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Theory of the Fully Ionized Plasma Column with External Particle Production

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The density and temperature distribution in a fully ionized cylindrical plasma with external particle production and without volume recombination are derived from the transport equations for the conservation of mass, momentum, and energy. It is found that the Schottky approach using a constant temperature and a simplified boundary condition is not applicable to this problem. The general calculation, which includes temperature variation and utilizes a more general boundary condition, produces results which cannot be represented in an analytical form but are determined by machine solutions. The results show density and temperature distributions which vary strongly with the discharge parameters. The general features of these distribution functions are discussed and interpreted in physical terms. The calculations also produce the maximum density n_0 at the edge of the core as the eigenvalue of the problem. The dependence of n_0 on the experimental parameters can be reasonably approximated by a simple analytical relation. A general similarity law relating the discharge parameters is given.

INTRODUCTION

THE theory of the self-sustained collision dominated column is well-known.[1–2] The particles in such a column are produced within the plasma volume by electron collisions with neutrals. This mechanism inherently requires high electron temperatures and, with that, an external electric field. Such an external field introduces difficulties which have been discussed elsewhere.[3–5]

In the recent past, therefore, experiments have been carried out which produce the charge carriers outside of the actual plasma column. This can be done, e.g., by contact ionization of atoms at a metal surface,[6–8] or by ionization in a hollow cathode.[9,10] Collimated by a magnetic field the carriers are then introduced in the axial direction into the center of the column where they form the effective particle source for the rest of the plasma volume.

It is the aim of this investigation to describe the plasma between the core and the wall. The limits of this description are set by the following model.

The discharge volume is bound by two infinite coaxial cylinders of radius r_0 and R, respectively, lying in a longitudinal magnetic field B. Within the smaller cylinder, of radius r_0, we have an ensemble of electrons and ions of temperature T_0. This core provides a radial-electron and ion-particle current of density Γ_0, which defines one of the boundary conditions of our problem. The electrons and ions move across the magnetic field towards the wall R. We assume that the effective mean free path of both particle kinds is much smaller than the extension $2R$ of the discharge vessel, and that the concept of quasi-neutrality is applicable. As we neglect volume recombination, all particles recombine at the insulated wall of the container.

BASIC EQUATIONS

We first tried to evaluate the distribution function in phase space from Boltzmann's equation using an expansion in special Laguerre polynomials. The general solution for the coefficients of this expansion is given by determinants which include heavy integral expressions. It has the decisive disadvantage of being practically unintelligible.

Accordingly, we base our calculations here on the transport equations for the mass, momentum, and energy. These equations read (see, e.g., reference 11)

$$\nabla \cdot \mathbf{\Gamma}_{\perp} = \alpha_n, \qquad (1)$$

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$$\mathbf{B}\times\boldsymbol{\Gamma}_{\pm}+\frac{\boldsymbol{\Gamma}_{\pm}}{\mu_{\pm}}+e\eta n_{\pm}(\boldsymbol{\Gamma}_{\pm}-\boldsymbol{\Gamma}_{\mp}) = \pm n_{\pm}\mathbf{X}-\nabla\left(\frac{\mathsf{P}_{\pm}}{e}\right), \quad (2)$$

$$\nabla(n\mathbf{v}v^2)_{\pm}\mp\frac{2n_{\pm}e\mathbf{v}d_{\pm}}{m_{\pm}}\cdot\mathbf{X} = \sum_i\frac{2n_{\pm}\nu_{\pm i}}{m_i+m_{\pm}}(m_i\bar{v}_i^2-m_{\pm}\bar{v}_{\pm}^2) - \sum_{i,x}\frac{2en_{\pm}\nu_{\pm xi}V_{ix}}{m_{\pm}}, \quad (3)$$

where $\mathbf{X}$ is the electric field; n, particle density; $\boldsymbol{\Gamma}$, particle current density; $\mathbf{v}_d$, drift velocity; μ, mobility due to neutral particle collisions; P, pressure tensor; $\mathbf{v}$, velocity; $\mathbf{B}$, magnetic field; e, elementary charge; m, particle mass; α_n, net particle production per unit time and volume; η, electron–ion interaction parameter; ν, collision frequencies; and V_x, excitation potential of level x. The bar indicates averages over the velocity space. The indexes $+$, $-$ refer to ions and electrons, respectively.

