

$$\approx \; t'' + T - \frac{s''}{s'''}\,(T + t''' - t_+''') \; . \qquad (C.13)$$

Also we can rewrite Eq. (C.11) as

$$D' \approx 2(-T)^{\frac{1}{2}}\,\frac{1}{2k_b}[4k''^2 k_b^2 - (u'' + 2E_b E'')^2]^{\frac{1}{2}}$$

$$= \frac{(-T)^{\frac{1}{2}}}{2m_a^2 k_b}\,[(\bar{s}''^2 - 4m_a^2 t'')(\bar{s}''^2 - 4m_a^2 t''') - (2m_a^2 u'' + s''s''')^2]^{\frac{1}{2}}$$

$$\approx \frac{(-T)^{\frac{1}{2}}}{m_a s'''}\,(-4m_a^2 t'' s'''^2 - 4m_a^2 t''' s''^2 - 4m_a^2 u'' s'' s''')^{\frac{1}{2}}$$

$$= 2[Tt'' + u''T\left(\frac{s''}{s'''}\right) + t'''T\left(\frac{s''}{s'''}\right)^2]^{\frac{1}{2}} \; . \qquad (C.14)$$

APPENDIX D. THE BOUNDARY FUNCTION

$\Psi_2(s''',s'',s',T;\ t''',t'',t';\ t_+''',t_+'',t_+';\ \epsilon'')$

The boundary function Ψ_2 is defined by Eq. (3.22) as

$$\Psi_2(s''',s'',s',T;\ t''',t'',t';\ t_+''',t_+'',t_+';\ \epsilon'')$$

$$\equiv \int d^4Q'\ \delta(Q'^2 - t')\,\delta[(Q + \triangle)^2 - t_+']\ \delta[(p_a + Q') - s']$$

$$\cdot\ \delta[(Q' - p_b)^2 - \epsilon'']\ \delta^+[(Q' - Q'')^2 - \mu'^2]\ , \qquad (D.1)$$

if we evaluate Ψ_2 in the same frame as chosen in Appendix B, i.e., in a rest frame of particle a such that

$$p_a = (m_a,0,0,0)\ ,$$

$$Q'' = (E'',0,0,k'')\ , \qquad k'' > 0\ ,$$

$$p_b = (E_b,k_b \sin\theta_b \cos\phi_b,\ k_b \sin\theta_b \sin\phi_b, k_b \cos\theta_b)\ ,$$

$$k_b > 0\ , \qquad (D.2)$$

$$Q' = (E',k' \sin\theta' \cos\phi',k' \sin\theta' \sin\phi',k' \cos\theta')\ ,$$

$$k' > 0\ ,$$

$$\triangle = (\triangle_0,\triangle_x,0,\triangle_z)\ , \qquad \triangle_x > 0\ ,$$

then from the definition of Ψ_2, Φ_2, and Ψ_1, and after manipulations similar to those in Appendix C, we will get

$$\Psi_2 = \Phi_2(s'',s',T;\ t'',t';\ t''_+,t'_+)\ \delta[(Q' - p_b)^2 - \epsilon'']$$

$$= \Phi_2(s'',s',T;\ t'',t';\ t''_+,t'_+)\ \delta[\epsilon'' - C - D\cos(\phi' - \phi_b)], \qquad (D.3)$$

with

$$\cos\phi' = \frac{t' + T - \left(\frac{s'}{s''}\right)\left(T + t'' - t''_+\right)}{2[Tt' + u'T\left(\frac{s'}{s''}\right) + t''T\left(\frac{s'}{s''}\right)^2]^{\frac{1}{2}}}, \qquad (D.4)$$

$$\cos\phi_b = \frac{t''' + T - t'''_+ - \frac{s'''}{s''}(T + t'' - t''_+)}{2\left[t'''T + u''T\left(\frac{s'''}{s''}\right) + Tt''\left(\frac{s'''}{s''}\right)^2\right]^{\frac{1}{2}}}, \qquad (D.5)$$

or

$$\Psi_2 = \Psi_1(s''',s'',s';\ t''',t'',t';\ \epsilon'') \cdot \delta[(Q' + \Delta)^2 - t'_+]$$

$$= \Psi_1(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$\cdot\ \delta(t' + T - t'_+ + 2E'\Delta_0 - 2k'\cos\Theta'\Delta_z - 2k'\sin\Theta'\Delta_x\cos\phi')$$

$$= \psi_1(s''',s'',s';\ t''',t'',t';\ \epsilon'')$$

$$\cdot\ \delta\Big\{t' - t'_+ + T - \left(\frac{s'}{s''}\right)(T + t'' - t''_+)$$

$$- 2[Tt' + u'T\left(\frac{s'}{s''}\right) - t''T\left(\frac{s'}{s''}\right)^2]^{\frac{1}{2}}$$

$$\cdot \cos[\cos^{-1}\left(\frac{\epsilon'' - c}{D}\right) + \phi_b]\Big\}, \qquad (D.6)$$

where Φ_2 and Ψ_1 have been discussed in Appendix C and Appendix B, respectively.

APPENDIX E. THE SPECTRUM FUNCTION $F(k^2, \underset{\sim}{k}_T{}^2)$

Let $N(k)dk$ be the number of final states, such that one of the secondary particles has its four momentum between k and $k + dk$. According to Eq. (5.16), (see Fig. 9)

$$N(k) = \int B_a(w',w,0;\ t',t)dt\ dw|\beta(t,t')|^2\ dt'\ dv\ B_a(v',v,0;\ t,t') \cdot I(w,v;\ t',t;\ k)\ \delta(k^2 - \mu_k{}^2)\ , \qquad (E.1)$$

where

$$w = (p_a + Q)^2, \quad v = (p_b - Q')^2, \quad t = Q^2, \quad k^2 = \mu_k{}^2,$$

$$w' = (p_a + Q')^2, \quad v' = (p_b - Q)^2, \quad t' = Q'^2$$

and

$$I(w,v;\ t',t;\ k) = \int d^4Q\ d^4Q'\ \delta(Q^2 - t)\ \delta(Q'^2 - t')\ \delta[(p_a + Q)^2 - w] \cdot \delta[(p_b - Q')^2 - v]\ \delta^4(Q' - Q - k)\ . \qquad (E.2)$$

