Title
THE EVALUATION OP PHASE SPACE INTEGRALS

Authors
Campbell, Graham H.
Lepore, Joseph V.
Riddell, Robert J.

Publication Date
1965-11-16

UCRL-16522

University of California

Ernest O. Lawrence Radiation Laboratory

THE EVALUATION OF PHASE SPACE INTEGRALS

DISCLAIMER

UCRL-16522

Phys. Rev.

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

THE EVALUATION OF PHASE SPACE INTEGRALS

Graham H. Campbell, Joseph V. Lepore, and Robert J. Riddell, Jr.

November 16, 1965

UCRL-16522

THE EVALUATION OF PHASE SPACE INTEGRALS*

Graham H. Campbell, Joseph V. Lepore, and Robert J. Riddell, Jr.

Lawrence Radiation Laboratory
University of California
Berkeley, California

November 16, 1965

ABSTRACT

The relativistic phase-space integral over the submanifold defined by total momentum zero and fixed total energy has been reduced to a single contour integration. The number of particles involved and their masses are arbitrary. It is shown that the contour integration may be readily approximated by the saddle-point technique and yields a result which is easily handled on a computer. In the non-relativistic and extreme relativistic limits this method leads to expressions for the phase space which may be obtained from the exact results for these cases by replacing gamma-function factors by the Stirling approximation.

I. INTRODUCTION

Interest in the evaluation of phase-space integrals involving the constraints of momentum and energy conservation arises from the study of multiple production of particles in high-energy nuclear collisions. A knowledge of the phase-space factor for a particular process allows the separation of the dynamical and kinematical features peculiar to the situation. For example, a knowledge of phase-space factors can be important for the determination of whether very-short-lived particles or "resonances" play a role in a particular reaction.

It has been shown that the general relativistic phase-space integrals are easily reduced to two integrations.[1] This note shows how still another integration may be performed.

II. REDUCTION OF THE INTEGRAL

The integral to be evaluated is

$$S_n(E) = \int \delta\left(E - \sum_{i=1}^{n} \omega_i\right) \delta\left(\sum \underset{\sim}{p}_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{\omega_i} , \tag{1}$$

where $\omega_i = (\underset{\sim}{p}_i{}^2 + m_i{}^2)^{1/2}$. This is a Lorentz-invariant quantity which we shall evaluate in the center-of-mass frame. If we insert a Fourier representation of the δ function we may write

$$S_n(E) = \frac{1}{(2\pi)^4} \int_{\infty - i\epsilon}^{\infty - i\epsilon} d\alpha \; e^{i\alpha E} \int d^3\underset{\sim}{\lambda} \prod_{i=1}^{n} J(\underset{\sim}{\lambda},\alpha,m_i), \tag{2}$$

where

$$J(\underset{\sim}{\lambda},\alpha,m) = \int \frac{d^3 p}{\omega} \; e^{i(\underset{\sim}{\lambda}\cdot\underset{\sim}{p}-\alpha\omega)} . \tag{3}$$

The variable α has been given a small negative imaginary part to make the integration over momenta well defined. After the trivial angular integrations are performed we can write

$$J(\underset{\sim}{\lambda},\alpha,m) = -\frac{2\pi}{\lambda} \frac{d}{d\lambda} I(\lambda,\alpha,m) , \tag{4}$$

where

$$I(\lambda,\alpha,m) = \int_{-\infty}^{\infty} \frac{dp}{\omega} \; e^{i(\lambda p-\alpha\omega)} . \tag{5}$$

If we now let

$$p = m \sinh\theta ,$$

$$\omega = m \cosh\theta ,$$

$$\alpha = (\alpha^2 - \lambda^2)^{1/2} \cosh \psi ,$$

and

$$\lambda = (\alpha^2 - \lambda^2)^{1/2} \sinh \psi ,$$

then[2]

$$I(\lambda,\alpha,m) = \int_{-\infty}^{\infty} d\theta \;\; \exp[-im(\alpha^2 - \lambda^2)^{1/2} \cosh(\theta - \psi)] \qquad (6)$$

