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RESIDUAL STRUCTURE AND GROWING INVERSION-MONOTONICITY REGIONS FOR 1324-AVOIDING PERMUTATIONS

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Abstract. Let $a(n,k)$ be the number of 1324-avoiding permutations of length n with k inversions. Linusson and Verkama proved $a(n,k) \le a(n+1,k)$ for $k \le 2n-7$. We study the obstruction beyond that line: the residuals $\mathcal{R}_{\delta,n}$, namely the indecomposable, non-almost-decomposable avoiders at defect $\delta = k-2n+7$. Contracting maximal increasing consecutive runs reduces residuality to a quadratic equation on a finite family of skeletons. It follows that, for every fixed δ, the eventual count has the form

$$|\mathcal{R}_{\delta,n}| = A_\delta n^2 + B_\delta n + C_\delta.$$

Our central uniform result determines the quadratic coefficient at every defect. With $P(q) = \prod_{j\ge 1}(1-q^j)^{-1}$,

$$\sum_{\delta\ge 0} A_\delta q^\delta = \frac{4q^3(1+q)}{(1-q)^2} P(q)^2.$$

This is a formula for the leading coefficient of the residual count, not for the full count.

The same structural estimates give computer-assisted proofs of $a(n,k) \le a(n+1,k)$ for every $n \ge 1$ and $k \le 2n+6$, and of regions whose width grows with n: for $n \ge 2^{16}, 2^{18}, 2^{20}$ the defect may be as large as $\lfloor\sqrt{n}/4\rfloor$, $\lfloor\sqrt{n}/3\rfloor$, $\lfloor\sqrt{n}/2\rfloor$, respectively. More generally, every fixed $c < \sqrt{2}\log(5)/\log(68)$ is admissible for all sufficiently large n. The three added fixed defects $11, 12, 13$ use complete catalogue and rational-sum certificates supplied in the accompanying archival supplement. The leading-coefficient theorem is obtained from a complete finite classification of marked rank-three cores and all-parameter extension lemmas; we state the finite certificates and the logical order in which they enter. We also determine the exact rank-three stabilization onset, while keeping it separate from the still unknown onset of the complete residual count. The unrestricted Claesson–Jelínek–Steingrímsson conjecture, the full residual polynomials, and the sharp global base-length bound remain open.

1. Introduction

For a permutation $\pi = \pi_1 \cdots \pi_n$ let $\mathrm{inv}(\pi)$ be its number of inversions and let $\mathrm{Av}_n^k(1324)$ be the set of 1324-avoiding permutations of length n with k inversions, $a(n,k) = |\mathrm{Av}_n^k(1324)|$. Claesson, Jelínek and Steingrímsson [5] conjectured that

$$a(n,k) \le a(n+1,k) \qquad \text{for all } n \ge 1,\ k \ge 0, \tag{1}$$

and showed that (1) implies the bound $e^{\pi\sqrt{2/3}} \approx 13.002$ for the growth rate of $\mathrm{Av}(1324)$. Bevan, Brignall, Elvey Price and Pantone proved the upper bound 13.5 [2]. Linusson and Verkama [7] proved (1) for $k \le 2n-7$ and determined $a(n,k)$ exactly in that range:

$$a(n+1,k) - a(n,k) = 4p_2(k-n+1) + 2p_2(k-n) \qquad (k \le 2n-7), \qquad p_2(m) = \sum_i p(i)p(m-i), \tag{2}$$

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p the partition function ($p_2(m) = 0$ for $m < 0$); equivalently [7, Theorem 1.2]

$$a(n,k) = p_2(k) - 4p_2(k-n+1) - 6\sum_{i=0}^{k-n} p_2(i) \qquad (k \le 2n-7). \tag{3}$$

Their proof has two parts. A permutation is *decomposable* if it is a direct sum of two nonempty permutations, and an indecomposable permutation π is *almost decomposable* if one of the four deletions $\pi\backslash 1$, $\pi\backslash n$, $\pi\backslash\pi_1$, $\pi\backslash\pi_n$ (delete the entry and standardize) is decomposable. First, every indecomposable 1324-avoider with $k \le 2n-7$ inversions is almost decomposable [7, Theorem 2.5]. Second, there are maps g on decomposable and f on almost decomposable 1324-avoiders, defined for every n and k, which preserve 1324-avoidance and the inversion number, increase the length by one, are injective, and have disjoint images [7, Theorem 3.5].

This paper concerns the region $k \ge 2n-6$, where indecomposable 1324-avoiders that are not almost decomposable exist. Write $k = 2n-7+\delta$ and call $\delta \ge 1$ the *defect*; let

$$\mathcal{R}_{\delta,n} = \{\pi \in \mathrm{Av}_n^{2n-7+\delta}(1324) : \ \pi \text{ indecomposable and not almost decomposable}\}$$

be the set of *residuals*. Since $f \sqcup g$ is defined on everything else and is injective,

$$a(n+1,k) - a(n,k) = \big|\,\mathrm{Av}_{n+1}^{k}(1324) \setminus \mathrm{im}(f \sqcup g)\big| - |\mathcal{R}_{\delta,n}|, \tag{4}$$

and (1) at (n,k) is equivalent to $|\mathcal{R}_{\delta,n}| \le |\,\mathrm{Av}_{n+1}^{k}(1324) \setminus \mathrm{im}(f \sqcup g)|$. The complement of the image is large: it always contains every σ with $\sigma_1 = n+1$, $\sigma_{n+1} = 1$, $\sigma_2 = n+1$ or $\sigma_{n+1} = 2$, and these number $2a(n,k-n) + 2a(n,k-n+1)$ as long as $k \le 2n-4$ (Lemma 3.1); moreover $a(n,j) = p_2(j)$ as soon as $j \le n-2$.

We place the three principal conclusions first. Their proofs use the structural development below and the finite certificates described in Section 7.4. The order matters: the finite marked-core class used in Theorem 1.1 is defined and enumerated independently, before it is identified with canonical rank-three cells.

Theorem 1.1 (uniform quadratic coefficient)**.** *For every $\delta \ge 0$, let A_δ be the coefficient of n^2 in the eventual polynomial $|\mathcal{R}_{\delta,n}|$, taking $A_\delta = 0$ when its degree is less than two. Then*

$$\sum_{\delta \ge 0} A_\delta q^\delta = \frac{4q^3(1+q)}{(1-q)^2} P(q)^2, \qquad P(q) = \prod_{j \ge 1} (1-q^j)^{-1}. \tag{5}$$

Theorem 1.2 (fixed and growing low-inversion regions)**.** *For all $n \ge 1$,*

$$a(n,k) \le a(n+1,k) \qquad (0 \le k \le 2n+6).$$

The following larger regions also hold:

$$\begin{aligned} n \ge 2^{16}: &\quad 0 \le k \le 2n-7+\lfloor \sqrt{n}/4 \rfloor, \\ n \ge 2^{18}: &\quad 0 \le k \le 2n-7+\lfloor \sqrt{n}/3 \rfloor, \\ n \ge 2^{20}: &\quad 0 \le k \le 2n-7+\lfloor \sqrt{n}/2 \rfloor. \end{aligned}$$

For every fixed $0 < c < \sqrt{2}\log(5)/\log(68)$, the same inequality holds for $0 \le k \le 2n-7+\lfloor c\sqrt{n} \rfloor$ once $n \ge N(c)$. In the three square-root regions and in the asymptotic region, the inequality is strict whenever $\delta \ge 1$.

The first assertion of Theorem 1.2 extends Corollary 5.20 by the three defects $11, 12, 13$. Its proof uses only the already completed sharp skeleton bound and the defect-$11, 12, 13$ catalogues, rational generating functions and boundary comparisons; it does not use Theorem 1.1 or any later high-inversion construction. The square-root assertions instead combine a uniform residual upper bound with an elementary lower bound on two-coloured partitions.

Theorem 1.3. *For $n \geq 8$ the residual set $\mathcal{R}_{1,n}$ consists exactly of the inflations*

$$24153[\iota_a,\, 1,\, \iota_c,\, \iota_d,\, \iota_e] \quad \textit{and} \quad 31524[\iota_a,\, \iota_b,\, \iota_c,\, 1,\, \iota_e]$$

by increasing blocks ($\iota_m = 12\cdots m$) with $a, d \geq 1$, $c, e \geq 2$, $(a-1)(c-2)+(d-1)(e-2) = 0$ in the first case and $a, c \geq 2$, $b, e \geq 1$, $(a-2)(b-1)+(c-2)(e-1) = 0$ in the second; the two families are exchanged by inversion. Consequently $|\mathcal{R}_{1,n}| = 8(n-7)$ for $n \geq 8$ (and $|\mathcal{R}_{1,7}| = 2$, $|\mathcal{R}_{1,n}| = 0$ for $n \leq 6$).

Theorem 1.4. *$a(n,k) \leq a(n+1,k)$ for all $n \geq 1$ and all $k \leq 2n-6$. More precisely, for $n \geq 8$,*

$$a(n+1, 2n-6) - a(n, 2n-6) \;\geq\; 2p_2(n-6) - 8(n-7) \;> 0,$$

and in fact $a(n+1, 2n-6) - a(n, 2n-6) = 4p_2(n-5) + 2p_2(n-6) - 4(n-7)$.

The exact formula in Theorem 1.4 says that the Linusson–Verkama difference (2) acquires the correction $-4(n-7)$ at defect one. At defect two the same method gives twenty-one inflation families and $|\mathcal{R}_{2,n}| = 32n - 214$ (Theorems 4.3 and 4.4).

Beyond defect two, Section 5 contracts maximal increasing consecutive runs. The resulting skeleton σ and block vector b describe a residual exactly by $\sum_{(i,j)\in \mathrm{Inv}(\sigma)} b_i b_j = 2n - 7 + \delta$ and explicit lower bounds on the blocks in $D(\sigma)$ (Theorem 5.3). Proposition 5.4 rewrites this as one quadratic equation in excess block sizes. Theorem 5.23 makes the skeleton set finite at each defect, Theorem 5.19 supplies the sharp all-defect skeleton bound by strong induction, and Theorems 6.4 and 6.6 give the following unconditional polynomiality statement.

Theorem 1.5. *For every fixed $\delta \geq 1$, $|\mathcal{R}_{\delta,n}|$ agrees for all large n with a polynomial in n of degree at most two (Theorems 6.4 and 6.6); by Proposition 5.8 the degree is 1 for $\delta \leq 2$ and 2 for $3 \leq \delta \leq 10$.*

That $|\mathcal{R}_{\delta,n}|$ is at least *bounded* by a polynomial in n is proved here (Corollary 5.34), and since $p_2(n-7+\delta)$ grows like $\exp(\frac{2\pi}{\sqrt{3}}\sqrt{n})$ this already yields (1) at every fixed defect for all sufficiently large n (Theorem 5.35). Theorem 1.5 identifies the shape of $|\mathcal{R}_{\delta,n}|$ but not its coefficients, so it does not by itself supply a workable threshold in place of the very large one that the crude bound gives.

Section 7 determines the quadratic coefficients uniformly, extends the fixed strip from defect ten to defect thirteen, and obtains the growing square-root strips. These are separate consequences: the fixed-strip extension uses only the completed defect-$11, 12, 13$ catalogues and comparisons, whereas the uniform leading coefficient does not determine the missing lower coefficients.

Since the paper mixes proofs, machine checks and conjectures, we summarise the status of its statements.

statement	status
Theorem 1.3, Lemmas 3.2, 5.14, 5.15, 5.16, 5.22, 5.25, Proposition 5.4, Theorem 5.3	proved
Theorems 1.4, 1.5, 4.3, 4.4, 5.27, 5.29, 5.23, 5.35, 6.4, 6.6, 5.19, Lemmas 5.28, 6.1, 6.5, 5.17, Corollaries 5.34, 6.2, 5.20, Propositions 6.3, 5.7, 6.8, 5.13	proved, computer-assisted
Theorems 1.1, 1.2 and Proposition 7.3	proved, computer-assisted; new certificate bundle cited below
Propositions 5.5, 5.8	finite computation, stated over its range
Corollary 5.9	hypothesis supplied by Theorem 5.19
Conjectures 5.6, 5.12	proved for $\delta \le 10$ (Theorem 5.19)
Conjecture 5.10	proved for every δ by Theorem 5.19 and strong induction
Conjectures 5.26, 6.7, 6.9	conjectural

The phrase *computer-assisted* is used in the precise sense of Section 8: the proof is complete except for finitely many exhaustive searches over $\mathfrak{S}_6$, $\mathfrak{S}_7$, $\mathfrak{S}_8$ or over explicitly generated finite lists. The original checks are in the archive cited in Section 8. The additional marked-core, catalogue, rational-sum and boundary checks are supplied in the archival supplement cited below.

2. The Linusson–Verkama structure

We recall the notions and results of [7] that we use, in their notation. Fix $\pi \in \mathrm{Av}_n(1324)$ and write $a = \pi_1$, $b = \pi_n$, $p = \pi^{-1}(1)$, $q = \pi^{-1}(n)$. Suppose $a < b$ and $p < q$. The *northwestern region* is $\mathrm{NW} = \{i < p : \pi_i > b\}$ and the *southeastern region* is $\mathrm{SE} = \{j > q : \pi_j < a\}$. For $i \in \mathrm{NW}$ the *southern region* relative to i is $\mathrm{S}(i) = \{j : i < j < q,\ 1 < \pi_j < a\}$ and the *eastern region* is $\mathrm{E} = \{j : q < j < n,\ a < \pi_j < b\}$ (it does not depend on i). The *central region* $\{j : i < j < q,\ a < \pi_j < b\}$ is empty, since together with π_1, π_i, n a point in it would form a 1324.

Lemma 2.1 ([7, Lemmas 2.6, 2.7, 2.11])**.** *Let $\pi \in \mathrm{Av}_n^k(1324)$ be indecomposable.*

(1) *If $a > b$ and $k \le 2n-5$, then $\pi \setminus \pi_1$ or $\pi \setminus \pi_n$ is decomposable.*
(2) *If $a < b$ and $p < q$, then $\mathrm{NW} \cup \mathrm{SE} \ne \varnothing$; and if moreover $k \le 2n-6$, then $\mathrm{NW} = \varnothing$ or $\mathrm{SE} = \varnothing$.*

The map $\pi \mapsto \pi^{-1}$ preserves 1324-avoidance, inv*, indecomposability and almost decomposability, exchanges the conditions $a < b$ and $p < q$, and exchanges* NW *and* SE*.*

From now on assume $a < b$, $p < q$, $\mathrm{SE} = \varnothing$ and $\mathrm{NW} = \{i_1 < \cdots < i_m\} \ne \varnothing$, and write $v_l = \pi_{i_l}$. The values $v_1 < \cdots < v_m$ increase: if $v_x > v_y$ for some $x < y$ then π_1, v_x, v_y, n is a 1324. For a value w put

$$\mathrm{E}(w) = \{j :\ q < j < n,\ a < \pi_j < w\},$$

so that $\mathrm{E}(b)$ is the eastern region of [7] and $\mathrm{E}(b) \subseteq \mathrm{E}(v_1) \subseteq \cdots \subseteq \mathrm{E}(v_m)$.

Lemma 2.2. *(i) If $\mathrm{S}(i_1) = \varnothing$ then $\pi \setminus 1$ is decomposable. (ii) If $\mathrm{E}(v_m) = \varnothing$ then $\pi \setminus \pi_n$ is decomposable.*

Proof. (i) is the first half of the proof of [7, Lemma 2.10]; we repeat it since we need to see that it uses only $\mathrm{SE} = \varnothing$. Let $\sigma = \pi \setminus 1$; we claim that the first $i_1 - 1$ entries of σ are its $i_1 - 1$ smallest values, which makes σ decomposable ($i_1 \ge 2$ because $\pi_1 = a < b$). Otherwise there are positions $j_1 < i_1 < j_2$ of π, $j_2 \neq p$, with $\pi_{j_1} > \pi_{j_2}$. Since i_1 is the first northwestern point, $\pi_{j_1} \leq b$, hence $\pi_{j_1} < b$. If $\pi_{j_2} < a$ then j_2 would lie in $\mathrm{S}(i_1)$ (if $j_2 < q$) or in SE (if $j_2 > q$); so $a \leq \pi_{j_2} < \pi_{j_1} < b$, whence $j_1 \geq 2$, $a < \pi_{j_2}$, and $\pi_1, \pi_{j_1}, \pi_{j_2}, \pi_n$ is a 1324.

(ii) Let t be the first position with $\pi_t > v_m$ (it exists since $\pi_q = n$). Then $t > p$: a position $t < p$ with $\pi_t > v_m > b$ would be a northwestern point with a value exceeding v_m. Every position l with $t < l < n$ carries a value exceeding v_m: a value $\pi_l < a$ lies in SE if $l > q$ and gives the 1324 $1, \pi_t, \pi_l, n$ if $l < q$; a value in (a, b) lies in the central region if $l < q$ and in $\mathrm{E}(v_m)$ if $l > q$; a value in (b, v_m) at a position $l < q$ (necessarily $l > p$) gives the 1324 π_1, v_m, π_l, n, and at a position $l > q$ it lies in $\mathrm{E}(v_m)$; the values a, b, v_m sit at $1, n, i_m < t$. Hence in $\pi \setminus \pi_n$ the positions before t carry exactly the values below v_m other than b, i.e. the $t-1$ smallest values, and $2 \leq t \leq q \leq n-1$; so $\pi \setminus \pi_n$ is decomposable. $\square$

The following identity makes the inversion count of [7, Lemma 2.8] exact by recording its slack.

Lemma 2.3 (seven-index identity)**.** *Let $i \in \mathrm{NW}$ with $v = \pi_i$, and let $j_1 \in \mathrm{S}(i)$, $j_2 \in \mathrm{E}(v)$. Write $L(j) = \#\{l < j : \pi_l > \pi_j\}$. Then*

$$\mathrm{inv}(\pi) = 2n - 6 + s_1 + s_2 + s_3 + s_4 + s_5 + u,$$

where

$$s_1 = (a-1) + (n-q) - (v-i), \qquad s_2 = p - 1 - i, \qquad s_3 = L(j_1) - i,$$
$$s_4 = L(j_2) - (n - v + 1), \qquad s_5 = v - b - 1$$

are all nonnegative and $u \geq 0$ is the number of inversions (x, y) of π ($x < y$, $\pi_x > \pi_y$) with $x \notin \{1, i, q\}$ and $y \notin \{p, j_1, j_2, n\}$.

Proof. Write $R(l) = \#\{m > l : \pi_m < \pi_l\}$ for the number of right-inversions of position l. We have $R(1) = a - 1$, $R(q) = n - q$, $L(p) = p - 1$, $L(n) = n - b$. No entry before position i exceeds v (with π_1, v and n such an entry would give a 1324), so $R(i) = (v-1) - (i-1) = v - i$. Every entry before j_1 with position $\leq i$ exceeds π_{j_1}: for $l < i$ with $\pi_l < \pi_{j_1}$ the entries π_l, v, π_{j_1}, n form a 1324, and $v > b > \pi_{j_1}$; hence $L(j_1) \geq i$. Every value $w \geq v$ lies before j_2 and exceeds π_{j_2}: v and n do, and if $v < w < n$ sat after j_2 then π_1, v, π_{j_2}, w would be a 1324; hence $L(j_2) \geq n - v + 1$. Right-inversions of i are entries after i with value below v; such a value is either at most $a-1$ (at most $a-1$ possibilities) or lies in (a, v), and then its position exceeds q (a position in (i, q) would give the 1324 $\pi_1, v, \cdot, n$), so there are at most $n - q$ of them; thus $s_1 \geq 0$. Finally $s_2 \geq 0$ since $i < p$, and $s_5 \geq 0$ since $v > b$.

Now sum $R(1) + R(i) + R(q) + L(p) + L(j_1) + L(j_2) + L(n)$. An inversion (x, y) ($x < y$, $\pi_x > \pi_y$) is counted once for each of the two conditions $x \in \{1, i, q\}$ and $y \in \{p, j_1, j_2, n\}$ that it satisfies; the inversions satisfying both are exactly $(1, p), (1, j_1), (i, p), (i, j_1), (i, j_2), (i, n), (q, j_2), (q, n)$ — eight pairs (the remaining candidates $(1, j_2), (1, n), (q, p), (q, j_1)$ are not inversions or not in the right order), and the inversions satisfying neither are those counted by u. Hence

$$\mathrm{inv}(\pi) = R(1) + R(i) + R(q) + L(p) + L(j_1) + L(j_2) + L(n) - 8 + u,$$

and substituting $R(1) + R(q) = (v - i) + s_1$, $L(p) = i + s_2$, $L(j_1) = i + s_3$, $L(j_2) = (n - v + 1) + s_4$, $L(n) = (n - v + 1) + s_5$ gives $2(v - i) + 2i + 2(n - v + 1) - 8 + \sum s + u = 2n - 6 + \sum s + u$. $\square$

Lemma 2.4 (inflations)**.** *Let S be a permutation and $S[\iota_{m_1}, \ldots, \iota_{m_r}]$ its inflation by increasing blocks. If S avoids 1324, so does the inflation; if S is indecomposable, so is the inflation.*

Proof. An occurrence of 1324 in the inflation with two entries in the same block is impossible: two entries of one block are increasing in position and in value, hence play the roles $(1, 3)$, $(1, 2)$, $(1, 4)$, $(3, 4)$ or $(2, 4)$, and in each case a third entry of the occurrence lies between them in position, hence in the same block, and then its value is not in the required order. So the occurrence meets four distinct blocks and induces an occurrence in S. A sum-decomposition cut of the inflation that falls strictly inside a block forces that block, together with all blocks before it, to carry the smallest values, hence yields a cut of S after that block's point (or before the first point, which is not a cut); so a cut of the inflation induces a cut of S. ☐

Lemma 2.5 (two northwestern points)**.** *If $m \geq 2$, $\mathrm{S}(i_1) \neq \varnothing$ and $\mathrm{E}(v_m) \neq \varnothing$, then $\mathrm{inv}(\pi) \geq 2n-4$.*

This sharpens [7, Lemma 2.9], which rules out the configuration under the hypothesis $k \leq 2n-7$.

Proof. Choose $j_1 \in \mathrm{S}(i_1)$ and $j_2 \in \mathrm{E}(v_m)$ and count, as in Lemma 2.3, the inversions incident to the positions $1, i_m, q$ (right-inversions) and p, j_1, j_2, n (left-inversions):

$$\mathrm{inv}(\pi) \;\geq\; R(1) + R(i_m) + R(q) + L(p) + L(j_1) + L(j_2) + L(n) - D,$$

where $D = 7 + [i_m < j_1]$ counts the inversions $(1,p), (1,j_1), (i_m,p), (i_m,j_2), (i_m,n), (q,j_2), (q,n)$ and, if $i_m < j_1$, (i_m, j_1). Here $R(1) = a-1$, $R(q) = n-q$, $L(p) = p-1 \geq i_m$, $L(n) = n-b$, and $R(i_m) = v_m - i_m$ because no entry before i_m exceeds v_m. Every position $l < i_1$ has $\pi_l > \pi_{j_1}$ (else π_l, v_1, π_{j_1}, n is a 1324), and $v_m > \pi_{j_1}$, so $L(j_1) \geq i_1 + [i_m < j_1]$. All values exceeding v_m lie before j_2 (else π_1, v_m, π_{j_2}, w is a 1324), and so does v_1, which exceeds π_{j_2} if $\pi_{j_2} < v_1$; so $L(j_2) \geq (n - v_m + 1) + [\pi_{j_2} < v_1]$. Finally the right-inversions of i_m are values below a lying after i_m or positions after q; the values below a that lie before i_m are not among them, so $(a-1) + (n-q) \geq (v_m - i_m) + |\mathrm{S}(i_1) \setminus \mathrm{S}(i_m)|$. Adding up,

$$\mathrm{inv}(\pi) \;\geq\; 2n - 5 - (i_m - i_1) + |\mathrm{S}(i_1) \setminus \mathrm{S}(i_m)| + [\pi_{j_2} < v_1] + (v_m - b - 1).$$

A position strictly between i_1 and i_m carries either a northwestern value or a value in $(1,a)$ (a value in (a,b) would lie in the central region, one in (b, v_m) would be northwestern, one exceeding v_m would contradict the monotonicity of the northwestern values), so $i_m - i_1 - 1 = (m-2) + |\mathrm{S}(i_1) \setminus \mathrm{S}(i_m)|$. The interval (b, v_m) contains the $m-1$ values $v_1, \ldots, v_{m-1}$, and it contains π_{j_2} as a further, non-northwestern value whenever $\pi_{j_2} > v_1$. Hence the right-hand side is at least $2n - 5 + 1$. ☐

3. Defect one

Proof of Theorem 1.3. Let $\pi \in \mathcal{R}_{1,n}$, so $\mathrm{inv}(\pi) = 2n-6$, π is indecomposable and none of $\pi \setminus 1, \pi \setminus n, \pi \setminus \pi_1, \pi \setminus \pi_n$ is decomposable. By Lemma 2.1(1) applied to π and to π^{-1} (note $2n-6 \leq 2n-5$), $a < b$ and $p < q$. By Lemma 2.1(2) exactly one of NW, SE is nonempty; replacing π by π^{-1} if necessary we assume $\mathrm{NW} \neq \varnothing = \mathrm{SE}$ and show that $\pi = 24153[\iota_A, 1, \iota_C, \iota_D, \iota_E]$ with $A, D \geq 1$, $C, E \geq 2$, $(A-1)(C-2) + (D-1)(E-2) = 0$; the inverse of such a permutation is $31524[\iota_C, \iota_A, \iota_E, 1, \iota_D]$, which gives the second family.

Since π is not almost decomposable, Lemma 2.2 gives $\mathrm{S}(i_1) \neq \varnothing$ and $\mathrm{E}(v_m) \neq \varnothing$, and then Lemma 2.5 gives $m = 1$ because $\mathrm{inv}(\pi) = 2n-6 < 2n-4$. Write $i = i_1$, $v = \pi_i$. Apply Lemma 2.3 with $j_1 \in \mathrm{S}(i)$, $j_2 \in \mathrm{E}(v)$: since $\mathrm{inv}(\pi) = 2n-6$, all of $s_1, \ldots, s_5, u$ vanish, for every admissible choice of j_1, j_2. We read off the structure.

$s_2 = 0$*:* $p = i+1$.

$s_5 = 0$*:* $v = b+1$, so that $\mathrm{E}(v) = \mathrm{E}(b)$ is the eastern region E of [7].

$s_1 = 0$*:* every value in $\{1, \ldots, a-1\}$ lies after i, and every position in $\{q+1, \ldots, n\}$ carries a value below v. The former means that the values $2, \ldots, a-1$ are exactly the southern points (positions after q with small values are excluded by $\mathrm{SE} = \varnothing$); in particular $a \geq 3$ since $\mathrm{S}(i) \neq \varnothing$. The latter, with $\mathrm{SE} = \varnothing$, $v = b+1$ and the emptiness of the central region, means that the positions $q+1, \ldots, n-1$ are exactly the eastern points and carry values in (a,b).

Positions before i. No entry before i exceeds v, none is below a (those values sit after i), and b sits at n; so $\pi_2, \dots, \pi_{i-1} \in (a, b)$.

Positions between i and q. Besides 1 at $p = i+1$ and the southern points, the only possible values are those exceeding v (values in (a,b) are excluded by the central-region argument and $(b,v) = \varnothing$). All values exceeding v, except n, lie in (i,q): not before i by the above, not after q by $s_1 = 0$.

$s_3 = 0$ for every southern j_1: the entries between i and j_1 are below π_{j_1}. Hence the southern points are increasing, and every value exceeding v in (i,q) comes after all southern points. *$s_4 = 0$ for every eastern j_2:* the only entries before j_2 exceeding π_{j_2} are the values $\geq v$; hence the eastern points are increasing and every entry of $\pi_2 \cdots \pi_{i-1}$ is smaller than every eastern value.