The boundary function $I(w,v;\ t',t;\ k)$ has been evaluated in Ref. 9; the result is

$$I(w,v;\ t',t;\ k) = \frac{1}{8s'''}\ T[-t - (\mu_k{}^2 + \underset{\sim}{k}_T)\ x(1 + z);\ -t' - (\mu_k{}^2 + \underset{\sim}{k}_T{}^2)\ z(1 + x);\ \underset{\sim}{k}_T{}^2], \qquad (E.3)$$

where T is the usual triangle function given by

$$T(a;\ b;\ c) = -\frac{\Theta(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac)}{(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac)^{\frac{1}{2}}} \qquad (E.4)$$

and

$$x = \frac{w}{s_1} , \qquad z = \frac{v}{s_2} \tag{E.5}$$

(we will define s_1 and s_2 later), and k_T is the c.m. transverse component of the four-momentum k with respect to the incident direction. Now let k_L and k_0 be the c.m. longitudinal component and the time component of k, respectively; we further define

$$k_+ = k_0 + k_L , \qquad k_- = k_0 - k_L , \tag{E.6}$$

and

$$s_1 = (p_a + k)^2 , \qquad s_2 = (p_b + k)^2 . \tag{E.7}$$

Then, as shown in Ref. 9, we get

$$s_1 = k_+(s''')^{\frac{1}{2}} , \qquad s_2 = k_-(s''')^{\frac{1}{2}} , \qquad s_1 s_2 = k_+ k_- s''' = (\mu_k^{\,2} + \underset{\sim}{k}_T^{\,2}) s''' \tag{E.8}$$

and

$$s_1 = (p_a + k)^2 = (p_a + q' - Q)^2 = w' + t - w + m_a^{\,2} - t' + \mu_k^{\,2}$$

$$\approx w' - w .$$

So

$$w' \approx s_1 + w = s_1(1 + x) . \tag{E.9}$$

Similarly we can prove

$$v' \approx s_2 + v = s_2(1 + z) . \tag{E.10}$$

By using Eqs. (2.36), (E.5), (E.8), (E.9), (E.10) together with Eq. (E.1) we get

$$N(k) = \frac{1}{8s'''} \iint \left(\frac{w'}{s_0}\right)^{\alpha(0)+1} \left(\frac{v'}{s_0}\right)^{\alpha(0)+1}$$

$$\cdot\; b_a^{\alpha}\left(\frac{w}{w'}, v;\; t', t\right)\; b_a^{\alpha}\left(\frac{v}{v'}, 0;\; t, t'\right) |\beta(t,t')|^2$$

$$\cdot\; T[-t - (\mu_k^2 + \underset{\sim}{k}_T^2)\, x(1+z);\; -t' - (\mu_k^2 + \underset{\sim}{k}_T^2)\, z(1+x);\; \underset{\sim}{k}_T^2]$$

$$\cdot\; dt\, dt'\, dw\, dv \quad . \tag{E.11}$$

If we define

$$N(k) \equiv \left(\frac{s'''}{s_0}\right)^{\alpha(0)} F \;, \tag{E.12}$$

then

$$F \equiv \left(\frac{\mu_k^2 + k_T^2}{s_0}\right)^{\alpha(0)+1} \iint |\beta(t,t')|^2\, dt\, dt'$$

$$\cdot \iint dx\, dz[(1+x)(1+z)]^{\alpha(0)+1}$$

$$\cdot\; b_a^{\alpha}\left(\frac{x}{1+x}, 0;\; t', t\right)\; b_b^{\alpha}\left(\frac{z}{1+z}, 0;\; t, t'\right)$$

$$\cdot\; T[-t - (\mu_k^2 + \underset{\sim}{k}_T^2)\, x(1+z);\; -t' - (\mu_k^2 + \underset{\sim}{k}_T^2)\, z(1+x);\; \underset{\sim}{k}_T^2] \;. \tag{E.13}$$

APPENDIX F. THE BOUNDARY FUNCTION $\Psi_3(s''',s'',s',s;\ t''',t'',t',t;\ \epsilon'',\epsilon',\epsilon_{12})$

The boundary function Ψ_3 is defined by Eq. (6.19) as

$$\Psi_3 \equiv \Psi_3(s''',s'',s',s;\ t''',t'',t',t;\ \epsilon',\epsilon'',\epsilon_{12})$$

$$\equiv \int d^4Q\ \delta(Q^2 - t)\ \delta[(p_a + Q)^2 - s]\ \delta[(Q - Q'')^2 - \epsilon']$$

$$\cdot\ \delta^+[(Q - Q')^2 - \mu^2]\ \delta[(Q - p_b)^2 - \epsilon_{12}]\ . \qquad \text{(F.1)}$$

Let us evaluate this in the rest frame of particle a, where

$$p_a = (m_a,0,0,0)\ ,$$

$$Q' = (E',0,0,k')\ , \qquad k' > 0\ ,$$

$$Q'' = (E'',k''\sin\Theta'',0,k''\cos\Theta'')\ , \qquad k'' > 0\ ,$$

$$p_b = (E_b,k_b\sin\Theta_b\cos\phi_b,k_b\sin\Theta_b\sin\phi_b,\ k_b\cos\Theta_b)\ ,$$

$$k_b > 0\ ,$$

$$Q = (E,k\sin\Theta\cos\phi,k\sin\Theta\sin\phi,\ k\cos\Theta)\ ,$$

$$k > 0\ .$$

Now Eq. (F.1) can be written as

$$\Psi_3 \equiv \Theta[(s')^{\frac{1}{2}} - (s)^{\frac{1}{2}} - \mu] \int_0^\infty k^2\,dk \int_{-1}^{+1} d\cos\Theta \int_0^{+\infty} dE \int_0^{2\pi} d\phi$$

$$\cdot\ \delta[E^2 - (k^2 + t)]\ \delta(m_a{}^2 + t - s + 2m_aE')$$

$$\cdot\ \delta(t + t' - \mu^2 - 2EE' + 2kk'\cos\Theta)$$

$$\cdot\ \delta[t + t'' - \epsilon' - 2EE'' + 2kk''(\cos\Theta''\cos\Theta + \sin\Theta''\sin\Theta\cos\phi)]$$

Equation (F.2) Continued.

Equation (F.2) Continued.

$$\cdot\ \delta[t+t'''-\epsilon_{12}-2EE_b+2kk_b(\cos\Theta_b\cos\Theta+\sin\Theta_b\sin\Theta\cos(\phi-\phi_b)]$$

$$= \Psi_1(s'',s',s;\ t'',t',t;\ \epsilon')\ \delta(-\epsilon_{12}+C''+D''), \qquad \text{(F.2)}$$

with

$$C'' = t + t''' - 2EE_b + 2kk_b \cos\Theta\cos\Theta_b, \qquad \text{(F.3)}$$

$$D'' = 2kk_b \sin\Theta\sin\Theta_b \cos(\phi - \phi_b)\ . \qquad \text{(F.4)}$$