$$= -i\pi \; H_0^{(2)}[m(\alpha^2 - \lambda^2)^{1/2}] . \qquad (7)$$

Therefore

$$J(\underset{\sim}{\lambda},\alpha,m) = 2\pi^2 im(\alpha^2 - \lambda^2)^{-1/2} \; H_1^{(2)}[m(\alpha^2 - \lambda^2)^{1/2}] \qquad (8)$$

and

$$S_n(E) = \frac{1}{4\pi^3} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha \int_0^{\infty} \lambda^2 \, d\lambda \; e^{i\alpha E} \; f[(\alpha^2 - \lambda^2)^{1/2}] , \qquad (9)$$

where

$$f(z) = \prod_{k=1}^{n} \left[\frac{2\pi^2 im_k}{z} \; H_1^{(2)}(m_k z) \right] . \qquad (10)$$

The sole role of ϵ in this equation is to define the continuation of the square root. If we had carried the analysis to this point using an arbitrary Lorentz frame this expression would be a Fourier transformation in a space with a timelike dimension. Thus the following steps seem to be a generalization of a theorem of Bochner's on Fourier transformations of radial functions.[3]

We shall now replace the α integration by an integration over $\rho = (\alpha^2 - \lambda^2)^{1/2}$. This will necessitate a separate consideration of different domains of integration. We write

$$S_n(E) = \frac{1}{4\pi^3}\left\{\int_0^R d\lambda \int_{C_1} d\alpha + \int_R^\infty d\lambda \int_{C_2} d\alpha\right\} \lambda^2 e^{i\alpha E} f[(\alpha^2 - \lambda^2)^{1/2}] . \tag{11}$$

In the first integral in Eq. (11), $\lambda \leq R$. In this case, we first distort the original α contour so that it follows the real axis except for an arc below the singular points at $\pm\lambda$ (see Fig. 1). For the variable ρ, there will be a corresponding contour traced out in the ρ plane as α follows its contour (see Fig. 2). For convenience, we choose the arc in the ρ plane as a half circle of radius R, so the α traces out an arc which is similar to an ellipse. In the second integral, $\lambda \geq R$, and here the α contour is distorted to follow the real axis except for separate arcs under the singularities at $\alpha = \pm\lambda$ (see Fig. 3). As before, we have a corresponding path in ρ and we choose the nonstraight portions to be circles of radius R (see Fig. 4).[4] We shall denote the contributions to S_n from the various pieces of the α contours simply by the labels as indicated in Figs. 1-4. The integrals can now all be expressed in terms of a real independent variable, but we must treat each one separately, taking into account the analytic continuation in the appropriately cut plane for the various functions involved.

Let us consider, for example, the calculation of S_I:

$$S_I = \frac{1}{4\pi^3}\int_R^\infty d\lambda \int_{-\infty}^{-(\lambda^2+R^2)^{1/2}} d\alpha\, \lambda^2 e^{i\alpha E} f[(\alpha^2 - \lambda^2)^{1/2}] . \tag{12}$$

If we now choose ρ as the integration variable, and introduce the variable r, using

$$\rho = (\alpha^2 - \lambda^2)^{1/2} = -r , \quad \text{for} \quad r > 0 ;$$

and (13)

$$\alpha = (\rho^2 + \lambda^2)^{1/2} = -(r^2 + \lambda^2)^{1/2} ,$$

we have

$$S_I = - \frac{1}{4\pi^3} \int_R^\infty d\lambda \int_{-\infty}^{-R} d\rho \, \rho\lambda^2(\lambda^2 + r^2)^{-1/2} \exp[-iE(\lambda^2 + r^2)^{1/2}]f(\rho) . \tag{14}$$

The phase of ρ is $-\pi$. If we now make the same change of variable in $\mathcal{S}_I$, interchange order of integration in both S_I and $\mathcal{S}_I$, and add the two, we have, in terms of the variable $r = |(\lambda/r)^2 + 1|^{1/2}$,

$$S_I + \mathcal{S}_I = \frac{1}{4\pi^3} \int_{-\infty}^{-R} d\rho \, f(\rho)\rho^3 \int_1^\infty du(u^2 - 1)^{1/2} \, e^{-iEru} . \tag{15}$$

