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Affine Solutions of Two Dimensional Magnetohydrodynamics and Related Quadratically Coupled Transport Equations

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by

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This dissertation is dedicated to my partner Rosie and the best of boys Basil.

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Experience

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Analyzed state of the art deep learning architectures and their relation to dynamical systems.
Discovered periodic solutions to Magnetohydrodynamic equations.
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Established global and local-in-time existence to a system of quadratically coupled hyperbolic PDEs.

Deep Learning Intern — June 2018 - December 2018
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Presentations

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Abstract

Affine Solutions of Two Dimensional Magnetohydrodynamics and Related Quadratically Coupled Transport Equations

by

Jay Roberts

This is a dissertation on the motion of incompressible charged and non charged particles in a fluid. Specifically, we are concerned with the affine motion of such two dimensional fluids. The physical quanitities of the fluid are derived in terms of the deformation gradient which reduces the Incompressible Euler Equations (EE) and the Incompressible Ideal Magnetohydrodynamical (MHD) equations to ordinary differential equations on $\mathrm{SL}(2, \mathsf{R})$. The EE and MHD become the equations of a free particle and harmonic oscillator, respectively, constrained to $\mathrm{SL}(2, \mathsf{R})$ with the magnetic field strength acting as a bifurcation parameter between the two types of dynamics. We analyze the geometry of $\mathrm{SL}(2, \mathsf{R})$ and completely characterize the behavior of all affine solutions.

Inspired by the decay of the pressure for affine solutions to EE we analyze a related system of quadratically coupled transport equations. By smoothing the equation we show local well posedness in a generalized Sobolev space along with coupled energy estimates for a low and high energy. These estimates are inherited by the non-smoothed solution which, along with weighted energy estimates, allow us to show global well posedness for small data.

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Chapter 1

Introduction

The first portion of this work is devoted to the study of affine solutions to the Incompressible Euler Equations (EE) and the incompressible equations of Magnetohydrodynamics (MHD). Without getting bogged down in the derivation or boundary conditions, these are saved for a later chapter, the EE and MHD equations are

$$\begin{cases} \partial_t u + u \cdot \nabla u &= -\nabla p \\ \nabla \cdot u &= 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t u + u \cdot \nabla u &= -\nabla p + (\nabla \times B) \times B \\ \partial_t B + u \cdot \nabla B &= \nabla \times (B \times u) \\ \nabla \cdot u = \nabla \cdot B &= 0 \end{cases}, \tag{1.2}$$

Here u, B, and p, are the fluid velocity, its magnetic field, and pressure, respectively. The physical fluids we are interested in are those which are surrounded by a vacuum with no external forces present and are moving at non relativistic speeds. Importantly, these fluids have a free boundary which moves along with the fluid.

These models are used for various astrophysical plasmas such as coronal magnetic loops and solar coronal flux tubes. In two dimensions they can be used to model laboratory plasmas with toroidal symmetry such as tokamaks [30, 11]. We will be studying the two dimensional version of these equations as the analysis becomes much more tractable while still admitting rich dynamics that has been seen to qualitatively mirror the long term behavior of the full three dimensional problems in certain instances [35].

Our work begins with a review of the fluid mechanical setting for the problem and a layout of the physical assumptions about the fluid and its magnetic field. We then begin the business of establishing the boundary conditions that are required for (1.1) and (1.2) to model plasmas surrounded by a vacuum. Here is where we review much of the work on the equations and emphasize some differences between the equations' well posedness theory. Both EE and MHD require constraints on the acceleration at the boundary of the fluid. This constraint comes in the form of the Rayleigh Taylor Sign Condition on the normal derivative of the total pressure at the boundary. However, for the MHD this is not enough. The magnetic field's boundary conditions depend on the assumptions of how the magnetic field travels with the plasma boundary and its coupling with the magnetic field of the vacuum. Even after imposing these we need to make additional boundary constraints on the size of the pullback of our magnetic field in material coordinates in order to uniquely identify the magnetic field.

Before we derive the dynamics of the affine motions we devote a chapter to the space of solutions. Our motions are identified with $\mathrm{SL}(2,\mathsf{R})$ which we view as a Riemannian manifold where the metric is the induced metric from the ambient Euclidean space M_2. This geometric point of view highlights the importance of the (*Lie*)subgroup $\mathrm{SO}(2,\mathsf{R})$. It is $\mathrm{SL}(2,\mathsf{R})^{0}s$ unique closed geodesic, and corresponds to absolute minimums of the potential energy associated to the MHD equation. This "neck" created by $\mathrm{SO}(2,\mathsf{R})$ is manifested in motion of the fluid as a limiting disk for the motions. That is, for a fixed initial volume of plasma there is a minimal volume disk that the plasma may, though many do not, reach. The family of motions which manage to squeeze down to this size form the more interesting of the motions we describe.

We also introduce various coordinates on $\mathrm{SL}(2,\mathsf{R})$. Our primary coordinates will emphasize the role that the magnitude of our path plays in the motion's dynamics. In fact, in these coordinates the metric is diagonal and we will be able to describe "generic" motions here. However, these coordinates will be a two branched cover of $SO(2, \mathsf{R})$ and so will be unable to capture the motions which shrink from an ellipse to a disk. This is overcome by restricting to two dimensional embedded submanifolds of $SL(2, \mathsf{R})$ where we will be able to fully describe such motions.

With the solution manifold in hand we will derive the equations of affine dynamics and the corresponding physical quantities. The velocity, pressure, and magnetic field of the fluid are all given in terms of the path in $SL(2, \mathsf{R})$. In particular the magnetic field decomposes into an initial, material, magnetic field which is then carried along by the path. We show that this initial magnetic field is the unique magnetic field which is compatible with affine motion. At this point we have fully translated the analysis of EE and MHD into a question about ODEs on $SL(2, \mathsf{R})$. Kinematically, these ODEs represent a free particle and harmonic oscillator constrained to $SL(2, \mathsf{R})$.

The dynamics of our affine solutions emphasize the advantage of working in two dimensions. For two by two matrices the cofactor operator is linear, in fact it is unitary, and so our ODE's have only a scalar nonlinearity. We derive various geometric invariants which allow us to write this nonlinearity as a function of just the magnitude of the path which justifies our choice of coordinates and reduces our six dimensional phase space to a two dimensional one. Next we handle the complicated work of carefully describing the dynamics for various relationships between the geometric and physical invariants of the system. After this detailed work we present an alternative description of the dynamics by taking advantage of the Hamiltonian structure. Since this section essentially restates the results of the previous ones we omit some of the formal arguments in favor of a more illustrative description of the dynamics. The upshot of all this is a complete description of the (affine) motion of incompressible fluids with and without magnetic fields in two dimensions surrounded by a vacuum with a free boundary.

In the presence of the magnetic field generically the motion of the fluid is an ellipse which rotates and whose diameter expands and contracts periodically. The ellipse never becomes a disk and in coordinates we see the motion is quasi periodic. The magnitude has a period, and there are two “rotation” components which have their own periods. Here the vorticity plays a role in the size of all these periods along with another mystery parameter related to the geometry of $\mathrm{SL}(2,\mathbb{R})$. For special geometric arrangements these two invariants agree and the magnetic field strength acts a a bifurcation parameter. If the magnetic field is strong with respect to the vorticity then the magnitude is still periodic but every ellipse now shrinks to a disk. At which point the major and minor axis of the ellipse exchange roles, leading to a magnitude with period that is half the period of the magnitude-velocity pair in phase space. In the case where the magnetic field is small with respect to the vorticity there is a critical energy for which solutions are oscillating ellipses which limit in time to a constant rotating disk. Energies lower than this critical energy behave the same way as the generic case. Higher energies exhibit the double period behavior seen in the strong magnetic field case.

For EE solutions generically are rotating ellipses whose area grows like t^2. Eventually these motions slow their spin and expand along an axis. There are also motions which begin as rotating ellipses and then decay to a rotating disk.

This domain spreading leads us to the second focus. In [14] the authors showed that for compressible fluids the Sobolev Space of affine solutions was in a sense stable. That is they constructed global in time solutions which started near affine solutions and remained close to these affine counterparts. If we let F be the deformation gradient of our motion then in material coordinates EE can be written

$$D_t u + \frac{1}{\det(F)}\operatorname{cof}(F)\,\nabla_y p = 0. \tag{1.3}$$

The deformation gradient of an affine motion is precisely the path in $\mathrm{SL}(2,\mathbb{R})$ and so if F remains near such a motion we expect its determinant, or at least a piece of it, to decay like t^2. Further, since we are in two dimensions the cofactor operator is linear so (1.3) resembles a type of quadratically coupled transport equation. It was also noted in [18] and [27] that this type of coupling occurs in the Alfve'n waves which propogate in MHD [2]. Though these are quadratically coupled wave equations.

In our work we consider a one dimensional quasi-linear model problem

$$\partial_t u + A\partial_x u = u_i B_i(t)\partial_x u.$$

The components $B_i(t)$ satisfy certain decay conditions based on whether the wave packets resulting from the propagation matrix, A, are interacting for a long or short time. For instance when A is diagonal the wave packets are just the components of u. In this case B_{112} need not decay as fast as $B_{111}(t)$ since for the former the packets will cross paths for a short time where as the latter are self interacting and so they never disperse. The reason for adding decay on the non interacting terms is to further mirror the decay of solutions to the three dimensional wave equation.

In order to show global existence of small solutions we use an energy splitting technique, see for instance [6, 34, 38]. The result comes from coupling the high and low energy estimates together with a low weighted energy. This allows us to show that the high energy grows polynomially in time with a rate that is controlled by the size of the initial data in the weighted and non weighted lower energies. The weighted estimates allow us to improve our low energy estimates and are gotten by a commuting vector field technique similar to those in [23]. For energy which is sufficiently small in the low weighted and non weighted energy these growth controls will allow us to bound the low energy, on a compact time interval, uniformly in time.

To make use of this bound, and in fact to properly justify the energy estimates, we need a well developed local theory. The first part of the local section is devoted to showing local existence with a precise continuation condition. The strategy is to smooth the equations and show existence in the appropriate Sobolev space. We then derive the analog of the energy estimates needed for global existence but now we must take care of the commutators which arise from the smoothing operator. Convergence of these approximations is messy but follows rather directly from another energy splitting. With our approximations in hand we can bequeath the smooth energy estimates to our actual solutions. These are then improved through coupling with the weighted energies giving us global existence.

Chapter 2

Magnetohydrodynamic Equations

We start with the fluid mechanical set up of the problem. First we establish the reference domains, their motions, coordinates and the various differential operators we will need throughout this chapter. Then we will describe forces we are interested in. Starting with the equations of state for the fluid and then coupling the fluid to Maxwell's equations. From here we use various physical assumptions on our body and its magnetic and electric field to simplify the equation arrive at the Ideal MHD equations for compressible fluids.

2.1 Fluid Mechanics Set Up

Throughout this chapter $\mathcal{B}$ will be our **reference domain**. We assume that $\mathcal{B}$ is a compact subset of $\mathbb{R}^d$ where $d = 2$ or 3. The boundary, $\partial\mathcal{B}$, will be assumed smooth though in most cases C^2 would be sufficient.

Definition 2.1. A **motion** of a reference domain $\mathcal{B}$ is a smooth function

$$x : \mathcal{B} \times I \to \mathbb{R}^d$$

where I is a subinterval of $\mathbb{R}$ and for all $t \in I$ the function $x(\cdot, t)$ is a diffeomorphism onto its image.

The reference domain can also be thought of as labeling the fluid particles along their trajectory. The actual position in space occupied by the fluid will be referred to as the ***spatial domain occupied by the uid*** and is

$$\Omega(t) = x(\mathit{B}, t). \tag{2.1}$$

When we describe a physical quantity in terms of the spatial domain, x variables, we say we using the ***spatial*** (Eulerian) description of the fluid. If we use the reference body, y variables, we say we are using the ***material*** (Lagrangian) description of the fluid. The physics of the fluid are generally most easily described using the spatial coordinates but for free boundary problems it can be more convenient to use the material coordinates.

Since our motion is injective at a fixed time the function $x(\ , t)$ has an inverse which we call the ***place*** of a fluid at point x, specifically

$$\mathsf{p} : \Omega(\ ) \quad I \ ! \ \mathit{B}$$

where $\mathsf{p}(x(y, t), t) = y$.

Definition 2.2. The **spatial velocity** of a motion x of a body B is

$$u(x, t) = \frac{\partial}{\partial t}\left[x(y, t)\right]\Big|_{y=\mathsf{p}(x;t)}.$$

We will now barrel through the various differential operators needed to describe transported quantities in the fluid. The reader is referred to [13] for a detailed derivation of the transport theorems.

Definition 2.3. The **material time derivative** of a function f in the spatial coordinates, x, is

$$D_t f = \partial_t f + (u \cdot \nabla_x) f,$$

where

$$(u \cdot \nabla_x) f = D_x f \cdot u.$$

Definition 2.4. The **deformation gradient** of a motion x is

$$F(y,t) = D_y x(y,t).$$

We now describe the basic equations of conservation and motion. Let $\rho : \Omega(\cdot) \times I \to \mathbb{R}^+$ be the density of the fluid, then conservation of mass and momentum are given by

$$\begin{aligned} D_t\rho + \rho\, \nabla_x \cdot u = 0 \\ \rho D_t u = f_{int} + f_{ext}. \end{aligned}$$

Where f_{int} and f_{ext} are the forces resulting from the fluid itself and from external forces respectively. An ***Eulerian*** fluid is one where the only internal force is conservative with potential p that we call the ***pressure***. The Equations of motion for such a fluid are

$$\begin{cases} D_t\rho + \rho\, \nabla_x \cdot u &= 0 \\ \rho\, D_t u + \nabla_x p &= f \end{cases}. \tag{2.3}$$

We have added a forcing term to the conservation of momentum equation to reflect the fact that there will be more internal forces on our fluid due to the effects of the electric and magnetic fields. A fluid is ***incompressible*** if its motion is volume preserving. That is if the deformation gradient satisfies $\det(F) = 1$. This is manifested in the spatial velocity by the divergence condition

$$\nabla_x \cdot u = 0.$$

Incorporating this divergence condition makes the conservation of mass in (2.3) simplify to $\rho(x,t) = \rho_0$ for some $\rho_0 \in \mathbb{R}^+$. Without loss of generality we can assume the density of our fluid is 1. So we arrive at the familiar incompressible Euler equations (EE)

$$\begin{cases} D_t u + \nabla_x p &= f \\ \nabla_x \cdot u &= 0 \end{cases}$$

The boundary conditions for these type of free boundary problems firstly require that the boundary of the fluid move with the fluid

$$(1, u(x,t)) \cdot n(x,t) = 0 \quad x \in \partial\Omega(t),$$

where $n(x,t)$ is the space time outward normal vector to the hyper-surface $I \times \Omega(\cdot)$. Since the fluid is surrounded by vacuum we require that $p(x,t) = 0$ on the boundary of our fluid domain.