Just as in Appendix B, at high energy, we can show that

$$C'' \approx t + t''' + u\left(\frac{s'''}{s'}\right) + \bar{\epsilon}''\left(\frac{s}{s'}\right) + 2t'\left(\frac{s}{s'}\right)\left(\frac{s'''}{s'}\right)$$

$$= t + t''' + u\left(\frac{s'''}{s''}\right)\left(\frac{s''}{s'}\right) + \bar{\epsilon}''\left(\frac{s}{s'}\right) + 2t'\left(\frac{s}{s'}\right)\left(\frac{s''}{s'}\right)\left(\frac{s'''}{s''}\right) \qquad \text{(F.5)}$$

and

$$D'' \approx 2\left\{\left[t + u\left(\frac{s}{s'}\right) + t'\left(\frac{s}{s'}\right)^2\right]\left[t''' + \bar{\epsilon}''\left(\frac{s'''}{s'}\right) + t'\left(\frac{s'''}{s'}\right)^2\right]\right\}^{\frac{1}{2}}$$

$$\cdot\ \cos\left[\cos^{-1}\left(\frac{\epsilon' - C'}{D'}\right) - \phi_b\right], \qquad \text{(F.6)}$$

with

$$C' = t + t'' - 2EE'' + 2kk''\cos\Theta\cos\Theta''$$

$$\approx t + t'' + u\left(\frac{s''}{s'}\right) + u'\left(\frac{s}{s'}\right) + 2t'\left(\frac{s}{s'}\right)\left(\frac{s''}{s'}\right)\ , \qquad \text{(F.7)}$$

$$D' = 2kk'' \sin\Theta \sin\Theta''$$

$$\approx 2\left\{\left[t + u\left(\frac{s}{s'}\right) + t'\left(\frac{s}{s'}\right)^2\right]\left[t'' + u'\left(\frac{s''}{s'}\right) + t'\left(\frac{s''}{s'}\right)^2\right]\right\}^{\frac{1}{2}}, \tag{F.8}$$

$$\cos\phi_b \approx \frac{u'' - u'\left(\frac{s'''}{s'}\right) - \bar{\epsilon}''\left(\frac{s''}{s'}\right) - 2t'\left(\frac{s''}{s'}\right)\left(\frac{s'''}{s'}\right)}{2\left[t'' + u'\left(\frac{s''}{s'}\right) + t'\left(\frac{s''}{s'}\right)^2\right]^{\frac{1}{2}}\left[t''' + \bar{\epsilon}''\left(\frac{s'''}{s'}\right) + t'\left(\frac{s'''}{s'}\right)^2\right]^{\frac{1}{2}}}, \tag{F.9}$$

and

$$u = \mu^2 - t - t', \quad u' = \mu'^2 - t' - t'', \quad u'' = \mu''^2 - t'' - t''', \tag{F.10}$$

$$\bar{\epsilon}' = \epsilon' - t - t'', \quad \bar{\epsilon}'' = \epsilon'' - t' - t''' . \tag{F.11}$$

FOOTNOTES AND REFERENCES

* This work was supported in part by the U. S. Atomic Energy Commission.

† This paper is a revised version of an unpublished manuscript written in November 1968.

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11. After the preliminary version of this paper was finished, the author saw lecture notes by Professor F. E. Low (Brookhaven National Laboratory 50162, T-527) in July 1969. He briefly mentioned the nonweak coupling case (at forward direction only), but the way Low included Toller-angle dependence is quite different from the method we use in Sec. 6 of this paper. The angle which Low called ϕ is not a short-range correlation, and if the kernel depends on this angle it will not have the scaling property unless all the subenergies are large.

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FIGURE CAPTIONS

Fig. 1. The production process $a + b \to 0 + 1 + \cdots + (n+1)$.

Fig. 2. (a) $A_{ab}^{\ n}(p_a, p_b;\ 0)$: the contribution to the absorptive part A_{ab} at forward direction from the n+2-particle intermediate state.

(b) $B_a^{\ n}(p_a, p_b;\ Q_{n+1};\ 0)$: the contribution to the modified absorptive part B_a at forward direction from the n+2-particle intermediate state.

(c) $|G_b(Q_{n+1}, p_b)|^2$: the absolute value squared of the external coupling constant G_b.

Fig. 3. The schematic representations of (a) $B_a(p_a, p_b;\ Q'';\ 0)$, (b) $B_a^{\ 0}(p_a, p_b;\ Q'';\ 0)$, (c) $B_a(p_a, Q'';\ Q';\ 0)$, and (d) $K(Q', Q'', p_b)$.

Fig. 4. The unitarity diagram in terms of invariant variables, where $s''' = (p_a + p_b)^2$, $s'' = (p_a + Q'')^2$, $s' = (p_a + Q')^2$, $\epsilon'' = (Q' - p_b)^2$, $\mu''^2 = (Q'' - p_b)^2$, $t''' = Q'''^2 = p_b^{\ 2}$, $t'' = Q''^2$, $t' = Q'^2$, $m_a^{\ 2} = p_a^{\ 2}$, and $\mu'^2 = (Q' - Q'')^2$.

Fig. 5. Schematic representation of $A_{ab}^{\ n}(p_a, p_b;\ \Delta)$ and Eq. (3.2).

Fig. 6. Schematic representation of (a) $B_a(p_a, p_b;\ Q'';\ \Delta)$, (b) $B_a^{\ 0}(p_a, p_b;\ Q'', \Delta)$, (c) $B_a(p_a, Q'', Q';\ \Delta)$, and (d) $K(Q', Q'', p_b;\ \Delta)$.

Fig. 7. Diagram used in calculating inelasticity in Sec. V.

Fig. 8. Diagram used in calculating the spectra of secondaries.

Fig. 9. Diagram used in calculating the spectra of secondaries.

Fig. 10. Schematic representation of (a) $B_a(p_a,p_b; Q',Q'')$, (b) $B_a^{\,0}(p_a,p_b; Q',Q'')$, (c) $B_a(p_a,Q''; Q,Q')$, and (d) $K(Q,Q',Q'',p_b)$.

Fig. 11. The modified absorptive part $B_a(s''',s'',s'; t''',t'',t'; \epsilon'')$, and the definitions of $s = (p_a + Q)^2$, $\epsilon' = (Q - Q'')^2$, $\mu^2 = (Q - Q')^2$, $t = Q^2$, and $\epsilon_{12} = (Q - p_b)^2$.