We must now face the problem, which we ignored above, that the u integration is divergent. However, it is a limiting case of a convergent integral, namely that in which r has a negative imaginary part. This is because we are dealing with generalized functions (recall the Fourier representation for the δ function introduced into Eq. (2)). We shall define this function and others to follow as the limit of the generalized function as parameters approach their final value through values which make the integral convergent. These statements applied here to the u integration actually refer to the method by which the original δ function was represented. With this intepretation we now have[5]

$$S_I + \mathcal{S}_I = \left(\frac{1}{4\pi^3}\right) \int_{-\infty}^{-R} d\rho \, f(\rho)\rho^3 \left[\frac{i\pi}{2Er} \; H_1^{(2)}(Er)\right] . \qquad (16)$$

An exactly analogous calculation, this time using

$$\rho = (\alpha^2 - \lambda^2)^{1/2} = r , \qquad \text{for} \quad r > 0 ,$$

and (17)

$$\alpha = (\rho^2 + \lambda^2)^{1/2} = (\lambda + r^2)^{1/2}$$

yields[5]

$$S_{IV} + \mathcal{S}_{II} = \left(\frac{1}{4\pi^3}\right) \int_{R}^{\infty} d\rho \, f(\rho)\rho^3 \left[\frac{-i\pi}{2Er} \; H_1^{(1)}(Er)\right] . \qquad (18)$$

Similar calculations using

$$\rho = (\alpha^2 - \lambda^2)^{1/2} = -ir , \qquad \text{for} \quad r > 0 ,$$

and (19)

$$\alpha = (\rho^2 + \lambda^2)^{1/2} = \mp(\lambda^2 - r^2)^{1/2}$$

(the minus sign for S_{II} , plus for S_{III}), followed by introduction of the variable $u = |(\lambda/r)^2 - 1|^{1/2}$, gives

$$S_{II} = \frac{1}{4\pi^3} \int_{-iR}^{-i\infty} d\rho \, f(\rho)\rho^3 \int_0^{\infty} du(u^2 + 1)^{1/2} \, e^{-iEru} ,$$

and (20)

$$S_{III} = \frac{1}{4\pi^3} \int_{-i\infty}^{-iR} d\rho \, f(\rho)\rho^3 \int_0^{\infty} du(u^2 + 1)^{1/2} \, e^{-iEru} .$$

Now, in a manner consistent with our interpretation of the u integrals, we can deform the u contours through the convergent quadrant to the imaginary axis, avoiding the branch points at $\pm i$ in the appropriate manner. Then using the variable $v = iu$ in S_{II} and $v = -iu$ in S_{III} we find that the contributions to the two integrals from the region $0 \leqslant v \leqslant 1$ cancel, while the remainder give[6]

$$S_{II} + S_{III} = -\frac{1}{2\pi^3}\int_{-iR}^{-i\infty} d\rho\, f(\rho)\rho^3 \int_1^{\infty} dv(v^2 - 1)^{1/2}\, e^{-Erv}$$

$$= \frac{1}{2\pi^3}\int_{-iR}^{-i\infty} d\rho\, f(\rho)\rho^3 \left[-\frac{K_1(Er)}{Er}\right] . \qquad (21)$$

To evaluate the loop integrals (Figs. 1 and 2) we let

$$\alpha = -(\lambda^2 + R^2\, e^{2i\theta})^{1/2} \qquad (22)$$

in both L_I and $\mathcal{L}_I$, and take θ as the new variable. In both cases θ ranges from 0 to $+\pi/2$. We may now add L_I and $\mathcal{L}_I$, with the result that λ ranges from zero to infinity. We can now go back to $\rho = Re^{i\theta}$, and we set

$$L_I + \mathcal{L}_I = \frac{1}{4\pi^3}\int_{-R}^{-iR} d\rho\, \rho f(p) \int_0^{\infty} d\lambda\, \lambda^2(\lambda^2 + \rho^2)^{-1/2} \exp[iE(\lambda^2 + \rho^2)^{1/2}] . \qquad (23)$$

In this expression the phase of the square root is to be chosen so that it becomes a negative real quantity as ρ becomes real and negative. Thus the function represented by the λ integration is the analytic continuation of the corresponding function in $S_I + \mathcal{S}_I$ (see Eq. (14)) as ρ follows the circular arc from $-R$ to $-iR$. The corresponding calculation for $L_{II} + \mathcal{L}_{II}$ yields the corresponding continuation of $S_{IV} + \mathcal{S}_{II}$ with the net result that the R limits on the remaining ρ integration in both $S_I + \mathcal{S}_I$ and $S_{IV} + \mathcal{S}_{II}$ can be replaced by $-iR$ upon adding the contribution of the loop integrals.