Equations (1) to (3) represent six simultaneous differential equations. The problem is simplified by the lack of a neutral gas component, and by the concept of quasi-neutrality, which means

$$\mu_+,\ \mu_-\to\infty, \quad (4)$$

$$n_+\approx n_-=n. \quad (5)$$

In addition to the differential equations (1) to (3), we have boundary conditions defined by the experimental setup.

At the edge of the core r_0, the temperatures and current densities are prescribed by the effective particle source

$$r=r_0\to T_-=T_+=T_0,\qquad \Gamma_{r-}=\Gamma_{r+}=\Gamma_0. \quad (6)$$

Since the outer wall of the plasma container is insulated, the electron and ion current at this wall are equal:

$$r=R\to\Gamma_{r+}=\Gamma_{r-}. \quad (7)$$

Also at the sheath edge of the plasma close to the wall (R) the current continuity must be satisfied.[12] To formulate this condition it is necessary to define what we mean by the edge of the sheath in our special case. The description of the plasma by a diffusion process is correct down to the point where we are about one effective ion mean free path away from the wall. From this point on, the motion of the ions is better described by the laws of free fall, and we will therefore use this point to define the beginning of the sheath. It is designated by an index s. According to our general assumptions, the extension of the sheath is small in comparison to the radius R.

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Therefore the current continuity at the sheath edge can be stated in the form[12]

$$r_0\Gamma_0/R=\tfrac{1}{4}(n\bar{v}_{r+})_s, \quad (8)$$

where $\bar{v}_{r+}$ is the average radial ion velocity. Since we have no net volume particle production this formula is quite obvious.

Because the sheath edge is practically at $r=R$, the boundary conditions of our problem may be summarized by

$$r=r_0\to T_-=T_+=T_0,\quad \Gamma_{r-}=\Gamma_{r+}=\Gamma_0, \quad (8a)$$

$$r=R\to\Gamma_{+r}=\Gamma_{-r},\quad \frac{r_0\Gamma_0}{R}=\tfrac{1}{4}n_R\bar{v}_{R+}. \quad (8b)$$

These are five conditions. Remembering the assumption of quasi-neutrality (5), we see that our problem is of the fourth order. Consequently one condition determines the eigenvalue, the particle density n_0 at the edge of the core.

GENERAL SOLUTION

We intend to find a steady-state solution of our problem which is cylinder symmetric and homogeneous along the z axis. Under these circumstances it follows, from Eqs. (1), (5), and (8b), that

$$\Gamma_{r+}=\Gamma_{r-}=\Gamma_{am} \quad (9)$$

throughout the whole discharge. The elimination of X_r from Eq. (2) using the scalar pressure approximation

$$\mathsf{P}_{\pm}=n_{\pm}kT_{\pm}\boldsymbol{\theta};\qquad \theta_{ik}=\delta_{ik}, \quad (10)$$

produces

$$\Gamma_{am}=-\frac{\eta n}{B^2}\frac{d}{dr}[nk(T_++T_-)]. \quad (11)$$

With that, the steady-state continuity equation requires

$$(r\eta nk/B^2)(d/dr)[nk(T_++T_-)]=-\Gamma_0r_0. \quad (12)$$

At this point one might be tempted to apply the Schottky approximation which replaces the energy balance by the assumption of constant temperatures and approximates the boundary condition (8b) by $n_{+R}=n_{-R}\approx 0$. The density distribution $n(r)$ an the eigenvalue n_0 is then readily derived to be

$$n = n_0[\ln (R/r)/\ln (R/r_0)]^{\frac{1}{2}} \tag{13}$$

and

$$n_0 = [B^2\Gamma_0 r_0 \ln (R/r_0)/kT_0\eta(T_0)]^{\frac{1}{2}}. \tag{14}$$

However, it seems doubtful whether Schottky's assumptions for the self-sustained column are applicable here for the following reasons.