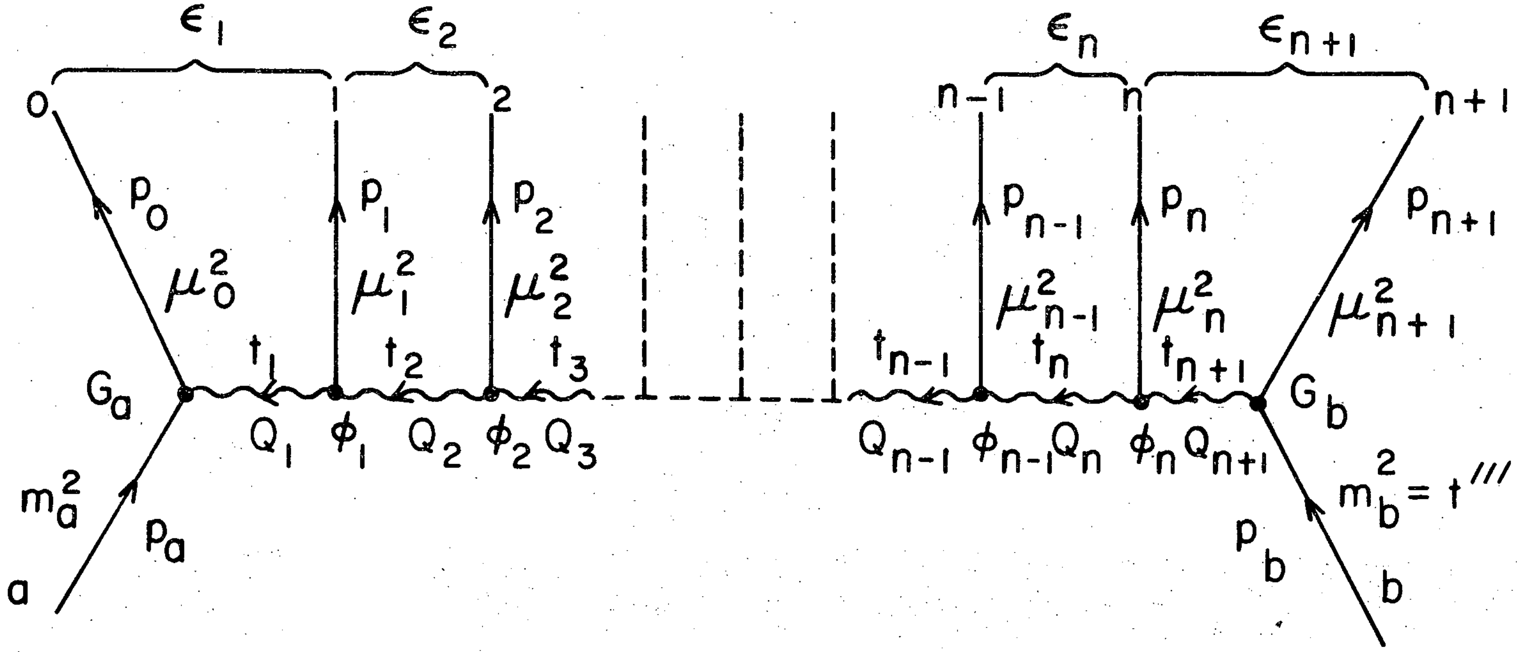

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Fig. 1.

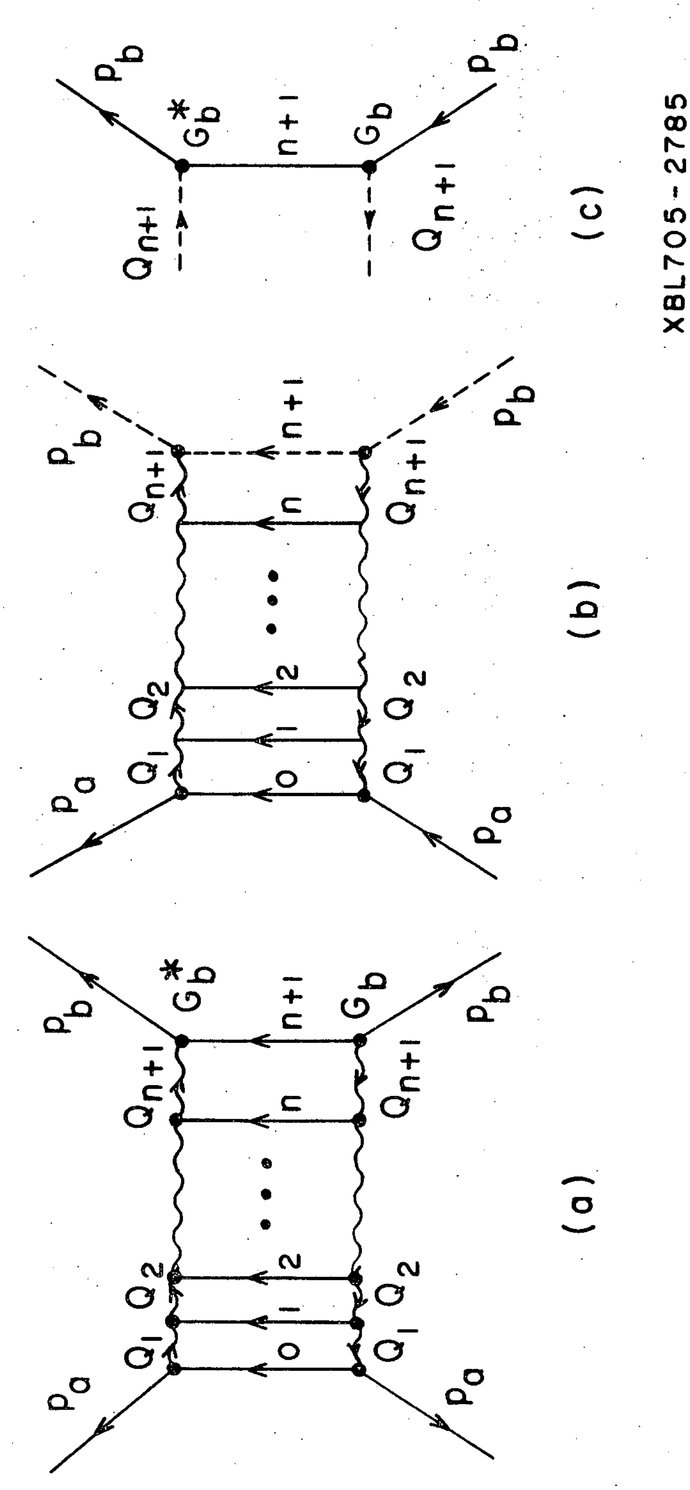
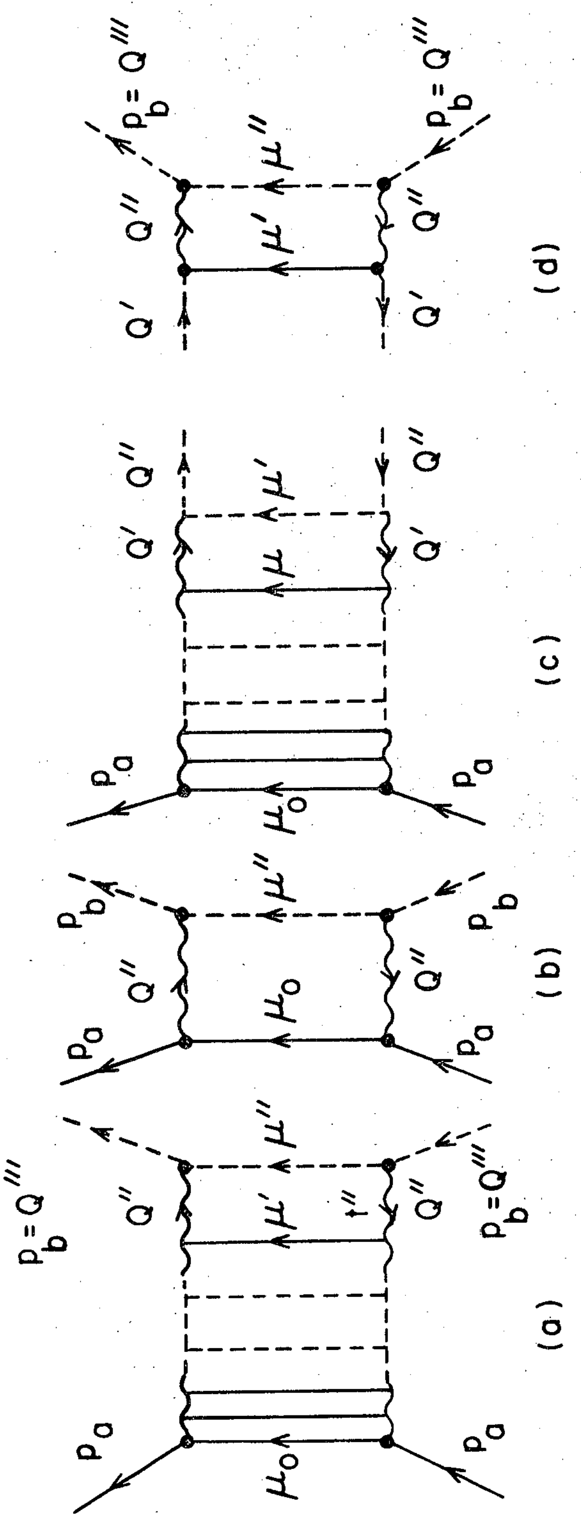
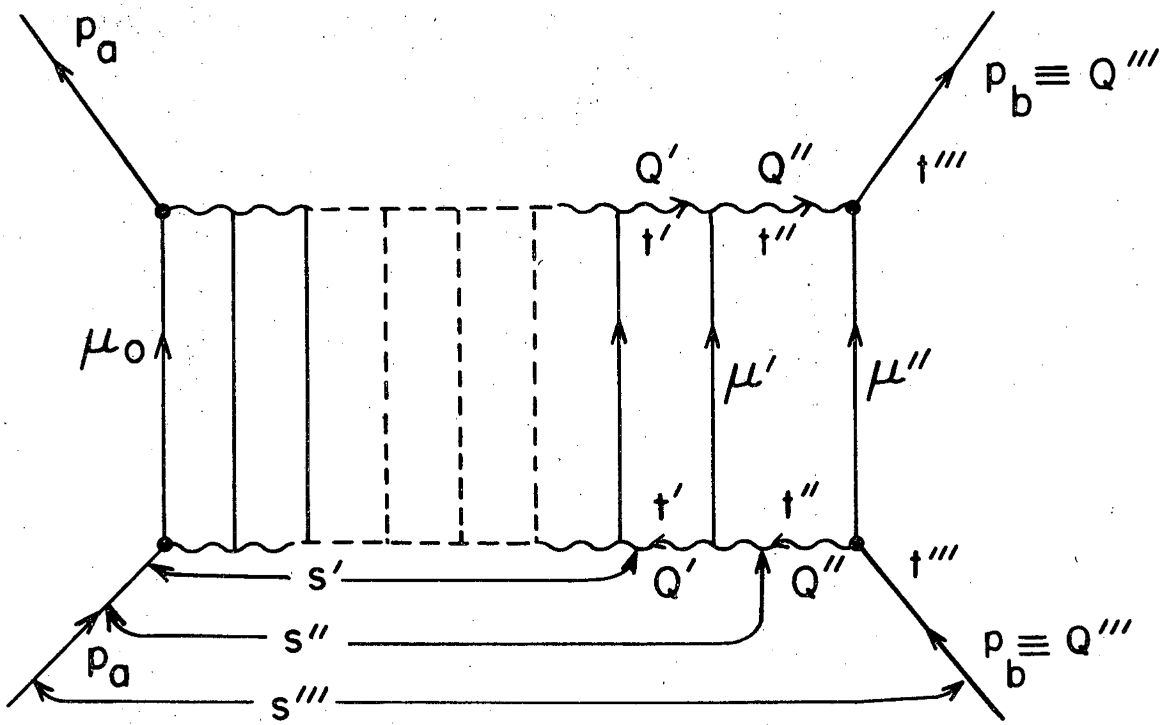