We note that for well posedness it was required that the initial pressure satisfy the Rayleigh-Taylor sign condition, $D_n p(\cdot, t) < 0$, on the boundary of $\Omega(0)$. Here D_n is the directional derivative in the direction of the normal to the boundary of $\Omega(t)$. So we have

$$\begin{cases} D_n p &< 0 \\ p &= 0 \end{cases} \quad x \in \partial\Omega(t).$$

For initial data satisfying the Rayleigh-Taylor sign condition, local well-posedness for the incompressible free boundary Euler equations with bulk vorticity was established [5, 8] and for the incompressible free boundary MHD problem in [12], [40]. The use of affine deformations is a well-established tool in continuum mechanics, first introduced in the context of the vacuum free boundary incompressible Euler system in [37, 33].

2.2 Deriving MHD

In order to incorporate the magnetic effects let B, E, j, and σ be the magnetic field, electric field, current, and electric conductivity. The force which the medium experiences is given by ***Ampere Force***:

$$f_A = \frac{1}{c} j \times B.$$

The new quantities are also supplemented by Maxwell's Equations

$$\nabla_x \times B = \frac{4\pi}{c} j \tag{2.5a}$$
$$\nabla_x \cdot B = 0$$
$$\partial_t B = -c \nabla_x \times E \tag{2.5b}$$

Equation (2.5a) allows us to eliminate the current, j, from our balance of momentum

$$D_t u + \nabla_x p = \frac{1}{4\pi}(\nabla_x \times B) \times B.$$

The motion of the medium gives rise to its own electric field $u \times B/c$ which we can relate to the current, j, through ***Ohm's law*** if the following conditions hold:

1. The magnetic field is weak.
2. The motion is moving at non-relativisitic speeds.
3. The plasma is perfectly conducting.

The relationship is

$$j = \sigma(E + \frac{1}{c} u \times B),$$

where σ is the electric conductivity of the plasma, which is perfectly conducting so this becomes the ***Ideal Ohm's Law***

$$E + \frac{1}{c} u \times B = 0.$$

We can substitute this into (2.5b) to get the dynamics of the magnetic field

$$D_t B = (B \cdot \nabla_x)u - B \nabla_x \cdot u$$
$$= (B \cdot \nabla_x)u.$$

$$\begin{cases} D_t u + \nabla_x(p + \frac{1}{2}|B|^2) &= (B \cdot \nabla_x)B \\ D_t B &= (B \cdot \nabla_x)u \\ \nabla_x \times u = \nabla_x \times B &= 0 \end{cases} \tag{2.6}$$

Notice we have collected the conservative portion of the force coming from the magnetic field, $\nabla_x(\frac{1}{2}|B|^2)$ which we refer to as the **magnetic pressure**, onto the left of the equation. We put this and the usual pressure together into what we will refer to as the **total pressure** of our fluid

$$q = p + \tfrac{1}{2}|B|^2.$$

2.3 Boundary Conditions

The boundary conditions for a fluid in vacuum are of particular importance from both a modeling and existence theory perspective. For EE it was shown by Ebin [10] that a necessary condition for stability of the equations was the so called Rayleigh-Taylor sign condition for the pressure at the boundary

$$\partial_n p \le -\epsilon < 0$$

on the boundary of $\Omega(t)$. Where ∂_n is the derivative in the direction of the outward normal of $\partial\Omega(t)$.

In the context of MHD an analog of the Rayleigh-Taylor sign condition was show to also be necessary for stability [16]. Instead of the actual pressure, the total pressure, q, which also must satisfy

$$\partial_n q \qquad \epsilon < 0.$$

The result was shown by using a geometric approach similar to the one used in [5]. Hao and Luo then showed [17] that this modified Rayleigh-Taylor sign condition is in fact necessary for well posedness of in two dimensions.

To show this they make use of the fact that in 2 dimensions MHD admits purely rotating disk solutions. These types of solutions are not possible in three dimensions. They then construct a sequence of initial data which converge to these rotating solutions but whose motions diverge in some appropriate analog of H for $\mu > 2$.

We make the usual assumption that the boundary of the plasma is a perfect conductor and so

$$B \quad n = 0,$$

on $\partial\Omega(t)$ and where n is the outward unit normal to the boundary.

Physically the energy which we are interested in is

$$\int_{(t)} \left(\tfrac{1}{2}|u|^2 + \tfrac{1}{2}|B|^2\right) dx,$$

which we note is the standard kinetic and potential energy for such a system. Putting everything together and adding in that the outside is a vacuum and so the pressure there should vanish gives us the boundary conditions for the plasma-vacuum interface

$$\begin{cases} q & = & 0 \\ \partial_n q & < & 0 \\ B \quad n & = & 0 \end{cases} . \tag{2.7}$$

These boundary conditions differ from those in for example [16] in that we require the total pressure vanish at the boundary which will conserve energy. It was noted in [16] that requiring $|B| = cosnt$ on the boundary in (2.7) also conserves energy and captures the case where the magnetic field vanishes at the boundary that was used by Hu and Wang [20, 19] to show global existence of weak solutions to the compressible MHD equations in a bounded domain. However, it can also represent more general and perhaps physically realistic magnetic fields. We note it was later show in [15] that this was not necessary and it is not used in the well posedness of the linearized and nonlinear MHD in [12, 40] where the vacuum colinearity condition of [32, 29, 42] is of more importance.

We take a different approach. As opposed to controlling the magnetic field on the free boundary of the fluid we pull the control back onto the material boundary. The flux condition of (2.11) implies that the magnetic field is in fact a vector field on $\partial\Omega(t)$, and so to analyze it in material coordinates we ought to consider its pull back to the boundary of the material domain. That is we should control $|F^{-1}B(x(y,t),t)|$, $y \in \partial\mathcal{B}$. specifically,

$$|F^{-1}B(y,t)| = c_0 \quad y \in \partial\mathcal{B}. \tag{2.8}$$

Notice this allows for a a magnetic field with varying norm in the fluid domain.

Past the interface we have the vacuum and must decide how we will handle its magnetic field. Since by definition it has nothing to carry the charge its electric field, $\hat{E}$, must be stationary. This assumption gives rise to the ***pre-Maxwellian*** vacuum dynamics

$$\nabla_x \cdot \hat{B} = 0, \qquad \nabla_x \times \hat{B} = 0, \qquad \partial_t \hat{B} = -\nabla_x \times \hat{E}, \qquad \nabla_x \cdot \hat{E} = 0,$$

where $\hat{B}$ is the magnetic field of the vacuum.

To impose continuity across the plasma-vacuum interface we impose various jump conditions. We let f^+, f^- refer to the plasma and vacuum value of a physical quantity f, respectively, and define the interface jump value of f to be

$$[[f]] = f^+ - f^- .$$

Our first condition is the balance of total pressure across the plasma-vacuum interface

$$[[q]] = 0,$$

and our second requirement is that the net magnetic flux across the interface vanishes, i.e.

$$[[B \cdot n]] = 0, \tag{2.9}$$

where again n is the normal to the boundary of $\Omega(t)$.

The perfect conduction assumption of (2.7) together with (2.9) implies that on the vacuum side of the interface

$$\hat{B} \cdot n = 0.$$

We could then restrict ourselves to magnetic fields in the vacuum that match our plasmas' magnetic field's norm at the interface, that is

$$|\hat{B}| = |B|,$$

as was done in [16, 15] which would decouple the magnetic fields or we could require the same norm matching but for the pullback of the verctor fields on the material boundary like in equation (2.8) and get the same result.

We note that Trakhinin in [42] and together with Secchi in [32] showed that the linearized compressible MHD were ill-posed in the plasma-vacuum interface case if the magnetic field in the plasma and vacuum failed a colinearity condition

$$|\hat{B} - B| = \delta > 0.$$

However, we do not explicitly find the vacuum magnetic field and so this comparison is left to future work.

Finally, we have our incompressible MHD equations which we repeat here for easy of reference.

$$\begin{cases} D_t u + \nabla_x q &= (B \cdot \nabla_x) B \\ D_t B &= (B \cdot \nabla_x) u \qquad x \in \Omega(t) \\ \nabla_x \cdot u &= \nabla_x \cdot B = 0 \end{cases} \tag{2.10}$$

subject to the boundary conditions

$$\begin{cases} \partial_n q &< 0 \\ q &= 0 \\ B \cdot n &= 0 \end{cases}, \qquad x \in \partial\Omega(t) \tag{2.11}$$

and the material boundary condition

$$|F^{-1}(y,t)B(x(y,t),t)| = c_0 \quad y \in \partial\mathcal{B}$$

Chapter 3

Function Space of Motions

Here we will discuss the space in which our motions will reside. We work in $\mathrm{SL}(2,\mathsf{R}) \subset \mathsf{M}^2$ and so we describe the inner-product space M^2 along with various operators. Then we describe the actual motion space $\mathrm{SL}(2,\mathsf{R})$ with the induced metric given by its inclusion in M^2. Finally, we rewrite the equations of MHD (2.6) in material coordinates as a final. We will begin with a quick review of some matrix calculus. Then we translate the equations into material coordinates and derive the physical quantities in terms of the affine motion.

3.1 Matrix Inner Product Space and Groups

Definition 3.1. By M^2, we denote the set of 2×2 matrices over R with the Euclidean inner product

$$\langle A, B\rangle = \sum_{i,j} A_{ij} B_{ij} = \operatorname{tr} A^{\top} B$$

and norm

$$|A|^2 = \langle A, A\rangle .$$

The inner product satisfies the permutation relations

$$\langle AB, C \rangle = \langle B, A^{\top} C \rangle = \langle A, C B^{\top} \rangle . \tag{3.1}$$

For all $A, B, C \in \mathsf{M}^n$. If $A, B \in \mathsf{M}^n$ then

$$|AB| \leq |A||B|. \tag{3.2}$$

The following basis will be useful for our derivation of the Affine MHD equations and in our study of the geometry of SL$(2, \mathbb{R})$.

Definition 3.2. Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

then the **usual basis** of M^2 is $\beta = \{I, Z, K, M\}$. For the representation of a matrix in this basis we write $[A]_\beta$.

Some properties of β and useful quantities' expression in this basis are given below.

Lemma 3.1

The ususal basis is an orthogonal basis of M^2, has only idempotent elements of norm $\sqrt{2}$, and if $A, B \in \mathsf{M}^2$ have usual basis representations

$$[A]_\beta = (a_i) \quad [B]_\beta = (b_i),$$

then

$$\det(A) = a_0^2 + a_1^2 - (a_2^2 + a_3^2), \tag{3.3a}$$

$$\langle A, B \rangle = 2[A]_\beta \cdot [B]_\beta , \tag{3.3b}$$

$$\tfrac{1}{2}|A|^2 = |[A]_\beta|^2. \tag{3.3c}$$

where $\langle \cdot, \cdot \rangle$ and $|[A]_\beta|$ are the usual $\mathbb{R}^4$ innerproduct and norm.

Proof. The properties of β, orthogonality, idempotentency, and size of its elements, are trivial.

For Equation (3.3a):

$$\begin{aligned}\det([A]_\beta) &= \det \begin{bmatrix} a_0 + a_2 & a_1 + a_3 \\ a_1 + a_3 & a_0 - a_2 \end{bmatrix} \\ &= a_0^2 - a_2^2 - (a_3^2 - a_1^2) \\ &= a_0^2 + a_1^2 - (a_2^2 + a_3^2).\end{aligned}$$

For (3.3b) since β is an orthogonal basis and each element has norm $\sqrt{2}$

$$\begin{aligned}\langle A, B \rangle &= \operatorname{tr}\left((a_0 I + a_1 Z + a_2 K + a_3 M)(b_0 I + b_1 Z + b_2 K + b_3 M)\right) \\ &= 2 \sum_i a_i b_i.\end{aligned}$$

(3.3c) follows from (3.3b).

$\square$

The cofactor operator will play a leading role in our analysis of affine motions.

Definition 3.3. Let $A \in \mathbb{M}^n$ and define its ij minor $\tilde{A}_{i;j}$ to be the determinant of the $(n-1) \times (n-1)$ matrix gotten by removing the i^{th} row and j^{th} column of A. Then its **cofactor matrix**, $\operatorname{cof}(A)$, is defined to be

$$[\operatorname{cof}(A)]_{ij} = (-1)^{i+j} \tilde{A}_{i;j}.$$

Lemma 3.2

The map

$$\operatorname{cof} : \mathbb{M}^2 \to \mathbb{M}^2$$

is an automorphism of the algebra M^2 and is represented as conjugation by the basis vector Z. Further, if A is invertible then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{cof}(A)^{\top}. \tag{3.4}$$

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$\begin{aligned} \mathrm{cof}(A) &= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \\ &= -ZAZ \\ &= ZAZ^{-1}. \end{aligned}$$

Conjugation is always an automorphism. The inverse formula (3.4) is a quick computation.

□

The first useful fact about the usual basis is that it diagonalizes the cofactor operator.

