A Localization Theorem for Derived Loop Spaces and Periodic Cyclic Homology

by

Harrison I-Yuan Chen

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor David Nadler, Chair
Professor Denis Auroux
Professor Richard Borcherds
Professor Ling Hon Lam

Summer 2018

A Localization Theorem for Derived Loop Spaces and Periodic Cyclic Homology

Abstract

A Localization Theorem for Derived Loop Spaces and Periodic Cyclic Homology

by

Harrison I-Yuan Chen

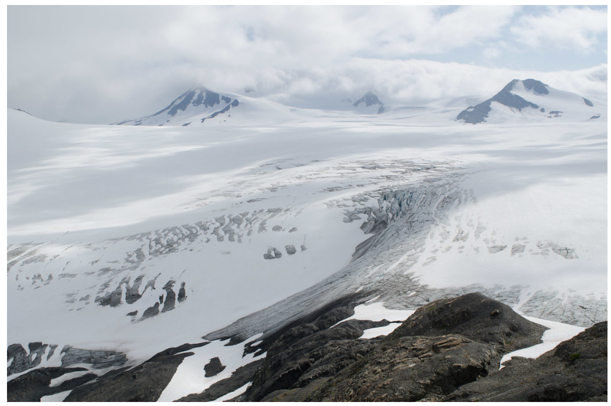
Doctor of Philosophy in Mathematics

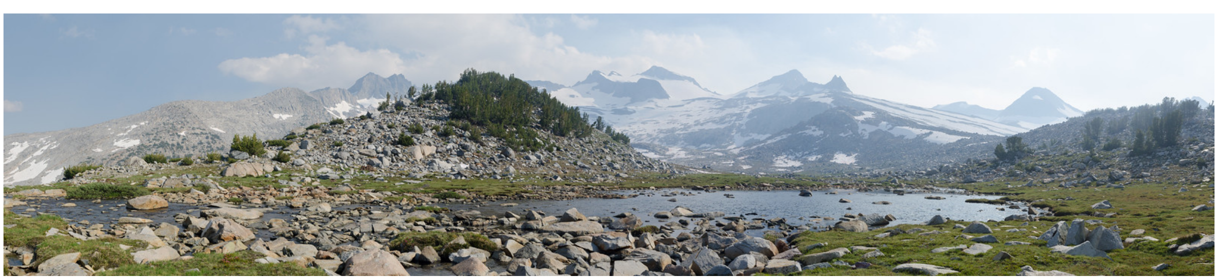
University of California, Berkeley

Professor David Nadler, Chair

Motivated by a theorem in the K-theoretic setting relating the localization of $K_0(X/T)$ over a closed point $z \in \mathrm{Spec}(K_0(BT))$ to the Borel-Moore homology of the fixed points $H^{BM}_\bullet(X^z; \mathbb{C})$, we prove an equivariant localization theorem for smooth quotient stacks by reductive groups G in the setting of derived loop spaces and periodic cyclic homology, realizing a Jordan decomposition of loops described by Ben-Zvi and Nadler. We show that the derived loop space $\mathcal{L}(X/G)$ is a family of twisted unipotent loop spaces over $\mathrm{Aff}(\mathcal{L}(BG)) = G//G$; more precisely, the fiber over a formal neighborhood of a semisimple orbit $[z] \in G//G$ is the unipotent loop space of the classical fixed points with a twisted S^1-action. We further study the relationship between unipotent loop spaces and formal loop spaces, and prove that their Tate S^1-invariant functions are isomorphic. Applying a theorem of Bhatt identifying derived de Rham cohomology with Betti cohomology, we obtain an equivariant localization theorem for periodic cyclic homology in the smooth case, identifying the completion of $HP(\mathrm{Perf}(X/G))$ at $z \in G//G$ with the 2-periodic equivariant singular cohomology of the z-fixed points $H^\bullet(X^z/G^z; k)((u))$.

I dedicate this work to the mountains. To the dominating shadow of Mount Shasta at dawn, to the morning fog breaking under the rime-covered summit of Lassen Peak, to the clouds of Taiwan pillowing into the valleys of Mount Jade, to Moose Lake and its granite shores glowing red under the light of the setting sun, to canyon dryfalls obstructing the path to Wahguyhe Peak, to sun cups on Mount Lyell melting into the soggy banks of an unnamed lake, to the lichened rock fields littering the gentle slopes of Mount Galen, to the swaying tufts of tall grass on the bare dome of Sutton Mountain.





Contents

Acknowledgments

I would like to express my immense gratitude to my advisor, David Nadler. The content of this work arose from questions he posed to me, and through our many conversations his ideas have undoubtedly seeped into every crevice of this work. Beyond mathematics, David was a thoughtful listener and offered much-needed encouragement in spite of my reticence. I feel fortunate to have been his student, and for the many facets of his support I am deeply thankful.

I would also like to thank David Ben-Zvi, Bhargav Bhatt, Daniel Halpern-Leistner, Anatoly Preygel, and Pavel Safranov for the mathematical insights which they have shared with me, as well as Shishir Agrawal, Gurbir Dhillon, Benjamin Gammage, Daniel Lowengrub, and Gus Schrader for the numerous mathematical discussions we have had throughout the years. I've learned much from these mathematicians and hope to continue to do so in the future.

I thank Jamal for feeding me during my time at Berkeley; I will miss his lunch plates wherever I go. I thank all my friends here who have kept me in good company throughout the years; amongst them Shishir Agrawal, Anna Lieb, Daniel Lowengrub, Eugenia Rosu, Gus Schrader, Minseon Shin, Ethan van Andel and Markus Vazquez for their always friendly faces within the department, and my roommates Catherine Chen, Gina Noh, Annie Ouyang and Markus Vazquez for making a place feel like home. I thank my aunt Danchu Yu and uncle Ron Chu for their hospitality and for trusting me with their house key. It occurs to me that this key is the sole item which I have always carried with me throughout my seven years in Berkeley.

I thank my parents, Harry Chen and Hanfei Yu, for always encouraging me to pursue my interests and for their patience towards an often difficult son. I remember how my mother had taught me arithmetic as a child via games of counting (and eating) grapes, a lesson surrounding the logic behind a restaurant's slogan as we drove to Burlington, and how my father sketched secants of curves limiting to tangents on the back of a narrow junk-mail coupon. I also thank my brothers Timothy and Winston, who have inspired me in ways they may not know.

Finally, I would like to thank Shu Tan, for her love which has never wavered.

Chapter 1

Introduction

Many phenomena in representation theory can be realized via categorical invariants applied to geometric objects: Borel, Weil and Bott realized representations of a complex reductive group via equivariant sheaves on the flag variety, Beilinson and Bernstein realized representations of a finite-type Lie algebra via D-modules on the flag variety, Springer realized representations of Weyl groups via Borel-Moore homology on the nilpotent cone, and Ginzburg, Kazhdan and Lusztig realized representations of the affine Hecke algebra, a q-deformation of the group algebra of the extended affine Weyl group which specializes at prime powers to Iwahori-Hecke algebras, via the 0-truncated algebraic K-theory on the Steinberg stack.

The last result in this list is a "decategorified" statement, obtained by applying an "underived" invariant to a category of interest. That is, the Grothendieck group of the derived category of coherent sheaves on the Steinberg stack $K_0(\mathrm{Coh}(\mathrm{St}/G_{\mathbb{G}_m}))$ has a $K_0(\mathrm{Coh}(BG_{\mathbb{G}_m}))$-algebra structure with multiplicative structure given by convolution and linear structure by pullback; in fact, it is a central algebra over $K_0(\mathrm{Coh}(BG_{\mathbb{G}_m}))$. This convolution algebra is identified with the affine Hecke algebra, which leads to a classification of irreducible representations of the affine Hecke algebra. A central part of its story is a localization theorem in K-theory, which relates the localizations and specializations of $K_0(\mathrm{Coh}(\mathrm{St}/G_{\mathbb{G}_m}))$ at fixed parameters of the center $\mathrm{Spec}(K_0(\mathrm{Coh}(BG_{\mathbb{G}_m}))) = G//G \times \mathbb{G}_m$ to the equivariant and ordinary cohomology of fixed points on the Steinberg stack.

This thesis grew out of an interest in developing a parallel theory of the affine Hecke algebra in the setting of cyclic homology, which leads directly to the consideration of derived loop spaces, whose global functions and circle action compute Hochschild and cyclic homology. In this thesis we prove a foundational result parallel to the localization theorem described above: a localization theorem in periodic cyclic homology. We further outline the broad program that we hope to achieve, and provide several toy examples.

Central to our argument is the following idea, which we learned from David Ben-Zvi and David Nadler. Traditionally, Hochschild homology has been computed algebraically via a cyclic bar complex on generators of a category. It is an invariant of small stable ∞-

categories, or equivalently, compactly generated presentable ∞-categories. However, the theory of dualizable ∞-categories under the Lurie tensor product allows us to generalize this notion to dualizable categories. Due to geometric descriptions of the dualizing structure of categories of sheaves on derived stacks X, the Hochschild homology can be naturally identified with global functions or distributions on the derived loop space [8]:

$$HH(\mathrm{Perf}(X)) = \mathcal{O}(\mathcal{L}X), \qquad HH(\mathrm{Coh}(X)) = \omega(\mathcal{L}X).$$

When X is a derived scheme, we can explicitly identify the loop space $\mathcal{L}X$ with the shifted odd tangent bundle via a derived variant of the Hochschild-Kostant-Rosenberg theorem since derived loops on X are Zariski local. Introducing the S^1-action, results in [10] [6] [51] allow us to identify the periodic cyclic homology with 2-periodic (derived) de Rham (co)chain complexes:

$$HP(\mathrm{Perf}(X)) \simeq C^\bullet_{\mathrm{dR}}(X^{an}; k) \otimes_k k((u)) \qquad HP(\mathrm{Coh}(X)) \simeq C_{\bullet,\mathrm{dR}}(X^{an}; k) \otimes_k k((u))$$

where $u \in C^\bullet(BS^1; k)$ is the degree 2 Chern class. Our ultimate goal is to describe the periodic cyclic homology in the case when X is taken to be a quotient stack.

A difficulty in understanding the Hochschild homology of derived stacks is its failure to be smooth local. On the other hand, as argued in [6], sheaves and functions on formal loop spaces (i.e. loop spaces completed at constant loops) are smooth local. Our aim in this note is to bridge this gap by understanding the global loop space $\mathcal{L}(X/G)$ as a family of twisted formal loop spaces over G/G, which itself can be understood as a family of unipotent loop spaces over its affinization $G//G$. This idea that first appeared in [7] under the banner of a Jordan decomposition for loop spaces and was employed to study categorical Langlands parameters for representations of real reductive groups.

A localization theorem for equivariant cohomology has been known for some time and appears in various forms, for example in [26]. Briefly, the philosophy is that given a topological space X with a topological torus T action, the equivariant cohomology $H^\bullet_T(X)$ is a module over $H^\bullet_T(\mathrm{pt}) = H^\bullet(BT) = k[[\mathfrak{t}]]$ where $\mathfrak{t}$ is placed in cohomological degree 2. The localization theorem states that over the generic point $H^\bullet(BT)_{loc}$ we have an isomorphism

$$H^\bullet_T(X)_{loc} \simeq H^\bullet(X^T) \otimes H^\bullet(BT)_{loc}.$$

A variant of this equivariant localization was proven for algebraic K-theory by Thomason in [57] [58] (also in [16]). In this setting, we let T be an abelian reductive group and X a variety with a T-action. In this case, $K_0(X/T)$ is a module over polynomial functions on the torus $K_0(BT) = k[T]$, and we have for any closed point $t \in T$ an isomorphism $K_0(X/T)_t \simeq K_0(X^t/T)_t$ after passing to the local ring $K_0(BT)_t = \mathcal{O}_{T,t}$. In particular if t is generic we have

$$K_0(X/T)_t \simeq K_0(X^t/T)_t \simeq K_0(X^t) \otimes K_0(BT)_t.$$

In good cases (e.g. cellular fibrations), specializing at the point a yields an isomorphism with Borel-Moore homology.

Our result applies the same philosophy in the setting of derived loop spaces and periodic cyclic homology, and we generalize the philosophy to the case of a nonabelian reductive group. The statements we prove have been previously developed in the setting of smooth quotient stacks with finitely many orbits (so that the loop space is an underived stack) in [7], and with special attention to the case $B\backslash G/B$ in Theorem 3.5. Similar statements also appear in [29] for the case of a cohomologically proper quotient stack X/G in Lemma 4.11 and Proposition 4.12.

In this note, we restrict to the case when X is smooth, although in the case of periodic cyclic homology we hope for a generalization to the singular case as well. The following theorem appears in the main text as Theorem 3.3.7 and Corollary 3.4.4.

Theorem 1.0.1 (Equivariant localization for derived loop spaces)**.** *Let G be reductive group acting on a smooth variety X, and $z \in G$ a semisimple element. For a certain twisting of the S^1 action, we have functorial S^1-equivariant isomorphisms*

$$\widehat{\mathcal{L}}(\pi_0(X^z)/G^z) \to \widehat{\mathcal{L}}^z(X/G) := \widehat{\{G \cdot z\}}/G \times_{\mathcal{L}(BG)} \mathcal{L}(X/G),$$

$$\mathcal{L}^u(\pi_0(X^z)/G^z) \to \mathcal{L}^{u,z}(X/G) := \widehat{\{G \cdot z\}}//G \times_{G//G} \mathcal{L}(X/G).$$

Our result easily implies the following interpretation of derived fixed points, which also appears as Corollary 1.12 in [3].

Corollary 1.0.2. *Let X be a smooth variety with an action of reductive G, and $z \in G$ semisimple. We have a natural identification of the derived z-fixed points:*

$$\mathcal{L}(\pi_0(X^z)) \to X^z := \mathcal{L}(X/G) \times_{BG} \{z\}.$$

Proof. Loop spaces commute with fiber products, and in general we have $Y = (Y/G) \times_{BG}$ pt. $\square$

We prove in Theorem 3.6.1 the following identification of Tate global functions on formal loop spaces and unipotent loop spaces for (possibly singular) quotient stacks.

Theorem 1.0.3. *For a X a variety acted on by an affine algebraic group G, the pullback on global derived functions*

$$\mathcal{O}(\mathcal{L}^u(X/G))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}$$

is an isomorphism. In particular, if U is a unipotent group acting on X, we have

$$HP(\mathrm{Perf}(X/U)) \simeq HP(\mathrm{Perf}(X)).$$

Using results in [10] [6] to identify global functions on formal loop spaces with singular cohomology, we prove Theorem 3.6.5, the promised equivariant localization result for periodic cyclic homology.

Theorem 1.0.4 (Equivariant localization for periodic cyclic homology)**.** *Let G be a reductive group acting on a smooth variety X. The periodic cyclic homology $HP(\mathrm{Perf}(X/G))$ is naturally a module over $HP(\mathrm{Perf}(BG)) = k[G//G]((u))$. For a closed point $z \in G//G$, we have a functorial (with respect to pullback) identification of the formal completion at z with a 2-periodicization of the singular cohomology of the fixed points*

$$HP(\mathrm{Perf}(X/G))_{\widehat{z}} \simeq H^\bullet((X^z)^{an}/(G^z)^{an}; k) \otimes_k^! k((u))$$

naturally as a module over $HP(\mathrm{Perf}(BG))_{\widehat{z}} \simeq H^\bullet(B(G^z)^{an}; k) \otimes_k^! k((u))$.

Coupling this with Corollary 1.0.2, we obtain an identification of the (derived) specialization of periodic cyclic homology at $z \in G$ with non-equivariant cohomology of the fixed points.

Corollary 1.0.5. *Let X be a smooth variety with an action of reductive G, and $z \in G$ semisimple, and k_z the skyscraper sheaf at $[z] \in G//G$. We have an isomorphism*

$$HP(\mathrm{Perf}(X/G)) \otimes_{k[G/G]((u))} k_z((u)) \simeq H^\bullet((X^z)^{an}; k)((u)).$$

We note that in the case that G is a torus, stronger statements remain true when we replace formal neighborhoods with local rings, recovering the equivariant localization in [16] for K-theory in the setting of periodic cyclic homology. However, for a nonabelian reductive group G, this fails even in the case when X is a point. For details, see Remark 3.3.9 and Example 3.3.4.

Finally, in the third chapter we provide a few examples and sketch some arguments to demonstrate a few applications we hope this project will be able to realize. In particular, assuming a still-conjectural devissage theorem for the periodic cyclic homology of quotient stacks, we prove that the periodic cyclic homology of the Steinberg stack is a 2-periodic version of the affine Hecke algebra.

Chapter 2

Background

The goal of this chapter is to collect various fundamental results and conventions in derived algebraic geometry, much of which are already well-documented in the literature. For more detailed and precise exposition, we refer the readers to [12] [18] [44] [45] [39] [50] [52] [60] [61]. The reader should be warned that we often sacrifice precision for ease of exposition; in particular, we will mostly black box ∞-categorical constructions and refer to [44] and [45] for precise statements. While perhaps unsatisfying, we feel this is appropriate because we never need to access these constructions directly.

Let us first establish notation. We will let k denote an algebraically closed field of characteristic zero, and in this note we work over k. All gradings follow cohomological conventions (i.e. differentials increase degree). Unless otherwise stated, all functors and categories are derived, e.g. for an affine scheme $X = \mathrm{Spec}(A)$, $\mathrm{QCoh}(X)$ denotes the category of unbounded complexes of A-modules localized with respect to quasi-isomorphisms. We use pt to denote the point scheme $\mathrm{Spec}(k)$.

By ∞-*category* we mean an $(\infty, 1)$-category, and we do not specify a particular model. We let $\mathbf{S}$ denote the ∞-category of ∞-*groupoids* or *spaces* and we will take for granted that the category of ∞-categories is enriched in $\mathbf{S}$. We let $\mathbf{Cat}$ denote the ∞-category of all infinity categories, $\mathbf{Pr}_k$ the full ∞-subcategory of presentable stable k-linear ∞-categories, $\mathbf{Pr}_k^L$ (respectively, $\mathbf{Pr}_k^R$) the ∞-category of presentable stable k-linear ∞-categories whose 1-morphisms are functors which are left (respectively, right) adjoints, and $\mathbf{Pr}_k^{L,\omega}$ the subcategory whose morphisms also preserve compact objects. The category $\mathbf{cat}$ will denote the ∞-category of small ∞-categories, $\mathbf{st}_k$ the full subcategory of k-linear stable ∞-categories with morphisms exact functors, and $\mathbf{st}^{id}$ the full subcategory of idempotent-complete categories. We let $\mathbf{Mod}_k$ or $\mathbf{Mod}_S$ denote the category of dg-modules over k or S-module spectra, where S is a ring spectrum. For $\mathbf{C} \in \mathbf{Pr}_k^L$, we let $\mathbf{C}^\omega \in \mathbf{st}_k$ denote its compact objects. For $\mathbf{C} \in \mathbf{st}_k$, we let $\mathrm{Ind}(\mathbf{C}) \in \mathbf{Pr}_k^L$ denote its ind-completion.

We let $\mathbf{DRng}$ denote the ∞-category of derived rings; during our exposition we do not insist on a particular model, but later when we perform calculations we will always take $\mathbf{DRng}$

to be the ∞-category of dg algebras over k. In this note we adopt the point of view (as in [23]) that (derived) stacks are prestacks which satisfy certain sheaf conditions, and that a *prestack* is an ∞-functor $\mathbf{Aff}^{op} := \mathbf{DRng} \to \mathbf{S}$.

We always use cohomological grading conventions, and HH will always denote the cochain complex of Hochschild chains rather than its cohomology groups. If we need to refer to the actual Hochschild homology we will denote it by $H^\bullet(HH)$. We refer to the nh cohomology group of a complex V by $H^n(V)$ or, viewing it as a spectrum, $\pi_{-n}(V)$.

2.1 Infinity categories

The main references for this section are [45] and [44].

Definition 2.1.1 ($(\infty, 1)$-categories)**.** Roughly speaking, an $(\infty, 1)$-*category* (which we will abbreviate as ∞-*category*) is a category enriched in spaces. The points of the Hom-spaces should be thought of as 1-morphisms; the paths between points 2-morphisms, and so-on. There are various models for ∞-categories, and they all possess model structures for which they are Quillen equivalent: categories enriched in spaces, Segal categories, A_∞-categories, and quasicategories. In this work we will not directly access infinity categories and do not require particular constructions; see [45] for a development of the theory of quasicategories.

An infinity category where all morphisms are invertible is known as an $(\infty, 0)$-*category* or ∞-*groupoid*, or more informally a *space*. Given a topological space X, we obtain an $(\infty, 0)$-category in the following way: the points $x \in X$ are the objects. For two points $x, y \in X$, the 1-morphisms are given by the set of paths $\gamma : [0, 1] \to X$ from x to y. For two 1-morphisms $\gamma, \delta : x \to y$, the 2-morphisms are the homotopies between these paths in X. The 3-morphisms are homotopies between homotopies, and so-on.

Definition 2.1.2 (Stable $(\infty, 1)$-categories)**.** An ∞-category is called *pointed* if it has a zero object (i.e. an object which is both initial and final, in an ∞-categorical sense, meaning that all mapping spaces in to and out of the zero object are contractible). If it exists, the zero object is unique up to equivalence; further note that the zero object is not an additional piece of the structure and that in all future references to it will allow for any choice of zero object. A *triangle* is a commuting square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

A triangle is a *fiber sequence* if it is a pullback square and a *cofiber sequence* if it is a pushout square. In this case we say X is a *fiber* or *cocone* of g and Z is a *cofiber* or *cone* of f. Note that cones and cocones are not unique (only up to homotopy), but they can be determined functorially. The *suspension functor* Σ is given by $\mathrm{cone}(X \to 0)$ and *loop functor* or *desuspension* Ω is given by $\mathrm{cocone}(0 \to X)$.

An ∞-category is *stable* if (a) it is pointed, (b) every morphism has a cone and cocone, and (c) every fiber sequence is a cofiber sequence and vice versa. A functor between two stable ∞-categories is called *exact* ([44] 1.1.4.1) if the following equivalent conditions hold: (a) it is left exact, i.e. it commutes with finite limits, (b) it is right exact, i.e. it commutes with finite colimits, (c) it takes zero to zero, and fiber sequences to fiber sequences. The homotopy category of a stable $(\infty, 1)$-category is naturally a triangulated category; in this way we can think of stable ∞-categories as a infinity-categorical analogue of abelian categories.

Definition 2.1.3 (Idempotent-completion and ind-completion)**.** Let κ be an infinite cardinal. We will generally take $\kappa = \aleph_0 = \omega$ to be the uncountable cardinal, but we state definitions in greater generality. An ∞-category $\mathbf{C}$ is called *small* if its objects form a set (as opposed to a proper class). A colimit is called κ*-filtered* if it is the colimit over a κ-filtered category. An ∞-category is κ*-filtered* ([45] 5.3.1.7) if for every κ-small simplicial set (i.e. a simplicial whose component sets have cardinality strictly smaller than κ) and every map $f : K \to \mathbf{C}$, there is an extension of f to the right cone of K (see Notation 1.2.8.4 of [45]).

There are two completion operations we will commonly use on small categories, both defined via the Yoneda embedding $\mathbf{C} \to \mathrm{PreSh}(\mathbf{C}) = \mathbf{Fun}(\mathbf{C}, \mathbf{S})$ (which is fully faithful by [45] Proposition 5.1.3.1). We define the *idempotent completion* $\mathrm{Idem}(\mathbf{C})$ by the smallest full subcategory of $\mathrm{PreSh}(\mathbf{C})$ containing the essential image of $\mathbf{C}$ and also retracts of objects in the essential image (see Proposition 5.1.4.2 in [45]). The idempotent completion is again a small stable ∞-category.

We define the κ*-ind-completion* $\mathrm{Ind}_\kappa(\mathbf{C})$ to be the full subcategory of $\mathrm{PreSh}(\mathbf{C})$ containing the essential image and κ-filtered colimits of objects in the essential image (via [45] Definition 5.3.5.1 and the characterization in Propositions 5.3.5.4). In a precise sense (see [45] 5.3.5.10), the ind-completion is the category obtained by freely generating under filtered colimits; that is, it has a universal property. Objects in $\mathrm{Ind}(\mathbf{C})$ can be represented by filtered colimits in $\mathbf{C}$, and we have the usual identification[1] $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(\mathrm{colim}_i X_i, \mathrm{colim}_j Y_j) = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathbf{C}}(X_i, Y_j)$. If we do not specify, we will assume that $\kappa = \omega$, the countable infinite ordinal, e.g. $\mathrm{Ind}_\omega = \mathrm{Ind}$. Any ind-complete category is also idempotent complete[2]. By [45] Lemma 5.4.2.4, $\mathrm{Idem}(\mathbf{C})$ are the compact objects of $\mathrm{Ind}(\mathbf{C})$.

By [44] Proposition 1.1.4.1, stable ∞-categories admit finite limits and coimits; in particular, if $\mathbf{C}$ is stable, then $\mathrm{Ind}(\mathbf{C})$ contains *all* small colimits. For details, see [45] Section 5.3. If $\mathbf{C}$ is a stable ∞-category, the idempotent completion and ind-completion have some nice

[1] By [45] Proposition 5.3.5.5, objects in the essential image of $\mathbf{C}$ under the Yoneda embedding are compact in $\mathrm{Ind}(\mathbf{C})$. Thus the left hand side is $\mathrm{Hom}_{\mathrm{PreSh}(\mathbf{C})}(\mathrm{colim}_i \mathrm{Hom}(X_i, -), \mathrm{colim}_j (Y_j, -)) = \mathrm{colim}_j \mathrm{Hom}_{\mathrm{PreSh}(\mathbf{C})}(\lim_i \mathrm{Hom}(X_i, -), \mathrm{Hom}(Y_j, -))$, and the final identification since Hom commutes colimits in the first variable with limits.

[2] This is not as obvious as it sounds; see [45] Corollary 4.4.5.16.

behavior. By Lemma 1.2.4.6 in [44], if $\mathbf{C}$ is a stable ∞-category, it is idempotent complete if and only if its homotopy category is (as a 1-category)[3].

Remark 2.1.4. The idempotent completion is meant to fix the following defect: if we have a idempotent $e : M \to M$, we would like to be able to think about e as a projector onto a subobject $X \subset M$. In order to do this we need to be able to realize the object X (and the quotient $M/X \simeq Y$ as an object of the category, using only the basic operations provided. When $\mathbf{C}$ is an abelian category, this is possible by taking kernels and cokernels. In the derived setting, we do not have the kernels and cokernel operations, only cones (and cocones). For example, if $M = X \oplus Y$, where X and Y are distinct indecomposable objects in a semisimple category, any attempt at doing so with a finite number of such operations will fail by additivity of triangles (for example, $\mathrm{cone}(e) \simeq Y \oplus Y[-1]$). However, we can achieve this goal with an infinite colimit by "pushing one of the terms off to infinity," i.e. the cone of $Y \oplus Y[-1] \to Y \oplus Y[-1]$ is $Y \oplus Y[-2]$, and iterating this n times we can obtain an object $Y \oplus Y[-n]$, and taking $n \to \infty$ gives us Y in the colimit. For a less wishy-washy treatment, see Section 4.4.5 in [45].

Definition 2.1.5 (Compactness, continuity)**.** We say a functor is κ-*continuous*[4] if it preserves κ-small filtered colimits (i.e. colimits whose indexing diagram is a proper set). By the ∞-categorical right adjoint theorem ([45] Corollary 5.5.2.9), a functor is continuous if and only if it admits a right adjoint.

An object $X \in \mathbf{C}$ of an ∞-category which admits κ-filtered colimits is κ-*compact* if $\mathrm{Map}_{\mathbf{C}}(X, -)$ preserves κ-filtered colimits. A functor *preserves κ-compact objects* if it takes κ-compact objects to κ-compact objects.

Definition 2.1.6 (Accessibility, presentability)**.** An ∞-category $\mathbf{C}$ is κ-*accessible* if it is the κ-ind-completion of a small ∞-category $\mathbf{C}^{\kappa}$, which necessarily becomes its subcategory of κ-compact objects. We say a category is *accessible* if it is κ-accessible for some regular cardinal κ. There are also intrinsic characterizations described in Section 5.4.2 of [45] which roughly say that $\mathbf{C}$ is accessible if and only if it is generated under κ-small filtered colimits by its full subcategory of κ-compact objects, and that the full subcategory of compact objects is essentially small[5]. By Proposition 5.4.3.4 of [45], a small ∞-category is accessible if and only if it is idempotent complete.

An ∞-category is *presentable* if it is accessible and has all small colimits. We denote by $\mathbf{Pr}^L$ is the ∞-category of presentable ∞-categories (where we only allow invertible 2-morphisms) and also require that the functors be continuous (equivalently, by adjoint functor

[3] See Warning 1.2.4.8 in [44] for a counterexample when $\mathbf{C}$ is not stable.

[4] Note that there are two opposite conventions: one convention where continuous means colimit-preserving functors and one where it means limit-preserving functors. We will prefer the former as most functors we are interested in will be colimit-preserving.

[5] This means that the number of equivalence classes in $\mathbf{C}$ is small and the homotopy groups of Hom-sets is are small.

theorem, they admit right adjoints). We define $\mathbf{Pr}^L_k$ to be the full subcategory of k-linear stable categories.

An ∞-category is *κ-compactly generated* if it is κ-accessible and presentable (i.e. if it is κ-accessible and has small colimits). We say a category is *compactly generated* if it is so for $\kappa = \omega$. One important fact is that the κ-small colimit of κ-compact objects is also κ-compact.

Remark 2.1.7. As the cardinal κ gets bigger, it is easier to be κ-small, the condition for a colimit to be κ-filtered becomes stricter, there are fewer κ-filtered colimits, it is easier to be κ-continuous, Ind_κ becomes smaller, and it is easier to be κ-compact. In particular, note that not every presentable category is compactly generated, since $\mathrm{Ind}_\kappa(\mathbf{C}^0) \not\simeq \mathrm{Ind}_\omega(\mathbf{C}^0)$.

Remark 2.1.8 (Functors with continuous right adjoints)**.** Suppose (F, G) are a pair of adjoint functors between presentable categories; in particular, F preserves colimits. By Proposition 5.5.7.2 of [45], if G is continuous (i.e. commutes with filtered colimits), then F preserves compact objects. If the source category of F is compactly generated, then the converse is also true.

Definition 2.1.9 (k-linear (stable) categories)**.** In Section 6 of [41], the notion of a presentable k-linear (stable) ∞-category is introduced for k a classical ring, and defined to be the category of presentable category which is a module over the ∞-category of k-modules. By Remark 6.5 of *loc. cit.* , every k-linear category is stable. We write by $\mathbf{Pr}^L_k$ the category of presentable k-linear stable ∞-categories.

Remark 2.1.10 (Equivalence between stable $(\infty, 1)$-categories and dg categories)**.** Since we work over a field of characteristic zero, it will be useful to work with dg categories rather than stable ∞-categories. To any dg category, there is an operation called the *dg-nerve* (see Construction 1.3.1.6 in [44]) which produces a stable ∞-category. It is shown in Corollary 5.5 of [18] that the ∞-category of small idempotent-complete k-linear stable ∞-categories and the ∞-category of dg categories over k localized with respect to Morita equivalences (see also [12] Definition 2.14 and 4.21). In particular, the latter category has a model structure where fibrant replacements are given by idempotent-complete pretriangulated hulls.

The *pretriangulated hull* of a dg category $\mathbf{C}$ is defined to be the smallest full dg-subcategory of $\mathbf{C}^{op} - \mathbf{mod}$ which contains the essential image of the Yoneda map $\mathbf{C} \to \mathbf{C}^{op} - \mathbf{mod}$ and is closed under shifts and cones. A dg category does not, a priori, have a cone operation, whereas a stable $(\infty, 1)$-category does. That is, a priori a dg category need not have a (de)suspension operation, nor (co)cones, that we expect in stable ∞-categories. The pretriangulated hull is the universal way to remedy this, i.e. by replacing a dg category $\mathbf{C}$ with the closure of its essential image under the Yoneda embedding under shifts and cones, gives a pretriangulated dg category. Further, under this correspondence, the category of perfect $\mathbf{C}$-modules can be thought of as the idempotent completion, and the category of $\mathbf{C}$-modules can be thought of as the ind-completion.

Remark 2.1.11. By [44] Theorem 1.1.4.4 and Proposition 1.1.4.6, $\mathbf{cat}_{st}^{ex}$ admits small limits and small filtered colimits. By [45] Proposition 5.5.3.13 and Theorem 5.5.3.18 (with Corollary 5.5.3.4), $\mathbf{Pr}^L$ admits small limits and small colimits. By [45] Proposition 5.5.3.8, if $\mathbf{C}, \mathbf{D} \in \mathbf{Pr}^L$, then $\mathbf{Fun}^L(\mathbf{C}, \mathbf{D}) \in \mathbf{Pr}^L$.

Definition 2.1.12. The category $\mathbf{Pr}^L$ is equipped with a monoidal structure called the *Lurie tensor product*, constructed in Section 4.8 of [44]. It can be thought of as an ∞-analogue of the Deligne tensor product. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Pr}^L$; we denote the Lurie tensor product by $\mathbf{C} \otimes \mathbf{D}$, and it is equipped with a canonical functor

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$$

$$(X, Y) \mapsto X \boxtimes Y.$$

It satisfies the universal property that it is initial amongst functors out of $\mathbf{C} \times \mathbf{D}$ which preserves small colimits in each variable. Proposition 4.8.1.17 of *loc. cit.* gives an explicit realization

$$\mathbf{C} \otimes \mathbf{D} \simeq \mathbf{Fun}^R(\mathbf{C}^{op}, \mathbf{D})$$

which is again a presentable category by Lemma 4.8.1.16. In particular, by [45] Proposition 5.5.3.8,the Lurie tensor product makes $\mathbf{Pr}^L$ into a closed monoidal category with internal mapping object $\mathbf{Fun}^L(-,-)$. Moreover, the Lurie tensor product induces a tensor product on S-module categories for any ring spectrum S (see Propositions 4.8.2.10 and 4.8.2.18 in [44]).

For more exposition on dualizable objects in a monoidal category, see Section 2.3 of [46].

Definition 2.1.13. Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal ∞-category with monoidal unit $1 \in \mathbf{C}$. An object $X \in \mathbf{C}$ is *dualizable* if there is an object $X^\vee$, a coevaluation map $\eta : 1 \to X \otimes X^\vee$ and an evaluation map $\epsilon : X^\vee \otimes X \to 1$ such that the diagrams

$$A^\vee \otimes 1 \xrightarrow{\mathrm{id}\otimes\eta} A^\vee \otimes (A \otimes A^\vee) \xrightarrow{\alpha^{-1}} (A^\vee \otimes A) \otimes A^\vee \xrightarrow{\epsilon\otimes\mathrm{id}} 1 \otimes A^\vee, \qquad A^\vee \otimes 1 \xrightarrow{\ell^{-1}\circ r} 1 \otimes A^\vee$$

$$1 \otimes A \xrightarrow{\eta\otimes\mathrm{id}} (A \otimes A^\vee) \otimes A \xrightarrow{\alpha} A \otimes (A^\vee \otimes A) \xrightarrow{\epsilon\otimes\mathrm{id}} A \otimes 1, \qquad 1 \otimes A \xrightarrow{r^{-1}\circ\ell} A \otimes 1$$

commute, where α is the associator and ℓ, r the left and right unitors for the symmetric monoidal structure. If X is dualizable, then the dual is unique up to unique isomorphism (see Remark 2.3.3. of [46]).

The following is proven in Theorem D.7.0.7 in [47] and Chapter I.1 Proposition 7.3.2 [23].

Proposition 2.1.14. *If $\mathbf{C} \in \mathbf{Pr}_k^L$ is compactly generated, then it is dualizable. In particular, if $\mathbf{C} = \mathrm{Ind}(\mathbf{C}^0)$, then $\mathbf{C}^\vee = \mathrm{Ind}(\mathbf{C}^{0,op})$, and the evaluation map is given by ind-completion via universal properties of the Yoneda pairing $\mathrm{Hom}(-,-) : \mathbf{C}^{0,op} \times \mathbf{C} \to \mathbf{Vect}_k$. Furthermore, we have isomorphisms*

$$\mathbf{Fun}_k^L(\mathbf{C}, \mathbf{C}) \simeq \mathbf{Fun}_k^L(\mathbf{C}, \mathbf{Vect}_k) \otimes \mathbf{C} \simeq \mathbf{C}^\vee \otimes \mathbf{C}.$$

2.2 t-structures

In this section we review some technical but easy material on t-structures for the reader's convenience and reference. For more details, the reader should consult [44], [51] and the appendix of [9].

Definition 2.2.1. Let $\mathbf{D}$ be an ∞-category, and $\mathbf{C} \subset \mathbf{D}$ a subcategory. We say that $\mathbf{C}$ is a *localization* of $\mathbf{D}$ if the inclusion functor has a left adjoint. The corresponding *localization functor* $L : \mathbf{D} \to \mathbf{D}$ is the composition of the left adjoint with the inclusion, and there is a natural map of functors $\mathrm{id}_{\mathrm{D}} \to L$. The subcategory $\mathbf{C}$ can be recovered as the essential image of L. Equivalently, $L : \mathbf{D} \to \mathbf{D}$ is a localization functor if there is a natural transformation $\mathrm{id}_{\mathrm{D}} \to L$ inducing an equivalence $L \to L^2$. Dually, we say $\mathbf{C} \subset \mathbf{D}$ is a *colocalization* of $\mathbf{D}$ if the inclusion functor has a right adjoint, and denote the colocalization C. See Section 5.2.7 in [45] for details.

Definition 2.2.2. A *t-structure* on a triangulated category $\mathbf{C}$ is a pair of full subcategories $\mathbf{C}^{\geq 0}$ and $\mathbf{C}^{\leq -1}$ such that

(a) The categories $(\mathbf{C}^{\leq -1}, \mathbf{C}^{\geq 0})$ are orthogonal; i.e. if $X \in \mathbf{C}^{\leq -1}$ and $Y \in \mathbf{C}^{\geq 0}$, then $\mathrm{Hom}_{\mathrm{C}}(X, Y) = 0$.

(b) The functor $[1]$ preserves $\mathbf{C}^{\leq -1}$ and the functor $[-1]$ preserves $\mathbf{C}^{\geq 0}$.

(c) For any $Z \in \mathbf{C}$, there is a $X \in \mathbf{C}^{\leq -1}$ and $Y \in \mathbf{C}^{\geq 0}$ such that $X \to Z \to Y$ is exact.

A t-structure on a stable ∞-category is a pair of such subcategories that induces a t-structure on the homotopy category.

Note that the axioms for a t-structure do not provide for canonical truncation functors. However, these canonical functors do exist, and are constructed in the following proposition.

Proposition 2.2.3. *The categories $(\mathbf{C}^{\leq -1}, \mathbf{C}^{\geq 0})$ are mutually orthogonal categories, i.e.*

$$\mathbf{C}^{\leq -1, \perp} = \mathbf{C}^{\geq 0}, \qquad {}^{\perp}\mathbf{C}^{\geq 0} = \mathbf{C}^{\leq -1}.$$

Furthermore, the inclusion functors have adjoints, i.e. we have the adjoint pairs $(\tau^{\geq 0}, \iota^{\geq 0})$ *and* $(\iota^{\leq -1}, \tau^{\leq -1})$*. In particular, we have a localization sequence*

$$\mathbf{C}^{\leq -1} \longrightarrow \mathbf{C} \xrightarrow{L} \mathbf{C}^{\geq 0}$$

realizing $\mathbf{C}^{\geq 0}$ *as a localization of* $\mathbf{C}$ *with localization functor* $\iota^{\geq 0} \circ \tau^{\geq 0}$*. Equivalently, there is a colocalization sequence*

$$\mathbf{C}^{\geq 0} \longrightarrow \mathbf{C} \xrightarrow{C} \mathbf{C}^{\leq -1}$$

realizing $\mathbf{C}^{\leq -1}$ *as a colocalization of* $\mathbf{C}$ *with colocalization functor* $\iota^{\leq -1} \circ \tau^{\leq -1}$*.*

Proof. The first sentence follows from property (3) in the definition of t-structures. The rest is Proposition 1.2.1.5 and Corollary 1.2.1.6 in [44]. □

Definition 2.2.4. Let $\mathbf{C}$ be a category with a t-structure. We say objects in $\mathbf{C}^{\leq 0}$ are *connective*, and objects in $\mathbf{C}^{\geq 0}$ are *coconnective*. We say objects which are in $\mathbf{C}^{\leq n}$ for some n are *eventually connective* and objects in $\mathbf{C}^{\geq -m}$ for some m are *eventually coconnective*.[6]

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between categories with t-structures. We say F is *left t-exact* if it takes $\mathbf{C}^{\geq 0}$ to $\mathbf{D}^{\geq 0}$ and *right t-exact* if it takes $\mathbf{C}^{\leq 0}$ to $\mathbf{D}^{\leq 0}$. We say F is *t-exact* if it is both right and left t-exact.