The contour of the integral for $S_{II} + S_{III}$ may be deformed, by Jordan's lemma, to go to infinity along the real axis and may then be combined with the other integrals. Upon replacing r by ρ times an appropriate phase factor we have

$$S_I + \mathcal{S}_I + L_I + \mathcal{L}_I + \frac{S_{II} + S_{III}}{2} = \frac{1}{4\pi^3} \int_{-\infty}^{-iR} d\rho\, \rho^2\, f(\rho) \left\{ \frac{i\pi}{2E} \left[H_1^{(2)}(E\rho e^{i\pi}) + H_1^{(2)}(E\rho) \right] \right\}$$

and (24)

$$S_{IV} + \mathcal{S}_{II} + L_{II} + \mathcal{L}_{II} + \frac{S_{II} + S_{III}}{2} = \frac{1}{4\pi^3} \int_{-iR}^{\infty} d\rho\, \rho^2\, f(\rho) \left\{ \frac{-i\pi}{2E} \left[H_1^{(1)}(E\rho) + H_1^{(2)}(E\rho) \right] \right\} .$$

The sum of the Hankel functions yields $-2J_1(E\rho)$ in the first case and $2J_1(E\rho)$ in the second, so we have

$$S_n(E) = \frac{-i}{(2\pi)^2 E} \int d\rho\, \rho^2\, f(\rho)\, J_1(E\rho) , \qquad (25)$$

where $f(\rho)$ is given by Eq. (10). The ρ contour runs from $-\infty$ to ∞ below the origin. If this contour is chosen to be symmetric under $\rho \rightarrow e^{-i\pi} \rho^*$, then it can be shown that the contribution from the left half of the contour is the negative complex conjugate of the contribution from the right half. Thus the above expression for S_n is real, as is required.

We note that we may also write

$$S_n(E) = \frac{-i}{2(2\pi)^2 E} \int d\rho\, \rho^2 f(\rho) H_1^{(1)}(E\rho) , \qquad (26)$$

since the contour integral in which $H_1^{(2)}$ replaces $H_1^{(1)}$ vanishes because the integrand is analytic in the entire lower half plane.. This form is convenient for consideration of the nonrelativistic limit.

III. EVALUATION OF THE INTEGRAL

The integrand of the expression for S_n (Eq. (25)) is, apart from constant, real factors,

$$g(\rho) = -i\rho^2 J_1(E\rho) \prod_{k=1}^{n} \frac{H_1^{(2)}(m_k\rho)}{-i\rho} . \qquad (27)$$

This function has a single saddle point on the negative imaginary axis because it takes the form

$$g(-iy) = y^2 I_1(Ey) \prod_{k=1}^{n} \left[\frac{2}{\pi y} K_1(m_k y)\right] . \qquad (28)$$

Paths for constant phase in $g(\rho)$ will generally connect the consecutive zeroes of $J_1(E\rho)$, and there will be saddle points for each of these segments. Since $g(\rho)$ alternates in sign on different sections of these portions of the contour, large cancellations are expected, and, further, the integrand will have its largest value at the saddle point on the negative imaginary axis. Thus the dominant contribution to the integral, taken along a path of constant phase, comes from the neighborhood of the saddle point on the negative imaginary axis. We may therefore approximate S_n by the standard saddle-point technique. Thus if we let $\rho = -iy$, we find that the saddle point is located at the solution of

$$\frac{1-2n}{y} + \frac{EI_0(Ey)}{I_1(Ey)} - \sum_{k=1}^{n} \frac{m_k K_0(m_k y)}{K_1(m_k y)} = 0 . \qquad (29)$$

Let y_0 be this root. Then

$$S_n(E) \approx \frac{y_0{}^2 I_1(Ey_0)}{(2\pi)^{3/2} E} \prod_{k=1}^{n} \left[\frac{4\pi m_k}{y_0} K_1(m_k y_0)\right]$$

$$\times \left\{\frac{4n-2}{y_0{}^2} + E^2 + \sum_{k=1}^{n} m_k{}^2 - \left[\frac{E I_0(Ey_0)}{I_1(Ey_0)}\right]^2 - \sum_{k=1}^{n} \left[\frac{m_k K_0(m_k y_0)}{K_1(m_k y_0)}\right]^2\right\}^{-1/2} \cdot \qquad (30)$$

The accuracy of this approximation can be judged by a comparison with the exact results for the nonrelativistic and extreme relativistic limits.