Fig. 2.



XBL705-2782

Fig. 3.



XBL705-2781

Fig. 4.

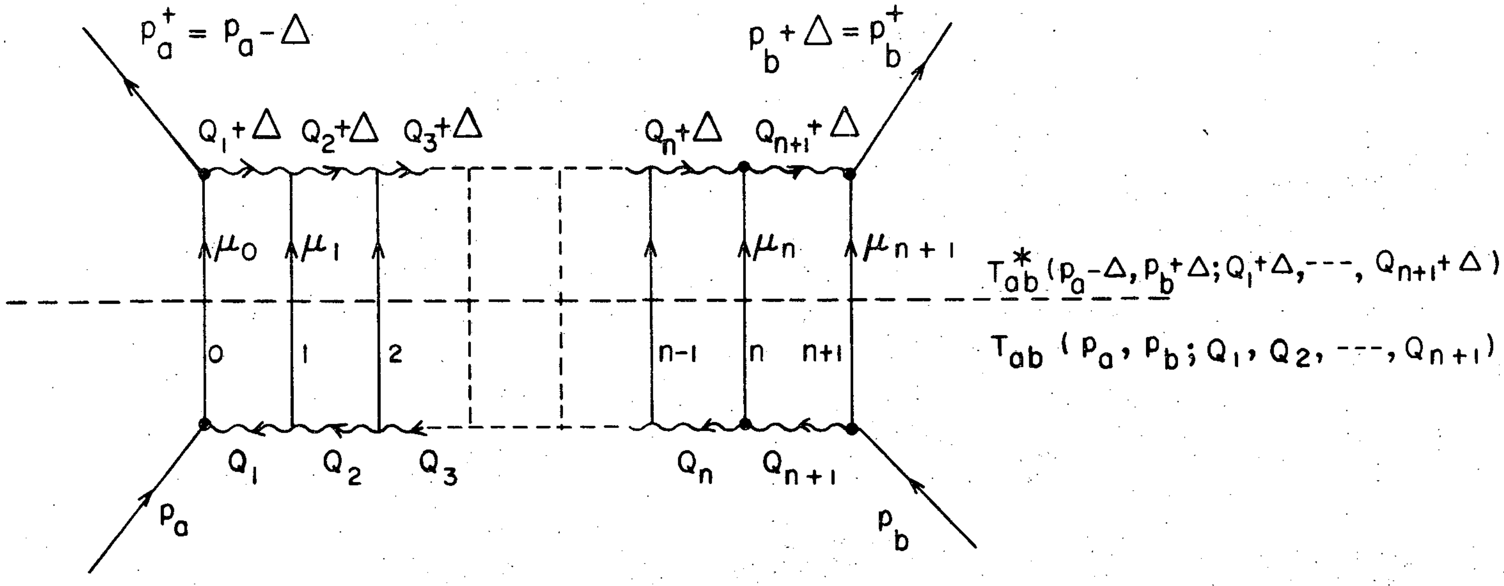


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Fig. 5.