Lemma 3.3

In the basis β

$$[\mathrm{cof}] = \mathrm{diag}(1, 1, -1, -1),$$

and satisfies

$$\langle A, \mathrm{cof}(A) \rangle = 2 \det(A). \tag{3.5}$$

Moreover, the determinant map $\det : \mathsf{M}^2 \to \mathsf{R}$ is C^∞ and

$$\frac{\partial}{\partial A} \det A = \mathrm{cof} A. \tag{3.6}$$

Proof. For the diagonalization result we can apply the conjugation result in Lemma 3.2 to each basis vector. Notice

$$\begin{aligned}
\mathrm{cof}(I) &= I \\
\mathrm{cof}(Z) &= -ZZZ = Z, \\
\mathrm{cof}(K) &= -ZKZ = ZM = -K, \\
\mathrm{cof}(M) &= -ZMZ = -ZK = -M,
\end{aligned}$$

as required.

For (3.5) we work in β coordinates and use (3.3b)

$$\begin{aligned}
\tfrac{1}{2}\langle A, \mathrm{cof}(A)\rangle &= [A]^T \mathrm{diag}(1, 1, -1, -1)[A] \\
&= a_0^2 + a_1^2 - a_2^2 - a_3^2 \\
&= \det(A).
\end{aligned}$$

The gradient result (3.6) follows from the conjugation representation of cofactor and differentiating (3.5). ☐

Another reason the usual basis is preferred to the standard basis is its connection to the complex coordinate representation of M^2.

Definition 3.4. Given a matrix $A \in \mathsf{M}^2$ we define its complex coordinates $z, w \in \mathsf{C}$ by

$$Ax = zx + w\bar{x}.$$

Where on the left we do the usual matrix vector product and on the right we use the usual embedding of R^2 into C. For a representation of a matrix in these coordinates we write $[A]_{\mathsf{C}} = (z, w)$.

To connect the complex coordinates with the usual basis we will use various representations of elements of C as vectors in R^2 and special subsets of M^2.

Lemma 3.4

Let

$$z = a + ib \quad w = u + iv$$

be complex numbers. Then, when viewed as complex numbers, the following products are all equivalent

$$zw = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} u & v \\ v & u \end{bmatrix}.$$

Proof. This equality of zw with the matrix product at the end is simply the fact that the map

$$a + ib \ ! \ \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

is a a homomorphism of C into M^2. To get the middle equality we simply compute

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au \quad bv \\ av + bu \end{bmatrix},$$

which when be viewed as an element of C embedded in M^2 becomes

$$\begin{bmatrix} au \quad bv & (av + bu) \\ (av + bu) & au \quad bv \end{bmatrix}.$$

$\square$

Lemma 3.4 allows us to connect the usual basis representation with the complex coordinates. Specifically

$$
\begin{aligned}
Ax &= (a_0 I + a_1 Z + a_2 K + a_3 M)x \\
&= a_0 \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + a_1 \begin{bmatrix} -x_1 \\ x_0 \end{bmatrix} + a_2 \begin{bmatrix} x_0 \\ -x_1 \end{bmatrix} + a_3 \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 & -x_1 \\ x_1 & x_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \\
&= (a_0 + ia_1)x + (a_2 + ia_3)\bar{x}.
\end{aligned}
$$

It is quick to see that this representation is also unique. The next lemma gives a dictionary of common matrix quantities in terms of their complex coordinates.

Lemma 3.5

Let $A \in \mathbb{M}^2$ have complex representation (z, w), then

$$
\begin{aligned}
A^\top &= (\bar{z}, w), && (3.7a) \\
\det(A) &= |z|^2 - |w|^2, && (3.7b) \\
\tfrac{1}{2}|A|^2 &= |z|^2 + |w|^2, && (3.7c) \\
\operatorname{cof}(A) &= (\bar{z}, -w),
\end{aligned}
$$

and if A is invertible

$$
A^{-1} = \frac{1}{|z|^2 - |w|^2}(\bar{z}, -w).
$$

Proof. Notice that the only basis vector of β which is not symmetric is Z and so transposition only acts on the z complex coordinate; moreover, it corresponds to conjugation in the complex coordinates, giving us (3.7a). The determinant relation (3.7b) and norm relation (3.7c) come from (3.3a) and (3.3c) respectively.

□

We now describe a few important subgroups of $\mathbb{M}^2$.

Definition 3.5. The **special linear group** is given by

$$\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathbb{M}^2 : \det A = 1\}$$

and the **special orthogonal** group is

$$\mathrm{SO}(2,\mathbb{R}) = \{U \in \mathrm{SL}(2,\mathbb{R}) : U^{-1} = U^{\top}\}.$$

It is clear that the action of $\mathrm{SO}(2,\mathbb{R})$ on $\mathbb{R}^2$ is an isometry and it turns out this property extends to its action $\mathbb{M}^2$.

Lemma 3.6

For all $A \in \mathbb{M}^2$ and $U, V \in \mathrm{SO}(2,\mathbb{R})$,

$$|UAV| = |A|.$$

The left and right action of $\mathrm{SO}(2,\mathbb{R})$ on $\mathbb{M}^2$ and on $\mathrm{SL}(2,\mathbb{R})$ is an isometry.

Proof. Any $U \in \mathrm{SO}(2,\mathbb{R})$ has complex representation $U = zx + 0\bar{x}$ and since $\det(U) = 1$ (3.7b) $|z| = 1$.

Let $A \in \mathbb{M}^2$ have complex representation $Ax = ux + v\bar{x}$, then

$$UA = zux + zv\bar{x}$$

and by (3.7c)

$$\begin{aligned} \tfrac{1}{2}|UA|^2 &= |zu|^2 + |zv|^2 \\ &= |u|^2 + |v|^2 \\ &= \tfrac{1}{2}|A|^2, \end{aligned}$$

and similarly for $|UAV|$. □

The subgroup $\mathrm{SO}(2,\mathsf{R})$ will play a special role in the sequel. Here is the first of several characterizations that we shall repeatedly use.

Definition 3.6. Define the one-parameter family of rotations

$$U(\sigma) = \exp(\sigma Z) = \begin{bmatrix} \cos\sigma & \sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix}, \quad \sigma \in \mathsf{R}.$$

As we saw in Lemma 3.3 if $U \in \mathrm{SO}(2,\mathsf{R})$ then it has complex representation $[U(\sigma)]_{\mathsf{C}} = (e^{i}\ , 0)$. Moreover, the elements of $\mathrm{SL}(2,\mathsf{R})$ which commute with Z have complex representation $[A]_{\mathsf{C}} = (z, 0)$ with $|z| = 1$ giving us the characterization of $\mathrm{SO}(2,\mathsf{R})$:

$$\mathrm{SO}(2,\mathsf{R}) = \{A \in \mathrm{SL}(2,\mathsf{R}) : [A, Z] = 0\}.$$

Lemma 3.7

Elements of $\mathrm{SO}(2,\mathsf{R})$ are norm minimizers in $\mathrm{SL}(2,\mathsf{R})$. There holds

$$\min\{|A|^2 : A \in \mathrm{SL}(2,\mathsf{R})\} = 2$$

and

$$\mathrm{SO}(2,\mathsf{R}) = \{A \in \mathrm{SL}(2,\mathsf{R}) : \tfrac{1}{2}|A|^2 = 1\}.$$

With the notation of Definition 3.6, we have

$$\mathrm{SO}(2,\mathsf{R}) = \{U(\sigma) : \sigma \in \mathsf{R}\}. \tag{3.8}$$

Finally,

$$\mathrm{SO}(2,\mathsf{R}) = \{A \in \mathrm{SL}(2,\mathsf{R}) : [A, Z] = 0\}. \tag{3.9}$$

Proof. Let A have complex representation $Ax = zx + w\bar{x}$ with $|z| = r$ and $|w| = \rho$. Since $A \in \mathrm{SL}(2, \mathbb{R})$ we have by (3.7b)

$$r^2 - \rho^2 = 1 \tag{3.10}$$

and the quantitiy we would like to minimize is $\frac{1}{2}|A|^2 = r^2 + \rho^2$ whose level sets in the (r, ρ) plane are circles. In the $r - \rho$ pane the unit circle is tagnent to the hyperbola (3.10) and so the smallest magnitude is $\frac{1}{2}|A|^2 = 1$, which by (3.7c) this implies $\rho = 0$ and $r = 1$ so $A = zx$ for a unit length z which implies that $A \in \mathrm{SO}(2, \mathbb{R})$.

□

Narrowing the result of Lemma 3.6 is the fact that the Z basis vector represents an orthogonal rotation in $\mathbb{M}^2$. Specifically

$$\begin{aligned} \langle A, ZA \rangle &= \left\langle Z^{\top} A, A \right\rangle \\ &= -\langle A, ZA \rangle \end{aligned}$$

and so by similar argument we have the orthogonality relations

$$\langle A, ZA \rangle = \langle ZA, AZ \rangle = 0$$

3.2 The Geometry of $\mathrm{SL}(2, \mathbb{R})$

Here we collect some basic facts about the geometry of $\mathrm{SL}(2, \mathbb{R})$. We will view it as an embedded submanifold of $\mathbb{M}^2$ and moreover a Lie Group with the corresponding Lie Algebra $\mathfrak{sl}(2, \mathbb{R})$. Since $\mathrm{SL}(2, \mathbb{R})$ is a level set of the determinant map, Lemma 3.3 allows us to characterize its tangent space as

$$T_A\mathrm{SL}(2,\mathbb{R}) = \{B \in \mathbb{M}^2 : \langle B, \mathrm{cof}A\rangle = \mathrm{tr}\, BA^{-T} = 0\}.$$

We use the notation $A^{-\top} = (A^{-1})^{\top}$.

Definition 3.7. Define the **special linear Lie algebra**

$$\mathfrak{sl}(2,\mathbb{R}) = T_I\mathrm{SL}(2,\mathbb{R}) = \{L \in \mathbb{M}^2 : \mathrm{tr}\, L = 0\} = \mathrm{span}\{K, M, Z\}.$$

Further given $A \in \mathrm{SL}(2,\mathbb{R})$, we define the **unit normal vector field**

$$N(A) = |A|^{-1}\mathrm{cof}A = |A|^{-1}A^{-\top}. \tag{3.11}$$

Lemma 3.8

$A \in \mathrm{SL}(2,\mathbb{R})$ is normal to $T_A\mathrm{SL}(2,\mathbb{R})$ if and only if $A \in \mathrm{SO}(2,\mathbb{R})$.

Proof. This follows from Lemmas 3.7 and Definition 3.7. □

To abbreviate notation we denote the tangent bundle of $\mathrm{SL}(2,\mathbb{R})$ by

$$\mathcal{D} = \{(A,B) \in \mathbb{M}^2 \times \mathbb{M}^2 : A \in \mathrm{SL}(2,\mathbb{R}),\ B \in T_A\mathrm{SL}(2,\mathbb{R})\}.$$

It is clear that $\mathcal{D}$ is a smooth 6-dimensional embedded submanifold of $\mathbb{M}^2$. Specifically 1 is a regular value of the det map and so the by the local immersion theorem $\mathrm{SL}(2,\mathbb{R})$ is an embedded submanifold of $\mathbb{M}^2$. Using the standard embedding of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathbb{M}^2$ and taking the product topology on $\mathcal{D}$ gives the result. The reader is referred to [43].

Definition 3.8. Given $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$ and Z as in Definition 3.2, define

$$
\begin{aligned}
\tau_1(A) &= ZA + AZ, \\
\tau_2(A) &= ZA - AZ, \\
\tau_3(A) &= \frac{|A|^2}{|A|^4 - 4}\left(A - \frac{2}{|A|^2}\mathrm{cof}A\right).
\end{aligned}
$$

We also define $\hat{\tau}_i(A) = \tau_i(A)/|\tau_i(A)|$.

The choice of normalization for $\tau_3(A)$ is motivated by Lemma 3.12 below. In situations when the base point $A \in \mathrm{SL}(2,\mathbb{R})$ is fixed, we shall occasionally find it convenient to write simply τ_i instead of $\tau_i(A)$.

Lemma 3.9

The functions $\tau_i : \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}) \to T_A\mathrm{SL}(2,\mathbb{R})$ are smooth tangent vector fields for which: if $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then

1. $g_{ij}(A) = \langle \tau_i(A), \tau_j(A) \rangle$ defines the metric on $T_A\mathrm{SL}(2,\mathbb{R})$ in local coordinates relative to the basis $\{\tau_i(A)\}$, and $g(A)$ is given by

$$
g(A) = \mathrm{diag}\left[2|A|^2 + 4,\ \ 2|A|^2 - 4,\ \ \frac{|A|^2}{|A|^4 - 4}\right], \tag{3.12}
$$

2. the set $\{\tau_i(A)\}_{i=1}^3$ spans $T_A\mathrm{SL}(2,\mathbb{R})$,

3. and for any $B \in T_A\mathrm{SL}(2,\mathbb{R})$, we have

$$
B = \sum_i c_i\, \tau_i(A), \qquad \text{with} \qquad c_i = \langle B, \tau_i(A) \rangle / g_{ii}(A).
$$

Proof. The orthogonality results follow from the relations in Lemma 3.3 and 3.1 and orthogonality together with a dimension count gives us the spanning property. □

Corollary 3.10

If $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then the set $\{\hat{\tau}_i(A)\}_{i=1}^3$ is an orthonormal frame in $T_A\mathrm{SL}(2,\mathbb{R})$.

A quick application of Lemma 3.3 gives us the following lemma.

Lemma 3.11

The cofactor map acts on the tangent basis as follows:

$$
\begin{aligned}
\operatorname{cof}\tau_1(A) &= \tau_1(A), \\
\operatorname{cof}\tau_2(A) &= -\tau_2(A), \\
\operatorname{cof}\tau_3(A) &= -\frac{2}{|A|^2}\tau_3(A) + \frac{1}{|A|}N(A).
\end{aligned}
$$

The next lemma gives a convenient set of local coordinates for $\mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$.