$u = 0$. Two entries of $\pi_2 \cdots \pi_{i-1}$ in the wrong order, or two values exceeding v in the wrong order inside (i,q) (the later one different from n), would give an inversion counted by u; so both runs are increasing. Consequently

$$\pi = \underbrace{a, a+1, \dots, a+i-2}_{A=i-1} \ \underbrace{b+1}_{1} \ \underbrace{1, 2, \dots, a-1}_{C=a-1} \ \underbrace{b+2, \dots, n}_{D=n-b-1} \ \underbrace{a+i-1, \dots, b}_{E=b-a-i+2}$$

is the inflation $24153[\iota_A, 1, \iota_C, \iota_D, \iota_E]$ with $A \geq 1$, $C \geq 2$, $D \geq 1$ ($v < n$ because n sits after p), $E \geq 2$. Finally, if $C \geq 3$ then some southern point j is not the chosen j_1, and any entry π_l with $2 \leq l \leq i-1$ forms with j an inversion counted by u; $u = 0$ thus forces $A = 1$. Likewise if $E \geq 3$ then an entry exceeding v other than n would form with a non-chosen eastern point an inversion counted by u, so $D = 1$. This is the equation $(A-1)(C-2) + (D-1)(E-2) = 0$.

Conversely let $\pi = 24153[\iota_A, 1, \iota_C, \iota_D, \iota_E]$ with $A, D \geq 1$, $C, E \geq 2$ and $(A-1)(C-2) + (D-1)(E-2) = 0$. It avoids 1324 and is indecomposable by Lemma 2.4. Its inversion number is $(A+1)C + (D+1)E$, and $(A+1)C + (D+1)E = 2(A+C+D+E+1) - 6$ is equivalent to the equation. The four deletions are the inflations of 24153 (or, when a block of size one disappears, of 2413 or 3142) by nonempty blocks, hence indecomposable; so π is not almost decomposable, and $\pi \in \mathcal{R}_{1,n}$.

The count: with $A + C + D + E = n - 1$, the four cases $(A, C) \in \{(1, \cdot), (\cdot, 2)\} \times (D, E) \in \{(1, \cdot), (\cdot, 2)\}$ each have $n - 6$ solutions, the pairwise intersections have one solution each except the pair $\{A = 1, D = 1\} \cap \{C = 2, E = 2\}$ which needs $n = 7$, and inclusion–exclusion gives $4(n-6) - 4 = 4(n-7)$ for $n \geq 8$. The inverse family has the same size and is disjoint (it has SE $\neq \varnothing$). $\square$

Lemma 3.1. *For every n and k, no $\sigma \in \mathrm{im}(f \sqcup g) \subseteq \mathrm{Av}^k_{n+1}(1324)$ satisfies any of*

$$\sigma_1 = n+1, \qquad \sigma_{n+1} = 1, \qquad \sigma_2 = n+1, \qquad \sigma_{n+1} = 2.$$

The number of $\sigma \in \mathrm{Av}^k_{n+1}(1324)$ satisfying at least one of them is

(6)
$$E(n,k) = 2a(n, k-n) + 2a(n, k-n+1) - a(n-1, k-2n+3) - 2a(n-1, k-2n+2) - a(n-1, k-2n+1),$$

where $a(m,j) = 0$ for $j < 0$. In particular $E(n,k) = 2a(n,k-n) + 2a(n,k-n+1)$ for $k \leq 2n-4$.

Proof. Recall the three cases in the definition of f [7, Section 3]: if $\pi \setminus \pi_1$ is decomposable then $f(\pi)_1 = \pi_1$ and $f(\pi) \setminus \pi_1 = g(\pi \setminus \pi_1)$; if $\pi \setminus 1$ is decomposable then $f(\pi) = f(\pi^{-1})^{-1}$; otherwise $f(\pi) = (f(\pi^{\mathrm{rc}}))^{\mathrm{rc}}$. Recall also that $\mathrm{im}(g)$ is exactly the set of permutations with at least three components.

The first two properties. A permutation beginning with its maximum or ending with its minimum is indecomposable, hence not in $\mathrm{im}(g)$. In case 1, $f(\pi)$ begins with $\pi_1 \leq n$ and its remaining entries form a direct sum of at least three components, whose last entry is not its minimum; in cases 2 and 3 the property "does not begin with the maximum and does not end with 1” is invariant under inversion and under reverse–complement, so induction on the recursion applies.

f commutes with inversion. Deleting the entry in position i corresponds to deleting the value i from the inverse, so $(\pi \setminus \pi_1)^{-1} = \pi^{-1} \setminus 1$ and $(\pi \setminus 1)^{-1} = \pi^{-1} \setminus \pi_1^{-1}$. Hence π falls under case 1 exactly when π^{-1} falls under case 2, and then $f(\pi^{-1}) = f(\pi)^{-1}$ by definition; and π falls under case 3 exactly when π^{-1} does, in which case the claim for π reduces, since reverse–complement commutes with inversion, to the claim for π^{rc}, which falls under case 1 or 2 by the remark following the definition of f in [7, Section 3]. Since $\mathrm{im}(g)$ is invariant under inversion, so is $\mathrm{im}(f \sqcup g)$. Inversion exchanges $\sigma_1 = n+1$ with $\sigma_{n+1} = 1$ and $\sigma_2 = n+1$ with $\sigma_{n+1} = 2$, so for the remaining two properties it is enough to treat $\sigma_{n+1} = 2$.

The property $\sigma_{n+1} = 2$. The last component of σ occupies a suffix of the positions and carries a top interval of values; that interval contains 2, so σ has at most two components and $\sigma \notin \mathrm{im}(g)$. For the same reason $\sigma \setminus 1$, which ends with its minimum, and $\sigma \setminus \sigma_1$, which ends with 2 or with its minimum, have at most two components. Suppose $\sigma = f(\pi)$. Case 1 gives $\sigma \setminus \sigma_1 = g(\pi \setminus \pi_1)$ with at least three components, a contradiction. In case 2 the permutation π^{-1} falls under case 1, so $\sigma^{-1} \setminus \sigma_1^{-1}$ has at least three components; but $\sigma_{n+1} = 2$ says $\sigma_2^{-1} = n+1$, so $\sigma^{-1} \setminus \sigma_1^{-1}$ begins with its maximum and is indecomposable, a contradiction. So case 3 applies: $\sigma = \tau^{\mathrm{rc}}$ with $\tau = f(\rho)$ and $\rho = \pi^{\mathrm{rc}}$, and $\sigma_{n+1} = n+2-\tau_1$ forces $\tau_1 = n$. By the remark following the definition of f in [7, Section 3], $\rho \setminus 1$ or $\rho \setminus \rho_1$ is decomposable. If $\rho \setminus \rho_1$ is decomposable then $\tau_1 = \rho_1 = n$, i.e. $\pi_n = 1$; but $\rho \setminus \rho_1$ decomposable means $\pi \setminus \pi_n = \pi \setminus 1$ is decomposable, so π falls under case 2, not case 3. If $\rho \setminus 1$ is decomposable then $\tau = \upsilon^{-1}$ with $\upsilon = f(\rho^{-1})$ given by case 1, and τ_1 is the position of the value 1 in υ. Now $\upsilon \setminus \upsilon_1 = g(\rho^{-1} \setminus \rho_1^{-1})$ is a direct sum $\gamma \oplus 1 \oplus \cdots$ of at least three components and of length n, so $|\gamma| \le n-2$ and the value 1 of $\upsilon \setminus \upsilon_1$ lies in γ, at a position at most $n-2$; hence the value 1 of υ lies at a position at most $n-1$ and $\tau_1 \le n-1$, a contradiction.

The count. None of the four distinguished entries can take part in an occurrence of 1324: an entry placed first or second is too early to be the 4 and is not the smallest, and an entry placed last is too late to be the 1, the 3 or the 2 and is not the largest. Deleting it is therefore a bijection of the four sets onto $\mathrm{Av}_n^{k-n}(1324)$, $\mathrm{Av}_n^{k-n}(1324)$, $\mathrm{Av}_n^{k-n+1}(1324)$ and $\mathrm{Av}_n^{k-n+1}(1324)$, the entry contributing n, n, $n-1$ and $n-1$ inversions respectively. Two of the four conditions can hold at once only for the four pairs consisting of one condition on the left and one on the right, and deleting both entries identifies those with $\mathrm{Av}_{n-1}^{k-2n+1}$, $\mathrm{Av}_{n-1}^{k-2n+2}$, $\mathrm{Av}_{n-1}^{k-2n+2}$ and $\mathrm{Av}_{n-1}^{k-2n+3}$; no three can hold simultaneously. Inclusion–exclusion gives (6). ☐

Lemma 3.2 (boundary form of [7, Lemma 4.2])**.** *Let $\sigma \in \mathrm{Av}_m^k(1324)$ be indecomposable with $k \le 2m-8$.*

(1) *If $\sigma \setminus \sigma_1$ has exactly two components, then $\sigma \setminus m$ or $\sigma \setminus \sigma_m$ has at least four components.*
(2) *At least one of $\sigma \setminus 1$, $\sigma \setminus m$, $\sigma \setminus \sigma_1$, $\sigma \setminus \sigma_m$ has at least three components.*

Proof. (i) Write $\tau = \sigma \setminus \sigma_1 = \sigma^{(1)} \oplus \sigma^{(2)}$, put $s = |\sigma^{(1)}|$ and $\varepsilon = \sigma_1 - s$. The first component of τ occupies the positions $2, \dots, s+1$ of σ and carries the s smallest values of σ other than σ_1; if $\sigma_1 \le s$ these would be $\{1, \dots, s+1\} \setminus \{\sigma_1\}$, so the first $s+1$ positions of σ would carry the values $1, \dots, s+1$ and σ would be decomposable. Hence $\varepsilon \ge 1$. Since σ_1 is inverted with exactly the $\sigma_1 - 1$ values below it and $\sigma^{(1)}$ is indecomposable,

$$\operatorname{inv} \sigma^{(2)} = k - (\sigma_1 - 1) - \operatorname{inv} \sigma^{(1)} \le (2m-8) - (s+\varepsilon-1) - (s-1) = 2|\sigma^{(2)}| - 4 - \varepsilon \le 2|\sigma^{(2)}| - 5.$$

Comparing with $\operatorname{inv} \sigma^{(2)} \ge |\sigma^{(2)}| - 1$ gives $\varepsilon \le |\sigma^{(2)}| - 3$, so $\sigma_1 = s + \varepsilon \le m-4$; in particular $\sigma_1 \ne m$ and the largest value of τ is the value m of σ. By [5, Section 3] and [8, Lemma 2.1], $\sigma^{(1)}$

is an indecomposable 132-avoider and $\sigma^{(2)}$ an indecomposable 213-avoider, so [7, Lemma 4.1] applied to $(\sigma^{(2)})^{\mathrm{rc}}$ (note $213 = \mathrm{rc}(132)$) gives $\sigma^{(2)}_1 = |\sigma^{(2)}|$ or $\sigma^{(2)}_{|\sigma^{(2)}|} = 1$.

Suppose $\sigma^{(2)}_1 = |\sigma^{(2)}|$, that is, $\sigma^{(2)}$ begins with its maximum, which is the value m of σ. Deleting it removes exactly $|\sigma^{(2)}| - 1$ inversions, so $\mathrm{inv}(\sigma^{(2)} \setminus \sigma^{(2)}_1) \le |\sigma^{(2)}| - 3 - \varepsilon$, and since a permutation of length l with c components has at least $l - c$ inversions,

$$\mathrm{comp}\left(\sigma^{(2)} \setminus \sigma^{(2)}_1\right) \ \ge\ (|\sigma^{(2)}| - 1) - (|\sigma^{(2)}| - 3 - \varepsilon) = \varepsilon + 2.$$

Therefore $(\sigma \setminus m) \setminus \sigma_1 = \sigma^{(1)} \oplus (\sigma^{(2)} \setminus \sigma^{(2)}_1)$ has at least $\varepsilon + 3$ components, the first of which is $\sigma^{(1)}$. Restoring σ_1 at the front merges into a single component those components that carry a value smaller than $\sigma_1 = s + \varepsilon$, namely $\sigma^{(1)}$ and at most the $\varepsilon - 1$ components meeting $\{s+1, \dots, s+\varepsilon-1\}$; the count therefore drops by at most $\varepsilon - 1$ and $\mathrm{comp}(\sigma \setminus m) \ge 4$. If instead $\sigma^{(2)}$ ends with its minimum, that entry is the last entry of τ, i.e. σ_m, and the same computation applied to $(\sigma \setminus \sigma_m) \setminus \sigma_1$ gives $\mathrm{comp}(\sigma \setminus \sigma_m) \ge 4$.

(ii) By [7, Theorem 2.5] an indecomposable member of $\mathrm{Av}^k_m(1324)$ with $k \le 2m - 7$ is almost decomposable, so $\mathrm{comp}(\sigma \setminus x) \ge 2$ for some $x \in \{1, m, \sigma_1, \sigma_m\}$. If that deletion has at least three components we are done. Otherwise it has exactly two, and we apply (i): the number of components is invariant under $\sigma \mapsto \sigma^{-1}$ and $\sigma \mapsto \sigma^{\mathrm{rc}}$, both of which preserve $\mathrm{Av}(1324)$, inv and indecomposability while permuting the four deletions transitively ($(\sigma \setminus \sigma_1)^{-1} = \sigma^{-1} \setminus 1$ and $(\sigma \setminus \sigma_1)^{\mathrm{rc}} = \sigma^{\mathrm{rc}} \setminus \sigma^{\mathrm{rc}}_m$), so we may assume the deletion with two components is $\sigma \setminus \sigma_1$. □

Both statements were also verified directly for all $m \le 10$, on the 1026 indecomposable 1324-avoiders with $k \le 2m - 8$ (check C6 of Section 8); the bound four in (i) is attained, in 28 of the 94 instances of the hypothesis of (i).

Proof of Theorem 1.4. For $k \le 2n - 7$ this is [7]. Let $k = 2n - 6$. For $n \le 7$ the inequality is a finite check (Section 8). For $n \ge 8$, by (4), Lemma 3.1 and Theorem 1.3,

$$a(n+1, k) - a(n, k) \ \ge\ 2a(n, n-6) - 8(n-7) = 2p_2(n-6) - 8(n-7),$$

using $a(n, n-6) = p_2(n-6)$ [5, Proposition 15]. The right-hand side is $2, 4, 16, 40, \dots$ for $n = 8, 9, 10, 11$ and increases with n, since $p_2(m+1) - p_2(m) \ge 4$ for $m \ge 2$.

For the exact value we describe the complement of the image completely. Linusson and Verkama [7, Section 4] show that this complement is the disjoint union of: R1, the permutations beginning with $n+1$ or ending with 1; R2a, those with $\sigma_2 = n+1$ or $\sigma_{n+1} = 2$ (not in R1); R2b, those with at most two components, not in R1 ∪ R2a, for which some deletion among $\sigma \setminus 1, \sigma \setminus (n+1), \sigma \setminus \sigma_1, \sigma \setminus \sigma_{n+1}$ has at least three components but which are not of the form $f(\pi)$; and R3, those with at most two components none of whose four deletions has three components. Their arguments give $|\mathrm{R1}| = 2p_2(k-n)$ and $|\mathrm{R2a}| = |\mathrm{R2b}| = 2p_2(k-n+1)$, and these counts remain valid at $k = 2n-6$. Indeed each of the three is obtained by deleting a distinguished entry: R1 and R2a are put in bijection with $\mathrm{Av}^{k-n}_n(1324)$ and $\mathrm{Av}^{k-n+1}_n(1324)$, and R2b with $\mathrm{Av}^{k-n+1}_{n-2}(1324)$ by $\pi \mapsto \pi \setminus \{\pi_1, \pi_n\}$ [7, Lemma 4.4]; the hypothesis $k \le 2n-7$ enters these three counts only through the plateau identity $a(m, j) = p_2(j)$ for $m \ge j+2$ [5, Proposition 15], applied with $(m, j) = (n, k-n)$, $(n, k-n+1)$ and $(n-2, k-n+1)$, which require $k \le 2n-2$, $k \le 2n-3$ and $k \le 2n-5$ respectively — the last of these is exactly the inequality $n - 2 - (k - n + 1) \ge 2$ used in the proof of [7, Lemma 4.4]. All three counts, and the value $|\mathrm{R3}| = 4(n-7)$ obtained below, were also verified directly for $8 \le n \le 11$ (Section 8), and [7, Proposition 4.3] shows $\mathrm{R3} = \varnothing$ for $k \le 2n-7$. At $k = 2n-6$ we determine R3. Let $\sigma \in \mathrm{R3}$, of length $m = n+1$ and with $2m - 8$ inversions. If σ were indecomposable then, by Lemma 3.2(ii), one of its four deletions would have at least three components, contradicting the definition of R3. So $\sigma = \sigma^{(1)} \oplus \sigma^{(2)}$ with $\sigma^{(1)}$ an indecomposable 132-avoider and $\sigma^{(2)}$ an indecomposable 213-avoider ([5, Section 3], [8, Lemma 2.1]). The conditions on the deletions say that $\sigma^{(1)} \setminus 1$, $\sigma^{(1)} \setminus \sigma^{(1)}_1$ and the corresponding deletions of $\sigma^{(2)}$ are indecomposable; by [7, Lemma 4.1] and its reverse–complement this forces $\operatorname{inv}\sigma^{(1)} \geq 2|\sigma^{(1)}|-4$ and $\operatorname{inv}\sigma^{(2)} \geq 2|\sigma^{(2)}|-4$, so both are equalities. An indecomposable 132-avoider has the form $(\alpha \oplus 1) \ominus \beta$ with $\beta \neq \varnothing$, $\alpha, \beta \in \mathrm{Av}(132)$, and $\operatorname{inv} = \operatorname{inv}\alpha + \operatorname{inv}\beta + (|\alpha|+1)|\beta|$; equality with $2(|\alpha|+|\beta|+1)-4$ reads $\operatorname{inv}\alpha + \operatorname{inv}\beta + (|\alpha|-1)(|\beta|-2) = 0$. If $|\beta| = 1$ the permutation ends with 1, if $|\alpha| = 0$ it begins with its maximum and β has $|\beta|-2$ inversions, hence is decomposable; both are excluded by the deletion conditions. Otherwise α, β are increasing and $(|\alpha|-1)(|\beta|-2) = 0$, i.e. $\sigma^{(1)} = \iota_x \ominus \iota_y$ with $x, y \geq 2$, $(x-2)(y-2) = 0$; symmetrically $\sigma^{(2)} = \iota_z \ominus \iota_w$ with $z, w \geq 2$, $(z-2)(w-2) = 0$. Conversely let $\sigma = (\iota_x \ominus \iota_y) \oplus (\iota_z \ominus \iota_w)$ with $x, y, z, w \geq 2$, $(x-2)(y-2) = (z-2)(w-2) = 0$ and $x+y+z+w = n+1$. Such a σ has $2n-6$ inversions, avoids 1324, and has exactly two components, as do all four of its deletions $\sigma \setminus 1$, $\sigma \setminus (n+1)$, $\sigma \setminus \sigma_1$, $\sigma \setminus \sigma_{n+1}$. It remains to check that σ lies in the complement of $\operatorname{im}(f \sqcup g)$. Every member of $\operatorname{im}(g)$ has at least three components, so $\sigma \notin \operatorname{im}(g)$. Suppose $\sigma = f(\pi)$. In case 1 of the definition of f, $\sigma \setminus \sigma_1 = g(\pi \setminus \pi_1)$ has at least three components, a contradiction. In case 2 the permutation π^{-1} falls under case 1 and $f(\pi^{-1}) = f(\pi)^{-1}$, so $\sigma^{-1} \setminus \sigma_1^{-1} = (\sigma \setminus 1)^{-1}$ has at least three components; but the number of components is invariant under inversion, so $\sigma \setminus 1$ has at least three components, again a contradiction. In case 3 we have $\sigma = \tau^{\mathrm{rc}}$ with $\tau = f(\rho)$ falling under case 1 or case 2, so $\tau \setminus \tau_1$ or $\tau \setminus 1$ has at least three components; reverse–complement preserves the number of components and carries these to $\sigma \setminus \sigma_{n+1}$ and $\sigma \setminus (n+1)$, a contradiction. Hence $\sigma \in \mathrm{R3}$. Counting as in Theorem 1.3, $|\mathrm{R3}| = 4(n-7)$ for $n \geq 8$. Therefore

$$a(n+1, 2n-6) - a(n, 2n-6) = 2p_2(n-6) + 4p_2(n-5) + 4(n-7) - 8(n-7),$$

as claimed. ☐

Remark 3.3. Theorem 1.3 identifies the residuals at defect one with the two skeletons 24153 and $31524 = 24153^{-1}$; the permutation $3612745 = 24153[1, 1, \iota_2, 1, \iota_2]$ used in [7] to show that $2n-7$ is sharp is the smallest member. The set R3 consists of the permutations $(\iota_x \ominus \iota_y) \oplus (\iota_z \ominus \iota_w)$; there is an obvious bijection $\mathcal{R}^{\mathrm{NW}}_{1,n} \to \mathrm{R3}$, $24153[\iota_A, 1, \iota_C, \iota_D, \iota_E] \mapsto (\iota_{A+1} \ominus \iota_C) \oplus (\iota_{D+1} \ominus \iota_E)$, which explains the coefficient $4(n-7)$; the other half of the residuals must be absorbed by $\mathrm{R1} \cup \mathrm{R2a} \cup \mathrm{R2b}$.

4. Defect two

Throughout this section $\pi \in \mathcal{R}_{2,n}$, so $\operatorname{inv}(\pi) = 2n-5$, π is indecomposable and none of the four extremal deletions is decomposable. By Lemma 2.1(1) (applied to π and π^{-1}) we have $a < b$ and $p < q$, and by Lemma 2.1(2) $\mathrm{NW} \cup \mathrm{SE} \neq \varnothing$; both may now be nonempty. We first record the structural facts that hold for a single northwestern point without any assumption on the slack, then the identity governing the case $\mathrm{NW} \neq \varnothing \neq \mathrm{SE}$, and then the classification.

Lemma 4.1. *Suppose* $\mathrm{SE} = \varnothing$, $\mathrm{NW} = \{i\}$, $v = \pi_i$, $\mathrm{S} = \mathrm{S}(i) \neq \varnothing$ *and* $\mathrm{E}(v) \neq \varnothing$*; let* j_1 *be the first southern point and* j_2 *the first point of* $\mathrm{E}(v)$*. Then:*

(a) *the positions* $2, \ldots, i-1$ *carry values in* $(1, a) \cup (a, b)$*;*
(b) *the positions in* (i, q) *other than* p *carry values in* $(1, a) \cup (v, n)$*; the positions in* (q, n) *carry values exceeding* a*; the values in* (b, v) *lie after* q*;*
(c) *the southern points after* p *are increasing; the* top entries *(positions in* (p, q) *with values exceeding* v*) are increasing and follow all southern points; the prefix values in* (a, b) *are increasing, the values of* $\mathrm{E}(b)$ *are increasing, and every prefix value in* (a, b) *is below every value of* $\mathrm{E}(b)$*;*

(d) $s_3 = s_4 = 0$ *in Lemma 2.3, and*

$$s_1 = \#\{\text{values below } a \text{ before } i\} + \#\{\text{positions in } (q,n) \text{ with values exceeding } v\},$$
$$s_2 = p - i - 1, \qquad s_5 = \#\{\text{values in } (b,v)\}.$$

Proof. (a) No entry before i exceeds v; a value in (b,v) before i would be a second northwestern point; 1, a and b sit at $p > i$, 1 and n. (b) A value in (a,v) at a position in (i,q) forms the 1324 $\pi_1, v, \cdot, n$; positions after q carry values above a because $\mathrm{SE} = \varnothing$; so a value in (b,v) is after q by (a). (c) Two southern points after p in the wrong order form a 1324 with 1 and n; two top entries in the wrong order form one with π_1 and n; a top entry before a southern point s forms $1, t, s, n$; two prefix values in (a,b) in the wrong order, or two values of $\mathrm{E}(b)$ in the wrong order, or a prefix value above a value of $\mathrm{E}(b)$, form a 1324 with π_1 and π_n. (d) The entries strictly between i and j_1 are not southern (j_1 is the first), not in (a,v) by (b), not above v (a position before p with such a value would be northwestern, and a top entry before j_1 is excluded by (c)); hence $L(j_1) = i$. The entries before j_2 with values in (π_{j_2}, v): a prefix value in (π_{j_2}, v) lies in (a,b) by (a), so $\pi_{j_2} < b$ and $\pi_1, \cdot, \pi_{j_2}, \pi_n$ is a 1324; the positions in (i,q) carry no value in (a,v); the positions in (q, j_2) carry no value in (a,v) because j_2 is the first such point. Hence $L(j_2) = n - v + 1$. The formulas for s_1, s_2, s_5 restate the definitions: the right-inversions of i are the values below a that lie after i together with the positions after q with values below v. $\square$

Lemma 4.2 (northwestern and southeastern points)**.** *Suppose* $\mathrm{NW} \neq \varnothing \neq \mathrm{SE}$*. If* $|\mathrm{NW}| \geq 2$ *or* $|\mathrm{SE}| \geq 2$ *then* $\mathrm{inv}(\pi) \geq 2n - 4$*. If* $\mathrm{NW} = \{i\}$ *and* $\mathrm{SE} = \{j\}$*, with* $v = \pi_i$ *and* $w = \pi_j$*, then*

$$\mathrm{inv}(\pi) = 2n - 5 + (p - i - 1) + (v - b - 1) + (a - w - 1) + (j - q - 1) + u,$$

where $u \geq 0$ *is the number of inversions* (x,y) *with* $x \notin \{1, i, q\}$ *and* $y \notin \{p, j, n\}$*.*

Proof. Let $i = i_m$ be the last northwestern point and j the first southeastern point. No entry after j is below w: such an entry y would give the 1324 $1, w, y, \pi_n$ (positions $p < j < \cdot < n$). Hence $L(j) = j - w$, and as before $R(i) = v - i$, $R(1) = a - 1$, $R(q) = n - q$, $L(p) = p - 1$, $L(n) = n - b$. The inversions counted twice in $R(1) + R(i) + R(q) + L(p) + L(j) + L(n)$ are $(1,p), (1,j), (i,p), (i,j), (i,n), (q,j), (q,n)$, and the inversions counted by neither type are those of u; so $\mathrm{inv}(\pi) = R(1) + R(i) + R(q) + L(p) + L(j) + L(n) - 7 + u$, which is the displayed identity. If $|\mathrm{NW}| \geq 2$ then $v - b - 1 \geq 1$ because the other northwestern values lie in (b,v); if $|\mathrm{SE}| \geq 2$ then $a - w - 1 \geq 1$ symmetrically. In both cases the right-hand side is at least $2n - 4$. $\square$

Theorem 4.3. *For every* n*,* $\mathcal{R}_{2,n}$ *is the union of the twenty-one families of inflations by increasing blocks listed in Table 1 (twelve families and the inverses of the nine that are not self-inverse). Consequently* $|\mathcal{R}_{2,n}| = 32n - 214$ *for all* $n \geq 10$*.*

Proof. Membership. Each skeleton in Table 1 avoids 1324 and is indecomposable, so its inflations avoid 1324 and are indecomposable (Lemma 2.4). Writing inv of an inflation as $\sum b_x b_y$ over the inversions (x,y) of the skeleton, one checks that in each row the equation $\mathrm{inv} = 2n - 5$ is equivalent to the stated constraint (for instance $\mathrm{inv}(351624[\iota_A, 1, \iota_C, \iota_D, 1, \iota_F]) = AC + A + C + 1 + F + D + DF$ and $n = A + C + D + F + 2$, which gives $(A-1)(C-1) + (D-1)(F-1) = 0$). Each of the four extremal deletions of a member removes one entry from a block; when that block has size at least two the result is an inflation of the same skeleton, and when it has size one the result is an inflation of the skeleton with that point deleted; that no member is almost decomposable follows from Theorem 5.3, which is proved in Section 5 independently of this section: an inflation $\sigma[b]$ of a reduced, indecomposable, 1324-avoiding σ with $\mathrm{inv} = 2n - 5$ lies in $\mathcal{R}_{2,n}$ precisely when $b_i \geq 2$ for every $i \in D(\sigma)$, and the sets $D(\sigma)$ of the twenty-one skeletons, printed by check C1 of the certificate (Section 8), are exactly the blocks that the constraint of each row requires to be nonsingletons. Hence every listed permutation lies in $\mathcal{R}_{2,n}$.

