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Derivation Approaches for the Theis (1935) Equation

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Introduction

The Theis (1935) equation, which is a solution of the partial differential equation (pde) describing the drawdown caused by radial flow to a well pumped at a constant rate in a confined, homogeneous, isotropic, and infinite aquifer, is one of the best known results of groundwater hydraulics. C.V. Theis recognized the mathematical analogy between heat conduction in solids and groundwater flow and derived his equation from a solution of the pde for the temperature distribution caused by a line heat source. This commentary presents several approaches that have been followed to derive the Theis (1935) equation: (1) analogies to solutions of heat-conduction problems, (2) solutions based on initial guesses, (3) the Laplace transform, and (4) a hybrid method of separation of variables. The extension of the Theis equation to the case of a finite-diameter well is also included.

The Governing Equations

The two-dimensional pde for drawdown in a confined, homogeneous, isotropic, and laterally infinite aquifer caused by a well pumped at a constant rate is:

$$\frac{\partial^2 s(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial s(r,t)}{\partial r} = \frac{S}{T}\frac{\partial s(r,t)}{\partial t} \tag{1}$$

where $s(r,t)$ is the drawdown at distance r from the well and time t since the start of pumping, and T and S are the aquifer's transmissivity and storage coefficient, respectively.

The initial condition is:

$$s(r,0) = 0 \quad \text{for } r \geq 0 \tag{2}$$

There are two spatial boundary conditions that can be specified conveniently. The first results from the fact that the drawdown diminishes continuously with distance from the pumped well, leading to:

$$s(r \to \infty, t) = 0 \quad \text{for } 0 \leq t < \infty \tag{3}$$

The second boundary condition is a statement of the mass balance at the well established between the pumping rate (Q) and the removal of ground water by elastic decompression of the aquifer (the well has negligible radius r_w and therefore negligible storage in this classical formulation):

$$Q = -2\pi T r \frac{\partial s(r,t)}{\partial r}\Big|_{r \to r_w = 0} \text{ for } 0 \leq t < \infty \tag{4}$$

Equations 1 through 4 were first formulated in an analogous mathematical structure to that of the pde and associated initial and boundary conditions describing the temperature distribution in an infinite solid caused by a line heat sink (e.g., Carslaw and Jaeger 1959). For ground water, they appear to have been first published by Jacob (1940). Muskat (1937) wrote the pde (Equation 1) using fluid density as the dependent variable (instead of drawdown) in compressible-fluid flow in porous media.

Solutions Based on Analogies to Heat-Conduction Problems

The Case of a Well of Negligible Diameter

The similarity between the mathematical formulation of a particular heat-conduction problem and that describing radial ground water flow was recognized by Theis (1935), who exploited it to solve the latter problem. Specifically, Theis (1935) relied on a result by Carslaw (1921) relating the change of temperature ($d\,\tau$) induced in a solid of infinite dimensions (with thermal diffusivity k, units of length squared over time) after a time t and a distance r from a (line) heat source (λ, units of temperature times length squared) applied over a time interval dt:

$$d\tau = \frac{\lambda}{4\pi\, kt} e^{-\frac{r^2}{4kt}}\, dt \tag{5}$$

Equation 5 was reported first by Fourier (1822). Supporting Information Appendix A contains a derivation of Equation 5 for drawdown using the Fourier transform. The temperature distribution ($\tau(r,t)$) in the solid for a constant heat source (λ) follows from the convolution of Equation 5:

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$$\tau(r,t) = \frac{\lambda}{4\pi k}\int_0^t \frac{e^{-\frac{r^2}{4k(t-t')}}}{t-t'}dt' \tag{6}$$

A change of variable $v = r^2/4k(t-t')$transforms Equation 6 into the following equation for the temperature distribution [noting that (1) $v = \infty$ when $t' = t$, (2) $v = r^2/4kt$ when $t' = 0$, and (3) $dv/v = dt'/(t-t')$]:

$$\tau(r,t) = \frac{\lambda}{4\pi k}\int_{r^2/4kt}^{\infty} \frac{e^{-v}}{v}dv$$

$$= \frac{\lambda}{4\pi k}\left(-Ei\left(-u = -\frac{r^2}{4kt}\right)\right) \tag{7}$$

in which the exponential integral function *–Ei(-u)* is given by the following series (see equation 8.214.1 in Gradshteyn and Ryzhik [1994]) and $C = 0.577215 \ldots$ is Euler's constant:

$$-Ei(-u) = -C - \ln(u) - \sum_{m=1}^{\infty}\frac{(-1)^m u^m}{m(m!)} \tag{8}$$

From Equation 7 it can be seen that $\tau(r,0) = 0$ and $\tau(r\to\infty, t) = 0$; so, the analogous boundary conditions (Equations 2 and 3) are satisfied. Also, taking the derivative of Equation 7 gives $-2\pi rk(\partial\tau/dr)|_{r\to 0} = \lambda$, which is the boundary condition for temperature at the location of the heat source, a statement of the conservation of heat relying on Fourier's (1822) law of heat conduction. This can be seen to be analogous to the boundary condition (Equation 4) in which drawdown replaces temperature, the thermal diffusivity k plays the role of the hydraulic diffusivity *T/S,* and the heat source λ is the hydraulic equivalent *Q/S*. Using this analogy to transpose Equation 7 to represent the solution for drawdown in the case of radial groundwater flow to a well leads directly to the Theis (1935) equation:

$$s(r,t) = \frac{Q}{4\pi T}W\left(u = \frac{r^2 S}{4Tt}\right) \tag{9}$$

in which the well function *W(u)* equals the exponential integral (Equation 8); that is, *W(u) = –Ei(–u)*.

The Case of a Well of Finite Diameter

The solution to the transient, radial, flow problem (Equations 1 through 4) modified to account for a well with finite radius $r_w$was reported by Papadopulos and Cooper (1967). The mathematical structure of the transient, radial, flow problem to a well of finite diameter is analogous to that of a heat-conduction problem solved by Carslaw and Jaeger (1959). The latter authors solved the problem of heat conduction in an infinite plate caused by a constant heat source emanating from a cylinder of finite diameter. Papadopulos and Cooper (1967) relied on the Carslaw and Jaeger (1959) solution to arrive at their expression. Details of the solution to the finite-radius problem are provided in Supporting Information Appendix B (Equations B15 through B22).

Solutions Based on Initial Guesses

Jacob (1940) guessed that the drawdown gradient was (note: the time variable in Equations 10 through 12 is represented by t'):

$$\frac{\partial s(r,t')}{\partial r} = -2c\frac{e^{-\frac{r^2 S}{4Tt'}}}{r} \tag{10}$$

in which c was a constant to be determined from the well-boundary condition (Equation 4). Substitution of Equation 10 into Equation 1 leads to the following equation for the change in drawdown with respect to time:

$$\frac{\partial s(r,t')}{\partial t'} = \frac{Q}{4\pi Tt'}e^{-\frac{r^2 S}{4Tt'}} \tag{11}$$

Jacob (1940) integrated Equation 11 with respect to time to determine the drawdown at a specified time t:

$$s(r,t) = \frac{Q}{4\pi T}\int_0^t \frac{e^{-\frac{r^2 S}{4Tt'}}}{t'}dt' \tag{12}$$

The change of variable $v = r^2 S/(4Tt')$transforms Equation 12 into the following expression [noting that (1) $v = \infty$ when $t' = 0$, (2) $v = r^2 S/4Tt$ when $t' = t$, and (3) $dv/v = -dt'/t'$]:

$$s(r,t) = \frac{Q}{4\pi T}\int_{u=r^2S/4Tt}^{\infty}\frac{e^{-v}}{v}dv = \frac{Q}{4\pi T}W\left(u = \frac{r^2 S}{4Tt}\right) \tag{13}$$

which is the Theis solution (Equation 9).

Li (1972) reported a derivation of the Theis equation based on an initial guess of the form of the solution in a manner resembling the approach of Jacob (1940).

The Theis Equation Derived Using Laplace-Transform Theory

Hantush (1964) demonstrates the use of the Laplace transform to solve ground water problems, including the problem of radial flow to a well in a leaky aquifer, which is an extension of the Theis (1935) problem. Verruijt (1982) reported the solution to the flow problem (Equations 1 through 4) (written in terms of hydraulic head) by means of the Laplace transform.