Lemma 3.12

Define a mapping $A : \mathbb{R}^2 \times [1, \infty) \to \mathbb{M}^2$ by

$$A(s) = U(s_1 + s_2)\, H(s_3)\, U(s_1 - s_2), \quad s = (s_1, s_2, s_3) \in \mathbb{R}^2 \times [1, \infty)$$

with $U(\sigma)$ as in Definition 3.6 and

$$H(\sigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} (\sigma+1)^{1/2} & (\sigma-1)^{1/2} \\ (\sigma-1)^{1/2} & (\sigma+1)^{1/2} \end{bmatrix}.$$

Then

1. the range of A is equal to $\mathrm{SL}(2,\mathbb{R})$,
2. $\frac{1}{2}|A(s)|^2 = s_3$,
3. $A(s) \in \mathrm{SO}(2,\mathbb{R})$ if and only if $s_3 = 1$,
4. the restriction

$$A : \mathbb{R}^2 \times (1, \infty) \to \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$$

is a local diffeomorphism with

$$\partial_i \mathcal{A}(s) = \tau_i(\mathcal{A}(s)), \quad i = 1, 2, 3.$$

Proof. Since $\det \mathcal{A}(s) = \det H(s_3) = 1$, we see that $\mathcal{A}(s) \in \mathrm{SL}(2,\mathbb{R})$, for every $s \in \mathbb{R}^2 \times [1, \infty)$. Moreover, by Lemma 3.6, $|\mathcal{A}(s)|^2 = |H(s_3)|^2 = 2s_3$, so $\mathcal{A}(s) \in \mathrm{SO}(2,\mathbb{R})$ if and only if $s_3 = 1$, by Lemma 3.7.

Let $A \in \mathrm{SL}(2,\mathbb{R})$. Using the polar decomposition, we can find a $U \in \mathrm{SO}(2,\mathbb{R})$ such that $A = U\sqrt{A^\top A}$. Since $\sqrt{A^\top A}$ is a symmetric matrix in $\mathrm{SL}(2,\mathbb{R})$, there exists $V \in \mathrm{SO}(2,\mathbb{R})$ such that

$$V(\sqrt{A^\top A})V^\top = \operatorname{diag}\left[\alpha, \ 1/\alpha\right] = D, \quad \text{with} \quad \alpha \geq 1.$$

Finally, taking

$$W = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

we have $W \in \mathrm{SO}(2,\mathbb{R})$ and $WDW^\top = H(\sigma)$, for $\sigma = (\alpha^2 + \alpha^{-2})/2 \in [1, \infty)$. Thus, we see that

$$A = (UVW)H(\sigma)(W^\top V^\top),$$

with UVW, $W^\top V^\top \in \mathrm{SO}(2,\mathbb{R})$. By Lemma 3.7, this shows that the mapping

$$\mathcal{A} : \mathbb{R}^2 \times [1, \infty) \to \mathrm{SL}(2,\mathbb{R})$$

is surjective.

We next verify the formulas for the derivatives. Since

$$U'(\sigma) = ZU(\sigma) = U(\sigma)Z,$$

we find that

$$\partial_i A(s) = \tau_i(A(s)), \quad i = 1, 2.$$

A simple calculation yields

$$H'(\sigma) = \tau_3(H(\sigma)).$$

Therefore, by Lemmas 3.6, 3.7 and 3.3, we have that

$$\partial_3 A(s) = U(s_1 + s_2)\tau_3(H(s_3))U(s_1 - s_2) = \tau_3(A(s)).$$

Finally, by Lemma 3.9, $\{\tau_i(A)\}_{i=1}^3$ is a frame in $T_A\mathrm{SL}(2,\mathbb{R})$, if $A \in \mathrm{SL}(2,\mathbb{R})\backslash\mathrm{SO}(2,\mathbb{R})$. Thus, we see that the mapping

$$A : \mathbb{R}^2 \times (1, \infty) \to \mathrm{SL}(2,\mathbb{R}) \backslash \mathrm{SO}(2,\mathbb{R})$$

is locally invertible. □

Corollary 3.13

In local coordinates, $\frac{1}{2}|A(s)|^2 = s_3$, and hence, the metric $g(A(s))$ is a function only of s_3. It has the form

$$g(A(s)) = \mathrm{diag}\left[4(s_3 + 1), \quad 4(s_3 - 1), \quad \frac{s_3}{2(s_3^2 - 1)}\right].$$

With abuse of notation, we shall sometimes write $g(s_3)$ instead of $g(A(s))$. It will turn out that the main coordinate describing the motion is s_3. The previous corollary then identifies $H(\sigma)$, a hyperbolic rotation, as the main term in the description of the motion in local coordinates.

This decompostiiont into tow "circular" coordinates and one "hyperbolic", or magnitude, coordiantes is a double edged sword. On the one hand, this isolation of the magnitude in one coordinate is why the metric is diagonal. On the other as $c \to \infty$ then the tori from before collapse to $\mathrm{SO}(2,\mathbb{R})$ in a doubly wrapped way. That is, these coordiantes are a two branched cover of $\mathrm{SO}(2,\mathbb{R})$. An unfortunate fact that makes its wrath known by the singularity our metric faces as we approach it.

Lemma 3.14

For $s \in \mathbb{R}^2 \times [1, \infty)$, the coordinate map can also be expressed as

$$\mathcal{A}(s) = \left(\frac{s_3 + 1}{2}\right)^{1/2} U(2s_1) + \left(\frac{s_3 - 1}{2}\right)^{1/2} U(2s_2)\, M,$$

and the normalized tangent vector fields have the form

$$\begin{aligned}
\hat{\tau}_1(\mathcal{A}(s)) &= \tfrac{1}{\sqrt{2}} U(2s_1) Z, \\
\hat{\tau}_2(\mathcal{A}(s)) &= -\tfrac{1}{\sqrt{2}} U(2s_2) K, \\
\hat{\tau}_3(\mathcal{A}(s)) &= \frac{1}{2}\left[\left(\frac{s_3 - 1}{s_3}\right)^{1/2} U(2s_1) + \left(\frac{s_3 + 1}{s_3}\right)^{1/2} U(2s_2) M\right].
\end{aligned}$$

Proof. The first statement follows from Lemma 3.12 by writing

$$H(\sigma) = \left(\frac{\sigma + 1}{2}\right)^{1/2} I + \left(\frac{\sigma - 1}{2}\right)^{1/2} M,$$

and then using the fact that

$$M\,U(\theta) = U(-\theta)\,M.$$

Differentiating the new expression for $A(s)$ with respect to s yields alternate expressions for $\tau_i(A(s))$. The formulas for $\hat{\tau}_i(A(s))$ follow after normalization. □

We also note that Lemma 3.14 provides an extension of the normalized tangent vectors $\hat{\tau}_i(A)$ to $T_A\mathrm{SL}(2,\mathbb{R})$ for $A \in \mathrm{SO}(2,\mathbb{R})$. In this case, the parameter s_2 is independent of A and simply rotates the frame $\{K, M\}$.

Lemma 3.15

In the local coordinates of Lemma 3.12, the Christoffel symbols depend only on s_3, and they have the form

$$\Gamma^i_{jk}(s_3) = \tfrac{1}{2} g^{ii}(s_3)[\delta_{j3} g'_{ki}(s_3) + \delta_{k3} g'_{ij}(s_3) - \delta_{i3} g'_{jk}(s_3)], \quad s_3 > 1,$$

where $g'(s_3)$ indicates the derivative in s_3.

Proof. Since the metric is diagonal and depends only on s_3, the result follows from the general formula

$$\begin{aligned} \Gamma^i_{jk}(s_3) &= \tfrac{1}{2} \sum_{\ell} g^{i\ell}(s_3)[\partial_j g_{k\ell}(s_3) + \partial_k g_{\ell j}(s_3) - \partial_\ell g_{jk}(s_3)] \\ &= \tfrac{1}{2} g^{ii}(s_3)[\delta_{j3} g'_{ki}(s_3) + \delta_{k3} g'_{ij}(s_3) - \delta_{i3} g'_{jk}(s_3)]. \end{aligned}$$

□

Lemma 3.16

The orthogonal projection of $\mathbb{M}^2$ onto $T_A\mathrm{SL}(2,\mathbb{R})$ is given by

$$P(A) = I \quad N(A) \quad N(A).$$

Definition 3.9. Given $(A, B) \in D$, we define the shape operator

$$S(A)B = \quad dN(A)B = \quad \sum_{a,b} B_{ab} \frac{\partial}{\partial A_{ab}} N(A).$$

Lemma 3.17

The shape operator may be expressed in the form

$$S(A)B = \quad \frac{1}{|A|} P(A) \mathrm{cof} B.$$

Moreover, for each $A \in \mathrm{SL}(2, \mathbb{R})$, the shape operator is symmetric on $T_A \mathrm{SL}(2, \mathbb{R})$.

Proof. By direct computation and Lemma 3.3, we have

$$\begin{aligned}
S(A)B &= \quad \sum_{ab} B_{ab} \frac{\partial}{\partial A_{ab}} \frac{\mathrm{cof} A}{|A|} \\
&= \frac{\mathrm{cof} B}{|A|} + \frac{\mathrm{cof} A}{|A|^3} \langle A, B \rangle \\
&= \frac{1}{|A|} \left(\mathrm{cof} B \quad \left\langle \frac{A}{|A|}, B \right\rangle \frac{\mathrm{cof} A}{|A|} \right) \\
&= \frac{1}{|A|} \left(\mathrm{cof} B \quad \langle N(A), \mathrm{cof} B \rangle N(A) \right) \\
&= \frac{1}{|A|} P(A) \mathrm{cof} B.
\end{aligned}$$

From this formula, we see that $S(A)$ maps into $T_A \mathrm{SL}(2, \mathbb{R})$, and by Lemma 3.3 , the verification of symmetry is immediate. □

Lemma 3.18

If $A \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R})$, then the vectors $\{\tau_i(A)\}$ are principal directions in $T_A \mathrm{SL}(2, \mathbb{R})$ with corresponding principal curvatures

$$\frac{1}{|A|}, \quad \frac{1}{|A|}, \quad \frac{2}{|A|^3}.$$

Proof. The principal curvatures and directions are the eigenvalues and eigenvectors of the shape operator. So this is an immediate consequence of Lemmas 3.11 and 3.17. □

Definition 3.10. The second fundamental form

$$\Pi(A) : T_A\mathrm{SL}(2,\mathbb{R}) \times T_A\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$$

is defined by

$$\Pi(A)[B_1, B_2] = \langle S(A)B_1, B_2 \rangle .$$

Lemma 3.19

For vector fields $V(A)$, $W(A)$, the Riemannian connection ∇ is given by

$$\nabla_{V(A)} W(A) = P(A) \sum_{a,b} V_{ab}(A) \frac{\partial}{\partial A_{ab}} W(A).$$

Lemma 3.20

For vector fields $V(A)$, $W(A)$, $Y(A)$, the curvature tensor is the map given by

$$\begin{aligned} Y(A) \mapsto\ & R[V(A), W(A)]Y(A) \\ &= \Pi(A)[W(A), Y(A)]S(A)V(A) - \Pi(A)[V(A), Y(A)]S(A)W(A). \end{aligned}$$

Corollary 3.21

Relative to the orthonormal basis $\{\hat{\tau}_i(A)\}$, the curvature tensor has the coordinates

$$\langle R(\hat{\tau}_i, \hat{\tau}_j)\hat{\tau}_k, \hat{\tau}_\ell \rangle = R_{ijk\ell} = \lambda_i \lambda_j (\delta_{jk}\delta_{i\ell} - \delta_{ik}\delta_{j\ell}),$$

where $\{\lambda_i\}$ are the principal curvatures.

3.3 Material Coordinates

Though we could have translated our equations (2.10) into material coordinates immediately, the results of the previous section will allow us to write them in a more useful form. The use of material coordinates for the free boundary problem of fluids is commonly used in both the Euler and MHD context. To name just a few see [35, 14, 8, 7, 12] for the Euler equation examples and [15, 16, 17] for MHD. However, it is noted in [32] that the material description can make analyzing contact discontinuities, such as current vortex sheets, difficult. Since we are not concerned with these the material coordinates will do great.

In material coordinates the material time derivative D_t is simply a time derivative. The other differential operators will follow from the chain rule. Recall F is our deformation gradient $D_y x(y, t)$. So

$$\begin{aligned} (B \cdot \nabla_x)B &= D_x B B \\ &= D_y B F^{-1} B \\ &= (F^{-1}B \cdot \nabla_y)B \end{aligned}$$

and similarly $(B \cdot \nabla_x)u = (F^{-1}B \cdot \nabla_y)u$. The conservation of mass and transport of the magnetic field in (2.10) become

$$\begin{aligned} D_t u + F^{-\top} \nabla_y p &= (F^{-1}B \cdot \nabla_y)B \\ D_t B - (F^{-1}B \cdot \nabla_y)u &= 0. \end{aligned}$$

For the divergence conditions we have

$$\begin{aligned}\nabla_x \cdot u &= \operatorname{tr} D_x u \\ &= \operatorname{tr} D_y u\, F^{-1}.\end{aligned}$$

The motion is incompressible so $\det(F) = 1$. By Lemma 3.3 $F^{-\top} = \operatorname{cof}(F)$. By definition 3.1

$$\nabla_x \cdot u = \langle D_y u, \operatorname{cof} F \rangle .$$

Putting all this together gives us MHD in material coordinates

$$\begin{cases} D_t u + \operatorname{cof}(F)\nabla_y p &= (F^{-1}B \cdot \nabla_y)B \\ D_t B - (F^{-1}B \cdot \nabla_y)u &= 0 \\ \langle D_y u, \operatorname{cof} F\rangle = \langle D_y B, \operatorname{cof} F\rangle &= 0 \end{cases} . \tag{3.13}$$

The motion is continuous so $x(\partial \mathcal{B}, t) = \partial\Omega(t)$. Therefore our total pressure condition is

$$q(y,t) = 0$$

for $y \in \partial\mathcal{B}$. The condition on the pullback fo the magnetic field is already in material coordinates.