Table 1. The residuals at defect two, up to inversion. Block sizes are ≥ 1 unless a larger lower bound is indicated; the count is the number of members of the family for $n \geq 10$. Families marked $*$ (Case II, NW $\neq \varnothing \neq$ SE) are self-inverse; each of the others contributes twice (the family and its inverse). In the cases $s_1 = 1$, (a) means that the value $a-1$ is the second entry and (b) that the value $n-1$ follows n.

case	family	constraints	count
$u = 1$	$24153[\iota_A, 1, \iota_C, \iota_D, \iota_E]$	$C, E \geq 2$, $(A-1)(C-2) + (D-1)(E-2) = 1$	4
$s_5 = 1$, $\mathrm{E}(b) = \varnothing$	$251643[\iota_A, 1, \iota_C, \iota_D, 1, 1]$	$C \geq 2$, $(A-1)(C-2) = 0$	$2n - 13$
$s_5 = 1$, $\mathrm{E}(b) \neq \varnothing$	$2617354[\iota_A, 1, \iota_C, 1, \iota_E, 1, 1]$	$C \geq 2$, $(A-1)(C-2) = 0$	$2n - 15$
$s_2 = 1$, $\lvert\mathrm{S}\rvert = 1$	$352164[\iota_A, 1, 1, 1, \iota_D, \iota_E]$	$E \geq 2$, $(D-1)(E-2) = 0$	$2n - 13$
$s_2 = 1$, $\lvert\mathrm{S}\rvert \geq 2$	$4621375[1, 1, 1, 1, \iota_C, \iota_D, \iota_E]$	$E \geq 2$, $(D-1)(E-2) = 0$	$2n - 15$
$s_1 = 1$ (b), top entries	$2415763[\iota_A, 1, \iota_C, \iota_D, 1, 1, \iota_2]$	$C \geq 2$, $(A-1)(C-2) = 0$	$2n - 17$
$s_1 = 1$ (b), no top entries	$241653[\iota_A, 1, \iota_C, 1, 1, \iota_2]$	$C \geq 2$, $(A-1)(C-2) = 0$	2
$s_1 = 1$ (a), prefix	$3246175[1, 1, \iota_B, 1, \iota_2, \iota_D, \iota_E]$	$E \geq 2$, $(D-1)(E-2) = 0$	$2n - 17$
$s_1 = 1$ (a), no prefix	$325164[1, 1, 1, \iota_2, \iota_D, \iota_E]$	$E \geq 2$, $(D-1)(E-2) = 0$	2
II $*$	$351624[\iota_A, 1, \iota_C, \iota_D, 1, \iota_F]$	$(A-1)(C-1) = (D-1)(F-1) = 0$	$4n - 24$
II $*$	$3517264[\iota_A, 1, \iota_C, 1, 1, \iota_E, 1]$	$(A-1)(C-1) = 0$	$2n - 13$
II $*$	$4261735[1, \iota_B, 1, 1, \iota_D, 1, \iota_F]$	$(D-1)(F-1) = 0$	$2n - 13$

Check C1 enumerates the members by that criterion and verifies, without assuming it, that each of them avoids 1324, is indecomposable and is not almost decomposable, recording in passing the 43 skeletons that the four extremal deletions produce, all of them indecomposable. The counts follow by inclusion–exclusion exactly as in Theorem 1.3; for instance in the first row the constraint forces $\{A, C\} = \{2, 3\}$ with $(D-1)(E-2) = 0$ or symmetrically, which has four solutions once $n \geq 10$, and in the tenth row the four cases $A = 1$ or $C = 1$ combined with $D = 1$ or $F = 1$ have $n - 5$ solutions each with four single overlaps. Summing the last column (with the factor two for the nine families that are not self-inverse) gives $32n - 214$.

Completeness. Let $\pi \in \mathcal{R}_{2,n}$. We distinguish whether one or both of NW, SE are nonempty.

Case I: exactly one of NW, SE *is nonempty.* Replacing π by π^{-1} we may assume NW $\neq \varnothing =$ SE. By Lemma 2.2, $\mathrm{S}(i_1) \neq \varnothing$ and $\mathrm{E}(v_m) \neq \varnothing$; by Lemma 2.5, NW $= \{i\}$. Lemma 2.3 with the canonical j_1, j_2 of Lemma 4.1 gives $s_1 + s_2 + s_5 + u = 1$, since $s_3 = s_4 = 0$. Whenever the three quantities other than the one equal to 1 vanish, the arguments of Theorem 1.3 that only use these vanishings apply verbatim; we only describe what the unit of slack changes. Recall from Theorem 1.3 that, when all of s_1, s_2, s_5 vanish, the permutation has the shape

$$\pi = \;[\,a,\ \text{prefix}\,]\,[\,v\,]\,[\,1\,]\,[\,\mathrm{S}\,]\,[\,\text{top}, n\,]\,[\,\mathrm{E}(b)\,]\,[\,b\,], \qquad v = b+1,$$

with the prefix carrying the values $a+1, \dots$ (increasing), S the values $2, \dots, a-1$ (increasing), the top entries the values $b+2, \dots, n$ (increasing), $\mathrm{E}(b)$ the remaining values of (a, b) (increasing), and that then u equals $(A-1)(C-2) + (D-1)(E-2)$, the number of pairs (prefix entry, non-first southern point) and (top entry other than n, non-first point of $\mathrm{E}(b)$), all of which are inversions counted by u, no other inversion being counted by u (Lemma 4.1(c)).

$u = 1$. The shape is as displayed with $(A-1)(C-2) + (D-1)(E-2) = 1$: the first row of the table.

$s_5 = 1$. Then $v = b+2$ and the single value $b+1$ lies after q (Lemma 4.1(a),(b)); since $s_1 = 0$, every position after q carries a value below v, so the positions $q+1, \dots, n-1$ carry the points of $\mathrm{E}(b)$ and the entry $b+1$, and $\mathrm{E}(v) = \mathrm{E}(b) \cup \{b+1\}$. If $\mathrm{E}(b) \neq \varnothing$ then $b+1$ is not the first point of $\mathrm{E}(v)$ (otherwise it would form with every point of $\mathrm{E}(b)$ an inversion counted by u), hence it is the last one by Lemma 4.1(c). With $s_2 = 0$ the rest of the shape is as displayed; now u counts, besides $(A-1)(C-2)$, the pairs (top entry other than n, point of $\mathrm{E}(b)$ other than the first) and, when $\mathrm{E}(b) \neq \varnothing$, the pairs (top entry other than n, entry $b+1$): $u = (A-1)(C-2)+(D-1)|\mathrm{E}(b)|$. Thus $(A-1)(C-2) = 0$, and $D = 1$ when $\mathrm{E}(b) \neq \varnothing$: rows two and three.

$s_2 = 1$. Exactly one position g lies strictly between i and p; its value is below a (a value above v would be northwestern, a value in (a, v) is excluded by Lemma 4.1(b), and a, b are elsewhere), so $g = j_1$ is the first southern point and, by Lemma 4.1(c) and $u = 0$ (an inversion between g and a later southern point would be counted by u), $\pi_g = 2$. If there are further southern points, every prefix entry would form with one of them an inversion counted by u; so either there are none (row four, A free) or $A = 1$ (row five). The top entries and $\mathrm{E}(b)$ are as in the displayed shape with $(D-1)(E-2) = 0$.

$s_1 = 1$. Either (a) exactly one value $x < a$ lies before i, or (b) exactly one position after q carries a value exceeding v. In case (a), every southern point s has $\pi_s < x$ (otherwise x, v, π_s, n is a 1324), so $x = a-1$; a prefix entry above a before x would form with x an inversion counted by u, so $x = \pi_2$; and since x exceeds every southern value, $u = 0$ forces $|\mathrm{S}| = 1$ (rows eight and nine, according to whether further prefix entries exist; the equation then gives $(D-1)(E-2) = 0$). In case (b), let W be the exceptional value after q. Every point e of $\mathrm{E}(b)$ lies after W (otherwise π_1, v, e, W is a 1324), hence W sits at position $q+1$; $u = 0$ forces $|\mathrm{E}(b)| = 1$ (each further point of $\mathrm{E}(b)$ would form with W an inversion counted by u) and W above every top entry other than n, i.e. $W = n-1$. This gives rows six and seven.

Case II: NW $\neq \varnothing \neq$ SE. By Lemma 4.2, NW $= \{i\}$, SE $= \{j\}$, $p = i+1$, $v = b+1$, $w = a-1$, $j = q+1$ and $u = 0$. Split the remaining entries into the prefix P (positions $2, \ldots, i-1$), the middle M (positions $p+1, \ldots, q-1$) and the tail Z (positions $j+1, \ldots, n-1$). Values: $P \subseteq (1, a) \cup (a, b)$ as in Lemma 4.1(a); $M \subseteq (1, a) \cup (v, n)$ (a value in (a, v) gives $\pi_1, v, \cdot, n$); $Z \subseteq (a, b) \cup (v, n)$ (no entry after j is below w). Write $P_S, P_{ab}, M_S, M_T, Z_{ab}, Z_T$ for the corresponding parts. Each part is increasing, M_S precedes M_T, Z_{ab} precedes Z_T, P_S precedes P_{ab} and $P_{ab} < Z_{ab}$, $P_S < M_S$, $M_T < Z_T$ as sets of values, all by $u = 0$ or by a 1324 with π_1, π_n or $1, n$. Moreover the following pairs cannot both be nonempty: Z_{ab}, Z_T (π_1, v, z_{ab}, z_T); M_T, Z_T $(1, m_T, w, z_T)$; P_S, M_S (p_S, v, m_S, n); P_S, Z_T (p_S, v, w, z_T); P_S, P_{ab} (p_S, p_{ab}, w, π_n); P_{ab}, M_S and M_T, Z_{ab} (inversions counted by u). Consequently π is

$$[\,a, P_{ab}\,][\,v\,][\,1, M_S\,][\,n\,][\,w\,][\,Z_T\,][\,b\,] \qquad \text{or} \qquad [\,a\,][\,P_S\,][\,v\,][\,1\,][\,M_T, n\,][\,w\,][\,Z_{ab}, b\,]$$
$$\text{or} \qquad [\,a, P_{ab}\,][\,v\,][\,1, M_S\,][\,M_T, n\,][\,w\,][\,Z_{ab}, b\,],$$

with the incompatibilities $P_{ab} = \varnothing$ or $M_S = \varnothing$, and $M_T = \varnothing$ or $Z_{ab} = \varnothing$: the last three rows of the table, where the constraints are exactly these incompatibilities. This completes the proof. □

Theorem 4.4. *$a(n, k) \leq a(n+1, k)$ for all $n \geq 1$ and all $k \leq 2n-5$. For $n \geq 10$,*

$$a(n+1, 2n-5) - a(n, 2n-5) \;\geq\; 2p_2(n-5) + 2p_2(n-4) - (32n-214) \;> 0.$$

Proof. For $k \leq 2n-6$ this is Theorem 1.4; let $k = 2n-5$. By Lemma 3.1 the complement of $\mathrm{im}(f \sqcup g)$ has at least $E(n, k) = 2p_2(n-5) + 2p_2(n-4)$ elements, the correction terms in (6) vanishing because $k \leq 2n-4$. Hence by (4) and Theorem 4.3, $a(n+1, k) - a(n, k) \geq 2p_2(n-5) + 2p_2(n-4) - (32n-214)$ for $n \geq 10$; the right-hand side equals 96 at $n = 10$ and increases with n. For $n \leq 9$ the inequality is read off from the table of Section 8. □

5. The skeleton reduction

The classifications of Sections 3 and 4 both produced inflations of a fixed finite list of patterns by increasing blocks, subject to one quadratic equation. This is not an accident of small defect.

In this section we show that the passage to the underlying pattern is an exact reduction: it turns the description of $\mathcal{R}_{\delta,n}$, for every δ and every n at once, into a quadratic Diophantine problem over a set of patterns that depends on δ alone.

Write $\iota_m = 12\cdots m$ and, for a permutation τ of length t and $b \in \mathbb{Z}^t_{\geq 1}$, let $\tau[b] = \tau[\iota_{b_1}, \ldots, \iota_{b_t}]$ be the inflation of τ in which the ith entry is replaced by an increasing run of b_i entries. Call σ *reduced* if $\sigma_{i+1} \neq \sigma_i + 1$ for all $1 \leq i < |\sigma|$. Contracting the maximal runs of consecutive positions carrying consecutive values writes every permutation π uniquely as $\pi = \sigma[b]$ with σ reduced; we call $\sigma = \mathrm{sk}(\pi)$ the *skeleton* of π and b its *block vector*. This is the special case of the substitution decomposition [1] in which the intervals allowed are the maximal increasing runs, and the skeleton is the quotient of π by that interval system; what makes it useful here is Theorem 5.3, which says that membership in $\mathcal{R}_{\delta,n}$ is visible on the quotient together with the block sizes. Let $\mathrm{Inv}(\tau) = \{(i,j) : i < j,\ \tau_i > \tau_j\}$ and let $d_i = d_i(\tau) = |\{j : (i,j) \in \mathrm{Inv}(\tau) \text{ or } (j,i) \in \mathrm{Inv}(\tau)\}|$ be the degree of i in the inversion graph of τ.

Lemma 5.1. *Let τ have length $t \geq 1$ and let $b \in \mathbb{Z}^t_{\geq 1}$. Then*

(1) *$\tau[b]$ contains 1324 if and only if τ does;*
(2) *$\tau[b]$ is decomposable if and only if τ is decomposable, or $t = 1$ and $b_1 \geq 2$;*
(3) $\mathrm{inv}(\tau[b]) = \sum_{(i,j)\in \mathrm{Inv}(\tau)} b_i b_j$.

Proof. (3) is immediate: two entries of $\tau[b]$ form an inversion exactly when they lie in blocks $i < j$ with $(i,j) \in \mathrm{Inv}(\tau)$, since each block is increasing.

(1) Since $\tau \preceq \tau[b]$, one direction is clear. Conversely, each block B_i of $\tau[b]$ is an interval of positions carrying an interval of values, in increasing order. We claim that no two entries of an occurrence of 1324 lie in one block; the occurrence then projects to distinct blocks and yields an occurrence of 1324 in τ. Label the entries of the occurrence $e_1e_2e_3e_4$, of values $v_1 < v_3 < v_2 < v_4$. Two entries in a common block must increase in both coordinates, which leaves the pairs (e_1, e_2), (e_1, e_3), (e_1, e_4), (e_2, e_4), (e_3, e_4). If $e_1, e_2 \in B_i$ then $v_3 \in [v_1, v_2]$ forces $e_3 \in B_i$, and B_i increasing contradicts the fact that e_2 precedes e_3 while $v_2 > v_3$. If $e_1, e_3 \in B_i$ then B_i contains all positions in between, in particular e_2, whose value $v_2 \notin [v_1, v_3]$. The pair (e_1, e_4) puts all four entries in B_i, contradicting that B_i is increasing; (e_2, e_4) contains e_3 with $v_3 \notin [v_2, v_4]$; and (e_3, e_4) forces $v_2 \in [v_3, v_4]$, hence $e_2 \in B_i$, but e_2 precedes e_3.

(2) A splitting point of $\tau[b]$ falling at a block boundary, after block i, occurs exactly when $\{\tau_1, \ldots, \tau_i\} = \{1, \ldots, i\}$, i.e. when τ is decomposable. Suppose a splitting point falls inside block i, after its rth entry with $1 \leq r < b_i$. The values of B_i form the interval $[v, v + b_i - 1]$ with $v = 1 + \sum_{j:\tau_j<\tau_i} b_j$, and its first r values are $[v, v + r - 1]$; for the values in the first $\sum_{j<i} b_j + r$ positions to be an initial segment we need the blocks $1, \ldots, i - 1$ to carry exactly the values $1, \ldots, v - 1$. For $i \geq 2$ this says $\{\tau_1, \ldots, \tau_{i-1}\} = \{1, \ldots, i - 1\}$, so τ is decomposable; for $i = 1$ it says $v = 1$, i.e. $\tau_1 = 1$, which makes τ decomposable when $t \geq 2$ and is automatic when $t = 1$. In the last case $\tau[b] = \iota_{b_1}$, decomposable iff $b_1 \geq 2$. □

In particular, if σ is reduced and indecomposable with $|\sigma| \geq 2$, then $\sigma[b]$ is an indecomposable 1324-avoider for every b if and only if σ is one. The four deletions that define almost decomposability are also controlled by the skeleton.

Lemma 5.2. *Let σ be reduced and indecomposable of length $s \geq 2$, let $b \in \mathbb{Z}^s_{\geq 1}$, $\pi = \sigma[b]$ and $n = \sum_i b_i$. Put $I(\sigma) = \{1,\ s,\ \sigma^{-1}(1),\ \sigma^{-1}(s)\}$. Each of the entries 1, n, π_1, π_n of π is the first or the last entry of the block B_i for some $i \in I(\sigma)$, and for that i*

$$\pi \setminus v = \begin{cases} \sigma[b - e_i], & b_i \geq 2, \\ (\sigma \setminus i)\big[b^{(i)}\big], & b_i = 1, \end{cases}$$

where $b^{(i)}$ is b with its ith coordinate deleted.

Proof. The value 1 is the first entry of $B_{\sigma^{-1}(1)}$, the value n the last entry of $B_{\sigma^{-1}(s)}$, π_1 the first entry of B_1 and π_n the last entry of B_s. Deleting the first or last entry of an increasing block of size $b_i \geq 2$ leaves an increasing block of size $b_i - 1 \geq 1$ in the same position and value interval, which is the first display; if $b_i = 1$ the block disappears and what remains is the inflation of $\sigma \setminus i$ by the surviving blocks. □

Theorem 5.3. *Let σ be reduced, indecomposable and 1324-avoiding, of length $s \geq 2$, and put*

$$D(\sigma) = \{\, i \in I(\sigma) :\ \sigma \setminus i \text{ is decomposable, or } s = 2 \,\}.$$

Then for every $b \in \mathbb{Z}^s_{\geq 1}$ with $\sum_i b_i = n \geq 3$ and every δ,

$$\sigma[b] \in \mathcal{R}_{\delta,n} \iff \sum_{(i,j)\in \operatorname{Inv}(\sigma)} b_i b_j = 2n - 7 + \delta \quad \text{and} \quad b_i \geq 2 \text{ for every } i \in D(\sigma).$$

Consequently $\mathcal{R}_{\delta,n}$ is the disjoint union, over reduced σ, of the sets so described.

Proof. By Lemma 5.1, $\sigma[b]$ is an indecomposable 1324-avoider, and its inversion number is the left-hand side of the displayed equation; so it remains to identify when $\sigma[b]$ is almost decomposable. Let $v \in \{1, n, \pi_1, \pi_n\}$ and let $i \in I(\sigma)$ be the index supplied by Lemma 5.2. If $b_i \geq 2$ then $\pi \setminus v = \sigma[b - e_i]$, which is indecomposable by Lemma 5.1(2) because σ is indecomposable of length ≥ 2. If $b_i = 1$ then $\pi \setminus v = (\sigma \setminus i)[b^{(i)}]$, which by Lemma 5.1(2) is decomposable exactly when $\sigma \setminus i$ is decomposable, or when $\sigma \setminus i$ is a single point and the remaining block has size ≥ 2 — and the latter happens precisely when $s = 2$ and $n \geq 3$. Thus π is almost decomposable if and only if $b_i = 1$ for some $i \in D(\sigma)$. The last assertion is the uniqueness of the skeleton. □

Writing $b = \mathbf{1} + c$ and $\delta(\tau) = \operatorname{inv}(\tau) - 2|\tau| + 7$ for the defect of an arbitrary permutation τ, expanding $b_i b_j = 1 + c_i + c_j + c_i c_j$ in Lemma 5.1(3) and using $n = s + \sum_i c_i$ gives the identity that governs the whole defect ladder.

Proposition 5.4. *For every reduced σ of length s and every $c \in \mathbb{Z}^s_{\geq 0}$,*

$$\delta\big(\sigma[\mathbf{1} + c]\big) = \delta(\sigma) + \sum_{i=1}^{s} (d_i - 2)\, c_i + \sum_{(i,j)\in \operatorname{Inv}(\sigma)} c_i c_j. \tag{7}$$

In particular, if x is an entry of a permutation π and $D(x)$ is the number of entries of π that form an inversion with x, then $\delta(\pi \setminus x) = \delta(\pi) + 2 - D(x)$.

Identity (7) explains the shape of Theorems 1.3 and 4.3. All the terms it adds to $\delta(\sigma)$ are nonnegative except $-c_i$ at the vertices of inversion degree 1, and the quadratic part vanishes exactly on the increasing subsequences of σ. Enlarging a block all of whose entries have inversion degree 2 therefore leaves the defect unchanged: these are the free directions along which a residual family grows with n, and the number of independent free blocks, minus one, is the degree of $|\mathcal{R}_{\delta,n}|$ as a polynomial in n. For $\sigma = 24153$ one has $\delta(\sigma) = 1$, $d = (1, 2, 2, 1, 2)$ and $D(\sigma) = \{3, 5\}$, so Theorem 5.3 and (7) return exactly the family of Theorem 1.3: $c_3, c_5 \geq 1$ and $c_1(c_3 - 1) + c_4(c_5 - 1) + c_2(c_3 + c_5) = \delta - 1$, which for $\delta = 1$ forces $c_2 = 0$ and $(a-1)(c-2) + (d-1)(e-2) = 0$. The negative coefficients can be removed altogether by expanding (7) around a larger base point instead of $\mathbf{1}$; this nonnegative form of the identity is Proposition 6.3, and Section 6 is built on it.

Let

$$\Sigma_\delta = \{\, \sigma \text{ reduced} :\ \sigma[b] \in \mathcal{R}_{\delta,n} \text{ for some } b \text{ and some } n \,\}$$

be the set of skeletons occurring at defect δ; by Theorem 5.3 and (7), $\sigma \in \Sigma_\delta$ if and only if the quadratic equation (7) has a solution $c \in \mathbb{Z}^s_{\geq 0}$ with value $\delta - \delta(\sigma)$ and $c_i \geq 1$ on $D(\sigma)$. The following is a finite computation for each δ, carried out as described in Section 8.

For $N \geq 6$ write $\Sigma_\delta^{(N)} = \{\mathrm{sk}(\pi) : \pi \in \mathcal{R}_{\delta,n}$ for some $6 \leq n \leq N\}$, so that Σ_δ is the union of $\bigcup_N \Sigma_\delta^{(N)}$ with the finitely many skeletons of the residuals of length at most 5 and, by Theorem 5.23, is finite with all its members of length at most $8\delta + 25$. The following records what the enumeration of Section 8 determines.

Proposition 5.5. *Let $1 \leq \delta \leq 10$. The sets $\Sigma_\delta^{(N)}$ are equal for $\lambda(\delta) \leq N \leq 22$, where*

$$\lambda(\delta) = 7,\ 9,\ 11,\ 12,\ 13,\ 14,\ 15,\ 15,\ 16,\ 16 \qquad (\delta = 1, \dots, 10)$$

is the length of the shortest residual realising the last skeleton to appear; their common size and maximal length are listed below. They were collected from the complete lists of residuals of length at most 22, that is from 10 210 333 permutations, and the full list of the 86 770 skeletons is archived with the paper. The lengths that enter are $6 \leq n \leq 22$: the residuals of length at most 5 contribute, beyond the skeletons already listed, exactly three further ones, namely 321 at $\delta = 5$, 4321 at $\delta = 5$ and $\delta = 6$, and 54321 at $\delta = 7$. All three are decreasing, and each is realised only by residuals of length at most 5, so none of them occurs in $\Sigma_\delta^{(N)}$ for any N at those defects (check C10).

δ	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\lvert\Sigma_\delta^{(22)}\rvert$	2	21	113	422	1234	2941	6295	12495	22978	40269
$\max\{\lvert\sigma\rvert : \sigma \in \Sigma_\delta^{(22)}\}$	5	7	9	11	13	13	14	15	15	16
$2\delta + 3$	5	7	9	11	13	15	17	19	21	23

In particular every $\sigma \in \Sigma_\delta^{(22)}$ satisfies $|\sigma| \leq 2\delta + 3$, with a margin that grows from $\delta = 6$ on, and $-1 \leq \delta(\sigma) \leq \delta$ for $\delta \leq 8$.

The stabilisation of $\Sigma_\delta^{(N)}$ over the computed range is evidence, not proof, that the catalogue is complete: Theorem 5.23 bounds the length of a member of Σ_δ by $8\delta + 25$, which at $\delta = 10$ is 105, and we have not enumerated the reduced permutations of length between 17 and 105 to rule out further skeletons; what the computation shows is that none of them is realised by a residual of length at most 22. We therefore record the completeness separately.

Conjecture 5.6. $\Sigma_\delta = \Sigma_\delta^{(22)}$ *for $1 \leq \delta \leq 10$.*

Here and below the three decreasing skeletons of Proposition 5.5 are understood to be adjoined to $\Sigma_\delta^{(22)}$ at $\delta = 5, 6, 7$; they are inessential, since none of them is the skeleton of a residual of length at least 6 and so none contributes to $|\mathcal{R}_{\delta,n}|$ for $n \geq 6$. Everything below that uses the catalogue is stated either over the finite range in which it was verified (Proposition 5.8) or conditionally (Corollary 5.9); no unconditional statement in this paper depends on Conjecture 5.6.

Conjecture 5.6 is not independent of the length bounds conjectured elsewhere in this paper: over the range computed here it follows from either of two of them.

Proposition 5.7. *Let $1 \leq \delta \leq 10$. Each of Conjecture 5.10, which bounds the length of a member of Σ_δ by $2\delta + 3$, and Conjecture 6.7 of Section 6, which bounds it by $\delta + 8$, implies Conjecture 5.6 at that δ: every reduced indecomposable 1324-avoider of length at most $\max(2\delta + 3, \delta + 8)$ that satisfies the membership criterion of Proposition 6.3 belongs to $\Sigma_\delta^{(22)}$ or is one of the three decreasing skeletons 321 ($\delta = 5$), 4321 ($\delta = 5, 6$), 54321 ($\delta = 7$) of Proposition 5.5 (check C10).*

Proof. By Proposition 6.3, $\sigma \in \Sigma_\delta$ if and only if $h = \delta - \delta_0 \geq 0$ and $Q(x) = h$ has a solution $x \in \{0, \dots, h\}^s$. The base residual of such a σ has length $L \geq |\sigma|$ and defect $\delta_0 \leq \delta$, so Conjecture 6.7 gives $|\sigma| \leq L \leq \delta_0 + 8 \leq \delta + 8$, while Conjecture 5.10 gives $|\sigma| \leq 2\delta + 3$; under either one the whole of Σ_δ is obtained by running that bounded search on every reduced indecomposable 1324-avoider of length $s \leq \max(2\delta + 3, \delta + 8)$, a bound that differs from $2\delta + 3$ only for $\delta \leq 4$, where it is $9, 10, 11, 12$. Those candidates may be generated with the pruning $\mathrm{inv}(\sigma) \leq 2s - 7 + \delta$, which is nondecreasing along the insertion recursion: for $s \geq 3$, $\delta_0 - (\mathrm{inv}(\sigma) - 2s + 7) = \sum_{i \in D(\sigma)} (d_i - 2) + e(D(\sigma)) \geq 0$, because a vertex of $D(\sigma)$ of degree one has $a_i \geq 0$ only if its unique neighbour also lies in $D(\sigma)$, and for $s \geq 3$ distinct such vertices contribute distinct edges to $e(D(\sigma))$; hence $\sigma \in \Sigma_\delta$ forces $\delta \geq \delta_0 \geq \mathrm{inv}(\sigma) - 2s + 7$, while the one case $s = 2$, where $\sigma = 21$, $\ell = (2, 2)$ and $\delta_0 = 3 < \mathrm{inv}(\sigma) - 2s + 7 = 4$, is added by hand. Carrying out the enumeration (check C10) returns, for each $\delta \leq 10$, exactly $\Sigma_\delta^{(22)}$ together with the three listed skeletons. ☐

Enumerating the solutions of (7) over the catalogue Σ_δ reproduces $\mathcal{R}_{\delta,n}$ set by set (Section 8) and gives the counting function in closed form.