The following definitions are mild conditions on t-structures that will hold for most the categories we consider.

Definition 2.2.5. A t-structure on a presentable stable ∞-category $\mathbf{C}$ is *accessible* if one of the following equivalent conditions hold: (a) $\mathbf{C}^{\leq -1}$ is presentable, (b) $\mathbf{C}^{\leq -1}$ is accessible, (c) $\mathbf{C}^{\geq 0}$ is presentable, (d) $\mathbf{C}^{\geq 0}$ is accessible, (e) the truncation $\iota^{\leq -1} \circ \tau^{\leq -1} : \mathbf{C} \to \mathbf{C}$ is accessible, and (f) the truncation $\iota^{\geq 0} \circ \tau^{\geq 0} : \mathbf{C} \to \mathbf{C}$ is accessible.

Proof. See [44] Definition 1.4.4.12 and Proposition 1.4.4.13. □

Remark 2.2.6. Since $(\tau^{\geq 0}, \iota^{\geq 0})$ and $(\iota^{\leq -1}, \tau^{\leq -1})$ are adjoint pairs, if the t-structure is accessible, then each of the four aforementioned functors are accessible.

Definition 2.2.7. The t-structure is *compatible with filtered colimits* if $\mathbf{C}$ has filtered colimits and $\mathbf{C}^{\geq 0}$ is closed under filtered colimits in $\mathbf{C}$.

When the t-structure is accessible, we have the following equivalent conditions.

[6] Note that sometimes connective objects are called *bounded above* and coconnective objects *bounded below*. The use of the former terminology has the benefit that it is consistent with both homological and cohomological grading conventions.

Proposition 2.2.8. *Let* $\mathbf{C}$ *be a stable presentable* ∞*-category with an accessible* t*-structure (i.e.* $\mathbf{C}^{\geq 0}$ *is also presentable). The following are equivalent:*

(i) $\mathbf{C}^{\geq 0}$ *is closed under filtered colimits in* $\mathbf{C}$*,*
(ii) $\iota^{\geq 0}$ *is continuous,*
(iii) $L^{\geq 0} = \iota^{\geq 0}\tau^{\geq 0}$ *is continuous,*
(iv) $L^{\leq -1} = \tau^{\leq -1}\iota^{\leq -1}$ *is continuous,*
(v) $\tau^{\leq -1}$ *is continuous.*

Proof. This is (a part of) Lemma 6.1.1 in [9], which is stated without proof, which we will provide. Since we assume $\mathbf{C}$ is presentable, the truncation and inclusion functors are accessible, so we may freely use adjoint functor theorems. Conditions (i) and (ii) are simply restatements of each other. For (iii), since $\tau^{\geq 0}$ is a left adjoint, it is already continuous, so (ii) implies (iii); the converse follows since if $X_i \in \mathbf{C}^{\geq 0}$, we have $\tau^{\geq 0}\iota^{\geq 0}$ is the identity functor, which must commute with colimits, so that

$$\iota^{\geq 0} \operatorname{colim} X_i = \iota^{\geq 0} \operatorname{colim} \tau^{\geq 0}\iota^{\geq 0} X_i = \iota^{\geq 0}\tau^{\geq 0} \operatorname{colim} \iota^{\geq 0} X_i = \operatorname{colim} \iota^{\geq 0}\tau^{\geq 0}\iota^{\geq 0} X_i = \operatorname{colim} \iota^{\geq 0} X_i.$$

Conditions (iii) and (iv) are equivalent, since we have an exact sequence for any object X

$$L^{\leq -1}X \to X \to L^{\geq 0}X.$$

Taking $X = \operatorname{colim} X_i$, and using the fact that exact triangles commute with colimits (being a colimit itself), the claim follows. Finally, the equivalence of (iv) and (v) follows in the same way as the equivalence of (ii) and (iii). Note that the rest of Lemma 6.1.1 in [9] follows via the consideration in Remark 2.1.8, but we will not consider this. □

The next two propositions allow us to transport t-structures across functors.

Proposition 2.2.9. *Let* $\mathbf{C}_i$ *be a diagram of stable presentable* ∞*-categories and exact continuous functors, where each* $\mathbf{C}_i$ *has an accessible* t*-structure and each of the transition functors are left* t*-exact and are left adjoints (i.e. continuous). Take* $\mathbf{C} = \lim_{i\in I} \mathbf{C}_i$*. Then, there is a unique accessible* t*-structure on* $\mathbf{C}$ *making the evaluation functors* $\mathbf{C} \to \mathbf{C}_i$ *left* t*-exact. Furthermore, if the* t*-structures on* $\mathbf{C}_i$ *are compatible with filtered colimits, right-complete or left-complete, then so is the* t*-structure on* $\mathbf{C}$ *(respectively).*

Proof. The conditions define a full subcategory $\mathbf{C}^{\leq -1}$; we need to show that the inclusion has a right adjoint, and that the resting subcategories satisfy the third criterion of Proposition 1.2.1.15 of [44]. The first criterion follows since $\mathbf{C}^{\leq -1}$ is presentable since the transition functors are continuous and the t-structure accessible. The latter criterion follows by exactness of the transition functors. If the t-structures on $\mathbf{C}_i$ are compatible with filtered colimits, then so is the one on $\mathbf{C}$ since the transition functors are continuous and the t-structures accessible. Right and left completeness follow since limits commute. □

Proposition 2.2.10. *Let $G : \mathbf{D} \to \mathbf{C}$ be an exact functor between presentable stable ∞-categories which is a right adjoint, and suppose that $\mathbf{C}$ has an accessible t-structure. There is a unique accessible t-structure on $\mathbf{D}$ such that G is left t-exact, i.e. $X \in \mathbf{D}^{\geq 0}$ if and only if $G(X) \in \mathbf{C}^{\geq 0}$.*

Dually, if $F : \mathbf{C} \to \mathbf{D}$ is an exact functor between stable ∞-categories which is a left adjoint, and $\mathbf{D}$ has a t-structure, there is a unique t-structure on $\mathbf{C}$ obtained by insisting that F is right t-exact.

Proof. We only prove the first statement. The second statement follows by the same argument using the adjoint functor theorem for left adjoints. The category $\mathbf{D}^{\geq 0}$ as defined above is accessible since $\mathbf{C}^{\geq 0}$ is and G is an accessible functor by the adjoint functor theorem. Since G is a right adjoint, it commutes with limits (taken in $\mathbf{D}$ and $\mathbf{C}$); in particular, the inclusion $\iota_{\mathrm{D}}^{\geq 0}$ tautologically commutes with limits by construction, so it has a left adjoint and is therefore a localization. One can then verify the third criterion of Proposition 1.2.1.16 in [44], which follows by exactness. $\square$

We now discuss left and right completion of categories. These will play a role when we try to understand categories of quasicoherent sheaves on stacks via derived categories of abelian categories.

Definition 2.2.11. Let $\mathbf{C}$ be a stable ∞-category with a t-structure. The subcategory of *eventually connective* or *bounded above* or *right bounded* objects is the union

$$\mathbf{C}^{-} = \bigcup \mathbf{C}^{\geq -n}$$

and the subcategory of *eventually coconnective* or *bounded below* or *left bounded* objects is the union

$$\mathbf{C}^{+} = \bigcup \mathbf{C}^{\leq n}.$$

The subcategory of *bounded* objects is given by $\mathbf{C}^{b} = \mathbf{C}^{+} \cap \mathbf{C}^{-}$.

The *left completion* of $\mathbf{C}$, denoted $\widehat{\mathbf{C}}$, is defined by the limit

$$\widehat{\mathbf{C}} := \lim_{n \to \infty} \mathbf{C}^{\geq -n} = \lim \left(\cdots \to \mathbf{C}^{\geq -1} \to \mathbf{C}^{\geq 0} \right).$$

By Proposition 1.2.1.17 in [44], the left-completion is stable and has a canonical t-structure. Further, there is a canonical functor $\mathbf{C} \to \widehat{\mathbf{C}}$ which is exact and induces an equivalence $\mathbf{C}^{+} \to \widehat{\mathbf{C}}^{+}$. We say that $\mathbf{C}$ is *weakly left complete* if $\bigcap \mathbf{C}^{\leq -n} \simeq 0$, i.e. if for any given object $X \in \mathbf{C}$, the map $X \to \lim X^{\geq -n}$ is an equivalence. By Proposition 1.2.1.19 of [44], if $\mathbf{C}$ has countable products and $\mathbf{C}_{\geq 0}$ is stable under countable coproducts, then weakly left complete implies left complete.

Dually, we define the *right completion* of $\mathbf{C}$ by the limit

$$\lim_{n \to \infty} \mathbf{C}^{\leq n} = \lim \left(\cdots \to \mathbf{C}^{\leq 1} \to \mathbf{C}^{\leq 0} \right)$$

Most categories we work with will already be right complete, so we do not introduce notation for this. A category is *right complete* if the analogous canonical map is an equivalence.

Remark 2.2.12. There is an equivalence between left bounded categories with a t-structure and left complete categories with a t-structure, and dually for right bounded and right complete categories. Note this means that the assignments $\mathbf{C} \mapsto \widehat{\mathbf{C}}$ and $\mathbf{C} \mapsto \mathbf{C}^+$ are inverse functors; it does not mean that $\widehat{\mathbf{C}} \simeq \mathbf{C}^+$.

The *unbounded derived category* $D(\mathbf{A})$ of a Grothendieck abelian category $\mathbf{A}$ is constructed in Section 1.3.4 and 1.3.5 of [44],. The condition of being a Grothendieck abelian category appears to largely deal with set-theoretic issues arising from localization. We have the following in Propositions 1.3.5.9 and 1.3.5.21, and Remark 1.3.5.23.

Proposition 2.2.13. *Let $\mathbf{A}$ be a Grothendieck abelian category. Then, $D(\mathbf{A})$ is stable and presentable. Furthermore, it has a natural t-structure which is accessible, right complete, and compatible with filtered colimits. Furthermore, if $\mathbf{C}$ is a stable ∞-category with an accessible t-structure compatible with filtered colimits, then $\mathbf{C}^\heartsuit$ is a Grothendieck abelian category.*

Remark 2.2.14. It is possible to define[7] the bounded below or bounded above derived categories $D^+(\mathbf{A})$ and $D^-(\mathbf{A})$ if $\mathbf{A}$ has enough injective and projective objects respectively. For details, see Section 1.3.2 of [44].

If $\mathbf{C}$ is a stable ∞-category with a t-structure such that $\mathbf{C}^\heartsuit$ has enough injective objects, then by Theorem 1.3.3.2 of [44] there is a canonical t-exact functor $D^+(\mathbf{C}^\heartsuit) \to \mathbf{C}$ corresponding to the identity functor on $\mathbf{C}^\heartsuit$. We state the following without proof.

Proposition 2.2.15. *Let $\mathbf{C}$ be a stable ∞-category with an accessible t-structure which is compatible with filtered colimits such that $\mathbf{C}^\heartsuit$ has enough injective objects. The canonical functor*

$$F : D^+(\mathbf{C}^\heartsuit) \to \mathbf{C}^+$$

is an equivalence when $\mathbf{C}$ has the following property: for any injective object I,

$$\mathrm{Hom}_{\mathbf{C}}(Y, I[n]) = 0$$

for all $Y \in \mathbf{C}^\heartsuit$ and $n > 0$.

Example 2.2.16. Let $\mathbf{C} = k[\epsilon]$, where $|\epsilon| = -1$, and define a t-structure such that the forgetful functor $k[\epsilon]$-mod $\to k$-mod is t-exact. This t-structure does not satisfy the conditions in the above proposition, since $\mathbf{C}^\heartsuit$ is the abelian category of k-vector spaces and $\mathrm{Hom}^\bullet_{k[\epsilon]}(k, k) = k[u]$ where $|u| = 2$, i.e. has positive cohomology groups.

[7]Explicitly, $D^+(\mathbf{A})$ is defined to be the dg-nerve (see Construction 1.3.1.16 in [44], this is a way to associate a stable ∞-category to a dg category) of the dg category of bounded below complexes of projective objects.

Now, let $\mathbf{C} = k[\eta]$ where $|\eta| = 1$. We insist only that the forgetful functor is right t-exact, and its left orthogonal $(k[\eta]\text{-mod})^{\leq -1}$ is defined to be the subcategory generated by objects of the form $k[\eta] \otimes_k V$ where $V \in (k\text{-mod})^{\leq -1}$; see Propositions 4.5.2 and 4.5.4 in [42] for details. In this case, $\mathbf{C}^\heartsuit$ is the abelian category of $k[\eta]$-modules, which is equivalent to the category of continuous $k[[x]]$-modules, and this category satisfies the conditions in the above proposition.

Example 2.2.17. This can be used to show that if X is a classical QCA stack, then $\mathrm{QCoh}(X)$ is the left-completion of $D^-(\mathrm{QCoh}(X)^\heartsuit)$.

Proposition 2.2.18. *Let* $\mathbf{A}$ *be an Grothendieck abelian category generated by a set of objects with finite cohomological dimension. That is, there is a set* G *of objects if every* $X \in \mathbf{A}$ *can be written as the quotient of a filtered colimit (equivalently, direct sum) of objects in* G*, i.e.* $X = \mathrm{coker}(\bigoplus G_\alpha \to \bigoplus G_\beta)$*, and the functor* $R^i \mathrm{Hom}_{\mathrm{A}}(T, -)$ *vanishes for large* i*, for each* $T \in G$*. Then,* $D(\mathbf{A})$ *is left complete.*

Example 2.2.19. If $\mathbf{A}$ is the abelian category of A-modules for a classical ring A, it is clear that $D(\mathbf{A})$ is left complete since $\mathbf{A}$ is generated by the free object A, and $R^i \mathrm{Hom}_{\mathrm{A}}(A, -)$ vanishes for $i > 0$. If G is a reductive group acting on a quasiprojective scheme X, then $\mathrm{QCoh}(X/G)^\heartsuit$ can also be seen to satisfy this property. Namely, it is known that equivariant locally free resolutions exist for quasiprojective schemes (see, for example, [16]) implying that equivariant locally free sheaves form a generating set. Further, because G is reductive, the G-invariants functor is t-exact and $R^i \mathrm{Hom}_{X/G}(\mathcal{E}, -)$ vanishes for $i > 0$.

2.3 Derived stacks and prestacks

Definition 2.3.1 (Derived stacks)**.** A (higher) *classical stack* is a functor of (ordinary) 1-categories $\mathbf{Aff} \to \mathbf{Space}$ satisfying a certain sheaf property, where the category of affine schemes is isomorphic to the opposite category of the category of rings: $\mathbf{Aff} \simeq \mathbf{Rng}^{op}$. The category of ∞-stacks has internal mapping objects (see [61] Section 3.2, Example 5). The sheaf property depends on a choice of Grothendieck topology, usually taken to be the étale or fppf topology. The sheaf condition for a functor F requires that for an (étale or fppf) atlas $U \to X$, $F(X)$ is isomorphic to the homotopy colimit of the diagram

$$\cdots \Rrightarrow F(U \times_X U) \rightrightarrows F(U).$$

The category of classical stacks remedies the failure of the category of schemes to contain certain colimits.

A *derived ring* over a field k of characteristic zero, denoted $\mathbf{DRng}$, is one of the following: (1) a simplicial ring, (2) a connective differential-graded ring, or (3) a connective E_∞ ring spectrum. In Proposition 7.1.4.6 and 7.1.4.11 of [44] it is proven that the category of

E_1-algebras is equivalent to the category of dg algebras over any ring, but the category of E_∞-algebras is equivalent to commutative dg algebras only over a $\mathbb{Q}$-algebra. The category of derived rings forms an $(\infty, 1)$-category. We define the category of *affine derived schemes*, denote **Aff**, to be the opposite category of the category of derived rings.

A *prestack* X is an $(\infty, 1)$-functor $X : \mathbf{DRng} = \mathbf{Aff}^{op} \to \mathbf{S}$ from the category of derived rings to the category of spaces.

A *derived stack* (or for us, just a *stack*) is prestack satisfying derived étale descent. The derived étale topology on derived rings is generated by the following: a map $U \to X$ is a derived étale cover if it is of finite presentation and if $\mathbb{L}_{U/X} \simeq 0$ and the induced map on zeroth homotopy groups $\pi_0(U) \to \pi_0(X)$ is a surjective morphism of classical schemes (see Section 3.2 of [60] for details). One important property of derived stacks is that they have internal mapping objects which satisfy the adjunction $\mathrm{Map}(X \times Y, Z) \simeq \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ (see Section 4.3, Example 4 of [61]). Derived stacks remedy the failure of pullback squares in the category of classical stacks to satisfy base change.

The category of stacks is very general; we often wish to focus on stacks which are in some sense understandable inductively via schemes. A *k-Artin stack* X is a stack such that the diagonal map is representable by a $(k-1)$-Artin stack, and which admits an atlas by a scheme U, i.e. a (representable by schemes) smooth map $U \to X$, and such that the restriction of X to n-coconnective dg algebras is k-truncated[8]. We define the category of 0-Artin stacks to be the category of algebraic spaces.

A *Laumon-Moret-Bailly stack* or *LM-algebraic stack* [37] is an Artin 1-stack (a) whose diagonal map is quasi-separated, quasi-compact, and representable by algebraic spaces and (b) admits an atlas by a (derived) algebraic space. Note that in this case, the smooth atlas is automatically representable by algebraic space[9] and further X admits an atlas by a scheme. A *Drinfeld-Gaitsgory algebraic stack* or *DG-algebraic stack* [21] is an LM-algebraic stack whose diagonal map is representable by schemes. A *geometric stack* is a DG-algebraic stack whose diagonal map is affine.

An LM-algebraic stack X over a field of characteristic zero is *quasicompact with affine automorphism groups*, or more concisely, *QCA* (see [21] for details), if it is quasicompact, the automorphism groups of its geometric points are affine, and the classical inertia stack of X is of finite presentation over X. A morphism is QCA if every base change to an affine scheme is an (algebraic) QCA stack.

A *(separated) derived scheme* is a derived stack X such that (a) the diagonal map is schematic and affine, and the base change of the diagonal map to any affine scheme T is a

[8]This means that it takes values in the subcategory of $(n+k)$-groupoids in the category of ∞-groupoids. Equivalently, its values have vanishing homotopy groups in degrees greater than $n+k$.

[9]This is a standard technique; if S is any algebraic space, then we need for $U \times_X S$ to be an algebraic space. On the other hand, $U \times_X S = X \times_{X \times X} (U \times S)$, so by representability of the diagonal we have that $T \times_X S \to T \times S$ is representable by algebraic spaces, but $T \times S$ is an algebraic space so $T \times_X S$ is.

closed embedding[10] and (b) admits a Zariski atlas, i.e. there are affine derived schemes U_i and open embeddings $f_i : U_i \to X$ such that the base change of the f_i to any affine derived scheme form a Zariski cover on π_0.

Further discussion of n-geometric stacks can be found in [54].

Example 2.3.2 (Derived schemes)**.** The definition of derived schemes is somewhat non-constructive. In [41], the notion of a spectral scheme is introduced as an explicit construction. Our examples will, instead, be differential-graded schemes, discussed in [17]: a *dg scheme* is a scheme X equipped with a connective complex of differential-graded $\mathcal{O}_X$-algebras $\mathcal{O}_X^\bullet$ such that the structure map $\mathcal{O}_X \to H^0(\mathcal{O}_X^\bullet)$ is surjective. A morphism of dg schemes $f : X \to Y$ is given by a map of underlying schemes $\pi_0(f)$ and a map of sheaves of differential-graded $\mathcal{O}_X$-algebras $\pi_0(f)^*\mathcal{O}_Y^\bullet \to \mathcal{O}_X^\bullet$. To describe the S-points of a dg scheme X, we need to describe the space $X(S)$. This can be done using the usual simplicial model category structure on dg algebras discussed in [31], since the pullback of $\mathcal{O}_X^\bullet$ to S is a dg S-algebra. If $\pi_0(X)$ is separated, then a dg scheme determines a derived scheme.

Remark 2.3.3. If X is a classical stack (i.e. a functor $\mathbf{Aff}^{cl} \to \mathbf{S}$), it can be considered as a derived stack via left Kan extension, i.e. left adjoint to the restriction functor $\mathbf{Fun}(\mathbf{Aff}, \mathbf{S}) \to \mathbf{Fun}(\mathbf{Aff}^{cl}, \mathbf{S})$. Explicitly, if S is a derived algebra, we have

$$X(S) := \underset{\substack{S \to S' \\ S' \text{ classical}}}{\operatorname{colim}} X(S').$$

Since the inclusion of classical affine schemes into affine schemes is fully faithful, this left Kan extension is fully faithful. Note that the restriction functor also has a right adjoint extension:

$$X(\pi_0(S)) = \lim_{\substack{S' \to S \\ S' \text{ classical}}} X(S')$$

which is not fully faithful in general. Note that if S is a classical scheme, then the left Kan extension of S is just S considered as a derived scheme where $\pi_n(S) = 0$ for $n > 0$.

2.4 Formal and derived completions

Finally, we will review the notion of formal completions of prestacks and derived completion on Artin stacks. We summarize the main points of Chapter 4 of [43], Section 3.4 of [11], Section 15.80 of the [55], and Chapter 6 of [24].

[10] A map of affine derived schemes is a closed embedding if it is on π_0.

Definition 2.4.1. Let $f : X \to Y$ be a map of derived stacks (or more generally, prestacks). The *formal completion* of f, written $\widehat{Y_X}$, has a functor-of-points whose S-points are given by diagrams

$$\begin{array}{ccc} \pi_0(S)^{red} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & Y \end{array}$$

defining it as a prestack.

Lemma 2.4.2. *The formal completion of a map $X \to Y$ only depends on $\pi_0(X)^{red} \to Y$. In particular, if $Z \to Y$ is a closed embedding, then $\widehat{Z_Y} \times_Y X = \widehat{X_{\pi_0(Z \times_Y X)^{red}}}$.*

Proof. This follows directly from the functor-of-points characterization of formal completions, and that $\pi_0(X)^{red}$ is the universal stack that factors any map from a classical reduced scheme $\pi_0(S)^{red}$. □

Example 2.4.3. The formal odd tangent bundle $\widehat{\mathbb{T}}_X[-1]$ is the completion of the inclusion of the zero section $X \to \mathbb{T}_X[-1]$. The corresponding filtration on $\mathcal{O}(\widehat{\mathbb{T}}_X[-1])$ is the Hodge filtration.

Definition 2.4.4. Let $f : A \to B$ be a map of derived rings. Following [40], we say that f is *étale* if the induced map $\pi_0(A) \to \pi_0(B)$ is étale and for every $n \in \mathbb{Z}$, the map $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$ is an isomorphism of abelian groups. A map of derived schemes is étale if it is for a Zariski cover.

Proposition 2.4.5. *Let X, Y, Z be stacks admitting deformation theory (e.g. quotient stacks). The relative cotangent complex vanishes $\mathbb{L}_{X/Y} \simeq 0$ if $f : X \to Y$ is étale. In particular, for any map $Z \to X$, we have an isomorphism of formal completions $\widehat{X}_Z \to \widehat{Y}_Z$.*

Proof. The first sentence is Proposition 2.22 in [40]. For the second, by the exact triangle for cotangent complexes we have a natural isomorphism $\mathbb{L}_{Z/Y} \simeq \mathbb{L}_{Z/X}$ under $\mathbb{L}_Z$, and note that the formal completion of a map $Z \to X$ is a colimit of square-zero extensions controlled by the the map between cotangent complexes $\mathbb{L}_Z \to \mathbb{L}_{Z/X}$ (see Chapter IV.5 in [23]). □

We now discuss how to compute the derived completion of a quasicoherent sheaf.

Definition 2.4.6. Let A be a connective dg ring, and fix an ideal $I \subset \pi_0(A)$. We define the full subcategory $A\text{-mod}_{nil}$ of I*-nilpotent* objects consisting of those modules on which I acts locally nilpotently, i.e. for each cycle $m \in H^\bullet(M)$ there is some power of I which annihilates m. By Proposition 4.1.12 and 4.1.15 in [43], the inclusion $A\text{-mod}_{nil} \hookrightarrow A\text{-mod}$ is continuous and preserves compact objects; therefore it has a continuous right adjoint Γ_I, which we call the *local cohomology functor*. We define the full subcategory $A\text{-mod}_{loc}$ of

I-local objects as the right orthogonal to A-mod$_{nil}$, and the full subcategory A-mod$_{cpl}$ of *I-complete* modules to be the right orthogonal to A-mod$_{loc}$. The subcategory A-mod$_{cpl}$ has an equivalent characterization as those modules such that the derived limit

$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$

is zero for all $x \in I$. By Proposition 4.2.2 of [43], the inclusion of the complete objects has a left adjoint, which we call the (derived) *completion functor* and denote $\widehat{(-)}$.

The following is Proposition 4.2.5 in [43].

Proposition 2.4.7. *The composition of left adjoints*

$$A\text{-mod}_{nil} \hookrightarrow A\text{-mod} \xrightarrow{\widehat{(-)}} A\text{-mod}_{cpl}$$

is an equivalence. Consequently, the composition of its right adjoints

$$A\text{-mod}_{cpl} \hookrightarrow A\text{-mod} \xrightarrow{\Gamma} A\text{-mod}_{nil}$$

is also an equivalence.

The derived completion and local cohomology functors can each be computed in two dual ways. The following can be found as Propositions 15.80.10 and 15.80.17 in [55] and in a global form as Proposition 3.4.12 in [11] and Proposition 6.7.4 in [24]. The statements on local cohomology are well known (and which we will not use).

Proposition 2.4.8. *Choose generators $f_1, \dots, f_r$ of $I \subset \pi_0(A)$. The derived completion of an A-module M can be computed*

$$\widehat{M} = \lim_n M \otimes_{\pi_0(A)} K_n^\bullet$$

where $K_n^\bullet$ is the Koszul complex for the sequence $f_1^n, \dots, f_r^n \in \pi_0(A)$, i.e. the polynomial dg algebra whose underlying graded algebra is $K_n^\bullet := \pi_0(A)[\epsilon_1, \dots, \epsilon_r]$ where $|\epsilon_i| = -1$, and whose differential is generated by $d(\epsilon_i) = f_j^n$. It can also be computed

$$\widehat{M} = R\operatorname{Hom}_{\pi_0(A)}(G^\bullet, M)$$

where $G^\bullet$ denotes the "local Koszul complex"

$$\pi_0(A) \longrightarrow \prod \pi_0(A)[\tfrac{1}{f_i^n}] \longrightarrow \prod_{i,j} \pi_0(A)[\tfrac{1}{f_i^n}, \tfrac{1}{f_k j^n}] \longrightarrow \cdots \longrightarrow \pi_0(A)[\tfrac{1}{f_1^n}, \dots, \tfrac{1}{f_r^n}] \ .$$

Likewise, we can compute the local cohomology of M by

$$\Gamma_I(M) = \operatorname{colim} \operatorname{Hom}_{\pi_0(A)}(K_n^\bullet, M),$$

$$\Gamma_I(M) = G^\bullet \otimes^L_{\pi_0(A)} M.$$

The latter formula is the calculation of local cohomology via a Cech resolution with supports on an affine scheme.

Remark 2.4.9. The above story is globalized in [24] in the following way. Let X be a dg scheme, and $Z \subset X$ a classical closed subscheme. There is a functor $\widehat{i}^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\widehat{X}_Z)$ with a fully faithful (continuous) left adjoint $\widehat{i}_?$ whose essential image is the category of quasicoherent sheaves supported on Z, and a fully faithful (non-continuous!) right adjoint $\widehat{i}_*$ whose essential image is the category of quasicoherent sheaves complete with respect to the ideal sheaf for Z. There is an exact triangle of functors arising from the localization functor $\widehat{i}_?\widehat{i}^*$ (whose essential image is cocomplete):

$$\Gamma_Z = \widehat{i}_?\widehat{i}^* \to \mathrm{id}_{\mathrm{QCoh}(X)} \to j_*j^* = (-)|_U.$$

In particular, the functor on the left is local cohomology, and the functor on the right is restriction to U. The (non-continuous) functor $\widehat{i}_*\widehat{i}^*$ is the (derived) completion.

The following lemma is likely well-known, but we could not find a reference.

Lemma 2.4.10. *Let X be a derived scheme, $i : Z \subset X$ a closed subscheme and $j : U = X - Z \to X$ its complement. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of quasicoherent sheaves on X. If the derived completion $\widehat{\phi}_Z$ and the restriction $\phi|_U$ are isomorphisms, then ϕ is an isomorphism.*

Proof. Using the above exact triangle, to show that ϕ is an isomorphism, it suffices to show that $\Gamma_Z(\phi)$ is an isomorphism, or equivalently, that $\Gamma_Z(\mathrm{cone}(\phi)) = 0$. To this end, note that $\widehat{\mathrm{cone}(\phi)} = \widehat{i}_*\widehat{i}^* \mathrm{cone}(\phi) = 0$, and that $\widehat{i}_*$ is fully faithful, so that $\widehat{i}^* \mathrm{cone}(\phi) = 0$, so that $\Gamma_Z(\mathrm{cone}(\phi)) = \widehat{i}_?\widehat{i}^* \mathrm{cone}(\phi) = 0$. ☐

Example 2.4.11. The above is not true for non-derived completions. For example, take $X = \mathbb{A}^1$, $Z = \{0\}$, and $\phi : 0 \to M = k[x, x^{-1}]/k[x]$ (one thinks of M as the module of distributions supported at zero). Since M is supported at zero, $M|_U = 0$, and since $x^k M = M$ for all k, $\widehat{M}_Z = 0$, but ϕ is not an isomorphism. On the other hand, the derived completion of M is $l[[x]][1]$.

2.5 Quasicoherent and ind-coherent sheaves

Remark 2.5.1. We begin with a general remark. There are, in some sense, two approaches to the theory of derived categories of quasicoherent sheaves on a derived stack X. The approach taken in [23] produces the "correct" categories at the expense of being inexplicit. That is, the category of quasicoherent sheaves on an affine derived scheme $\mathrm{Spec}(R)$ is defined to be the ∞-category of complexes of R-modules localized with respect to quasi-isomorphisms, and the category $\mathrm{QCoh}(X)$ is defined to be the ∞-limit over quasicoherent sheaves on its S-points. This category inherits a natural t-structure from the t-structures on $\mathrm{QCoh}(S)$ and is automatically left-complete with respect to this t-structure; it is generally not true, however, that $\mathrm{QCoh}(X)$ is the derived category of its heart.

On the other hand, one might prefer to choose a site and define the abelian category $\mathrm{QCoh}(X)^\heartsuit$ to be a certain subcategory of sheaves of $\mathcal{O}_X$-modules on the site. Olsson [50] gives an explicit construction of the category $\mathrm{QCoh}(X)^\heartsuit$ when X is a *classical* Artin stack using the lisse-étale topology. This construction works only for classical Artin stacks since it realizes $\mathrm{QCoh}(X)$ as the left-completion of the derived category of an abelian category of quasicoherent sheaves $\mathrm{QCoh}(X)^\heartsuit$, generalizing the classical construction of the derived category of quasicoherent sheaves on non-affine schemes. The category $\mathrm{QCoh}(X)$ above can be recovered as the left-completion of $D^+(\mathrm{QCoh}(X)^\heartsuit)$.

Remark 2.5.2. The approach via abelian categories is bound to fail in the derived setting, since even for derived schemes it is not true that the stable ∞-category of quasicoherent sheaves is the derived category of an abelian category. However, an approach via ∞-topoi is currently in development by Lurie in [47]. As of the time of this writing, the story has been laid out for derived Deligne-Mumford stacks but not for Artin stacks.

Definition 2.5.3. Let S be an affine derived scheme; we define $\mathrm{QCoh}(S)$ to be S-mod. When A is a dg algebra, we mean the dg category of complexes of S-modules localized with respect to quasi-isomorphisms. When A is a simplicial algebra, we mean the stabilization of the simplicially enriched category of simplicial A-modules localized with respect to weak equivalences. This category is equipped with the usual t-structure inherited from the category of complexes of A-modules. Given a map $f : S' \to S$ of derived schemes, there is a natural pullback $f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S')$ given by derived tensor product, and a pushforward $f_* : \mathrm{QCoh}(S') \to \mathrm{QCoh}(S)$ given by the restriction of scalars functor, which together form an adjoint pair (f^*, f_*). The pullback is left t-exact (i.e. preserves connective objects) and the pushforward is right t-exact (i.e. preserves coconnective objects).

When X is a prestack, we define

$$\mathrm{QCoh}(X) = \lim_{\substack{x:S\to X \\ S \text{ affine}}} \mathrm{QCoh}(S)$$

where the limit is taken inside the ∞-category of stable ∞-categories (or dg categories) via $*$-pullback functors. Tautologically, if $X' \to X$ is a map of prestacks, then we have a pullback functor $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X')$. Its right adjoint is defined to be the pushforward functor, but it is not continuous in general. By Theorem 1.4.2 of [21], the pushforward is continuous for QCA morphisms. The category $\mathrm{QCoh}(X)$ inherits a t-structure by insisting that x^* is left t-exact for every point $x : S \to X$.

Remark 2.5.4. Let us somewhat unwind what this ∞-limit means. An object of the diagram category is an affine dg scheme S along with map $x : S \to X$. The automorphisms of such an object form a pointed space with base point $x \in X(S)$, and for any map $f : S' \to S$, we have a map of spaces $X(f) : X(S) \to X(S')$. A quasicoherent sheaf $\mathcal{F} \in \mathrm{QCoh}(S)$ is the data of a compatible collection of $\mathrm{QCoh}(S)$-valued local systems $\mathcal{F}_S$ on the spaces $X(S)$ for each affine scheme S. That is, (a) for every $x \in X(S)$, the stalk $\mathcal{F}_x \in \mathrm{QCoh}(S)$ is a quasicoherent complex on S and (b) for every path in $X(S)$ from x to y, we have a map $\mathcal{F}_x \to \mathcal{F}_y$ and (c) for every 2-simplex in $X(S)$, we have a degree -1 map with prescribed boundary and so on.

Denote by $X(f)^{-1}$ the natural pullback of local systems from $X(S')$ to $X(S)$ and $f^* : \mathrm{QCoh}(S') \to \mathrm{QCoh}(S)$ the quasicoherent pullback; for the compatibility we require an equivalence $X(f)^{-1}\mathcal{F}_{S'} \simeq f^*\mathcal{F}_S$ of $\mathrm{QCoh}(S')$-valued local systems over $X(S)$, i.e. for all $x \in X(S)$, we have equivalences $\mathcal{F}_{X(f)(x)} \simeq f^*\mathcal{F}_x$. One should also account for higher data, but we will not in this sketch.

The following can be found in [23]; we repeat it for convenience.

Proposition 2.5.5. *The canonical functor*

$$D(\mathrm{QCoh}(X)^{\heartsuit})^+ \to \mathrm{QCoh}(X)^+$$

is an equivalence for a quasicompact algebraic stack with affine diagonal. In particular, since $\mathrm{QCoh}(X)$ *is left-complete, we have an equivalence on left completions*

$$\lim_{n\to\infty} D(\mathrm{QCoh}(X)^{\heartsuit})^{\geq -n} \to \mathrm{QCoh}(X).$$

Proof. We show that $\mathrm{QCoh}(X)$ satisfies the conditions of Proposition 2.2.15. Since X has affine diagonal, we can choose a *classical* smooth (in particular, flat) affine atlas $f : S \to X$; since the diagonal is affine, f is also affine. In particular, f_* is t-exact by affineness and f^* is t-exact by flatness, and define functors on the heart. Note that if X is not classical, we can not in general find a flat atlas by a classical scheme.

Let $M \in \mathrm{QCoh}(X)^{\heartsuit}$ be any object and $I \in \mathrm{QCoh}(S)^{\heartsuit}$ an injective object with a monomorphism $f^*M \hookrightarrow I$. Since f was chosen to be flat, $M \hookrightarrow f^*M$ is a monomorphism, so that $M \hookrightarrow f_*I$ is also a monomorphism. Further, we have $\mathrm{Hom}_{\mathrm{QCoh}(X)}(Y, f_*I[n]) = \mathrm{Hom}_{\mathrm{QCoh}(S)}(f^*Y, I[n]) = 0$ for $n > 0$, satisfying the required conditions. □

Remark 2.5.6. In [50], the abelian category of quasicoherent sheaves on a stack is defined. Its derived category is not left-complete, and is therefore not equivalent to the ∞-categorical definition of $\mathrm{QCoh}(X)$ we take above. Similarly, the comments after Theorem 1.1 of [27] indicate further results in this direction.

Example 2.5.7. If S is a classical dg scheme, then $\mathrm{QCoh}(S)$ is easily verified to be the derived category of its heart. On the other hand, if S is a dg algebra, this is seen to be false in any example: take $S = \mathrm{Spec}(k[\epsilon])$ where $|\epsilon| = -1$. The derived category of the heart is k-mod.

Remark 2.5.8. In [50], the class of classical stacks for which $\mathrm{QCoh}(X)$ is constructed is somewhat larger; in particular, their construction works for any 1-stack whose (a) diagonal is representable by schemes, quasi-compact and quasi-separated and (b) which admits an atlas by an algebraic space. We do not know whether the above proposition can be extended to this case.

Example 2.5.9 ($X = BG$)**.** Let us identify $\mathrm{Perf}(BG)$ by brute force as the dg category of G-representations. First, note that BG as a derived stack is realized via a left Kan extension, i.e. a colimit. Since BG has a simplicial presentation, i.e. is the totalization of a simplicial diagram whose terms are classical schemes, we find that $\mathrm{QCoh}(BG)$ where BG is considered as a derived stack is equivalent to the category where BG is considered as a classical stack.

Let V be a perfect complex of G-representations; we will associate to it an object of $\mathrm{Perf}(BG)$. Note that since BG is a classical stack, we only need to address the pullback of V to classical schemes S. An S-point $x : S \to BG$ is determined by a G-torsor P over S; we define x^*V to be the sheaf of sections of the vector bundle $P \times^G V^* \to S$. This is clearly functorial under pullback.

Conversely, to an object of $\mathcal{F} \in \mathrm{Perf}(BG)$, we can assign a complex of G-representations as follows. Let $p : \mathrm{pt} \to BG$ be the trivial torsor, and let $p^*\mathcal{F} = V$ be the resulting complex of vector spaces. For each $g \in G$, we obtain an automorphism of the point p, inducing an automorphism of V. This defines the G-action on V.

The following notion was defined and studied in [5].

Definition 2.5.10. A derived stack X is *perfect* if it has affine diagonal and if $\mathrm{QCoh}(X)$ is compactly generated by $\mathrm{Perf}(X)$.

In Section 3.3 of *loc. cit.* and the discussion in Section 7 of [28], the following examples of perfect stacks are established.

Theorem 2.5.11. *The following are examples of perfect stacks over a field of characteristic zero: (a) quasicompact and separated schemes, (b) the quotient stack X/G where X is (derived) quasiprojective and G is affine, (c) mapping stacks $\mathrm{Map}(S, X)$ where S is a space finite in CW-complexes and X is perfect, (d) fiber products of perfect stacks, (e) quasiprojective stacks over a perfect stack, (f) any stack possessing the resolution property, i.e. the property that any coherent sheaf is the quotient of a vector bundle.*

Example 2.5.12 (Non-quasicompact scheme)**.** Let $\mathbb{Z}$ be considered as a non-quasicompact scheme, i.e. the disjoint union of infinitely many copies of $\mathrm{Spec}(k)$. Note that since $\mathbb{Z}$ is not affine, we do not define $\mathrm{IndCoh}(\mathbb{Z})$ as the ind-completion of $\mathrm{Coh}(\mathbb{Z})$; instead, it is defined to be $\mathrm{Ind}(\mathrm{Coh}(U))$ on affine opens U and patched together via the !-restriction. In particular, since $\mathbb{Z}$ is zero-dimensional and smooth, $*$-restriction equals !-restriction and $\mathrm{IndCoh}(\mathbb{Z}) = \mathrm{QCoh}(\mathbb{Z})$. Further, again by smoothness, $\mathrm{Perf}(\mathbb{Z}) = \mathrm{Coh}(\mathbb{Z})$.