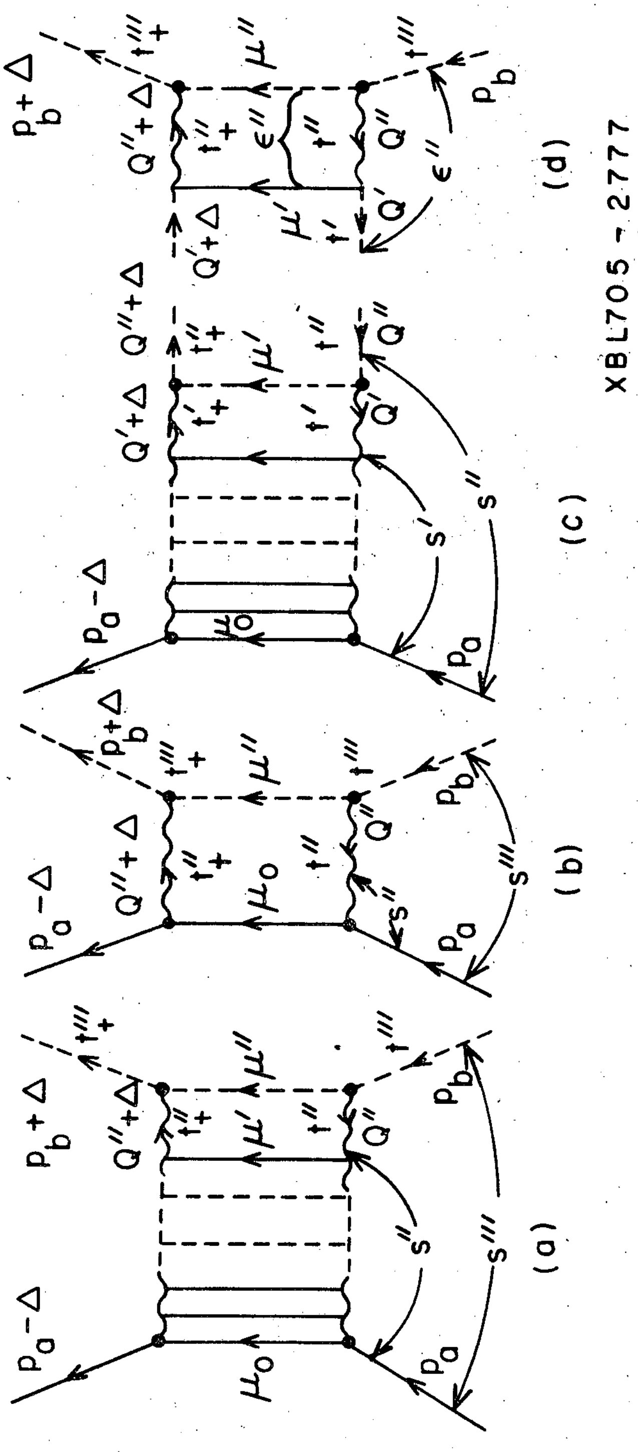


Fig. 6.