Proposition 5.8. *For $1 \leq \delta \leq 10$ and $n_0(\delta) \leq n \leq 30$, $|\mathcal{R}_{\delta,n}| = P_\delta(n)$ with*

δ	n_0	$P_\delta(n)$	δ	n_0	$P_\delta(n)$
1	8	$8n - 56$	6	15	$168n^2 - 888n - 2067$
2	10	$32n - 214$	7	16	$392n^2 - 2874n - 93$
3	12	$4n^2 + 59n - 549$	8	17	$840n^2 - 7564n + 8430$
4	13	$20n^2 + 40n - 1105$	9	18	$1692n^2 - 17695n + 32934$
5	14	$64n^2 - 162n - 1813$	10	19	$3244n^2 - 38176n + 92439$

The degree is 1 for $\delta \leq 2$ and 2 for $3 \leq \delta \leq 10$; the guess $\lfloor (\delta+1)/2 \rfloor$ made in an earlier version of this paper is therefore wrong from $\delta = 5$ on. The reason is visible in (7): the free blocks form an increasing subsequence of σ all of whose entries have inversion degree 2, and by Theorem 6.6 no residual, at any defect, has four independent such blocks.

Corollary 5.9. *Suppose $|\mathcal{R}_{\delta,n}| = P_\delta(n)$ for all $n \geq n_0(\delta)$ and $1 \leq \delta \leq 10$. Then $a(n,k) \leq a(n+1,k)$ for all $n \geq 1$ and all $k \leq 2n+3$.*

Proof. For $k \leq 2n - 5$ this is Theorem 4.4. Let $k = 2n - 7 + \delta$ with $3 \leq \delta \leq 10$. By (4) and Lemma 3.1,

$$a(n+1,k) - a(n,k) \;\geq\; E(n,k) - P_\delta(n),$$

and $E(n,k)$ is evaluated from the Linusson–Verkama formula (3). One checks that $E(n,k) > P_\delta(n)$ for all $n \geq N(\delta)$ with

$$N(3) = 12, \quad N(4) = 14, \quad N(5) = 16, \quad N(6) = 18,$$
$$N(7) = 20, \quad N(8) = 21, \quad N(9) = 23, \quad N(10) = 24,$$

the difference being increasing in n in each case. For $n < N(\delta)$ the inequality $a(n+1,k) \geq a(n,k)$ is read off from the exhaustive enumeration of Section 8: the table of $a(n,k)$ reaches $n = 26$ and $k \leq 55$, so it contains both sides of (1) for every $n \leq 25$ and every $k \leq 2n + 3$. ☐

The one hypothesis of Corollary 5.9 is that the polynomials of Proposition 5.8 persist beyond $n = 30$; Theorem 5.19 below supplies it, and Corollary 5.20 restates the conclusion unconditionally. We record here what else would have sufficed, since the alternatives apply at every defect rather than only at $\delta \leq 10$. That $|\mathcal{R}_{\delta,n}|$ is eventually *some* polynomial of degree at most two is proved below (Theorems 6.4 and 6.6); by the criterion recorded after Theorem 6.4 it is P_δ exactly when no skeleton outside the catalogue $\Sigma_\delta^{(22)}$ contributes a triple with $F \neq \varnothing$ at defect δ. By Theorem 5.3 this would also follow from Conjecture 5.6, which turns it into a fixed finite list of quadratic equations; and by Proposition 5.7 it therefore follows already from either Conjecture 5.10 or Conjecture 6.7, restricted to $\delta \le 10$, so that one conjectured length bound—on skeletons, or on base residuals—would by itself have made Corollary 5.9 unconditional. By Proposition 6.8 the second alternative reduces further, to the local statement Conjecture 6.9. Theorem 5.23 below shows that every member of Σ_δ has length at most $8\delta + 25$, so Σ_δ is finite and can in principle be determined by a finite search; the constant is far from what the data suggest, and we record the sharp form.

Conjecture 5.10. *Every $\sigma \in \Sigma_\delta$ has length at most $2\delta + 3$; equivalently, every residual π satisfies $\delta(\pi) \ge \lceil(|\mathrm{sk}(\pi)| - 3)/2\rceil$.*

A linear bound is proved in Theorem 5.23; what is conjectured here is the constant, which Proposition 5.5 confirms for the skeletons of the residuals of length at most 22 at each $\delta \le 10$. Write $\nu(s)$ for the least defect of a residual whose skeleton has length s, and $\mu(s)$ for the least defect of a residual of length s that is itself reduced, so that $\nu(s) \le \mu(s)$ and the conjecture reads $\nu(s) \ge \lceil(s-3)/2\rceil$. Both quantities range over residuals of every length, so a search of bounded length can only bound them from above; write $\nu_{22}(s)$ and $\mu_{22}(s)$ for the same minima taken over the residuals of length at most 22, so that $\nu(s) \le \nu_{22}(s)$. Direct enumeration (Section 8) gives

s	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\lceil(s-3)/2\rceil$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9
$\nu_{22}(s)$	1	2	2	3	3	4	4	5	5	7	8	10	–	–	–	–	–
$\mu_{22}(s)$	3	2	2	3	3	4	4	5	5	7	8	10	11	–	–	–	

A dash records that the search range contains no residual with a skeleton of that length: the enumeration covers every residual of length at most 22 and defect at most 10, and none of them has a skeleton longer than 16. What the table verifies is therefore not a lower bound on ν but the absence of a counterexample: no residual of length at most 22 and defect at most 10 violates $\nu(s) \ge \lceil(s-3)/2\rceil$, and for $5 \le s \le 16$ the value $\nu_{22}(s)$ meets the conjectured bound with the margins shown. The last row comes from a separate enumeration of reduced permutations, and its dashes record that no reduced residual of that length occurs at the defects enumerated there. The two quantities agree for $6 \le s \le 16$, the bound is attained for $5 \le s \le 13$, and the margin grows from $s = 14$ on. The extremal configurations are rigid: for each s the minimum is attained by exactly eight reduced residuals, two orbits under the group generated by inversion and reverse–complement, for instance

$$\begin{aligned}
s = 13,\ \delta = 5 :&\quad 4, 3, 5, 7, 2, 8, 10, 1, 11, 13, 12, 9, 6,\\
s = 15,\ \delta = 8 :&\quad 4, 3, 5, 7, 2, 8, 10, 1, 11, 13, 15, 14, 12, 9, 6,\\
s = 17,\ \delta = 11 :&\quad 5, 4, 6, 3, 7, 9, 2, 10, 12, 1, 13, 15, 17, 16, 14, 11, 8.
\end{aligned}$$

Each is a staircase: a decreasing run of small values interleaved with increasing runs of large ones, closed by a decreasing tail. Conjecture 5.10 is a sharpening of [7, Theorem 2.5], which is the statement $\nu(s) \ge 1$ for $s \ge 5$: it asserts that forbidding two adjacent entries of consecutive value costs one further inversion for every two entries of the skeleton. Together with Theorem 5.3 it would make $|\mathcal{R}_{\delta,n}|$ a finite computation for each δ, and hence turn Proposition 5.8 and Corollary 5.9 into theorems at every defect; Theorem 5.19 does this for $\delta \le 10$.

The deletion rule of Proposition 5.4 reduces Conjecture 5.10 to a local statement. Call the deletion of an entry x from a residual π *admissible* if $\pi \setminus x$ is again a residual and

$$2D(x) \ \ge\ 4 + \big(|\mathrm{sk}(\pi)| - |\mathrm{sk}(\pi \setminus x)|\big), \tag{8}$$

$D(x)$ being the number of entries of π that form an inversion with x.

Proposition 5.11. *If every residual of length at least* 10 *admits an admissible deletion, then Conjecture 5.10 holds.*

Proof. Induct on $n = |\pi|$. For $n \leq 9$ the inequality $\delta(\pi) \geq \lceil(|\mathrm{sk}\pi| - 3)/2\rceil$ is a finite check: there are 84 998 residuals of length at most 9, of every defect, and all of them satisfy it. Let $n \geq 10$, let x be an admissible deletion and put $s = |\mathrm{sk}\pi|$, $s' = |\mathrm{sk}(\pi \setminus x)|$, $t = s - s'$. By Proposition 5.4, $\delta(\pi) = \delta(\pi \setminus x) + D(x) - 2$, and by induction $\delta(\pi \setminus x) \geq \lceil(s' - 3)/2\rceil$. Now $\lceil(s-3)/2\rceil - \lceil(s-t-3)/2\rceil \leq \lceil t/2\rceil$, while (8) says $D(x) - 2 \geq \lceil t/2\rceil$ because $D(x)$ is an integer. Adding the two inequalities gives $\delta(\pi) \geq \lceil(s-3)/2\rceil$. □

Conjecture 5.12. *Every residual of length at least* 10 *admits an admissible deletion.*

Much of Conjecture 5.12 is immediate. If $\pi = \sigma[b]$ has a block i whose entries have inversion degree 2, i.e. $\sum_{j \in N(i)} b_j = 2$ for the neighbourhood $N(i)$ of i in the inversion graph of σ, and if $b_i \geq 3$, or $b_i = 2$ and $i \notin D(\sigma)$, then deleting an entry of that block leaves the skeleton unchanged, preserves the defect and, by Theorem 5.3, gives a residual: the deletion is admissible with $t = 0$. The content of the conjecture is therefore concentrated on the reduced residuals and their small inflations. It has been checked on all 33 342 781 residuals of length at most 30 and defect at most 10: exactly 74 admit no admissible deletion, all of them of length 6, 7, 8 or 9 and of defect at most 5, the smallest being $3, 5, 1, 6, 2, 4$ at defect two and $3, 5, 1, 6, 4, 2$ at defect three.

The same observation, applied at the base point of Section 6, disposes of every residual that is long for its defect.

Proposition 5.13. *Every residual of length* $n \geq 8\delta + 30$ *admits an admissible deletion. Consequently Conjecture 5.12 at defect δ is a statement about the finitely many residuals of length at most* $8\delta + 29$*, and for* $\delta \leq 10$ *it remains open only in the range* $25 \leq n \leq 8\delta + 29$*.*

Proof. Let $\pi \in \mathcal{R}_{\delta,n}$, let $\sigma = \mathrm{sk}(\pi)$ and write $\pi = \sigma[\ell + x]$ with ℓ the base block vector and $x \in \mathbb{Z}^s_{\geq 0}$ of Proposition 6.3; let (T, x_T, F) be the triple attached to x in the proof of Theorem 6.4, so that $T \cup F$ is the support of x and $t = \sum_{i \in T} x_i$.

If $F = \varnothing$ then the support of x is T, so $n = L + \sum_i x_i = L + t$, and the last paragraph of that proof bounds $L + t \leq 8\delta + 29$. Thus $n \geq 8\delta + 30$ forces $F \neq \varnothing$.

So let $i \in F$. By the definition of F we have $a_i = 0$ and no neighbour of i in the inversion graph of σ lies in the support of x; hence $x_j = 0$ for every $j \in N(i)$, and the inversion degree in π of each entry of the ith block is

$$\sum_{j \in N(i)} (\ell_j + x_j) \;=\; \sum_{j \in N(i)} \ell_j \;=\; a_i + 2 \;=\; 2.$$

Moreover i lies in the support, so $x_i \geq 1$ and the ith block of π has size $b_i = \ell_i + x_i \geq \ell_i + 1 \geq 2$. If $i \in D(\sigma)$ then $\ell_i = 2$ and $b_i \geq 3$; if $i \notin D(\sigma)$ then $b_i \geq 2$. Either way the pair (π, i) falls under the case treated above: deleting an entry z of that block leaves a block of size $b_i - 1 \geq \ell_i$, which is nonempty and is a nonsingleton whenever $i \in D(\sigma)$, so by Theorem 5.3 the result is again a residual and its skeleton is still σ. Therefore $|\mathrm{sk}(\pi)| - |\mathrm{sk}(\pi \setminus z)| = 0$ while $D(z) = 2$, and (8) reads $4 \geq 4 + 0$: the deletion of z is admissible.

For the last sentence, (8) was tested on all 33 342 781 residuals of length at most 30 and defect at most 10 (Section 8), and the only failures have length at most 9; so at defect $\delta \leq 10$ nothing is open below $n = 31$, and nothing above $n = 8\delta + 29$. □

Before recording what a sweep of the residual lists says about that gap, we prove three of the bounds it produces.

Lemma 5.14. *Let $\pi \in \mathcal{R}_{\delta,n}$ and let c be an entry such that $\pi \setminus c$ is decomposable (a cut vertex of the inversion graph). Then, after replacing π by its reverse–complement if necessary,*

$$\pi = \ \alpha_{\mathrm{b}} \ \ c \ \ \alpha_{\mathrm{a}} \ \ L \ \ S,$$

where the four segments are nonempty, every entry of α_{b} exceeds every entry of α_{a}, the values of $\alpha_{\mathrm{b}}\alpha_{\mathrm{a}}$ are $\{1, \dots, m\}$, every entry of L exceeds c and every entry of S lies strictly between m and c, $|\alpha_{\mathrm{a}}| \geq 2$ and $|S| \geq 2$. In particular c is inverted exactly with the entries of $\alpha_{\mathrm{a}} \cup S$, and c is the only entry of π whose deletion is decomposable.

Proof. Let c be at position p with value v, and let m be an index at which $\pi \setminus c$ splits, so that the first m entries of $\pi \setminus c$ carry the values $1, \dots, m$. If $p \leq m+1$ and $v \leq m+1$, the first $m+1$ positions of π would carry $\{1, \dots, m+1\}$, and if $p \geq m+2$ and $v \geq m+1$ its first m positions would carry $\{1, \dots, m\}$; both contradict the indecomposability of π. Hence either $p \leq m+1$ and $v \geq m+2$, or $p \geq m+2$ and $v \leq m$; reverse–complement exchanges the two cases and preserves residuals and cut vertices, so we assume the first. Then the first $m+1$ positions of π carry $\{1, \dots, m\} \cup \{v\}$; write α for the pattern of the m entries other than c among them and β for the pattern of the entries at positions $m+2, \dots, n$, whose values are $\{m+1, \dots, n\} \setminus \{v\}$. Since c is not extremal, $p \geq 2$ and $v \leq n-1$, so α has an entry before c and β has an entry ℓ with value above v. The entries of α precede those of β and are smaller, so π splits at m unless c is inverted with some entry of α, which forces $p \leq m$.

Four applications of 1324-avoidance. (1) If e_1 precedes c and e_3 follows c within α with $e_1 < e_3$, then e_1, c, e_3, ℓ is an occurrence of 1324; so every entry of α before c exceeds every entry of α after c. Call these two parts α_{b} and α_{a}. (2) If an entry s of β with value below v precedes an entry ℓ' of β with value above v, then e_1, c, s, ℓ' is an occurrence of 1324 for any $e_1 \in \alpha_{\mathrm{b}}$; so $\beta = LS$ with L the entries above v and S those below. (3) Any occurrence of 132 in α extends by an entry of β, and any occurrence of 213 in β extends by an entry of α, to an occurrence of 1324; these facts are not needed below. Thus $\pi = \alpha_{\mathrm{b}}\, c\, \alpha_{\mathrm{a}}\, L\, S$ with α_{a} nonempty (as $p \leq m$), L nonempty and $S \ni m+1$ nonempty, and c is inverted exactly with $\alpha_{\mathrm{a}} \cup S$.

Now the residual conditions. If α_{a} were a single entry, it would carry the value 1 and its deletion would isolate α_{b}, whose entries are inverted with nothing outside α_{a}; so $|\alpha_{\mathrm{a}}| \geq 2$. If S were a single entry, it would be π_n and its deletion would isolate L, whose entries are inverted only with S; so $|S| \geq 2$. Finally let $c' \neq c$. Every entry of α_{b} is inverted with every entry of α_{a}, c with every entry of $\alpha_{\mathrm{a}} \cup S$, and every entry of L with every entry of S, so the segments form a chain $\alpha_{\mathrm{b}} - \alpha_{\mathrm{a}} - c - S - L$ of complete bipartite connections. Deleting c' cannot empty α_{a} or S, and if it empties the end segment α_{b} or L the rest of the chain is unaffected; so the inversion graph of $\pi \setminus c'$ stays connected and c' is not a cut vertex. $\square$

Lemma 5.15. *Let π be a 1324-avoiding permutation of length $n \geq 6$. An entry x of inversion degree three whose deletion shortens the skeleton by three lies at position 2 or 3 with $x < \pi_1$, or at position $n-2$ or $n-1$ with $x > \pi_n$; there is at most one such entry on each side, hence at most two in all.*

Proof. Let $x = \pi_p$ have value v. The skeleton shortens by three exactly when x is a run of length one, its neighbours in position carry consecutive values $\pi_{p-1} = a$, $\pi_{p+1} = a+1$, and the values $v-1$ and $v+1$ occupy two adjacent positions; in particular $1 < p < n$ and $v \notin \{a, a+1\}$.

Suppose $v > a+1$. If some entry $z > v$ followed x, then $a, v, a+1, z$ would be an occurrence of 1324 (with $\pi_{p+1} = a+1$ playing the part of 2); so every entry after x is smaller than v and inverted with x, which gives $D(x) \geq n-p$. The value $v+1$ therefore precedes x and is inverted with it, so $D(x) \geq n-p+1$ and $D(x) = 3$ forces $p \in \{n-2, n-1\}$. For $p = n-2$ all $n-v$ values above v precede x, so $v = n-1$; for $p = n-1$ exactly two entries above v precede x. Both cannot occur: $p = n-2$ gives $\pi_{n-2} = n-1$, and $p = n-1$ would need $\pi_{n-1} > \pi_{n-2} + 1 = n$.

Suppose $v < a$. If some entry $w < v$ preceded π_{p-1}, then $w, a, v, a+1$ would be an occurrence of 1324; so every entry before x exceeds v and is inverted with it, $v-1$ follows x, and $D(x) \geq p$ forces $p \in \{2,3\}$, with $v = 3$ for $p = 2$ and $v = 2$ for $p = 3$. Both cannot occur, since $p = 2$ gives $\pi_3 = \pi_1 + 1 \geq 5$ while $p = 3$ needs $\pi_3 = 2$. The two cases are exchanged by reverse–complement, and for $n \geq 6$ the positions $\{2,3\}$ and $\{n-2, n-1\}$ are disjoint. □

The third bound needs no residual hypothesis either, and counts cut vertices without assuming that there is only one.

Lemma 5.16. *The inversion graph of an indecomposable* 1324*-avoiding permutation has at most three cut vertices: at most one entry that is neither first, last, smallest nor largest, together with possibly the smallest entry and the last entry, or, after reverse–complement, the first entry and the largest one.*

Proof. Let H be indecomposable and 1324-avoiding, of length N, and let c be a cut vertex, at position p with value v, so that $H \setminus c$ splits at some index m. Exactly as in the proof of Lemma 5.14, either $p \leq m+1$ and $v \geq m+2$, or $p \geq m+2$ and $v \leq m$, and reverse–complement exchanges the two cases; in the first case the first $m+1$ positions carry $\{1, \ldots, m\} \cup \{v\}$, and $p \leq m$ because c must be inverted with an entry of α.

Suppose first that some cut vertex c is not extremal, so $p \geq 2$ and $v \leq N-1$ in the first case. The two applications (1) and (2) of 1324-avoidance in the proof of Lemma 5.14 used only these two facts, so $H = \alpha_{\mathrm{b}}\, c\, \alpha_{\mathrm{a}}\, L\, S$ with all four segments nonempty and with the complete bipartite connections α_{b}–α_{a}, c–$(\alpha_{\mathrm{a}} \cup S)$ and L–S. Deleting an entry $c' \neq c$ keeps this chain connected unless it empties α_{a} or S; so the cut vertices of H are c, the entry of α_{a} if $|\alpha_{\mathrm{a}}| = 1$ (it carries the value 1), and the entry of S if $|S| = 1$ (it is π_N). In particular c is the only non-extremal cut vertex and there are at most three in all.

Suppose next that every cut vertex is extremal. If π_1 is a cut vertex, then $p = 1$ forces the first case, so $H = \pi_1\, \alpha\, \beta$ with α carrying $\{1, \ldots, m\}$ at positions $2, \ldots, m+1$ and $v = \pi_1 \geq m+2$. The entry π_1 is inverted with every entry of α, and every sum component of β contains a value below v, since otherwise it would be inverted with nothing outside itself and H would be decomposable; hence in $H \setminus 1$ every entry of $\alpha \setminus \{1\}$ and every component of β is joined to π_1, and the value 1 is not a cut vertex. Applying this to the inverse permutation (inversion preserves 1324-avoidance, indecomposability and the inversion graph, and exchanges π_1 with the value 1) shows that if 1 is a cut vertex then π_1 is not; and reverse–complement gives the same for π_N and N. So at most two extremal entries are cut vertices. □

For a residual π and an extremal entry y, the permutation $\pi \setminus y$ is an indecomposable 1324-avoider, so Lemma 5.16 bounds by three the entries x whose deletion turns y into a cut vertex; this is the bound $4 \cdot 3 = 12$ used in the proof of Theorem 5.19 below.

The last ingredient bounds the entries of inversion degree two that shorten the skeleton.

Lemma 5.17. *Let π be a residual of length $n \geq 8$. Call a maximal set of entries of inversion degree two with a common inversion neighbourhood a* degree-two class*; by Lemma 6.5 it is a run. Then π has at most three degree-two classes.*

Proof. Suppose π has four classes. Choosing one entry from each gives four entries of degree two with pairwise distinct neighbourhoods; the four entries and their at most eight neighbours form a pattern of length at most twelve in which their degrees and neighbourhoods are preserved, as in the proof of Theorem 6.6, so by the finite statement C8 two of them are inverted; call the classes R and R'. Then $R \subseteq N(R')$ and $R' \subseteq N(R)$, so both classes have at most two entries; if both had two, $R \cup R'$ would be a connected component of the inversion graph, and $n = 4$.

Two single entries. Suppose $R = \{x\}$ and $R' = \{y\}$ with x before y, so $x > y$, and write $N(x) = \{y, a\}$, $N(y) = \{x, b\}$. An entry $w \notin \{a, b\}$ strictly between x and y in position would satisfy $w > x$ and $w < y$; so there is none. If a preceded x then $a > x > y$ and a would be inverted with y, so $a = b$ and $\{x, y\}$ would be a connected component of $G(\pi) \setminus a$, contradicting Lemma 5.14, whose two components have at least three entries. So $a \neq b$, a follows x, with $a < y$ if it lies between x and y and $y < a < x$ if it follows y; symmetrically b precedes y, with $b > x$ if it lies between and $y < b < x$ if it precedes x. Every other entry before x is smaller than y and every other entry after y is larger than x, and the only values strictly between y and x are those of a and b. If some entry $w < y$ preceded x and some entry $u > x$ followed y, then w, x, y, u would be an occurrence of 1324; so one of the two sides is empty, and by reverse–complement we may assume that no entry other than a follows y. Then the only entry that can exceed x is b, and only if b lies between x and y; in that case $b = n$ and $x = n - 1$, and since no other value lies between y and x, either a lies between as well and $x = y + 1$, or a follows y and $a = y + 1$. In the first case the orders $n-1, a, n, n-2$ and $n-1, n, a, n-2$ make $\pi \setminus \pi_n$ end with its maximum or put x in the run $(n-1)\, n$; in the second case π ends with $n-1, n, n-3, n-2$ and again x lies in a run. So b precedes x and $x = n$. If a followed y, the values a, b would be $n-1, n-2$ in some order: $a = n-2$ puts $y = n-3$ in the run $(n-3)\,(n-2)$, and $a = n-1$ makes $\pi \setminus n$ end with its maximum. So a lies between x and y with $a < y$, hence $y = n-2$ and $b = n-1$, and $a \neq n-3$ since otherwise a, y would form a run. So

$$\pi = P \;\; n \;\; a \;\; (n-2), \qquad n-1 \in P, \qquad a \leq n-4,$$

and every entry of P other than $b = n-1$ is smaller than $n-2$. Write $P = P_1\, b\, P_2$; then P_2 is nonempty (else $b\,x$ is a run), every entry of P_1 exceeds every entry of P_2 (else w_1, b, w_2, n is a 1324), and within $P \setminus b$ every entry above a precedes every entry below a (else w, p, a, y is a 1324). Hence $P = H_1\, L_1\, b\, H_2\, L_2$ with H_i above a and L_i below a, and $L_1 = \varnothing$ or $H_2 = \varnothing$. Now $N(b) = P_2 \cup \{a, y\}$ and $N(a) = H_1 \cup H_2 \cup \{x, b\}$, where $H_1 \cup H_2$ contains $n-3$; so neither a nor b has degree two, and the third and fourth classes lie in P. Counting inversions segment by segment, an entry of H_1 has degree $|L_1| + |P_2| + 1$ plus its inversions inside H_1, an entry of L_1 has degree $|H_1| + |P_2|$ plus those inside L_1, an entry of H_2 has degree $|P_1| + |L_2| + 2$ plus those inside H_2, and an entry of L_2 has degree $|P_1| + |H_2| + 1$ plus those inside L_2. Entries of degree two in two of these segments are impossible for $n \geq 7$: H_1 with L_1, H_1 with H_2, H_2 with L_2 and L_1 with H_2 are excluded outright by the formulas and by $L_1 = \varnothing$ or $H_2 = \varnothing$; H_1 with L_2 forces $|P_1| = |P_2| = 1$ and $n = 6$; and L_1 with L_2 forces $H_1 = H_2 = \varnothing$, contradicting $n - 3 \in H_1 \cup H_2$. So at most one further class exists in two different segments. Two classes inside one segment would need two entries of degree two with different neighbourhoods, hence an inversion inside the segment, which the formulas allow only in L_1 with $H_1 = \varnothing$ and $|P_2| = 1$, or in L_2 with $P_1 = H_2 = \varnothing$; in the first case $P_2 = H_2 = \{n-3\}$ would lie below $L_1 \subseteq P_1$, contradicting $L_1 < a$, and in the second $H_1 \cup H_2$ would be empty. So π has at most three classes.

A pendant pair. Otherwise $R' = \{r_1, r_2\}$ has two entries and $R = \{x\}$ with $N(x) = R'$ and $N(R') = \{x, z\}$. Then $\{x, r_1, r_2\}$ is a connected component of $G(\pi) \setminus z$, so z is a cut vertex and, by Lemma 5.14 and up to reverse–complement, $\pi = \alpha_{\mathrm{b}}\, z\, \alpha_{\mathrm{a}}\, L\, S$ with $\{x, r_1, r_2\}$ equal to $\alpha_{\mathrm{b}} \cup \alpha_{\mathrm{a}}$ or to $L \cup S$. In the first case $\alpha_{\mathrm{b}} = \{x\}$ and $\alpha_{\mathrm{a}} = R'$, so $\pi = 3\, v\, 1\, 2\, L\, S$ with $v \geq 6$; the classes are $\{3\}$, $\{1, 2\}$, at most one class inside L (an entry of L has degree two only if $|S| = 2$ and it is inverted with nothing else in L, and all such entries share the neighbourhood S) and at most one class inside S (only if $|L| = 1$). Four classes therefore force $|L| = 1$ and $|S| = 2$, that is $n = 7$. The second case is the mirror image, $L = \{x\}$ and $S = R'$, where four classes force $|\alpha_{\mathrm{b}}| = 1$ and $|\alpha_{\mathrm{a}}| = 2$, again $n = 7$. For $n \geq 8$ there are at most three classes. $\square$

The residuals 3612745 (a pendant pair) and 351624 (two single entries) show that the bound fails at $n = 7$ and $n = 6$.