On the other hand, the compact objects of $\mathrm{QCoh}(\mathbb{Z})$ is the subcategory of complexes with finite cohomological support and finite geometric support in $\mathbb{Z}$. In particular, $\mathcal{O}_{\mathbb{Z}}$ is not compact, so $\mathbb{Z}$ is not perfect. Explicitly, the identity map is an element of

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{Z})}(\mathcal{O}_{\mathbb{Z}}, \mathcal{O}_{\mathbb{Z}}) = \mathrm{Hom}_{\mathrm{QCoh}(\mathbb{Z})}(\mathcal{O}_{\mathbb{Z}}, \bigoplus_{n \in \mathbb{Z}} k_n)$$

which does not factor through any finite sum. In particular,

$$\mathrm{QCoh}(\mathbb{Z}) \not\simeq \mathrm{Ind}(\mathrm{Perf}(\mathbb{Z})) \qquad \mathrm{IndCoh}(\mathbb{Z}) \not\simeq \mathrm{Ind}(\mathrm{Coh}(\mathbb{Z})).$$

Example 2.5.13 (Non-separated scheme)**.** Let X be the double affine line, i.e. $X = \mathbb{A}^1 \cup_{\mathbb{G}_m} \mathbb{A}^1$. Let $x_1, x_2 \in X$ be the two doubled origins. The skyscraper sheaf k_{x_1} is not perfect, since for any perfect complex the stalk at x_1 is equal to the stalk at x_2. On the other hand, it is compact, since it is compact on restriction to any affine open.

Further examples of stacks for which $\mathrm{QCoh}(X)$ is not compactly generated in positive characteristic can be found in [1].

Remark 2.5.14. Let X be a QCA stack. It is known that $\mathrm{Perf}(X)$ are the compact objects of $\mathrm{QCoh}(X)$, but it is not known whether $\mathrm{QCoh}(X)$ is compactly generated. However, it is proven in Theorem 4.3.1 of [21] that $\mathrm{QCoh}(X)$ is dualizable when X is eventually coconnective. The proof uses the fact that in this case, $\mathrm{QCoh}(X)$ is a retract of $\mathrm{IndCoh}(X)$ (in particular, since $\mathcal{O}_X$ is coherent when X is eventually coconnective), and Theorem 3.3.5, which shows that $\mathrm{IndCoh}(X)$ is compactly generated by $\mathrm{Coh}(X)$.

In [22] the theory of ind-coherent sheaves is developed for derived schemes and stacks. We will introduce the definitions here. Although we do not work directly with ind-coherent sheaves, their study is important for our future work.

Definition 2.5.15. Let S be an affine scheme. We define the category of *ind-coherent sheaves* on S to be

$$\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S)).$$

Let $f : S' \to S$ be a map of dg schemes. If f is proper (i.e. proper on underlying classical schemes), then we obtain a functor $f_* : \mathrm{IndCoh}(S') \to \mathrm{IndCoh}(S')$ which is continuous and preserves compact objects. Therefore, it has a continuous right adjoint, which we denote $f^! : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S')$. When f is an open immersion, we also have a continuous functor f_* which does not preserve compact objects but also commutes with limits; in this case, it has a left adjoint which we denote $f^!$ (also f^*). By Theorem 5.2.2 of [22], via Nagata's compactification theorem, this defines for any map f of dg schemes a well-defined pullback $f^!$ and pushforward f_*.

When X is a prestack, we then define

$$\mathrm{IndCoh}(X) = \lim_{\substack{x:S\to X\\ S \text{ affine}}} \mathrm{IndCoh}(S)$$

where the limit is taken inside the ∞-category of stable ∞-categories (or dg categories) via !-pullback functors.

Remark 2.5.16 (Relationship with homotopy category of injectives)**.** A *model* for $\mathrm{IndCoh}(X)$ should be a category $\mathbf{C}$ equipped with a fully faithful functor $\mathbf{C} \to \mathrm{Fun}(\mathrm{Coh}(X)^{op}, \mathbf{Ch})$ whose essential image coincides with the ind-completion of $\mathrm{Coh}(X)$. When X is a classical noetherian scheme, the homotopy category of injectives $\mathbf{K}_{inj}(X)$ (for us, the dg category of injective complexes) is such a model with structure functor $\mathcal{I}^\bullet \mapsto \mathrm{Hom}(-, \mathcal{I}^\bullet)$. This Hom is the literal Hom-complex of complexes (i.e. we do not invert quasi-isomorphisms). Likewise, a model for $\mathrm{IndCoh}(X)^\vee$ should be a category equipped with a fully faithful functor $\mathbf{C} \to \mathrm{Fun}(\mathrm{Coh}(X), \mathbf{Ch})$. Neeman and Murfet [49] [48] prove that the mock category of projectives does the trick, with realization functor $\mathcal{P}^\bullet \mapsto \mathrm{Hom}(\mathcal{P}^\bullet, -)$.

Example 2.5.17. Recall that when $R = k[x]/x^2$, the skyscraper module $k_0 = R/x$ has an infinite injective resolution $0 \to R \to R \to \cdots$. In particular, there is a short exact sequence of modules

$$0 \to k_0 \to R \to k_0 \to 0$$

which induces a nonzero map $k_0 \to k_0[1]$, which on injective resolutions looks like

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & \cdots \end{array}$$

Thus we have a filtered diagram of modules $k_0 \to k_0[1] \to k_0[2] \to \cdots$ whose colimit is the acyclic injective complex unbounded in both directions:

$$\cdots \to R \to R \to R \to \cdots.$$

This complex is *not* isomorphic to zero in $\mathrm{IndCoh}(X)$ because in each finite stage of the colimit the module is nonzero; more precisely, colimit has endomorphisms given by

$$\lim_n \operatorname*{colim}_m \mathrm{Hom}_R(k_0[n], k_0[m]) \simeq \lim_n \operatorname*{colim}_m k \otimes \delta_{m\geq n} = k$$

which is nonzero.

Definition 2.5.18 (Comparison functors)**.** There is a canonical object $p^!k = \omega_X \in \mathrm{IndCoh}(X)$ for any prestack X, where $p : X \to \mathrm{pt}$ is the structure map. It is analogous to $p^*k = \mathcal{O}_X \in \mathrm{QCoh}(X)$, but in general it is not an algebra object but a coalgebra object. The category $\mathrm{QCoh}(X)$ is monoidal, and $\mathrm{IndCoh}(X)$ has a continuous action of the monoidal category $\mathrm{QCoh}(X)$. This action defines a functor $\Phi : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$ for any prestack X.

When S is a dg scheme, we have a comparison functor $\Psi : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$ induced by ind-completion of the inclusion $\mathrm{Coh}(S) \to \mathrm{QCoh}(S)$. For a general prestack, it is not true that $\mathrm{Ind}(\mathrm{Coh}(X)) = \mathrm{IndCoh}(X)$; the latter is defined via Kan extension.

We have the following Propositions 1.2.4 and 1.3.4 in [22].

Proposition 2.5.19. *Let S be a dg scheme. The functor $\Psi^{\geq n} : \mathrm{IndCoh}(S)^{\geq n} \to \mathrm{QCoh}(S)^{\geq n}$ is an equivalence for every n. The category $\mathrm{QCoh}(S)$ is left complete, and Ψ realizes $\mathrm{QCoh}(S)$ as the left completion of $\mathrm{IndCoh}(S)$.*

Remark 2.5.20 (Convergence)**.** Let A be a connective dg algebra. It is known that the Postnikov tower

$$A = \lim \tau^{\geq -n} A = \lim(\cdots \to \tau^{\geq -2} A \to \tau^{\geq -1} A \to \tau^{\geq 0} A)$$

converges, i.e. that the dg scheme $\mathrm{Spec}(A)$ is *convergent* as a prestack. Note that if $S = \mathrm{Spec}(A)$, the indexing is reversed:

$$S = \mathrm{colim}\, \tau^{\leq n} S = \mathrm{colim}(\tau^{\leq 0} S \to \tau^{\leq 1} S \to \tau^{\leq 2} S \to \cdots)$$

and the maps are closed immersions (and therefore proper). An equivalent way to express convergence of dg schemes is that every dg scheme S is an ind-object of eventually coconnective dg schemes.

The following is Proposition 4.3.4 in [22].

Proposition 2.5.21. *We have that the natural functor*

$$\mathrm{colim}\, \mathrm{IndCoh}(\pi_{\leq n}(S)) \to \mathrm{IndCoh}(A)$$

is an equivalence.

Example 2.5.22. Take $A = k[u]$ with $|u| = -2$, and $S = \mathrm{Spec}(A)$. Since every finitely generated complex of A-modules has a finite resolution, the category $\mathrm{Perf}(S)$ is the dg category of finitely generated A-modules. On the other hand, $\mathrm{Coh}(S)$ is the category of finitely generated A-modules with coherent cohomology. We find, then that $\mathrm{IndCoh}(S)$ consists of complexes of A-modules on which u locally acts by torsion.

We observe that the convergence property above is false for quasicoherent sheaves. The colimit of the categories

$$k[u]/u\text{-mod} \to k[u]/u^2\text{-mod} \to \cdots$$

is the category of $k[u]$-modules where u acts by torsion, i.e. $\mathrm{IndCoh}(S)$ and not $\mathrm{QCoh}(S)$.

The following is Theorem 3.3.5 in [21].

Proposition 2.5.23. *Let X be a QCA stack. The category* $\mathrm{IndCoh}(X)$ *is compactly generated by* $\mathrm{Coh}(X)$.

2.6 Group actions on categories and equivariant objects

We will give a bird's eye overview of categories with actions of ∞-groups; unfortunately, we do not know of a more comprehensive or detailed reference. In this note we will only apply the case when $G = S^1$.

Definition 2.6.1. An *∞-group G* is a (∞-)group object[11] in the ∞-category of spaces $\mathbf{S}$. Let G be an ∞-group, and let $\mathbf{B}_*G$ denote the ∞-category with a single object, whose space of morphisms is G; this can be thought of a pointed version of the space BG realized as an ∞-category. A G-action on a small k-linear stable ∞-category is a functor $\mathbf{B}G \to \mathbf{st}_k$; equivalently, it is a category $\mathbf{C} \in \mathbf{st}_k$ with a map of ∞-groups $a : G \to \mathrm{Aut}(\mathbf{C})$. We define the *$G$-invariants* of a category $\mathbf{C}$ with a G-action by

$$\mathbf{C}^G = \lim_{\mathrm{B}G} \mathbf{C}$$

where we regard the G-action on $\mathbf{C}$ as a $\mathbf{B}G$-indexed diagram in $\mathbf{st}_k$ (which has all limits by Proposition 1.1.4.4 of [44]).

Let $\mathbf{st}_k^G$ denote the ∞-category of small stable k-linear ∞-categories equipped with a G-action as above. Then, we obtain a functor of G-invariants

$$(-)^G : \mathbf{st}_k^G \to \mathbf{st}_k.$$

This functor has a left adjoint, $\mathrm{triv} : \mathbf{st}_k \to \mathbf{st}_k^G$ which assigns to a category $\mathbf{C}$ the trivial G-action. This adjunction has counit, defining a functor $\mathbf{C}^G \to \mathbf{C}$ in $\mathbf{st}_k^G$ (where $\mathbf{C}^G$ is equipped with the trivial G-action).

Remark 2.6.2. Let G be a topological group which is a finite CW complex; equip the category of finite chain complexes $\mathbf{Vect}_k^\omega$ with the trivial G-action. Then,

$$\mathbf{Vect}_k^{\omega G} \simeq C_\bullet(G; k)\text{-mod}$$

[11] See [45] Definition 7.2.2.1 and Proposition 7.2.2.4.

where $C_\bullet(G;k)$ is the group algebra whose product comes from that of G. This follows from an argument similar to Lemma 3.10 in [6]. Choose a map $p : \mathrm{pt} \to BG$ making BG a pointed space in derived stacks. Note that every map $S \to BG$ factors through $\mathrm{pt} \to BG$ since BG is the locally constant stack with value BG. Then, we have

$$(\mathbf{Vect}_k^\omega)^G \simeq \mathrm{Coh}(BG) = \lim_{BG} \mathrm{Coh}(\mathrm{pt}).$$

and $p^* : \mathrm{Coh}(BG) = \mathrm{Perf}(BG) \to \mathrm{Perf}(\mathrm{pt})$ is identified with the counit map $(\mathbf{Vect}_k^\omega)^G \to \mathbf{Vect}_k^\omega$. The Barr-Beck argument in Lemma 3.10 of [6] identifies

$$\mathbf{Vect}_k^G \simeq \mathrm{QCoh}(BG) = \lim_{BG} \mathrm{QCoh}(\mathrm{pt}) = \mathcal{O}(G)\text{-comod}\,.$$

Since G acts trivially on $\mathbf{Vect}_k$, all functors in the limit are t-exact, and by the arguments in Section 4 of [51] the t-coherent objects in $\mathbf{Vect}_k^G$ is equivalent to the category $(\mathbf{Vect}_k^\omega)^G$. Since G is a finite CW complex, $\mathcal{O}(G) \simeq C^\bullet(G;k)$ has finite-dimensional and bounded cohomology, and we have an equivalence between the category of finite $\mathcal{O}(G) \simeq C^\bullet(G;k)$-comodules and the category of finite $C_\bullet(G;k)$-modules.

Remark 2.6.3. If G is *connected*, the Koszul duality of [26] between $C^\bullet(BG;k)$-mod and $C_\bullet(G;k)$-mod asserts an equivalence

$$(C^\bullet(BG;k)\text{-perf})^{op} \underset{R\,\mathrm{Hom}_{C_\bullet(G;k)}(C_\bullet(EG;k),-)}{\overset{C^\bullet(EG;k)\otimes^L_{C^\bullet(BG;k)}-}{\rightleftarrows}} C_\bullet(G;k)\text{-mod}_{fg}$$

arising from the calculation that

$$R\,\mathrm{Hom}_{C_\bullet(G;k)}(k,k) \simeq R\,\mathrm{Hom}_{C_\bullet(G;k)}(C_\bullet(EG;k),k) \simeq C^\bullet(BG;k).$$

The assumption that G is connected guarantees that the augmentation module k generates $C_\bullet(G;k)$-mod. Taking ind-completions, we find that $\mathrm{Ind}(C^\bullet(BG;k)\text{-perf}) = C^\bullet(BG;k)\text{-mod}$.

Example 2.6.4. The above is false if G is not connected. For example, the heart of $\mathrm{Coh}(B(\mathbb{Z}/2\mathbb{Z}))$ is the category of $\mathbb{Z}/2\mathbb{Z}$-representations, but $C^\bullet(B(\mathbb{Z}/2\mathbb{Z});k) \simeq k$ rationally.

Remark 2.6.5. More generally, with the above assumptions, if $\mathbf{C}$ is equipped with the trivial G-action, then we have natural equivalences

$$\mathbf{C}^G \simeq \mathbf{C} \otimes C_\bullet(G;k)\text{-mod}_{fg} \simeq \mathbf{C} \otimes C^\bullet(BG;k)\text{-perf}\,.$$

Definition 2.6.6. Let X be a prestack. A G-action on X is a factorization of its functor of points $X(-) : \mathbf{Aff}^{op} \to \mathbf{S}$ through the ∞-category of spaces with a G-action, i.e. the ∞-category of spaces over BG.

Proposition 2.6.7. *Let X be a prestack with a G-action for an ∞-group G. Then,* $\mathrm{QCoh}(X)$ *and* $\mathrm{Perf}(X)$ *have natural G-actions. Furthermore, the structure sheaf $\mathcal{O}_X$ has a canonical G-equivariant structure, i.e. the object $\mathcal{O}_X \in \mathrm{QCoh}(X)$ lifts to an object $\mathcal{O}_X^G \in \mathrm{QCoh}(X)^G$ (and $\mathrm{Perf}(X)^G$, if $\mathcal{O}_X$ is perfect) and is functorial with respect to pullback along G-equivariant maps of prestacks with a G-action.*

Remark 2.6.8. Before giving a mostly tautological proof, we give an informal discussion of the definition of $\mathrm{QCoh}(X)$ for X a prestack. Let S be an affine derived scheme; we define[12] $\mathrm{QCoh}(S)$ to be S-mod. When X is a prestack, we define

$$\mathrm{QCoh}(X) = \lim_{\substack{x:S\to X \\ S \text{ affine}}} \mathrm{QCoh}(S)$$

where the limit is taken inside the ∞-category of stable ∞-categories (or dg categories). Let us somewhat unwind what this means. An object of the diagram category is an affine dg scheme S along with map $x : S \to X$. The automorphisms of such an object form a pointed space with base point $x \in X(S)$, and for any map $f : S' \to S$, we have a map of spaces $X(f) : X(S) \to X(S')$.

A quasicoherent sheaf $\mathcal{F} \in \mathrm{QCoh}(S)$ is the data of a compatible collection of $\mathrm{QCoh}(S)$-valued local systems $\mathcal{F}_S$ on the spaces $X(S)$ for each affine scheme S. That is, (a) for every $x \in X(S)$, the stalk $\mathcal{F}_x \in \mathrm{QCoh}(S)$ is a quasicoherent complex on S and (b) for every path in $X(S)$ from x to y, we have a map $\mathcal{F}_x \to \mathcal{F}_y$ and (c) for every 2-simplex in $X(S)$, we have a degree -1 map with prescribed boundary and so on.

Denote by $X(f)^{-1}$ the natural pullback of local systems from $X(S')$ to $X(S)$ and $f^* : \mathrm{QCoh}(S') \to \mathrm{QCoh}(S)$ the quasicoherent pullback; for the compatibility we require an equivalence $X(f)^{-1}\mathcal{F}_{S'} \simeq f^*\mathcal{F}_S$ of $\mathrm{QCoh}(S')$-valued local systems over $X(S)$, i.e. for all $x \in X(S)$, we have equivalences $\mathcal{F}_{X(f)(x)} \simeq f^*\mathcal{F}_x$. One should also account for higher data, but we will not do so here.

Proof. By the above definition, it is clear that $\mathrm{QCoh}(X)$ has a G-action. A G-action on X is a collection of compatible G-actions on $X(S)$ for all $S \in \mathbf{Aff}$, which evidently induces a map from G to automorphisms of the category of $\mathrm{QCoh}(S)$-valued local systems on $X(S)$, which defines a G-action on $\mathrm{QCoh}(S)$. Furthermore, we define the category of *perfect complexes* $\mathrm{Perf}(X)$ to be the compact objects of $\mathrm{QCoh}(X)$. Since $\pi_0(G)$ acts by equivalences, its action preserves compact objects, so the G-action restricts to $\mathrm{Perf}(X)$. Finally, we define $\mathcal{O}_X \in \mathrm{QCoh}(X)$ to be the assignment for $S \in \mathbf{Aff}$ of the constant local system on $X(S)$ with

[12]For example, when $S = \mathrm{Spec}(A)$ for a connective dg algebra or simplicial algebra A, we define $\mathrm{QCoh}(S)$ to be (a) if A is a dg algebra, the dg derived category of A-modules (i.e. the localization of the dg category of A-modules with respect to quasi-isomorphisms) or (b) if A is a simplicial algebra, the stabilization of the (simplicially enriched) category of simplicial A-modules localized with respect to weak equivalences. These two incarnations of quasicoherent sheaves are Quillen equivalent by the Dold-Kan correspondence.

value $\mathcal{O}_S \in \mathrm{QCoh}(S)$. It is clear that $\mathcal{O}_S$ is canonically a fixed point, and since the G-action is already canonically trivial, it has a G-equivariant structure. □

Remark 2.6.9. If X is a classical stack (i.e. a functor $\mathbf{Aff}^{cl} \to \mathbf{S}$), it can be considered as a derived stack via a Kan extension. Explicitly, if S is a derived algebra, we have

$$X(S) := \lim_{\substack{S' \to S \\ S' \text{ classical}}} X(S') = X(\pi_0(S')).$$

By this description, it is clear that for a classical stack X, $\mathrm{QCoh}(X)$ is determined by its pullbacks to classical schemes S.

Example 2.6.10. Consider $X = S^1$ as the locally constant derived stack with value $S^1 \in \mathbf{S}$. The category $\mathrm{QCoh}(S^1)$ assigns to every connected $S \in \mathbf{Aff}$ the space $X(S) = S^1$. A $\mathrm{QCoh}(S)$-valued local system on S^1 is equivalent to a sheaf on S^1 with a distinguished automorphism. Altogether, we have that an object of $\mathrm{QCoh}(S^1)$ consists of a k-vector space along with an invertible linear map, i.e. a $\mathbb{Z}$-representation. This is expected as $S^1 = B\mathbb{Z}$. See also Lemma 3.17 of [6] (which does not apply to the present example as S^1 is not simply connected).

Remark 2.6.11. For X a stack with a G-action, the G-equivariant structure on $\mathcal{O}_X$ is always canonically trivial, but the G-equivariant structure on $\mathcal{O}(X)$ may be nontrivial. For example, take $X = \mathcal{L}(B\mathbb{G}_a) \simeq \mathbb{G}_a \times B\mathbb{G}_a$; the S^1-equivariant structure on $\mathcal{O}(\mathcal{L}(B\mathbb{G}_a)) \simeq \mathcal{O}(\mathbb{G}_a) \oplus \Omega^1(\mathbb{G}_a)[-1]$ is given by contraction with the Euler vector field.

Example 2.6.12. Let $G = \mathbb{Z}$. A $\mathbb{Z}$-action on a k-linear dg category $\mathbf{C}$ is the data of an autofunctor $F : \mathbf{C} \to \mathbf{C}$. Objects in the category of invariants $\mathbf{C}^{\mathbb{Z}}$ are pairs (X, α) where $X \in \mathbf{C}$ is an underlying object and $\alpha_X : F(X) \simeq X$ is an isomorphism; morphisms are given by the usual intertwiners.

2.7 Circle actions and mixed complexes

Remark 2.7.1. Let $G = S^1$; an S^1-action on a k-linear dg category $\mathbf{C}$ is given by a map of ∞-groups $S^1 \to \mathbf{Aut}(\mathbf{C})$. We often seek a smaller, more explicit "formal" model. Theorem 5.2.0.4 of Preygel's thesis [53] allows us to do just this; in particular, a S^1-action on a dg category $\mathbf{C}$ is determined up to equivalence by a $k[z, z^{-1}]-$ linear structure on $\mathbf{C}$, i.e. a universal automorphism of the identity functor (which is just the assignment of a universal automorphism α_X for each object $X \in \mathbf{C}$). The objects of the equivariant category $\mathbf{C}^{S^1}$ can naively be thought of as pairs (X, ϵ_X) where $X \in \mathbf{C}$ is the underlying object and ϵ_X is a nullhomotopy of $1 \simeq \alpha_X$. We remain vague about what form this homotopy might take, but will provide examples below, and refer the reader to Lemma 5.2.0.6 of [53].

Example 2.7.2 (Matrix factorizations)**.** This example is discussed in Section 6 of [52]. Let M be a smooth classical scheme. By Lemma 6.1.1 of *loc. cit.* , the set of S^1-actions on $\mathrm{Coh}(M) = \mathrm{Perf}(X)$ are in bijection with $\mathbb{G}_m(M) = \Gamma(M, \mathcal{O}_M)^\times$, i.e. $f \in \mathbb{G}_m(M)$ acts by multiplication by f on any given sheaf. An object of $\mathrm{Coh}(M)^{S^1}$ or $\mathrm{Perf}(M)^{S^1}$ is a coherent complex of sheaves $\mathcal{F}^\bullet$ along with a nullhomotopy of the endomorphism $1 - f$; an object admits an S^1-equivariant structure if and only if its cohomology sheaves are supported on the zeroes of $1 - f$, i.e. it is in the essential image of $\mathrm{Coh}(M_1)$ under pushforward. Such an object can have multiple equivariant structures.

Example 2.7.3. Let $\mathbf{C} = k\text{-perf} = \mathbf{Vect}_k^\omega$, the category of finite-dimensional chain complexes over k; S^1-actions on $\mathbf{C}$ are in correspondence with $k^\times$. Only the S^1-action corresponding to $1 \in \mathbb{G}_m$, i.e. the trivial action, admits equivariant objects; any other S^1-action results in $(\mathbf{Vect}_k^\omega)^{S^1} = 0$.

Taking the trivial S^1-action, by Remark 2.6.2, $(\mathbf{Vect}^\omega)^{S^1} \simeq C_\bullet(S^1; k)\text{-mod}$. We can replace this with a smaller model; in a general situation, given a quasi-isomorphism $A \to B$ of dg algebras (or small dg categories), the adjoint functors $(- \otimes_B A, \mathrm{Res}_B^A)$ form a Quillen equivalence. In the case of a circle $G = S^1$, the algebra of chains $A = C_\bullet(S^1; k)$ is formal, so we can take $B = H_\bullet(S^1; k) \simeq k[\epsilon]$ where $|\epsilon| = -1$ is a choice of generating 1-chain.

Definition 2.7.4. Let $\mathbf{C} \in \mathbf{Pr}_k^L$ be a compactly generated k-linear stable ∞-category, equipped with an S^1-action. We define

$$\mathbf{C}^{\omega S^1} := \mathrm{Ind}((\mathbf{C}^\omega)^{S^1}),$$
$$\mathbf{C}^{\omega\,\mathrm{Tate}} := \mathbf{C}^{\omega S^1} \otimes_{(k\text{-perf})^{\omega S^1}} k((u))\text{-mod}\,.$$

Remark 2.7.5. We make the above definition because in general, taking G-invariants may not commute with ind-completion: $\mathbf{C}^{S^1} \not\simeq \mathrm{Ind}((\mathbf{C}^\omega)^{S^1})$. For example, take $\mathbf{C} = \mathbf{Vect}_k$ with the trivial S^1-action. The Koszul duality functor discussed in Section 3 of [52] defines two different equivalences (the f subscript indicates chain complexes with finite-dimensional total cohomology):

$$k[[u]]\text{-perf} \underset{R\,\mathrm{Hom}_{k[\epsilon]}(k,-)}{\overset{k\otimes^L_{k[[u]]}-}{\rightleftarrows}} k[\epsilon]\text{-mod}_f,$$

$$k[[u]]\text{-mod}_f \underset{k\otimes^L_{k[\epsilon]}-}{\overset{R\,\mathrm{Hom}_{k[[u]]}(k,-)}{\rightleftarrows}} k[\epsilon]\text{-perf}\,.$$

Taking ind-completions, we have

$$k[[u]]\text{-mod} \underset{R\,\mathrm{Hom}_{k[\epsilon]}(k,-)^\vee}{\overset{k\otimes^L_{k[[u]]}(-)^\vee}{\rightleftarrows}} \mathrm{Ind}(k[\epsilon]\text{-mod}_f),$$

$$k[[u]]\text{-mod}_{tors} \underset{k\otimes^L_{k[\epsilon]}-}{\overset{R\,\mathrm{Hom}_{k[[u]]}(k,-)}{\rightleftarrows}} k[\epsilon]\text{-mod}\,.$$

Let us illustrate this with some explicit examples. Let M be the augmentation module for $k[\epsilon]$. It is coherent (i.e. finite) but not perfect. Under the first Koszul duality equivalence, it is sent to the free module $k[[u]]$. Under the ind-completion of the second equivalence, writing k as the colimit of $C_\bullet(S^1;k)$-modules $C_\bullet(S^{2n+1};k)$, it is sent to the module of distributions $\mathrm{colim}_n(k[u]/u^n)^* = k((u))/k[[u]]$.

Or, let $N = \mathrm{colim}_n\, k[-n]$ be a presentation via a colimit of coherent modules of the $k[\epsilon]$-complex (where the maps are multiplication by ϵ)

$$\cdots \to k[\epsilon][2] \to k[\epsilon] \to k[\epsilon][-2] \to \cdots\,.$$

Its image under the first equivalence is $\mathrm{colim}_n\, k[[u]][n] \simeq k((u))$, whereas its image under the second equivalence is zero.

In terms of S^1-actions, the first Koszul duality equivalence above realizes

$$\mathbf{Vect}_k^{\omega S^1} := \mathrm{Ind}((\mathbf{Vect}_k^{\omega})^{S^1}) \simeq k[[u]]\text{-mod}$$

and the second realizes

$$\mathbf{Vect}_k^{S^1} \simeq k[[u]]\text{-mod}_{tors}\,.$$

Applying the Tate construction, we find that $\mathbf{Vect}_k^{\omega\,\mathrm{Tate}} \simeq k((u))\text{-mod}$, whereas $\mathbf{Vect}_k^{\mathrm{Tate}} \simeq 0$.

We are generally interested in taking S^1-invariants at the level of compact objects and small categories. On the other hand, working in this setting is somewhat restrictive, since many functors will not preserve compact objects. A workaround is discussed in Section 4 of [51] via t-structures.

Definition 2.7.6. A *mixed complex* (V, d, ϵ) is a chain complex V with internal differential d (of degree 1) and a k-linear map $\epsilon : V \to V$ of degree -1 such that $d(\epsilon) = 0$ (i.e. $d\epsilon + \epsilon d = 0$) and $\epsilon^2 = 0$. Equivalently, a mixed complex is a $H_\bullet(S^1;k)$-module where $H_\bullet(S^1;k) \simeq k[\epsilon]$ is the commutative dg algebra generated by a single generator in degree -1. We consider a mixed complex as an object of $\mathbf{Vect}_k^{\omega S^1} \simeq \mathrm{Ind}(k[\epsilon]\text{-mod}_f)$, i.e. as a filtered colimit of its finite subcomplexes.

Definition 2.7.7. The S^1*-invariants* of a mixed complex (V, d, ϵ) are defined by

$$V^{S^1} = R\,\mathrm{Hom}_{C_\bullet(S^1;k)}(C_\bullet(ES^1;k), V) \simeq R\,\mathrm{Hom}_{k[\epsilon]}(k, V) = (V[[u]], d + u\epsilon)$$

where, given a space X with an S^1-action, the action of $C_\bullet(S^1;k)$ on $C_\bullet(X;k)$ is by the sweep action of [26]. It can be expressed as a limit

$$V^{S^1} = \lim_n R\,\mathrm{Hom}_{C_\bullet(S^1;k)}(C_\bullet(S^{2n+1}/S^1;k), V) \simeq \lim_n (V[u]/u^n, d + u\epsilon) = (V[[u]], d + u\epsilon)$$

where here we choose a presentation of the homotopy quotient

$$BS^1 = ES^1/S^1 \simeq S^\infty/S^1 \simeq \operatorname{colim}_n S^{2n+1}/S^1 \simeq \operatorname{colim}_n(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* \simeq \operatorname{colim}_n \mathbb{CP}^{n+1}$$

as a colimit of ordinary quotients by free actions of S^1 on odd spheres. Since

$$R\operatorname{Hom}_{C_\bullet(S^1;k)}(k,k) \simeq C^\bullet(BS^1;k)^{op},$$

V^{S^1} is naturally a module over $C^\bullet(BS^1;k) = k[[u]]$ where $|u| = 2$.

Definition 2.7.8. The *Tate construction* on (V, d, ϵ) is defined by

$$V^{\mathrm{Tate}} = V^{S^1} \otimes_{k[[u]]} k((u)) = (V((u)), d + u\epsilon).$$

Remark 2.7.9. The S^1-invariants of a mixed complex V^{S^1} are an object of $\mathbf{Vect}^{\omega S^1}$. On the other hand, the S^1-coinvariants V_{S^1} are an object of $\mathbf{Vect}^{S^1}$; we expect this since coinvariants commutes with ind-completion.

2.8 Hochschild homology and cyclic homology

In this section we give a brief overview of the basic definitions of Hochschild homology and cyclic homology for convenience, for the most part following [8]. A discussion of circle actions can be found in Section 6 of [52]. Further discussion of circle actions can be found in [51]

Definition 2.8.1. Let $\mathbf{Cat}_\otimes$ be a symmetric monoidal ∞-category with monoidal unit $1_\otimes$, and $X \in \mathbf{Cat}_\otimes$ a 1-dualizable object with dual $X^\vee$, coevaluation $\eta : 1_\otimes \to X \otimes X^\vee$ and evaluation $\epsilon : X^\vee \otimes X \to 1_\otimes$. We define the *dimension* of X by

$$\dim(X) = \epsilon \circ \eta \in \operatorname{End}_{\mathsf{C}_\otimes}(1_\otimes).$$

and if ϕ is an endomorphism of X, we define its *trace* to be

$$\operatorname{tr}(\phi) = \epsilon \circ (\phi \otimes 1) \circ \eta \in \operatorname{End}_{\mathsf{C}_\otimes}(1_\otimes).$$

Note that $\dim(X) = \operatorname{tr}(\mathrm{id}_X)$. If $F : X \to Y$ is a morphism with a right adjoint G, then we can define

$$\dim(F): \ \dim(X) \xrightarrow{\gamma} \operatorname{tr}(G \circ F) \xrightarrow{\simeq} \operatorname{tr}(F \circ G) \xrightarrow{\upsilon} \dim(Y)$$

where γ is the counit of the adjunction, υ the unit, and the middle isomorphism induced by cyclic symmetry of traces.

Remark 2.8.2. Equivalently, using Lurie's proof of the Cobordism Hypothesis [46], there is an equivalence between n-dualizable objects $X \in \mathbf{Cat}_{\otimes}$ and framed extended $\mathbf{Cat}_{\otimes}$-valued n-dimensional topological field theories $\mathcal{Z}_X$; for a 1-dualizable object X we define the dimension by

$$\dim(X) = \mathcal{Z}_X(S^1).$$

By this definition, there is evidently an S^1-equivariant structure on $\dim(X)$.

We will define the Hochschild homology of a category to be its dimension; we first need to define a monoidal structure on ∞-categories.

Definition 2.8.3 (Lurie tensor product)**.** The category $\mathbf{Pr}^L$ is equipped with a monoidal structure called the *Lurie tensor product*, constructed in Section 4.8 of [44]. It can be thought of as an ∞-analogue of the Deligne tensor product. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Pr}^L$; we denote the Lurie tensor product by $\mathbf{C} \otimes \mathbf{D}$, and it is equipped with a canonical functor

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$$

$$(X, Y) \mapsto X \boxtimes Y.$$

It satisfies the universal property that it is initial amongst functors out of $\mathbf{C} \times \mathbf{D}$ which preserves small colimits in each variable. Proposition 4.8.1.17 of *loc. cit.* gives an explicit realization

$$\mathbf{C} \otimes \mathbf{D} \simeq \mathbf{Fun}^R(\mathbf{C}^{op}, \mathbf{D})$$

which is again a presentable category by Lemma 4.8.1.16. In particular, by [45] Proposition 5.5.3.8, the Lurie tensor product makes $\mathbf{Pr}^L$ into a closed monoidal category with internal mapping object $\mathbf{Fun}^L(-,-)$. Furthermore, by Propositions 4.8.2.10 and 4.8.2.18 in [44], the Lurie tensor product induces a tensor product on k-linear presentable categories $\mathbf{Pr}^L_k$.

The following is proven in Theorem D.7.0.7 in [47] and Chapter I.1 Proposition 7.3.2 [23].

Proposition 2.8.4. *If* $\mathbf{C} \in \mathbf{Pr}^L_k$ *is compactly generated, then it is dualizable. In particular, if* $\mathbf{C} = \mathrm{Ind}(\mathbf{C}^0)$*, then* $\mathbf{C}^\vee = \mathrm{Ind}(\mathbf{C}^{0,op})$*, and the evaluation map is given by ind-completion via universal properties of the Yoneda pairing* $\mathrm{Hom}(-,-) : \mathbf{C}^{0,op} \times \mathbf{C} \to \mathbf{Vect}_k$*. Furthermore, we have isomorphisms*

$$\mathbf{Fun}^L_k(\mathbf{C}, \mathbf{C}) \simeq \mathbf{Fun}^L_k(\mathbf{C}, \mathbf{Vect}_k) \otimes \mathbf{C} \simeq \mathbf{C}^\vee \otimes \mathbf{C}.$$

Remark 2.8.5 (Functoriality)**.** Morphisms in $\mathbf{Pr}^L_k$ between dualizable categories admitting continuous right adjoints are exactly those left adjoints which also preserve (ω-)compact objects, i.e. functors coming from a functor between small stable k-linear ∞-categories.

Remark 2.8.6. The dimension of a dualizable category takes values in chain complexes, i.e.

$$\mathbf{End}_{\mathrm{Cat}_\otimes}(1_\otimes) = \mathbf{Fun}^L_k(\mathbf{Vect}_k, \mathbf{Vect}_k) \simeq \mathbf{Vect}_k.$$

That is, every such endofunctor F commuting with colimits is determined by its value $F(k)$ since every vector space can be written as a colimit (possibly over a large cardinal) of the one-dimensional vector space k.

Definition 2.8.7. Let $\mathbf{C} \in \mathbf{st}_k$ be a small stable k-linear ∞-category. We define the *Hochschild homology* functor

$$HH := \dim \circ \mathrm{Ind} : \mathbf{st}_k \to \mathbf{Fun}^L_k(\mathbf{Vect}_k, \mathbf{Vect}_k) \simeq \mathbf{Vect}_k$$

i.e. $HH(\mathbf{C})$ is the image of k under the composition

$$\mathbf{Vect}_k \xrightarrow{\mathrm{coev}} \mathrm{Fun}^L_k(\mathbf{C}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{C}^\vee \otimes \mathbf{C} \xrightarrow{\mathrm{ev}} \mathbf{Vect}_k.$$

Let $\mathbf{C} \in \mathbf{Pr}^{L,\omega}_{k,\otimes}$ be a dualizable presentable stable k-linear ∞-category, and consider only functors in $\mathbf{Pr}^L_k$ which preserve (ω-)compact objects. We define the *Hochschild homology* functor

$$HH := \dim : \mathbf{Pr}^{L,\omega}_{k,\otimes} \to \mathbf{Fun}^L_k(\mathbf{Vect}_k, \mathbf{Vect}_k) \simeq \mathbf{Vect}_k.$$

Note that if $\mathbf{C}$ is compactly generated, then $HH(\mathbf{C}) = HH(\mathbf{C}^\omega)$; there is no ambiguity, and the latter definition is strictly more general.

We define the *(negative) cyclic homology*[13] and *periodic cyclic homology* by

$$HC(\mathbf{C}) := HH(\mathbf{C})^{S^1}$$

$$HP(\mathbf{C}) := HH(\mathbf{C})^{\mathrm{Tate}} := HC(\mathbf{C})^{S^1} \otimes_{C^\bullet(BS^1;k)} C^\bullet(BS^1;k)_{loc}$$

where $C^\bullet(BS^1;k)_{loc} \simeq k((u))$ is obtained by inverting the Chern class $|u| = 2$.

Example 2.8.8 (Algebraic examples)**.** Let A be a dg algebra (or more generally, a dg category) over k, and $\mathbf{C} = A$-mod the category of left dg-modules over A. By the dg Morita theory of [62], continuous functors A-mod $\to B$-mod are given by objects of $A^{op} \otimes B$-mod. The coevaluation k-mod $\to A \otimes A^{op}$-mod is given the identity functor $A \otimes_k -$, where A is considered as a bimodule over itself, and the evaluation map is given by $- \otimes_{A \otimes A^{op}} A$. In particular, the Hochschild homology is given by the usual Hochschild homology

$$HH(A\text{-mod}) = HH(A;A) = A \otimes_{A \otimes A^{op}} A$$

and similarly for cyclic and periodic cyclic homology.

[13] Historically, this has been referred to as the negative cyclic homology. We opt not to use this term due possible confusion owing to to differences in our grading conventions.

Remark 2.8.9 (Cyclic bar complex)**.** Applying the above example to a small dg category recovers the cyclic bar complex of [35], i.e. the chain complex of the simplicial set $C^\bullet(\mathbf{C})$ whose n-simplices are given by

$$C^n(\mathbf{C}) = \coprod_{X_0,\dots,x_n \in \mathrm{Ob}(\mathrm{C})} \mathrm{Hom}_{\mathrm{C}}(X_0, X_n) \otimes \mathrm{Hom}_{\mathrm{C}}(X_n, X_{n-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathrm{C}}(X_1, X_0)$$

where the face maps are given by composition and the degeneracy maps by the identity homomorphism as usual. The Connes B operator can be defined in the usual way. One can obtain smaller models by taking objects from a set of compact generators (see Theorem 5.2 of [35]) rather than all of $\mathrm{Ob}(\mathbf{C})$; for example, if $\mathbf{C} = A\text{-perf}$, then the free module A is a compact generator and one recovers the classical cyclic bar complex $C^\bullet(A; A)$.