In the case of the self-sustained positive column, the electron temperature is defined by the energy gain in the longitudinal electric field, and by the energy loss through collisions with neutral atoms. As both these quantities do not depend on the radial coordinate, the assumption of constant electron temperature is reasonable.

In the fully ionized column with external particle production, the particles enter the discharge volume with equal energy. Moving across the magnetic field, they interact directly or via the ambipolar electric field, exchanging energy in a rather complicated way. Here the assumption of constant temperature is not obvious.

We therefore try to include the temperature variation. Again we have Eq. (12). When the temperature dependence

$$\eta = \eta_0/(T_-/T_0)^{\frac{3}{2}} \tag{15}$$

is included, it reads

$$n\frac{d}{dr}[n(T_+ + T_-)] = -\frac{\Gamma_0 r_0 B^2}{\eta_0 k}\left(\frac{T_-}{T_0}\right)^{\frac{3}{2}}\frac{1}{r}. \tag{16}$$

If we further use in Eq. (3) the relations

$$\nu_{+-} = \frac{e^2 n\eta}{m_+}; \qquad \nu_{-+} = \frac{e^2 n\eta}{m_-}, \tag{17}$$

and the approximation

$$\nabla(n\mathbf{v}_\pm v_\pm^2) = \frac{3k}{m_\pm}\nabla(\mathbf{\Gamma}_r T_\pm), \tag{18}$$

we find

$$\frac{2e^2\eta n^2}{m_+ + m_-}(T_- - T_+) = \nabla(\mathbf{\Gamma}_r T_+) - \frac{2e}{3k}\mathbf{X}\cdot\mathbf{\Gamma}_r, \tag{19a}$$

$$\frac{2e^2\eta n^2}{m_+ + m_-}(T_+ - T_-) = \nabla(\mathbf{\Gamma}_r T_-) + \frac{2e}{3k}\mathbf{X}\cdot\mathbf{\Gamma}_r + \sum_x \frac{2e}{3k} n\nu_{-x}V_{+x}. \tag{19b}$$

Adding these equations, we have

$$\nabla[\mathbf{\Gamma}_r(T_+ + T_-)] = -\sum_x \frac{2e}{3k} n\nu_{-x}V_{+x}. \tag{20}$$

This equation states that the divergence of the energy current is equal to the energy loss due to ion excitation and ionization. Since in the plasma under consideration the temperatures are too low to cause such excitation to an appreciable extent, we have

$$(T_+ + T_-)\nabla\cdot\mathbf{\Gamma}_r + \mathbf{\Gamma}_r\cdot\text{grad}(T_+ + T_-) = 0, \tag{21}$$

and therefore

$$T_+ + T_- = 2T_0. \tag{22}$$

Making use of Eqs. (22) and (16), we find

$$(d/dx)(z^2) = -C_1(1 - \tfrac{1}{2}y)^{\frac{3}{2}}/x, \tag{23}$$

where the abbreviations are

$$y = \frac{T_+}{T_0}; \qquad z = \frac{n}{n_0}; \qquad x = \frac{r}{R}; \qquad C_1 = \frac{2^{\frac{3}{2}}RB^2\Gamma_0 x_0}{\eta(T_0)T_0 k n_0^2}. \tag{24}$$

The second relation for y and z, Eq. (19a), reads, with the abbreviations (24),

$$\frac{4e^2R\eta n^2}{m_+ + m_-}(1 - y) = \Gamma_r\left(\frac{dy}{dx} - \frac{2eX_rR}{3kT_0}\right). \tag{25}$$