On examination of Eq. (28) one finds that the saddle point occurs as a result of a balance between the decreasing functions in the product over k and the increasing term, $y^2 I_1(Ey)$. As the energy is reduced the minimum is reached for larger and larger y. Thus, to obtain the exact nonrelativistic limit, the predominant contribution to the integral will arise from large values of ρ and we may use the asymptotic expansions for the Hankel functions in Eq. (26). After some rearrangement we find[7]

$$S_n \approx (2\pi)^{(3n-5)/2} P^{1/2} E^{-3/2} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(1,\ell)}{(2E)^{\ell}}$$

$$\times \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \left[\prod_{k=1}^{n} \frac{(1,j_k)}{(2m_k)^{j_k}}\right] \left\{ i^{[(5n-5)/2]-\ell-\Sigma j_k} \int d\rho \, e^{iT\rho} \right.$$

$$\left. \times \rho^{[-(3n-2)/2]-\ell-\Sigma j_k} \right\} . \qquad (31)$$

We have used the abbreviation P for the product of the masses and will use M for their sum. The kinetic energy is T (= E - M) . Now the term in braces is simply

$$\frac{T^{\frac{3}{2}n - \frac{5}{2} + \ell + \Sigma j_k}}{\Gamma(\frac{3}{2}n - \frac{3}{2} + \ell + \sum_k j_k)} . \qquad (32)$$

Thus we have obtained an asymptotic expansion of S_n in powers of T (if the inverse powers of E are so expanded). The first term of this expansion is

$$S_n^{NR} = (2\pi)^{(3n-3)/2} P^{1/2} M^{-3/2} \frac{T^{(3n-5)/2}}{\Gamma[(3n-3)/2]} . \qquad (33)$$

This agrees with the nonrelativistic phase space computed by more elementary methods.

In this limit we can examine the accuracy of the saddle-point approximation. The saddle point is located at

$$\rho_0 = -\frac{3}{2} i(n-1)/T . \qquad (34)$$

Application of the saddle-point method to the integral yields the above result with the Γ function replaced by Stirling's approximation to it. For n = 2 this approximation is in error by about 6%, and the error decreases as n increases. We note that this estimate applies only when E_{ρ_0} and $m_k\rho_0$ are all large, that is,

$$\frac{3n-3}{2} \frac{m_k}{T} \gg 1 ; \qquad (35)$$

otherwise the saddle point of the original integral would not occur in the asymptotic region of the Hankel functions.

We can also easily determine the asymptotic value of the phase space in the extreme relativistic case. In this case, we expect that ρ_0 will be very small. Thus we can expand the $H_1^{(2)}(m_k\rho)$ for small values of the argument. If we assume that E is large, however, and use the asymptotic form of $H_1^{(1)}(E)$, we then deduce that

$$\rho_0 = -i(2n - 3/2)/E . \tag{36}$$

Thus $E\rho_0$ will be large only if $2n - 3/2 \gg 1$. On the other hand, we can carry out the integration in Eq. (25) without expanding $J_1(\rho E)$, and we find

$$S_n^{ER} = \frac{2^{n-1}}{(n-1)!(n-2)!} E^{2n-4} . \tag{37}$$

This result agrees with the saddle-point approximation again to the extent that the factorials are replaced by the Stirling approximation. In this case the Stirling approximation is not as good as before, and an 11% error is found for $n = 3$. The ratio of the approximate to the exact result approaches 1 with reasonable rapidity as n increases. This is to be expected, for as n increases the saddle point of this integral moves into the asymptotic region of J_1 where the saddle point method was applied to find S_n .