Example 2.8.10. Let A be a regular commutative ring over k. Then, the Hochschild-Kostant-Rosenberg theorem identifies $HH(A; A) \simeq \Omega^{-\bullet}_{A/k}$, and the mixed complex structure is given by the de Rham differential $B = d_{dR}$. Thus, $HP(A\text{-mod}) = H^\bullet_{dR}(X; k)((u))$.

Example 2.8.11 (Geometric examples)**.** By [5], when X is a perfect stack (e.g. a quotient stack of a derived quasiprojective scheme by an affine group in characteristic zero), then $\mathrm{QCoh}(X)$ is compactly generated by $\mathrm{Perf}(X)$ and we have isomorphisms

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \simeq \mathrm{Fun}^L_k(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

where the functors on the right are given by integral transforms. Explicitly, we identify

$$\mathrm{QCoh}(X) \xrightarrow{\simeq} \mathrm{QCoh}(X)^\vee$$

on compact objects $\mathcal{K} \in \mathrm{Perf}(X)$ by

$$\mathcal{K} \mapsto \Gamma(X, \mathcal{K} \otimes -) \simeq \Gamma(X, \mathcal{H}om_X(\mathcal{K}^\vee, -)).$$

Letting $p : X \to \mathrm{Spec}(k)$ be the map to a point and $\Delta : X \to X \times X$ the diagonal, the coevaluation is given by the functor $\Delta_* p^*$ and the evaluation by $p_* \Delta^*$. In particular, we find that the Hochschild homology is

$$HH(\mathrm{Perf}(X)) \simeq p_* \Delta^* \Delta_* p^* k = \Gamma(X, \Delta^* \Delta_* \mathcal{O}_X) \simeq \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) = \mathcal{O}(\mathcal{L}X)$$

with the last isomorphism arising via base change.

Remark 2.8.12. If X is QCA but not perfect, then $\mathrm{QCoh}(X)$ is dualizable, so that

$$HH(\mathrm{QCoh}(X)) = \mathcal{O}(\mathcal{L}X)$$

by a similar argument. It is unclear, in this case, how to compute $HH(\mathrm{Perf}(X))$, and whether it agrees with $HH(\mathrm{QCoh}(X))$.

Remark 2.8.13 (Monoidal structure in Hochschild homology)**.** Let $\mathbf{C}$ be a small monoidal ∞-category, i.e. an algebra object in the category of ∞-categories. In particular, there is a functor

$$\mathbf{C} \otimes \mathbf{C} \to \mathbf{C}$$

which is continuous in each factor. Since the categories $\mathbf{C}$ are small, the monoidal product preserves compact objects on ind-completions, and we obtain a lax monoidal structure on Hochschild homology

$$HH(\mathbf{C}) \otimes HH(\mathbf{C}) \to HH(\mathbf{C})$$

defining an algebra structure on $HH(\mathbf{C})$. In fact, by Theorem 4.2.5 of [38], this map is an isomorphism.

Example 2.8.14 (Two different monoidal structures)**.** Take A to be any associative dg-algebra, and consider $A \otimes A^{op}$-mod, the category of A bi-modules. This is a monoidal category under convolution, even if A is noncommutative (and therefore A-mod has no monoidal structure). Then, $HH((A \otimes A^{op})\text{-mod})$ is an algebra object in chain complexes.

For example, take $A = k^n$. There are two monoidal structures on $A \otimes A^{op}$-mod. The first is just by pointwise multiplication (which comes from the tensor product monoidal structure on modules over the commutative ring k^n), and the other by convolution (i.e.composition). As a dg vector space, it is a simple calculation to compute the Hochschild homology $\mathrm{HH}(A \otimes_k A) = k^{n^2}$. It has two multiplicative structures: one is pointwise multiplication corresponding to the tensor structure on k^n-mod, and the other is matrix multiplication coming from convolution.

2.9 Hochcschild homology is a localizing invariant

The results in this section have, in some form or another, been documented in the literature, for example in Section 1.5 in [2], Section 5.2.7 of [45], [36] or Section 5 of [12].

Definition 2.9.1 (Localization in stable presentable ∞-categories)**.** Let $\mathbf{B}$ be a presentable stable ∞-category, and $\mathbf{C} \subset \mathbf{B}$ a full presentable (i.e. closed under direct sums) subcategory such that the inclusion has a left adjoint ℓ called the *localization functor*, which is automatically continuous. Let $\mathbf{A} = {}^{\perp}\mathbf{C}$ be the left orthogonal, i.e. the full subcategory consisting of $X \in \mathbf{C}$ such that

$$\mathrm{Hom}_{\mathrm{B}}(X, C) = 0 \quad \forall C \in \mathbf{C} \Rightarrow X \simeq 0.$$

The adjunction allows us to rephrase this: the left orthogonal consists of $X \in \mathbf{C}$ such that $\mathrm{Hom}_{\mathrm{C}}(\ell(X), C) = 0$ for all $C \in \mathbf{C}$, i.e. such that $\ell(X) \simeq 0$.

The inclusion of **A** into **B** is automatically continuous (being a left orthogonal) and thus admits a right adjoint *colocalization*[14] c, which one can prove (see Lemma 3.1.5 in [2]) is continuous. Thus we have a diagram

$$\mathbf{A} \underset{c}{\overset{i}{\rightleftarrows}} \mathbf{B} \underset{j}{\overset{\ell}{\rightleftarrows}} \mathbf{C}$$

which we call a *short exact sequence of dg categories*. Note that all functors here are continuous (j is by assumption).

Definition 2.9.2 (Exact sequences in small stable ∞-categories)**.** Let $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ be a sequence of small stable ∞-categories which are idempotent complete. We say that the sequence is *exact* if the ind-completion is exact in the above sense. Note that the ind-completion of all functors preserve compact objects by construction, and are continuous by construction, and therefore by the adjoint functor theorem they all admit continuous right adjoints, recovering the localization diagram above. We say the sequence is *split exact* the functors admit right adjoint splittings; a split exact sequence is also known as a semiorthogonal decomposition, where the right orthogonal to **A** is realized as a subcategory of **B** by the right adjoint to the quotient functor $\mathbf{B} \to \mathbf{C}$.

Definition 2.9.3. Let $E : \mathbf{st}_k \to \mathbf{Vect}_k$ be a functor from k-linear stable ∞-categories to dg k-vector spaces. We say that E is an *additive invariant* if sends split exact sequences (i.e. semiorthogonal decompositions) to exact triangles. We say that E is a *localizing invariant* if it takes exact sequences to exact triangles.

Example 2.9.4. Let $\mathbf{B} = \mathrm{Coh}(\mathbb{P}^1)$, the derived category of coherent sheaves on $\mathbb{P}^1$. We will use the notation $\langle X_1, \dots, X_r\rangle$ to denote the subcategory generated by the objects X_i. It is well known that the following is a semi-orthogonal decomposition

$$\langle \mathcal{O}_{\mathbb{P}^1}(-1)\rangle \to \mathrm{Coh}(\mathbb{P}^1) \to \langle \mathcal{O}_{\mathbb{P}^1}\rangle$$

whereas, given a choice of point $x \in \mathbb{P}^1$ and skyscraper sheaf k_x,

$$\langle k_x\rangle \to \mathrm{Coh}(\mathbb{P}^1) \to \mathrm{Coh}(\mathbb{P}^1 - \{x\})$$

is not, since the pushforward functor $j : \mathrm{QCoh}(\mathbb{P}^1 - \{x\}) \to \mathrm{QCoh}(\mathbb{P}^1)$ does not preserve compact objects.

[14] This nomenclature might be confusing. We say the data described, i.e. a fully faithful functor i with a right adjoint and a functor ℓ with a fully faithful right adjoint, is a localization sequence. A colocalization sequence is a localization sequence where we replace the word "right" with "left." The functors j and c define a "dual" colocalization sequence. The usual recollement diagram is obtained if the localization sequence depicted is also a colocalization sequence, i.e. if i has a left adjoint and if j has a fully faithful left adjoint.

Example 2.9.5. Let X be a QCA stack, Z a closed substack (necessarily representable by schemes), and $U = X - Z$. In general, there is a non-split exact sequence

$$\mathrm{Coh}_Z(X) \to \mathrm{Coh}(X) \to \mathrm{Coh}(U)$$

where $\mathrm{Coh}_Z(X)$ denotes coherent sheaves on X supported on Z.

Remark 2.9.6. The adjunction unit $\ell \circ j \to 1_{\mathsf{C}}$ and counit $1_{\mathsf{A}} \to c \circ i$ are isomorphisms. On the other hand, the compositions $j \circ \ell$ and $i \circ c$ are idempotent, and we have maps

$$\eta : 1_{\mathsf{B}} \to j \circ \ell$$

$$\epsilon : i \circ c \to 1_{\mathsf{B}}$$

such that

$$i \circ c \to 1_{\mathsf{B}} \to j \circ \ell$$

is an exact triangle in $\mathbf{End}^L(\mathbf{B})$. In particular, $j \circ \ell = \mathrm{cone}(\epsilon)$ and $i \circ c[-1] = \mathrm{cone}(\eta)$.

Example 2.9.7. Take $X = \mathbb{A}^1$ and $Z = \{0\}$, so $U = \mathbb{G}_m$. The functor $i \circ c$ is the local cohomology functor, and the functor $j \circ \ell$ is the restriction and then pushforward. Evaluating the above exact triangle in $\mathbf{End}^L(\mathrm{Coh}(\mathbb{A}^1))$ on the sheaf $M = k[x]$, we obtain the exact triangle

$$k((x))/k[[x]][-1] \to k[x] \to k[x, x^{-1}].$$

Rotating the triangle, we obtain the following exact sequence

$$k[x] \to k[x, x^{-1}] \to k[x, x^{-1}]/k[x]$$

Lemma 2.9.8. *Let $F \to G \to H$ be an exact sequence of endofunctors of a dualizable k-linear presentable ∞-category $\mathbf{C}$. Then, we have an exact triangle in $\mathbf{Vect}_k$*

$$\mathrm{tr}(F) \to \mathrm{tr}(G) \to \mathrm{tr}(H)$$

where tr *is the monoidal trace in $\mathbf{Pr}^L_k$.*

Proof. By Theorem 3.8 and Lemma 3.3 (which shows that triangulated categories arising from dg categories are algebraic) of [35], there is a small dg-category $\mathbf{R}$ such that $\mathbf{C} \simeq \mathbf{R} - \mathbf{mod}$, and by Corollary 3.3 of [12], the short exact sequence $F \to G \to H$ corresponds to a short exact sequence of $\mathbf{R}$-bimodules $M \to N \to P$. Applying the exact functor $- \otimes_{\mathbf{R} \otimes^L \mathbf{R}^{op}} \mathbf{R}$ preserves exactness of this triangle. □

Proposition 2.9.9. *Let $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ be a localization sequence of ∞-categories; by definition the arrows are continuous (since they have right adjoints). Then,*

$$HH(\mathbf{A}) \to HH(\mathbf{B}) \to HH(\mathbf{C})$$

is a (split) exact triangle.

Proof. Let us adopt the notation above. The sequence of maps on Hochschild homology is given by

$$\mathrm{tr}(1_{\mathsf{A}}) \to \mathrm{tr}(c \circ i) \simeq \mathrm{tr}(i \circ c) \to \mathrm{tr}(1_{\mathsf{B}}) \to \mathrm{tr}(j \circ \ell) \simeq \mathrm{tr}(\ell \circ j) \to \mathrm{tr}(1_{\mathsf{C}})$$

Since $1_{\mathsf{A}} \to c \circ i$ and $\ell \circ j \to 1_{\mathsf{C}}$ are isomorphisms, we only have to show that the sequence of endofunctors on **B**

$$\mathrm{tr}(i \circ c) \to \mathrm{tr}(1_{\mathsf{B}}) \to \mathrm{tr}(j \circ \ell)$$

is exact. The proposition follows from the previous lemma. □

Corollary 2.9.10. *There is an S^1-equivariant map of algebras*

$$\mathrm{ch} : K_\bullet(\mathbf{C}) \to HH(\mathbf{C})$$

where $K_\bullet(\mathbf{C})$ is endowed with the trivial S^1-action. This map is functorial in the sense that it is a morphism between the functors $K, HH : \mathbf{st}^{ex} \to \mathbf{Mod}_k$.

Proof. First, let us prove that Hochschild homology is a localizing invariant in the language of [12], where it is shown that non-connective K-theory is the universal spectra-valued localizing invariant of small stable ∞-categories (though we will work in the dg setting). A functor from the category of stable ∞-categories to spectra (in our case, chain complexes over k) is a *localizing invariant* if (a) it inverts Morita equivalences, (b) preserves filtered colimits and (c) sends exact sequences to cofiber sequences. The previous proposition proves property (c). We defined Hochschild homology using large ind-completed presentable categories in $\mathbf{Pr}^L$ rather than their small categories in $\mathbf{st}^{ex}$; the Hochschild homology of a small stable category is defined to be the Hochschild homology of its ind-completion. In particular, since the ind-completion of a category is equivalent to the ind-completion of its idempotent completion, and idempotent completions are fibrant replacements under the Morita model structure of dg categories, (a) is automatic, and (b) is a consequence of the fact that Hochschild homology can be computed via a categorical cyclic nerve complex, which is built from tensor products, which commute with filtered colimits.

To complete the argument, by Theorem 1.1 in [12], there is a factorization

$$\begin{array}{ccc} \mathbf{st}^{ex} & \xrightarrow{U} & \mathbf{M}_{loc} \\ & \searrow^{HH} & \downarrow^{HH'} \\ & & \mathbf{Mod}_k \end{array}$$

Furthermore, the initial object $\mathrm{QCoh}(\mathrm{pt}) \in \mathbf{st}^{ex}$ is mapped to an initial object under U, since U preserves filtered colimits, and by Theorem 1.3 in *loc. cit.* , this initial object corepresents the nonconnective K-theory functor $K : \mathbf{st}^{ex} \to \mathrm{QCoh}(\mathrm{pt})$. To see that the map is a map of algebras, note Theorem 5.8 in [13] and the earlier observation that Hochschild homology is lax monoidal (in fact, strict symmetric monoidal). □

Remark 2.9.11. The difference between Hochschild homology of small categories versus large categories plays a role when considering the trace from K-theory. In particular, K-theory is a universal invariant for small stable ∞-categories, not large (presentable) categories. The Chern character induces a natural transformation

$$K(\mathbf{C}^0) \to HH(\mathrm{Ind}(\mathbf{C}^0))$$

where $\mathbf{C}^0$ is a small stable ∞-category. If we take instead a presentable stable ∞-category $\mathbf{C}$, unless we know it is compactly generated, it does not necessarily make sense to talk about the trace as above. For example, it is not known in general for X a QCA stack that $\mathrm{QCoh}(X)$ is compactly generated by $\mathrm{Perf}(X)$ (or compactly generated at all). On the other hand, in this note we are largely interested in cases where X is a quotient stack of a quasiprojective scheme by an affine group, for which $\mathrm{QCoh}(X)$ is known to be compactly generated by $\mathrm{Perf}(X)$.

Remark 2.9.12 (Monoidal structures in K-theory)**.** Following the discussion in [13], the (connective and non-connective) K-theory spectrum of a stable monoidal ∞-category is lax but not strongly monoidal, i.e. there is a map

$$K(\mathbf{C}) \otimes K(\mathbf{C}) \to K(\mathbf{C})$$

which is not an equivalence.

Remark 2.9.13 (Cyclic homology is not a localizing invariant)**.** On the other hand, (negative) cyclic homology and periodic cyclic homology are *not* localizing invariants, since their constructions involve an infinite limit which may not commute with filtered colimits. However, the formation of S^1 and Tate invariants on a mixed complex are exact, so they still send localization sequences to exact triangles, and also invert Morita equivalences. Further, the trace map to Hochschild homology induces a trace map to both cyclic and periodic cyclic homology. In particular, we can write

$$V^{S^1} = \lim_n R\,\mathrm{Hom}_{C_\bullet(S^1;k)}(C_\bullet(S^{2n+1};k), V) = \lim_n (V[u]/u^n, d + u\epsilon)$$

Each finite stage of the above limit is a localizing invariant and thus receives a trace from K-theory, and by universal property of the limit cyclic homology also receives a trace from K-theory.

2.10 Calculations of Hochschild homology

The functorial definition of the trace from algebraic K-theory to Hochschild homology has the following explicit realization on any given object.

Definition 2.10.1 (Chern characters of compact objects)**.** Let $\mathbf{C}$ be a presentable k-linear ∞-category. A compact object $X \in \mathbf{C}$ defines a compact-object preserving functor $c_X : \mathbf{Vect}_k \to \mathbf{C}$ by $V \mapsto V \otimes X$, and thus induces a map on Hochschild homology, defining a map of chain complexes $k \to HH(\mathbf{C})$. The *trace* or *Chern character* of X is

$$\mathrm{tr}(X) = HH(c_X)(1).$$

Example 2.10.2 (Group representations)**.** Let $\mathbf{C} = \mathrm{Coh}(BG)$ where G is reductive. The Hochschild homology is given by the composition of the coevaluation with the evaluation

$$\mathbf{Vect}_k \to \mathrm{Coh}(B(G \times G)) \to \mathbf{Vect}_k.$$

The integral kernel for the identity functor is the $G\times G$-representation $k[G] \in \mathrm{Coh}(B(G\times G))$, and evaluation is the G-invariants functor under the diagonal G-action. By the Peter-Weyl theorem,

$$k[G] \simeq \bigoplus_{\lambda \in \mathrm{Irr}(G)} V_\lambda \boxtimes V_\lambda^*$$

so we can identify the Hochschild homology

$$HH(\mathrm{Coh}(BG)) \simeq k[G]^G \simeq \bigoplus_{\lambda \in \mathrm{Irr}(G)} k \cdot \mathrm{id}_{V_\lambda}.$$

The trace of a G-representation V is computed as follows: the functor $- \otimes_k V$ has right adjoint $\mathrm{Hom}_G(V, -)$, giving us the sequence of maps

$$k \to \mathrm{tr}_{\mathrm{Vect}_k}(\mathrm{Hom}_G(V,V) \otimes -) \simeq \mathrm{tr}_{\mathrm{Coh}(BG)}(\mathrm{Hom}(V,-) \otimes V) \to HH(\mathrm{Coh}(BG))$$

which can be identified

$$k \to \mathrm{Hom}_k(V,V)^G \simeq (V^* \otimes V)^G \to HH(\mathrm{Coh}(BG)).$$

Writing $V = \bigoplus_{\lambda \in \mathrm{Irr(G)}} L_\lambda \otimes V_\lambda$ where L_λ is a vector space, this sequence can be written

$$k \to \bigoplus_{\lambda \in \mathrm{Irr}(G)} L_\lambda \to \bigoplus_{\lambda \in \mathrm{Irr}(G)} k.$$

The value of 1 under this composition is

$$(\dim(L_\lambda))_{\lambda \in \mathrm{Irr}(G)} \in \bigoplus_{\lambda \in \mathrm{Irr}(G)} k.$$

Example 2.10.3 (Cohomological Chern character)**.** We can use functoriality to determine the *Chern character* or *trace* of a compact object in $\mathrm{QCoh}(X)$, i.e. $\mathcal{E} \in \mathrm{Perf}(X)$. Let $p : X \to \mathrm{pt}$; such an object defines a functor

$$f_* = p^* - \otimes_{\mathcal{O}_X} \mathcal{E} : \mathbf{Vect}_k \to \mathrm{QCoh}(X)$$

which has right adjoint

$$f^! = \Gamma(X, - \otimes_{\mathcal{O}_X} \mathcal{E}^\vee) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -) : \mathrm{QCoh}(X) \to \mathbf{Vect}_k$$

Note that we require that $\mathcal{E}$ is compact (perfect) so that it we can define the right adjoint; the functor it defines is on cocomplete categories. Explicitly, we can understand this as the chain by

$$\dim(\mathrm{Perf}(\mathrm{pt})) \to \mathrm{tr}(f^! f_*) \simeq \mathrm{tr}(f_* f^!) \to \dim(\mathrm{Perf}(X))$$

$$k \xrightarrow{\eta} \Gamma(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \xrightarrow{\simeq} \Gamma(X, \Delta^* \mathcal{E} \boxtimes \mathcal{E}^\vee) \xrightarrow{\epsilon} \Gamma(X, \Delta^* \Delta_* \mathcal{O}_X) = \mathcal{O}(\mathcal{L}X).$$

This Chern character, like the usual one, is multiplicative.

Theorem 2.10.4 (Theorem 4.5 in [15])**.** *Suppose that X is a smooth scheme. The trace map*

$$K_0(X) \to H^0(\mathcal{O}(\mathcal{L}X)) \simeq \bigoplus_i H^i(X, \Omega^i_X)$$

agrees with the classical Chern character.

Remark 2.10.5 (Homological Chern character)**.** There is also a notion of homological Chern character, although it is more difficult to compute. Note that the monoidal structure $\otimes^!$ on ind-coherent sheaves does not preserve compact objects, so there is no algebra structure on $HH(\mathrm{Coh}(X))$; however, there is an action of $\mathrm{Perf}(X)$ on $\mathrm{Coh}(X)$, inducing a $HH(\mathrm{Perf}(X))$-module structure on $HH(\mathrm{Coh}(X))$.

Let $X = \mathbb{P}^1/\mathbb{G}_m$; we compute K_0 and HH of $\mathrm{Coh}(X)$.

Example 2.10.6 (K-theory)**.** In K-theory, we find $K_0(\mathbb{P}^1/\mathbb{G}_m) \simeq k[z, z^{-1}, w, w^{-1}]$. Let $\mathcal{L}$ be a line bundle on X. At each (fixed point) pole we have a representation of $\mathbb{G}_m$, say of weight a and b. It turns out that this determines the twist on the line bundle, so $\mathcal{L} = \mathcal{O}_X(a-b)$. The $\mathbb{G}_m$-action determines a choice of (opposite) orientation for each pole. Indeed, if x and y are local coordinates on $\mathbb{P}^1$ with respect to these poles (such that $xy = 1$ on the intersection), with $\deg(x) = 1$ and $\deg(y) = -1$, then the line bundle may be trivialized by $k[x]\sigma_x$ and $k[y]\sigma_y$. If $\deg(\sigma_x) = a$ and $\deg(\sigma_y) = b$, then necessarily by degrees, we have on the intersection that $\sigma_x = y^{b-a}\sigma_y$ up to a constant, determining the twist on the line bundle. This line bundle, for example, may be represented by $z^a w^b$.

There is the Euler short exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 2} \to \mathcal{O}(2) \to 0$$

This sequence can be made equivariant:

$$0 \to \mathcal{O}(1,1) \to \mathcal{O}(2,1) \oplus \mathcal{O}(1,0) \to \mathcal{O}(1,-1) \to 0$$

The above tells us that we have a map $k[z^{\pm}, w^{\pm}]/(z-1)(w-1) \to K_0(\mathbb{P}^1/\mathbb{G}_m)$ which is surjective. To see that it is injective, consider the diagram

$$\begin{array}{ccc} & & \mathbb{Z}[z^{\pm}] \simeq K_0(\mathbb{A}^1/\mathbb{G}_m) \\ & \overset{w=1}{\nearrow} & \uparrow \\ \mathbb{Z}[z^{\pm}, w^{\pm}]/(z-1)(w-1) & \twoheadrightarrow & K_0(\mathbb{P}^1/\mathbb{G}_m) \\ & \underset{z=1}{\searrow} & \downarrow \\ & & \mathbb{Z}[w^{\pm}] \simeq K_0(\mathbb{A}^1/\mathbb{G}_m) \end{array}$$

Any word sent to zero must be in the ideal $(w-1)$ and the ideal $(z-1)$ (i.e. the kernel of the diagonal maps).

Example 2.10.7 (Hochschild homology)**.** Since $\mathbb{P}^1$ has finitely many $\mathbb{G}_m$-orbits, we find that $\mathcal{L}(\mathbb{P}^1/\mathbb{G}_m)$ is $\mathbb{P}^1$ with a $\mathbb{G}_m$ nodally intersecting at the poles, and the $\mathbb{G}_m$-equivariant global functions on this space are $k[z, z^{-1}, w, w^{-1}]/(z-1)(w-1)$, agreeing with the above. In particular, in this example the Hochschild homology was much easier to compute than K-theory.

We now consider the example $X = \mathbb{A}^2/\mathbb{G}_m$.

Example 2.10.8 (K-theory)**.** Every module over $R = k[x, y]$ has a finite equivariant free resolution, so $K_0(\mathbb{A}^2/\mathbb{G}_m) \simeq \mathbb{Z}[z, z^{-1}]$, where z corresponds to the trivial line bundle with $\mathbb{G}_m$-weight 1.

Example 2.10.9 (Hochschild homology)**.** It's not hard to check that the geometric points of $\mathcal{L}(\mathbb{A}^2/\mathbb{G}_m)$ have two connected components: $\mathbb{A}^2$ and $\mathbb{G}_m$, which meet at $\{0\} \in \mathbb{A}^2$ and $\{1\} \in \mathbb{G}_m$. We can explicitly write down, using the Koszul resolution, that while $\mathcal{L}(\mathbb{A}^2/\mathbb{G}_m)$ is a proper derived scheme, its derived structure has cohomological weight 1. Explicitly, $\mathcal{O}(\mathcal{L}(\mathbb{A}^2/\mathbb{G}_m) \times_{\mathbb{G}_m} \mathrm{pt})$ is the dg complex

$$k[x, y, z^{\pm}] \xrightarrow{\begin{pmatrix} z^{-1}y - y \\ x - z^{-1}x \end{pmatrix}} k[x, y, z^{\pm}] \xrightarrow{\begin{pmatrix} x - z^{-1}x & y - z^{-1}y \end{pmatrix}} k[x, y, z^{\pm}]$$

Note that H^{-1} has a nontrivial cycle $\begin{pmatrix} y \\ -x \end{pmatrix}$ of degree 1, corresponding to

$$y\, dx - x\, dy \in \Omega^1_{\mathbb{A}^2}$$

and that $H^{-1} \simeq k[x, y, z^{\pm}]/(z-1)$ is supported at $z = 1$, the irreducible component isomorphic to $\mathbb{A}^2$. These cycles are the 1-forms which take value zero on the vector field generated by the $\mathbb{G}_m$-scaling action; the corresponding "geometric points" of H^{-1} are vector fields "modulo" those generated by the $\mathbb{G}_m$-scaling. However, since $\deg(x, y, dx, dy) = 1$, all of the non-zero vanish when we take $\mathbb{G}_m$-invariants, i.e. $\mathcal{O}(\mathcal{L}(\mathbb{A}^2/\mathbb{G}_m)) \simeq \mathcal{O}(\mathbb{G}_m)$. Similarly, we find that $\mathcal{O}(\mathcal{L}(\mathbb{A}^n/\mathbb{G}_m)) = \mathcal{O}(\mathbb{G}_m) = k[z, z^{-1}]$.

We now compute the circle action on $HH(\mathrm{Coh}(B\mathbb{G}_a))$ explicitly. We expect, more generally, that for a unipotent group U there is a description of the circle action on $HH(\mathrm{Coh}(BU))$ via contraction with a scaling action on the Eilenberg-Zilber complex, but we are not able to produce it.

Lemma 2.10.10. *Since $\mathbb{G}_a$ is abelian, there is an isomorphism*

$$\mathcal{L}(B\mathbb{G}_a) \simeq \mathbb{G}_a \times B\mathbb{G}_a.$$

Fix coordinates x, η , with $|x| = 0$ and $|\eta| = 1$, such that $\mathcal{O}(\mathbb{G}_a) = k[x]$ and $\mathcal{O}(B\mathbb{G}_a) = k[\eta]$. The Connes B-operator on $\mathcal{O}(\mathcal{L}(B\mathbb{G}_a)) = k[x, \eta]$ is given by

$$B(x) = \eta.$$

Consequently, $HP(\mathrm{Perf}(B\mathbb{G}_a)) = k((u))$.

Proof. In [32] it is shown that $\mathcal{O}(\mathcal{L}(B\mathbb{G}_a)) \simeq k[x, \eta]$ where $|x| = 0$ is a coordinate for $\mathcal{O}(\mathbb{G}_a)$ and $|\eta| = 1$ is a coordinate for $\mathcal{O}(B\mathbb{G}_a)$. It remains to compute the mixed complex structure. The first three terms of the twisted atlas are

$$\mathbb{G}_a \longleftarrow\!\!\!\!\longleftarrow \mathbb{G}_a \times \mathbb{G}_a \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_a$$

where the face maps send

$$(x, y) \mapsto x + y \quad y + x$$

$$(x, y, z) \mapsto (x + y, z) \quad (x, y + z) \quad (z + x, y)$$

On functions, the first three terms of the bar complex which computes the cohomology of $\mathbb{G}_a/\mathbb{G}_a$ is

$$k[x] \rightrightarrows k[x] \otimes k[x] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; k[x] \otimes k[x] \otimes k[x]$$

All maps are algebra morphisms. The first level of coface maps send

$$x \mapsto 1 \otimes x + x \otimes 1 \qquad x \otimes 1 + 1 \otimes x$$

and the second level of coface maps send

$$x \otimes 1 \mapsto x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 \qquad x \otimes 1 \otimes 1 \qquad x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x$$

$$1 \otimes x \mapsto 1 \otimes 1 \otimes x \qquad 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \qquad 1 \otimes x \otimes 1$$

Taking alternating sums, we find that the first differential is zero, and the second differential has kernel $(1 \otimes x)(x \otimes 1 + 1 \otimes x)^n$ for $n \geq 0$. The Connes B-operator is $N^\sharp s^\sharp (1-t)^\sharp$ where s is the extra degeneracy map, and N is the norm (which is the identity map at the zeroth level). The extra degeneracy sends $x \otimes 1 \mapsto 0$ and $1 \otimes x \mapsto x$, and thus B sends

$$(1 \otimes x)(x \otimes 1 + 1 \otimes x)^n \mapsto (1 \otimes x - x \otimes 1)(x \otimes 1 + 1 \otimes x)^n \mapsto x^{n+1}$$

Thus the mixed differential sends $\eta \mapsto x$. □

Finally, we compute examples of the Chern character with values in $HH(\mathrm{Coh}(\mathbb{P}^1))$. We do so in two ways; first, geometrically taking values in Dolbeault cohomology, and second, by choosing generators of the category $\mathrm{Coh}(\mathbb{P}^1)$.

Example 2.10.11 (Projective space via Dolbeault cohomology)**.** The loop space $\mathcal{O}(X)$ when X is projective can be understood as a shifted version of Dolbeault cohomology where $H^p(X, \Omega^q_X)$ is in degree $q - p$. We can compute easily that $\mathcal{O}(\mathcal{L}\mathbb{P}^n)) = k^{n+1}$. We focus on the example $X = \mathbb{P}^1$ so that

$$HH(\mathrm{Perf}(\mathbb{P}^1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq k^2.$$

Our goal is to compute the trace map for line bundles on $\mathbb{P}^1$.

To do this, we compute the derived global sections of the map of sheaves (in the derived category of sheaves on X):

$$\mathcal{O}_X \to \mathcal{E} \otimes \mathcal{E}^\vee \simeq \Delta^*(\mathcal{E} \boxtimes \mathcal{E}^\vee) \to \Delta^*\Delta_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{L}X} \simeq \mathcal{O}_X \oplus \Omega^1_X[1]$$

The most important map is the middle map. As written, it is a global section of the sheaf $\mathcal{H}om_X(\Delta^*\mathcal{E} \boxtimes \mathcal{E}^\vee, \Delta^*\Delta_*\mathcal{O}_X)$. However, we will need to understand it as the image of the evaluation map on sheafy Hom:

$$\mathcal{H}om_{X \times X}(\mathcal{E} \boxtimes \mathcal{E}^\vee, \Delta_*\mathcal{O}_X) \to \Delta_* \mathcal{H}om_X(\Delta^*\mathcal{E} \boxtimes \mathcal{E}^\vee, \Delta^*\Delta_*\mathcal{O}_X).$$

These sheaves as supported on the diagonal in $X \times X$, and thus we can think of them as sheaves on the formal completion of the diagonal, and in particular we can take a Cech cover of the diagonal rather than all of $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us fix some coordinates. Take $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(n)$. Let x, y be coordinates on the open charts for the first $\mathbb{P}^1$ factor, and x', y' on the second factor (i.e. $xy = 1$ and $x'y' = 1$ on the intersections). Let e_x, e_y denote local trivializing sections for $\mathcal{O}_{\mathbb{P}^1}(n)$ and $f_{x'}, f_{y'}$ denote local trivializing sections for $\mathcal{O}_{\mathbb{P}^1}(-n)$. We have a Cech resolution of $\mathcal{O}_{\mathbb{P}^1}(n) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n)$:

$$\mathcal{O}(n) \boxtimes \mathcal{O}(-n) \longrightarrow k[x, x']e_x f_{x'} \oplus k[y, y']e_y f_{y'} \longrightarrow \frac{k[x, y, x', y']}{xy = 1, x'y' = 1} e_x f_{x'}$$

where $e_x = y^n e_y$ and $f_{x'} = y'^{-n} f_{y'}$. Over the diagonal open affines, the trivializing section satisfy the relation

$$e_x f_{x'} = y^n y'^{-n} e_y f_{y'}$$

and as we expect, this sheaf has no global sections.

For $\Delta_* \mathcal{O}_X$, we take Beilinson's resolution of the diagonal, and then the Cech resolution of that to get a double complex:

$$\begin{array}{ccccc}
\mathcal{O} \boxtimes \mathcal{O} & \longrightarrow & k[x,x']c_x c_{x'} \oplus k[y,y']c_y c_{y'} & \longrightarrow & \frac{k[x,y,x',y']}{xy=1,x'y'=1} c_x c_{x'} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{O}(-1) \boxtimes \Omega^1(-1) & \longrightarrow & k[x,x']s_x t_{x'} \oplus k[y,y']s_y t_{y'} & \longrightarrow & \frac{k[x,y,x',y']}{xy=1,x'y'=1} s_x t_{x'}
\end{array}$$

Here, we have the relations $c_x = c_y$ and $c_{x'} = c_{y'}$ (i.e. these are just the constant sections); the relations $s_x = y^{-1} s_y$ and $t_{x'} = y'^{-1} t_{y'}$. The vertical maps send

$$s_x t_{x'} \mapsto (x - x')c_x c_{x'}$$

$$s_y t_{y'} \mapsto (y' - y)c_y c_{y'}$$

We are interested in where the element $e_x f_{x'} + e_y f_{y'}$ in the first Cech complex should go to in the second Cech complex. We need the map to be a map of chain complexes, i.e. we need the following square to commute

$$\begin{array}{ccc}
k[x,x']e_x f_{x'} \oplus k[y,y']e_y f_{y'} & \longrightarrow & \frac{k[x,y,x',y']}{xy=1,x'y'=1} e_x f_{x'} \\
\downarrow & & \downarrow \\
k[x,x']c_x c_{x'} \oplus k[y,y']c_y c_{y'} \oplus \frac{k[x,y,x',y']}{xy=1,x'y'=1} s_x t_{x'} & \longrightarrow & \frac{k[x,y,x',y']}{xy=1,x'y'=1} c_x c_{x'}
\end{array}$$

Going around clockwise, we compute

$$e_x f_{x'} + e_y f_{y'} \mapsto e_x f_{x'} - x^n x'^{-n} e_x f_{x'} \mapsto (1 - x^n x'^{-n}) c_x c_{x'}$$

Going around counterclockwise, we compute (with p a stand-in for the target in the third summand):

$$e_x f_{x'} + e_y f_{y'} \mapsto c_x c_{x'} + c_y c_{y'} + p s_x t_{x'} \mapsto (x - x') p c_x c_{x'}$$

Thus, $(x - x')p = (1 - x^n x'^{-n})$, so

$$p = -y\left(1 + \frac{x}{x'} + \frac{x^2}{x'^2} + \cdots + \frac{x^{n-1}}{x'^{n-1}}\right)$$

Now, when we pull back to the diagonal, this becomes

$$p s_x t_x = -ny s_x t_x = -nx s_y t_y \in H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$$

Thus, the Chern character takes $\mathcal{O}(n) \mapsto (1, -n)$, like the classical Chern character.

Example 2.10.12 (Projective space via Kronecker quiver)**.** It is well-known that the category of coherent sheaves on $\mathbb{P}^1$ is equivalent to representations of the Kronecker quiver:

$$\bullet \rightrightarrows \bullet$$

Thus, we can compute the Hochschild homology of $\mathrm{Coh}(\mathbb{P}^1)$ by the classical Hochschild homology of this quiver algebra. It is convenient to use the following *standard resolution* for quiver algebras. Let E denote the subalgebra of edges and V the subalgebra of vertices of a quiver path algebra A, and take

$$0 \to A \otimes_V E \otimes_V A \to A \otimes_V A$$

$$1 \otimes \gamma \otimes 1 \mapsto 1 \otimes \gamma - \gamma \otimes 1$$

as a projective $A \otimes A$ resolution of A. Sometimes, this is written instead

$$0 \to \bigoplus_{\gamma : i \to j \in E} Ae_j \otimes e_i A \to \bigoplus_{i \in V} Ae_i \otimes e_i A$$

Writing it this way makes it obvious that it is a projective (in fact, free) resolution.[15] Further, if the quiver has no cycles, then the first term of this resolution vanishes after applying $- \otimes_{A \otimes A} A$ and the zeroth term becomes V. Thus, we find that

$$HH_\bullet(\mathrm{Coh}(\mathbb{P}^n) = k^{\oplus n+1}.$$

To compute the trace, recall how the category $\mathrm{Coh}(\mathbb{P}^1)$ is equivalent to representations on the Kronecker quiver. Let us choose $\mathcal{O}(-1)$ and $\mathcal{O}$ as our generators for the category. Then, the equivalence functor takes

$$\mathcal{F} \mapsto \; R\,\mathrm{Hom}(\mathcal{F}, \mathcal{O}(-1)) \rightrightarrows R\,\mathrm{Hom}(\mathcal{F}, \mathcal{O})$$

For example,

$$\mathcal{O}(n) \mapsto \; R\Gamma(\mathbb{P}^1, \mathcal{O}(-1-n)) \rightrightarrows R\Gamma(\mathbb{P}^1, \mathcal{O}(-n))$$

The trace morphism simply computes the (signed) dimension on each vertex. Thus, we find that

$$\mathrm{tr}(\mathcal{O}(n)) = (-n, 1-n)$$

[15] Here is the rest of the proof. Clearly, $A \otimes_V A \to A$ is surjective. To see that $A \otimes_V E \otimes_V A \to A \otimes_V A$ is injective, grade the left tensor factor of A such that the vertices have degree zero and edges have degree one, and the entire right tensor factor is degree zero. Then, the map is a degree one map. One can check explicitly then that it is injective on its homogeneous parts, and thus is injective in general. To see that it is exact in the middle, we impose the relations generated by the image. We can use these relations to move all nontrivial paths to the right factor, i.e. reduce any expression to a degree zero expression. It's easy to check that the map is injective for degree zero maps, so we are done.

Note that the sum of the components computes the rank of the bundle. This basis differs from our previous example by an automorphism of $GL_2(\mathbb{Z})$, i.e. the map $HH(\mathrm{Coh}(\mathbb{P}^1)) \to HH(\mathrm{Coh}(Q))$ is given, in the chosen basis, by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Example 2.10.13 (Not an isomorphism of algebras)**.** It is well-known that $\mathrm{Coh}(\mathbb{P}^1) = \mathrm{Perf}(\mathbb{P}^1)$ is equivalent to $\mathrm{Rep}(Q)$ where Q is the Kronecker quiver, and we observe above that both have Hochschild homology isomorphic to k^2. There is a natural basis on each side. For $HH(\mathrm{Perf}(\mathbb{P}^1))$, we can take the constant function $1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and the canonical distribution $\omega \in H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$. For $\mathrm{Rep}(Q)$, each basis vector corresponds to a vertex of the quiver, which in turn correspond to generating objects $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$. However, the equivalence $\mathrm{Perf}(\mathbb{P}^1) \simeq \mathrm{Rep}(Q)$ is not monoidal, equipping $\mathrm{QCoh}(\mathbb{P}^1)$ with the usual tensor product of quasicoherent sheaves and $\mathrm{Rep}(Q)$ with the tensor product of representations. In turn, they induce non-isomorphic algebra structures on k^2: $HH(\mathrm{QCoh}(\mathbb{P}^1))$ is given by $(a, b) \cdot (a', b') = (aa', ab' + a'b)$, whereas the multiplication in $HH(\mathrm{Rep}(Q))$ is $(a, b) \cdot (a', b') = (aa', bb')$. They cannot be isomorphic because the former algebra has only one idempotent but the latter has two.