Lemma 5.18 (change of skeleton length under deletion)**.** *If π' is obtained from a permutation π by deleting one entry and standardizing, then $|\mathrm{sk}(\pi)| - |\mathrm{sk}(\pi')| \le 3$.*

Proof. Call adjacent positions carrying consecutive increasing values a bond, and write $b(\pi)$ for the number of bonds. Each maximal increasing consecutive run of length r contributes $r-1$ bonds, so $|\mathrm{sk}(\pi)| = |\pi| - b(\pi)$. Deleting an entry of value v can create a bond only in one of two ways. Its predecessor and successor in position can become adjacent, giving at most one new bond. Alternatively an already adjacent pair can become consecutive in value only if its old values were $v-1, v+1$, again giving at most one new bond. Every other surviving pair keeps its bond status. Some old bonds may disappear, hence $b(\pi') - b(\pi) \le 2$ and $|\mathrm{sk}(\pi)| - |\mathrm{sk}(\pi')| = 1 + b(\pi') - b(\pi) \le 3$. ☐

Theorem 5.19. *Every residual of length $n \ge 23$ admits an admissible deletion. Consequently Conjecture 5.12 holds for every residual of defect at most* 10*, and Conjecture 5.10, Conjecture 5.6 and the polynomials of Proposition 5.8 hold for $1 \le \delta \le 10$.*

Proof. Let $\pi \in \mathcal{R}_{\delta,n}$ and let x be an entry whose deletion is not admissible. Either $\pi \setminus x$ is not a residual, or $2D(x) < 4 + t$ with $t = |\mathrm{sk}\pi| - |\mathrm{sk}(\pi \setminus x)| \le 3$. The final inequality is Lemma 5.18.

If $\pi \setminus x$ is decomposable, x is a cut vertex of $G(\pi)$: at most one entry by Lemma 5.14. If $\pi \setminus x$ is indecomposable but not a residual, one of its four extremal entries y' has $(\pi \setminus x) \setminus y'$ decomposable. If x is itself extremal in π this accounts for at most four entries. Otherwise y' is an extremal entry y of π, the graph $G(\pi) \setminus y$ is connected because π is a residual, and x is a cut vertex of it; since $\pi \setminus y$ is an indecomposable 1324-avoider, Lemma 5.16 allows at most three such x for each of the four choices of y. So at most $1 + 4 + 12 = 17$ entries fail the first condition.

For the second condition, $D(x) \ge 2$ by Corollary 6.2, and $D(x) \ge 4$ never fails. An entry of degree two fails only when $t \ge 1$, which requires it to be a run of length one, hence a degree-two class of size one; by Lemma 5.17 there are at most three such entries (for $n \ge 8$). An entry of degree three fails only when $t = 3$: at most two by Lemma 5.15. So at most $17 + 3 + 2 = 22$ entries fail, and for $n \ge 23$ some deletion is admissible.

For $10 \le n \le 30$ and $\delta \le 10$ the existence of an admissible deletion was verified directly (check C12: all 33 342 781 residuals of length at most 30). Hence Conjecture 5.12 holds at defect at most 10; Proposition 5.11, whose induction stays within defect at most 10 because a deletion never raises the defect, gives Conjecture 5.10 there; Proposition 5.7 gives Conjecture 5.6; and Theorem 6.4 with the complete catalogue gives $|\mathcal{R}_{\delta,n}|$ as the sum (12), which agrees with $P_\delta(n)$ for all $n \ge n_0(\delta)$ (check C9, extended to $n \le 60$, beyond the largest $L + t$ of any triple). ☐

Corollary 5.20. *$a(n,k) \le a(n+1,k)$ for all $n \ge 1$ and all $k \le 2n+3$.*

Proof. Theorem 5.19 supplies the hypothesis of Corollary 5.9. ☐

Remark 5.21. The bound of 22 in Theorem 5.19 is a sum of five, and the data say that each of the five is generous. For a residual π of length n let u be the number of entries whose deletion is not a residual and w the number of entries that fail (8), so that an admissible deletion exists as soon as $u + w \le n - 1$: on all 776 860 residuals with $10 \le n \le 16$ and on uniform samples of 20 000 for each $17 \le n \le 20$ — so at defect at most eight — one finds $u \le 6$ and $w \le 3$, with $u + w \le 9$, the value 9 being attained at every length from 10 to 16 (check C12 of Section 8).

Two further remarks on Conjecture 5.12. First, the connected components of the inversion graph of a permutation are its sum-indecomposable components [6], [10, Proposition 2.1], so

$\pi \setminus x$ is indecomposable exactly when x is not a cut vertex of that graph; the requirements on an admissible deletion are therefore a degree condition and a 2-connectivity condition on the four extremal entries. Secondly, the deletion is not the only route to finiteness: the identity of Lemma 2.3 says that seven entries of π meet all but u of its inversions, and a bound of that shape controls the skeleton directly. For a set T of positions write $e(\pi \setminus T)$ for the number of inversions of π with neither end in T.

Lemma 5.22. *For every permutation π and every set T of positions,*

$$|\mathrm{sk}(\pi)| \;\leq\; 4|T| + 4\,e(\pi \setminus T) + 1.$$

Proof. Let F be the set of positions outside T whose entry is inverted with the entry of another position outside T, so that $|F| \leq 2e(\pi \setminus T)$, and let Z be the set of the remaining positions. The entries at the positions of Z are pairwise non-inverted, so they increase in position and in value; list them as $z_1 < \cdots < z_r$.

Fix $t \in T$, at position P with value V; we index the elements of Z by l and reserve m for positions of π. The entry at z_l is inverted with t exactly when $z_l < P$ and $\pi_{z_l} > V$, or $z_l > P$ and $\pi_{z_l} < V$. Put $\alpha = \#\{l : z_l < P\}$ and $\beta = \min\{l : \pi_{z_l} > V\}$. The first condition holds for $\beta \leq l \leq \alpha$ and the second for $\alpha < l < \beta$, so at most one of the two sets is nonempty and each is an interval of indices. Hence the map $l \mapsto \{t \in T :\ t \text{ inverted with } z_l\}$ is constant on the parts of a partition of $\{1, \ldots, r\}$ into at most $2|T| + 1$ intervals.

Suppose two consecutive *positions* m and $m+1$ both lie in Z, say $m = z_l$ and $m + 1 = z_{l+1}$, and the indices $l, l+1$ lie in the same part. Their entries are then inverted with exactly the same entries of π, and not with each other, so $\pi_m < \pi_{m+1}$. If $\pi_{m+1} > \pi_m + 1$, choose a value w with $\pi_m < w < \pi_{m+1}$; its position is neither m nor $m+1$, and if it precedes m then w is inverted with π_m and not with π_{m+1}, while if it follows $m+1$ the opposite holds. Either way the two entries have different inversion sets, a contradiction. So $\pi_{m+1} = \pi_m + 1$ and the run decomposition has no break at m.

Every break at a position m therefore has $m \in T \cup F$ or $m + 1 \in T \cup F$, or has $m, m+1 \in Z$ lying in different parts. The first alternative occurs at most $2|T \cup F| \leq 2|T| + 4e(\pi \setminus T)$ times and the second at most $2|T|$ times, so $|\mathrm{sk}(\pi)| - 1 \leq 4|T| + 4e(\pi \setminus T)$. ☐

The constants are not optimal: $|\mathrm{sk}(\pi)| \leq 3|T| + 3e(\pi \setminus T) + 1$ has no counterexample among the 2.7 million pairs (π, T) tested in Section 8, with 76 cases of equality, whereas $2|T| + 4e + 1$ already fails for $\pi = 1324$ and $T = \{2\}$. Using the sharper form would replace the bound $8\delta + 25$ of Theorem 5.23 by $6\delta + 19$; we prove and use only the weaker one.

Lemma 2.3 produces such a set whenever its configuration exists, namely when $\mathrm{NW} \neq \varnothing$ and $\mathrm{S}(i) \neq \varnothing \neq \mathrm{E}(\pi_i)$ for some $i \in \mathrm{NW}$: there $T = \{1, i, q, p, j_1, j_2, n\}$ and $e(\pi \setminus T) = u \leq \delta - 1$. That configuration is not always available — at higher defect the hypotheses $a < b$ and $p < q$ of Lemma 2.1 may fail, and even when they hold the two regions may both be nonempty — but a bounded cover always is.

Theorem 5.23. *Every residual $\pi \in \mathcal{R}_{\delta,n}$ has a set T of at most $\delta + 7$ entries with $e(\pi \setminus T) \leq \delta - 1$. Consequently*

$$|\mathrm{sk}(\pi)| \leq 8\delta + 25,$$

so Σ_δ consists of reduced permutations of length at most $8\delta + 25$ and is finite.

The consequence is Lemma 5.22: $4(\delta + 7) + 4(\delta - 1) + 1 = 8\delta + 25$. The proof of the first assertion occupies the rest of the section and splits into five cases according to the sign pattern of π and the shape of its middle part.

The two sign conditions $a < b$ and $p < q$ of Lemma 2.1 are exchanged by inversion, so a residual that fails one of them satisfies $\pi_1 > \pi_n$ after replacing π by π^{-1}. That branch splits into three cases, of which we can settle two.

Proposition 5.24. *Let $\pi \in \mathcal{R}_{\delta,n}$ with $a = \pi_1 > b = \pi_n$, put $\Delta = a - b$, and let σ be the pattern of $\pi_2 \cdots \pi_{n-1}$, of length $m = n - 2$. Then π_1 and π_n are inverted with every other entry of π, they meet exactly $n - 2 + \Delta$ inversions, and*

$$\operatorname{inv}(\sigma) = m - 3 + \delta - \Delta. \tag{9}$$

Moreover:

(i) *if $\sigma_1 > \sigma_m$, the four entries $\pi_1, \pi_2, \pi_{n-1}, \pi_n$ meet all but at most $\delta - \Delta - 1 - (\sigma_1 - \sigma_m) \leq \delta - 3$ of the inversions of π;*

(ii) *if the largest entry of σ precedes its smallest, then π_1, π_n and the largest and the smallest of $\pi_2, \ldots, \pi_{n-1}$ meet all but at most $\delta - \Delta - 2 \leq \delta - 3$;*

(iii) *otherwise $\sigma_1 < \sigma_m$ and the smallest entry of σ precedes the largest; and if σ has entries in both its northwestern and its southeastern region, then $\operatorname{inv}(\sigma) \geq 2m - 5$ by the counting in the proof of* [7, Lemma 2.7]*, so that $n \leq \delta - \Delta + 4$.*

In particular the conclusion of Theorem 5.23 holds with four entries in the cases (i) and (ii). A negative right-hand side in (i) or (ii) — which happens for small δ, since $\Delta \geq 1$ — is to be read as saying that the case in question does not occur.

Proof. Since $b < a$, every value is smaller than a or larger than b, so every entry other than π_1 and π_n is inverted with one of them. The entry π_1 has $a - 1$ right-inversions and π_n has $n - b$ left-inversions, and the pair $(1, n)$ is an inversion counted in both; so together they meet $(a - 1) + (n - b) - 1 = n - 2 + \Delta$ inversions, and (9) follows from $\operatorname{inv}(\pi) = 2n - 7 + \delta$.

(i) Inside σ the first entry has $\sigma_1 - 1$ right-inversions and the last has $m - \sigma_m$ left-inversions, and the pair of them is an inversion counted in both. Hence at most $\operatorname{inv}(\sigma) - (\sigma_1 - 1) - (m - \sigma_m) + 1 = \delta - \Delta - 1 - (\sigma_1 - \sigma_m)$ inversions of π avoid all four entries, and this is at most $\delta - \Delta - 2 \leq \delta - 3$ because $\sigma_1 > \sigma_m$ and $\Delta \geq 1$.

(ii) Let p' and q' be the positions in σ of its smallest and its largest entry, so $q' < p'$ by hypothesis. They have $p' - 1$ and $m - q'$ inversions inside σ respectively, and the pair (q', p') is an inversion counted in both, so at most $\operatorname{inv}(\sigma) - (p' - 1) - (m - q') + 1 = \delta - \Delta - 1 - (p' - q') \leq \delta - \Delta - 2 \leq \delta - 3$ inversions of π avoid the four entries.

(iii) is the remaining case, together with the quoted inversion count for a permutation with points in both regions. ☐

Among the 564 634 residuals of length between 9 and 18 and defect at most 8 with $\pi_1 > \pi_n$, case (i) accounts for 74 909, case (ii) for 183 200 and case (iii) for 306 525; in case (iii) all but 442 have an empty northwestern or southeastern region for σ, and each of those 442 satisfies $n \leq \delta - \Delta + 4$, as Proposition 5.24(iii) requires. Identity (9) was checked on all of them. The trichotomy is recorded because cases (i) and (ii) need only four entries; Corollary 5.30 and Proposition 5.32 settle the whole branch at once, in all three cases, by splitting instead on whether σ is decomposable.

The tool for that is a statement about 1324-avoiders of near-minimal inversion number, which has a companion we can prove outright. Write $G(\pi)$ for the inversion graph of π: its vertices are the positions and its edges the inversions, so that the induced subgraph on a set of positions is the inversion graph of the corresponding pattern.

Lemma 5.25. *The inversion graph of a* 1324*-avoiding permutation contains no induced path on six vertices.*

Proof. By a theorem of Brignall and Vatter [4, Proposition 1.16], the permutations whose inversion graph is a path are exactly the *increasing oscillations*, that is, the sum-indecomposable permutations order isomorphic to subsequences of $2, 4, 1, 6, 3, 8, 5, \dots$; there are exactly two of each length at least three, and the list begins 2413, 3142, 24153, 31524, 241635, 315264. (That 24153 and 31524 are the only permutations whose inversion graph is P_5 is also recorded in [3, p. 4].) An induced path on six vertices of $G(\pi)$ is the inversion graph of the pattern of π on those six entries, so such a π would contain 241635 or 315264; but the entries in positions $1, 2, 5, 6$ of the first and in positions $2, 3, 4, 5$ of the second form occurrences of 1324. □

That the two length-five permutations are exactly the skeletons of Theorem 1.3 is not a coincidence: both statements say that a path of five inversions is the longest rigid chain a 1324-avoider can carry.

Conjecture 5.26. *Let τ be an indecomposable* 1324*-avoiding permutation of length m with* $\mathrm{inv}(\tau) = m - 1 + \gamma$*, so that γ is the cyclomatic number of its inversion graph in the sense of* [10]*. Then three entries of τ meet all but at most γ of its inversions; and $\gamma + 2$ entries meet all of them.*

The case $\gamma = 0$ of both halves is a theorem, and it comes with a complete classification. The permutations in question — the sum-indecomposable ones with $m - 1$ inversions, equivalently those whose inversion graph is a tree — are known: by [10, Propositions 5.2 and 5.4] they are exactly the unimodal cycles of $[m]$, and their inversion graphs are caterpillars. Avoiding 1324 collapses the caterpillar to a double star. Recall that a tree has a vertex cover of size two exactly when it has no three pairwise disjoint edges, by Kőnig's theorem. Write $S(2, 2, 1)$ for the tree on six vertices consisting of a centre with two paths of length two and one pendant edge attached, and $S(2, 2, 2)$ for the tree on seven vertices with three paths of length two attached to a centre.

Theorem 5.27. *Let τ be an indecomposable* 1324*-avoiding permutation of length m with* $\mathrm{inv}(\tau) = m - 1$*. Then two entries of τ meet all of its inversions. Moreover τ is one of*

$$21[\iota_{m-1}, 1], \quad 21[1, \iota_{m-1}], \quad 2413[\iota_t, 1, 1, \iota_{m-2-t}], \quad 24153[\iota_t, 1, 1, \iota_{m-3-t}, 1]$$

or the inverse of one of the last two, where $t \geq 1$ and the remaining block is nonempty; there are exactly $4(m - 3)$ of them for $m \geq 4$.

Proof. The inversion graph $G = G(\tau)$ has m vertices and $m - 1$ edges and is connected because τ is indecomposable, so it is a tree. By Lemma 5.25 it has no induced path on six vertices; a tree of diameter at least five contains such a path, so the diameter of G is at most four. If the diameter is at most three then G is a star or a double star and two vertices cover it. So assume the diameter is four and let c be a centre, every vertex being within distance two of c.

Suppose two vertices do not suffice. By Kőnig's theorem G has three pairwise disjoint edges. Every edge of G joins c to a vertex at distance one, or a vertex at distance one to a vertex at distance two, and at most one of the three edges meets c. If none does, the three edges are $a_i b_i$ with $d(c, a_i) = 1$ and $d(c, b_i) = 2$, the a_i distinct; since each b_i is a leaf of G, the induced subgraph on $\{c, a_1, b_1, a_2, b_2, a_3, b_3\}$ is $S(2, 2, 2)$. If one of them is ca_3, the induced subgraph on $\{c, a_1, b_1, a_2, b_2, a_3\}$ is $S(2, 2, 1)$. In either case, since an induced subgraph of G on a set of positions is the inversion graph of the corresponding pattern, τ would contain a pattern whose inversion graph is $S(2, 2, 1)$ or $S(2, 2, 2)$. A check of the 720 and the 5040 permutations of lengths six and seven shows that $S(2, 2, 1)$ is the inversion graph of exactly two of them, 251364 and 314625, which contain 1324 in positions $1, 2, 4, 5$ and $2, 3, 5, 6$ respectively, and that $S(2, 2, 2)$ is the inversion graph of none. This contradiction proves the first assertion.

For the classification write $\tau = \rho[b]$ with ρ reduced of length r, put $b = \mathbf{1} + c$ and let E be the inversion set of ρ and d_i the degrees in it. Expanding $\mathrm{inv}(\tau) = \sum_{(i,j) \in E} b_i b_j$ and $m = r + \sum_i c_i$

in $\mathrm{inv}(\tau) = m-1$ gives

$$\sum_i (d_i-1)c_i + \sum_{(i,j)\in E} c_i c_j \;=\; r-1-|E|.$$

The left-hand side is nonnegative because $d_i \geq 1$, and $|E| \geq r-1$ because ρ is indecomposable, so both sides vanish: E is a tree, $c_i = 0$ at every vertex of degree at least two, and $c_i c_j = 0$ along every edge. In particular ρ is itself an indecomposable 1324-avoider with $r-1$ inversions, and it is reduced, so by the first assertion and Lemma 5.22, applied with T a vertex cover of size two and hence $e(\rho \backslash T) = 0$, we get $r = |\mathrm{sk}(\rho)| \leq 4\cdot 2+1 = 9$. The indecomposable 1324-avoiders with $r-1$ inversions number $12, 16, 20, 24$ for $r = 6, 7, 8, 9$ and none of them is reduced, so $r \leq 5$ and $\rho \in \{21, 2413, 3142, 24153, 31524\}$. Inflating only at the leaves of the corresponding tree, and not at two ends of an edge, yields exactly the list; the count is $2+2(m-3)+2(m-4) = 4(m-3)$. $\square$

The second half of Conjecture 5.26 follows from Theorem 5.27 by an induction that deletes one entry at a time. The step needs a vertex whose removal keeps the inversion graph connected and destroys a cycle, and such a vertex always exists.

Lemma 5.28. *Let G be the inversion graph of a permutation. If G is connected and contains a cycle, then G has a vertex of degree at least two that is not a cut vertex.*

Proof. Suppose every vertex of degree at least two is a cut vertex. A leaf block of the block tree of G contains a vertex that is not a cut vertex, and if that block were 2-connected such a vertex would have degree at least two; so every leaf block is a single edge, and in particular G has at least two blocks. Since G has a cycle, some block B is 2-connected, and by assumption every vertex of B is a cut vertex of G.

Let C be a shortest cycle of B. It is chordless, hence an induced subgraph of G, hence the inversion graph of a pattern of the permutation; and no inversion graph has an induced cycle of length five or more [10, Proposition 2.3], so C has length three or four; the only permutations realizing C_3 and C_4 are 321 and 3412. For each $v \in C$ pick a neighbour w_v of v lying in a component of $G \setminus v$ that does not meet B. The w_v are distinct; none is adjacent to a vertex of C other than v, and no two of them are adjacent, since either would place them in the block B. The induced subgraph of G on $C \cup \{w_v\}$ is therefore a triangle or a four-cycle with one pendant vertex at each of its vertices. Neither of those two graphs is the inversion graph of a permutation, as a check of the 720 permutations of length six and the 40320 of length eight shows. $\square$

Theorem 5.29. *Let τ be an indecomposable* 1324*-avoiding permutation of length m with $\mathrm{inv}(\tau) = m-1+\gamma$. Then some $\gamma+2$ entries of τ meet all of its inversions.*

Proof. Induction on γ, the case $\gamma = 0$ being Theorem 5.27. Let $\gamma \geq 1$. The inversion graph G of τ is connected with m vertices and $m-1+\gamma$ edges, so it has a cycle, and Lemma 5.28 supplies a vertex x of degree $d \geq 2$ that is not a cut vertex. Then $\tau \setminus x$ is an indecomposable 1324-avoider of length $m-1$ with $m-1+\gamma-d = (m-1)-1+\gamma'$ inversions, where $\gamma' = \gamma-d+1 \leq \gamma-1$ and $\gamma' \geq 0$ because $G \setminus x$ is connected. By induction $\gamma'+2$ entries of $\tau \setminus x$ meet all of its inversions; adjoining x gives $\gamma'+3 \leq \gamma+2$ entries meeting all inversions of τ. $\square$

This settles the branch $\pi_1 > \pi_n$ whenever the middle part is indecomposable.

Corollary 5.30. *Let $\pi \in \mathcal{R}_{\delta,n}$ with $\pi_1 > \pi_n$ and suppose the pattern σ of $\pi_2 \cdots \pi_{n-1}$ is indecomposable. Then at most $\delta+1$ entries of π meet all of its inversions, and $|\mathrm{sk}(\pi)| \leq 4\delta+5$.*

Proof. By Proposition 5.24, $\mathrm{inv}(\sigma) = m-1+\gamma$ with $m = n-2$ and $\gamma = \delta-\Delta-2 \geq 0$. By Theorem 5.29 some $\gamma+2 = \delta-\Delta$ entries of σ meet all of its inversions; adjoining π_1 and π_n, which are inverted with every other entry, gives a set T with $|T| = \delta-\Delta+2 \leq \delta+1$ and $e(\pi \setminus T) = 0$. Lemma 5.22 gives $|\mathrm{sk}(\pi)| \leq 4(\delta+1)+1$. $\square$

The first half of Conjecture 5.26 — three entries suffice when γ inversions may be left over — remains open; it was verified for every indecomposable 1324-avoider of length at most 12 with $\gamma \le 6$, where the least size of a vertex cover is $2, 3, 4, 4, 5, 5, 6$ for $\gamma = 0, \dots, 6$, independently of m once $m \ge 8$. Theorem 5.29 is all that Corollary 5.30 needs.

Proposition 5.31. *Assume the first half of Conjecture 5.26 and let $\pi \in \mathcal{R}_{\delta,n}$ with $\pi_1 > \pi_n$ be such that the pattern σ of $\pi_2 \cdots \pi_{n-1}$ is indecomposable. Then five entries of π meet all but at most $\delta - 3$ of its inversions.*

Proof. By Proposition 5.24 the entries π_1 and π_n meet $n - 2 + \Delta$ inversions and $\mathrm{inv}(\sigma) = m - 3 + \delta - \Delta$, where $m = n - 2$. Thus $\mathrm{inv}(\sigma) = m - 1 + \gamma$ with $\gamma = \delta - \Delta - 2$, and $\gamma \ge 0$ because the inversion graph of the indecomposable σ is connected. Three entries of σ therefore meet all but at most $\gamma \le \delta - 3$ of the inversions of σ, and together with π_1, π_n they meet all but at most $\delta - 3$ of the inversions of π. □

Corollary 5.30 is weaker in the number of entries but unconditional, and it is what the finiteness of Σ_δ requires. When σ is decomposable neither statement applies, but the same five entries — π_1, π_n and three of the middle entries — leave at most $\delta - 3$ inversions on each of the 49 699 residuals with $\pi_1 > \pi_n$ and $10 \le n \le 16$ that were sampled, whether or not σ is indecomposable.

The decomposable case is settled by the residual conditions themselves, which force the outer components of σ to be large and the inner ones to be cheap.

Proposition 5.32. *Let $\pi \in \mathcal{R}_{\delta,n}$ with $a = \pi_1 > b = \pi_n$ and $\Delta = a - b$, and suppose $\sigma = C_1 \oplus \cdots \oplus C_r$ with $r \ge 2$. Then $|C_1| \ge b$, $|C_r| \ge n - a + 1$, the components $C_2, \dots, C_{r-1}$ have total length at most $\Delta - 3$ and carry at most $\delta - 4$ inversions in all, and $r \le \Delta - 1$. Moreover π_1, π_n and vertex covers of C_1 and C_r form a set T with $|T| \le \delta + 2$ and $e(\pi \setminus T) \le \delta - 4$.*

Proof. The inversion graph of $\pi \setminus \pi_1$ is that of σ together with the entry π_n, which is inverted with exactly the entries of value greater than b. A component of σ all of whose values lie below b would be isolated, so every component contains a value exceeding b; the components carry increasing intervals of values, so this constrains only C_1, and the $b-1$ middle values below b give $|C_1| \ge b$. Applying the same argument to $\pi \setminus \pi_n$ gives $|C_r| \ge n-a+1$. Since $\sum_i |C_i| = n-2$, the remaining components have total length at most $(n-2) - b - (n-a+1) = \Delta - 3$; in particular $\Delta \ge 3$ and $r \le \Delta - 1$.

By (9), $\sum_i \mathrm{inv}(C_i) = n-5+\delta-\Delta$, while $\mathrm{inv}(C_1) \ge |C_1|-1 \ge b-1$ and $\mathrm{inv}(C_r) \ge |C_r|-1 \ge n-a$; subtracting leaves at most $\delta - 4$ inversions in $C_2, \dots, C_{r-1}$. Writing $\gamma_i = \mathrm{inv}(C_i) - |C_i| + 1 \ge 0$ we get $\sum_i \gamma_i = r - 3 + \delta - \Delta \le \delta - 4$, so by Theorem 5.29 the components C_1 and C_r have vertex covers of sizes $\gamma(C_1) + 2$ and $\gamma(C_r) + 2$, of total size at most δ. Adjoining π_1 and π_n gives $|T| \le \delta + 2$. An inversion of π missing T involves neither π_1 nor π_n, hence lies inside a single component of σ, and not inside C_1 or C_r; so $e(\pi \setminus T)$ is at most the number of inversions of $C_2, \dots, C_{r-1}$. □

It remains to treat the residuals with $a < b$ and $p < q$. Recall from Lemma 2.1 that inversion preserves these two conditions and exchanges NW with SE, and that $\mathrm{NW} \cup \mathrm{SE} \ne \varnothing$; so we may assume $\mathrm{NW} \ne \varnothing$.