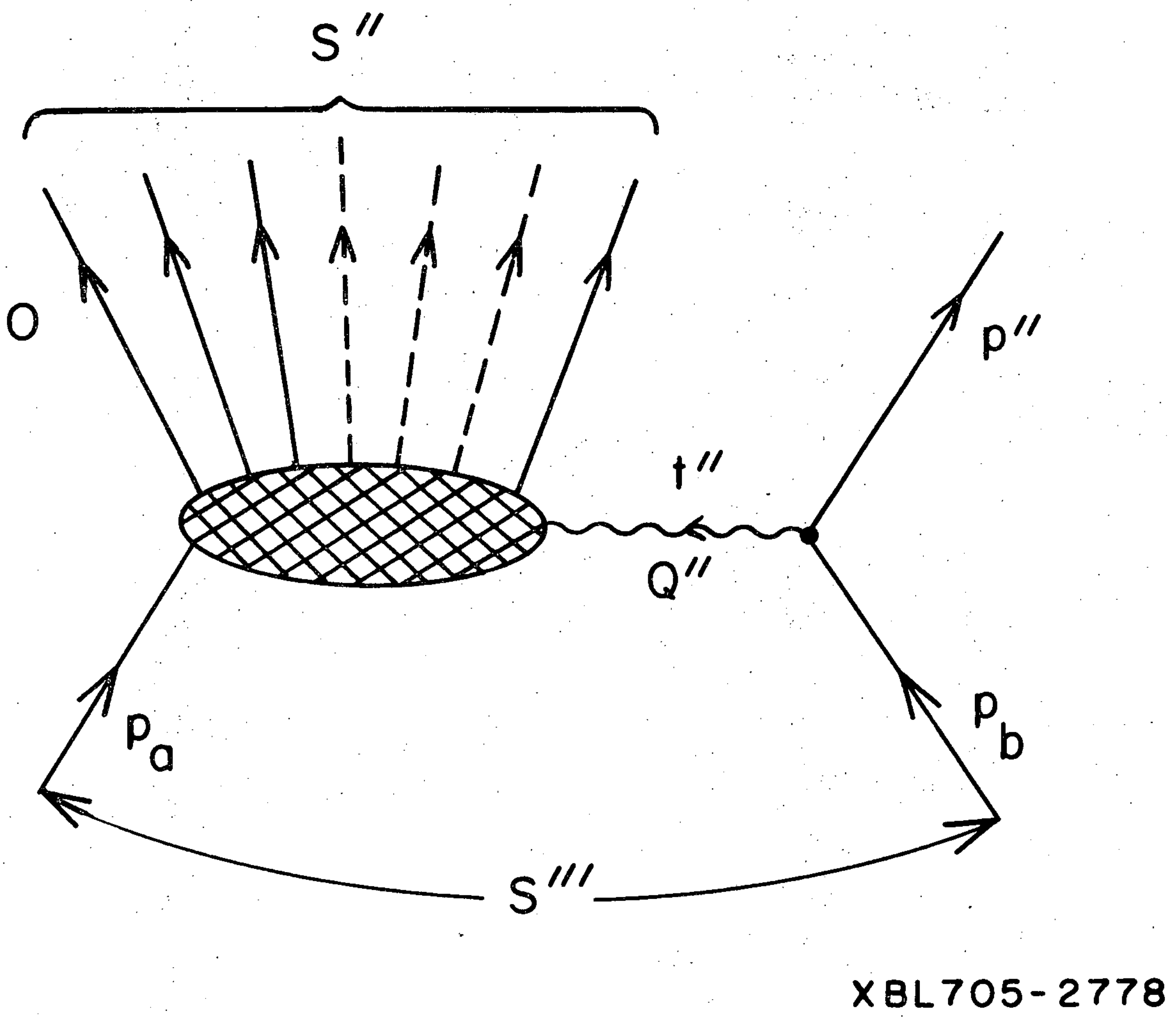
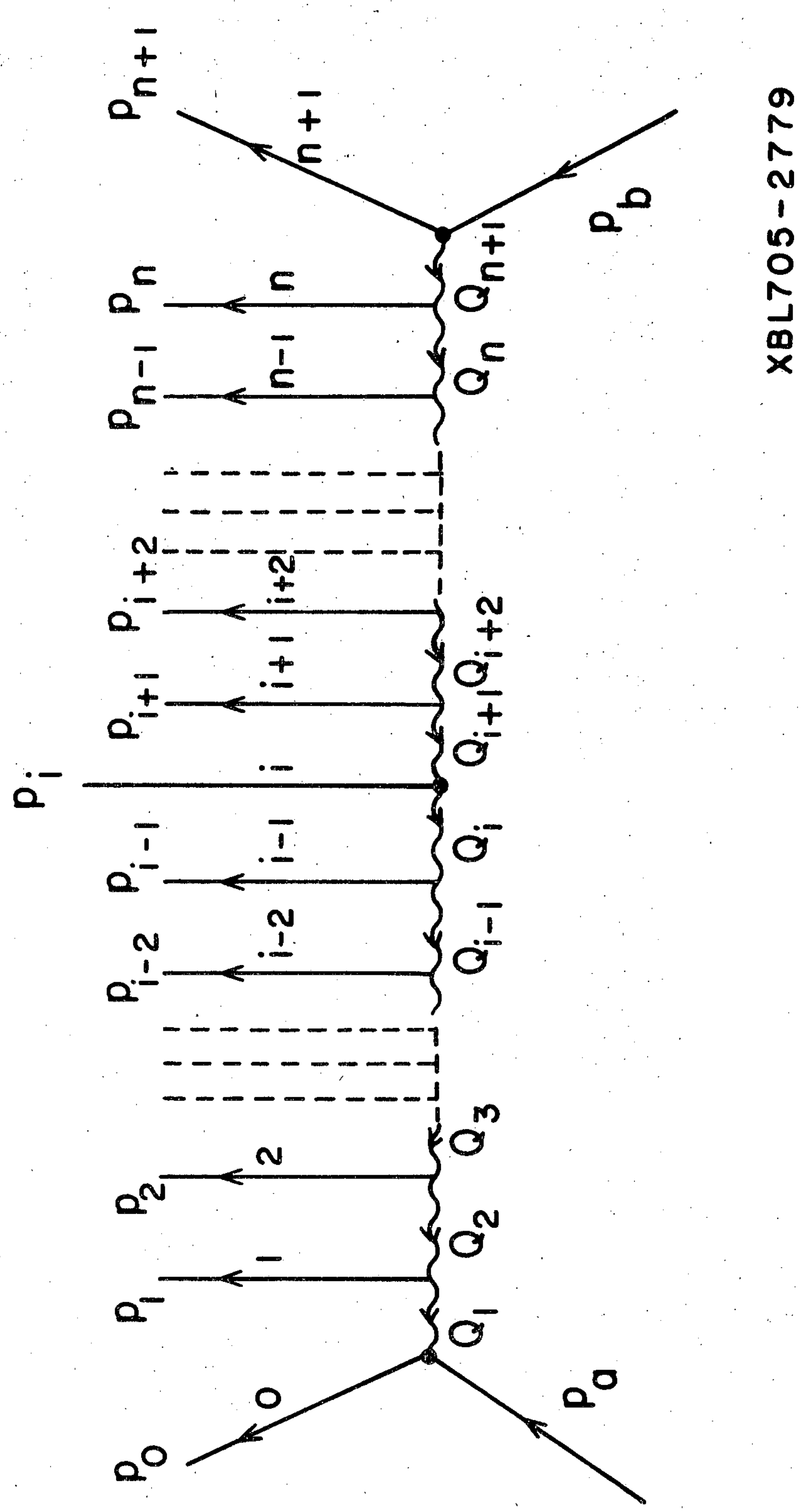
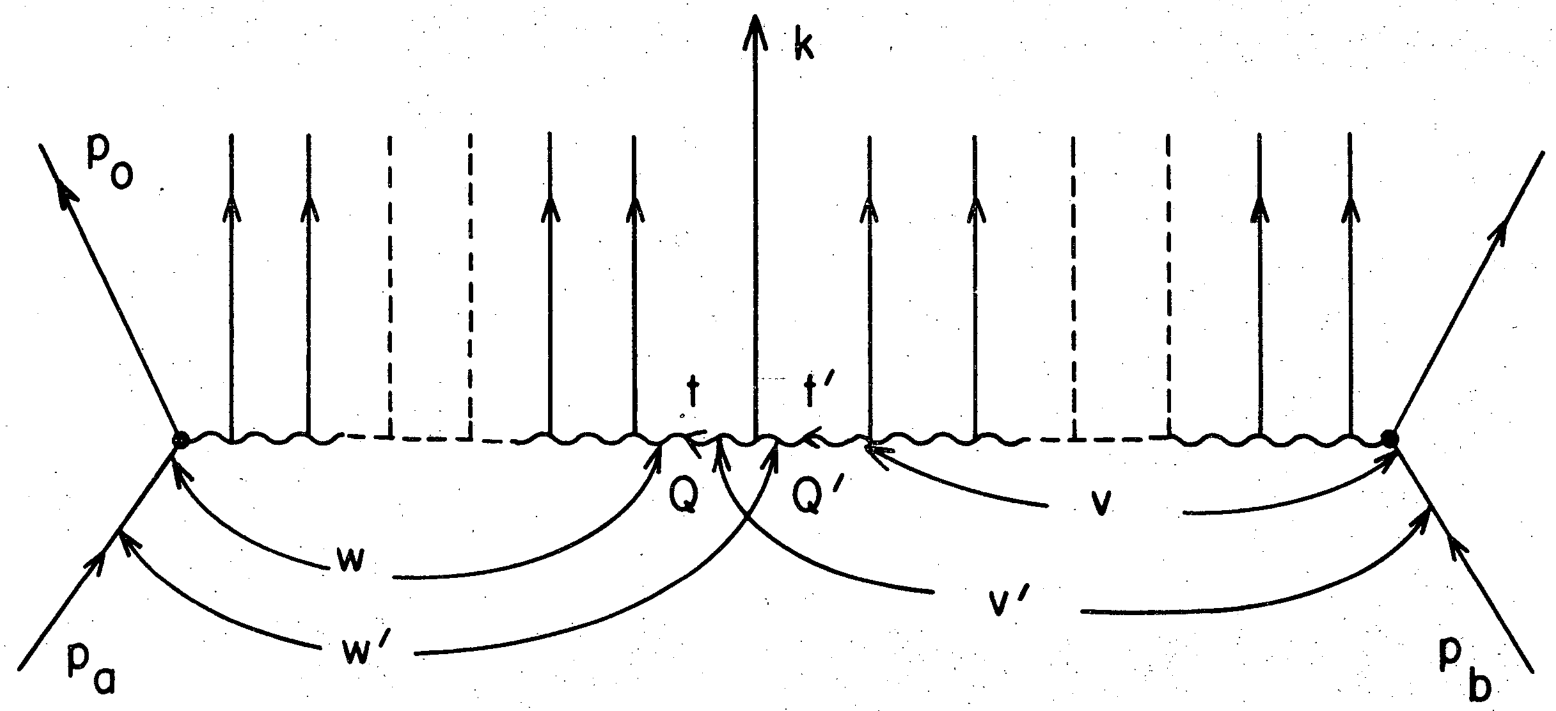