Chapter 3

Equivariant Localization for Derived Loop Spaces and Periodic Cyclic Homology

In this chapter we prove the main result of this thesis: a equivariant localization formula for periodic cyclic homology and derived loop spaces.

3.1 Derived loop spaces and its variants

An in-depth discussion of derived loop spaces can be found in [6]; we will summarize the main definitions and prove some basic lemmas in the case of the loop space of an algebraic or geometric stack. Note that we often refer to derived loop spaces as simply loop spaces.

Definition 3.1.1. We consider the higher derived stack S^1 as a locally constant sheaf on **Aff** with value the topological circle S^1. Its affinization is the shifted affine line $B\mathbb{G}_a = \operatorname{Spec} C^\bullet(S^1; k)$ and the map $S^1 = B\mathbb{Z} \to B\mathbb{G}_a$ is induced by the (algebraic) map of abelian groups $\mathbb{Z} \to \mathbb{G}_a$. The stack $B\mathbb{G}_a$ is not an affine scheme since $C^\bullet(BS^1; k)$ is not connective, but it still has a well-defined functor of points; it is an example of a *coaffine stack* (in the language of [42]) or an *affine stack* (in the language of [59]). Explicitly, by Lemma 2.2.5 in [59] or the introduction to Section 4 of [42], it is the left Kan extension[1] of the classical stack[2] sending an affine scheme $S = \operatorname{Spec}(R)$ to the Eilenberg-Maclane space $K(R, 1)$ where R is considered as an abelian group under addition. In particular, the affinization map on S-points is given by Eilenberg-Maclane spaces $K(1, \mathbb{Z}) \to K(1, S)$ where we consider S as an abelian group under addition. We fix an isomorphism $C^\bullet(S^1; k) \simeq k[\eta]$ where $|\eta| = 1$.

Definition 3.1.2. Let X be a derived stack. We have the following variants of derived loop spaces.

[1] That is, the (fully faithful) left adjoint to the restriction of a prestack (i.e. a functor **DRng** $\to$ **S**) to a classical prestack (i.e. a functor **Rng** $\to$ **S**).

[2] In fact, coaffine stacks are always left Kan extensions of classical stacks.

- The *(derived) loop space* of X is the derived mapping stack

$$\mathcal{L}(X) := \mathrm{Map}(S^1, X) \simeq X \times_{X\times X} X.$$

  The second presentation is a consequence of the presentation of S^1 as the suspension of S^0, i.e. the homotopy pushout $S^1 \simeq \mathrm{pt} \coprod_{S^0} \mathrm{pt} = \Sigma S^0$, and the property that derived mapping stacks take coproducts in the source to products. The evaluation map $p : \mathcal{L}X \to X$ realizes the loop space as a relative group stack over X.

- The *formal loop space* $\widehat{\mathcal{L}}(X)$ is the completion of $\mathcal{L}(X)$ along the closed substack of constant loops; the evaluation realizes the formal loop space as a relative (formal) group stack over X.

- The *unipotent loop space* $\mathcal{L}^u(X)$ is the derived mapping stack

$$\mathcal{L}^u(X) := \mathrm{Map}(B\mathbb{G}_a, X)$$

  and the affinization map $S^1 \to B\mathbb{G}_a$ defines a map $\mathcal{L}^u X \to \mathcal{L}X$. There is a natural $\mathbb{G}_a \rtimes \mathbb{G}_m$-action on $\mathcal{L}^u X$ arising from the natural $\mathbb{G}_m$-action on $B\mathbb{G}_a$.

- We define the *odd tangent bundle*, a linearized form of the loop space,

$$\mathbb{T}_X[-1] := \mathrm{Spec}_X \mathrm{Sym}_X \mathbb{L}_X[1]$$

  where $\mathbb{L}_X$ is the relative spectrum of the derived symmetric powers[3] of the cotangent complex. There is a projection $q : \mathbb{T}_X[-1] \to X$ and a zero section $c : X \to \mathbb{T}_X[-1]$ induced by the structure and augmentation maps respectively. We write $\widehat{\mathbb{T}}_X[-1]$ for the odd tangent bundle completed at its zero section. Both $\mathbb{T}_X[-1]$ and $\widehat{\mathbb{T}}_X[-1]$ are equipped with the natural scaling $\mathbb{G}_m$-action on fibers.

Example 3.1.3. If $X = \mathrm{Spec}(A)$, then the derived loop space

$$\mathcal{L}(X) = \mathrm{Spec}(A \otimes_{A\otimes_k A^{op}} A) = \mathrm{Spec}(C^\bullet(A; A))$$

is the derived spectrum of the cyclic bar complex equipped with the shuffle product. The rotation S^1-action has a combinatorial realization via the cyclic structure on the cyclic bar complex (see [38] and [33]).

[3] We define the relative spectrum as follows: for an algebra object $\mathcal{A} \in \mathrm{QCoh}(X)$, we define the S-points for $\mathrm{Spec}_X \mathcal{A}$ as pairs (η, δ) where $\eta \in X(S)$ and $\delta : S \to \mathrm{Spec}\, \eta^*\mathcal{A}$ which are compatible under the projection; note that $\eta^*\mathcal{A}$ is an algebra since S is an affine derived scheme and pullback preserves the monoidal structure on quasicoherent sheaves. The symmetric algebra functor Sym_X is left adjoint to the forgetful functor from the category of augmented commutative unital associative algebra objects of $\mathrm{QCoh}(X)$, which exists by the adjoint functor theorem.

Remark 3.1.4. The following perspective may be helpful. The bar resolution $B^\bullet(A)$ for A as an $A \otimes A^{op}$-module is obtained by tensoring A with the map of simplicial complexes $I \to \mathrm{pt}$, where the unit interval I is presented by a simplicial set with two 0-simplices and one non-degenerate 1-simplex,

$$B^\bullet(A) = A \otimes I \to A \qquad X \simeq \mathrm{Map}_{DSt}(I, X) = \mathrm{Spec}(A \otimes I).$$

The cyclic bar complex $C^\bullet(A) = B^\bullet(A) \otimes_{A \otimes A^{op}} A$ is obtained by gluing the two 0-simplices of I, i.e. it is the chain complex associated to the tensor product of A with the presentation of S^1 by one 0-simplex and one non-degenerate 1-simplex:

$$C^\bullet(A) = A \otimes S^1 \qquad \mathcal{L}(X) = \mathrm{Map}_{DSt}(S^1, \mathrm{Spec}(A)) = \mathrm{Spec}(A \otimes S^1).$$

This induces on X the structure of a cocyclic scheme, and $\mathcal{O}(X)$ the structure of a cyclic algebra.

Example 3.1.5. If X is a stack, then $\pi_0(\mathcal{L}(X))$ is the (classical) inertia stack of X. In particular, let $X = BG$; then $\mathcal{L}(BG) = G/G$ is the stacky adjoint quotient. Note that $\mathcal{L}(BG) = BG \times_{BG \times BG} BG$ is classical since the diagonal map is flat. The S^1-equivariant structure on $\mathcal{O}(G/G)$ has a description in terms of the a cyclic algebra (see Section 7.3.3 of [38]) arising from the cyclic structure on the simplicial Cech diagram for the atlas $G \to G/G$.

Proposition 3.1.6 (Loop space of a quotient stack)**.** *The loop space of a quotient stack $\mathcal{L}(X/G)$ can be computed by the G-equivariant fiber product*

$$\begin{array}{ccc} \mathcal{L}(X/G) & \longrightarrow & (X \times G)/G \\ \big\downarrow & & \big\downarrow{\scriptstyle a \times p} \\ X/G & \xrightarrow{\Delta} & (X \times X)/G \end{array}$$

where G acts on $X \times X$ and $X \times G$ diagonally.

Proof. Note that $X/G \times X/G \simeq (X \times X)/(G \times G)$ with action $(g_1, g_2) \cdot (x_1, x_2) = (g_1 x_1, g_2 x_2)$. We write

$$\frac{X}{G} \times_{\frac{X \times X}{G \times G}} \frac{X}{G} = \frac{X \times G}{G \times G} \times_{\frac{X \times X}{G \times G}} \frac{X \times G}{G \times G}$$

where the map $X \times G \to X \times X$ sends $(x, g) \mapsto (x, gx)$ and the action of $G \times G$ on $X \times G$ is $(g_1, g_2) \cdot (x, g) = (g_1 x, g_2 g g_1^{-1})$. By the "two-out-of-three" lemma for pullback squares applied to

$$\begin{array}{ccccc} \mathcal{L}(X/G) & \longrightarrow & (X \times G)/G & \longrightarrow & (X \times G)/(G \times G) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X/G & \longrightarrow & (X \times X)/G & \longrightarrow & (X \times X)/(G \times G) \end{array}$$

the claim follows. $\square$

Remark 3.1.7. Forgetting G-equivariance, the loop space $\mathcal{L}(X/G)$ consists of pairs $(x, g) \in X \times G$ such that $g \cdot x = x$. Ignoring issues of equivariance, the geometric fiber of the map $\mathcal{L}(X/G) \to \mathcal{L}(BG) = G/G$ over $g \in G$ is its fixed points X^g. The geometric fiber of the evaluation map $\mathcal{L}(X/G) \to X/G$ over $x \in X$ is the stabilizer of x.

Example 3.1.8 (Odd tangent bundle of smooth quotient stacks)**.** In the case of X/G where X is smooth, we have that

$$\mathbb{L}_{X/G} = (\Omega^1_X \to \mathfrak{g}^* \otimes \mathcal{O}_X)$$

where the internal differential d is the Cartan differential, so that

$$\mathrm{Sym}^n(\mathbb{L}_{X/G}[1]) = \mathrm{Sym}^n(\Omega^1_X \to \mathfrak{g}^* \otimes \mathcal{O}_X) \simeq (\Omega^n_X \to \mathfrak{g}^* \otimes \Omega^{n-1}_X \to \cdots \to \mathrm{Sym}^n(\mathfrak{g}^*) \otimes \mathcal{O}_X)$$

and

$$p_* \mathcal{O}_{\widehat{\mathbb{T}}_{X/G}[-1]} = \left(\lim_k \bigoplus_{i \geq k} \frac{\mathrm{Sym}^\bullet \mathfrak{g}^*}{\mathrm{Sym}^{\geq k} \mathfrak{g}^*} \otimes \Omega^i_X[i], d \right).$$

This example can also be carried out when X is not smooth, replacing Ω^1_X with $\mathbb{L}_X$.

We first prove some technical facts, which may be skipped on a first reading. Note that a quasi-compact geometric stack is automatically QCA in the sense of [21]. If X is an algebraic stack then X admits a cover by a disjoint union of affine schemes; if X is quasi-compact this disjoint union can be taken to be finite, so that X admits a cover by an affine scheme. If X is geometric (i.e. has affine diagonal), then

$$S \times_X T = (S \times T) \times_{X \times X} X$$

is affine for any affine schemes S, T.

Lemma 3.1.9. *Let X be an algebraic stack. Then $\mathcal{L}X$ is an algebraic stack. If X is geometric, then $\mathcal{L}X$ is geometric. If X is geometric and quasi-compact, then so is $\mathcal{L}X$.*

Proof. Assume X is algebraic. That $\mathcal{L}X$ is algebraic follows from the fact that $\mathcal{L}X = \mathrm{Map}(S^1, X)$ is a finite limit, and any finite limit of algebraic stacks is algebraic. An algebraic stack with a cover by a scheme $U \to X$ is geometric if and only if $U \times_X U = (U \times U) \times_{X \times X} X$ is affine. Assume X is geometric, so that $U \times_X U$ is affine. In particular, $U \times_{X \times X} X$ is a cover for $\mathcal{L}X$, and we have

$$(U \times_{X \times X} X) \times_{\mathcal{L}X} (U \times_{X \times X} X) = U \times_X (\mathcal{L}X \times_{\mathcal{L}X} (U \times_{X \times X} X)) = (U \times_X U) \times_{X \times X} X$$

which is affine since $U \times_X U$ is affine and the diagonal map is affine, so $\mathcal{L}X$ is geometric. Assume X is also quasi-compact; then it admits a cover by an affine U, and $U \times_{X \times X} X$ is also affine since the diagonal is affine. $\square$

Lemma 3.1.10. *Let X be an algebraic stack. Then the inclusion of constant loops $X \to \mathcal{L}(X)$ is a (schematic) closed immersion.*

Proof. Since the diagonal map $X \to X \times X$ is representable by schemes, so is the evaluation map $\mathcal{L}(X) \to X$. Let $U \to X$ be an atlas for X with U a scheme; its base change along the evaluation map gives a cover by a scheme $U \times_{X \times X} X \to \mathcal{L}X$. In particular, by the two-out-of-three property of Cartesian squares, the left square is Cartesian

$$\begin{array}{ccccc} U & \longrightarrow & U \times_{X \times X} X & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{L}X & \longrightarrow & X \end{array}$$

i.e. the base change of the inclusion of constant loops $X \to \mathcal{L}X$ along an atlas is a scheme, so it is schematic. It is a closed embedding since any map of derived schemes which admits a retract is a closed embedding, and $U \to U \times_{X \times X} X$ admits a retract by universal property.

We provide a proof for the last claim. It suffices to assume all schemes are classical, since the the property of being a closed immersion depends only on classical schemes and the property of admitting a retract is preserved by π_0. Let $f : Z \to Y$ be a map of schemes admitting a retract. We can verify that f is a closed immersion affine locally on Y, so assume Y is affine. It is a closed immersion if $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_Z$ is surjective. Since Y is affine, this is equivalent to $\mathcal{O}(Y) \to \mathcal{O}(Z)$ being surjective, which follows since the composition on global functions $\mathcal{O}(Z) \to \mathcal{O}(Y) \to \mathcal{O}(Z)$ is the identity. □

Definition 3.1.11. Let X be a prestack, and $x : S \to X$ be an S-point where S is an affine derived scheme. The *group of based loops* at x, denoted $\Omega(X, x)$, is the ∞-group object in prestacks over S defined by the pullback $\mathcal{L}X \times_X S$, or equivalently, the pullback

$$\begin{array}{ccc} \Omega(X,x) & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle x} \\ S & \xrightarrow{x} & X. \end{array}$$

The group structure given by the simplicial diagram with terms $S \times_X \cdots \times_X S$, i.e. the group multiplication is given by convolution and inversion by switching the factors in the fiber product. If X is algebraic, then based loops at any point $x \in X(S)$ is an ∞-group object in derived schemes over S, and if X is geometric, it is an ∞-group object in affine derived schemes over S. If $f : X \to Y$ is a map of prestacks, with $x \in X(S)$, then there is a map of ∞-groups $\Omega(f, x) : \Omega(X, x) \to \Omega(Y, f(x))$. We define the *unipotent based loops* of X, denoted $\Omega^u(X, x)$, by the fiber product $\mathcal{L}^u(X) \times_X S$; there is a natural map $\Omega^u(X, x) \to \Omega(X, x)$. Note that the unipotent based loops do not form a group.

Example 3.1.12. Let X be an (affine) derived scheme and $x \in X(k)$ a geometric point. Then, $\Omega(X, x) = \mathbb{T}_{X,x}[-1] = \operatorname{Spec}\operatorname{Sym} x^*\mathbb{L}_X[1]$ is the odd tangent space at $x \in X(k)$. The comultiplication on functions is given by the natural comultiplication on the symmetric algebra and antipode map by the sign morphism.

Proposition 3.1.13. *For any prestack X, there is a natural bijection between S-points $x \in X(S)$ along with a based loop $\gamma \in \Omega(X, x)(S)$ and S-points of $\mathcal{L}X$. Similarly, there is a natural bijection between S-points $x \in X(S)$ along with a based loop $\alpha \in \operatorname{Hom}_{grp,S}(\mathbb{G}_a \times S, \Omega(X, x))$ and S-points of $\mathcal{L}^u X$.*

Proof. For the first claim, note we the based loops can be realized by the fiber product

$$\begin{array}{ccccc} \Omega(X,x) & \longrightarrow & \mathcal{L}X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle x} \\ S & \xrightarrow{x} & X & \longrightarrow & X \times X. \end{array}$$

By universal property, a map $S \to \Omega(X, x)$ is equivalent to a map $S \to \mathcal{L}X$; all fitting into the above diagram. For the second claim, note by universal property that we obtain a map

$$\mathbb{G}_a \times S = S \times_{B\mathbb{G}_a \times S} S \to \Omega(X, x) = S \times_X S$$

where $S \to B\mathbb{G}_a \times S$ is the canonical cover. Taking higher fiber products, we obtain a map of ∞-groups $\mathbb{G}_a \times S \to \Omega(X, x)$ over S. Conversely, given a map of ∞-groups, by universal property of totalization, we obtain a map $B\mathbb{G}_a \times S \to X$ since $x : S \to X$ defines an augmentation of the simplicial diagram for the ∞-group $\Omega(X, x)$ over S. □

Definition 3.1.14. Let $X = \operatorname{Spec}(R)$ be an affine scheme with a $\mathbb{G}_m$-action. We say the $\mathbb{G}_m$-action is *contracting* if it acts by only non-positive weights on R. In this case, the fixed point locus is $Y = \operatorname{Spec}(R^{\mathbb{G}_m})$, and we say the $\mathbb{G}_m$-action contracts to Y. In particular, there are maps $Y \to X \to Y$. More generally, let X be a prestack with a $\mathbb{G}_m$-action, equipped with a $\mathbb{G}_m$-equivariant affine map $p : X \to Y$ where Y is given the trivial action. We say the $\mathbb{G}_m$-action *contracts* to Y if for any affine S and map $S \to Y$, the induced $\mathbb{G}_m$-action on $S \times_Y X$ contracts to S. In particular, this implies there is also a $\mathbb{G}_m$-equivariant section $Y \to X$.

Lemma 3.1.15. *Let X be a quasi-compact geometric stack. The $\mathbb{G}_m$-actions on $\mathcal{L}^u X$ and $\mathbb{T}_X[-1]$ contract to the fixed point locus of constant loops.*

Proof. The claim for $\mathbb{T}_X[-1]$ is by definition. For the unipotent loop space, take $x \in X(S)$. It suffices to show that the induced $\mathbb{G}_m$-action on $\Omega^u(X, x)$ is contracting. This follows from the description of $\Omega^u(X, x) = \operatorname{Hom}_{grp,S}(\mathbb{G}_a, \Omega(X, x))$, and the contracting $\mathbb{G}_m$-action on $\mathbb{G}_a$. □

We now examine some relationships between the above variants of loop spaces which are not immediately obvious. The following is essentially Remark 6.11 in [6], but will not be used in our results.

Definition 3.1.16. Let X be a quasi-compact geometric stack. We define a natural $\mathbb{G}_m$-equivariant map

$$\upsilon : \mathcal{L}^u X \to \mathbb{T}_X[-1]$$

as follows. Since X is geometric, the evaluation map $p : \mathcal{L}X \to X$ is affine, and the ideal sheaf for the constant loops $X \subset \mathcal{L}X$ defines a filtration on $p_* \mathcal{O}_{\mathcal{L}X}$. The usual Rees construction produces a family over $\mathbb{A}^1/\mathbb{G}_m$ whose generic fiber is $\mathcal{L}X$ and whose special fiber is $\mathbb{T}_X[-1]/\mathbb{G}_m$. Likewise, the evaluation map $q : \mathcal{L}^u X \to X$ is ind-affine by Lemma A.8.13 in [19], and the $\mathbb{G}_m$-scaling action is compatible with the filtration. Thus, by functoriality we obtain a map $\mathcal{L}^u(X)/\mathbb{G}_m \to \mathbb{T}_X[-1]/\mathbb{G}_m$.

Definition 3.1.17. Let X be a quasicompact geometric stack. There is an *exponential map*

$$\exp : \widehat{\mathbb{T}}_X[-1] \to \widehat{\mathcal{L}}X$$

defined as follows. Proposition 4.4 of [6] defines for a derived scheme U a Hochschild-Kostant-Rosenberg map

$$\exp : \widehat{\mathbb{T}}_X[-1] = \mathbb{T}_X[-1] \to \widehat{\mathcal{L}}(X) = \mathcal{L}(X)$$

which is an isomorphism by Theorem 6.9 of *loc. cit.* . Choose an atlas $U \to X$ where U is an affine scheme; the associated simplicial Cech diagram $U_\bullet$ also consists of affine schemes, and by universal property of the totalization we have a map

$$\mathrm{Spec}_X \mathrm{Tot}(\mathcal{O}_{\widehat{\mathbb{T}}_{U_\bullet}[-1]}) = \mathrm{Spec}_X \mathrm{Tot}(\mathcal{O}_{\widehat{\mathcal{L}}U_\bullet}) \to \widehat{\mathcal{L}}X.$$

Using the fact that the assignment of cotangent complexes is a (non-quasicoherent) sheaf on the smooth site, Theorem 6.6 of [6] identifies the left-hand side with $\widehat{\mathbb{T}}_X[-1]$. The exponential map is proven to be an equivalence in Corollary 6.1 of *loc. cit.* .

Remark 3.1.18. When X is a smooth affine scheme, the Hochschild-Kostant-Rosenberg map is well-known. When X is an affine (derived) scheme, a version of the Hochschild-Kostant-Rosenberg map can be written down in the same fashion by replacing $\mathcal{O}(X)$ with a semi-free dg-resolution and replacing the module of differentials Ω^1_X with the cotangent complex $\mathbb{L}_X$. When X is a smooth scheme, the Hochschild-Kostant-Rosenberg map can be explicitly realized via the completed bar complex of [64], which defines the sheafy cyclic bar complex as a complex of sheaves on X rather than $X \times X$. We are not aware of a reference which has extended these results to the case of a non-smooth (dg derived) scheme X. When X has the resolution property (e.g. when X is quasiprojective), such a generalization appears straightforward.

In [6], it is shown that for X a scheme, $\widehat{\mathcal{L}}(X) = \mathcal{L}^u(X) = \mathcal{L}(X)$. This is not true for stacks, but we will now show that for a schematic map $f : X \to Y$, the formal and unipotent loops of X are loops in X whose images in Y are formal and unipotent respectively.

Proposition 3.1.19. *Suppose that $f : X \to Y$ is a map of algebraic stacks representable by schemes. Then,*

$$\widehat{\mathcal{L}}(X) = \widehat{\mathcal{L}}(Y) \times_{\mathcal{L}(Y)} \mathcal{L}(X).$$

Proof. It suffices to show that the closed classical substack $\pi_0(Y \times_{\mathcal{L}(Y)} \mathcal{L}(X))$ has the same reduced points as $X \subset \mathcal{L}(X)$. To do this, it suffices to check on geometric points. Consider the diagram of *classical pullbacks*

$$\begin{array}{ccccc} \Omega(X,x) & \longrightarrow & \Omega(Y,f(x)) & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle x} \\ \operatorname{Spec} k & \xrightarrow{x} & f^{-1}(f(x)) & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} k & \xrightarrow{f(x)} & Y. \end{array}$$

Since f is schematic, $x : \operatorname{Spec} k \to f^{-1}(f(x))$ is a closed embedding of schemes (since it as a map of schemes admitting a retract; see Lemma 3.1.10). Since Y is an algebraic stack, $\Omega(Y, f(x)))$ is a scheme, and so $\Omega(X, x) \to \Omega(Y, f(x))$ is a closed embedding of schemes and a map of affine (classical) group schemes. The preimage of the identity is thus the identity, so constant loops in $\mathcal{L}Y$ are preimages of constant loops in $\mathcal{L}X$. □

Example 3.1.20 (Quotient stacks)**.** In the case of quotient stacks, we have a map $\mathcal{L}(X/G) \to \mathcal{L}(BG) = G/G$. The above proposition says that $\widehat{\mathcal{L}}(X/G)$ is the completion of $\mathcal{L}(X/G)$ at the closed substack of points lying over $\{e\}/G \subset G/G$.

We will need to consider certain derived linear groups over derived schemes. They will all take the following form.

Definition 3.1.21. Let S be an affine derived scheme, i.e. $S = \operatorname{Spec}(R)$. Let G' be a linear algebraic group over k, and denote $G'_S := G' \times_k S$. Let G' act on a derived k-scheme X, and let $x : S \to X$ be an S-point of X. We call the derived affine group

$$G = \{x\} \times_X G'_S$$

a *derived linear stabilizer group*. The *formal unipotent cone* of G'_S is the inverse image of the formal neighborhood of the unipotent cone $U \subset G$ over k, and likewise for G.

Example 3.1.22. Let $S = \mathrm{Spec}(R)$. The automorphism groups of S-points of quotient stacks by affine algebraic groups are derived linear stabilizer groups. More explicitly, for $p : X/G \to BG$ and $x \in X/G(S)$, we have a Cartesian diagram

$$\begin{array}{ccc} \Omega(X/G, x) & \longrightarrow & \Omega(BG, p(x)) = G \times S \\ \downarrow & & \downarrow \\ S & \xrightarrow{\quad x \quad} & X \times S. \end{array}$$

The following is well-known, but we provide a brief argument as we could not find a reference in the literature.

Proposition 3.1.23. *Let H, G be groups in derived schemes over k (a field of characteristic zero). Then, we can identify the mapping stack*

$$\mathrm{Map}(BH, BG) = \mathrm{Hom}_{grp}(H, G)/G$$

where $\mathrm{Hom}_{grp}(H, G)$ is an ind-scheme and G acts by the adjoint action.

Proof. By Lemma A.8.13 in [19], $\mathrm{Hom}_{grp}(H, G)$ is an ind-scheme (the argument easily generalizes to the case of derived group schemes). We define a map $\Phi : \mathrm{Map}(BH, BG) \to \mathrm{Hom}_{grp}(H, G)/G$ and $\Psi : \mathrm{Hom}_{grp}(H, G)/G \to \mathrm{Map}(BH, BG)$ and leave the verification that they are *strict* (i.e. not just quasi) inverses to the reader. On S-points, $\mathrm{Map}(BH, BG)(S) = \mathrm{Map}(BH \times S, BG)$; the value of this sheaf at $S' \to S$ consists of a functor $F_{S'}$ from H-torsors over S' to G-torsors over S'. We define (note that $\mathrm{Hom}_{grp}(H, G)(S) = \mathrm{Hom}_{S-grp}(H \times S, G \times S)$, where $S - grp$ indicates a map of groups over S)

$$\Phi(F)(S') = \left\{ \begin{array}{ccc} F(H \times S') & \xrightarrow{\phi} & \mathrm{Hom}_{grp}(H(S'), G(S')) \\ \Big\downarrow {\scriptstyle G-\mathrm{torsor}} & & \\ S' & & \end{array} \right\}$$

where ϕ is defined as follows. There is a canonical isomorphism $\mathrm{Aut}_{S'}(S' \times H) = H(S')$, and for $h \in \mathrm{Aut}_S(S \times H)$, $F(h)$ is an automorphism of $F(S' \times H)$, given by the action by some g_h. We define $\phi(x)(h) = x \cdot g_h$ (note our torsors are right-torsors). The map Ψ is defined as follows. The S-points of $\mathrm{Hom}_{grp}(H, G)$ consist of a G-torsor $P \to S$ with a G-equivariant map $\phi \in \mathrm{Hom}_{Grp}(H(S), G(S))$. Then, $\Psi(P, \phi)$ is the functor which takes a H-torsor $Q \to S$ to $Q \times^H P$ where H acts on P via ϕ. □

Definition 3.1.24. A map of prestacks $X \to Y$ is a *monomorphism*, i.e. X is a *substack* of Y, if for any affine derived scheme S and $y \in Y(S)$ the fiber product $\{y\} \times_{Y(S)} X(S)$ is contractible (in the category of spaces).

Remark 3.1.25. The following proof will assume the existence of a theory of Hopf objects in an arbitrary symmetric monoidal ∞-category.

Proposition 3.1.26. *Let X be a geometric stack. The map $\mathcal{L}^u X \to \mathcal{L}X$ is a monomorphism, i.e. unipotence of a loop is a property and not a structure.*

Proof. Let $S = \mathrm{Spec}(R)$. Consider an S-point of $\mathcal{L}X$, i.e. an $x \in X(S)$ and $g \in \Omega(X,x)(S)$. We wish to show that $\mathcal{L}^u X(S) \times_{\mathcal{L}X(S)} \{(x,g)\}$ is contractible. Equivalently, we wish to show that $\Omega^u(X,x)(S) \times_{\Omega(X,x)(S)} \{g\}$ is contractible, i.e. that the space of maps $\mathbb{G}_a \times S \to \Omega(X,x)$ such that $(1,s) \mapsto g$ is contractible. Since our objects are affine, we can work with global functions. That is, we wish to show that the sequence of spaces

$$\mathrm{Hom}_{\mathrm{Hopf(QCoh}(S))}(p_*\mathcal{O}_{\Omega(X,x)}, \mathcal{O}_{S\times\mathbb{G}_a}) \to \mathrm{Hom}_{\mathrm{Coalg(QCoh}(S))}(p_*\mathcal{O}_{\Omega(X,x)}, \mathcal{O}_{S\times\mathbb{G}_a})$$

$$\to \mathrm{Hom}_R(R[\Omega(X,x)], R[x])$$

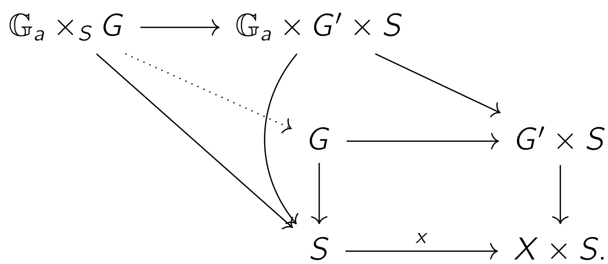
has contractible fibers. We claim that the first map has contractible fibers (i.e. the forgetful functor from Hopf objects to coalgebra objects is faithful) and that the second map is an equivalence (i.e. $\mathcal{O}_S \otimes_k \mathcal{O}(\mathbb{G}_a)$ is the cofree coalgebra object in $\mathrm{Coalg(QCoh}(S)))$. We will assume the existence of a theory of Hopf objects in a symmetric monoidal ∞-category which provides the first claim. The second claim follows from the calculation in Lemma 1.12 of [25]. ☐

Proposition 3.1.27. *Let G be a linear algebraic group over an affine k-scheme S. Then, there is a natural G-equivariant equivalence*

$$\mathrm{Hom}_{grp,S}(\mathbb{G}_a \times S, G) \simeq \widehat{U}.$$

Proof. Evaluation at the identity provides a map $\mathrm{Hom}_{grp,S}(\mathbb{G}_a \times S, G) \to G$; we claim its set-theoretic image lies in $U \subset G$, so that the map factors through $\widehat{U}$. To check this, it suffices to check along geometric points of S, i.e. we can assume $S = k$, and now the claim follows since the image of any unipotent k-point must also be unipotent.

The inverse map is produced by the exponential map as follows. We take as a given that such an exponential map is constructed for classical affine algebraic groups over k, i.e. we have a map $\mathbb{G}_a \times \widehat{U}_{G'} \to G'$. This defines a map $\mathbb{G}_a \times \widehat{U}_{G'_S} \to G'_S$. To lift this to a map $\mathbb{G}_a \times_S \widehat{U}_G \to G$, we use the universal property of fiber products and the fact that if an S-point $g \in G'(S)$ fixes $x \in X(S)$, then so does $g^t \in G'(S)$ for $t \in \mathbb{G}_a(S)$. More precisely, the following diagram commutes, inducing the desired exponential map



That these two maps are inverse equivalences is due to Proposition 3.1.26. ☐

Proposition 3.1.28 (Unipotent loops of quotient stacks)**.** *Let G be a classical affine algebraic group over k. Then,*

$$\mathcal{L}^u(BG) = \widehat{U}/G$$

where U is the unipotent cone of G. Furthermore, if G acts on a scheme X, then $\mathcal{L}^u(X/G)$ is computed by the pullback square

$$\begin{array}{ccc} \mathcal{L}^u(X/G) & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow \\ \widehat{U}/G & \longrightarrow & G/G. \end{array}$$

Proof. The first claim is a combination of Propositions 3.1.23 and 3.1.27. For the second claim, let $x \in (X/G)(S)$ and $p(x) \in BG(S)$. Since p is schematic, $\Omega(X/G, x) \to \Omega(BG, p(x)) \simeq G \times S$ is a (derived) closed immersion. The formal neighborhood of based unipotent loops is closed under inverse image. ☐

3.2 Twisted circle actions on loop spaces

Definition 3.2.1 (Twisted S^1-actions)**.** We focus on the case of a quotient stack X/G by a linear algebraic group. We fix the following notation for various circle actions on X/G. Let $z \in Z(G)$ be a central element of G.

- The *loop rotation action* on $\mathcal{L}(X/G)$ on the loop space is denoted ρ.
- The *z-twisted action* on BG or X/G for a scheme X on which z acts trivially is denoted σ_z or simply σ. This action arises on BG by the group homomorphism $\mathbb{Z} \times G \to G$ sending $(n, g) \mapsto z^n g$. On functor-of-points, recall that the S-points of X/G is the

category whose objects are diagrams

$$(X/G)(S) = \left\{ \begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle p} & & \\ S & & \end{array} \right\}$$

where $P \to S$ is a (right) G-torsor and $P \to X$ is G-equivariant. The morphisms are given torsors maps intertwining ϕ. The S^1-action determines a universal automorphism of the identity functor which multiplies on the right by z; this is a well-defined map of torsors since $z \in G$ is central, and a well-defined automorphism of ϕ since z acts trivially on X.

- The *z-twisted rotation* on $\mathcal{L}(X/G)$ is the diagonal of the $S^1 \times S^1$-action $\rho \times \mathcal{L}\sigma_z$, and is denoted τ_z or τ. Note that it makes sense to talk about the diagonal since ρ and $\mathcal{L}\sigma$ commute: ρ commutes with any group action of the form $\mathcal{L}\gamma$, where γ is an S^1-action on X/G.

Remark 3.2.2. We can characterize the z-twisted rotation on $\mathcal{L}(X/G)$ in the following way. Since z acts on X trivially and is central in G, we have a "multiplication by z" map of stacks $\mu_z : \mathcal{L}(X/G) \to \mathcal{L}(X/G)$, which is an isomorphism identifying the formal neighborhood of the fiber in X/G over $e \in G$ with that over $z \in G$, but is not S^1-equivariant under the naive rotation actions. The z-twisted rotation τ_z on $\mathcal{L}(X/G)$ is obtained by transporting the rotation action via μ_z, i.e. it can be characterized by making the diagram commute:

$$\begin{array}{ccc} S^1 \times \mathcal{L}(X/G) & \xrightarrow{a_\tau} & \mathcal{L}(X/G) \\ \simeq \downarrow {\scriptstyle \rho \times \mu_z} & & \simeq \downarrow {\scriptstyle \mu_z} \\ S^1 \times \mathcal{L}(X/G) & \xrightarrow{a_\rho} & \mathcal{L}(X/G). \end{array}$$

Example 3.2.3. On $\mathrm{Perf}(G/G)$, the rotation ρ acts on fibers over $g \in G$ by g; the z-twisting $\mathcal{L}\sigma_z$ acts on fibers over any $g \in G$ by z, and the twisted rotation τ_z acts on fibers over $g \in G$ by $gz = zg$.

Example 3.2.4. Let $G = T$ be a torus (in particular, every $t \in T$ is central). We can explicitly describe the S^1-actions on linear categories

$$\mathrm{Perf}(\mathcal{L}(BT)) = \mathrm{Perf}(T \times BT) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Perf}(T)$$

where Λ is the character lattice of T. We have a basis z^λ of $k[T]$ where we range over $\lambda \in \Lambda$. The rotation S^1-action ρ acts on the λ-summand by z^λ. The t-twisting action σ_t

on BT acts on the λ-summand by the scalar $z^\lambda(t)$. The t-twisted rotation τ_t acts on the λ-summand by $z^\lambda(t)z^\lambda$.

Let us take the t-twisted rotation τ_t (which reduces to the usual rotation for $t = e$). We have, via the category[4] PreMF in [52],

$$\mathrm{Perf}(\mathcal{L}(BT))^{S^1} = \bigoplus_{\lambda \in \Lambda} \mathrm{PreMF}(T, 1 - z^\lambda(t) \cdot z^\lambda),$$

$$\mathrm{Perf}(\widehat{\mathcal{L}}(BT))^{S^1} = \bigoplus_{\substack{\lambda \in \Lambda \\ \lambda(t) = 1}} \mathrm{PreMF}(\widehat{T}, 1 - z^\lambda).$$

For the second identity, the zeros of $1 - z^\lambda(t)z^\lambda$ meet the constant loops if and only if $z^\lambda(t) = 1$. After passing to the Tate category under the rotation action, we note that the zero locus of $1 - z^\lambda$ is smooth (it is a subgroup of T) of codimension 1 unless $\lambda = 0$ (in which case it has codimension zero and must be derived). Therefore,

$$\mathrm{Perf}(\mathcal{L}(BT))^{\mathrm{Tate}} = \bigoplus_{\lambda \in \Lambda} \mathrm{MF}(T, 1 - z^\lambda(t) \cdot z^\lambda) = \mathrm{Perf}(T) \otimes_k k(u)),$$

$$\mathrm{Perf}(\widehat{\mathcal{L}}(BT))^{\mathrm{Tate}} = \bigoplus_{\substack{\lambda \in \Lambda \\ \lambda(t) = 1}} \mathrm{MF}(\widehat{T}, 1 - z^\lambda) = \mathrm{Perf}(\widehat{T}) \otimes_k k((u)).$$

Note that the Tate categories do not depend on the twisting at all (but the S^1-invariant categories do).

Lemma 3.2.5. *Let $z \in G$ be a central element of a reductive group, and suppose z acts on a derived scheme Y trivially. The z-twisted (non-rotation) S^1-equivariant structure on $\mathcal{O}(Y/G)$ is zero. In particular, the (non-rotation) S^1-equivariant structure $\mathcal{L}\sigma_z$ on $\mathcal{O}(\mathcal{L}(X/G))$ is zero.*

Proof. The S^1-equivariant structure is given by the Connes B-operator. Note that z acts on $\mathcal{L}(X/G) \times_{BG} \mathrm{pt} = X \times_{X \times X} X \times G$ trivially, so that the second claim follows from the first. To this end, note that the map $Y/G \to \mathrm{Aff}(Y)/G$ is S^1-equivariant, so we may replace Y with its affinization. In this case, we can write down the usual simplicial presentation for the atlas $Y \to Y/G$; the S^1-action endows this simplicial presentation with the structure of a cyclic presentation. Explicitly, via the formulas in Section 7.3.3 of [38], the nth level is $Y \times G^{\times n}$, and the rotation sends $(y, g_1, \ldots, g_n) \mapsto (y, z(g_1 \cdots g_n)^{-1}, g_1, g_2, \ldots, g_{n-1})$. In particular, since z acts on Y trivially, the S^1-action does not depend on Y, so it suffices to check when $Y = \mathrm{pt}$. In this case, $\mathcal{O}(BG) = k$ has no higher cohomology since G is reductive, so the S^1-equivariant structure is zero. □

[4]For M a scheme and $f : M \to \mathbb{G}_m$, the category $\mathrm{PreMF}(M, f)$ is the category $\mathrm{Perf}(M \times_{\mathbb{G}_m} \{1\})$ with an extra $k[u]]$-linear structure acting by cohomological operators.

Corollary 3.2.6. *Suppose $z \in G$ is a central element of a reductive group and acts on a scheme X trivially. Then,*

$$\mathcal{O}(\mathcal{L}(X/G))^{S^1,\tau} \simeq \mathcal{O}(\mathcal{L}(X/G))^{S^1,\rho}$$

as objects in the equivariant category $\mathbf{Vect}_k^{\omega S^1}$. The same holds for formal and unipotent loop spaces.