The Laplace method of solution starts by taking the Laplace transform of the drawdown with respect to time (with $\theta > 0$):

$$F(r,\theta) = \int_0^{\infty} e^{-\theta t}s(r,t)dt \tag{14}$$

Taking the Laplace transform of Equations 1 through 4 produces an ordinary differential equation (ODE) with two boundary conditions (see Supporting Information Appendix B):

$$\frac{d^2F(r,\theta)}{dr^2}+\frac{1}{r}\frac{dF(r,\theta)}{dr}=\frac{S}{T}\theta F(r,\theta) \quad (15)$$

$$F(r\to\infty,\theta)=0 \quad (16)$$

$$r\frac{dF(r,\theta)}{dr}\bigg|_{r\to 0}=-\frac{Q}{2\pi T\theta} \quad (17)$$

The second step in the Laplace-transform method is to solve the problem constituted by Equations 15 through 17 to obtain $F(r,\theta)$ using standard theory for ODEs (see Supporting Information Appendix B), and letting $z(r,\theta)=i\sqrt{r^2S\theta/T}$, $i^2=-1$):

$$F(r,\theta)=\frac{Q}{2\pi T\theta}K_0\left[\left|z(r,\theta)\right|\right] \quad (18)$$

where $K_0[z(r,\theta)]$ is the modified Bessel function (see Gradshteyn and Ryzhik [1994], equation 8.447.3) and:

$$|z(r,\theta)|=\sqrt{\frac{r^2S\theta}{T}} \quad (19)$$

The final step in the Laplace-based method of solution is to apply the inverse Laplace transform to $F(r,\theta)$ and obtain the drawdown $s(r,t)$. The inverse Laplace transform of Equation 18 has been tabulated in equation 5.16.41 of Erdélyi (1954):

$$s(r,t)=\frac{Q}{2\pi T}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\theta t}\frac{K_0\left[\left|z(r,\theta)\right|\right]}{\theta}d\theta\right)$$

$$=\frac{Q}{4\pi T}\left\{-Ei\left[-u=-\frac{r^2S}{4Tt}\right]\right\} \quad (20)$$

This completes the derivation of the Theis equation, because $W(u)=-Ei(-u)$ (see Equation 9). Hantush (1964) also tabulates the inverse Laplace transform of Equation 18, but in a more general form (position 12 of table on p. 303 of Hantush [1964]).

Solution by a Hybrid Method of Separation of Variables

The method of separation of variables expresses the field variable (drawdown or hydraulic head) as a linear combination of the product of functions of the radial coordinate times functions of time. By separating variables in Equation 1, the proposed drawdown equation becomes (see Supporting Information Appendix C):

$$s(r,t)=\sum_{n=0}^{\infty}\left[A_n\,J_0(\alpha_n r)e^{-\alpha_n^2\phi t}+B_nY_0(\alpha_n r)e^{-\alpha_n^2\phi t}\right] \quad (21)$$

in which $\phi=T/S$; $A_n, B_n, \alpha_n, n=0, 1, 2, 3,\ldots$, are coefficients to be determined from the initial and boundary conditions (Equations 2 through 4), and J_0, Y_0 are Bessel equations of the first and second kind, respectively (see equations 8.441.1 and 8.444.1 in Gradshteyn and Ryzhik [1994]). The reduction of Equation 21 to the Theis equation (Equation 9) does not appear feasible, but Hermance (1999) introduced a hybrid version of the method of separation of variables in which the sum involving the Bessel function Y_0 was dropped from Equation 21, and the remaining sum involving the Bessel function J_0 was replaced by an integral. Specifically, Hermance (using hydraulic head instead of drawdown) proposed the following integral representation of the hydraulic head:

$$h(r,t)=\int_0^\infty A(\alpha)J_0(\alpha r)e^{-\alpha^2\phi t}d\alpha \quad (22)$$

Thereafter, Hermance (1999) applied Hankel transforms and convolution theory to reduce Equation 22 to the Theis equation (Equation 9). The main justification for Hermance's hybrid method appears to have been an intuition that it could lead to the Theis equation. In this respect, the inspiration of this approach and its mathematical principles lack the rigor of methodological approaches that have wide applicability, the Laplace method being one of them.

Conclusion

Several solution approaches that have been used in the past to derive the Theis (1935) equation have been reviewed in this commentary. The most rigorous of these appears to be the approach based on the Laplace-transform method.

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Supporting Information

Additional Supporting Information may be found in the online version of this commentary:

Appendix A. The Derivation of an Analog of Equation 5 for Drawdown

Appendix B. The Laplace-Transform Method

Appendix C. A Hybrid Method of Separation of Variables

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