IV. APPLICATIONS AND VARIATIONS

Extensive numerical calculations have been made with the phase-space or "statistical" model by use of this approximation.[8] Numerous comparisons with the exact two- and three-body phase space have confirmed the error estimates presented in Section III.

The noncovariant form of the phase-space integral may be treated by similar methods to those used for the covariant form. The noncovariant form is

$$S_n(E) = \int \delta\left(E - \sum_{i=1}^{n} \omega_i\right) \delta^{(3)}\left(\sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} d^3 \underset{\sim}{p}_i \, . \qquad (38)$$

This has previously been reduced to[1]

$$S_n = \frac{1}{4\pi^3} \int_{-\infty - i\infty}^{\infty - i\epsilon} d\alpha \int_0^{\infty} d\lambda\, \lambda^2 \alpha^n e^{i\alpha E} \prod_{k=1}^{n} \left[\frac{2\pi^2 m_k}{\alpha^2 - \lambda^2} H_2^{(2)}\left(m_k(\alpha^2 - \lambda^2)^{1/2}\right)\right] . \qquad (39)$$

The only essential difference between this and Eq. (9) is the factor α^n . These integrals define generalized functions; hence we may replace this factor by n-fold differentiation with respect to the energy. Then, using the results of the covariant calculation, we may write

$$S_n(E) = \frac{-i}{(2\pi)^2} \int d\rho\, \rho^2 f(\rho) \left(\frac{d}{dE}\right)^n \frac{J_1(E\rho)}{\rho}, \qquad (40)$$

where

$$f(\rho) = \prod_{k=1}^{n} \left[\frac{2\pi^2 m_k}{i\rho^2} H_2^{(2)}(m_k \rho) \right] . \tag{41}$$

Although the saddle-point method is in principle applicable to this integral it is not convenient for numerical approximation because the form of the integrand depends upon n .

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U. S. Atomic Energy Commission.

1. Joseph V. Lepore and Richard N. Stuart, Phys. Rev. 94, 1724 (1954).
2. G. N. Watson, Theory of Bessel Functions (Second Edition) (Macmillan, New York, 1948), p. 180.
3. S. Bochner and K. Chandrasekharan, Fourier Transforms (Princeton University Press, Princeton, 1949), p. 69.
4. In Figs. 3 and 4 the contours are shown displaced from the cuts for clarity, although in fact they are to be taken on the cuts.
5. Watson, op. cit., p. 167.
6. Watson, op. cit., p. 172.
7. Watson, op. cit., p. 198. The symbols (ν,m) are defined there.
8. Graham H. Campbell, Statistical-Model Calculations of High Energy Reactions, UCRL-16315, September 1965.

FIGURE CAPTIONS

Fig. 1. Integration contour in the α plane for $\lambda < R$.

Fig. 2. Integration contour in the ρ plane for $\lambda < R$.

Fig. 3. Integration contour in the α plane for $\lambda > R$.

Fig. 4. Integration contour in the ρ plane for $\lambda > R$.