Proof. There is a $S^1 \times S^1$-equivariant structure on $\mathcal{O}(\mathcal{L}(X/G))$, i.e. $\mathcal{O}(\mathcal{L}(X/G))$ has the structure of a $H_\bullet(S^1 \times S^1; k) = k[\epsilon_1, \epsilon_2]$-module where $|\epsilon_1| = |\epsilon_2| = -1$. Let the first factor be the loop rotation ρ and the second the twisting $\mathcal{L}\sigma$. Then, the S^1-equivariant structure corresponding to the diagonal τ is the mixed complex $\mathcal{O}(\mathcal{L}(X/G))$ with the action of $\epsilon_1 + \epsilon_2$. However, we've shown in the previous lemma that $\epsilon_2 = 0$. ☐

3.3 Localization for formal loop spaces

The following construction defines a notion of formal loops near a semisimple orbit $z \in G$.

Construction 3.3.1 (Formal loops over $z \in G$)**.** Let X be a derived scheme, G a reductive group acting on X, and $z \in G$ a semisimple element; we define $\widehat{\mathcal{L}}^z(X/G)$ as follows. Denote by $\mu_z : G^z/G^z \to G^z/G^z$ the multiplication by z. We define $\widehat{\mathcal{L}}^z(BG) := \widehat{\mathcal{L}}(BG^z)$ and a natural map $\widehat{\mathcal{L}}^z(BG) \to \mathcal{L}(BG)$ by the composition

$$\widehat{\mathcal{L}}^z(BG) := \widehat{\mathcal{L}}(BG^z) \xrightarrow{\mu_z} \mathcal{L}(BG^z) \longrightarrow \mathcal{L}(BG).$$

We define the *formal loops over z*, denoted $\widehat{\mathcal{L}}^z(X/G)$, by the pullback

$$\begin{array}{ccc} \widehat{\mathcal{L}}^z(X/G) & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{L}}^z(BG) & \longrightarrow & \mathcal{L}(BG). \end{array}$$

This construction is functorial in schematic maps between stacks X/G representable (by schemes) over BG.

Remark 3.3.2. Unwinding the definition, we find that the pullback can be separated into two stages

$$\begin{array}{ccccc} \widehat{\mathcal{L}}^z(X/G) & \longrightarrow & \mathcal{L}(X/G^z) & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{L}}^z(BG) & \longrightarrow & \mathcal{L}(BG^z) & \longrightarrow & \mathcal{L}(BG). \end{array}$$

The first (rightmost) pullback square arises by applying the loop space to the fiber product

$$X/G^z = X/G \times_{BG} BG^z.$$

The second arises by completion at the closed point $z \in \mathcal{L}(BG^z) = G^z/G^z$.

The following assertion proves that the right maps in the above remark are étale, and therefore isomorphisms on formal neighborhoods.

Proposition 3.3.3. *Let G be a reductive group, $z \in G$ a semisimple element and G^z its centralizer. The map $\mathcal{L}(BG^z) \to \mathcal{L}(BG)$ is locally étale at z.*

Proof. Let us recall the set-up of the étale slice theorem as in [20]. Let G be a reductive group acting on an affine variety X, and $x \in X$ a closed point such that the stabilizer $Z_G(x)$ is reductive. We define a map $\phi : X \to T_x(X)$ as follows. Let $\mathfrak{m}$ be the maximal ideal for $x \in X$; the quotient map to the cotangent space has a $Z_G(x)$-equivariant splitting $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}$ since $Z_G(x)$ is reductive, defining a map $\mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \to k[X]$. Geometrically, this means choosing functions $f_1, \ldots, f_n \in k[X]$ vanishing at x whose differentials generate the cotangent space at x, and defining the map $\phi : X \to T_x(X)$ by evaluation

$$y \mapsto \sum f_i(y) \left. \frac{d}{df_i} \right|_{y=x},$$

in a $Z_G(x)$-equivariant manner. The étale slice is the inverse image $\phi^{-1}(N)$ where $N \subset T_x(X)$ is any normal subspace to $T_x(G \cdot x) \subset T_x(X)$, and the theorem tells us that the map $G \times^{Z_G(x)} \phi^{-1}(N) \to X$ is étale.

Specializing to our situation, where G acts on itself by the adjoint action, we produce the G^z-equivariant map $\phi : G \to T_z(G)$ as follows. Affine locally at z, we can choose generators $f_1, \ldots, f_n$ such that

$$k[G]/(f_1, \ldots, f_r) = k[G^z],$$

and the vanishing of $df_1, \ldots, df_r$ determines $\mathfrak{g}^z \subset T_z(G) \simeq \mathfrak{g}$. Thus it suffices to show that $\mathfrak{g}^z$ is a normal subspace to $T_z(G \cdot z)$, since $\phi^{-1}(\mathfrak{g}^z) = G^z$ by construction. On the other hand, we have a natural isomorphism $G \cdot z \simeq G/G^z$, inducing $T_z(G \cdot z) \simeq \mathfrak{g}/\mathfrak{g}^z$, which produces a splitting of $T_z(G \cdot z) \subset T_z(G)$ whose kernel is $\mathfrak{g}^z$. Explicitly, z is semisimple and acts on $\mathfrak{g}$, so $\mathfrak{g}$ decomposes into z-eigenspaces; $\mathfrak{g}^z$ is the trivial eigenspace and $\mathfrak{g}/\mathfrak{g}^z$ is isomorphic to the sum of all other eigenspaces. □

Example 3.3.4. Let G be a simple reductive algebraic group and choose $z \in G$ regular semisimple. Its centralizer is a torus T and the map $G \times^T T^{reg} \to G^{rs}$ is étale with fiber $W_T = N(T)/T$.

We now construct the map that realizes the localization on formal loops.

Construction 3.3.5 (Localization map and formal localization map)**.** We define *global* and *formal localization maps*

$$\ell_z : \mathcal{L}(\pi_0(X^z/G^z)) \to \mathcal{L}(X/G) \qquad \widehat{\ell}_z : \widehat{\mathcal{L}}(\pi_0(X^z)/G^z) \to \widehat{\mathcal{L}}^z(X/G)$$

as follows. Let Y be a derived G^z-scheme such that the action of z is trivial, and

$$f : (G \times^{G^z} Y)/G = Y/G^z \to X/G$$

a representable map of quotient stacks (to define ℓ_z, just take $Y = \pi_0(X^z)$). We denote $p : X/G \to BG$ and $q : X/G^z \to BG^z$. We have S^1-equivariant pullback squares

$$\begin{array}{ccccc}
\mathcal{L}(Y/G^z) & \xrightarrow[\simeq]{} & \mathcal{L}(Y/G^z)^\rho & & \\
\downarrow & \searrow^{\ell_z} & \downarrow{\scriptstyle \mathcal{L}f} & & \\
\mathcal{L}(X/G^z)_{shift} & \xrightarrow[\simeq]{} & \mathcal{L}(X/G^z)^\rho & \xrightarrow[\text{étale}]{} & \mathcal{L}(X/G)^\rho \\
\downarrow & & \downarrow{\scriptstyle \mathcal{L}q} & & \downarrow{\scriptstyle \mathcal{L}p} \\
\mathcal{L}(BG^z)^\tau & \xrightarrow[\simeq]{\mu_z} & \mathcal{L}(BG^z)^\rho & \xrightarrow[\text{étale}]{G^z \subset G} & \mathcal{L}(BG)^\rho.
\end{array}$$

The subscript in $\mathcal{L}(X/G^z)_{shift}$ is only to emphasize that the map $\mathcal{L}(X/G^z) \to \mathcal{L}(BG^z)^\tau$ on the left cannot be identified with the loop space of a map $X/G^z \to BG^z$; it is "shifted" so that the constant loops live over $z^{-1} \in G^z$. On the other hand, by Remark 3.2.2 the left map $\mathcal{L}(Y/G^z) \to \mathcal{L}(BG^z)$ can be identified with the map on loops induced by $Y/G^z \to BG^z$ since z acts trivially on Y. This entire diagram can be completed at constant loops on $\mathcal{L}(BG^z)^\tau$, equivalently loops near $t \in \mathcal{L}(BG^z)^\rho$ to obtain the map $\widehat{\ell}_z$

$$\begin{array}{ccccc}
\widehat{\mathcal{L}}(Y/G^z) & \xrightarrow[\simeq]{} & \widehat{\mathcal{L}}^z(Y/G^z)^\rho & & \\
\downarrow & \searrow^{\widehat{\ell}_z} & \downarrow{\scriptstyle \widehat{\mathcal{L}}^z f} & & \\
\widehat{\mathcal{L}}^z(X/G^z)_{shift} & \xrightarrow[\simeq]{} & \widehat{\mathcal{L}}^z(X/G^z)^\rho & \xrightarrow[\simeq]{} & \widehat{\mathcal{L}}^z(X/G)^\rho \\
\downarrow & & \downarrow{\scriptstyle \widehat{\mathcal{L}}^z q} & & \downarrow{\scriptstyle \widehat{\mathcal{L}}^z p} \\
\widehat{\mathcal{L}}(BG^z)^\tau & \xrightarrow[\simeq]{\widehat{\mu_z}} & \widehat{\mathcal{L}}^z(BG^z)^\rho & \xrightarrow[\simeq]{} & \widehat{\mathcal{L}}^z(BG)^\rho.
\end{array}$$

This construction is functorial in schemes over BG.

Remark 3.3.6. One can check that ℓ_z can be identified with the map (breaking S^1-symmetry):

$$\left(\begin{array}{ccc}
 & & G \times^{G^z} (\pi_0(X^z) \times G^z) \\
 & & \downarrow \\
G \times^{G^z} \pi_0(X^z) & \longrightarrow & G \times^{G^z} (\pi_0(X^z) \times \pi_0(X^z))
\end{array}\right) \longrightarrow \left(\begin{array}{ccc}
 & & X \times G \\
 & & \downarrow \\
X & \longrightarrow & X \times X
\end{array}\right)$$

where the top map sends $(h, x, g) \mapsto (h \cdot x, hgzh^{-1})$.

Theorem 3.3.7. *Let X be a smooth variety with an action of a reductive group G, and $z \in G$ semisimple. The S^1-equivariant map*

$$\widehat{\ell}_z : \widehat{\mathcal{L}}(\pi_0(X^z)/G^z) \to \widehat{\mathcal{L}}^z(X/G)$$

is an isomorphism.

Proof. By the discussion in Definition 3.3.5 we only need to show that $\mathcal{L}^z f$ is an isomorphism at local rings over $z \in G^z$, i.e. replacing G with G^z, we can assume G is a reductive group acting on a scheme X, and that $z \in G$ is central. We wish to show that $\widehat{\mathcal{L}}^z(\pi_0(X^z)/G) \to \widehat{\mathcal{L}}^z(X/G)$ is an isomorphism (note there is no "shift by z" in this set-up). The map $\widehat{\mathcal{L}}^z$ is an isomorphism on $\pi_0(-)^{red}$ by the discussion in Lemma 3.1.6, so we need only show it is a derived isomorphism. It thus suffices to show that $\mathcal{L}^z f : \mathcal{L}^z(\pi_0(X^z/G)) \to \mathcal{L}^z(X/G)$, after forgetting equivariance, induces an isomorphism on local rings for all closed points $x \in \pi_0(X^z)$. We will use the essential fact that X^z is smooth when $z \in G$ is a semisimple element of a reductive group: in this case, we can choose a torus T containing z and and apply the étale slice theorem to a T-closed affine open cover.

Let us first discuss the localization that we carry out. Forget all group actions; we wish to localize the diagram

$$\begin{array}{ccc} \mathcal{L}(\pi_0(X^z)/G) \times_{BG} \text{pt} & \longrightarrow & \mathcal{L}(X/G) \times_{BG} \text{pt} \\ & \searrow & \downarrow \\ & & G \end{array}$$

at points $(x, z) \in \pi_0(\mathcal{L}(\pi_0(X^z)/G))$ upstairs and show that the resulting map on local rings is an isomorphism. More explicitly, let $A = \mathcal{O}_{X,x}$ be the local ring with maximal ideal $\mathfrak{m}$, $J \subset \mathfrak{m} \subset A$ the ideal for $\pi_0(X^z) \subset X$ and $B = \mathcal{O}_{G,z}$ be the local ring at $z \in G$. The coaction induces a map $c : A \to A \otimes B$. We can compute

$$(\mathcal{L}(X/G) \times_{BG} \text{pt})_{loc} = A \otimes^L_{A\otimes A} (A \otimes B)$$

$$(\mathcal{L}(\pi_0(X^z)/G) \times_{BG} \text{pt})_{loc} = A/J \otimes^L_{A/J\otimes A/J} (A/J \otimes B)$$

with the map between them being the naive one (induced by the quotient $A \to A/J$). By smoothness of X and X^z, the diagonal embeddings are local complete intersections, and $X^z \subset X$ is also a local complete intersection. In particular by Nakayama's lemma we can produce generators $x_1, \ldots, x_n$ of $\mathfrak{m}$ whose linear span is G-closed, such that $J = (x_1, \ldots, x_r)$, the linear span of $x_1, \ldots, x_r$ forms a subrepresentation, and furthermore $x_i \otimes 1 - 1 \otimes x_i$ forms a regular sequence for the diagonal[5]. Thus, taking semi-free Koszul resolutions for A as an $A \otimes A$-module and A/J as an $A/J \otimes A/J$-module, we have the derived tensor products

$$A \otimes^L_{A\otimes A} (A \otimes B) = ((A \otimes B)[\epsilon_1, \ldots, \epsilon_n], d_{int}(\epsilon_i) = c(x_i) - x_i)$$

[5]In particular, since A and A/J are Cohen-Macaulay, any minimal generating set is automatically regular.

$$A/J \otimes^L_{A/J\otimes A/J} (A/J \otimes B) = ((A/J \otimes B)[\epsilon_{r+1}, \dots, \epsilon_n], d_{int}(\epsilon_i) = c(x_i) - x_i)$$

where $|\epsilon_i| = -1$. The map

$$A \otimes^L_{A\otimes A} A \otimes B \to A/J \otimes^L_{A/J\otimes A/J} (A/J \otimes B)$$

sends $\epsilon_i \mapsto 0$ for $i \leq r$ and $\epsilon_i \mapsto \epsilon_i$ when $i > r$. We wish to show that the derived equations imposed on $\mathcal{L}(X/G)$ via the Koszul resolution $(d_{int}(\epsilon_1), \dots, d_{int}(\epsilon_r))$ and the classical equations imposed on $\mathcal{L}(\pi_0(X/G))$ via quotient $(x_1, \dots, x_r)$ differ by an (invertible) change of variables, so that this map is a quasi-isomorphism.

Write $c(x_i) = \sum_{i,j=1}^s e_{ij} x_j$ where $e_{ij} \in B$ and let $E = (e_{ij})$ denote the corresponding matrix, so that the two coordinates we wish to relate are related by the matrix $E - I$. We claim this matrix is invertible. It suffices to show that its evaluation at $z \in G$ is invertible, i.e. to show that $E(z) = \mathrm{ev}_z \circ c : J/J^2 \to J/J^2$ has no eigenvector with eigenvalue 1 in its induced action on the conormal space J/J^2 of $X^z \subset X$ at z. That z acts on the normal spaces of X^z by eigenvalues not equal to 1 is a property of the z-fixed point variety. □

Corollary 3.3.8. *Let X, Y, Z be smooth varieties, with maps $f : X \to Z$ and $g : Y \to Z$. Then*

$$\widehat{\mathcal{L}}(\pi_0(X^z) \times_{\pi_0(Z^z)} \pi_0(Y^z)) \simeq \widehat{\mathcal{L}}^z(X \times_Z Y)$$

where all fiber products are derived.

Proof. This follows immediately since loop spaces commute with fiber products. □

Remark 3.3.9. Note that in the case $G = T$ is a torus, the étale factor of the map

$$\ell_z : \mathcal{L}(\pi_0(X^z)/G^z) \to \mathcal{L}(X/G)$$

is trivial, so ℓ_z is an isomorphism on local rings (rather than formal neighborhoods) over $z \in G/G$. This recovers a statement similar to the K-theoretic equivariant localization. In fact, we have something even stronger: the following lemma shows how to find a Zariski open neighborhood $U \subset T$ of z on which ℓ_z induces an isomorphism. That is, only finitely many subgroups of T appear as stabilizers of the action of T on a fixed X, and ℓ_z is an isomorphism on the localization of T obtained by deleting those subgroups which do not contain z, which parallels the equivariant localization argument for equivariant cohomology in [26].

Lemma 3.3.10 (Finiteness of stabilizer subgroups)**.** *Let T be a torus acting on a (quasi-compact) variety X. Only finitely many subgroups of T may appear as stabilizers of this action.*

Proof. We can work affine locally, since X has a finite T-closed Zariski cover, and may also assume that X is connected. If every point of X has stabilizer of equal dimension to T, the possible stabilizer subgroups are in bijection with a subset of the (obviously finite) set of subgroups of the (finite) component group T/T°. If there is a point $x \in X$ whose stabilizer $T^x \subset T$ has strictly smaller dimension, then by the Luna slice theorem (note that the stabilizer T^x is reductive since every subgroup is) there is a locally closed subvariety $V \subset X$ such that the map $a : T \times^{T^x} V \to X$ is étale and dominant. Any stabilizer of a point in the image of a must be a subgroup of T^x, so the problem reduces to considering the action of T^x on V, and the action of T on the compliment of $V \subset X$ (which is a closed subvariety, therefore affine, of strictly smaller dimension). Note that in both cases, the dimension of the variety decreases, and that the claim is obviously true for zero-dimensional varieties, so the lemma follows by induction. □

3.4 Localization for unipotent loop spaces

We now carry out the constructions in the previous section for unipotent loop spaces, and prove an analogous result.

Construction 3.4.1 (Unipotent loops over $z \in G$)**.** Let X be a derived scheme, G a reductive group acting on X, and $z \in G$ a semisimple element; we define $\mathcal{L}^{u,z}(X/G)$ as follows. First define $\mathcal{L}^{u,z}(BG) := \mathcal{L}^u(BG^z)$ and a natural map $\mathcal{L}^{u,z}(BG) \to \mathcal{L}(BG)$ by the composition

$$\mathcal{L}^{u,z}(BG) = \mathcal{L}^u(BG^z) \xrightarrow{\mu_z} \mathcal{L}^u(BG^z) \longrightarrow \mathcal{L}(BG^z) \longrightarrow \mathcal{L}(BG)$$

We define the *unipotent loops over* z, denoted $\mathcal{L}^{u,z}(X/G)$, by the pullback

$$\begin{array}{ccc} \mathcal{L}^{u,z}(X/G) & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow \\ \mathcal{L}^{u,z}(BG) & \longrightarrow & \mathcal{L}(BG) \end{array}$$

This construction is functorial in schematic maps between stacks X/G representable (by schemes) over BG.

Proposition 3.4.2. *Let $a : G/G \to G//G$ be the affinization, and let $a(z) = [z] \in G//G$. The map above induces an isomorphism on completions*

$$\mathcal{L}^{u,z}(BG) \simeq \widehat{a^{-1}([z])}/G \subset \mathcal{L}(BG).$$

In particular, the map $\mathcal{L}^{u,z}(BG) \to \mathcal{L}(BG)$ factors isomorphically through $\widehat{\mu^{-1}([z])}/G$.

Proof. This follows from Proposition 3.1.28 and the observation that the classical reduced fiber over $[z] \in G$ is isomorphic to the unipotent cone of G^z/G^z. That is, if $u \in G^z$ is unipotent, then $uz = zu \in G$ has the same eigenvalues as z, so $a(z) = a(zu)$, and by Jordan composition any $y \in \mu^{-1}([z])$ can be written uniquely in this way. Furthermore, $G^{uz} \subset G^z$, so letting U be the unipotent elements of G^z, we have that the fibers of the map $G \times^{G^z} U \to G$ over points in its image $\mu^{-1}([z])$ are singleton points. □

Construction 3.4.3 (Unipotent localization map)**.** We define a *unipotent localization map*

$$\ell_z^u : \mathcal{L}^u(\pi_0(X^z)/G^z) \to \mathcal{L}^{u,z}(X/G)$$

as follows, using the same set-up as Definition 3.3.5. The diagram can be completed at unipotent loops on $\mathcal{L}(BG^z)^\tau$:

$$\begin{array}{ccccc}
\mathcal{L}^u(Y/G^z) & \xrightarrow[\simeq]{} & \mathcal{L}^{u,z}(Y/G^z)^\rho & \overset{\ell_z^u}{\cdots\cdots\rightarrow} & \\
\downarrow & & \downarrow{\scriptstyle \mathcal{L}^{u,z}f} & & \\
\mathcal{L}^{u,z}(X/G^z)_{shift} & \xrightarrow[\simeq]{} & \mathcal{L}^{u,z}(X/G^z)^\rho & \xrightarrow[\simeq]{} & \mathcal{L}^{u,z}(X/G)^\rho \\
\downarrow & & \downarrow{\scriptstyle \mathcal{L}^{u,z}q} & & \downarrow{\scriptstyle \mathcal{L}^{u,z}p} \\
\mathcal{L}^u(BG^z)^\tau & \xrightarrow[\simeq]{\mu_z^u} & \mathcal{L}^{u,z}(BG^z)^\rho & \xrightarrow[\simeq]{\text{Prop 3.4.2}} & \mathcal{L}^{u,z}(BG)^\rho
\end{array}$$

This construction is functorial in schemes over BG.

Corollary 3.4.4. *Let X be a smooth variety with an action of a reductive group G. The S^1-equivariant map*

$$\ell_z^u : \mathcal{L}^u(\pi_0(X^z)/G^z) \to \mathcal{L}^{u,z}(X/G)$$

is an isomorphism.

Proof. By Definition 3.4.3, the map is an isomorphism on π_0. Let $z \in G$ be a central semisimple element, and $u \in G$ be any unipotent element. The proof of Theorem 3.3.7 goes through with the following modification: we have to check local rings at $(x, zu) \in X \times G$ where $x \in \pi_0(X^{zu}) \subset \pi_0(X^z)$, and $u \in G$ is unipotent. The same argument goes through, as we only require semisimplicity to show that $\widehat{\mathcal{L}}(X/G^z) \to \widehat{\mathcal{L}}(X/G)$ is an isomorphism, and we only require centrality in defining the shift map. □

3.5 Tate S^1-equivariant functions on formal loop spaces compute Betti cohomology

We now set out to prove the equivariant localization theorem for periodic cyclic homology. We first establish the following general assertion for geometric stacks. It is well-known by experts and is essentially a simple corollary of results in [10], [30] and [6]. A general discussion can also be found in the introduction of [34]. We first introduce some technical notions needed to phrase the result in the 2-periodic setting. Recall the following notions for vector spaces from [4].

Definition 3.5.1. A *linear topological vector space* is a a vector space V which admits a topology for which the vector space operations are continuous, and such that there is a system of neighborhoods at 0 consisting of subspaces. In this case, the topology is generated by this system at 0 and translations under addition. The *completion* $\widehat{V}$ of V is the limit over the system of neighborhoods U_α:

$$\widehat{V} := \lim_{0 \in U_\alpha} V/U_\alpha$$

We say the topology is *complete* if the natural map $V \to \widehat{V}$ is an isomorphism. Let V_1, V_2 be linear topological vector spaces. We define the !-tensor product $V_1 \otimes^! V_2$ with topology by the basis consisting of open sets of the form $U_1 \otimes V_2 + V_1 \otimes U_2$, where $U_1 \subset V_1$ and $U_2 \subset V_2$ are opens.

These notions generalize immediately to chain complexes, where we replace the notion of subspace with subcomplex. In this case, the complexes term-wise satisfy the Mittag-Leffler condition and therefore $\lim^1 = 0$, so that the naive notion of completion is "automatically derived."

Remark 3.5.2. Let X be a possibly singular scheme, and $X \subset M$ an embedding into a smooth scheme. The *derived de Rham complex* on a scheme X, defined in [10], is the complex of quasicoherent sheaves built from (derived) exterior powers of $\mathbb{L}_X$ on X, equipped with the de Rham differential. The main theorem of [10] produces a quasi-isomorphism between the Hodge-completed[6] derived de Rham complex of X and the de Rham complex of M completed at the closed subscheme X (also considered as an abelian sheaf on X). Hartshorne [30] proved that the hypercohomology of this latter complex computes Betti cohomology.

These complexes arise in the context of loop spaces follows. For X a scheme, we have that $\widehat{\mathcal{L}}X = \mathcal{L}X \simeq \mathbb{T}_X[-1]$. Though the adic filtration corresponding to the closed immersion of constant loops defines an analogue of the Hodge filtration, these *negatively*-graded differential forms are already complete with respect to this filtration. On the other hand, the negative cyclic homology $(HH(\mathrm{Perf}(X))[[u]], d+u\epsilon)$ is completed with respect to the *noncommutative Hodge filtration* defined by $F^k HN(\mathrm{Perf}(X)) = u^k HN(\mathrm{Perf}(X))$. The $\otimes^!$-tensor product of these filtrations produces a 2-periodic version of the Hodge filtration.

Proposition 3.5.3. *Let X be a geometric stack with a smooth cover by a variety. There is a natural isomorphism*

$$\mathcal{O}(\widehat{\mathcal{L}}(X))^{\mathrm{Tate}} \simeq C^\bullet_{dR}(X^{an}; k) \otimes^! k((u))$$

where $C^\bullet_{dR}(X; k)$ denotes the Hodge-completed de Rham cochains on X.

[6]Completing with respect to the Hodge filtration (i.e. the "rows" in the spectral sequence) makes a difference when X is singular, since the total complex may be unbounded in a positive and a negative direction: the de Rham complex extends in the positive direction, whereas a semifree resolution extends in the negative direction.

Proof. By Theorem 6.9 of [6], the exponential map $\widehat{\mathbb{T}}_X[-1] \to \widehat{\mathcal{L}}(X)$ is an filtration-preserving isomorphism, so we compute $\mathcal{O}(\widehat{\mathbb{T}}_X[-1])^{\mathrm{Tate}}$ instead. We first show that the mixed differential on

$$\mathcal{O}(\widehat{\mathbb{T}}_X[-1]) = \bigoplus_{i=0}^{\infty} \Gamma(X, \Omega^i_X)[i]$$

is the de Rham differential. Sheaves (and functions) on formal loop spaces and odd tangent bundles are smooth local, and since $\Gamma(X, -)$ commutes with limits, it commutes with both S^1-invariants and totalization. We can compute the S^1-invariant global functions via global sections of the following (non-quasicoherent) complex (where $U^{\times_X n}$ denotes the nth stage in the simplicial complex associated to a smooth cover $U \to X$ by a scheme)

$$\begin{aligned} \mathcal{O}(\widehat{\mathbb{T}}_X[-1])^{S^1} &\simeq \mathrm{Tot}(\mathcal{O}(\mathbb{T}_{U^{\times_X n}}[-1])^{S^1}) \simeq \mathrm{Tot}(\mathcal{O}(\mathbb{T}_{U^{\times_X n}}[-1]))^{S^1} \\ &\simeq \Gamma(X, \mathrm{Sym}_X \mathbb{L}_X[1])^{S^1} \simeq (\Gamma(X, \lim_j \mathrm{Sym}_X^{\leq j} \mathbb{L}_X[1][[u]]), u \cdot d_{dR}). \end{aligned}$$

This completes the claim.

Next, note that the Tate construction can be realized as a limit followed by a colimit (i.e. a localization of the S^1-invariants) or a colimit followed by a limit (i.e. a limit of the S^1-coinvariants under the periodicity operator). In particular, using the second description, we find that $\mathcal{O}(\widehat{\mathbb{T}}_X[-1])^{\mathrm{Tate}}$ is the completion of the topological chain complex

$$(\Gamma(X, \mathrm{Sym}_X \mathbb{L}_X[1]) \otimes^! k[u, u^{-1}], u \cdot d_{dR})$$

where (ignoring the differential; one verifies that the differential is compatible with the following topology) we equip $\Gamma(X, \mathrm{Sym}_X \mathbb{L}_X[1])$ with the topology induced by the $\mathrm{Sym}_X^{\geq 1} \mathbb{L}_X[1]$-adic topology on hypercohomology, and $k[u]$ with the u-adic topology. Forgetting the topology, this complex evidently has the same underlying chain complex as $C^\bullet_{dR}(X^{an}; k) \otimes^! k((u))$. We claim their topologies agree. The topology on $\mathcal{O}(\widehat{\mathcal{L}}(X))^{\mathrm{Tate}}$ has basis given by opens $U_{ij} := \Gamma(X, \mathrm{Sym}_X^{\geq i} \Omega^1_X[1] u^{\geq j})$. The topology on $C^\bullet_{dR}(X^{an}; k) \otimes^! k((u))$ is given by opens $V_{ij} := \Gamma(X, \mathrm{Sym}_X^{\geq i} \Omega^1_X[-1] u^{\geq j})$. We see that $U_{i,j+i} = V_{i,j}$, so the topologies agree, so the completions agree. □

Remark 3.5.4. The above proposition is false if we do not consider the topologies. For example, take $X = B\mathbb{G}_m$. Then, we have $\mathcal{O}(\widehat{\mathcal{L}}(B\mathbb{G}_m)) = k[[[t]]$ where $|t| = 0$, and in particular, $H^0(\mathcal{O}(\widehat{\mathcal{L}}(B\mathbb{G}_m))^{\mathrm{Tate}}) = k[[t]]$. On the other hand, $H^\bullet(B\mathbb{G}_m; k) \simeq k[s] = k[[s]]$ where $|s| = 2$, so $H^0(H^\bullet(B\mathbb{G}_m; k)((u))) = k[su^{-1}]$.

3.6 Equivariant localization for periodic cyclic homology

We prove the equivariant localization theorem for periodic cyclic homology, assuming the following theorem, which is proven as Theorem 3.7.15.

Theorem 3.6.1. *Let X be an algebraic space with an action of an affine algebraic group G. The natural map*

$$\mathcal{O}(\mathcal{L}^u(X/G))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}$$

is an isomorphism.

We first need to introduce a few technical notions regarding mixed complexes, with the aim of proving that in our situation formal completions commute with the Tate construction on Hochschild homology. Analogous results and arguments can be found in [34].

Definition 3.6.2. Let (V, d, ϵ) be a mixed complex. We define

$$V^{\prod \mathrm{Tate}} = (\prod_k V u^k, d + u\epsilon)$$

where $|u| = 2$. Note that $V^{\prod \mathrm{Tate}}$ cannot carry a multiplicative structure. We also define

$$V^{\oplus \mathrm{Tate}} = (\bigoplus_k V u^k, d + u\epsilon) = (V[u^{-1}, u], d + u\epsilon).$$

There are natural maps

$$V^{\oplus \mathrm{Tate}} \to V^{\mathrm{Tate}} \to V^{\prod \mathrm{Tate}}.$$

Remark 3.6.3. Lemma 2.6 of [34] shows that the Tate construction preserves quasi isomorphisms, essentially because it is computed via the right spectral sequence. On the other hand, the other variants above do not preserve quasi-isomorphisms. In particular, they are not well-behaved in the derived category.

Lemma 3.6.4. *Let (V, d, ϵ) be a mixed complex. If V is cohomologically bounded below, then $V^{\oplus \mathrm{Tate}} \to V^{\mathrm{Tate}}$ is a quasi-isomorphism. If V is cohomologically bounded above, then $V^{\mathrm{Tate}} \to V^{\prod \mathrm{Tate}}$ is a quasi-isomorphism.*

Proof. The proof of the first statement appears as Corollary 2.7 in [34] (note that they use homological grading conventions). For the second, note that although the product-Tate construction does not respect quasi-isomorphisms, the usual Tate construction does, so we can still replace V with a quasi-isomorphic mixed complex. In particular, replace V via quasi-isomorphism with a complex which vanishes in sufficiently large degrees. Then, $V^{\mathrm{Tate}} \to V^{\prod \mathrm{Tate}}$ is an isomorphism, since they differ only in direct sum versus direct product in the u^k component for k small, but since V vanishes in large degree, in each cohomological degree the u^k component vanishes for sufficiently small k. □

Theorem 3.6.5 (Equivariant localization for periodic cyclic homology)**.** *Let G be a reductive group acting on a smooth quasi-projective variety X. The periodic cyclic homology $HP(\mathrm{Perf}(X/G))$ is naturally a module over $HP(\mathrm{Perf}(BG)) = k[G//G]((u))$. For a closed* *point $z \in G//G$, we have a functorial (with respect to pullback) identification of the formal completion at z with a 2-periodicization of the singular cohomology of the fixed points*

$$HP(\mathrm{Perf}(X/G))_{\widehat{z}} \simeq H^\bullet((X^z)^{an}/(G^z)^{an}; k) \otimes_k^! k((u))$$

naturally as a module over $HP(\mathrm{Perf}(BG))_{\widehat{z}} \simeq H^\bullet(B(G^z)^{an}; k)^! \otimes_k k((u))$.

Proof. First, note that though the map of stacks $G/G \to G//G$ is not representable by schemes, we can still calculate (directly via its functor of points) that classical fiber product

$$\pi_0(G/G \times_{G//G} \{z\}) = \{G \cdot z\}/G$$

is a closed substack of G/G. In particular, by its functor of points description, the fiber product with respect to the formal completion is exactly $\widehat{\{G \cdot z\}}/G$, and by Proposition 3.4.2 we have that

$$\mathcal{L}(X/G) \times_{G//G} \widehat{\{z\}} = \mathcal{L}^{u,z}(X/G)$$

and in particular, since pushforward commutes with limits, we have that $\mathcal{O}(\mathcal{L}^{u,z}(X/G)) \simeq \mathcal{O}(\mathcal{L}(X/G))_{\widehat{z}}$.

We wish to show that the Tate construction commutes with completion, i.e.

$$HP(\mathrm{Perf}(X/G))_{\widehat{z}} = (\mathcal{O}(\mathcal{L}(X/G))^{\mathrm{Tate}})_{\widehat{z}} \simeq (\mathcal{O}(\mathcal{L}(X/G))_{\widehat{z}})^{\mathrm{Tate}}.$$

This claim turns out to be rather delicate, and occupies most of this proof.

First, we claim that the explicit realization of Hochschild homology $HH(\mathrm{Perf}(X/G))$ via the cyclic bar complex of Example 2.8.9 is a $k[G]^G$-linear mixed complex, i.e. a mixed complex where both the internal and mixed differentials are $k[G]^G$-linear. By Proposition 5.1.28 of [16], if X is quasiprojective (this is the only place we use quasiprojectiveness), then $\mathrm{Perf}(X/G)$ is generated by locally free objects; write the cyclic bar complex using the generating set of all equivariant locally free sheaves on X. Its $k[G]^G$-linear structure arises via the shuffle product as follows. Via the Peter-Weyl theorem there is a natural isomorphism $k[G]^G \simeq \bigoplus_{V \mathrm{irrep.}} k$, so that $c \in \mathrm{Hom}_{\mathrm{Perf}(BG)}(V_\lambda, V_\lambda)$ acts by sending elements of the cyclic Hom-tensor with objects $X_0, \dots, X_n$ to the cyclic Hom-tensor with objects $X_0 \otimes V_\lambda, X_1 \otimes V_\lambda, \dots, X_n \otimes V_\lambda$ by

$$f_0 \otimes \cdots \otimes f_n \mapsto (f_0 \otimes c) \otimes (f_1 \otimes 1) \otimes \cdots \otimes (f_n \otimes 1) = (f_0 \otimes 1) \otimes \cdots \otimes (f_{n-1} \otimes 1) \otimes (f_n \otimes c).$$

This action respects the face maps and degeneracy maps by general construction. Note that since the X_i are locally free, there are no higher Ext sheaves.

By Theorem 1.4.2 of [21], $HH(\mathrm{Perf}(X/G))$ is cohomologically bounded above. Recall that the $\prod$ Tate construction does not behave well with respect to quasi-isomorphisms, so we need to fix this particular model computing $HH(\mathrm{Perf}(X/G))$. By Lemma 3.6.4, the map

$$HP(\mathrm{Perf}(X/G)) \to HH(\mathrm{Perf}(X/G))^{\prod \mathrm{Tate}}$$

is a quasi-isomorphism. In particular, the right-hand side is entirely built out of products, so it commutes with limits such as derived completion. More precisely, letting $K_n^\bullet$ be the Koszul complex computing derived completion,

$$HP(\mathrm{Perf}(X/G)_{\widehat{z}} \simeq HH(\mathrm{Perf}(X/G))^{\Pi\,\mathrm{Tate}})_{\widehat{z}} \simeq \lim_n((\prod_k HH(\mathrm{Perf}(X/G))u^k, d+u\epsilon)\otimes_{k[G]^G} K_n^\bullet).$$

Since the mixed complex structure on $HH(\mathrm{Perf}(X/G))$ is $k[G]^G$-linear and $K_n^\bullet$ is semi-free of bounded cohomological amplitude, this is equal to

$$\lim_n(\prod_k HH(\mathrm{Perf}(X/G))u^k \otimes_{k[G]^G} K_n^\bullet, d+u\epsilon) = \lim_n HH(\mathrm{Perf}(X/G)\otimes_{k[G]^G} K_n^\bullet)^{\Pi\,\mathrm{Tate}}.$$

Now, since products commute with limits, and $HH(\mathrm{Perf}(X/G))\otimes_{k[G]^G} K_n^\bullet$ is still bounded above, this is equivalent to

$$(HH(\mathrm{Perf}(X/G))_{\widehat{z}})^{\Pi\,\mathrm{Tate}} \simeq HH(\mathrm{Perf}(X/G)_{\widehat{z}})^{\mathrm{Tate}}.$$

By Corollary 3.4.4, Theorem 3.6.1, Corollary 3.2.6, and Proposition 3.5.3, we have

$$HP(\mathrm{Perf}(X/G))_{\widehat{z}} \simeq \mathcal{O}(\mathcal{L}^{u,z}(X/G))^{\mathrm{Tate}} \simeq \mathcal{O}(\mathcal{L}^u(\pi_0(X^z)/G^z))^{\mathrm{Tate},\tau}$$

$$\simeq \mathcal{O}(\mathcal{L}^u(\pi_0(X^z)/G^z))^{\mathrm{Tate},\rho} \simeq \mathcal{O}(\widehat{\mathcal{L}}(\pi_0(X^z)/G^z)) \simeq H^\bullet((X^z)^{an}/(G^z)^{an};k)((u)).$$

Functoriality follows by functoriality of Theorem 3.3.7 and since f^* preserves compact objects in quasi-coherent sheaves. □

Example 3.6.6 (Toy example)**.** Let $G = \mathbb{G}_m = \mathrm{Spec}\, k[z,z^{-1}]$ act on $X = \mathbb{A}^1 = \mathrm{Spec}\, k[x]$ by scaling, i.e. the $\mathbb{G}_m$-weight $|x| = 1$. The loop space can be calculated directly

$$\mathcal{L}(\mathbb{A}^1/\mathbb{G}_m) = \frac{\mathrm{Spec}\, k[z,z^{-1},x]/\langle x(z-1)\rangle}{\mathbb{G}_m}$$

and has no derived structure (since G acts on X by finitely many orbits). The Hochschild homology and periodic cyclic homology can also be calculated directly

$$HH(\mathrm{Perf}(\mathbb{A}^1/\mathbb{G}_m)) = (k[z,z^{-1},x]/x(z-1))^{\mathbb{G}_m} = k[z,z^{-1}]$$

$$HP(\mathrm{Perf}(\mathbb{A}^1/\mathbb{G}_m)) = k[z,z^{-1}]((u))$$

as the only possible S^1-action on a $HH(\mathrm{Perf}(\mathbb{A}^1/\mathbb{G}_m))$ is trivial, being concentrated in a single cohomological degree. It is clear that completing at any $z_0 \in \mathbb{G}_m$ gives, for $t = z - z_0$ and $|t| = 0$,

$$HP(\mathrm{Perf}(\mathbb{A}^1/\mathbb{G}_m))_{\widehat{z}} \simeq k[[t]]((u)).$$

On the other hand, we can compute $H^\bullet((X^z)^{an}; k)((u))$ for each z_0. For $z = z_0 \neq 1$, the fixed points $\pi_0(X^{z_0})/G^{z_0} = \{0\}/\mathbb{G}_m \simeq BS^1$, whose 2-periodic cohomology is $H^\bullet(BS^1; k)((u)) = k[[s]]((u))$ with $|s| = 2$. For $z = 1$ the fixed points are $\mathbb{A}^1/\mathbb{G}_m \simeq \mathbb{C}/S^1 \simeq BS^1$, and the same argument applies. Note that the identification $k[[s]]((u)) \simeq k[[t]]((u))$ is by the relationship $tu = s$; in particular it is necessary that we formally invert u for the rings $k[[t]]((u))$ and $k[[s]]((u))$ to be isomorphic even abstractly. The discrepancy between the cohomological degrees of t and s is a manifestation of the Koszul duality shearing discussed in [6].