From Eq. (25) we eliminate the radial-field component. For this purpose we use the momentum equations (2) in the form

$$\Gamma_{\theta-}B - \Gamma_r/\mu_- = nX_r + (d/dr)(nkT_-/e), \tag{26a}$$

$$\Gamma_{\theta-} = -\{B\mu_-/[1 + e\eta n(\mu_+ + \mu_-)]\}\Gamma_r, \tag{26b}$$

which gives

$$-\Gamma_r\frac{1 + \mu_-^2B^2 + e\eta n(\mu_+ + \mu_-)}{\mu_-[1 + e\eta n(\mu_+ + \mu_-)]} = nX_r + \frac{d}{dr}\left(\frac{nkT_-}{e}\right). \tag{27}$$

Further, by using Eqs. (4) and (11), it follows from (27) that

$$neX_r = \frac{1}{\mu_+ + \mu_-}\left[\mu_-\frac{d}{dr}(nkT_+) - \mu_+\frac{d}{dr}(nkT_-)\right] \tag{28}$$

or

$$\frac{eX_rR}{kT_0} = \frac{dy}{dx} + \frac{1}{z}\frac{dz}{dx}g(y) \tag{29}$$

with

$$g(y) = y\frac{1 - \kappa[(2 - y)/y]^{\frac{3}{2}}}{1 + \kappa[(2 - y)/y]^{\frac{3}{2}}}; \qquad \kappa = \frac{\mu_{+0}}{\mu_{-0}}. \tag{30}$$

Introducing (11) and (29) in Eq. (25), we finally arrive at the following two simultaneous differential equations:

$$\frac{d}{dx} z^2 = -C_1 \frac{(1 - \frac{1}{2}y)^{\frac{3}{2}}}{x}, \tag{31a}$$

$$C_2(1 - y) = \frac{1}{z}\frac{dz}{dx}\left[-\frac{dy}{dx} + 2g(y)\,\frac{1}{z}\frac{dz}{dx}\right], \tag{31b}$$

where C_1 and C_2 are defined by

$$C_1 = \frac{2^{\frac{3}{2}}\Gamma_0 r_0 B^2}{kT_0 \eta_0 n_0^2}, \qquad C_2 = \frac{6e^2B^2R^2}{kT_0(m_+ + m_-)}. \tag{31c}$$

With the new abbreviations the boundary conditions (8) can be written in the form

$$x = x_0 \rightarrow y = 1;\; z = 1;\; \Gamma_{r+} = \Gamma_{r-} = \Gamma_\theta, \tag{32a}$$

$$x = 1 \rightarrow z = \frac{4x_0\Gamma_0}{n_0}\left(\frac{m_+}{3kT_0 y}\right)^{\frac{1}{2}};\quad \Gamma_{r+} = \Gamma_{r-}. \tag{32b}$$

Equations (31) together with the boundary conditions (32) define the electron temperature T_-, the ion temperature T_+, the particle density n and the eigenvalue n_0.

SIMILARITY LAW

The coefficients of the differential equations (31) and the boundary conditions (32) include the parameters of our problem only in the combinations

$$\frac{\Gamma_0 r_0}{T_0^{5/2}};\quad \frac{B}{T_0}\frac{R^2}{m_+};\quad \frac{n_0}{T_0^2}\frac{R}{m_+^{1/2}}. \tag{33}$$

Similar discharges have identical relative values of density, temperature, etc., at homolog points. Here this is true if the quantities (33) are identical. Consequently, for similar discharges, a variation of one of the parameters (T_0, B, R, r_0, Γ_0, m_+) prescribes the necessary changes for all the other parameters according to the following scheme:

	T_0	B	$R/(m_+)^{\frac{1}{2}}$	$\Gamma_0 r_0$
$T_0 \rightarrow$	$\propto T_0$	$\propto T_0^{3/2}$	$\propto T_0^2$	$\propto T_0^{5/2}$
$B \rightarrow$	$\propto B^{-2/3}$	$\propto B$	$\propto B^{-4/3}$	$\propto B^{-5/3}$
$R/(m_+)^{\frac{1}{2}} \rightarrow$	$\propto [R/(m_+)^{\frac{1}{2}}]^{\frac{1}{2}}$	$\propto [R/(m_+)^{\frac{1}{2}}]^{-3/4}$	$\propto [R/(m_+)^{\frac{1}{2}}]$	$\propto [R/(m_+)^{\frac{1}{2}}]^{5/4}$
$\Gamma_0 r_0 \rightarrow$	$\propto (\Gamma_0 r_0)^{2/5}$	$\propto (\Gamma_0 r_0)^{-3/5}$	$\propto (\Gamma_0 r_0)^{-4/5}$	$\propto (\Gamma_0 r_0)$