The remaining boundary conditions require having the normal to the boundary of $\Omega(t)$. Recall the boundary is the unit circle so for any $y \in \partial\mathcal{B}$ the tangent vector $v(y)$ to $\partial\mathcal{B}$ at y is Zy. Thus

$$\begin{aligned} n(x(y,t)) &= Z D_y x(y,t) v(y) \\ &= ZFZy \\ &= -\operatorname{cof}(F)y. \end{aligned}$$

The Rayleigh-Taylor sign condition then becomes

$$
\begin{aligned}
\nabla_x p(x(y,t)) \cdot n(x(y,t)) &= \operatorname{cof}(F)\nabla_y q(y,t) \cdot \operatorname{cof}(F)y \\
&= \nabla_y q(y,t) \cdot \operatorname{cof}(F^T F)y < 0,
\end{aligned}
$$

and the zero magnetic flux condition becomes

$$
B(y,t) \cdot \operatorname{cof}(F)y = 0
$$

All together the boundary conditions in material coordinates

$$
\left\{
\begin{array}{rcl}
q(y,t) &=& 0 \\
|F^{-1}B(y,t)| &=& \kappa_0 \\
\nabla_y p \cdot \operatorname{cof}(F^T F)y &<& 0 \\
B(y,t) \cdot \operatorname{cof}(F)y &=& 0
\end{array}
\right. \qquad y \in \partial B.
$$

Chapter 4

Equations of Affine Motion

We now specify that our reference domain, $\mathcal{B}$, from the previous section will be the unit ball in $\mathbb{R}^d$.

Definition 4.1. An **incompressible affine motion** defined on the unit ball reference domain, $\mathcal{B}$, is a one-parameter family of volume preserving diffeomorphism of the form

$$x(t,y) = A(t)y \quad y \in \mathcal{B}, t \in \mathbb{R},$$

with

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2).$$

The use of affine deformations is well established in continuum mechanics, see for instance [28]. They were first introduced in the context of the vacuum free boundary incompressible Euler system by Sideris [37] and later expanded on in [35, 31]. In the case of compressible Euler equations Hadžić and Jang [14] were able to show that perturbations, in an appropriate Sobolev space, of these affine solutions in fact remain global in time large solutions to the equations. Moreover, the growth of the fluid domain is tied to the growth of the domain given in [37]. This result was extended in [33].

Under affine motions our fluid domain, $\Omega(t)$, is an ellipse centered at the origin with principal axes determined by the eigenvectors and eigenvalues of the positive definite symmetric (stretch) matrix $(A(t)A(t)^\top)^{1/2}$. The velocity of the fluid is given in material coordinates simply by

$$u(y,t) = \dot{A}(t)y,$$

and in spatial coordinates

$$u(x,t) = \dot{A}(t)A^{-1}(t)x.$$

We work in tangent bundle $\mathcal{D}$ since keeping track of not just the deformation gradient, but also the velocity gradient will be useful.

Definition 4.2. Define the mapping $L : \mathcal{D} \to \mathfrak{sl}(2,\mathbb{R})$ by

$$L(A,B) = BA^{-1}.$$

For our affine motion, the velocity gradient $D_x u = L(A(t), \dot{A}(t))$.

Refocusing on our derivation we now substitute our affine versions of u into (3.13), which gives us

$$\begin{cases} \ddot{A}(t)y + \mathrm{cof}(A)\nabla_y q &= D_y B A^{-1} B \\ D_t B - \dot{A}A^{-1}B &= 0 \\ \left\langle \mathrm{cof}(A), \dot{A} \right\rangle = \langle \mathrm{cof} A, D_y B\rangle &= 0 \end{cases},$$

4.1 Deriving Affine Versions of Physical Quantities

We will refer to the velocity, total pressure, and magnetic field as the *physical triple* of equation (2.10), or more often (3.13) since we will work almost exclusively in material coordinates.

Lemma 4.1

The solution to the magnetic field equation in (3.13) for affine motions are of the form

$$B(t, y) = \kappa_0 A(t) b(y),$$

where

$$\begin{cases} \nabla_y \cdot b = 0 & y \in \mathcal{B} \\ b \cdot y = 0 & y \in \partial\mathcal{B} \\ |b(y)| = 1 & y \in \partial\mathcal{B} \end{cases},$$

and $\kappa_0 \in \mathbb{R}$.

Proof. Notice that

$$D_t B = \kappa_0 \dot{A}(t) b(y) = \kappa_0 \dot{A} A^{-1} A b = \dot{A} A^{-1} B$$

and

$$\begin{aligned} \langle \operatorname{cof}(A), D_y B \rangle &= \langle \operatorname{cof}(A), A D_y b \rangle \\ &= \operatorname{tr}(A^{-1} A D_y b) \\ &= \operatorname{tr}(D_y b) \\ &= \nabla_y \cdot b. \end{aligned}$$

The boundary condition

$$\begin{aligned} B \cdot \operatorname{cof}(A) y &= A(t) b(y) \cdot \operatorname{cof}(A) y \\ &= y^\top \operatorname{cof}(A)^\top A b(y) \\ &= y^\top I b(y). \end{aligned}$$

The final boundary condition follows from the material boundary condition (2.8). $\square$

A special class of simple affine motions will be excluded temporarily from the derivation but we will later show that the dynamics described below can encompass certain forms of these motions.

Definition 4.3. If $x(t, y) = A(t)y$ is an affine motion where $A^\top A$ is constant then we say that the motion is **rigid**.

We recall that $A^\top A$ encodes the principle axis of the fluid ellipse and so if this is constant then the motion can consist only of isometries of $\mathbb{R}^2$.

Before completing our full derivation we need the following lemma that will allow us to specify the material magnetic field required for (3.13) to admit affine solutions.

Lemma 4.2

Let v be a vectorfield on $\bar{B}$ satisfy

$$\begin{aligned} |v| &= 1 \\ v \quad y &= 0 \end{aligned}$$

on ∂B. Then if $D_y v v = M_0 y$ for some y it must be the case that

$$M_0 = c_0 I$$

for some $c_0 \in \mathbb{R}$.

Proof. By our boundary assumptions on v we see for all $\hat{y} \in \partial B$

$$v(\hat{y}) = \quad Z\hat{y},$$

and the boundary of the unit disk is a level set for $\frac{1}{2}|v|^2$ so

$$D_y \left[\tfrac{1}{2}|v|^2\right]^\top (\hat{y}) = c(\hat{y})\hat{y}$$

for some scalar function $c(\cdot)$. Notice

$$D_y\left[\tfrac{1}{2}|v|^2\right] = D_y v^\top v$$

and so on the boundary

$$\begin{aligned} v^\top D_y v &= v^\top M_0 y \\ c(\hat{y})\hat{y} \cdot v &= (\ \ Z\hat{y}) \cdot M_0\hat{y} \\ 0 &= \ \ \hat{y} \cdot (M_0 Z\hat{y}), \end{aligned}$$

which implies $M_0 = c_0 I$ as required. □

We are now ready to derive the form of physical triples that arise from affine motions. We note that while the total pressure and spatial velocity are completely described there remains some ambiguity in the exact form of the material magnetic field.

Lemma 4.3

Let $x(y,t) = A(t)y$ be a non rigid affine motion. Then $x(t,y)$is a solution of (3.13) if and only if its physical triple is

$$u(t,y) = \ A'(t)y, \quad q(t,y) = \ \lambda(t)(1 \ \ |y|^2), \quad B(t,y) = \ \kappa_0 A(t) b(y),$$

where the material magnetic field satisfies

$$\begin{aligned} \nabla_y \cdot b &= 0 \\ D_y b b &= \kappa_1 y \end{aligned}$$

inside $\mathcal{B}$,

$$\begin{aligned} b \cdot y &= 0 \\ |b| &= 1 \end{aligned}$$

on ∂B.

$A(t)$ satisfies the ODE

$$A''(t) - \lambda(t)\mathrm{cof}(A)(t) = \kappa_1\kappa_0^2 A(t).$$

where $\kappa_i \in \mathbb{R}$.

Proof. For the forward direction we immediately have

$$u(t,y) = A'(t)y,$$

and by Lemma 4.1

$$B(t,y) = \kappa_0 A(t) b(y)$$

for a b which satisfies

$$\nabla_y \cdot b = 0 \quad y \in B, \qquad b \cdot \hat{y} = 0, \quad |b(\hat{y})| = 1 \quad \hat{y} \in \partial B.$$

Plugging these into (3.13) gives us

$$A''(t)y + \mathrm{cof}(A)\nabla_y q(t,y) = \kappa_0^2 A(t) D_y b b(y),$$

or

$$A^{-1}A''(t)y + \mathrm{cof}(A^\top A)(t)\nabla_y q(t,y) = \kappa_0^2 D_y b b(y). \tag{4.1a}$$

Differentiating (4.1a) with respect to y gives us

$$A^{-1}A''(t) + \mathrm{cof}(A^\top A)(t) D_y^2 q(t,y) = \kappa_0^2 D_y[D_y b b(y)]. \tag{4.1b}$$

Thus for any $y_1, y_2 \in B$

$$\operatorname{cof}(A^\top A)(t)(D_y^2 q(t,y_1) - D_y^2 q(t,y_2)) = 0,$$

implying that $D_y^2 q(y,t) = M_0(t)$. Where $M_0(t)$ is a path of symmetric 2×2 matrices. This makes (4.1b)

$$A^{-1}A''(t) + \operatorname{cof}(A^\top A)(t)M_0(t) = \kappa_0^2 D_y[D_y bb(y)]$$

which implies there is some $M_2 \in \mathbb{M}^2$ so that

$$A^{-1}A''(t) + \operatorname{cof}(A^\top A)(t)M_0(t) = M_2 = \kappa_0^2 D_y[D_y bb(y)] \tag{4.2}$$

We now focus on the total pressure. Integrating $D_y^2 q$ with respect to y gives

$$\nabla_y q = M_0(t)y + c_0(t).$$

Recall that the boundary conditions imply that the boundary is a level, null, set of the total pressure and so for all $\hat{y} \in \partial B$ $\nabla_y q = \tilde{\lambda}(t)\hat{y}$ for some $\tilde{\lambda}(t)$. Thus on the boundary

$$\begin{aligned} M_0(t)\hat{y} + c_0(t) &= \tilde{\lambda}(t)\hat{y} \\ (M_0(t) - \tilde{\lambda}(t)I)\hat{y} &= -c_0(t), \end{aligned}$$

which implies that $M_0(t) = \tilde{\lambda}(t)I$. By symmetry of I this implies

$$\begin{aligned} q(t,y) &= \tfrac{1}{2} y^T \tilde{\lambda}(t) y + c_1(t) \\ &= \lambda(t)(1 - |y|^2), \end{aligned}$$

where $\lambda(t)$ is a multiple of $\tilde{\lambda}(t)$. Plugging this and integrating the material magnetic field side of (4.2) gives us

$$A \ ^{1}A''(t)y + \lambda(t)\mathrm{cof}(A^{\top}A)(t)y = M_2 y + c_2$$

for some $c_2 \in \mathbb{R}^2$. Plugging in $y = 0$ shows us that $c_2 = 0$, then Lemma 4.2 implies that $M_2 = \kappa_1 I$ for some $c \in \mathbb{R}$.

The reverse direction is a simple computation. □

Notice that the total pressure satisfies the Rayleigh Taylor like sign condition when $\lambda(t) > 0$. We will show that the sign of $\lambda(t)$ is preserved by the and so this will reduce to a condition on the initial data.

Though the above will be sufficient to analyze the affine motion we would like to have a better idea of what the magnetic field is doing. Using the usual basis representation and our knowledge about what the self directional derivative of our material magnetic field must be we can arrive at a number of interesting relations for this field.

Lemma 4.4

The material magnetic field, $b(y)$, satisfies the

$$\det D_y b = \quad c \tag{4.3}$$

and the orthogonality relations

$$\langle Zb, \Delta b\rangle = \langle Zb, \partial_{12} b\rangle = \langle Zb, \quad b\rangle = 0$$

where the differential operators

$$\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \qquad = \frac{\partial^2}{\partial y_1^2} \quad \frac{\partial^2}{\partial y_2^2} \quad \partial_{12} = \frac{\partial^2}{\partial y_1 \partial y_2}$$

are applied component wise.

Proof. This result follows from the divergence free condition on b and some index manipulation. We let b^i_j be the derivative of the i^{th} component of the vectorfield b with respect to the j^{th} componety of y. Repeatetd indicies are summed over so our divergence free condition reads

$$b^l_l = 0$$

The equation defining M_0 in coordinates is thus

$$b^k b^i_k = m_{ik} y_k$$

differentiating the expression with respect to y^j gives

$$b^k_j b^i_k + b^k b^i_{kj} = m_{ij}$$

The first coordinate equation may be written

$$\begin{aligned} m_{ll} &= b^k_l b^l_k + b^k b^l_{kl} \\ &= b^k_l b^l_k + b^k \partial_k b^l_l \\ &= b^k_l b^l_k + 0 \\ &= (b^l_l)^2 + b^k_l b^l_k \quad b^l_l b^k_k \\ &= \quad 2(b^1_1 b^2_2 \quad b^2_1 b^1_2) \\ &= \quad 2\det(D_y b) \end{aligned}$$

We tackle the mixed derivatives next as it follows most similarly to the first. For this argument recall the divergence free condition $b^1_1 = \quad b^2_2$.

$$\begin{aligned} m_{11} \quad m_{22} = 0 &= b^k_1 b^1_k \quad b^k_2 b^2_k + b^k b^1_{k1} \quad b^k b^2_{k2} \\ &= (b^1_1)^2 \quad (b^2_2)^2 + b^k \partial_k (b^1_1 \quad b^2_2) \\ &= b^k \partial_k (b^1_1 \quad b^2_2) \\ &= b^1(b^1_{11} \quad b^2_{21}) + b^2(b^1_{12} \quad b^2_{22}) \\ &= \quad 2b^1 b^2_{12} + 2b^2 b^1_{12} \\ &= 2hZb, \partial_{12} b i \ . \end{aligned}$$

For the next notice

$$\begin{aligned} m_{12} + m_{21} = 0 &= b^k_1 b^2_k + b^k_2 b^2_k + b^k b^1_{k2} + b^k b^2_{k1} \\ &= (b^l_l)(b^n_k + b^k_n) + b^k b^1_{k2} + b^k b^2_{k1} \\ &= b^1(b^1_{12} + b^2_{11}) + b^2(b^1_{22} + b^2_{21}) \\ &= b^1(b^2_{11} \quad b^2_{22}) \quad b^2(b^1_{11} \quad b^1_{22}) \\ &= hZb, \quad bi \end{aligned}$$

The final component will follow similarly.

$$\begin{aligned} m_{12} + m_{21} = 0 &= \quad b^k_1 b^2_k + b^k_2 b^2_k \quad b^k b^1_{k2} + b^k b^2_{k1} \\ &= (b^l_l)(b^n_k \quad b^k_n) + b^k b^1_{k2} \quad b^k b^2_{k1} \\ &= b^1(b^1_{12} \quad b^2_{11}) + b^2(b^1_{22} \quad b^2_{21}) \\ &= \quad b^1(b^2_{11} + b^2_{22}) + b^2(b^1_{11} + b^1_{22}) \\ &= \quad hZb, \Delta bi \end{aligned}$$

as required.