Proposition 5.33. *Let $\pi \in \mathcal{R}_{\delta,n}$ with $a < b$, $p < q$ and $\mathrm{NW} \ne \varnothing$, and write $\mathrm{NW} = \{i_1 < \cdots < i_m\}$, $v_l = \pi_{i_l}$.*

(i) *If $\mathrm{SE} \ne \varnothing$, then for $i \in \mathrm{NW}$ and $j \in \mathrm{SE}$ the six positions $1, i, q, p, j, n$ meet all but at most $\delta - 2$ of the inversions of π.*

(ii) *If $\mathrm{SE} = \varnothing$ and $m = 1$, the seven positions of Lemma 2.3 meet all but at most $\delta - 1$.*

(iii) *If* $\mathrm{SE} = \varnothing$ *and* $m \geq 2$*, then with* $j_1 \in \mathrm{S}(i_1)$ *and* $j_2 \in \mathrm{E}(v_m)$ *the seven positions* $1, i_m, q, p, j_1, j_2, n$ *meet all but at most* $\delta - 3$*.*

Proof. (i) The count in the proof of [7, Lemma 2.7] — right-inversions of i, left-inversions of p and of n, right-inversions of 1 and of j, left-inversions of q, with five double-counted pairs removed — produces at least $2n-5$ distinct inversions, each incident to one of the six positions. Subtracting from $\mathrm{inv}(\pi) = 2n - 7 + \delta$ leaves at most $\delta - 2$.

(ii) Lemma 2.2 gives $\mathrm{S}(i_1) \neq \varnothing$ and $\mathrm{E}(v_m) \neq \varnothing$, and $m = 1$ makes these regions belong to the same northwestern point, so Lemma 2.3 applies. Every inversion counted by its remainder u has both ends outside $\{1, i, q\} \cup \{p, j_1, j_2, n\}$ or one end in that set but on the wrong side, so $e(\pi \setminus T) \leq u = \delta - 1 - \sum_l s_l \leq \delta - 1$.

(iii) This is the count in the proof of Lemma 2.5, which produces at least $2n-4$ inversions incident to the seven positions. ☐

Proof of Theorem 5.23. Suppose first $a > b$. If σ is indecomposable, Corollary 5.30 gives $|T| \leq \delta+1$ and $e(\pi \setminus T) = 0$; if it is decomposable, Proposition 5.32 gives $|T| \leq \delta+2$ and $e(\pi \setminus T) \leq \delta-4$. If $a < b$ and $p > q$, apply this to π^{-1}, which satisfies $a > b$ by Lemma 2.1 and is again a residual of the same defect; a set of positions of π^{-1} corresponds to the set of entries of π carrying those values, and inversion is an isomorphism of inversion graphs, so the cover transports. Finally, if $a < b$ and $p < q$ we may assume $\mathrm{NW} \neq \varnothing$ after inverting, and Proposition 5.33 gives a set of at most seven positions with $e(\pi \setminus T) \leq \delta - 1$. In every case $|T| \leq \max(7, \delta+2) \leq \delta + 7$ and $e(\pi \setminus T) \leq \delta - 1$, and Lemma 5.22 gives $|\mathrm{sk}(\pi)| \leq 4(\delta+7) + 4(\delta-1) + 1 = 8\delta + 25$. A reduced residual is its own skeleton, so Σ_δ consists of reduced permutations of length at most $8\delta+25$. ☐

Two consequences. The first is the polynomial bound in the form in which it is used.

Corollary 5.34. *For every* $\delta \geq 1$ *and every* n*,*

$$|\mathcal{R}_{\delta,n}| \;\leq\; (8\delta+26)!\, n^{8\delta+24}.$$

Proof. Every residual is $\sigma[b]$ for a unique reduced σ and a unique composition b of n into $|\sigma|$ parts. By Theorem 5.23 the skeleton has length at most $s = 8\delta + 25$, and the reduced permutations of length at most s number at most $\sum_{r \leq s} r! \leq (s+1)!$, while for each of them the compositions of n into $r \leq s$ parts number $\binom{n-1}{r-1} \leq n^{s-1}$. ☐

Theorem 5.35. *For every* $\delta \geq 1$ *there is an explicit* $N(\delta)$ *such that*

$$a(n, 2n-7+\delta) \;\leq\; a(n+1, 2n-7+\delta) \qquad \text{for all } n \geq N(\delta).$$

Proof. Write $k = 2n - 7 + \delta$ and $m = n - 7 + \delta$. By (4) the difference $a(n+1,k) - a(n,k)$ is the number of members of $\mathrm{Av}^k_{n+1}(1324)$ outside $\mathrm{im}(f \sqcup g)$ minus $|\mathcal{R}_{\delta,n}|$. We use a single one of the four classes of Lemma 3.1, so that no inclusion–exclusion is needed: the members of $\mathrm{Av}^k_{n+1}(1324)$ beginning with $n+1$ lie outside the image, and deleting that first entry, which is inverted with every other one, is a bijection onto $\mathrm{Av}^m_n(1324)$. Hence

$$a(n+1,k) - a(n,k) \;\geq\; a(n,m) - |\mathcal{R}_{\delta,n}|.$$

For $n \geq \delta$ we have $m \leq 2n - 7$, so (3) gives $a(n,m) = p_2(m) - c(\delta)$ with $c(\delta) = 4p_2(\delta - 6) + 6\sum_{i \leq \delta-7} p_2(i)$ independent of n (and $c(\delta) = 0$ for $\delta \leq 5$), while Corollary 5.34 gives $|\mathcal{R}_{\delta,n}| \leq (8\delta+26)!\, n^{8\delta+24}$. It is therefore enough to produce an $N(\delta)$ with

$$p_2(n-7+\delta) - c(\delta) \;>\; (8\delta+26)!\, n^{8\delta+24} \qquad \text{for all } n \geq N(\delta). \tag{10}$$

For this we use the elementary bound $p_2(j) \geq p(j) \geq 2^{\sqrt{j}-2}$, valid for $j \geq 1$: with $r = \lfloor \sqrt{j} \rfloor - 1$ every subset $S \subseteq \{1, \ldots, r\}$ has $\sum S \leq r(r+1)/2 < j - r$, so adjoining the part $j - \sum S$, which exceeds r, turns S into a partition of j into distinct parts, and distinct subsets give distinct partitions; hence $p(j) \geq 2^r \geq 2^{\sqrt{j}-2}$. The constant $c(\delta)$ is absorbed by a factor two: $c(\delta) \leq (8\delta+26)! \leq (8\delta+26)!\, n^{8\delta+24}$ for every $n \geq 1$, so (10) follows from $p_2(n-7+\delta) > 2\,(8\delta+26)!\, n^{8\delta+24}$ and therefore holds as soon as

$$H(n) := \bigl(\sqrt{n-7+\delta}-2\bigr)\log 2 - (8\delta+24)\log n - \log\bigl(2\,(8\delta+26)!\bigr) \ > \ 0.$$

Now $H'(n) = \log 2/\bigl(2\sqrt{n-7+\delta}\bigr) - (8\delta+24)/n > 0$ once $n > 2(8\delta+24)\sqrt{n-7+\delta}/\log 2$, which holds for all $n \geq \lceil (2(8\delta+24)/\log 2)^2 \rceil + \delta$. Beyond that point H is increasing, so the set of n where $H(n) > 0$ is a final segment and $N(\delta)$ may be taken to be its least element. For $\delta = 1, 2, 3$ this gives $N(\delta) = 548\,967$, $933\,556$ and $1\,437\,767$, the monotonicity of H setting in no later than at the explicit points $n = 8527$, $13\,323$ and $19\,185$ given by that bound, all three well below the thresholds. The programs of [9] evaluate the same function H that is displayed here. □

Both ingredients are very crude — the data of Proposition 5.8 say that $|\mathcal{R}_{\delta,n}|$ is in truth quadratic in n, and $2^{\sqrt{j}}$ is far below $p_2(j)$ — and the thresholds are correspondingly large. Replacing the elementary bound by exact partition numbers moves the comparison down by a factor of about forty: (10) is satisfied at $n = 12\,425$, $21\,466$ and $33\,491$ for $\delta = 1, 2, 3$ and fails at $n-1$ in each case (the program is in [9]). We claim nothing more about these three numbers: they are two directly verified evaluations of (10), not thresholds — we have not shown that (10) fails at every smaller n, nor that it holds at every larger one. The thresholds we prove are the ones in the theorem. For $\delta \leq 2$ Theorems 1.4 and 4.4 are of course much better, giving the inequality for every n. What Theorem 5.35 adds is that the Claesson–Jelínek–Steingrímsson inequality holds at *every* fixed defect once n is large, with no hypothesis.

6. Polynomiality of the residual counts

Theorem 5.23 makes Σ_δ finite, and Corollary 5.34 turns that into a polynomial *bound* on $|\mathcal{R}_{\delta,n}|$. In this section we show that $|\mathcal{R}_{\delta,n}|$ is, for every fixed δ and all large n, exactly a polynomial in n, of degree at most two, with an explicit description as a finite sum of binomial coefficients. The two ingredients are a lemma on entries of inversion degree one, which removes the only negative terms from the master identity, and a compression argument that reduces the degree bound to a finite check.

6.1. Entries of degree one.

Lemma 6.1. *Let σ be a 1324-avoiding permutation of length $s \geq 3$ and let the position i have inversion degree exactly one, its unique neighbour in the inversion graph being the position j. Then $j \in I(\sigma) = \{1, s, \sigma^{-1}(1), \sigma^{-1}(s)\}$ and $\sigma \setminus j$ is decomposable. In particular $j \in D(\sigma)$ whenever σ is reduced and indecomposable.*

Proof. Suppose first $j > i$. Every entry left of i is smaller than σ_i and every entry right of i other than σ_j is larger, since otherwise it would be inverted with σ_i. Hence the values below σ_i are the $i-1$ entries to the left of i together with σ_j, so $\sigma_i = i+1$ and positions $1, \ldots, i-1$ carry $\{1, \ldots, i\} \setminus \{\sigma_j\}$. If $\sigma_j \neq 1$ then the value 1 lies left of i, and every entry x to the right of j has value greater than $i+1$; so $1,\ \sigma_i,\ \sigma_j,\ x$ would be an occurrence of 1324. Therefore $j = s$ or $\sigma_j = 1$, and in both cases $j \in I(\sigma)$.

Delete j and standardize. If $\sigma_j = 1$, positions $1, \ldots, i$ carried $\{2, \ldots, i+1\}$ and now carry $\{1, \ldots, i\}$. If $j = s$ and $\sigma_j = v \in \{2, \ldots, i\}$, positions $1, \ldots, i$ carried $\{1, \ldots, i+1\} \setminus \{v\}$ and now carry $\{1, \ldots, i\}$. This is a proper prefix of $\sigma \setminus j$ unless $i = s-1$, in which case $j = s$, $\sigma_{s-1} = s$, and positions $1, \ldots, s-2$ carry $\{1, \ldots, s-2\}$ after the deletion; since $s \geq 3$ that prefix is proper. So $\sigma \setminus j$ is decomposable.

If $j < i$, apply reverse–complement, which fixes the pattern 1324, maps inversions to inversions, preserves $I(\sigma)$ and decomposability, and exchanges the two cases. □

Corollary 6.2. *Every entry of a residual is inverted with at least two other entries.*

Proof. A residual π is indecomposable, so no entry has degree zero (an isolated vertex of the inversion graph is a sum component). If an entry had degree one, Lemma 6.1 would give a neighbour $y \in \{\pi_1, \pi_n, 1, n\}$ with $\pi \setminus y$ decomposable, so π would be almost decomposable. $\square$

6.2. **The base residual and the nonnegative form of the master identity.** Let σ be reduced, indecomposable and 1324-avoiding, of length $s \geq 2$, and put

$$\ell_i = 1 + [\, i \in D(\sigma)\,] \qquad (1 \leq i \leq s), \qquad L = \sum_i \ell_i, \qquad \rho = \sigma[\ell], \qquad \delta_0 = \delta(\rho).$$

By Theorem 5.3, $\sigma[\ell + x]$ is a residual for every $x \in \mathbb{Z}^s_{\geq 0}$, and every residual with skeleton σ is of this form; we call ρ the *base residual* of σ. By [7, Theorem 2.5], $\delta_0 \geq 1$.

Proposition 6.3. *With $a_i = \sum_{j \in N(i)} \ell_j - 2$, where $N(i)$ is the neighbourhood of i in the inversion graph of σ,*

$$\delta\big(\sigma[\ell + x]\big) = \delta_0 + Q(x), \qquad Q(x) = \sum_{i=1}^{s} a_i x_i + \sum_{(i,j) \in \mathrm{Inv}(\sigma)} x_i x_j, \tag{11}$$

and $a_i \geq 0$ for every i. Consequently $\delta_0 \leq \delta$ for every $\sigma \in \Sigma_\delta$, and $\sigma \in \Sigma_\delta$ if and only if $h = \delta - \delta_0 \geq 0$ and $Q(x) = h$ has a solution $x \in \{0, 1, \ldots, h\}^s$.

Proof. Expanding $\mathrm{inv}(\sigma[\ell + x]) = \sum_{\mathrm{Inv}(\sigma)} (\ell_i + x_i)(\ell_j + x_j)$ and $n = L + \sum_i x_i$ in $\delta = \mathrm{inv} - 2n + 7$ gives (11). The number $a_i + 2 = \sum_{j \in N(i)} \ell_j$ is the inversion degree, in ρ, of each entry of the ith block, so $a_i \geq 0$ by Corollary 6.2. (Alternatively: if $a_i < 0$ then $\sigma[\ell + te_i]$ is a residual of defect $\delta_0 + ta_i$, which is nonpositive for large t, contradicting [7, Theorem 2.5].) Since every term of Q is nonnegative, $Q(x) = h$ forces $x_i \leq h$ at every i with $a_i \geq 1$ and $x_i x_j \leq h$ on every edge; a coordinate exceeding h therefore has $a_i = 0$ and all its neighbours at zero, contributes nothing to Q, and may be replaced by 0. $\square$

Proposition 5.4 is the case $\ell = \mathbf{1}$ of (11), in which the entries of degree one contribute the negative coefficients $d_i - 2 = -1$; the change of base point moves those entries' neighbours, which by Lemma 6.1 lie in $D(\sigma)$, to block size two, and the negative terms disappear. Note that the base point also covers $\sigma = 21$, where $\ell = (2, 2)$, $\rho = 3412$, $\delta_0 = 3$ and $Q(x) = x_1 x_2$.

6.3. **Eventual polynomiality.**

Theorem 6.4. *For every $\delta \geq 1$ there is an integer $n_1(\delta) \leq 8\delta + 30$ such that $|\mathcal{R}_{\delta,n}|$ agrees with a polynomial in n for all $n \geq n_1(\delta)$. Explicitly,*

$$|\mathcal{R}_{\delta,n}| = \sum_{\sigma \in \Sigma_\delta} \sum_{(T, x_T, F)} \binom{n - L - t - 1}{|F| - 1}, \tag{12}$$

where, for each σ with $h = \delta - \delta_0$, the inner sum runs over the finitely many triples consisting of a set $T \subseteq \{1, \ldots, s\}$, a vector $x_T \in \mathbb{Z}^T_{\geq 1}$ with $Q(x_T, 0) = h$ and $\sum_{i \in T} x_i = t \leq 2h$, and a set F of positions $i \notin T$ with $a_i = 0$ and $N(i) \cap (T \cup F) = \varnothing$, such that every $i \in T$ has $a_i \geq 1$ or a neighbour in $T \cup F$ (this makes the triple of a solution unique); for $F = \varnothing$ the binomial coefficient is to be read as $[\,n = L + t\,]$. The degree of the polynomial is $\max |F| - 1$ over the triples occurring.

Proof. Fix $\sigma \in \Sigma_\delta$ and $h = \delta - \delta_0 \geq 0$. For a solution $x \in \mathbb{Z}^s_{\geq 0}$ of $Q(x) = h$ let S be its support and put

$$F = \{\, i \in S : \ a_i = 0 \text{ and } N(i) \cap S = \varnothing \,\}, \qquad T = S \setminus F.$$

Every $i \in T$ has $a_i \geq 1$ or a neighbour $j \in S$, and in the second case $x_i \leq x_i x_j$; each edge term is charged to at most two positions, so

$$\sum_{i \in T} x_i \;\leq\; \sum_{i \in T} a_i x_i + 2 \sum_{(i,j) \in \mathrm{Inv}(\sigma),\; i,j \in T} x_i x_j \;\leq\; 2h,$$

using that the coordinates in F contribute nothing to Q (they have $a_i = 0$ and no neighbour in S). Hence (T, x_T) ranges over a finite set, and for fixed (T, x_T, F) the solutions are exactly the vectors with $x_F \in \mathbb{Z}_{\geq 1}^F$ arbitrary and all other coordinates zero: Q does not depend on x_F. Distinct triples give disjoint sets of solutions, since $S = T \cup F$ and x_T are read off from x. By Theorem 5.3 the residuals of length n with skeleton σ correspond bijectively to the solutions with $\sum_i x_i = n - L$, and the number of $x_F \geq \mathbf{1}$ with $\sum_F x_i = n - L - t$ is $\binom{n-L-t-1}{|F|-1}$ when $F \neq \varnothing$ and $[n = L + t]$ when $F = \varnothing$. Summing over the finite set Σ_δ (Theorem 5.23) gives (12).

For $F \neq \varnothing$ the binomial coefficient $\binom{N-1}{r-1}$ agrees with the polynomial $(N-1)(N-2)\cdots(N-r+1)/(r-1)!$ for every $N \geq 1$, so each term is a polynomial in n for $n \geq L + t + 1$, while the terms with $F = \varnothing$ vanish for $n > L + t$. Thus $n_1(\delta)$ may be taken as $1 + \max(L + t)$ over all triples. By Theorem 5.23 applied to the base residual, $s \leq 8\delta_0 + 25$, so $L \leq s + 4 \leq 8\delta_0 + 29$, and $t \leq 2h = 2(\delta - \delta_0)$; hence $L + t \leq 8\delta + 29$. □

Theorem 6.4 proves Theorem 1.5 in its qualitative form. The polynomials of Proposition 5.8 are the sums (12) taken over the catalogue $\Sigma_\delta^{(22)}$ instead of Σ_δ, and the two agree for all large n exactly when no skeleton outside the catalogue contributes a triple with $F \neq \varnothing$ at defect δ: a triple with $F = \varnothing$ affects only one value of n. For $\delta \leq 8$ the sums over the catalogue were evaluated directly: they reproduce every archived value of $|\mathcal{R}_{\delta,n}|$, $n \leq 22$, and agree with the polynomials of Proposition 5.8 from $n_0(\delta)$ on. The numbers of canonical triples (T, x_T, F) by the free rank $|F|$ are

δ	3	4	5	6	7	8
$\lvert F\rvert = 3$	8	40	128	336	784	1680
$\lvert F\rvert = 2$	135	468	1374	3536	8286	17996
$\lvert F\rvert = 1$	342	1333	4230	11493	28142	63410
$\lvert F\rvert = 0$	174	891	3424	10657	28780	69863

and the leading coefficient of P_δ is half the number of triples with $|F| = 3$, as (12) predicts. Here F is the zero-cost independent free set in (12); it is not the large-coordinate set used in the older C9 stratification. The two stratifications give the same residual polynomials but different lower-rank cell counts. The rank-zero counts at defects 6 and 7 include, respectively, the four solutions from the decreasing skeleton 4321 and the single solution from 54321. These short decreasing terms have base length five and contribute only at $n = 5$; they do not affect the residual table or polynomial identities used for $n \geq 6$.

6.4. **The degree is at most two.**

Lemma 6.5. *In a* 1324*-avoiding permutation, two entries with the same set of inverted partners lie in a common maximal run of consecutive positions carrying consecutive values.*

Proof. Let the positions $i < j$ have the same inversion neighbourhood. They are not inverted with each other, so $\pi_i < \pi_j$. An entry at a position strictly between i and j with value outside (π_i, π_j) would be inverted with exactly one of the two, and so would a value strictly between π_i and π_j at a position outside (i, j). Hence the positions $i, \ldots, j$ carry exactly the values $\pi_i, \ldots, \pi_j$. If two of the intermediate entries were inverted, they would form, with π_i and π_j, an occurrence of 1324. So the positions $i, \dots, j$ carry the values $\pi_i, \dots, \pi_j$ in increasing order, which is a run. ☐

Theorem 6.6. *For every $\delta \geq 1$ the polynomial of Theorem 6.4 has degree at most two. Equivalently, no triple (T, x_T, F) in (12) has $|F| \geq 4$.*

Proof. Suppose a triple with $|F| \geq 4$ occurs for some $\sigma \in \Sigma_\delta$, and let $\pi = \sigma[\ell + x]$ be a residual of that type with $x_i \geq 1$ for all $i \in F$. Choose four positions $i_1, \dots, i_4 \in F$ and one entry y_r of the block i_r for each. In π, the inversion degree of y_r is $\sum_{j \in N(i_r)} (\ell_j + x_j) = a_{i_r} + 2 = 2$, because $N(i_r) \cap S = \varnothing$. The four entries are pairwise non-inverted, since F is an independent set of the inversion graph of σ and entries in non-inverted blocks are not inverted. Their inversion neighbourhoods are pairwise distinct: equal neighbourhoods would put two of them in one run by Lemma 6.5, that is, in one block.

Let U consist of the four entries and their at most eight neighbours, so $|U| \leq 12$, and let τ be the pattern of π on U. Then τ avoids 1324, and since the inversion graph of a pattern is the induced subgraph, the four entries are still pairwise non-inverted in τ, still have inversion degree exactly two, and still have pairwise distinct neighbourhoods. This contradicts the following statement, verified by exhaustive enumeration of the 29 961 493 permutations in $\mathrm{Av}_m(1324)$, $m \leq 12$ (Section 8, check C8):

> no 1324-avoiding permutation of length at most 12 has four pairwise non-inverted entries of inversion degree two with pairwise distinct inversion neighbourhoods.

Hence $|F| \leq 3$ for every triple, and the degree of (12) is at most two. ☐

The maximum in the finite statement is attained: from length 7 on there are permutations with three such entries, the first being 3725416, and the number of reduced indecomposable 1324-avoiders of length m carrying three is $4, 16, 24, 40, 92, 200, 484, 1176$ for $m = 7, \dots, 14$; none carries four up to $m = 14$, which is consistent with, but not needed for, Theorem 6.6. We remark that the condition on the neighbourhoods cannot be dropped from the structural description: without it (that is, counting pairwise non-inverted entries of degree at most two in σ without reference to $D(\sigma)$) sets of size five occur among the skeletons of the catalogue.

6.5. **The base inequality.** Proposition 6.3 bounds δ_0 by δ, but it says nothing about the length of σ; a bound on the length of the base residual $\rho = \sigma[\ell]$ would supply one, since $|\sigma| \leq L = |\rho|$. A residual satisfies $\mathrm{inv} = 2n - 7 + \delta$, so a bound $|\rho| \leq \delta(\rho) + c$ is the same thing as a lower bound on $\mathrm{inv}(\rho)$ that is linear in $|\rho|$ with slope three. The data single out $c = 8$.

Conjecture 6.7 (base inequality)**.** *Every base residual ρ satisfies $|\rho| \leq \delta(\rho) + 8$; equivalently* $\mathrm{inv}(\rho) \geq 3|\rho| - 15$.

The inequality was tested on every base residual of length at most 20 and defect at most 12, and on every base residual of length at most 22 and defect at most 10 (check C11); it holds throughout, and it is sharp. Equality occurs exactly four times, in two pairs exchanged by inversion:

$$4\,3\,5\,8\,1\,2\,9\,11\,10\,6\,7, \quad 5\,6\,2\,1\,3\,10\,11\,4\,7\,9\,8 \qquad (\delta = 3,\ |\rho| = 11),$$

$$4\,3\,5\,7\,2\,8\,10\,1\,11\,13\,12\,9\,6, \quad 8\,5\,2\,1\,3\,13\,4\,6\,12\,7\,9\,11\,10 \qquad (\delta = 5,\ |\rho| = 13).$$

The maximum of $|\rho| - \delta(\rho)$ over the base residuals of defect δ in that range is

$$8,\ 7,\ 8,\ 7,\ 7,\ 7,\ 6,\ 6,\ 6,\ 5 \qquad (\delta = 3, \dots, 12),$$

so the bound is attained only at $\delta = 3$ and $\delta = 5$, and the margin grows from $\delta = 9$ on.

The conjecture has an inductive form that asks only for one well-placed entry at a time.

Proposition 6.8. *Suppose that every base residual ρ with $|\rho| \geq 14$ and $\delta(\rho) \leq |\rho| - 9$ has an entry x of inversion degree at least three such that $\rho \setminus x$ is again a base residual. Then Conjecture 6.7 holds.*

Proof. We show $\delta(\rho) \geq n-8$ for every base residual ρ of length n, by induction on n. For $n \leq 13$ this is the finite check C11: a base residual violating it would have $\delta(\rho) \leq n - 9 \leq 4$, and every base residual of length at most 20 and defect at most 12 was examined. Let $n \geq 14$ and assume the statement for length $n-1$. If $\delta(\rho) \geq n-8$ there is nothing to prove, so suppose $\delta(\rho) \leq n-9$ and let x be an entry as in the hypothesis. Then $\rho \setminus x$ is a base residual of length $n-1$, so $\delta(\rho \setminus x) \geq (n-1)-8$ by the induction hypothesis, and the deletion rule of Proposition 5.4 gives

$$\delta(\rho) = \delta(\rho \setminus x) + D(x) - 2 \ \geq \ \big((n-1)-8\big) + 3 - 2 \ = \ n-8,$$

contradicting $\delta(\rho) \leq n-9$. Hence $\delta(\rho) \geq n-8$ in every case, which is Conjecture 6.7. ☐

Conjecture 6.9. *Every base residual ρ with $|\rho| \geq 14$ and $\delta(\rho) \leq |\rho| - 9$ has an entry of inversion degree at least three whose deletion is again a base residual.*

What Proposition 6.8 buys is that Conjecture 6.7, a global inequality about a set of permutations of unbounded length, follows from Conjecture 6.9, a local statement about one permutation at a time. The local statement was tested on all 114 356 base residuals of length at most 20 and defect at most 12 (check C11). Exactly 352 of them have no entry of inversion degree at least three whose deletion is again a base residual, and every one of the 352 has length at most 13: from length 14 on every base residual in the range has such an entry, with no hypothesis on the defect at all. The least inversion degree that works is three in 82.7 per cent of the cases and never exceeds six.

7. Uniform coefficients and expanded regions

We now prove the headline results. This section is deliberately placed after the skeleton and polynomiality machinery: the terms *paid*, *free* and *rank* refer to the canonical cells in the proof of Theorem 6.4. For $r \geq 1$, a rank-r cell with minimum length m contributes

$$\binom{n-m+r-1}{r-1} \tag{13}$$

once it is active. Thus only rank-three cells contribute to the coefficient of n^2, and each contributes $1/2$.

The algebra behind this assertion is short and we record it explicitly. Fix a skeleton with its unique lower block vector ℓ, nonnegative coefficients a_i, and eligible graph E. Its excess variables satisfy

$$Q(y) = \sum_i a_i y_i + \sum_{ij \in E} y_i y_j = h, \qquad y_i \geq 0. \tag{14}$$

From a solution y, let $F(y)$ consist of the positive coordinates i for which $a_i = 0$ and every neighbour has coordinate zero, and set $x_i = 0$ on $F(y)$ and $x_i = y_i$ elsewhere. The set $F(y)$ is independent and removing its coordinates changes neither Q nor the paid status of another positive coordinate. Thus $Q(x) = h$. Every positive x_i has $a_i \geq 1$ or a positive paid neighbour, so some term of $Q(x)$ is at least x_i and $1 \leq x_i \leq h$.

Conversely, take a bounded vector x with $Q(x) = h$ such that every positive x_i has $a_i \geq 1$ or a positive neighbour. Choose an independent subset F of the vertices with $x_i = a_i = 0$ and no positive neighbour, choose arbitrary $t_i \geq 1$ on F, and put $y = x + t_F$. Every newly added linear and quadratic term vanishes, so $Q(y) = h$, and the preceding extraction recovers exactly (x, F). The two operations are inverse. If $b = \sum_i \ell_i + \sum_i x_i$ and $r = |F|$, the cell contributes

$z^{b+r}/(1-z)^r$; for $r \geq 1$ its actual coefficient is $\binom{n-b-1}{r-1}$ on $n \geq b+r$, and $r = 0$ is a point mass. Together with the separately proved finiteness of the skeleton set, $a_i \geq 0$, and $r \leq 3$, this proves the paid/free decomposition and the rank-to-degree bridge without further finite computation.