All other parameters being constant, we have similar discharges if

$$R \propto (m_+)^{\frac{1}{2}};\qquad \Gamma_0 \propto 1/r_0. \tag{34}$$

INTEGRATION AND RESULTS

Equations (31) and (32) do not allow an analytic solution. Machine solutions are complicated by the fact that we are dealing with a boundary–eigenvalue problem. However, as we have several parameters at our disposal, we can evade this difficulty by the following procedure.

The magnetic field B, the radius R, and the core temperature T_0 define the constant C_2. Choosing values of C_1, we integrate simultaneously Eqs. (31) starting from $z(x_0) = 1$ and $y(x_0) = 1$. At $x = 1$ we find values z_1 and y_1. By introducing these into Eqs. (32b), we have the relations

$$C_1 = 2^{\frac{3}{2}}\Gamma_0 r_0 B^2/kT_0\eta_0 n_0^2 \tag{35a}$$

and

$$z_1 = (4\Gamma_0 r_0/n_0 R)(m_+/3kT_0 y_1)^{\frac{1}{2}}, \tag{35b}$$

which define the parameter value $\Gamma_0 r_0$ and the eigenvalue n_0 belonging to these density and temperature distributions.

Examples of the results of such calculations are given in Figs. 1 and 2 for Cs.

In addition, Fig. 3 shows the eigenvalue n_0 as a function of the magnetic field B and the effective particle production.

DISCUSSION

The discussion uses the two parameters

$$p_1 = \Gamma_0 r_0/T_0^{5/2};\qquad p_2 = (BR)^2/T_0. \tag{36}$$

The characteristic features of the relative density distributions z shown in Figs. 1(a), (b), and (c) may be summarized as follows.

All distributions decrease from the edge of the core towards the wall, the slope $|dz/dx|$ being larger near the two limiting cylinders than in between. With increasing parameter value p_1 the relative density (and, according to Fig. 3, also the absolute density) increases in all cases. The influence of p_1 is stronger for small values of p_2. With increasing p_2 the relative density decreases. (However, this cannot be said of the absolute density, because—according to Fig. 3—n_0 increases with p_2). The densities at the walls have finite values.

These features may be qualitatively understood simply from the mass conservation law, which requires that the radial particle current shall be constant across the plasma volume.

If the diffusion coefficient were constant, the slope $|dz/dx|$ would decrease towards the wall in proportion to $1/r$. However, in the case of a fully ionized positive column in a longitudinal field the effective transverse diffusion coefficient is proportional to the particle density and inversely proportional to the square of the magnetic field