Fig. 7.



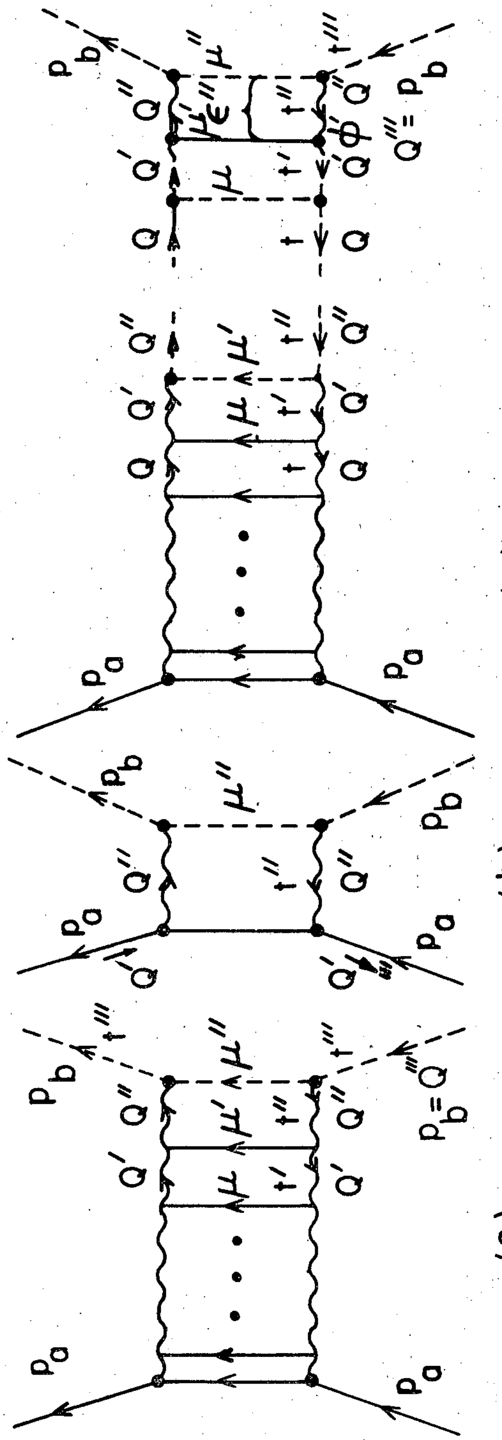
p_{n+1}
n+1
p_b
p_n
n
Q_{n+1}
p_{n-1}
n-1
Q_n
p_{i+2}
i+2
p_{i+1}
i+1
Q_{i+2}
p_i
i
Q_{i+1}
p_{i-1}
i-1
Q_i
p_{i-2}
i-2
Q_{i-1}
Q_3
p_2
2
Q_2
p_1
1
Q_1
p_0
0
p_a
XBL705-2779

Fig. 8.



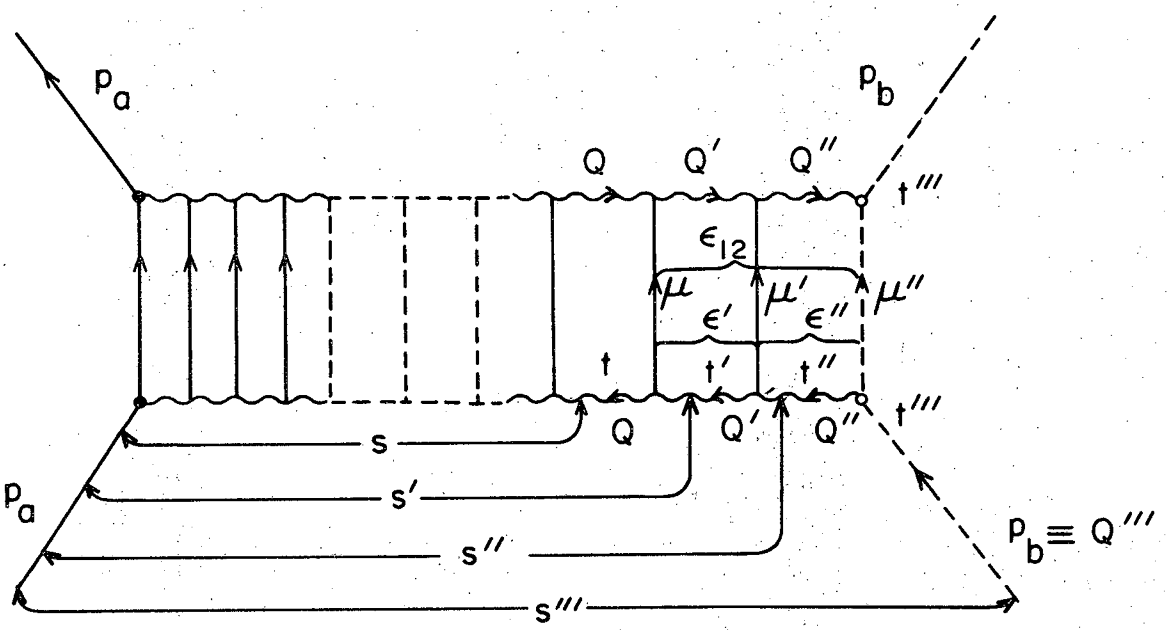
XBL705-2783

Fig. 9.



XBL705-2786

Fig. 10.



p_a
p_b
Q
Q′
Q″
t‴
ϵ_{12}
μ
μ′
μ″
ε′
ε″
t
t′
t″
t‴
Q
Q′
Q″
s
s′
p_a
s″
$p_b \equiv Q'''$
s‴
XBL705-2784

Fig. 11.

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