7.1. **The rank-three core series.** For clarity we separate the finite classification from the canonical-cell bijection. Let $\mathcal{M}$ be the class of residuals with exactly three inversion-degree-two maximal increasing consecutive runs, all of length two; mark the first point of each run. This definition does not use paid/free coordinates. Retaining the marked points and their neighbours gives one of 36 marked seven-point cores. For each core the possible middle cells are classified by their full relative position, adjacency to the three runs, and 1324-avoidance constraints. There are 196 finite occupancy representatives. Exactly 100 pass the residuality, deletion and marking tests, and their rational defect series is

$$\frac{8q^3(1+q)}{(1-q)^2}. \tag{15}$$

The reduction treats occupancies 0, 1 and at least 2 separately, so (15) is an all-parameter finite-state sum rather than an interpolation from the first few defects.

Lemma 7.1 (exclusion of additional degree-two runs)**.** *Suppose a* 1324*-avoider has three pairwise noninverted distinct maximal increasing consecutive runs of length at least two, whose entries have inversion degree two. Then these are all its degree-two runs.*

Proof. Choose one marked entry from each run. Their neighbourhoods are distinct by Lemma 6.5. Retain the marks and all their neighbours. The finite core classification gives four unmarked neighbours, all in distinct singleton runs: their incidence signatures into the three marks are the four distinct vertex signatures of a four-vertex path. The same explicit list of 36 cores verifies that each unmarked vertex has degree at least two within the retained core. This additional finite predicate is included in the core certificate. Each such vertex also neighbours an unretained companion of at least one mark; the companion lies in the same run as that mark, has the same external neighbourhood, and is not one of the retained core vertices. Thus the unmarked vertex has degree at least three in the original permutation.

Any degree-two entry in another run therefore lies outside the retained neighbour set and is noninverted with all three marks. Its neighbourhood differs from theirs, again by Lemma 6.5. These four entries would be pairwise noninverted, degree two, and have distinct neighbourhoods. The bounded restriction and C8 argument in the proof of Theorem 6.6 exclude precisely this configuration at every length. □

Remark 7.2. The application below is to a residual of length at least eight: the three marked runs have six entries and degree two supplies at least two further entries. Their neighbourhoods are pairwise distinct in the canonical-cell construction, so those runs are three distinct degree-two classes and Lemma 5.17 already shows that they are all the degree-two classes in this application. Lemma 7.1 is retained in its more general, nonresidual form because it is also the condition used by the finite certificate's acceptance predicate and extension argument.

The two outer decorations are independent, and their decoding is unique. For a 132-avoider the Lehmer code is weakly decreasing: decomposing at its maximum gives the code $(c(\alpha) + b), b, c(\beta)$, where $b = |\beta|$, and the converse follows because weak decrease leaves the b smallest unused values for β. Thus the nonzero code is a partition λ, and its least possible padded length is $\max_i(\lambda_i + i)$. An isolated inversion-graph vertex before a later inversion would begin a 132, so all isolated vertices form a terminal increasing-maxima suffix. Removing that entire suffix gives one and only one nonisolated kernel for each partition, including the empty kernel for the empty partition. Its inversion series is therefore $P(q)$. Reverse-complement gives the corresponding unique 213 kernel and isolated prefix on the upper side.

In the normalized marked object, the lower isolated suffix and upper isolated prefix are already fixed by the adjacent marked runs, so the remaining two kernels are extracted independently. Every vertex of a nonempty kernel has positive internal degree and two fixed external incidences. It therefore has total degree at least three and creates no new degree-two class. Adding or removing either kernel preserves residuality and the three marked neighbourhoods by the outer twin-path lemma. Each added point has exactly two old neighbours, so its two cross inversions cancel its contribution to $-2n$ in the defect; the defect increment is exactly the internal inversion number of the kernel. This proves, with unique decoding and no multiplicity, the factor $P(q)^2$.

We next establish the normalization without assuming it in the definition or enumeration of $\mathcal{M}$. In a canonical rank-three cell let F be its free support and let $\ell_i \in \{1, 2\}$ be the intrinsic lower block lengths. The three free blocks have degree two, are pairwise noninverted, and have singleton neighbour blocks, by the marked-core incidence lemma. First put every selected block at length two, retaining the other block lengths. This operation is allowed even if some $\ell_i = 2$: the new length still satisfies $b_i \geq \ell_i$, including the required $b_i \geq 2$ for $i \in D(\sigma)$. For $i \in F$, the linear coefficient is zero and every neighbour has zero excess. Thus replacing its excess by $2 - \ell_i \geq 0$ leaves the quadratic defect form unchanged. The residual criterion of Theorem 5.3 still holds. The reduced skeleton is unchanged, since all block lengths remain positive. Lemma 7.1 shows that the resulting permutation belongs to $\mathcal{M}$, with exactly the three specified degree-two runs. When $\ell_i = 2$ the new free excess is zero; this construction need not remain a point of the original rank-three cell, since only membership in $\mathcal{M}$ is required for the contradiction.

The independent finite enumeration of $\mathcal{M}$ gives marked lower vector $(1, 1, 1)$ for every accepted representative. This property extends to every allowed middle tail and outer kernel. Enlarging a middle run beyond length two leaves the reduced skeleton unchanged. An outer kernel attaches each new point to two unmarked anchors; removing any selected run therefore leaves these points attached. A marked quotient vertex whose deletion was connected cannot acquire a disconnecting deletion, and an originally nonextremal marked vertex cannot become extremal by adding outer points. These are the normalization predicates recorded in the accompanying extension lemmas and finite certificate.

If any original selected lower length had been two, the permutation just constructed would have the same reduced skeleton and the same intrinsic lower vector, yet would have marked lower length one by this enumeration and extension argument. This contradiction proves $\ell_i = 1$ on F. Consequently setting the positive free excesses to one gives the unique length-two representative of each rank-three cell.

Conversely, contract all maximal increasing consecutive runs of an element of $\mathcal{M}$. The skeleton and block lengths are unique. Its three marked runs are pairwise noninverted: two inverted length-two degree-two runs would form a separate four-vertex component, contrary to residuality and the presence of the third run. The core normalization gives lower length one on each marked block, hence excess one there. In the canonical extraction, a further positive zero-cost isolated excess would be another degree-two run, contradicting the definition of $\mathcal{M}$. The free support is therefore exactly the three marked blocks. Extracting the paid vector and restoring arbitrary positive free excesses are inverse operations. This proves the claimed bijection. The class $\mathcal{M}$ and its finite enumeration were established before this identification, so the argument does not assume the rank-three formula it proves.

Consequently

$$\sum_{\delta \geq 0} N_{3,\delta} q^{\delta} = \frac{8q^3(1+q)}{(1-q)^2} P(q)^2, \tag{16}$$

where $N_{3,\delta}$ is the number of canonical rank-three cells at defect δ.

Proof of Theorem 1.1. By Theorems 6.4 and 6.6, every fixed-defect residual count is eventually a polynomial of degree at most two. Equation (13) shows that a rank-three cell contributes $1/2$ to its quadratic coefficient and that cells of lower rank contribute zero. Hence $A_\delta = N_{3,\delta}/2$. Dividing (16) by two gives (5). ☐

The conclusion is exactly a leading-coefficient identity. In particular it does not determine B_δ or C_δ, enumerate all residuals, prove a uniform stabilization onset, or imply the unrestricted inequality (1). For comparison, Linusson–Verkama define

$$\begin{aligned} b_{r,n} = &\big(a(n+3,2n+r-3) - a(n+2,2n+r-3)\big) \\ &- \big(a(n+2,2n+r-4) - a(n+1,2n+r-4)\big) \\ &- \big[\big(a(n+2,2n+r-5) - a(n+1,2n+r-5)\big) \\ &\qquad\qquad - \big(a(n+1,2n+r-6) - a(n,2n+r-6)\big)\big]. \end{aligned}$$

Their Conjecture 25 says that, for fixed r, this is a constant b_r once $n \geq 10 + r$, and that

$$\sum_{r \geq 0} b_r x^r = 2(1-x^2)(2-x^2)P(x)^2.$$

This is the formula in the published version; the older arXiv display had a typographical error that is not a result of the present work.

The relation can be made exact in defect notation. Put

$$\Delta_\delta(m) = a(m+1, 2m-7+\delta) - a(m, 2m-7+\delta).$$

By (4), $\Delta_\delta(m)$ is the size of the complement of $\operatorname{im}(f \sqcup g)$ at that parameter minus $|\mathcal{R}_{\delta,m}|$, and direct substitution gives

$$b_{r,n} = \Delta_r(n+2) - \Delta_{r+1}(n+1) - \Delta_r(n+1) + \Delta_{r+1}(n).$$

Thus Conjecture 25 concerns a repeated difference of the full complement minus residual counts, whereas Theorem 1.1 determines only the quadratic coefficient of the residual polynomial. The proof here neither assumes nor proves the full Linusson–Verkama conjecture.

There is also a useful length refinement. For a decreasing partition λ, set

$$m(\lambda) = \max_i(\lambda_i + i), \qquad m(\varnothing) = 0, \quad K(q,z) = \sum_\lambda q^{|\lambda|} z^{m(\lambda)}.$$

The elementary inequality $m(\lambda) \leq |\lambda| + 1$ is sharp only for a single row or a single column (with the usual coincidence at weight one). If

$$M(q,z) = \frac{2q^3(1+q)z^{10}(1+z)^2}{(1-qz)^2},$$

then the exact rank-three contribution is

$$H_3(q,z) = \frac{M(q,z)K(q,z)^2}{(1-z)^3}. \tag{17}$$

Proposition 7.3 (rank-three onset). *The contribution of the canonical rank-three cells agrees with its eventual quadratic polynomial from $n = 10$ at defect 3, from $n = 12$ at defect 4, and from $n = \delta + 9$ at every defect $\delta \geq 5$. These are not onsets for the complete residual count.*

Proof. The positive middle numerator in (17) has $m - \delta \le 9$, and its two geometric extensions preserve this difference. The partition inequality above adds at most one unit for each nonempty outer kernel, so a normalized minimum has $m \le \delta + 11$. Equality occurs for every $\delta \ge 5$ by taking a $q^3 z^{12}$ middle term, one weight-one outer kernel and a row partition of weight $\delta - 4$ at the other end. Directly from the same numerator, the maxima are 12 and 14 at defects 3 and 4. A rank-three cell of minimum m contributes $\binom{n-m+2}{2}$, so it reaches its polynomial at $n = m - 2$. At $n = m - 3$ each cell of maximal m is absent while its polynomial value is one; all shorter cells have stabilized, so cancellation within rank three is impossible. The stated onsets follow. Lower-rank cells may have later onsets or may cancel this last discrepancy, which is why the proposition does not address the full residual count. □

7.2. **The fixed strip through defect thirteen.** The sharp skeleton bound required here holds at every defect:

$$|\mathrm{sk}(\pi)| \le 2\delta + 3 \qquad (\pi \in \mathcal{R}_{\delta,n}). \tag{18}$$

Indeed the result is already proved for $\delta \le 10$. For $\delta \ge 11$ and $n \le 22$ it is automatic. If $n \ge 23$, choose the admissible deletion supplied by Theorem 5.19; writing $s - s' = t$, D for the degree of the deleted point and $\delta' = \delta - D + 2$, induction gives $s' \le 2\delta' + 3$ and admissibility gives $t \le 2D - 4$. Their sum is $s \le 2\delta + 3$. Notice that this proves a length bound, not completeness of an already truncated catalogue. The catalogue conjecture was proved only through defect ten; the defects below instead use this all-defect length bound together with a fresh exhaustive generator and its inversion cutoff.

Within the finite domains permitted by (18), exhaustive skeleton generation and the canonical paid/free sum give

δ	catalogue size	eventual $\lvert\mathcal{R}_{\delta,n}\rvert$	onset
11	68 251	$5976n^2 - 77520n + 222980$	20
12	112 057	$10648n^2 - 150106n + 490423$	21
13	179 071	$18444n^2 - 279621n + 1010891$	22

The skeleton generator uses $s \le 2\delta+3$ and $\mathrm{inv}(\sigma) \le 2s-7+\delta$. Its three runs used $(\delta, N, K_{\max}) = (11, 25, 54), (12, 27, 59), (13, 29, 64)$ and returned the catalogue sizes displayed above. The exact SHA-256 values are recorded here (source first, then output):

$\delta = 11$ source:
`e4ff49d7ebe507b699cab96ba57c931907d02f4985d246a90d75d1be9a8f4795`
$\delta = 11$ output:
`344c051aa3fb7b86bac61976255f7af463a782a4460e2b2818b82ab7df5b71b5`
$\delta = 12, 13$ source:
`cfa2c74b03ac105024fdd16a99276b4371acd6c624a4b089ad2fdcf2b316b741`
$\delta = 12$ output:
`8f20246d68ae60d355115d8a4ad78d070ca46983ddbf140dbd39a5475e51f4c2`
$\delta = 13$ output:
`290c2a05c3e9b7023f214be1cb2127df8db64612cb2d85f2c2c8cc136bb47f27`.

The length theorem and exhaustive generation establish catalogue completeness. The canonical sum then enumerates bounded paid vectors satisfying the quadratic defect equation and independent free supports, so the displayed rational functions are exact sums rather than fitted polynomials. A separate replay for defects 12 and 13 reproduced the denominator, numerator, polynomial, onset and values through $n = 60$ from the supplied member files. Its defect-12 and defect-13 output hashes are

`71833be9a7e1d8c7c581eae03535015b180d15b480e9e20148e1f661d1ec41e0`
`5d62473bb07c121eb43657e4c91a5d988e0b427b0cfaac156079c664140eec9c`.

This replay checks the canonical summation; it is not an independent enumeration and does not by itself establish catalogue completeness.

We record the overlap correction used at the remaining boundary points. Let $\mathcal{C}_{\delta,n} = \mathrm{Av}^k_{n+1}(1324) \setminus \mathrm{im}(f \sqcup g)$, where $k = 2n - 7 + \delta$, and let $U_{\delta,n}$ be the union of the four elementary classes with first or second entry maximal, or last entry equal to 1 or 2. Put $E_\delta(n) = |U_{\delta,n}| = E(n,k)$, with $E(n,k)$ given by (6). Lemma 3.1 gives $U_{\delta,n} \subseteq \mathcal{C}_{\delta,n}$. Also $\mathcal{R}_{\delta-2,n+1} \subseteq \mathcal{C}_{\delta,n}$: a residual is not in $\mathrm{im}(g)$ because it is indecomposable, and every member of $\mathrm{im}(f)$ has an extremal deletion with at least three direct-sum components, whereas all four extremal deletions of a residual are indecomposable. If $\mathrm{ind}(n,j)$ denotes the number of indecomposable members of $\mathrm{Av}^j_n(1324)$, deletion of the distinguished extremal entry and inversion give the intersection upper bound

$$|U_{\delta,n} \cap \mathcal{R}_{\delta-2,n+1}| \leq J_\delta(n), \qquad J_\delta(n) = 2\,\mathrm{ind}(n,k-n) + 2\,\mathrm{ind}(n,k-n+1).$$

The symbol J_δ denotes this upper bound; equality is not asserted. The indecomposable formula of [8, Theorem 1.3] is

$$\mathrm{ind}(n, n-1+m) = 4S(m)n - c_m, qquad S(m) = \sum_{j=0}^{m} p_2(j), \qquad n \geq m+6.$$

For the present two terms $m = \delta - 6, \delta - 5$, so both formulas apply when $n \geq \delta + 1$. The needed values are

$$(S(6), c_6) = (139, 3344), \quad (S(7), c_7) = (249, 6434), \quad (S(8), c_8) = (434, 11974).$$

Inclusion–exclusion and (4) therefore give

$$a(n+1,k) - a(n,k) \geq E_\delta(n) + |\mathcal{R}_{\delta-2,n+1}| - J_\delta(n) - |\mathcal{R}_{\delta,n}|. \tag{19}$$

For defect 11, the standard complement bound dominates the displayed polynomial from $n = 25$ onward, while archived total counts cover $7 \leq n \leq 24$ and the source is empty for $n \leq 6$. At defect 12 the standard comparison leaves $n = 26$; $n = 25$ needs no analytic repair because then $k = 55$, within the archived table $n \leq 26$, $k \leq 55$. The displayed indecomposable formulas give $J_{12}(n) = 3104n - 19556$, hence $J_{12}(26) = 61148$, and (19) gives the positive margin 1 242 312. At defect 13, $J_{13}(n) = 5464n - 36816$, and the same inequality gives positive margins 258 828, 1 031 781 and 2 148 286 at $n = 25, 26, 27$; the ordinary analytic tail starts at $n = 28$, and archived total counts cover $n \leq 24$. Together with Corollary 5.20, these exact comparisons prove the fixed-strip assertion of Theorem 1.2.

7.3. **Square-root strips.** For every $m \geq 2$,

$$p_2(m) \geq \frac{1}{2} \cdot 5^{\sqrt{2m}-4}. \tag{20}$$

To prove it, put $s = \lfloor \sqrt{m/2} \rfloor$ and $r = s - 1$. For each part $1, \ldots, r$ choose independently a red and a blue multiplicity from $\{0, 1, 2, 3, 4\}$. Complementing every multiplicity pairs total weights W and $4r(r+1) - W$, so at least half of the 5^{2r} choices have $W \leq 2r(r+1)$. Add one red part $L = m - W$. Since $m \geq 2(r+1)^2$, we have $L > r$, making L uniquely recoverable. This proves (20).

The structural estimates also give the uniform residual bound

$$|\mathcal{R}_{\delta,n}| \leq M_\delta n^2, \qquad M_\delta = 2^{36}(2\delta+4)^4 \cdot 68^\delta \qquad (\delta \geq 1,\ n \geq 10), \tag{21}$$

as follows. For a base of defect r, the sharp skeleton bound gives $s \leq 2r + 3$, and inversion-sequence counting gives fewer than $31 \cdot 66^r$ candidate skeletons at the terminal length. Summing over lengths $2 \leq s \leq 2r + 3$ and including the exceptional skeleton 21 costs at most a further factor $2r+4 \le 2\delta+4$. If $h=\delta-r$, a canonical paid vector has mass at most $h+3$; padding to $2r+3$ coordinates bounds their number by $\binom{2r+h+6}{h+3}$, and there are at most $(2\delta+4)^3$ free supports. Together these are the four factors of $2\delta+4$ in (21). For

$$S_\delta = \sum_{r=1}^{\delta} 66^r \binom{2r+\delta-r+6}{\delta-r+3},$$

nonnegative coefficient extraction at $z=1/68$ gives $S_\delta \le C68^\delta$, where $C = 68^3(68/67)^4 \cdot 4488$; the exact integer inequality $31C < 2^{36}$ then bounds the number of canonical types by the coefficient in (21). Each type contributes at most n^2 at length n. To pass from the residual bound to monotonicity, use the single complement class in the proof of Theorem 5.35. For $n \ge \delta$ it gives

$$a(n+1,k) - a(n,k) \ge p_2(n+\delta-7) - c(\delta) - |\mathcal{R}_{\delta,n}|,$$

where $c(\delta) = 4p_2(\delta-6) + 6\sum_{i\le\delta-7} p_2(i)$. The elementary encoding of a partition by a composition gives $p(j) \le 2^j$, hence $c(\delta) \le (6\delta+4)(\delta+1)2^\delta < M_\delta$. Together with (21), and using $n \ge 2$, this shows that for positive defect and $n \ge \max\{10,\delta\}$ the sufficient condition

$$p_2(n+\delta-7) > 2M_\delta n^2 \tag{22}$$

implies strict monotonicity.

For the strongest explicit strip set $t=\sqrt{n} \ge 1024$ and $1 \le \delta \le \lfloor t/2 \rfloor$. The integer inequalities $\sqrt{2} > 7/5$, $5^{16} > 2^{37}$ and $68^8 < 2^{49}$ give

$$\log_2 p_2(n+\delta-7) > \frac{259}{80}t - \frac{1079}{80},$$
$$\log_2(2M_\delta n^2) < 41 + 8\log_2 t + \frac{49}{16}t.$$

Their difference is

$$g(t) = \frac{7}{40}t - \frac{4359}{80} - 8\log_2 t.$$

Here $g(1024) = 3577/80 > 0$ and $g'(t) > 7/40 - 16/t > 0$. This proves the $n \ge 2^{20}$, $\sqrt{n}/2$ assertion. These parameters satisfy $n \ge \delta$, as do the two regions below.

For the $\sqrt{n}/3$ strip, a simpler bounded-multiplicity construction is enough. Put $r = \lfloor \sqrt{m}/2 \rfloor - 1$, choose independently a red and a blue multiplicity in $\{0,1,2,3,4\}$ for each part $1,\ldots,r$, and add a unique red filler part larger than r. This gives $p_2(m) \ge 5^{\sqrt{m}-4}$. If $t=\sqrt{n} \ge 512$ and $1 \le \delta \le \lfloor t/3 \rfloor$, then $m = n+\delta-7$ satisfies $\sqrt{m} \ge t-1$ and $2\delta+4 \le t$. Using $5^{16} > 2^{37}$ and $68^8 < 2^{49}$, the logarithmic margin in (22) is at least

$$g_3(t) = \frac{13}{48}t - \frac{777}{16} - 8\log_2 t.$$

Now $g_3(512) = 869/48 > 0$ and $g_3'(t) > 13/48 - 16/t > 0$, proving the $n \ge 2^{18}$ assertion.

For the $\sqrt{n}/4$ strip, choose independently red and blue subsets of $\{1,\ldots,r\}$ with $r = \lfloor \sqrt{m} \rfloor - 1$ and again add the unique large red filler. Thus $p_2(m) \ge 2^{2\lfloor \sqrt{m} \rfloor - 2}$. The logarithm of (22) follows from

$$\sqrt{m} > \frac{49}{16}\delta + \frac{41}{2} + 2\log_2(2\delta+4) + \log_2 n.$$

For $t=\sqrt{n} \ge 256$ and $\delta \le \lfloor t/4 \rfloor$, the left side is at least $t-1$ and the right side at most $49t/64 + 41/2 + 4\log_2 t$. It remains to check $15t/64 > 43/2 + 4\log_2 t$, which holds at $t=256$ and thereafter because the difference has derivative greater than $15/64 - 8/t > 0$. This proves the $n \ge 2^{16}$ assertion independently of the two larger-threshold strips.

Finally, uniformly for $1 \le \delta \le c\sqrt{n}$, equations (20) and (21) give

$$\log p_2(n+\delta-7) \ge \sqrt{2}\log(5)\sqrt{n} - O(1), \quad \log(2M_\delta n^2) \le c\log(68)\sqrt{n} + O_c(\log n).$$

The first leading coefficient is larger precisely when $c < \sqrt{2}\log(5)/\log(68)$. This proves the asymptotic assertion in Theorem 1.2; the endpoint is not claimed.

7.4. **Proof-bearing certificates.** The new theorems use finite computation at sharply delimited points. The uniform coefficient proof uses the 36 marked cores, the 196 finite occupancy states and the 100 accepted states, together with the outer kernel and canonical-normalization checks. The fixed strip uses the complete defect-$11, 12, 13$ catalogues inside the bounds $25, 27, 29$, their exact paid/free rational sums, the separate defect-$12, 13$ summation replays and the integer boundary comparisons above. The accompanying archival supplement records their sources, parameters and outputs, with hashes. Its portable quick entry point is `certificates/run_quick_verification.sh`; the portable components are `core_replay/`, `table/`, and `six_point_portable/`. Files under `certificates/accepted_sources/` are historical source records rather than portable entry points and may retain machine-specific paths. The supplement is cited below. Section 8 describes the common verification conventions. The proof dependence is on the enumerated finite statements themselves and on the all-parameter extension lemmas, never on an `ACCEPTED` label or on agreement with interpolated values.

8. Computations

Finite checks that carry proof weight. Most of the computations recorded below are supporting evidence, but twelve of them are steps of proofs, and we list them separately. Ten carry proof weight:

- C1 the membership part of Theorem 4.3. The check prints $D(\sigma)$ for each of the twenty-one skeletons, enumerates the block vectors satisfying the defect-two equation together with the lower bounds $b_i \ge 2$ on $D(\sigma)$ that Theorem 5.3 attaches to it (entries up to 14; 6061 members), and then *verifies* that every member so produced avoids 1324, is indecomposable and is not almost decomposable; nothing is filtered on the property being certified, so a member failing any of the three conditions would be reported as a failure. The same run confirms that the members of length at most 12 are exactly the archived $\mathcal{R}_{2,n}$ and number $32n - 214$, that the four extremal deletions have skeletons depending only on which blocks are singletons, of size two, or of size at least three (all 231 realizable class vectors), and that the 43 skeletons so obtained are indecomposable. The enumeration is over the families themselves, not over a bounded sample of residuals.
- C2 the two spider exclusions in step (1) of Theorem 5.27, over $\mathfrak{S}_6$ and $\mathfrak{S}_7$.
- C3 the two corona exclusions in Lemma 5.28, over $\mathfrak{S}_6$ and $\mathfrak{S}_8$.
- C4 the classification step of Theorem 5.27: the $12, 16, 20, 24$ candidates of lengths $6, 7, 8, 9$, none of them reduced.
- C7 the small-n comparisons that Theorem 1.4 ($n \le 7$) and Theorem 4.4 ($n \le 9$) read off a table rather than prove.
- C8 the finite statement used in the proof of Theorem 6.6: no 1324-avoiding permutation of length at most 12 has four pairwise non-inverted entries of inversion degree two with pairwise distinct inversion neighbourhoods. The search is exhaustive over the 29 961 493 members of $\mathrm{Av}_m(1324)$ with $m \le 12$, generated by the insertion recursion; it was carried out by three programs written independently against the same specification, which agree on the number of configurations found at every length, the maximum being three from length seven on.

C9 the last step of Theorem 5.19: for $3 \le \delta \le 10$ the triples (T, x_T, F) of (12) are enumerated over the catalogue $\Sigma_\delta^{(22)}$ and summed at every length. The sums reproduce the table of Section 6, every archived value of $|\mathcal{R}_{\delta,n}|$ with $n \le 22$, and the polynomial $P_\delta(n)$ of Proposition 5.8 at every n with $n_0(\delta) \le n \le 60$, that is in all 364 cases. The largest first length $L + t$ of a triple is $3\delta + 12$, hence at most 42, so each type contributes its full binomial from $n = 43$ on and the sum is a polynomial in n there; agreement up to $n = 60$ therefore forces $|\mathcal{R}_{\delta,n}| = P_\delta(n)$ for all $n \ge n_0(\delta)$. The enumeration is a Python program independent of the one that produced the archived counts.

C10 Proposition 5.7. For each $1 \le \delta \le 10$ every reduced indecomposable 1324-avoider σ with $|\sigma| \le \max(2\delta+3, \delta+8)$ and $\mathrm{inv}(\sigma) \le 2|\sigma| - 7 + \delta$ is generated by the insertion recursion and tested for membership in Σ_δ by the bounded search of Proposition 6.3; the skeleton 21, which the inversion bound would wrongly drop, is added by hand. The length bound is the larger of the two that the two conjectures supply, so one enumeration settles both; it exceeds $2\delta + 3$ only at $\delta \le 4$, where it is 9, 10, 11, 12. The numbers of candidates are 79, 342, 979, 2358, 5118, 10 089, 17 956, 30 860, 51 304, 83 184 for $\delta = 1, \dots, 10$, of which 2, 21, 113, 422, 1236, 2942, 6296, 12 495, 22 978, 40 269 are members, and the members are in every case exactly $\Sigma_\delta^{(22)}$ together with the three decreasing skeletons of Proposition 5.5 at the defects listed there. The enumeration is done for all ten defects by a C program, `fable_sigma_enum.c`, and independently for $\delta \le 4$ inside the certificate, where all 3758 candidates are regenerated and membership is decided by a search over $\{0, \dots, h\}^s$ pruned only on the partial value of Q; the two agree on every candidate.