Example 3.6.7 (Flag variety)**.** Let $X = G/B$ be the flag variety with the usual action of G. Then, $X/G = BB$, so $\mathcal{L}(X/G) = B/B = \tilde{G}/G$ is the Grothendieck-Springer resolution. The fiber for the map $\tilde{G} \to G$ over any point $g \in G$ consists of the Borel subgroups containing g, i.e. the g-fixed points of G/B. We have that

$$HH(\mathrm{Perf}(BB)) = \mathcal{O}(\tilde{G}/G) = k[H] \leftarrow HH(\mathrm{Perf}(BG)) = \mathcal{O}(G/G) = k[H]^W$$

where H is the universal Cartan subgroup and W the universal Weyl group. Let $s \in G$ be a semisimple element, and $[s]$ its adjoint orbit. Completing at $[s] \in k[H]^W$, we have

$$HP(\mathrm{Perf}(BB))_{\widehat{[s]}} = \bigoplus_{|W\cdot s|} k[[\mathfrak{h}]] \leftarrow HP(\mathrm{Perf}(BG))_{\widehat{s}} \simeq k[[\mathfrak{h}]]^{W_{G^s}}.$$

On the left, the tangent space at all points of H are identified naturally with $\mathfrak{h}$ everywhere, and there are $|W \cdot s|$ preimages to the point $[s] \in H//W$. On the right, the tangent space at $[s]$ is identified with $\mathfrak{h}$ modulo W^s, the subgroup of W fixing s. In particular, the rank of $HP(\mathrm{Perf}(BB))_{\widehat{s}}$ over $HP(\mathrm{Perf}(BG))_{\widehat{s}}$ is $|W \cdot s| \cdot |W_{G^s}| = |W|$ by a theorem of Steinberg and Pittie.

On the other hand, the fixed points $(G/B)^s$ consist of Borels containing s; by conjugating, we can choose a torus such that $s \in T \subset B$. There is a G^s-action on $(G/B)^s$ and its stabilizer at every point is conjugate to B^s, but the action may not be transitive; thus, $(G/B)^s$ is the disjoint union of copies of G^s/B^s. To count the number of connected components we count T-fixed points: the T-fixed points of G/B are also s-fixed points, and furthermore each G^s/B^s contains $|W_{G^s}|$ such T-fixed points, so we have $|W|/|W_{G^s}|$ connected components. Finally, taking into account equivariance, we have

$$H^\bullet(\coprod_{|W|/|W\cdot s|} B(B^s)^{an}; k) = \bigoplus_{|W|/|W^{G^s}|} k[[\mathfrak{h}]] \leftarrow H^\bullet(B(G^s)^{an}; k) = k[[\mathfrak{h}]]^{W_{G^s}}$$

where $\mathfrak{h}$ is placed in cohomological degree 2.

3.7 Comparing global functions on unipotent and formal loop spaces

This subsection is devoted to the proof of Theorem 3.6.1. Central to our proof will be to use the fact that the map is a pro-graded isomorphism; the following lemma establishes a general situation when this is true. Let us first clarify what we mean by a pro-graded isomorphism, and why this notion is necessary.

Definition 3.7.1. A *pro-graded chain complex* V is an object of $\text{Pro}(\text{QCoh}(B\mathbb{G}_m))$; that is, it is a filtered limit of graded vector spaces[7]. Letting $\mathcal{L}_n$ denote the weight n twisting one-dimensional $\mathbb{G}_m$-representation, the *nth homogeneous part* functor is given by

$$(-)^{\text{wt}=n} := \text{ev} \circ \text{Pro}(\Gamma)(B\mathbb{G}_m, - \otimes \mathcal{L}_{-n}) : \text{Pro}(\text{QCoh}(B\mathbb{G}_m)) \to \mathbf{Vect}_k.$$

where the functor

$$\text{Pro}(\Gamma)(B\mathbb{G}_m, -) : \text{Pro}(\text{QCoh}(B\mathbb{G}_m)) \to \text{Pro}(\mathbf{Vect}_k)$$

is the functor induced on pro-completions from $\Gamma(B\mathbb{G}_m, -) : \text{QCoh}(B\mathbb{G}_m) \to \text{QCoh}(\text{pt})$ and the functor

$$\text{ev} : \text{Pro}(\mathbf{Vect}_k) \to \mathbf{Vect}_k$$

takes a limit diagram and evaluates it in $\mathbf{Vect}_k$ (which has all limits); it is right adjoint to the inclusion. The *underlying chain complex* is given by

$$\text{ev} \circ \text{Pro}(p^*) : \text{Pro}(\text{QCoh}(B\mathbb{G}_m)) \to \mathbf{Vect}_k$$

where $p : \text{pt} \to B\mathbb{G}_m$ is the usual atlas so that p^* is the forgetful functor. A map of graded chain complexes is a *pro-graded isomorphism* if it is an isomorphism on nth graded parts for all n.

Remark 3.7.2. We briefly remark on the necessity of this formalism, which describes a very simple idea. Let $p : \text{pt} \to B\mathbb{G}_m$ be the standard atlas; the pullback (forgetful functor) p^* does not commute with limits. Even worse, the category $\text{QCoh}(B\mathbb{G}_m)$ cannot differentiate between certain direct sums and direct products: its objects are $\mathbb{Z}$-graded chain complexes which are equal to the direct sum of their homogeneous pieces[8]. In particular, completing $\{0\} \subset \mathbb{A}^1$ with the usual scaling action,

$$\lim_{n, \text{QCoh}(\text{pt})} k[x]/x^n = k[[x]] \qquad \lim_{n, \text{QCoh}(B\mathbb{G}_m)} k[x]/x^n = k[x].$$

[7]By Proposition 1.1.3.6 of [44], this category is stable.

[8]To see this, note that objects of $\text{QCoh}(B\mathbb{G}_m)$ are chain complexes which are $\mathcal{O}(\mathbb{G}_m)$-coalgebras, i.e. equipped with a map $V \to V \otimes_k k[z, z^{-1}]$. In particular, tensors have finite rank, so any vector can only have finitely many homogeneous parts.

This presents a problem in situations where we wish to compare two (topological) vector spaces via a grading.

One way to remedy this is to keep track of the limit diagrams by working in the category $\mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m))$ and apply the evaluation functor in $\mathbf{Vect}_k = \mathrm{QCoh}(\mathrm{pt})$ rather than $\mathrm{QCoh}(B\mathbb{G}_m)$. The category $\mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m))$ contains more information than we need; in particular, we do not wish to track the topologies on various completed vector spaces. For our purposes, we can consider a more smaller, more familiar category via Carier duality: the category of $k^{\mathbb{Z}}$-modules. That is, these operations of interest are still available to us in $\mathrm{QCoh}(\mathrm{Spec}\, k^{\mathbb{Z}})$, compatibly with $\mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m))$.

The Cartier dual of $\mathbb{G}_m$ is $\mathbb{Z}$, so that there is an equivalence[9] $\mathrm{Perf}(B\mathbb{G}_m) \simeq \mathrm{Coh}_{prop}(\mathbb{Z})$, where $\mathrm{Coh}_{prop}(\mathbb{Z})$ is the category of sheaves on $\mathbb{Z}$ with proper (in this case, finite) support, inducing a functor $D : \mathrm{QCoh}(B\mathbb{G}_m) \to \mathrm{QCoh}(\mathbb{Z})$. There is a natural affinization map $r : \mathbb{Z} \to \mathrm{Spec}(k^{\mathbb{Z}})$, inducing a functor $\mathrm{Pro}(r_*) : \mathrm{Pro}(\mathrm{QCoh}(\mathbb{Z})) \to \mathrm{Pro}(\mathrm{QCoh}(\mathrm{Spec}(k^{\mathbb{Z}})))$. The composition defines a functor which we denote Ψ :

$$\mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m)) \xrightarrow{D} \mathrm{Pro}(\mathrm{QCoh}(\mathbb{Z})) \xrightarrow{\mathrm{Pro}(r_*)} \mathrm{Pro}(\mathrm{QCoh}(\mathrm{Spec}(k^{\mathbb{Z}}))) \xrightarrow{\mathrm{ev}} k^{\mathbb{Z}}\text{-mod}\,, \qquad \Psi = \mathrm{ev} \circ \mathrm{Pro}(r_*) \circ D$$

Composing Ψ with the global sections functors recovers the underlying vector space, and composing with the costalk at $n \in \mathrm{Spec}\, k^{\mathbb{Z}}$ recovers the nth homogeneous part. In particular, if we are interested in studying the underlying vector space of $V \in \mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m))$ via its homogeneous components, it suffices to consider it as an object of $\mathrm{QCoh}(\mathrm{Spec}(k^{\mathbb{Z}}))$.

We summarize the above discussion in the following proposition, and prove the various claims. Informally, it for the purpose of studying pro-graded isomorphisms, it suffices to work in $k^{\mathbb{Z}}$-mod rather than the larger category $\mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m))$.

Proposition 3.7.3. *We have commutative diagrams of functors*

$$\begin{array}{ccc} \mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m)) & \xrightarrow{\Phi} & k^{\mathbb{Z}}\text{-mod} \\ & {\scriptstyle \mathrm{ev}\circ\mathrm{Pro}(p^*)} \searrow \quad \swarrow {\scriptstyle \Gamma(\mathrm{Spec}\, k^{\mathbb{Z}}, -)} & \\ & \mathbf{Vect}_k & \end{array}$$

$$\begin{array}{ccc} \mathrm{Pro}(\mathrm{QCoh}(B\mathbb{G}_m)) & \xrightarrow{\Phi} & k^{\mathbb{Z}}\text{-mod} \\ & {\scriptstyle \mathrm{ev}\circ\Gamma(B\mathbb{G}_m, -\otimes\mathcal{L}_{-n})} \searrow \quad \swarrow {\scriptstyle \iota_n^!} & \\ & \mathbf{Vect}_k & \end{array}$$

[9] Explicitly, a graded vector space is sent to the sheaf on $\mathbb{Z}$ whose fiber over n is the nth homogeneous component.

Proof. The proposition follows from two claims: (1) that the diagrams above commute without the Pro, i.e. $p^* = \Gamma(\operatorname{Spec} k^{\mathbb{Z}}, -) \circ r_* \circ D$ and $\Gamma(B\mathbb{G}_m, -) = \iota_0^! \circ r_* \circ D$, and (2) that the evaluation functor $\mathrm{Pro}(k^{\mathbb{Z}}\text{-mod}) \to k^{\mathbb{Z}}\text{-mod}$ commutes with global sections and taking costalks. We first remark that the quotient ring for the closed subscheme $\{n\} \subset \operatorname{Spec} k^{\mathbb{Z}}$, which we denote $k_n \simeq k$, is equal to the localization of the ring at $n \in \mathbb{Z}$, and in particular is flat, so that the costalk functor $\iota_n^!(-) = \mathrm{Hom}_{k^{\mathbb{Z}}}(k_n, -)$ need not be derived.

The second claim follows since both the global sections functor is a left adjoint. For taking stalks, note that the costalk is equal to the stalk, and that the costalk is a right adjoint. The first claim can be directly verified: it suffices to consider modules since all functors are exact. In particular, if $V \in \mathrm{QCoh}(B\mathbb{G}_m)$, then it is a $k[z, z^{-1}]$-comodule, i.e. there is a map

$$V \to V \otimes k[z, z^{-1}] \simeq \bigoplus_n V_n z^n.$$

The functor D takes V to the complex on $\mathbb{Z}$ whose value on open affine $\{n\} \in \mathbb{Z}$ is V_n. The functor r_* takes $D(V)$ to $\bigoplus_n V_n$ where $k^{\mathbb{Z}}$ acts in the natural way. Finally, we see that the global sections are exactly $p^*V = \bigoplus V_n$ and the stalk $i_n^!(\bigoplus_n V_n) = \mathrm{Hom}_{k^{\mathbb{Z}}}(k_n, \bigoplus_n V_n) = V_n$. □

Definition 3.7.4. Let V be a $k^{\mathbb{Z}}$-module. The *support* of $v \in V$ is the closed subscheme defined by the annihilator ideal of v.

Remark 3.7.5. It is perhaps not surprising that $\mathrm{QCoh}(\operatorname{Spec} k^{\mathbb{Z}})$ is a convenient place to discuss such phenomena. In particular, points of $\operatorname{Spec} k^{\mathbb{Z}}$ correspond bijectively with filters S_Z on $\mathbb{Z}$. Recall that a *filter* on a set S is a collection of subsets closed under finite intersection and taking supersets. The bijection associates to a (prime) ideal $I \subset k^{\mathbb{Z}}$ the collection of subsets $\{\{n \in \mathbb{Z} \mid a_n = 0\} \mid a \in I\}$ where a_n denotes the nth component of $a \in k^{\mathbb{Z}}$. The closed point $\{n\}$ corresponds to the filter of subsets containing n.

For example, given a collection of vector spaces V_n, we can associate at two extremes the $k^{\mathbb{Z}}$-modules $\bigoplus V_n$ and $\prod V_n$. The support of $\bigoplus V_n$ corresponds to the filter of cofinite sets, and the support of $\prod V_n$ corresponds to the power set $2^{\mathbb{Z}}$.

Lemma 3.7.6. *A pro-graded isomorphism is injective.*

Proof. This is the easy fact that if a map of sheaves on $\mathrm{Spec}(k^{\mathbb{Z}})$ is zero on stalks at closed points, then it is zero, and the observation that on $\operatorname{Spec} k^{\mathbb{Z}}$, costalks and stalks coincide. □

Lemma 3.7.7. *Suppose that $f : V \to W$ is a pro-graded isomorphism of pro-graded vector spaces such that either V or W are supported at finitely many weights. Then f is an isomorphism on underlying vector spaces. More generally, let A be a sheaf of algebras on $\mathrm{Spec}(k^{\mathbb{Z}})$, and $f : V \to W$ a pro-graded isomorphism of sheaves of A-modules where W is generated by elements supported at finitely many weights. Then, f is an isomorphism on underlying vector spaces.*

Proof. For the first claim, the assumptions of the proposition imply that V and W have finite support, whose points consist entirely of closed points of $k^{\mathbb{Z}}$. A map being a pro-graded isomorphism means that it is an isomorphism at stalks of closed points.

For the second more general claim, note that if W is an A-module, and $w \in W$ is an element of finitely many weights, say $w = w_1 + \cdots + w_r$ where the w_i are homogeneous of weight c_i, then $w_i \in A \cdot w$ since $e_{c_i} \cdot w = w_i$. In particular, W having a set of generators supported at finitely many weights is equivalent to W having a set of homogeneous generators. Now, if $f : V \to W$ is a pro-graded isomorphism, then it is injective by the previous lemma. For surjectivity, note that for a given homogeneous $w \in W$, since f is a pro-graded isomorphism, we have a homogeneous $v \in V$ such that $f(v) = w$, and surjectivity follows since W is generated by homogeneous elements. □

The following lemma allows us to reduce statements in the derived category to statements in the abelian category.

Lemma 3.7.8. *A map $f : V \to W$ is a pro-graded quasi-isomorphism of pro-graded complexes if and only if each of the $H^i(f) : H^i(V) \to H^i(W)$ are pro-graded isomorphisms of modules.*

Proof. We can take a map $f : V \to W$ of complexes of $k^{\mathbb{Z}}$-modules. It is easy to verify that taking nth homogeneous parts (i.e. talking stalks via localization) is exact, so that if $H^n(f)$ is a pro-graded isomorphism, it is an isomorphism of modules and therefore f is a quasi-isomorphism. Conversely, the global sections functor is clearly exact. □

Definition 3.7.9. The category $\mathrm{Pro}(\mathrm{QCoh}(\mathbb{G}_m))$ has a monoidal structure via the dual Day convolution, defined by

$$(\lim_i V_i) \otimes (\lim_j V_j) := \lim_{i,j} (V_i \otimes V_j).$$

A *pro-graded* dg-algebra is an algebra object in pro-graded chain complexes. If A is a pro-graded dg-algebra, then $\mathrm{Spec}(A)$ is naturally a *dg-indscheme with a $\mathbb{G}_m$-action* in the sense of [24]. We will use the word *ind-stack* to mean a prestack which can be written as an inductive limit of closed embeddings of (derived) QCA stacks (in the sense of [21]); in practice we only need the case of a formal completion of a closed substack of a quotient stack.

Recall the definition of a contracting $\mathbb{G}_m$-action in Definition 3.1.14.

Lemma 3.7.10. *Let A be a noetherian weight $\mathbb{Z}^{\leq 0}$ pro-graded connective dg-algebra, which is generated in negative weights over its weight 0 part, and let $I = \pi_0(A^{\mathrm{wt}<0}) \subset \pi_0(A)$ be the classical augmentation ideal. The derived completion $A \to \widehat{A}_I$ is a pro-graded quasi-isomorphism. Globally, if X is an ind-stack with a representable contracting $\mathbb{G}_m$-action with fixed point locus $Z \subset X$, then $\mathcal{O}_X \to \mathcal{O}_{\widehat{X_Z}}$ is a pro-graded quasi-isomorphism of quasicoherent sheaves on X. In particular, $\mathcal{O}(X) \to \mathcal{O}(\widehat{X_Z})$ is a pro-graded isomorphism.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. By Proposition 3.4.12 in [11], we can compute the homotopy limit $\widehat{A}_I$ via the limit

$$\widehat{A}_I = \lim_n \left(A \otimes_{\pi_0(A)} K_n^\bullet \right)$$

where $K_n^\bullet$ is the Koszul complex for $f_1^n, \ldots, f_r^n \in \pi_0(A)$. Since $f_1, \ldots, f_r$ are of strictly negative weight, for large n, $(K_n^\bullet)^{\mathrm{wt} \leq k} = (\pi_0(A))^{\mathrm{wt} \leq k}$ for any k. Furthermore, homotopy limits can be computed in the derived category of k-complexes, and in particular we can compute the homotopy limit on each graded piece. Thus, $(\widehat{A}_I)^{\mathrm{wt}=k}$ is a computed by a limit which stabilizes at $A^{\mathrm{wt}=k}$, proving the claim. For the global claim where X is an ind-scheme, one can pass to an open affine $\mathbb{G}_m$-closed cover (which exists since $\mathbb{G}_m$ is a torus). For the global claim where X is an ind-stack, one can check the equivalence smooth locally by passing to an atlas on X. □

Recall that by Remark 6.11 of [6] that there are embeddings $\widehat{\mathcal{L}}(X/G) \hookrightarrow \mathcal{L}^u(X/G) \hookrightarrow \mathbb{T}_X[-1]$. Thus, formal loops and unipotent loops inherit compatible $\mathbb{G}_m$-actions and their functions are $\mathbb{Z}^{\leq 0}$ pro-graded. We have the following.

Lemma 3.7.11. *Let X be a geometric stack. The map*

$$\mathcal{O}(\mathcal{L}^u X) \to \mathcal{O}(\widehat{\mathcal{L}} X)$$

is a pro-graded (quasi-)isomorphism.

Proof. An argument is outlined in Corollary 2.7 of [7]; we will repeat it for convenience. By Lemma 3.1.9, $\mathcal{L}X$ is geometric. The formal loops $\widehat{\mathcal{L}}X$ are the completion of the unipotent loops $\mathcal{L}^u X$ along constant loops, and the action is contracting by Lemma 3.1.15. The statement follows by Lemma 3.7.10. □

Example 3.7.12. The pro-graded isomorphism of Lemma 3.7.11 may fail to be an isomorphism. For example, take $X = G/U$ where U is any unipotent subgroup of G; then we have that

$$\mathcal{L}((G/U)/G) = \mathcal{L}(BU) = U/U \to \mathcal{L}(BG) = G/G$$

has image inside the unipotent cone of G. In particular,

$$\mathcal{L}^u(BU) = \mathcal{L}(BU) = U/U \qquad \widehat{\mathcal{L}}(BU) = \widehat{\mathfrak{u}}/U.$$

For example, if $U = B\mathbb{G}_a$, then the map is

$$\mathcal{O}(\mathcal{L}^u(B\mathbb{G}_a)) = \mathcal{O}(\mathbb{G}_a \times B\mathbb{G}_a) = k[x, \eta] \to \mathcal{O}(\widehat{\mathcal{L}}(B\mathbb{G}_a)) = \mathcal{O}(\widehat{\mathbb{G}}_a \times B\mathbb{G}_a) = k[[x]][\eta]$$

is a pro-graded isomorphism but not an isomorphism, where $|x| = 0$ is a generator for $\mathcal{O}(\mathbb{G}_a)$ and $|\eta| = 1$ is a generator for $\mathcal{O}(B\mathbb{G}_a)$.

Using the fact that the map is a pro-graded isomorphism, we can show in the case of a unipotent group that the map on Tate-equivariant functions is an isomorphism by a finiteness argument. Essentially, we show that applying the Tate construction collapses enough of the target to produce an isomorphism. We include the following proposition as an easy precursor to the next one; it is not required in future arguments.

Corollary 3.7.13. *Let U be a unipotent algebraic group, and X an algebraic space with a U-action. Then, the natural map*

$$\mathcal{O}(\mathcal{L}^u(X/U))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/U))^{\mathrm{Tate}}$$

is an isomorphism. In particular, taking $X = \mathrm{pt}$,

$$\mathcal{O}(\mathcal{L}^u(BU))^{\mathrm{Tate}} = \mathcal{O}(U/U)^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(BU)) \simeq k((u))$$

is an isomorphism.

Proof. By Lemma 3.7.11, the map is a pro-graded isomorphism. Furthermore, applying the Tate construction, we have that $\mathcal{O}(\widehat{\mathcal{L}}(X/U)) \simeq H^\bullet(X;k)((u))$ since U is contractible, where u has cohomological degree 2 and weight 1. The statement follows from Lemma 3.7.7: since $H^\bullet(X;k)$ is finite-dimensional, it has a homogeneous basis. □

A tweaking of the above argument gives us the reductive case.

Corollary 3.7.14. *Let G be a reductive algebraic group, and X an algebraic space with a G-action. Then, the natural map*

$$\mathcal{O}(\mathcal{L}^u(X/G))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}$$

is an isomorphism.

Proof. We first show that $\mathcal{O}(\widehat{\mathcal{L}}(BG)) = \mathcal{O}(\mathcal{L}^u(BG))$. To this end, by Proposition 3.5.3 we have

$$\mathcal{O}(\widehat{\mathcal{L}}(BG))^{\mathrm{Tate}} \simeq H^\bullet(BG;k) \otimes^! ((u)) \simeq k[[\mathfrak{h}]]^W((u))$$

where $\mathfrak{h}$ is in cohomological degree zero and subcomplex $H^\bullet(BG;k)$ is given by $k[\mathfrak{h}u]^W$. By Proposition 3.1.28, we have

$$\mathcal{O}(\mathcal{L}^u(BG)) \simeq \lim_n \mathcal{O}(\mathfrak{g} \times_{\mathfrak{h}//W} \{0\}^{(n)})^G \simeq k[[\mathfrak{h}]]^W$$

so that $\mathcal{O}(\mathcal{L}^u(BG))^{\mathrm{Tate}} \simeq \mathcal{O}(\widehat{\mathcal{L}}(BG))^{\mathrm{Tate}} \simeq k[[\mathfrak{h}]]^W((u))$. In particular, the map

$$\mathcal{O}(\mathcal{L}^u(X/G))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}$$

is linear over $k[[\mathfrak{h}]]^W((u))$ in the pro-graded category, where $\mathfrak{h}$ has weight -1. We claim that

$$H^i(\mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}) \simeq H^i(H^\bullet(X/G;k) \otimes^! k((u)))$$

is finitely generated over $H^\bullet(BG;k) \simeq k[[\mathfrak{h}]]^W$ by weight-homogeneous generators; assuming the claim, the result follows from Lemma 3.7.10 and Lemma 3.7.7.

To see the claim, recall the topological chain complex in Proposition 3.5.3 computing Tate functions on formal loop spaces. The topology is invariant under the degree-weight shearing which multiplies the weight $-k$ part by u^k. In particular, $H^\bullet(X/G;k)$ is finitely generated over $H^\bullet(BG;k)$ by cohomologically homogeneous generators in weight 0; we can lift these generators $x_1, \ldots, x_r$ living in cohomological degrees $2d_1, \ldots, 2d_r$ to the chain complex. Since the chain complex and its completion are invariant under shearing, then $H^{2i}(\mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}})$ is generated by $x_i u^{i-d_i}$ (and similarly when i is odd), proving the claim. □

We can now prove Theorem 3.6.1, which we restate for convenience.

Theorem 3.7.15. *Let X be an algebraic space with an action of an affine algebraic group G. The natural map*

$$\mathcal{O}(\mathcal{L}^u(X/G))^{\mathrm{Tate}} \to \mathcal{O}(\widehat{\mathcal{L}}(X/G))^{\mathrm{Tate}}$$

is an isomorphism.

Proof. Every affine algebraic group G embeds as a subgroup of a reductive group K. Apply the previous corollary to $(X \times^G K)/K$. Note that $X \times^G K$ is not guaranteed to be a scheme, but is always an algebraic space. □

Chapter 4

Convolution Methods in Geometric Representation Theory

In this chapter we preview a possible application of our main theorem to geometric representation theory. Let G be a reductive group acting on smooth schemes X, Y and $f : X/G \to Y/G$ be a proper (representable) map of quotient stacks. Let $S = BG$. By the formalism of [21] applied to derived loop spaces, we have that

$$HH(\mathrm{Coh}(X \times_Y X)) = \mathrm{Ext}_{\mathrm{QCoh}(\mathcal{L}Y)}(\mathcal{L}f_*\mathcal{O}_{\mathcal{L}X}, \mathcal{L}f_*\mathcal{O}_{\mathcal{L}X})$$

Since f is proper, it preserves compact (i.e. coherent, which since X, Y are smooth, are the same as perfect) objects; we can work with small categories and the issue of functoriality in [51] do not appear. We thus find

$$HP(\mathrm{Coh}(X \times_Y X)) = \mathrm{Ext}_{\mathrm{QCoh}(\mathcal{L}Y)}(\mathcal{L}f_*\mathcal{O}_{\mathcal{L}X}, \mathcal{L}f_*\mathcal{O}_{\mathcal{L}X})^{\mathrm{Tate}}.$$

Note that $\mathrm{Coh}(X \times_Y X)$ acts on $\mathrm{Coh}(X)$ linearly over $\mathrm{Coh}(Y)$ via convolution, inducing an algebra structure on $HH(\mathrm{Coh}(X \times_Y X))$ and $HH(\mathrm{Coh}(Y))$-module over this algebra $HH(\mathrm{Coh}(X))$, and similarly for periodic cyclic homology. Similar patterns arise in K-theory, developed fully in [16].

4.1 Toy example: convolution pattern in Hochschild homology: $\mathbb{A}^1/\mathbb{G}_m$

We will perform an analysis of convolution structures in Hochschild homology in a toy example, and make comparisons to K-theory. For this section, take $f : X \to Y$, with $X = (\mathbb{A}^1 \cup \mathrm{pt})/\mathbb{G}_m$ and $Y = \mathbb{A}^1/\mathbb{G}_m$. Note that since $\mathbb{G}_m$ acts by finitely many orbits on everything in sight, the S^1-equivariant structure is necessarily trivial and thus there is nothing gained by passing to HP, i.e. $HP = HH \otimes_k k((u))$. For this section, we will only work with HH.

Example 4.1.1 (K-theory)**.** We naturally have $K(\mathbb{A}^1/\mathbb{G}_m) = \mathbb{Z}[z, z^{-1}]$, defined by the action of $\mathbb{G}_m$ on the zero fiber $k[t]/(t)$. This determines the $\mathbb{G}_m$-action on $k[x]$ – if z acts by degree n on k, then it acts by degree $n + r$ on $x^r \in k[x]$. We let z^n denote the sheaf $k[x]$ where $\deg(1) = n$. Likewise, $K(B\mathbb{G}_m) = \mathbb{Z}[z, z^{-1}]$. Thus, $K(X) = \mathbb{Z}[z, z^{-1}]^2$, and $K(X \times_Y X) = \mathbb{Z}[z, z^{-1}]^4$. There is only one interesting convolution sheaf on $X \times_Y X$, which is the one that "sends [pt] $\mapsto [\mathbb{A}^1]$ in $K(X)$. It acts by pushing forward a sheaf on the origin to $\mathbb{A}^1$. Let k be a sheaf on the origin of degree n; after pushing forward to $\mathbb{A}^1$, it has a free resolution of the form $k[x](n-1) \xrightarrow{x} k[x](n)$, i.e. $z^n - z^{n-1} = z^{n-1}(z-1)$. Thus, the convolution action is by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & (1-z^{-1})b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in Z[z]$.

Note that there is a short exact sequence

$$0 \to K_0(\{0\}/\mathbb{G}_m) \to K_0(\mathbb{A}^1/\mathbb{G}_m) \to K_0(\mathbb{P}^1) \to 0$$

where the first map $\mathbb{Z}[z, z^{-1}] \to \mathbb{Z}[z, z^{-1}]$ is multiplication by $1 - z^{-1}$, and the second map is evaluation at $z = 1$.

Example 4.1.2 (Hochschild homology)**.** We want to study the convolution given by $\mathcal{L}(f) : \mathcal{L}X \to \mathcal{L}Y$. Choose coordinates $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and $\mathbb{G}_m = \operatorname{Spec} k[z, z^{-1}]$, and define $R = k[x, z, z^{-1}]/x(z-1)$ with a $\mathbb{G}_m$-action $\deg(x) = 1$ and $\deg(z) = 0$, so that $\mathcal{L}(Y) = \operatorname{Spec}(R)/\mathbb{G}_m$ and $\mathcal{L}(X) = (\operatorname{Spec}(R) \cup \mathbb{G}_m)/\mathbb{G}_m$. Our goal is to compute

$$\omega_{\mathcal{L}Y}(\mathcal{L}(X \times_Y X)) = \mathcal{H}om_{\mathcal{L}Y}(\mathcal{L}f_*\mathcal{O}_{\mathcal{L}X}, \mathcal{L}f_*, \omega_{\mathcal{L}X})$$

This will act on the module $\mathcal{O}(\mathcal{L}X) = k[z, z^{-1}]^2$. We've already identified $\mathcal{O}_{\mathcal{L}X}$; to identify $\omega_{\mathcal{L}X}$, note that $\mathcal{L}X$ has two connected components: $\mathcal{L}Y$ and $\mathbb{G}_m \times B\mathbb{G}_m$. The map $\mathcal{L}Y \to \mathcal{L}Y$ is just the identity map, so the pushforward on distributions is more or less obvious. The map $\mathbb{G}_m \times B\mathbb{G}_m \to \mathcal{L}Y$ is given by closed immersion, and we can compute the shriek pullback by the adjunction

$$\operatorname{Hom}_R(R/t, R) \simeq \operatorname{Hom}_{R/t}(R/t, f^!R) = f^!R$$

There is a resolution of R/t by the complex

$$\begin{array}{ccccccc} & & & & & & R/t \\ & & & & & & \uparrow \simeq \\ \cdots \longrightarrow R & \xrightarrow{t} & R & \xrightarrow{z-1} & R & \xrightarrow{t} & R \end{array}$$

Its R-linear dual is quasi-isomorphic (under a noncanonical identification of functions and distributions) to functions on $\mathcal{L}Y$ supported on the set where $t = 0$, i.e. the ideal sheaf

$(z-1)$. Note that $(z-1) \simeq R/t$, again noncanonically. To compute $\mathcal{H}om_{\mathcal{L}Y}$ we need to take $\mathbb{G}_m$-equivariant homomorphisms of the relevant sheaves:

$$\omega_{\mathcal{L}Y}(\mathcal{L}(X \times_Y X)/\mathcal{L}Y) = \mathrm{Hom}_R(R \oplus R/t, R \oplus (z-1))^G \simeq \mathrm{Hom}_R(R \oplus R/t, R \oplus R/t)^G$$

twisting with a choice of isomorphism $(z-1) \simeq R/t$ to identify the source and the target (so that we have a monoidal product). To compute this, we need to again use the above resolution. We will choose to write this in "matrix form", i.e. think of a pair $(f, g) \in (R \oplus R/t)$ as a column vector $\begin{pmatrix} f \\ g \end{pmatrix}$. We find that

$$\omega_{\mathcal{L}Y}(\mathcal{L}(X \times_Y X)/\mathcal{L}Y) = \begin{pmatrix} R & (z-1) \\ R/t & R/t[[\beta]] \end{pmatrix}^G$$

where $\deg(\beta) = 2$, and $\mathrm{wt}(\beta) = 1$ (since β is multiplication by $(z-1)(t)$, which has degree 1). Note that while $(z-1) \simeq R/t$ as a module, the functions $f \in (z-1)$ represent multiplication by f. Further note that these matrices are *exactly* the endomorphisms

$$\omega(\mathcal{L}(X \times_Y X) \simeq \mathrm{End}_{\mathcal{L}Y}(\mathcal{L}f_*\omega_{\mathcal{L}X}) = \mathrm{End}_{G \ltimes R}(R \oplus R/t)$$

i.e. R-linear morphisms $R \to R/t$ are exactly given by $(z-1)$, and derived R-linear morphisms $R/t \to R/t$ are exactly given by $R/t[[\beta]]$, and further we insist that the maps are G-equivariant on both sides.

Finally, taking G-invariants, we find

$$\omega_{\mathcal{L}Y}(\mathcal{L}(X \times_Y X)/\mathcal{L}Y) \simeq \begin{pmatrix} k[z, z^{-1}] & (z-1) \\ k[z, z^{-1}] & k[z, z^{-1}] \end{pmatrix}$$

Specifying generic z recovers a matrix algebra, realizing the K-theoretic convolution algebra of $K(X \times_Y X)$ where we localize to a generic character, and specifying $z = 1$ recovers the K-theoretic action of $\mathbb{A}^1$ where we localize to $1 \in \mathbb{G}_m$, i.e. forget the equivariance entirely. The vanishing of $z - 1$ realizes the fact that the K-theoretic pushforward of sheaves on $\{0\}$ to $\mathbb{A}^1$ is zero.

4.2 The Hochschild homology of $\mathrm{Coh}(B\backslash G/B)$

Let G be a reductive group. Let $S = BG$, $X = BB$ and so $X \times_S X = B\backslash G/B$. We will first interpret the convolution patterns in K-theory, and then apply our previous results on convolution patterns in Hochschild homology. In particular, we will obtain a Weyl group action on $K_0(\mathrm{Perf}(BB)) = HH(\mathrm{Perf}(BB))$ linear over $K_0(\mathrm{Perf}(BG)) = HH(\mathrm{Perf}(BG))$ which does not come from a categorical action. Note that we will not take S^1-invariants; since S^1 acts trivially, it will merely clutter the notation.

Remark 4.2.1 (K-theory)**.** It is well-known that the representation ring of a reductive group is given by $k[H]^W$ where H is a universal Cartan (i.e. the quotient of a Borel by its unipotent radical). The stack $BB \simeq G\backslash(G/B)$ warrants a brief discussion; the flag variety is customarily denoted G/B, but can actually be defined without making a choice of Borel as the variety of Borel subgroups of G. Choosing a base point of this variety amounts to choosing a Borel, which induces an isomorphism with the homogeneous space G/B. The identification $K_0(\mathrm{Perf}(BB)) \simeq [H]$ depends on such an identification, and $H = B/U$. We choose once and for all a notion of fundamental Weyl chamber (determined either by an embedding of G/B into projective space or a choice of Borel); this gives us a length function $\ell : W \to \mathbb{Z}$.

By the pullback map, $K_0(\mathrm{Perf}(BB)) = k[H]$ is a module over $K_0(\mathrm{Perf}(BG)) = k[H]^W$, and $K_0(\mathrm{Perf}(B\backslash G/B))$ has a left and right action of $K_0(\mathrm{Perf}(BB)) = k[H]$. As a left-module, using the results of Section 5.2.18 of [16], we can determine that

$$K_0(\mathrm{Perf}(B\backslash G/B)) \simeq K_0(\mathrm{Perf}(T\backslash G/B)) \simeq k[H] \otimes k[W].$$

We let e^λ denote the basis element in $k[H]$ where $\lambda \in X^*(H)$ is a character of H. The pushforward functor $\mathrm{Perf}(BB) \to \mathrm{Perf}(BG)$ (which makes sense since f is proper and all stacks involved are smooth) induces a "trace" map on K_0; it can be explicitly computed by the Borel-Weil-Bott theorem as the map sending e^λ to $(-1)^{\ell(w)}\chi_{w\cdot\lambda}$, where $w \in W$ is chosen such that $w \cdot \lambda$ is dominant under the *twisted* Weyl action and $\chi_{w\cdot\lambda}$ is the character for the irreducible representation with highest weight $w \cdot \lambda$. By the Weyl character formula, we can compute the trace:

$$\tau : k[H] \to k[H]^W \qquad P \mapsto \Delta^{-1} \sum_{w\in W} (-1)^{\ell(w)} w \cdot (P) e^{w(\rho)}$$

where

$$\Delta = \prod_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2})$$

is the "Weyl denominator" (R^+ the set of positive roots), and $\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha$. In Proposition 6.1.19 of [16], it is shown that $BB \to BG$ satisfies the conditions of the Kunneth theorem (Theorem 5.6.1) of *loc. cit.* ; therefore we have

$$K_0(\mathrm{Perf}(B\backslash G/B)) \simeq k[H] \otimes_{k[H]^W} k[H]$$

By the base change formula, we have that the action of $K_0(\mathrm{Perf}(B\backslash G/B))$ on $K_0(\mathrm{Perf}(BB))$ is by

$$(f \otimes g) \cdot x = f\tau(gx).$$

We make a few stray observations:

- The trace map induces a nondegenerate pairing $K(BB) \otimes_{K(BG)} K(BB) \to K(BG)$, i.e.

$$k[H] \otimes_{k[H]^W} k[H] \to k[H]^W.$$

- Note that the $k[H]^W$-linearity of the action is expressed by the projection formula, i.e.

$$f_*(f^*V \otimes W) \simeq V \otimes f_*W$$

  where V is a G-representation and W is a B-representation, and the functor f_* is the induction from a B-representation to a G-representation. While this functor may be difficult to describe in general due to nontrivial extensions inducing complicated spectral sequences, on the level of K_0-groups it is not.

- Proposition 6.1.19 of [16] argues using an explicit basis of $k[H]$ over $k[H]^W$ (of rank $|W|$) by Steinberg and Pittie [56].

- By the Kunneth theorem in K-theory, we have an isomorphism

$$K_0(B\backslash G/B) \simeq \mathrm{End}_{k[H]^W}(k[H]).$$

  Since the affine Weyl group acts on the weight lattice, we have an embedding of the group algebra of the affine Weyl group $k[W^{aff}] \hookrightarrow K_0(B\backslash G/B)$. This map is not an isomorphism as we will see in the next example.

Example 4.2.2 ($G = SL_2$)**.** It is instructive to work out the above explicitly when $G = SL_2$. In this case, $k[H] = k[t, t^{-1}]$ and $k[H]^W = k[t + t^{-1}]$. First, the Steinberg-Pittie basis is given by $\{1, t^{-1}\}$. Furthermore one observes the following phenomena:

- The Weyl group (or any larger covering, like the braid group) does not act on the category $\mathrm{Coh}(BB)$ linearly over $\mathrm{Coh}(BG)$. For example, take $G = SL_2$, and let $w \in W$ be the nontrivial Weyl reflection. Suppose we had an autofunctor $w : \mathrm{Coh}(BB) \to \mathrm{Coh}(BB)$. That it is invertible means that it takes line bundles to line bundles; in particular $w(\mathcal{O}(n)) = \mathcal{O}(-n)[k]$ for some shift k. That it is $\mathrm{Coh}(BG)$-linear means that $w(p^*V) \simeq p^*V$ for every representation $V \in \mathrm{Coh}(BG)$. Now, $p^*\mathbb{C}^2$ sits inside a short exact sequence $0 \to \mathcal{O}(-1) \to p^*\mathbb{C}^2 \to \mathcal{O}(1) \to 0$; acting by w gives a short exact sequence $0 \to \mathcal{O}(1)[a] \to p^*\mathbb{C}^2 \to \mathcal{O}(-1)[b] \to 0$, but there cannot be such a short exact sequence. Note that the Weyl group *does* act on the non-equivariant category $\mathrm{Coh}(G/B)$, but not canonically. One can choose a torus $T \subset B$, and clearly $N(T)/T$ acts on the right.