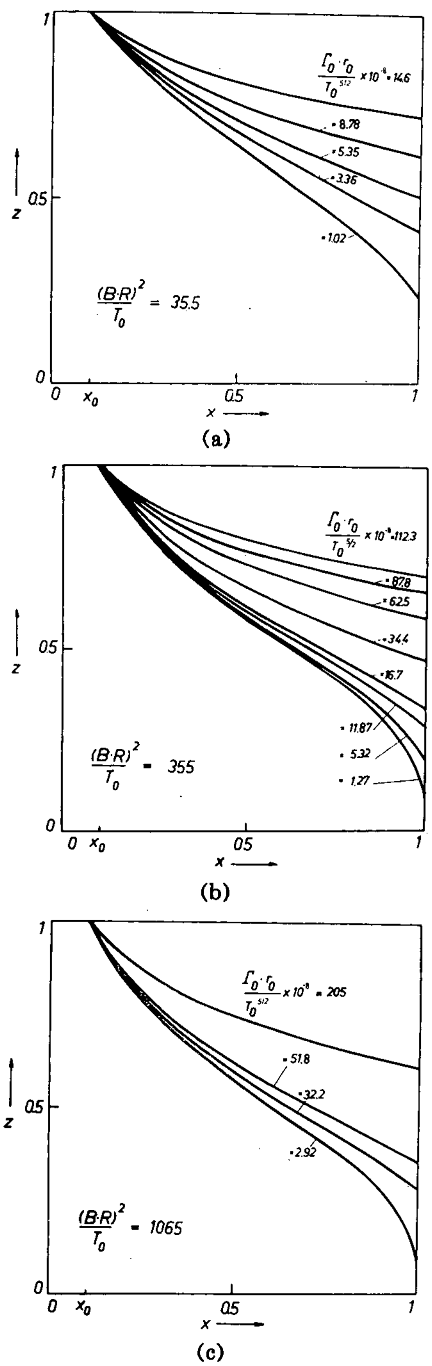

FIG. 1. Relative density distribution $z = n/n_0$ as a function of the relative axial distance $x = r/R$ for various parameter values of p_1 and p_2, calculated from Eqs. (31), (32) for the example of cesium. $p_1 = (BR)^2/T_0$ is measured in units [G^2 cm^2/°K], $p_1 = \Gamma_0 r_0/T_0^{5/2}$ is measured in units (cm sec · $°K^{5/2})^{-1}$.

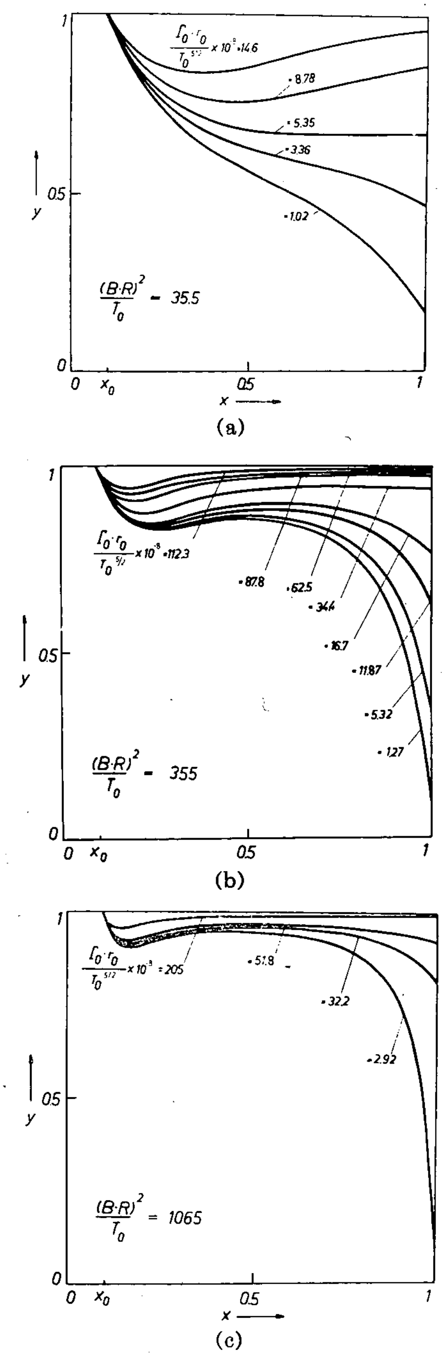

FIG. 2. Relative temperature distribution $y = T_+/T_0$ as a function of the relative radial distance $x = r/R$ for various parameter values of p_1 and p_2, calculated from Eqs. (31), (32) for the example of cesium. $p_2 = (BR)^2/T_0$ is measured in units [G^2 cm^2/°K], $p_1 = \Gamma_0 r_0/T_0^{5/2}$ is measured in units (cm sec$°K^{5/2})^{-1}$.