□

Using Lemma 4.4 we can pin down the sign of κ_1. The divergence free condition implies that $D_y b$ is trace free and so is in spanfZ, K, Mg. If $\kappa_1 \quad 0$ then by (4.3) $\det D_y b \quad 0$ which by the usual basis expression for the determinant in Lemma 3.1 implies that if

$$D_y b = a_1 Z + a_2 K + a_3 M$$

then

$$\det D_y b = a_1^2 \quad (a_2^2 + a_3^2)$$

and so $D_y b$ must be anti-symmetric. However, this would contradict the magnetic field conditions of Lemma 4.3 and so κ_1 must be strictly negative. Putting all of this together we get the following corollary.

Corollary 4.5

Let $x(y,t) = A(t)y$ be a non rigid affine motion. Then $x(t,y)$is a solution of (3.13) if and only if the physical triple is set as in Lemma 4.3 and $A(t)$ satisfies the ODE

$$A''(t) - \lambda(t)\mathrm{cof}(A)(t) = -\kappa A(t). \tag{4.4}$$

where $\kappa \geq 0$, and $\kappa = 0$ corresponds to the Affine Euler Equations.

Remark 4.1. The equations of motion (4.4) are the Euler-Lagrange equations associated to the Lagrangian $\mathsf{L} : \mathsf{M}^2 \times \mathsf{M}^2 \times \mathsf{R} \to \mathsf{R}$ given by

$$\mathsf{L}(A, A', \lambda) = \tfrac{1}{2}|A'|^2 - \tfrac{1}{2}c_0^2|A|^2 + \lambda(\det A - 1).$$

The scalar function $\lambda(t)$ in (4.4) is a Lagrange multiplier which will now be identified.

Definition 4.4. Given a parameter value $\kappa \geq 0$, define the Lagrange multiplier map $\Lambda_\kappa : D \to \mathsf{R}$ by

$$\Lambda_\kappa(A,B) = \frac{\mathrm{tr}[L(A,B)^2] + 2\kappa}{\mathrm{tr}\, A^{-\top}A^{-1}} = \frac{\mathrm{tr}[(BA^{-1})^2] + 2\kappa}{\mathrm{tr}\, A^{-\top}A^{-1}}.$$

Using the inner product introduced in 3.1 and results from Lemma 3.3 we can rewrite the Lagrange Multiplier as

$$\begin{aligned} \Lambda_\kappa(A,B) &= \frac{\mathrm{tr}[L(A,B)^2] + 2\kappa}{\mathrm{tr}\, A^{-\top}A^{-1}} \\ &= \frac{\left\langle L^T, L\right\rangle + \kappa}{\frac{1}{2}|\mathrm{cof}(A)|^2} \\ &= \frac{\frac{1}{2}\left\langle L^T, L\right\rangle + \kappa}{\frac{1}{2}|A|^2}. \end{aligned} \tag{4.5}$$

Lemma 4.6

Fix $\kappa \geq 0$. If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ satisfies

$$A''(t) + \kappa A(t) = \lambda(t)\, A(t)^{-\top}, \quad t \in \mathbb{R},$$

for some function $\lambda \in C^0(\mathbb{R}, \mathbb{R})$, then

$$\lambda(t) = \Lambda_\kappa(A(t), A'(t)).$$

Proof. Since

$$A'(t) = L(A(t), A'(t))A(t),$$

we have

$$A''(t) = L(A(t), A'(t))'\, A(t) + L(A(t), A'(t))\, A'(t).$$

It follows that

$$\begin{aligned} L(A(t), A'(t)) &= A''(t)A(t)^{-1} - L(A(t), A'(t))\, A'(t)A(t)^{-1} \\ &= \left(-\kappa A(t) + \lambda(t)\, A(t)^{-\top}\right) A(t)^{-1} - L(A(t), A'(t))^2 \\ &= -\kappa I + \lambda(t)\, A(t)^{-\top} A(t)^{-1} - L(A(t), A'(t))^2. \end{aligned}$$

Taking the trace, we obtain

$$\operatorname{tr} L(A(t), A'(t))' = \operatorname{tr} A(t)^{-\top} A(t)^{-1} \left[\lambda(t) - \Lambda_\kappa(A(t), A'(t))\right].$$

Since $(A(t), A'(t)) \in \mathcal{D}$, we have $\operatorname{tr} L(A(t), A'(t)) = 0$, which implies the result. □

Definition 4.5. Given a parameter value $\kappa \geq 0$, define the energy map $E_\kappa : \mathcal{D} \to [\kappa, \infty)$

by

$$E\ (A, B) = \tfrac{1}{2}|B|^2 + \tfrac{}{2}|A|^2.$$

We refer to $\frac{1}{2}|B|^2$ as the kinetic energy and $\frac{}{2}|A|^2$ as the potential energy.

By Lemma 3.7 the potential energy is minimized on $\mathrm{SO}(2, \mathbb{R})$, and so

$$E\ (A, B) \quad \tfrac{}{2}|A|^2 \quad \kappa,$$

for all $(A, B) \in D$.

Theorem 4.7

Given a parameter value $\kappa \quad 0$ and initial data

$$(A_0, B_0) \in D, \tag{4.6}$$

the initial value problem

$$A''(t) + \kappa A(t) = \Lambda\ (A(t), A'(t)) \operatorname{cof}(A), \tag{4.7}$$

$$(A(0), A'(0)) = (A_0, B_0) \tag{4.8}$$

has a unique global solution $A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$. Additionally, D is invariant:

$$(A(t), A'(t)) \in D, \quad \text{for all} \quad t \in \mathbb{R},$$

and the energy is conserved:

$$E\ (A(t), A'(t)) = E\ (A_0, B_0), \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof. First we show that the system has local in time solutions. For this we write it in system form as

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} B \\ \Lambda(A,B)\mathrm{cof}(A) \end{pmatrix} = G(A,B)$$

To see that G is locally Lipschitz on a compact subset of $\mathcal{D}$ let $(A_i, B_i) \in \mathcal{D}$ and notice

$$\begin{aligned} |G(A_1,B_1) - G(A_2,B_2)| &\leq |B_1 - B_2| + \kappa|A_1 - A_2| \\ &\quad + |\Lambda(A_1,B_1)\mathrm{cof}(A_1) - \Lambda(A_2,B_2)\mathrm{cof}(A_2)| \\ &\leq c|(A_1,B_1) - (A_2,B_2)| \\ &\quad + |\Lambda(A_1,B_1)\mathrm{cof}(A_1) - \Lambda(A_2,B_2)\mathrm{cof}(A_2)|. \end{aligned}$$

For the remaining term notice by linearity of cof we have

$$\begin{aligned} |\Lambda(A_1,B_1)\mathrm{cof}(A_1) - \Lambda(A_2,B_2)\mathrm{cof}(A_2)| &\leq |\Lambda(A_1,B_1)||\mathrm{cof}(A_1 - A_2)| \\ &\quad + |\Lambda(A_1,B_1) - \Lambda(A_2,B_2)||\mathrm{cof}(A_2)| \\ &\leq |\Lambda(A_1,B_1)||(A_1,B_1) - (A_2,B_2)| \\ &\quad + |\Lambda(A_1,B_1) - \Lambda(A_2,B_2)||A_2|. \end{aligned}$$

Using the representation of (4.5), the product bound (3.2), and the minimality condition of Lemma 3.7 we have for any $(A,B) \in \mathcal{D}$

$$\begin{aligned} \Lambda(A,B) = \frac{\langle L^T, L\rangle + \kappa}{\frac{1}{2}|A|^2} &\leq \frac{|L|^2 + \kappa}{\frac{1}{2}|A|^2} \\ &\leq \frac{|B|^2|A^{-1}|^2 + \kappa}{\frac{1}{2}|A|^2} \end{aligned}$$

Since $A \in \mathrm{SL}(2,\mathbb{R})$ $A^{-1} = \mathrm{cof}(A)^T$ and by Lemma 3.3 the cofactor operator is an isometry so we have

$$
\begin{aligned}
\Lambda\ (A,B) \quad & \frac{|B|^2|A|^2+\kappa}{\frac{1}{2}|A|^2} \\
& 2|B|^2+\frac{2\kappa}{|A|^2} \\
& 2|B|^2+\kappa.
\end{aligned}
$$

Thus the Lagrange Multiplier is bounded. A similar argument shows it is locally Lipschitz. Therefore if our path preserves D then we have local in time solutions to (4.4). Notice by Lemma 4.6 we see that with Λ as defined that our flow must remain in D.

Energy conservation is a quick computation.

$$
\begin{aligned}
E' &= \langle A', A''\rangle + \kappa\langle A, A'\rangle \\
&= \Lambda\ \langle A', \operatorname{cof}(A)\rangle \quad \kappa\langle A, A'\rangle + \kappa\langle A, A'\rangle \\
&= 0.
\end{aligned}
$$

It is then quick to see that conservation of the energy implies that $G(A,B)$ is uniformly Lipschitz and so we have global solutions. □

We conclude this section with a result that echos Arnold's Theorem [4] that solutions to the Incompressible Euler Equations form geodesics in the space of volume preserving diffeomorphisms. Our situation is slightly different in that our free boundary technically prevents this space from being an honest Lie Group, or group for that matter, but it is an interesting connection none the less.

Corollary 4.8

A curve $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a geodesic in $\mathrm{SL}(2,\mathbb{R})$ with the (induced) Euclidean metric if and only if it satisfies (4.7) with $\kappa = 0$. (We include constant solutions as geodesics.)

Proof. A geodesic curve is one for which $A'(t)$ is parallel along $A(t)$. That is

$$\frac{D_A}{dt} A'(t) = 0,$$

in which the covariant derivative along $A(t)$ is

$$\frac{D_A}{dt} = \nabla_{A'(t)} = P(A(t)) \frac{d}{dt}.$$

Thus, $A(t)$ is a geodesic if and only if

$$A''(t) = \lambda(t) N(A(t)),$$

for some scalar $\lambda(t)$. The result follows from Lemma 4.6. $\square$

Chapter 5

Invariant Quantities and Sets

5.1 Conserved Quantities

Aside from the energy of the system we have other conserved quantities. The collection of our three conserved quantities will give us foliations of our phase space that allows us to determine the dynamics from these invariants alone.

Definition 5.1. Define the maps $X_i : \mathcal{D} \to \mathbb{R}$, $i = 1, 2$, by

$$X_1(A, B) = \langle ZA, B \rangle \quad \text{and} \quad X_2(A, B) = \langle AZ, B \rangle .$$

We shall frequently write $X(A, B)$ for $(X_1(A, B), X_2(A, B))$.

These quantities, aside from being conserved, give us a nice characterization of $\mathrm{SO}(2, \mathbb{R})$. Let

$$S = \{A \in \mathrm{SL}(2, \mathbb{R}) : X_1(A, B) = X_2(A, B), \text{ for all } B \in T_A\mathrm{SL}(2, \mathbb{R})\}, \tag{5.1}$$

and notice

$$
\begin{aligned}
S &= \{A \in \mathrm{SL}(2,\mathbb{R}) : \langle [A,Z], B\rangle = 0, \text{ for all } B \in T_A\mathrm{SL}(2,\mathbb{R})\} \\
&= \{A \in \mathrm{SL}(2,\mathbb{R}) : [A,Z] = 0\}.
\end{aligned}
$$

Which by the characterization (3.9) implies that S is $\mathrm{SO}(2,\mathbb{R})$.

Theorem 5.1

If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a solution of the IVP (4.6), (4.7), (4.8), then the quantities

$$
E\ (A(t), A'(t)) \quad \text{and} \quad X_i(A(t), A'(t)), \quad i = 1, 2,
$$

are invariant.

Proof. The first statement is just conservation of energy which was already noted in Theorem 4.7.