- Even though this multiplication in K-theory cannot come from a categorical action, the identity functor still makes sense as an integral transform. Using the (equivariant) Beilinson resolution of the diagonal of $\mathbb{P}^1$, we find that $1 \in \mathrm{End}_{k[H]^W}(k[H])$ corresponds to $1 \otimes 1 - t^{-1} \otimes t^{-1}$. Through guesswork, one can compute that $w \in \mathrm{End}_{k[H]^W}(k[H])$ corresponds to $1 \otimes 1 - t \otimes t^{-1}$.

- By contrast, Bezrukavnikov and Riche proved that there is a braid group action on the category $\mathrm{Coh}(\tilde{\mathcal{N}}/G_{\mathbb{G}_m})$. In particular, the integral kernel corresponding to w is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \in \mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/SL_{2,\mathbb{G}_m})$.

- Note that $K_0(B\backslash G/B) \simeq \mathrm{End}_{k[H]^W}(k[H]) \not\simeq k[W^{aff}]$. We've identified the actions of 1 and w inside the convolution algebra $k[H]\otimes_{k[H]^W} k[H]$ above. We have the following $k[H]$-basis of $\mathbb{Z}[H]\otimes_{\mathbb{Z}[H]^W}\mathbb{Z}[H]$ (under the left action) by Steinberg-Pittie: $1\otimes 1$ and $1\otimes t^{-1}$. The $k[H]$-basis $1, w$ of $\mathbb{Z}[W^{aff}]$ can be expressed in terms of the Steinberg-Pittie one by the matrix $\begin{pmatrix} 1 & t^{-1} \\ 1 & t \end{pmatrix}$. This matrix is not invertible; its determinant is $t - t^{-1}$.

- Inverting $t - t^{-1}$ "corresponds" to restricting to regular semisimple elements of G. We can make this precise using loop spaces shortly.[1]

We need the following calculation, which is well-known to experts but we include for completeness and convenience.

Lemma 4.2.3. *Let $\mu : \tilde{G} \to G$ be the global Grothendieck-Springer resolution. We have*

$$\mu_*\mathcal{O}_{\tilde{G}} \simeq \mathcal{O}_{G\times_{H//W}H}$$

where μ_* is the derived pushforward. In particular, the higher pushforwards vanish.

Proof. First, we will show that the pushforward of $\mathcal{O}_{\tilde{G}}$ along μ has vanishing cohomology. Since G is affine, it suffices to show that the global sections functor has vanishing cohomology; equivalently, we can show that the pushforward $i_*\mathcal{O}_{\tilde{G}}$ along the embedding $i : \tilde{G} \to G\times G/B$ has vanishing higher global sections. We replace $i_*\mathcal{O}_{\tilde{G}}$ with its Koszul resolution

$$\mathcal{O}_{\tilde{G}} \simeq \left(p^*\Omega^d_{G/B} \to \cdots \to p^*\Omega^1_{G/B} \to p^*\mathcal{O}_{G/B}\right)$$

where $p : G\times G/B \to G/B$ is the projection and $d = \dim(G/B)$. Using the projection formula, the pushforward of this complex along p is

$$p_*\mathcal{O}_{\tilde{G}} \simeq \left(k[G]\otimes\Omega^d_{G/B} \to \cdots \to k[G]\otimes\Omega^1_{G/B} \to k[G]\otimes\mathcal{O}_{G/B}\right).$$

The global sections of this complex can be computed by a spectral sequence with $E_1^{p,q} = H^p(G/B, \Omega^q_{G/B})$, which is known to vanish unless $p = q$, for example by [63]. Thus, the spectral sequence degenerates at the E_2 page and $\Gamma^\bullet(\tilde{G}, \mathcal{O}_{\tilde{G}})$ is concentrated in degree zero.

[1] Alternatively, we can show this inside $\mathrm{End}_{\mathbb{Z}[H]^W}(\mathbb{Z}[H])$ by seeing that the Weyl character "trace" endomorphism of $k[H]$ cannot possibly be obtained by a sum affine Weyl group actions. Indeed, suppose $\mathrm{tr} = f(t) + g(t)w$. Since $\mathrm{tr}(t^{-1}) = f(t)t^{-1} + g(t)t = 0$, we have that $\mathrm{tr} = -t^2 g(t) + g(t)w$. Next, since $\mathrm{tr}(1) = -t^2 g(t) + g(t) = 1$, but $1 - t^2$ is not a unit.

To show that $\Gamma^0(\tilde{G}, \mathcal{O}_{\tilde{G}}) = \mathcal{O}(G \times_{H//W} H)$, we apply a standard fact in algebraic geometry: if $f : X \to Y$ is a proper birational map with Y normal and affine, then $f_*\mathcal{O}_X = \mathcal{O}_Y$. There is a map $\nu : \tilde{G} \to H$ which induces the desired map, which is an isomorphism on the regular semisimple locus and is proper by base change. We claim that $G \times_{H//W} H$ is normal. The base change of a normal variety along a normal morphism is normal; a morphism $f : X \to Y$ is normal if for every $x \in X$, f is flat at x and if the geometric fiber $f^{-1}(f(x))$ is normal at x. The claim follows, since $H \to H//W$ has finite fibers and is thus normal. □

Lemma 4.2.4. *Suppose Z is affine. Then there is a canonical isomorphism*

$$\Gamma(Z, f_*\mathcal{F} \otimes_{\mathcal{O}_Z} g_*\mathcal{G}) \simeq \Gamma(X, \mathcal{F}) \otimes_{\mathcal{O}(Z)} \Gamma(Y, \mathcal{G})$$

In particular,

$$\mathrm{Aff}(X \times_Z Y) \to \mathrm{Aff}(X) \times_Z \mathrm{Aff}(Y).$$

Proof. The map is obtained purely by universal property of fiber products and affinization. That is, to produce this map, it suffices to produce a map $\phi : X \times_Z Y \to \mathrm{Aff}(X) \times_Z \mathrm{Aff}(Y)$, which is produced by the diagram

$$\begin{array}{ccccc}
X \times_Z Y & \longrightarrow & & \longrightarrow & X \\
\downarrow & \searrow & & & \downarrow \\
 & & \mathrm{Aff}(X) \times_Z \mathrm{Aff}(Y) & \longrightarrow & \mathrm{Aff}(X) \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & \mathrm{Aff}(Y) & \longrightarrow & Z
\end{array}$$

In fact this map is an isomorphism[2] which more or less follows from the definitions. One possible argument is to use base change and the projection formula to show that $\phi_*\mathcal{O}_{X\times_Z Y} = \mathcal{O}_{\mathrm{Aff}(X)\times_Z \mathrm{Aff}(Y)}$; another is to take Cech covers $U \to X$ and $V \to Y$ and notice that $U \times_Z V$ is a Cech cover of $X \times_Z Y$. □

Remark 4.2.5. This result is not true if Z is the quotient of an affine by a reductive group. For example, take $X = Y = Z = BG$. It is not true that the invariants of a tensor product of representations is isomorphic to the tensor product of invariants.

Lemma 4.2.6. *Let H be the universal Cartan. The global functions on the Grothendieck-Springer stack are given by*

$$\mathcal{O}(\tilde{G}/G) = k[H]$$

and the global functions on the Steinberg stack are given by

$$\mathcal{O}(\mathrm{St}_G/G) = k[H] \otimes_{k[H]^W} k[H]$$

[2] This is not true if Z is nonaffine. For example, take $Z = \mathbb{P}^1$ and $X = Y = \mathrm{Tot}(\mathcal{O}(-1))$. Then $X \times_Z Y = \tilde{\mathfrak{sl}}_2$, so we know $\mathrm{Aff}(X \times_Z Y) = \mathfrak{sl}_2 \times_{\mathfrak{t}//W} \mathfrak{t}$, whose graded dimension does not agree with $\mathrm{Aff}(X) \times \mathrm{Aff}(Y)$.

Proof. We will use liberally the fact that for any scheme X, $\mathcal{O}(X/G) = \mathcal{O}(X)^G$, which follows from a standard descent argument. In particular, $\mathcal{O}(X/G) \simeq \mathcal{O}(\mathrm{Aff}(X)/G)$. For the Grothendieck-Springer resolution, we claim that $\mathrm{Aff}(\tilde{G}) = G \times_{H//W} H$. To see this, note that the map $\tilde{G} \to G \times_{H//W} H$ is birational and proper, and that $G \times_{H//W} H$ is normal. We claim that if $f : X \to Y$ is proper and birational and Y is normal, then $f_*\mathcal{O}_X = \mathcal{O}_Y$. Since f is birational, $f_*\mathcal{O}_X \subset \mathrm{Frac}(\mathcal{O}_Y)$. Since f is proper, $f_*\mathcal{O}_X$ is finitely generated as an $\mathcal{O}_Y$-module. Since Y is normal, any finitely generated $\mathcal{O}_Y$-module in $\mathrm{Frac}(\mathcal{O}_Y)$ is $\mathcal{O}_Y$, proving the claim. Thus, $\mathrm{Aff}(\tilde{G}) = G \times_{T//W} T$, and

$$\mathcal{O}(\tilde{G}/G) = \mathcal{O}(H//W \times_{H//W} H) = \mathcal{O}(T).$$

Alternatively, one can note that the projection $G \times_{H//W} H \to H$ is a categorical quotient, since the fiber of every closed point contains a unique closed orbit.

For the Steinberg variety, we apply the next lemma. In our case, one has that

$$\mathrm{Aff}(\mathrm{St}_G) = \mathrm{Aff}(\tilde{G}) \times_G \mathrm{Aff}(\tilde{G}) = H \times_{H//W} G \times_{H//W} H.$$

So, one has

$$\mathcal{O}(\mathrm{St}_G/G) = \mathcal{O}(\mathrm{Aff}(\mathrm{St}_G)/G) = \mathcal{O}(H \times_{H//W} H//W \times_{H//W} H) = \mathcal{O}(H) \otimes_{\mathcal{O}(H//W)} \mathcal{O}(H).$$

□

Remark 4.2.7 (Hochschild homology)**.** In Hochschild homology, we have $\mathcal{L}(BG) = G/G$, $\mathcal{L}(BB) = B/B \simeq (G \times^B B)/G = \tilde{G}/G$ and $\mathcal{L}(B\backslash G/B) = \mathrm{St}_G/G$. Since these are all smooth quotient stacks with no nontrivial moduli of orbits, the loop spaces are all classical. We have,

$$\omega(\mathrm{St}_G/G) \simeq \mathrm{End}_{\mathcal{O}_{G/G}}(\mu_*\mathcal{O}_{\tilde{G}/G}) \simeq \mathrm{End}_{\mathcal{O}_{G/G}}(\mathcal{O}_{(G\times_{H//W}H)/G})$$

and we wish to study the map

$$\mathrm{End}_{k[G]}(k[G \times_{H//W} H])^G \longrightarrow \mathrm{End}_{k[H]^W}(k[H])$$

Note that since $G \times_{H//W} H$ is flat over G, there are no higher Ext groups and this verifies in a different way that $\mathcal{L}(B\backslash G/B)$ has no derived structure. Since we are in an entirely classical situation, and since $k[G \times_{H//W} H]$ is generated as a $k[G]$-module by $k[H] \subset k[G \times_{H//W} H]^G$, the map above is injective. To see that it is surjective, we can explicitly produce a section of the above map as follows: for $\phi \in \mathrm{End}_{k[H]^W}(k[H])$, define $\tilde{\phi} \in \mathrm{End}_{k[G]}(k[G \times_{H//W} H])^G$ by

$$\tilde{\phi}(f \otimes g) = \phi(f) \otimes g$$

Since ϕ is linear over $k[H]^W$, this is well-defined, and it's clear that $\tilde{\phi}$ is G-equivariant.

We have that

$$\omega(\mathrm{St}_G/G) = \mathcal{O}(\mathrm{St}_G/G) \simeq \Gamma(G/G, \mu_*\mathcal{O}_{\tilde{G}/G} \otimes_{G/G} \mu_*\mathcal{O}_{\tilde{G}/G})$$
$$\simeq \Gamma(H//W, \mathcal{O}_H \otimes_{H//W} \mathcal{O}_H) \simeq k[H] \otimes_{k[H]^W} k[H].$$

The embedding

$$k[H] \otimes_{k[H]^W} k[H] \to \mathrm{End}_{k[H]^W}(k[H]) \qquad (f \otimes g)(x) = f \cdot \mathrm{tr}(g \cdot x)$$

is determined the trace map $\mathrm{tr} : \mu_*\omega_{\tilde{G}} \to \omega_G$. Explicitly, this trace map is

$$\mathrm{tr}(x) = \frac{1}{|W|} \sum_{w \in W} w \cdot x$$

Furthermore, it's clear that $k[W^{aff}] \subset \mathrm{End}_{k[H]^W}(k[H]) \simeq k[H] \otimes_{k[H]^W} k[H]$; for SL_2 we've explicitly written out the reflection operators in terms of the Steinberg-Pittie basis. On the other hand, the Steinberg-Pittie operators can be written in terms of the reflection and shift operators via the Weyl character formula, which has denominator Δ, which is the equation for the reflection hyperplanes in the character lattice. Thus, localizing away from the singular locus in G/G makes the above inclusion an equality.

4.3 The Hochschild homology of $\mathrm{Coh}(\mathcal{L}^u(B\backslash G/B))$

For this section and the next, we will let $G_{\mathbb{G}_m}$ denote $G \times \mathbb{G}_m$. We will assume the following unproven conjecture, whose proof has eluded us.

Conjecture 4.3.1 (Devissage)**.** Let X be a QCA stack, Z a closed reduced substack, and $i : Z \to Z'$ a nilthickening of Z in X; we write $i' : Z' \to X$ for the closed immersion. Then, the pushforward functors

$$HP(\mathrm{Coh}(Z)) \xrightarrow{HP(i_*)} HP(\mathrm{Coh}(Z')) \xrightarrow{HP(i'_*)} HP(\mathrm{Coh}_Z(X))$$

induce isomorphisms on periodic cyclic homology.

Remark 4.3.2. If we assume that X is a quotient stack of a quasiprojective scheme by a group, it is not difficult to reduce to the case when X is smooth. If X is not smooth, then embed X G-equivariantly into a smooth quotient stack M. We have a map of exact triangles

$$\begin{array}{ccccc} HP(\mathrm{Coh}_Z(X)) & \longrightarrow & HP(\mathrm{Coh}(X)) & \longrightarrow & HP(\mathrm{Coh}(X-Z)) \\ \downarrow & & \downarrow & & \downarrow \\ HP(\mathrm{Coh}_Z(M)) & \longrightarrow & HP(\mathrm{Coh}(M)) & \longrightarrow & HP(\mathrm{Coh}(M-Z)) \end{array}$$

Assuming the devissage result when for an embedding into smooth M, we have

$$\begin{array}{ccccc} HP(\mathrm{Coh}_Z(X)) & \longrightarrow & HP(\mathrm{Coh}_X(M)) & \longrightarrow & HP(\mathrm{Coh}_{X-Z}(M-Z)) \\ \downarrow & & \downarrow & & \downarrow \\ HP(\mathrm{Coh}(Z)) & \longrightarrow & HP(\mathrm{Coh}(M)) & \longrightarrow & HP(\mathrm{Coh}(M-Z)) \end{array}$$

Taking cones of the columns, we have an localization sequence

$$0 \longrightarrow HP(\mathrm{Coh}(M-X)) \longrightarrow HP(\mathrm{Coh}(M-X))$$

proving the claim.

Theorem 4.3.3. *Let* **H** *denote the affine Hecke algebra. We have isomorphisms of algebras*

$$\begin{array}{ccc} \mathbf{H} \otimes_k k((\beta)) & \xrightarrow{\simeq} & HP(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m}) \\ \uparrow & & \uparrow \\ k[q,q^{-1}][H]^W \otimes_k k((\beta)) & \xrightarrow{\simeq} & HP(\tilde{\mathcal{N}}/G_{\mathbb{G}_m}) \end{array}$$

Partial proof of theorem. We argue by producing a natural map

$$K_0(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m})) \to HP(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m}))$$

which factors the trace map from the connective K-theory *spectrum*

$$K_\bullet(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m})) \to HP(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m})).$$

and showing that this map is a quasi-isomorphism, and then applying Theorem 7.2.5 in [16]. We first need to deal with the fact that the treatment in [16] deals with the underived Steinberg variety $\mathrm{St}'_{\mathcal{N}} = \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$, which is a classical scheme, whereas our convolution formalism requires us to deal with the derived Steinberg variety $\mathrm{St}_{\mathcal{N}} = \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$. They do not differ by much; in particular, the following squares are Cartesian

$$\begin{array}{ccccc} \mathrm{St}_{\mathcal{N}} & \longrightarrow & \tilde{\mathcal{N}} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{St}'_{\mathcal{N}} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{h} \\ \downarrow & & \downarrow & & \\ \tilde{\mathcal{N}} & \longrightarrow & \mathfrak{g} & & \end{array}$$

Since $\nu : \tilde{\mathfrak{g}} \to \mathfrak{h}$ is smooth, the normal bundle to the fiber $\nu^{-1}(0) = \tilde{\mathcal{N}}$ inside $\mathfrak{g}$ is trivial with fibers isomorphic to $\mathfrak{h}$. In particular, since the top two squares together form a Cartesian square, and since the image of $\mathrm{St}'_{\mathcal{N}}$ in $\mathfrak{h}$ is $\{0\}$, we have

$$\mathrm{St}_{\mathcal{N}} \simeq \mathrm{St}'_{\mathcal{N}} \times \mathfrak{h}[-1]$$

where $\mathfrak{h}[-1] = \mathrm{Spec}_k \mathrm{Sym}_k \mathfrak{h}[1]$. It is well known that the K-theory of coherent sheaves is not sensitive to nilthickenings, essentially by the usual devissage argument for K-theory of coherent sheaves. For periodic cyclic homology, we do not currently have a devissage result for stacks. However, it is known that periodic cyclic homology is a $\mathbb{A}^1$-homotopy invariant. In particular, since there is an equivariant $\mathbb{A}^1$-homotopy contracting $\mathfrak{h}[1]$ to its zero-section, we can conclude that the pushforward functor

$$HP(\mathrm{Coh}(\mathrm{St}'_{\mathcal{N}})) \to HP(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}))$$

is a quasi-isomorphism without the full strength of devissage. In particular, from now on we can make our arguments replacing $\mathrm{St}_{\mathcal{N}}$ with $\mathrm{St}'_{\mathcal{N}}$.

Using the filtration of the Steinberg variety by irreducible components as in Lemma 7.6.11 of [16], which we will denote

$$Z_1 = \tilde{\mathcal{N}}/G_{\mathbb{G}_m} \subset Z_2 \subset \cdots \subset Z_k = Z = \mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m},$$

we can inductively compute $HP(\mathrm{Coh}(\mathrm{St}_{\mathcal{N}}/G_{\mathbb{G}_m}))$ by exact triangles. By Theorem 6.2.4 and the Cellular Fibration Lemma 5.5.1 of [16], the differential from K_1 to K_0 in the corresponding long exact sequence is zero. That is, there is an exact triangle of complexes

$$K_0(\mathrm{Coh}_{Z_{i-1}}(Z) \otimes k((u)) \to K_0(\mathrm{Coh}(Z_i)) \otimes k((u)) \to K_0(\mathrm{Coh}(Z_i - Z_{i-1}) \otimes k((u))$$

We wish to induce a natural isomorphism down the middle

$$\begin{array}{ccccc} K_0(\mathrm{Coh}_{Z_{i-1}}(Z))((u)) & \longrightarrow & K_0(\mathrm{Coh}(Z_i))((u)) & \longrightarrow & K_0(\mathrm{Coh}(Z_i - Z_{i-1}))((u)) \\ \downarrow & & \vdots & & \downarrow \\ HP(\mathrm{Coh}_{Z_{i-1}}(Z)) & \longrightarrow & HP(\mathrm{Coh}(Z_i)) & \longrightarrow & HP(\mathrm{Coh}(Z_i - Z_{i-1})) \end{array}$$

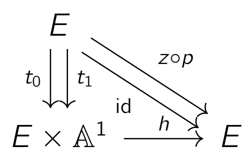
which factors the trace $K_0 \otimes k((u)) \to HP$. The left isomorphism is given by the still conjectural devissage result in periodic cyclic homology and by induction on i. Since $Z_i - Z_{i-1}$ is an equivariant vector bundle over BK where $K = B \cap w \cdot B$, by Lemma 4.3.4 and Lemma 4.3.5 the map on Hochschild homology

$$K_0(Z_i - Z_{i-1}) \to HH(\mathrm{Coh}(Z_i - Z_{i-1}) \simeq k[H]$$

is an isomorphism. Furthermore, it factors the natural map $K \to HC$, since HC has non-negative cohomological amplitude. Therefore the middle arrow factors $K \to HP$ and is also an isomorphism. $\square$

Lemma 4.3.4 (Periodic cyclic homology of vector bundles)**.** *Let $p : E \to X$ be a vector bundle over a stack X. Then, the functors $p^* : \mathrm{Perf}(X) \to \mathrm{Perf}(E)$ and $p^* : \mathrm{Coh}(X) \to \mathrm{Coh}(E)$ induce isomorphisms on periodic cyclic homology.*

Proof. Let $z : X \to E$ denote the zero section First note that $p^* : \mathrm{Coh}(X) \to \mathrm{Coh}(E)$ is well-defined since p is smooth, and $z^* : \mathrm{Coh}(E) \to \mathrm{Coh}(X)$ is well-defined since it is a local complete intersection. Let us prove the Coh case; the argument for Perf is the same but easier since we do not have to worry about functors being well-defined on compact objects. Consider the diagram



where $h : E \times \mathbb{A}^1$ is the homotopy contracting the fibers of the vector bundle. Note that the pullbacks t_0^*, t_1^* make sense as functors on derived categories of coherent sheaves since t_0, t_1 are local complete intersections (of codimension one). Since $\mathrm{Coh}(E \times \mathbb{A}^1) \simeq \mathrm{Coh}(E) \otimes \mathrm{Perf}(\mathbb{A}^1)$ (see Remark 1.1.6 in [9]) and periodic cyclic homology is $\mathbb{A}^1$-homotopy invariant (i.e.$HP(\mathbf{C} \otimes \mathrm{Perf}(\mathbb{A}^1)) \simeq HP(\mathbf{C})$ is an equivalence for any $\mathbf{C}$; see Theorem B (iii) of [14] for details) we have that $(z \circ p)^* : HP(\mathrm{Coh}(X)) \to HP(\mathrm{Coh}(E))$ is homotopic to the identity. The functor $p \circ z$ is the identity, and the statement follows. □

Lemma 4.3.5. *Let G be a reductive linear algebraic group, B, B' two Borel subgroups, and $K = B \cap B'$, and let H be the universal Cartan, canonically realized as the quotient of K, B or B' by their respective unipotent radicals. The natural map $\mathcal{O}(\mathcal{L}(BH)) \to \mathcal{O}(\mathcal{L}(BK))$ is an isomorphism.*

Proof. First, we claim that that $\mathcal{O}(\mathcal{L}(BK)) = \mathcal{O}(K/K)$ has cohomological amplitude zero. There is an isomorphism of stacks $K \backslash K \simeq G \backslash (G \times^K K)$, and if we forget the stacky G-quotient, there is a sequence of (G-equivariant) maps of schemes $G \times^K K \to G/K \to G/B$, where the first map is a K-bundle and the second map is an affine bundle with fiber B/K. Our strategy will be to apply two functors to the G-equivariant sheaf $\mathcal{O}_{G \times^K K}$: (1) the pushforward along the map $G/K \to G/B$ to the category of G-equivariant coherent sheaves on G/B, which is equivalent to the category B-representations, and then (2) push it forward along the map $BB \to BG \to \mathrm{pt}$, i.e. induce it to a G-representation using Borel-Weil-Bott and then take its trivial isotypic component.

This B-representation we consider is somewhat complex; since we are only interested in the trivial isotypic component of its induction to G, we can make a some simplifying reductions. First, it is a (infinite) direct sum of finite-dimensional representations, so we can focus on each of these finite-dimensional pieces. Any finite-dimensional B-representation has a finite Jordan-Holder series whose composition factors are one-dimensional, and one-dimensional representations are classified by the action of a torus $T \subset B$. In general, to compute the induction to G (i.e. global sections on the flag variety under Borel-Weil-Bott) of such a representation, we need to use a spectral sequence associated to the Jordan-Holder filtration whose boundary maps can be difficult to compute. Since we are only interested in the trivial isotypic component, we only need to keep track of where its corresponding weights appear in the spectral sequence, in particular we want to look for weights which are reflections of zero on the weight lattice along the shifted reflection hyperplanes. We will show that only the zero weights (and not its reflections) show up in as composition factors of this B-representation (i.e. not the shifted reflected zero weights), and thus the trivial-isotypic component of the spectral sequence degenerates, and the result follows.

In what follows, we will identify the B-representation and the filtration we use to compute its induction to G. By equivariance we can consider the B-equivariant sequence of maps $B \times^K K \to B/K \to \text{pt}$. Let us first identify B/K. Let $U \subset B$ be the unipotent radical and choose a subgroup $V \subset U$ such that $K \cap V = \{e\}$ and $KV = B$. For example, we can do this by choosing a torus $T \subset B$ such that $K = B \cap w \cdot B$, for $w \in N(T)$, and take $V = U \cap w \cdot B^{op}$. Then, the multiplication map $K \times V \to B$ is clearly K-equivariant; furthermore, each K-orbit in B intersects exactly one point in the image, so the map induces on quotients an isomorphism $V \simeq B/K$. However, this isomorphism is not B-equivariant; we cannot even naturally endow V with a B-action since V is not normalized by B. Our remedy will be instead to consider T-actions, which will suffice since our argument only depends on knowing the composition factors of B-representations, and these are determined by the action of a maximal torus $T \subset B$. The maximal torus T we chose in defining V will suffice; it normalizes V, so we can write $t \cdot vK = tvK = (tvt^{-1})tK = (tvt^{-1})K$.

Further, we claim that $B \times_K K \simeq B/K \times K$. In a general setting, a G-torsor $P \to X$ and a scheme Z with a G-action determines a Z-bundle over X by $P \times^G Z \to X$. Here, $B \times_K K \to B/K$ is a K-bundle which is determined by the K-torsor $B \to B/K$ and the adjoint action of K on itself. The isomorphism followed by the inclusion $B/K \to V \to B$ is a trivializing section (though not B-equivariant, as discussed). Collecting the above discussion, we have an isomorphism $V \times K \simeq B \times_K K$ which is T-equivariant, with T acting on $V \times K$ by simultaneous conjugation on each factor. In particular this implies that $\mathcal{O}(B/K \times K)$ has the same T-weights as $\mathcal{O}(B)$, which completes our argument that $\mathcal{O}(\mathcal{L}BG)$ has cohomological amplitude zero.

Finally, we claim that the map $\mathcal{O}(H/H) \to \mathcal{O}(K/K)$ is an isomorphism in degree zero; we can understand $\mathcal{L}(BK)$ more equivariantly as the subvariety of the global Steinberg St_G consisting of points (x, B, B') where B, B' have relative position given by w, and such that $x \in B \cap B'$. Then, if $U \subset B$ and $U' \subset B'$ are the respective unipotent radicals, and $B = w \cdot B'$, the map sends $(x, B, B') \in K$ to $x \mod U = w \cdot (x \mod U') \in H$. It is well-known that the closed G-orbits correspond bijectively to the closed points downstairs. □

Corollary 4.3.6. *The square*

$$\begin{array}{ccc} K_0(BH) & \longrightarrow & K_0(BK) \\ \downarrow & & \downarrow \\ HH(BH) & \longrightarrow & HH(BK) \end{array}$$

commutes, and all maps are isomorphisms.

Proof. The map $K(BH) \to HH(BH)$ factors through the truncation to K_0, since $HH(BH)$ has cohomological amplitude zero as shown above, and likewise for BK. We've just proven that the bottom arrow is an isomorphism, and it is well-known that the top and left arrows are also isomorphisms. □

Bibliography

[1] Jarod Alper, Jack Hall, and David Rydh. "A Luna étale slice theorem for algebraic stacks". In: (2015). arXiv: 1504.06467.

[2] D. Arinkin and D. Gaitsgory. "Singular support of coherent sheaves and the geometric Langlands conjecture". In: *Selecta Math. (N.S.)* 21.1 (2015), pp. 1–199. ISSN: 1022-1824. URL: https://doi.org/10.1007/s00029-014-0167-5.

[3] Dima Arinkin, Andrei Căldăraru, and Márton Hablicsek Hablicsek. "Formality of derived intersections and the orbifold HKR isomorphism". In: (2014). arXiv: 1412.5233v1.

[4] A. Beilinson. "Remarks on topological algebras". In: *Mosc. Math. J.* 8.1 (2008), pp. 1–20, 183. ISSN: 1609-3321.

[5] David Ben-Zvi, John Francis, and David Nadler. "Integral transforms and Drinfeld centers in derived algebraic geometry". In: *J. Amer. Math. Soc.* 23.4 (2010), pp. 909–966. ISSN: 0894-0347. URL: https://doi.org/10.1090/S0894-0347-10-00669-7.

[6] David Ben-Zvi and David Nadler. "Loop spaces and connections". In: *J. Topol.* 5.2 (2012), pp. 377–430. ISSN: 1753-8416. URL: https://doi.org/10.1112/jtopol/jts007.

[7] David Ben-Zvi and David Nadler. "Loop spaces and representations". In: *Duke Math. J.* 162.9 (2013), pp. 1587–1619. ISSN: 0012-7094. URL: https://doi.org/10.1215/00127094-2266130.

[8] David Ben-Zvi and David Nadler. "Nonlinear Traces". In: (2013). arXiv: 1305.7175v3.

[9] David Ben-Zvi, David Nadler, and Anatoly Preygel. "Integral transforms for coherent sheaves". In: (2013). arXiv: arXiv:1312.7164v1.

[10] Bhargav Bhatt. "Completions and derived de Rham cohomology". In: (2012). arXiv: 1207.6193v1.

[11] Bhargav Bhatt and Peter Scholze. "The pro-étale topology for schemes". In: *Astérisque* 369 (2015), pp. 99–201. ISSN: 0303-1179.

[12] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. "A universal characterization of higher algebraic K-theory". In: *Geom. Topol.* 17.2 (2013), pp. 733–838. ISSN: 1465-3060. URL: https://doi.org/10.2140/gt.2013.17.733.

[13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. "Uniqueness of the multiplicative cyclotomic trace". In: *Adv. Math.* 260 (2014), pp. 191–232. ISSN: 0001-8708. URL: `https://doi.org/10.1016/j.aim.2014.02.004`.

[14] Dan Burghelea and Crichton Ogle. "The Künneth formula in cyclic homology". In: *Math. Z.* 193.4 (1986), pp. 527–536. ISSN: 0025-5874. DOI: `10.1007/BF01160473`. URL: `https://doi.org/10.1007/BF01160473`.

[15] Andrei Căldăraru. "The Mukai Pairing II: The Hochschild-Kostant-Rosenberg Isomorphism". In: (2004). arXiv: `0308080v3`.

[16] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Reprint of the 1997 edition. Birkhäuser Boston, Inc., Boston, MA, 2010, pp. x+495. ISBN: 978-0-8176-4937-1. URL: `https://doi.org/10.1007/978-0-8176-4938-8`.

[17] Ionuţ Ciocan-Fontanine and Mikhail Kapranov. "Derived Quot schemes". In: *Ann. Sci. École Norm. Sup. (4)* 34.3 (2001), pp. 403–440. ISSN: 0012-9593. DOI: `10.1016/S0012-9593(01)01064-3`. URL: `https://doi.org/10.1016/S0012-9593(01)01064-3`.

[18] Lee Cohn. "Differential Graded Categories are k-linear Stable Infinity Categories". In: (2016). arXiv: `1308.2587v2`.

[19] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*. Second. Vol. 26. New Mathematical Monographs. Cambridge University Press, Cambridge, 2015, pp. xxiv+665. ISBN: 978-1-107-08723-1. DOI: `10.1017/CBO9781316092439`. URL: `https://doi.org/10.1017/CBO9781316092439`.

[20] Jean-Marc Drézet. "Luna's slice theorem and applications". In: *Algebraic group actions and quotients*. Hindawi Publ. Corp., Cairo, 2004, pp. 39–89.

[21] Vladimir Drinfeld and Dennis Gaitsgory. "On some finiteness questions for algebraic stacks". In: *Geom. Funct. Anal.* 23.1 (2013), pp. 149–294. ISSN: 1016-443X. URL: `https://doi.org/10.1007/s00039-012-0204-5`.

[22] Dennis Gaitsgory. "ind-coherent sheaves". In: *Mosc. Math. J.* 13.3 (2013), pp. 399–528, 553. ISSN: 1609-3321.

[23] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vols. I. and II.* Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017. ISBN: 978-1-4704-3569-1, 978-1-4704-3570-7.

[24] Dennis Gaitsgory and Nick Rozenblyum. "DG indschemes". In: *Perspectives in representation theory*. Vol. 610. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 139–251. URL: `https://doi.org/10.1090/conm/610/12080`.

[25] Ezra Getzler and Paul Goerss. "A Model Category Structure for Differential Graded Coalgebras". In: (1999). URL: `https://ncatlab.org/nlab/files/GetzlerGoerss99.pdf`.

[26] Mark Goresky, Robert Kottwitz, and Robert MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". In: *Invent. Math.* 131.1 (1998), pp. 25–83. ISSN: 0020-9910. URL: `https://doi.org/10.1007/s002220050197`.

[27] Jack Hall, Amnon Neeman, and David Rydh. "One positive and two negative results for derived categories of algebraic stacks". In: (2015). arXiv: `1405.1888v2`.

[28] Jack Hall and David Rydh. "Perfect complexes on algebraic stacks". In: (2017). arXiv: `1405.1887v3`.

[29] Daniel Halpern-Leistner and Daniel Pomerleano. "Equivariant Hodge theory and non-commutative geometry". In: (2016). arXiv: `1507.01924`.

[30] Robin Hartshorne. "On the De Rham cohomology of algebraic varieties". In: *Inst. Hautes Études Sci. Publ. Math.* 45 (1975), pp. 5–99. ISSN: 0073-8301. URL: `http://www.numdam.org/item?id=PMIHES_1975__45__5_0`.

[31] Kathryn Hess. "Rational homotopy theory: a brief introduction". In: *Interactions between homotopy theory and algebra*. Vol. 436. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 175–202. DOI: `10.1090/conm/436/08409`. URL: `https://doi.org/10.1090/conm/436/08409`.

[32] Jens Carsten Jantzen. *Representations of algebraic groups*. Second. Vol. 107. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xiv+576. ISBN: 0-8218-3527-0.

[33] John D. S. Jones. "Cyclic homology and equivariant homology". In: *Invent. Math.* 87.2 (1987), pp. 403–423. ISSN: 0020-9910. URL: `https://doi.org/10.1007/BF01389424`.

[34] Dmitry Kaledin. "Co-periodic cyclic homology". In: (2015). arXiv: `1509.08784v1`.

[35] Bernhard Keller. "On differential graded categories". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[36] Bernhard Keller. "On the cyclic homology of exact categories". In: *J. Pure Appl. Algebra* 136.1 (1999), pp. 1–56. ISSN: 0022-4049. URL: `https://doi.org/10.1016/S0022-4049(97)00152-7`.

[37] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208. ISBN: 3-540-65761-4.

[38] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by María O. Ronco. Springer-Verlag, Berlin, 1992, pp. xviii+454. ISBN: 3-540-53339-7. URL: https://doi.org/10.1007/978-3-662-21739-9.

[39] Jacob Lurie. "Derived Algebraic Geometry II: Noncommutative Algebra". In: (2007). arXiv: 0702299v5.

[40] Jacob Lurie. "Derived Algebraic Geometry IV: Deformation Theory". In: (2009). arXiv: 0709.3091v2.

[41] Jacob Lurie. "Derived Algebraic Geometry VII: Spectral Schemes". In: (2011). URL: http://www.math.harvard.edu/~lurie/.

[42] Jacob Lurie. "Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems". In: (2011). URL: http://www.math.harvard.edu/~lurie/.

[43] Jacob Lurie. "Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem". In: (2011). URL: http://www.math.harvard.edu/~lurie/.

[44] Jacob Lurie. *Higher Algebra*. 2017. URL: http://www.math.harvard.edu/~lurie/.

[45] Jacob Lurie. *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. URL: https://doi.org/10.1515/9781400830558.

[46] Jacob Lurie. "On the classification of topological field theories". In: *Current developments in mathematics, 2008*. Int. Press, Somerville, MA, 2009, pp. 129–280.

[47] Jacob Lurie. "Spectral Algebraic Geometry". In: (2018). URL: http://www.math.harvard.edu/~lurie/.

[48] Daniel Murfet. "The Mock Homotopy Category of Projectives and Grothendieck Duality". PhD thesis. Australian National University, 2007.

[49] Amnon Neeman. "The homotopy category of flat modules, and Grothendieck duality". In: *Invent. Math.* 174.2 (2008), pp. 255–308. ISSN: 0020-9910. URL: https://doi.org/10.1007/s00222-008-0131-0.

[50] Martin Olsson. "Sheaves on Artin stacks". In: *J. Reine Angew. Math.* 603 (2007), pp. 55–112. ISSN: 0075-4102. URL: https://doi.org/10.1515/CRELLE.2007.012.

[51] Anatoly Preygel. "Ind-coherent complexes on loop spaces and connections". In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 289–323. URL: https://doi.org/10.1090/conm/643/12902.

[52] Anatoly Preygel. "Thom-Sebastiani & Duality for Matrix Factorizations". In: (2011). arXiv: 1101.5834v1.

[53] Anatoly Preygel. *Thom-Sebastiani and Duality for Matrix Factorizations, and Results on the Higher Structures of the Hochschild Invariants*. Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2012, (no paging). URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:0828804.

[54] J. P. Pridham. "Presenting higher stacks as simplicial schemes". In: *Adv. Math.* 238 (2013), pp. 184–245. ISSN: 0001-8708. DOI: 10.1016/j.aim.2013.01.009. URL: https://doi.org/10.1016/j.aim.2013.01.009.

[55] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2018.

[56] Robert Steinberg. "On a theorem of Pittie". In: *Topology* 14 (1975), pp. 173–177. ISSN: 0040-9383. DOI: 10.1016/0040-9383(75)90025-7. URL: https://doi.org/10.1016/0040-9383(75)90025-7.

[57] R. W. Thomason. "Algebraic K-theory of group scheme actions". In: *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*. Vol. 113. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.

[58] R. W. Thomason. "Lefschetz-Riemann-Roch theorem and coherent trace formula". In: *Invent. Math.* 85.3 (1986), pp. 515–543. ISSN: 0020-9910. URL: https://doi.org/10.1007/BF01390328.

[59] Bertrand Toën. "Champs affines". In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 39–135. ISSN: 1022-1824. URL: https://doi.org/10.1007/s00029-006-0019-z.

[60] Bertrand Toën. "Derived algebraic geometry". In: *EMS Surv. Math. Sci.* 1.2 (2014), pp. 153–240. ISSN: 2308-2151. URL: https://doi.org/10.4171/EMSS/4.

[61] Bertrand Toën. "Derived algebraic geometry". In: *EMS Surv. Math. Sci.* 1.2 (2014), pp. 153–240. ISSN: 2308-2151. URL: https://doi.org/10.4171/EMSS/4.

[62] Bertrand Toën. "The homotopy theory of dg-categories and derived Morita theory". In: *Invent. Math.* 167.3 (2007), pp. 615–667. ISSN: 0020-9910. URL: https://doi.org/10.1007/s00222-006-0025-y.

[63] Rahbar Virk. "Tate classes, equivariant geometry and purity". In: (2015). arXiv: 1507.0727v2.

[64] Amnon Yekutieli. "The continuous Hochschild cochain complex of a scheme". In: *Canad. J. Math.* 54.6 (2002), pp. 1319–1337. ISSN: 0008-414X. DOI: 10.4153/CJM-2002-051-8. URL: https://doi.org/10.4153/CJM-2002-051-8.