[see Eq. (11)]. Therefore $|dz/dx|$ increases in regions of small particle density. Consequently, starting from the edge of the core, $|dz/dx|$ should be expected to decrease because of the increase in r, but then, approaching regions of low particle density, should increase again due to the decrease in the effective diffusion coefficient. This agrees well with the results. An increase in the magnetic field increases the parameter p_2. It decreases the diffusion coefficient, and therefore requires in general a larger slope $|dz/dx|$ and, with that, a decrease in the relative density—again in agreement with the results of our analysis. From an increase of p_1 we would expect—and Fig. 3 confirms this—an increase in the absolute density across the plasma volume. An increase of p_1 would also increase the effective diffusion coefficient, which results in a decrease of $|dz/dx|$ as demonstrated by the curves of Figs. 1(a) to (c).

As T_0 is a constant experimental parameter, the relative temperature distributions $y(x)$ shown in Figs. 2(a)–(c) are proportional to the absolute temperature distributions. We see that the temperature variation is not at all negligible. The ion temperature always decreases. In some cases it decreases monotonically towards the wall, but it can also show a minimum—or even a minimum and a maximum—as a function of x. As the parameter p_1 increases, the temperature decrease is reduced. This influence is stronger for smaller values of p_2. With increasing p_2 the temperature distribution $y(x)$ approaches a constant value, except for a decrease near the core edge and near the wall of the vessel.

Again these features can be qualitatively understood, remembering that two processes govern the change in temperature. There is the collective interaction of the particles via the space and wall charge (ambipolar field), which increases with the magnetic field. This interaction takes energy from the ions and gives it to the electrons. The other process—the energy exchange due to individual particle interactions—tends to decrease the temperature difference between ions and electrons.

At the edge of the core, where the two temperatures are identical, only the ambipolar field is in action, which causes a decrease in the ion temperature (and, with that, an increase in the electron temperature) as shown in all of the Figs. 2(a)–(c). This increase in the temperature difference brings the individual energy exchange of the unlike particles into play, which causes an increase in dy/dx. Remembering that the individual exchange varies in proportion to the particle density, we expect dy/dx to decrease again in the regions of low particle density, close to the wall. This is confirmed in Figs. 2(a) to (c), except in those cases where the decrease near the wall is small. An increase in p_1 causes an increase in the particle density, as described in the preceding paragraph. This favors the individual energy exchange, and consequently increases the ion temperature, in agreement with the calculated results. An increase in the magnetic field p_2 reduces the influence of heat conduction and collective interaction, and so favors the individual energy exchange which moves the temperature distributions $y(x)$ closer to $y = 1$. This is also demonstrated in Figs. 2(a) to (c).

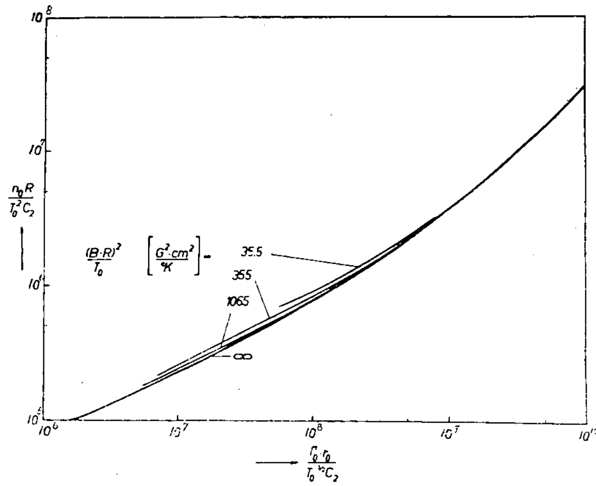

FIG. 3. n_0 as a function of B.

We note further that in the appropriate units chosen in Fig. 3 the dependence of the eigenvalue n_0 on $\Gamma_0 r_0$ is practically not influenced by the magnetic field. In a first approximation the relation may be represented in the double logarithmic plot by a straight line of slope $\frac{2}{3}$, which produces the analytical approximation

$$n_0 = (6e^2/km_+)^{\frac{1}{3}}(B\Gamma_0 r_0)^{\frac{2}{3}}R^{-\frac{1}{3}}. \tag{37}$$

Comparison of Eq. (37) with (14) demonstrates the inapplicability of Schottky's approach to the present problem.

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