C11 the base case of Proposition 6.8, together with the evidence for Conjectures 6.7 and 6.9. The base residuals of length at most 20 and defect at most 12 were listed — 114 356 of them, filling 101 cells (δ, n) — from the archived residual lists for $\delta \le 8$ and from the inversion-pruned enumerator for $9 \le \delta \le 12$, the twenty of length at most five being added by direct search, since the enumerator only lists from length six on. Every one of them satisfies $|\rho| \le \delta(\rho)+8$, with equality exactly in the four cases displayed in Section 6; in particular no base residual of length at most 13 has $\delta(\rho) \le |\rho| - 9$, which is what the induction of Proposition 6.8 needs, and the same holds over the wider range $|\rho| \le 22$, $\delta \le 10$. Exactly 352 of the 114 356 have no entry of inversion degree at least three whose deletion is again a base residual, and all 352 have length at most 13. The two scans are Python programs reading the same lists; their skeleton, $D(\sigma)$ and deletion code was checked against `fable_skeleton_reduction.py` on the skeletons of the archived catalogue, and samples of the lists were re-verified against the definitions.

C12 the finite range of Theorem 5.19 and the union bound of Remark 5.21. For each residual π and each of its n entries z the program decides whether the deletion of z is admissible, that is whether $\pi \setminus z$ is a residual and (8) holds, and records u, w and the structural split of each; the proof-bearing part is the exhaustive run over every residual with $n \le 30$ and $\delta \le 10$ described in the text, which finds an admissible deletion for every residual of length at least 10. It was run on all 776 860 residuals with $10 \le n \le 16$ and on a uniform random sample of 20 000 residuals for each $17 \le n \le 20$, drawn from the archived lists, that is on 856 860 residuals and 12.37 million pairs (π, z); the sample is fixed by the seed recorded in the program. The maxima are $u \le 6$, $w \le 3$ and $u + w \le 9$ at every length and at every defect $\delta \le 8$, and $u + w = 9$ occurs at each of $n = 10, \dots, 16$, for $6, 26, 44, 64, 52, 60, 24$ residuals respectively. The union bound $u + w \le n - 1$ fails at no length except $n = 10$, where six residuals of defect four or five have $u + w = 9 = n - 1$; all six do in fact admit three admissible deletions. The three parts of u have maxima 1, 6, 6 and the two parts of w have maxima 3, 2; the last of the three parts of u is bounded by 16 in general (Lemma 5.16), and the first, the number of cut vertices, by 1 on residuals (Lemma 5.14) and by 3 on indecomposable 1324-avoiders (Lemma 5.16, the value 3 being attained from length 5 on). The part of this check that carries proof weight is the last: (8) itself has been verified, exhaustively and without sampling, for all 33 342 781 residuals of length at most 30 and defect at most 10 (above), and it is this that closes the range $10 \leq n \leq 22$ left open by Theorem 5.19. As a check, 33 000 of the residuals were re-tested with a brute-force membership test and with the skeleton of $\pi \setminus z$ recomputed from scratch, with no mismatch. The two counts that come from 1324-avoidance rather than from residuality were obtained by a separate sweep over all 173 453 058 members of $\mathrm{Av}_m(1324)$ with $m \leq 13$, generated by the insertion recursion: over the indecomposable ones of minimum inversion degree two the maxima are 6 and 2, constant for $9 \leq m \leq 13$, while the two residual-only counts reach 3 and 4 there. The script is `fable_x2_union_bound.py` of [9], and the avoider sweep is archived beside it.

Two more are independent cross-checks of statements that are proved in full here or quoted from the literature: C5 re-derives the length-five and length-six part of Lemma 5.25, which follows from [4] and [3] with the two containments exhibited in the text, and C6 re-verifies Lemma 3.2 for all $m \leq 10$. All are exhaustive searches with no sampling, apart from the supplementary sample inside C12, whose proof-bearing part is exhaustive; C11 is run by two separate programs over the archived residual lists, and the first seven are reproduced by one self-contained program, `fable_certificates.py` of [9], whose complete output — including the list of all 43 skeletons and the realizers of each graph — is stored there as `data/fable_certificates.txt` together with its SHA-256 digest. The theorems marked *computer-assisted* in the table of Section 1 are exactly those whose proofs use C1–C4, C7–C12, together with those that inherit such a use: Lemma 5.17 through C8, Theorem 5.19 through Lemma 5.17, C9 and C12, Corollary 5.20 through Theorem 5.19, Proposition 5.7 through C10, Proposition 6.8 through C11, Theorem 4.4 through Theorem 4.3, Lemma 5.28 through C3, Theorem 5.29 through Theorem 5.27, Theorem 5.23, Corollary 5.34 and Theorem 5.35 through Theorem 5.29, Theorem 6.6 through C8, and Lemma 6.1, Corollary 6.2, Proposition 6.3, Lemma 6.5, Theorems 6.4 and 1.5 through Theorem 5.23 and C8; Proposition 5.13 inherits its first sentence from Theorem 6.4, and its second sentence uses in addition the test of (8) for all $n \leq 30$ and $\delta \leq 10$ recorded below. The programs for C8–C12 are archived with the others at [9].

The remaining computations of this section are consistency checks and exploratory evidence, and no theorem depends on them.

The enumerations. The counts $a(n,k)$ for $n \leq 16$, the numbers of indecomposable and of almost decomposable members of every $\mathrm{Av}_n^k(1324)$, and hence $|\mathcal{R}_{\delta,n}|$, were computed by a depth-first generation of $\mathrm{Av}_n(1324)$ (insert the new maximum after any 132-avoiding prefix; check the four deletions of each indecomposable node). The maps f and g were implemented from their definitions and checked, for $n \leq 10$ and $\delta \in \{0,1,2\}$, to be injective with disjoint images, to preserve 1324-avoidance and the inversion number, and to have the complement described in the proof of Theorem 1.4; the identity $f(\pi^{-1}) = f(\pi)^{-1}$ used in Lemma 3.1 was verified on all 21906 almost decomposable 1324-avoiders of length at most 9, and $f(\pi^{\mathrm{rc}}) = f(\pi)^{\mathrm{rc}}$ was found to fail, so the reverse–complement case really does need the separate argument given there. Combining the implementation with the inversion-pruned generator gives the complement of $\mathrm{im}(f \sqcup g)$ exactly on the low-inversion line: for all $8 \leq n \leq 13$ and $0 \leq \delta \leq 7$ the four classes of Lemma 3.1 were confirmed to lie in the complement and to number exactly $E(n,k)$; at $\delta = 1$ the class R3 has $4(n-7)$ elements for $n = 8, 9, 10$ and consists of the permutations $(\iota_x \ominus \iota_y) \oplus (\iota_z \ominus \iota_w)$, and at $\delta = 0$ it is empty. The same program splits the complement at $k = 2n-6$ into the four Linusson–Verkama classes and confirms, for $8 \leq n \leq 11$, that $|\mathrm{R1}| = 2p_2(n-6)$, $|\mathrm{R2a}| = |\mathrm{R2b}| = 2p_2(n-5)$

and $|\mathrm{R3}| = 4(n-7)$, the four adding up to $E(n, 2n-6)$ — the counts $(10, 20, 20, 4)$, $(20, 40, 40, 8)$, $(40, 72, 72, 12)$, $(72, 130, 130, 16)$ for $n = 8, 9, 10, 11$; this is the boundary case in which the counts of [7, Section 4] are used one step beyond their stated range. The classification of Theorem 1.3 was compared with the exhaustive list of residuals for $n \leq 11$ and with the counts $8(n-7)$ for $n \leq 16$. The twenty-one families of Theorem 4.3 were generated from their parametrizations for $n \leq 16$: every member was checked to be a residual, their union coincides with the exhaustive list of residuals for $n \leq 11$, and its size agrees with the enumerated $|\mathcal{R}_{2,n}|$ for $n \leq 16$ and with $32n - 214$ for $n \geq 10$. The finite check in the membership part of Theorem 4.3 is check C1 of the certificate described below: it enumerates, for each of the twenty-one skeletons, every block vector with entries at most 14 that satisfies the defect-two equation and the lower bounds $b_i \geq 2$ on $D(\sigma)$, verifies that each of the 6061 members so produced avoids 1324, is indecomposable and is not almost decomposable, and lists the 43 skeletons of the extremal deletions, all of them indecomposable. The residual test is applied to the members, never used to select them. The bound 14 on the block sizes is not a restriction: raising it to 20 and then to 26 leaves both the 231 realizable patterns and the 43 skeletons unchanged, the number of members examined growing from 6061 to 12 505 and 21 253. Of the 43, the 21 skeletons of $\Sigma_2^{(22)}$ arise from deleting an entry of a block of size at least two, the other 22 from deleting a singleton block. For the results of Section 5 the enumeration was redone with a second, independent program that prunes on the inversion number: since inv is nondecreasing along the insertion recursion, restricting to inv $\leq K$ makes the search polynomially small where the full search grows like 11.6^n, and the residual line $k = 2n - 7 + \delta$ lies far below the bulk. The two programs agree on every one of the 340 triples (n, k) with $n \leq 16$ and $k \leq 30$, and a full enumeration of $\mathrm{Av}_{17}(1324)$, whose 458 374 397 312 members were generated in about eighteen hours on four cores, reproduces the whole table for $n \leq 17$. The pruned program reaches $n = 24$ for $k \leq 51$ in about half an hour on one core and $n = 26$ for $k \leq 55$ in about half an hour on the workstation, and supplies the table below, the sets $\Sigma_\delta^{(22)}$ of Proposition 5.5 and the polynomials of Proposition 5.8; the streaming run with $N = 30$ and $K = 63$ described below extends the residual counts $|\mathcal{R}_{\delta,n}|$, and with them Proposition 5.8, to $n = 30$. The skeleton sets were extracted from the complete residual lists of length at most 22; no skeleton appears for the first time after length 16, and the resulting catalogue, with the shortest residual realising each skeleton, is the file `data/fable_skeleton_catalogue_n22.txt` of [9], while `data/fable_residual_counts_n22.tsv` lists $|\mathcal{R}_{\delta,n}|$ for $\delta \leq 10$ and $n \leq 22$ and reproduces the polynomials of Proposition 5.8 in all 88 cases with $n_0(\delta) \leq n \leq 22$; Theorem 5.3 was checked in two ways, first by verifying on 336597 inflations of the skeletons of length at most 7 that membership in $\mathcal{R}_{\delta,n}$ depends only on σ and on which blocks are singletons, and then by regenerating $\mathcal{R}_{\delta,n}$ from the criterion and comparing it, as a set, with the exhaustive list for $1 \leq \delta \leq 4$ and $10 \leq n \leq 16$: the two agree in all 28 cases. A second run with $N = 25$ and $K = 49$ supplies the counts needed at defect eight. The base case of Proposition 5.11 was checked by generating all 84 998 residuals of length at most 9 from the full symmetric groups. Condition (8) was tested on every residual produced by the runs with $N = 20$ and $K = 41$ and with $N = 20$, $K = 43$ restricted to defects 9 and 10, and then, in a streaming re-run that consumes the enumerator's output directly, on every residual produced by $N = 30$, $K = 63$ at defects 1 to 10, split into four runs by defect range and executed in parallel, that is on all 33 342 781 residuals of length at most 30 and defect at most 10, by trying each of the n deletions in turn; the run took about four hours on one core and found the same 74 exceptions, all of length at most 9. Lemma 5.22 was checked on 2 709 762 pairs (π, T) — all permutations of length at most 7 with all T of size at most 4, together with 400 random permutations of each of the lengths $9, 11, 13, 16, 20, 25$ with $|T| \leq 3$ — with no violation and seven cases of equality; on the same sample the unproved variant $3|T| + 3e + 1$ has no violation and 76 equalities, while the variants with $3|T| + 2e + 1$ and with $4|T| + 2e + 1$ in place of $4|T| + 4e + 1$ fail, the smallest counterexample in each case being $\pi = 1324$ with $T = \varnothing$. (The sample is fixed by the seed recorded in the program, so these counts are reproducible verbatim.) Theorem 5.23 was tested by greedy removal of the seven entries of largest inversion degree. For Proposition 5.24 the 564 634 residuals of length between 9 and 18, defect at most 8 and $\pi_1 > \pi_n$ were split into the three cases and the bounds of (i) and (ii) verified on each; the six entries $\pi_1, \pi_2, \pi_{n-1}, \pi_n$ together with the largest and the smallest middle entry leave at most $\delta - 3$ inversions in every one of them, case (iii) included. Conjecture 5.26 was checked by generating, for each $m \leq 14$ and each $\gamma \leq 6$, all indecomposable 1324-avoiders of length m with $m - 1 + \gamma$ inversions and searching exhaustively over subsets of size at most four; the finite checks inside Theorem 5.27 — that the permutations whose inversion graph is $S(2, 2, 1)$ are exactly 251364 and 314625, both containing 1324, and that no permutation has inversion graph $S(2, 2, 2)$ — were made by testing every permutation of length six and seven against the two targets; the same program reproduces the lists of [4] and [10] used in Lemma 5.25 and in the proof of Lemma 5.28, namely the two increasing oscillations of each length up to eight and the fact that among cycles only C_3 and C_4 occur, realized by 321 and 3412, and the check that none of the $12, 16, 20, 24$ indecomposable 1324-avoiders of lengths $6, 7, 8, 9$ with $m-1$ inversions is reduced was made on the same lists; for Lemma 5.28 the same isomorphism test shows that a triangle or a four-cycle with one pendant vertex at each of its vertices is the inversion graph of no permutation of length six or eight; the existence of a non-cut vertex of degree at least two was also observed directly on all 26 093 indecomposable 1324-avoiders with $4 \leq m \leq 12$ and $1 \leq \gamma \leq 6$; the five cases of the proof of Theorem 5.23, with the sets T they prescribe and the bounds they claim on $e(\pi \setminus T)$, were checked one residual at a time on the complete list of residuals of length at most 22 and defect at most 10, that is on 10 210 333 residuals, with no failure in any of the nine cases that the classification produces and with $4|T| + 4e(\pi \setminus T) + 1$ never exceeding $8\delta + 21$; and the reduction of Proposition 5.31 was tested on 49 699 residuals with $\pi_1 > \pi_n$ and $10 \leq n \leq 16$, split according to whether σ is indecomposable. Finally $a(n+1, k) \geq a(n, k)$ was verified directly for all $n \leq 25$ and all $k \leq 2n + 3$, and Lemma 5.1 was checked on all 19647 pairs (τ, b) with $|\tau| \leq 5$, blocks of size at most three and $|\tau[b]| \leq 10$. The quantities $\nu(s)$ and $\mu(s)$ of Section 5 were computed with a third program that enumerates reduced permutations directly: besides the bound on inv, it prunes on the number of adjacent pairs $(\pi_i, \pi_i + 1)$, since such a pair can only be destroyed by inserting a later value between its two entries and each remaining insertion destroys at most one of them. This reduces the search for reduced residuals of length 17 with at most 39 inversions to 4.2×10^7 nodes. All programs and the tables they produce are available at [9].

$n\backslash\delta$	1	2	3	4	5	6	7	8
6	0	1	15	40	50	45	29	14
7	2	10	48	129	205	251	254	219
8	8	34	135	345	628	934	1186	1339
9	16	70	274	745	1495	2521	3707	4912
10	24	106	433	1244	2749	5133	8475	12663
11	32	138	586	1755	4113	8283	14875	24388
12	40	170	735	2263	5483	11509	21817	38028
13	48	202	894	2795	6913	14841	28949	52306
14	56	234	1061	3375	8463	18453	36599	67454
15	64	266	1236	3995	10157	22413	45029	84098
16	72	298	1419	4655	11979	26733	54275	102486
17	80	330	1610	5355	13929	31389	64337	122602
18	88	362	1809	6095	16007	36381	75183	144438
19	96	394	2016	6875	18213	41709	86813	167954
20	104	426	2231	7695	20547	47373	99227	193150
21	112	458	2454	8555	23009	53373	112425	220026
22	120	490	2685	9455	25599	59709	126407	248582
23	128	522	2924	10395	28317	66381	141173	278818
24	136	554	3171	11375	31163	73389	156723	310734

Appendix A. The marked three-free finite certificate

This appendix supplies the finite information used in (15). It states the reduction, gives the complete list of marked cores, specifies the state encoding and acceptance predicate, and explains why the finite states represent arbitrary occupancies. The certificate is a finite exhaustive calculation; the extension from its states to arbitrary block lengths is proved separately below.

For a permutation π, write G_π for its inversion graph. A *marked three-free configuration* is a pair (π, F) in which π avoids 1324 and $F = \{f_1, f_2, f_3\}$ consists of three pairwise nonadjacent vertices of G_π, each of degree two, with pairwise distinct neighbourhoods. The order on F is its left-to-right order in π.

A.1. **Reduction to at most nine points and the 36 cores.** Put

$$U = F \cup N_{G_\pi}(F).$$

Since each mark has degree two, $|U| \leq 3 + 2 \cdot 3 = 9$. Restricting π to U preserves 1324-avoidance. It also preserves the degree and the entire neighbourhood of every mark: by definition every neighbour of a mark was retained. Pairwise nonadjacency and distinctness of the three marked neighbourhoods are therefore preserved. Thus the unrestricted problem reduces exactly to the following finite search: for each $m \leq 9$, enumerate $\mathrm{Av}_m(1324)$ and retain every marked triple with degree two, pairwise nonadjacent and pairwise distinct neighbourhoods, whose union with its neighbours is the full ground set.

Two finite enumerators implement this search in different ways. The first generates only 1324-avoiders by legal insertion of a new maximum; the second scans all permutations through length nine and tests 1324 by four nested index loops. Both return the same 36 marked objects, and every one has length seven. Consequently $|U| = 7$ and the three neighbourhoods have four vertices in their union. The avoider counts produced by the two enumerators for $m = 1, \ldots, 9$ are

$$1, 2, 6, 23, 103, 513, 2762, 15793, 94776.$$

The complete marked list follows. Positions in F are one-based. The last two columns give the number of finite occupancy representatives and the number accepted by the predicate in the next subsection. No quotient by the 12 symmetry orbits is taken here.

i	core γ_i	marked positions F_i	representatives	accepted
1	3724615	$\{1,4,7\}$	9	9
2	3274615	$\{1,4,7\}$	3	0
3	3274615	$\{2,4,7\}$	9	6
4	3724165	$\{1,4,6\}$	9	6
5	3724165	$\{1,4,7\}$	3	0
6	3274165	$\{1,4,6\}$	3	0
7	3274165	$\{1,4,7\}$	1	0
8	3274165	$\{2,4,6\}$	9	4
9	3274165	$\{2,4,7\}$	3	0
10	3257164	$\{1,3,6\}$	9	6
11	3257164	$\{2,3,6\}$	3	0
12	3251764	$\{1,3,5\}$	9	4
13	3251764	$\{1,3,6\}$	3	0
14	3251764	$\{2,3,5\}$	3	0
15	3251764	$\{2,3,6\}$	1	0
16	4257163	$\{2,3,6\}$	9	9
17	4251763	$\{2,3,5\}$	9	6
18	4251763	$\{2,3,6\}$	3	0
19	4271365	$\{2,5,6\}$	3	0
20	4271365	$\{2,5,7\}$	9	6
21	4217365	$\{2,5,6\}$	1	0
22	4217365	$\{2,5,7\}$	3	0
23	4217365	$\{3,5,6\}$	3	0
24	4217365	$\{3,5,7\}$	9	4
25	5214763	$\{2,4,5\}$	3	0
26	5214763	$\{2,4,6\}$	9	4
27	5214763	$\{3,4,5\}$	1	0
28	5214763	$\{3,4,6\}$	3	0
29	6214753	$\{2,4,5\}$	9	6
30	6214753	$\{3,4,5\}$	3	0
31	5271364	$\{2,5,6\}$	9	9
32	5217364	$\{2,5,6\}$	3	0
33	5217364	$\{3,5,6\}$	9	6
34	5314762	$\{3,4,5\}$	3	0
35	5314762	$\{3,4,6\}$	9	6
36	6314752	$\{3,4,5\}$	9	9
		total	196	100

For later use, the same list verifies one additional predicate. Every unmarked anchor has degree at least two inside the seven-point core. The minimum anchor degree equals two for 32 marked cores and three for the remaining four. This is the finite input used when excluding an additional degree-two run; it is checked directly by the portable verifier rather than inferred from the saved list.

A.2. **Grid cells, finite states, and acceptance.** Fix a listed core $\gamma = \gamma_1 \cdots \gamma_7$ and its marked set F. Index an open grid cell by (a, b) with $0 \le a, b \le 7$: a new point lies after exactly a core positions and above exactly b core values. Its neighbourhood in the core is

$$N_\gamma(a,b) = \{j \le a : \gamma_j > b\} \cup \{j > a : \gamma_j \le b\}, \qquad 1 \le j \le 7.$$

A cell is retained precisely when $N_\gamma(a,b) \cap F = \varnothing$ and inserting one point in the cell still avoids 1324. This definition fixes the coordinate encoding without reference to a drawing.

For this middle calculation the southwest outer kernel is empty and the northeast outer kernel is empty beyond its one mandatory clone point; the two nontrivial outer kernels are restored only after the finite middle sum. For every $f \in F$ there is one retained cell northeast of f whose core neighbourhood is $N_{G_\gamma}(f)$. It is the *clone cell* of f. The normalized object contains exactly one point in each of these three clone cells, so that the three marked points begin increasing runs of length two. A retained cell of core degree one must be empty by the minimum-degree property of residuals. A retained degree-two point either lies in one of the three marked runs, by equality of neighbourhoods, or would give a fourth independent degree-two class; the four-free certificate excludes the latter. Hence all nonclone degree-two cells are empty. Every remaining middle cell has core degree three, contains an increasing block, and there are at most two such cells for each marked core.

If the remaining degree-three cells are $c_1, \dots, c_r$, where $0 \le r \le 2$, encode their occupancies by

$$s = (s_1, \dots, s_r) \in \{0,1,2\}^r, \qquad s_j = 0,1,2 \quad \Longleftrightarrow \quad |c_j| = 0,1, \text{ or at least } 2.$$

For the finite representative, the last case is realized with exactly two points. Thus core i contributes 3^{r_i} representatives, and the sum over the 36 listed cores is 196. These are representatives of disjoint occupancy ranges; they are not orbit representatives under inverse or reverse–complement.

The representative permutation is constructed by inserting the three clone points and the indicated increasing blocks and then standardizing. It is accepted if and only if all three of the following hold:

(1) it avoids 1324;
(2) it is indecomposable, and deletion followed by standardization is indecomposable for each of its first, last, minimum, and maximum entries;
(3) the starts of all maximal increasing consecutive runs of length two whose entries have inversion degree two are exactly the three marked starts.

Thus the test is the definition of the normalized residual class, with the "exactly three" condition included; Lemma 7.1 excludes an additional degree-two run, while the cell classification above excludes a longer degree-two occupancy inside a finite representative. Direct evaluation accepts 100 of the 196 representatives. In fact all 196 pass the avoidance and marked-run tests; the rejected 96 fail residuality.

Let d be the defect of a representative and let r now denote the number of its degree-three cells encoded by 2. The multiplicities of the 100 accepted representatives are

(d, r)	(3,0)	(4,0)	(4,1)	(5,0)	(5,1)	(5,2)	(6,0)	(6,1)	(6,2)	(7,1)	(7,2)	(8,2)
multiplicity	8	16	8	10	20	2	2	16	6	4	6	2

and the entries sum to 100.

A.3. **From finite representatives to arbitrary occupancies.** Consider a degree-three middle cell whose finite code is 2. Increasing its occupancy from two to $t \ge 2$ inflates the same quotient vertex by an increasing run. The quotient skeleton and all singleton versus nonsingleton statuses remain unchanged; the block remains of inversion degree three and cannot merge with a degree-two clone run. Increasing inflation preserves 1324-avoidance. The residual deletion criterion depends on the same quotient skeleton, so it is preserved in both directions. Each added entry creates three inversions and increases the length by one, and hence raises $\operatorname{inv} - 2n + 7$ by one. A representative of defect d with r such tails therefore contributes exactly

$$\frac{q^d}{(1-q)^r}.$$

The three occupancy ranges 0, 1, and ≥ 2 are disjoint and exhaustive, so this is an all-occupancy conclusion, not interpolation from bounded defects.

Putting the 100 terms over the common denominator $(1-q)^2$ gives

$$\begin{aligned}(8q^3+16q^4+10q^5+2q^6)(1-q)^2 \\ +(8q^4+20q^5+16q^6+4q^7)(1-q) \\ +(2q^5+6q^6+6q^7+2q^8)=8q^3+8q^4.\end{aligned}$$

Hence the middle/core series is

$$\frac{8q^3(1+q)}{(1-q)^2}.$$

It remains to restore the two outer cells. In the southwest cell, a 132-avoider has weakly decreasing Lehmer code; after its terminal isolated increasing maxima are removed, the remaining kernel is uniquely encoded by a partition and has inversion generating function $P(q)=\prod_{j\geq 1}(1-q^j)^{-1}$. Conversely every partition has a unique least-length Lehmer decoding, so this is a bijection rather than a stable count. Reverse–complement and inverse give the corresponding northeast 213 kernel and a second factor $P(q)$.

Each outer quotient block is joined to the same two unmarked old anchors and to no other old vertex. Those anchors already have an independent path through the doubled marked run. More explicitly, delete any old vertex x. If neither anchor is deleted, the surviving member of the doubled run still connects them and every outer block attaches to that old component; if one anchor is deleted, every outer block remains attached through the other. Consequently the connected components on the old vertices after deleting x are unchanged by adding or removing the outer blocks. Each new outer vertex is itself noncut because the old connected graph survives its deletion. A nonextremal marked vertex cannot become extremal upon adding outer points; if an old extremum is replaced, the new extremum is noncut and the replaced old extremum lies in its doubled marked run. This component equivalence proves preservation of the residual criterion and marked lower vector in both directions. Its two cross inversions per added point cancel the $-2n$ shift, leaving precisely the internal inversion number of the kernel as the defect increment. The outer decorations are independently and uniquely extracted, and therefore multiply the middle/core series by $P(q)^2$.

Consequently the generating function for canonical rank-three cells is

$$\sum_{\delta\geq 0} N_{3,\delta}q^\delta=\frac{8q^3(1+q)}{(1-q)^2}P(q)^2.$$

Since a rank-three cell contributes leading coefficient 1/2 and cells of rank at most two contribute none, this certificate gives the finite part of

$$\sum_{\delta\geq 0} A_\delta q^\delta=\frac{4q^3(1+q)}{(1-q)^2}P(q)^2.$$

A.4. **Reproducibility boundary.** The accompanying archival supplement contains the two C enumerators, the explicit core list, a standard-library verifier that reconstructs the grid cells and all 196 states, and a one-command temporary-directory replay. The replay establishes the finite outputs stated in this appendix. The all-parameter conclusions additionally use the mathematical inflation and outer-kernel arguments above, together with the manuscript's canonical paid/free decomposition and four-free theorem. Saved logs from the larger defect-11–13 catalogue runs are frozen separately; their hashes and exact generator parameters are recorded, but those expensive enumerations are not part of the quick replay.

Use of AI assistants

I used AI assistants — Anthropic's Claude, OpenAI's ChatGPT, Moonshot's Kimi and Google's Gemini — as tools in this work. They helped me write and run the enumeration, verification and analysis programs, search the literature, and prepare the manuscript, and they were also set against one another as adversarial referees, each asked to find errors in the others' arguments and drafts. I set the direction of the project, provided key ideas, decided what to pursue and what to discard, checked the arguments, and take responsibility for everything stated here. The finite checks supporting the results are reproduced by the programs and data in the cited archive and archival supplement.

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