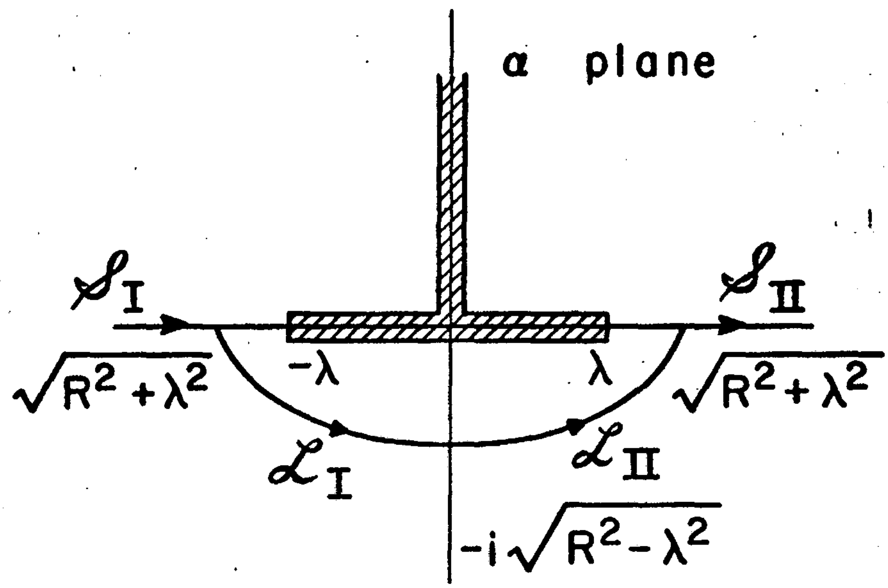

Fig. 1

MUB-8862

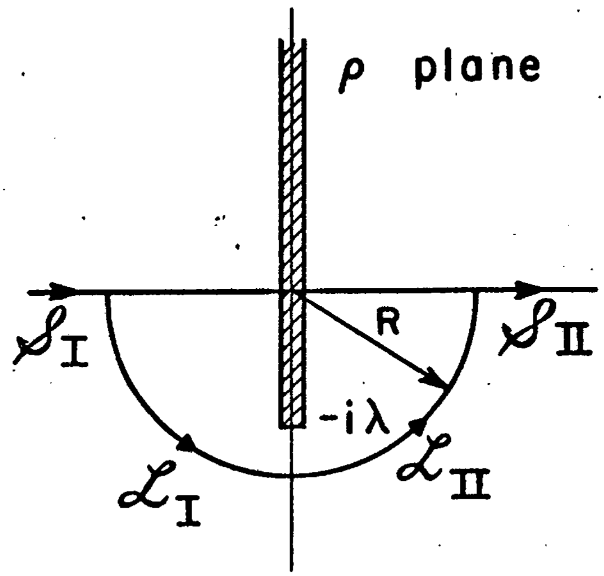

Fig. 2

MUB-8863

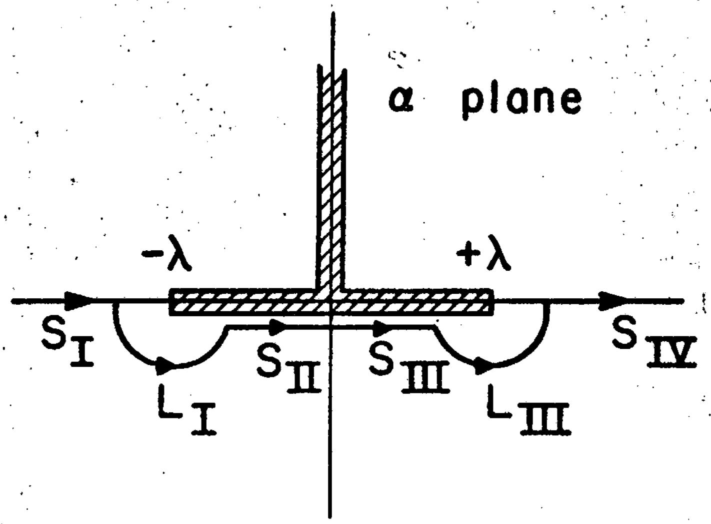

Fig. 3

MUB-8864

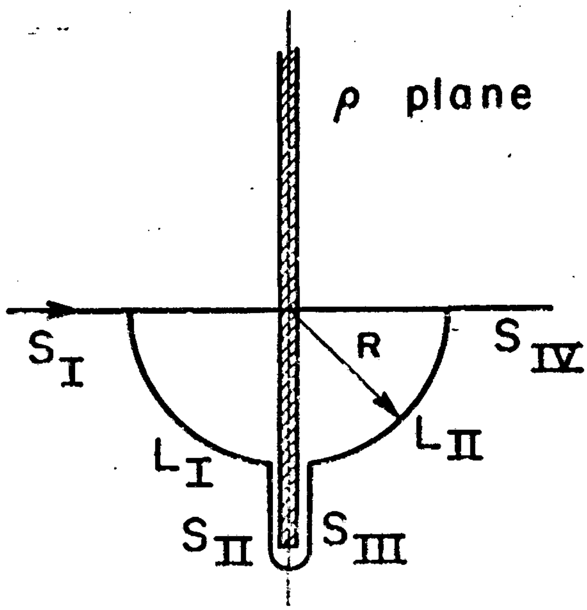

Fig. 4 MUB-8865