We can easily compute the derivatives:

$$
\begin{aligned}
\frac{d}{dt} X_1(A(t), A'(t)) &= \frac{d}{dt} \langle A'(t), ZA(t)\rangle \\
&= \langle A''(t), ZA(t)\rangle + \langle A'(t), ZA'(t)\rangle \\
&= \quad \kappa \langle A(t), ZA(t)\rangle + \Lambda\ (A(t), A'(t)) \langle \mathrm{cof} A(t), ZA(t)\rangle \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt} X_2(A(t), A'(t)) &= \frac{d}{dt} \langle A'(t), A(t)Z\rangle \\
&= \langle A''(t), A(t)Z\rangle + \langle A'(t), A'(t)Z\rangle \\
&= \quad \kappa \langle A(t), A(t)Z\rangle + \Lambda\ (A(t), A'(t)) \langle \mathrm{cof} A(t), A(t)Z\rangle \\
&= 0.
\end{aligned}
$$

□

Observe that by Lemma 3.6, the Lagrangian defined in Remark 4.1 is invariant under the left and right action of $\mathrm{SO}(2,\mathsf{R})$:

$$\mathsf{L}(A,B,\lambda)=\mathsf{L}(UA,UB,\lambda)=\mathsf{L}(AU,BU,\lambda),$$

for all $(A,B,\lambda)\in\mathsf{M}^2\times\mathsf{M}^2\times\mathsf{R}$ and $U\in\mathrm{SO}(2,\mathsf{R})$. Therefore, we can deduce the invariants $X_i(A,B)$ from Noether's theorem. For example, we have

$$X_1(A,B)=\frac{\partial}{\partial\sigma}\left\langle\frac{\partial\mathsf{L}(A,B)}{\partial B},U(\sigma)A\right\rangle\Big|_{\sigma=0}.$$

For 2d incompressible perfect fluids ($\kappa=0$) taking the curl of (2.4) in material coordinates shwo that the material vorticity, i.e. curl $u(t,x(t,y))$, is independent of time in general. However, this does not hold in general for MHD ($\kappa>0$). The following corollary shows that the material vorticity is conserved even for non zero κ in the affine case.

Corollary 5.2

If $A\in C^0(\mathsf{R},\mathrm{SL}(2,\mathsf{R}))\cap C^2(\mathsf{R},\mathsf{M}^2)$, then its vorticity $W(t)$ satisfies

$$W(t)\equiv\tfrac{1}{2}[L(A(t),A'(t))-L(A(t),A'(t))^\top]=\tfrac{1}{2}X_2(A(t),A'(t))Z.$$

If $A(t)$ is also solution of the system (4.7), then its vorticity is invariant and the corresponding affine motion is invariant if and only if $X_2=0$.

Proof. Since $W(t)$ is anti-symmetric, we can write $W(t)=\omega(t)Z$, for some scalar function $\omega(t)$. Then by Lemma 3.3 and the orthogonality relations 3.1 we have

$$
\begin{aligned}
2\,\omega(t) &= \omega(t)\langle Z, Z\rangle \\
&= \langle W(t), Z\rangle \\
&= \tfrac{1}{2}\left\langle L(A(t), A'(t)) - L(A(t), A'(t))^{\top}, Z\right\rangle \\
&= \langle L(A(t), A'(t)), Z\rangle \\
&= \left\langle A'(t)A(t)^{-1}, Z\right\rangle \\
&= \left\langle A'(t), Z A(t)^{-\top}\right\rangle \\
&= \langle A'(t), Z \operatorname{cof} A(t)\rangle \\
&= \langle A'(t), A(t) Z\rangle \\
&= X_2(A(t), A'(t)).
\end{aligned}
$$

This is invariant by Theorem 5.1, if $A(t)$ solves (4.7). □

5.2 Invariant Sets

Definition 5.2. Given parameter values $\kappa \quad 0$ and

$$(E, X) = (E, X_1, X_2) \in [\kappa, \infty) \quad \mathbb{R}^2,$$

define

$$\mathcal{D}(X) = \{(A, B) \in \mathcal{D} : X_i(A, B) = X_i,\ i = 1, 2\}.$$

and

$$\mathcal{D}\ (E, X) = \{(A, B) \in \mathcal{D} : E\ (A, B) = E,\ X_i(A, B) = X_i,\ i = 1, 2\}.$$

Lemma 3.9 allows us to express any tangent vector $B \in T_A \mathrm{SL}(2, \mathbb{R})$ in terms of the basis $\{\tau_i\}$ as

$$B = \sum_{i=1}^{3} c_i \tau_i(A), \quad \text{with} \quad c_1 = \frac{X_1 + X_2}{g_{11}(A)},\ c_2 = \frac{X_1 - X_2}{g_{22}(A)},\ c_3 \in \mathbb{R}. \tag{5.2}$$

In this case, $c_3 = \langle A, B \rangle$. Moreover, at a fixed $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, if we have $(A,B) \in \mathcal{D}(X)$ then $(A,B) \in \mathcal{D}(E,X)$ if and only if

$$E = \frac{\kappa}{2}|A|^2 + \frac{1}{2}\sum_{i=1}^{3} g_{ii}(A) c_i^2. \tag{5.3}$$

Therefore fixing an $X \in \mathbb{R}^2$ and an energy E gives us a unique, up to sign, tangent vector B. In the case where $\kappa = 0$ this implies that each invariant triple defines a vector field on $\mathrm{SL}(2,\mathbb{R})$, whose trajectories foiliate $\mathrm{SL}(2,\mathbb{R})$. Further the left and right action of $\mathrm{SO}(2,\mathbb{R})$ on $\mathcal{D}$ moves one between disjoint trajectories.

In the case where $\kappa > 0$ the situation is more complicated since when $\frac{1}{2}|A|^2 = \frac{1}{\kappa}E$ then $\mathcal{D}(E,X) = \{(A,0)\}$ and so necessarily $X = 0$, and if $\frac{1}{2}|A|^2 = \frac{1}{\kappa}E$ then $\mathcal{D}(E,X)$ is empty. Thus we can only hope to foiliate an open subset of $\mathrm{SL}(2,\mathbb{R})$ on which

$$\tfrac{1}{2}|A|^2 < \frac{1}{\kappa}E.$$

We now describe the invariant sets for $\mathrm{SO}(2,\mathbb{R})$.

Lemma 5.3

Fix $\kappa \geq 0$. Let $(E,X) \in [\kappa, \infty) \times \mathbb{R}^2$. Suppose that $(A,B) \in \mathcal{D}$ and $A \in \mathrm{SO}(2,\mathbb{R})$.

Then $(A,B) \in \mathcal{D}(X)$ if and only if $X_1 = X_2$ and

$$A = U(2s_1), \quad B = (\tfrac{1}{2}X_1)\, U(2s_1)\, Z + \beta\, U(2s_2)\, M, \tag{5.4}$$

with $\beta \geq 0$, $s_1, s_2 \in \mathbb{R}$.

Moreover, $(A,B) \in \mathcal{D}(E,X)$, if and only if $(A,B) \in \mathcal{D}(X)$ and

$$E = \kappa + \tfrac{1}{4}X_1^2 + \beta^2.$$

Proof. Using Lemma 3.12, write $A = A(s_1, 0, 1) = U(2s_1)$. By Lemma 3.14, for $B \in T_A\mathrm{SL}(2, \mathbb{R})$, we have

$$B = c_1 U(2s_1)Z \quad c_2 K + c_3 M.$$

Note that $\quad K = ZM$ and take $(c_3, c_2) = \beta(\cos 2s_2, \sin 2s_2)$. Then

$$c_2 K + c_3 M = \beta(\sin 2s_2 Z + \cos 2s_2 I)M = \beta U(2s_2)M,$$

so that

$$B = c_1 U(2s_1)Z + \beta U(2s_2)M.$$

Now since $A \in \mathrm{SO}(2, \mathbb{R})$ and $(A, B) \in D(X)$, we have

$$X_1 = X_2 = X_2(A, B) = \langle AZ, B\rangle = 2c_1,$$

which yields the formula (5.4).

By (5.4), we have

$$|B|^2 = \tfrac{1}{4}X_1^2|ZU(\sigma_1)|^2 + \beta^2|MU(\sigma_2)|^2 = 2(\tfrac{1}{4}X_1^2 + \beta^2),$$

and so if $(A, B) \in D\ (E, X)$, then

$$E = E\ (A, B) = \tfrac{1}{2}|B|^2 + {}_{2}|A|^2 = \kappa + \tfrac{1}{4}X_1^2 + \beta^2.$$

The converse statements are easily verified. □

Corollary 5.4

For every $X \in \mathbb{R}^2$, there exists $(A, B) \in D(X)$ such that $B \neq 0$.

Proof. This follows from Lemmas 3.9 and 5.3. □

5.3 The Nonlinearity, Revisited

The dynamics of (4.7) are linear except for the Lagrange Multiplier term. A priori this term depends on both the the position and velocity of the trajectory; however, after fixing initial invariant triples the nonlinearity only depends on the position and moreover only the magnitude of the position.

Lemma 5.5

Fix κ 0. If $(A,B) \in D$ (E,X), then

$$\Lambda\ (A,B) = \frac{2(\kappa - \det B)}{|A|^2} = \frac{4E - 2X_1X_2}{|A|^4}.$$

Proof. Let $(A,B) \in D$ (E,X) D, and put $L = L(A,B) = BA^{-1}$. Then since $(A,B) \in D$, $\operatorname{tr} L = \langle B, \operatorname{cof}(A)\rangle = 0$, and so the Cayley-Hamilton Theorem implies that

$$L^2 + (\det L)I = 0.$$

Taking the trace yields

$$\operatorname{tr} L^2 = -2\det L = -2\det B \det A^{-1} = -2\det B.$$

Also note that by Lemma 3.3, we have

$$\operatorname{tr} A^{-\top}A^{-1} = |A^{-\top}|^2 = |\operatorname{cof} A|^2 = |A|^2.$$

According to Definition 4.4, this shows that

$$\Lambda(A, B) = \frac{2(\kappa - \det B)}{|A|^2},$$

which is the first statement.

Therefore, the result will follow if we can verify that

$$\kappa - \det B = \frac{2E - X_1 X_2}{|A|^2}, \quad \text{for} \quad (A, B) \in \mathcal{D}(E, X). \tag{5.5}$$

To proceed, we temporarily assume that $A \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R})$. Using Lemmas 3.9 and 3.11, we have

$$\mathrm{cof} B = c_1 \tau_1(A) - c_2 \tau_2(A) - \frac{2}{|A|^2} c_3 \tau_3(A) + \frac{1}{|A|} N(A).$$

Therefore, by Lemmas 3.3 and 3.9, we find that

$$2 \det B = \langle \mathrm{cof} B, B \rangle = g_{11}(A) c_1^2 - g_{22}(A) c_2^2 - \frac{2}{|A|^2} g_{33}(A) c_3^2.$$

Combining this with (5.3) to eliminate the term with c_3, we obtain

$$\begin{aligned}
\kappa - \det B &= \frac{1}{|A|^2} \left(2E - \tfrac{1}{4} g_{11}(A)^2 c_1^2 + \tfrac{1}{4} g_{22}(A)^2 c_2^2 \right) \\
&= \frac{2E - X_1 X_2}{|A|^2},
\end{aligned}$$

as desired.

We now establish the identity (5.5) for $A \in \mathrm{SO}(2, \mathbb{R})$. In this case, we have that $|A|^2 = 2$, by Lemma 3.7, and $X_1 = X_2$ by Lemma 3.9, so we aim to show that

$$\kappa - \det B = E - \tfrac{1}{2} X_1^2.$$

This now follows from Lemma 5.3 since

$$
\begin{aligned}
2\det B &= \langle B, \operatorname{cof} B\rangle \\
&= \left\langle \tfrac{1}{2}X_1U(2s_1)Z + \beta U(2s_2)M, \tfrac{1}{2}X_1U(2s_1)Z - \beta U(2s_2)M \right\rangle \\
&= \tfrac{1}{4}X_1^2|U(2s_1)Z|^2 - \beta^2|U(2s_2)M|^2 \\
&= \tfrac{1}{2}X_1^2 - 2\beta^2 \\
&= -2E + 2\kappa + X_1^2.
\end{aligned}
$$

□

Corollary 5.6

For each $\kappa \geq 0$, the set

$$\{(A,B) \in D : \Lambda_\kappa(A,B) = 0\}$$

is invariant under the flow of (4.7).

Our invariant sets can be further described by the possible energy values obtained on them. By Corollary 5.4, $D(X) \neq \emptyset$, so for $X \in \mathbb{R}^2$ and $\kappa \geq 0$, we may define

$$e_\kappa(X) = \inf\{E_\kappa(A,B) : (A,B) \in D(X)\}.$$

Lemma 5.7

If $\kappa > 0$, then for any $X \in \mathbb{R}^2$, $e_\kappa(X) \geq \kappa$ with equality if and only if $X = 0$. Moreover, $E_\kappa(D(X)) = [e_\kappa(X), \infty)$.

Proof. We begin with the $\kappa = 0$ case. Fix $X \in \mathbb{R}^2$. Let A_j be a sequence in $\mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $|A_j| \to \infty$. Fix c_1, c_2 and take $c_3 = 0$ in (5.3). We obtain a sequence $(A_j, B_j) \in D(X)$ such that $E_0(A_j, B_j) \to 0$. Thus, we see that $e_0(X) = 0$.

Now, letting c_3 range over all values in $\mathbb{R}$, we observe that

$$(E_0(A_j, B_j), \infty) \subset E_0(\mathcal{D}(X)),$$

for each j. This shows that $(0, \infty) \subset E_0(\mathcal{D}(X))$.

If $X \neq 0$, then for all $(A, B) \in \mathcal{D}(X)$, $B \neq 0$ and thus $E_0(A, B) \neq 0$. This means that $E_0(\mathcal{D}(X)) = (0, \infty)$.

Finally, take $X = 0$. Since $(I, 0) \in \mathcal{D}(0)$, we see that $0 = E_0(I, 0) \in E_0(\mathcal{D}(0))$, and we conclude $E_0(\mathcal{D}(X)) = [0, \infty)$.

For the $\kappa > 0$ case. Since $E_\kappa(A, B) \geq \kappa$, for all $(A, B) \in \mathcal{D}$, we have that $e_\kappa(X) \geq \kappa$.

If $e_\kappa(X) = \kappa$, then for any $\varepsilon > 0$, there exists $(A, B) \in \mathcal{D}(X)$ such that $0 \leq E_\kappa(A, B) - \kappa < \varepsilon$. It follows that

$$|B|^2 \leq 2\varepsilon \quad \text{and} \quad |A|^2 \leq 2 + 2\varepsilon/\kappa.$$

Therefore, we see that for $i = 1, 2$,

$$|X_i| = |X_i(A, B)| \leq |A||B| \leq \varepsilon^{1/2}(2 + 2\varepsilon/\kappa)^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary, we get that $X = 0$.

On the other hand, if $X = 0$, then for any $A \in \mathrm{SO}(2, \mathbb{R})$, we have $(A, 0) \in \mathcal{D}(X)$. Thus, we see that

$$\kappa \leq e_\kappa(0) \leq E_\kappa(A, 0) = \kappa.$$

We have shown that $e_\kappa(X) = \kappa$ if and only if $X = 0$.

Take a sequence $(A_j, B_j) \in \mathcal{D}(X)$ with $E_\kappa(A_j, B_j) \searrow e_\kappa(X)$. By Lemmas 3.9and 5.3, we may assume without loss of generality that for each j, B_j lies in the span of $\tau_i(A_j)$, $i = 1, 2$. Since $\kappa > 0$, this sequence is bounded in $\mathcal{D}(X)$. By compactness we obtain an energy minimizer $(A, B) \in \mathcal{D}(X)$ where B lies in the span of $\tau_i(A)$, $i = 1, 2$. By considering the family $(A, B + B_1) \in D(X)$, where $\langle \tau_i(A), B_1 \rangle = 0$, $i = 1, 2$, we see that that $E\ (D(X)) = [e\ (X), \infty)$. ☐

Lemma 5.8

If $\kappa > 0$ and $X_1 = X_2$, then

$$e\ (X) = \begin{cases} \kappa + \frac{1}{4}X_1^2, & \frac{1}{8}X_1^2 \quad \kappa \\ (2\kappa)^{1/2}|X_1| \quad \kappa, & \frac{1}{8}X_1^2 \quad \kappa. \end{cases}$$

Proof. If $A \in \mathrm{SO}(2, \mathbb{R})$, then by Lemma 5.3, we see that

$$\min\{E\ (A, B) : B \in T_A\mathrm{SL}(2, \mathbb{R})\} = \kappa + \tfrac{1}{4}X_1^2.$$

If $A \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R})$, then by Lemma 3.12

$$\min\{E\ (A, B) : B \in T_A\mathrm{SL}(2, \mathbb{R})\} = {}_{2}|A|^2 + \frac{X_1^2}{|A|^2 + 2} \qquad f(|A|^2).$$

Taking the infimum over $\xi = |A|^2 > 2$, we obtain

$$\begin{aligned} &\inf\{E\ (A, B) : A \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R}),\ B \in T_A\mathrm{SL}(2, \mathbb{R})\} \\ &\qquad = \inf_{>2} f(\xi) = \begin{cases} \kappa + \frac{1}{4}X_1^2, & \frac{1}{8}X_1^2 \quad \kappa \\ \sqrt{2\kappa}|X_1| \quad \kappa, & \frac{1}{8}X_1^2 \quad \kappa. \end{cases} \end{aligned}$$

☐

Chapter 6

Reduced Hamiltonian

In this section we will derive the principle dynamics behind the affine motions in $\mathrm{SL}(2,\mathsf{R})$. The local coordinates given in Lemma 3.12 allow us to decompose any path into two rotational pieces and a stretch portion which is described by the magnitude of the path. Further, once we fix this invariant triple Lemma 5.5 determines the Lagrange Multiplier Λ up to this magnitude. Finally, the representation of tangent vectors given by (5.2) shows their coefficients in the $\tau_i(A)$ frame depend only on the magnitude as well. The strategy is to use the representation of tangent vectors given by (5.2) to get a geometric relationship between the magnitude and its velocity in terms of an invariant triple.

6.1 The Reduced Hamiltonian

In this preparatory section we introduce the reduced Hamiltonian and investigate its level curves in the phase plane. The connection with the dynamics will be made in future sections.

Definition 6.1. Given values $\kappa \geq 0$ and $(E, X) \in [0, \infty) \times \mathsf{R}^2$, define the polynomials

$$
\begin{aligned}
P(x;E,X) = \quad & 4\kappa x(x^2 - 1) + 4E(x^2 - 1) \\
& - \tfrac{1}{2}(X_1 - X_2)^2(x+1) - \tfrac{1}{2}(X_1 + X_2)^2(x - 1), \quad x \in \mathbb{R},
\end{aligned}
$$

and

$$
\Phi(x,y;E,X) = \frac{y^2}{2} - \frac{P(x;E,X)}{2x}, \quad (x,y) \in \mathbb{R}^2.
$$

The reader is cautioned that from now on x and y shall represent real numbers, and not spatial and material points, as in previous sections. We will see momentarily that if $(A,B) \in \mathcal{D}(E,X)$, then the point $(x,y) = (\frac{1}{2}|A|^2, \langle A, B\rangle)$ satsifies $\Phi(x,y;E,X) = 0$.

With the phase plane of the magnitude in mind we define the phase plane projection to be the mapping $\mathcal{P} : \mathcal{D} \to \mathbb{R}^2$ by

$$
\mathcal{P}(A,B) = \left(\tfrac{1}{2}|A|^2, \langle A, B\rangle\right).
$$

By Lemma 3.7 the range of $\mathcal{P}$ is

$$
\begin{aligned}
\mathcal{P}(\mathcal{D}) &= \{(x,y) \in \mathbb{R}^2 : x > 1\} \cup \{(1,0)\}, \\
\mathcal{P}(A,B) = (1,0) &\iff A \in \mathrm{SO}(2,\mathbb{R}).
\end{aligned}
\tag{6.1}
$$

For a fixed invariant triple $\mathcal{P}$ restricts to the vectorfield on $\mathrm{SL}(2,\mathbb{R})$ defined by these invatiants. The image of these vectorfields under $\mathcal{P}$ is our main concern. For fixed parameter values $\kappa \geq 0$ and $(E,X) \in [e(X), \infty) \times \mathbb{R}^2$, define

$$
\begin{aligned}
C(E,X) &= \mathcal{P}(\mathcal{D}(E,X)) \\
&= \{(x,y) = \mathcal{P}(A,B) \in \mathbb{R}^2 : (A,B) \in \mathcal{D}(E,X)\}.
\end{aligned}
$$

Lemma 6.1

Fix values $\kappa \geq 0$ and $(E,X) \in [e(X), \infty) \times \mathbb{R}^2$. There holds

$$C\ (E,X) \quad P(D).$$

A point (x,y) with $x>1$ belongs to $C\ (E,X)$ if and only if

$$\Phi\ (x,y;E,X)=0.$$

The point $(1,0)$ belongs to $C\ (E,X)$ if and only if

$$\Phi\ (1,0;E,X)=0 \quad \text{and} \quad \partial_x\Phi\ (1,0;E,X) \quad 0 \tag{6.2}$$

if and only if

$$X_1 = X_2, \quad \text{and} \quad E \quad \kappa + \tfrac{1}{4}X_1^2. \tag{6.3}$$

Proof. The first statement is a consequence of Definition 5.2.

Select any point $(x,y) = P(A,B)\ 2\ P(D)$, with $x>1$. By definition, $(x,y)\ 2$ $C\ (E,X)$ if and only if $(A,B)\ 2\ D\ (E,X)$. By Lemma 3.9, we find that

$$g(A) = \text{diag}\left[4(x+1), \quad 4(x \quad 1), \quad \frac{x}{2(x^2 \quad 1)}\right].$$

The third coordinate c_3 defined in Lemma 3.9 satisfies

$$c_3 = hA,Bi = y.$$

Therefore, Lemma 3.9 implies that $(A,B)\ 2\ D\ (E,X)$ if and only if

$$E = \kappa x + \frac{(X_1+X_2)^2}{8(x+1)} + \frac{(X_1 \quad X_2)^2}{8(x \quad 1)} + \frac{xy^2}{4(x^2 \quad 1)},$$

which is in turn equivalent to the desired result $\Phi\ (x,y;E,X)=0$.

Now suppose that $(1, 0) = \mathcal{P}(A, B) \in C\ (E, X)$. Then $A \in \mathrm{SO}(2, \mathbb{R})$ and $X_1 = X_2$, by 5.1 and 6.1. By the Cauchy-Schwarz inequality and Lemmas 3.6, 3.7, we have

$$X_1^2 = \langle ZA, B\rangle^2 \le |ZA|^2|B|^2 = |A|^2|B|^2 = 2|B|^2 = 4(E - (A, B) - \tfrac{1}{2}|A|^2) = 4(E - \kappa).$$

Thus, (6.3) is true.

Next, suppose that (6.3) holds. Choose $A \in \mathrm{SO}(2, \mathbb{R})$ and using Lemma 5.3 set

$$B = (\tfrac{1}{2}X_1)\ ZA + B_1,$$

with

$$\langle A, B_1\rangle = \langle ZA, B_1\rangle = 0 \quad \text{and} \quad \tfrac{1}{2}|B_1|^2 = E - \kappa - \tfrac{1}{4}X_1^2.$$

By Lemma 5.3, $(A, B) \in \mathcal{D}\ (E, X)$, and so $(1, 0) = \mathcal{P}(A, B) \in C\ (E, X)$.

It is immediate to verify the equivalence of (6.2) and (6.3). □

Lemma 6.2

A point $(x_0, y_0) \in C\ (E, X)$ is a critical point of the Hamiltonian $\Phi\ (x, y; E, X)$ if and only if

$$x_0 \ge 1, \quad y_0 = 0, \quad \text{and} \quad P\ (x_0; E, X) = P'(x_0; E, X) = 0. \qquad (6.4)$$

Proof. Suppose that $(x_0, y_0) \in C\ (E, X)$ is a critical point of $\Phi\ (x, y; E, X)$. Then $(x_0, y_0) \in \mathcal{P}(\mathcal{D})$, so $x_0 \ge 1$. By Lemma 6.1,

$$\Phi\ (x_0, y_0; E, X) = y_0^2/2 - P\ (x_0, E, X)/2x_0 = 0.$$

Critical points are characterized by the equations

$$\partial_x \Phi (x_0, y_0; E, X) = (P(x_0; E, X) - x_0 P'(x_0; E, X))/2x_0^2 = 0$$

and

$$\partial_y \Phi (x_0, y_0; E, X) = y_0 = 0.$$

Thus, we see that (6.4) holds.

If (6.4) holds, then

$$\Phi (x_0, y_0; E, X) = 0 \quad \text{and} \quad \nabla_{x,y} \Phi (x_0, y_0; E, X) = 0.$$

So (x_0, y_0) is a critical point of Φ, and by Lemma 6.1, $(x_0, y_0) \in C(E, X)$. □

Notice by Lemma 6.2, critical points in $C(E, X)$ correspond to double roots of $P(x; E, X)$, a nonzero polynomial of degree at most 3, so there can be at most one critical point of $P(x; E, X)$ for $(\kappa, E, X) \neq 0$.

Lemma 6.3

Fix $\kappa > 0$, $X \in \mathbb{R}^2$. The set $C(E, X)$ is a singleton if and only if $E = e(X)$. In this case, $C(e(X), X) = \{(x_0, 0)\}$, where $(x_0, 0)$ is a critical point of $\Phi(x, y; e(X), X)$ and a minimum in $P(D)$.

Proof. Suppose that $E < e(X)$. Then $C(E, X) = \emptyset$, and so by Lemma 6.1

$$\Phi (x, y; E, X) \neq 0, \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

Since $\kappa > 0$, we have $\Phi(x, 0; E, X) \to +\infty$, as $x \to +\infty$, and as a consequence

$$\Phi (x, y; E, X) > 0 \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

By continuity, we obtain

$$\Phi(x, y; e(X), X) \geq 0, \quad \text{for all} \quad (x, y) \in P(D).$$

Since

$$\Phi(x, y; e(X), X) = \tfrac{1}{2}y^2 + \Phi(x, 0; e(X), X),$$

we see that

$$\Phi(x, y; e(X), X) > 0, \quad \text{for all} \quad (x, y) \in P(D),\ y \neq 0.$$

Thus, by Lemma 6.1 we have that

$$C(e(X), X) \subset \{(x, 0) : x \geq 1\}.$$

Additionally, Lemma 5.7 assures us that $C(e(X), X) = P(D_k(e(X), X)) \neq \emptyset$.

If $(x_0, 0) \in C(e(X), X)$ for some $x_0 \geq 1$, then

$$0 = \Phi(x_0, 0; e(X), X) \leq \Phi(x, y; e(X), X), \quad (x, y) \in P(D).$$

This says that $(x_0, 0)$ is a minimum value for $\Phi(x, y; e(X), X)$. It follows that

$$\partial_x \Phi(x_0, 0; e(X), X) = 0, \quad \text{if} \quad x_0 > 1$$

and

$$\partial_x \Phi(x_0, 0; e(X), X) \geq 0, \quad \text{if} \quad x_0 = 1.$$

On the other hand, if $x_0 = 1$, then by Lemma 6.1,

$$\partial_x \Phi \ (x_0, 0; e \ (X), X) \quad 0.$$

We conclude that

$$\partial_x \Phi \ (x_0, 0; e \ (X), X) = 0, \quad \text{for all} \quad x_0 \quad 1.$$

This shows that $(x_0, 0)$ must be a critical point of $\Phi \ (x, y; e \ (X), X)$. Since $\kappa > 0$, there can be only one critical point in $C \ (e \ (X), X)$ and so this set is a singleton.

Now suppose that $C \ (E, X)$ is a singleton. By Lemma 6.1 and definition of $\Phi \ (x, y; E, X)$, if $(x, y) \in C \ (E, X)$, then $(x, \ \ y) \in C \ (E, X)$. So it must be that

$$C \ (E, X) = \{(x_0, 0)\}, \quad \text{for some} \quad x_0 \quad 1.$$

By Lemma 6.1, we have that

$$\{x > 1 : \Phi \ (x, y; E, X) = 0\} \quad C \ (E, X) = \{(x_0, 0)\}.$$

This implies that $\Phi \ (x, y; E, X)$ does not vanish on the connected open set

$$\{(x, y) : x > 1, \ x \neq x_0\}.$$

Using the fact that $\lim_{x \to \infty} \Phi \ (x, 0; E, X) = \infty$, we conclude that

$$\Phi \ (x, y; E, X) > 0 \quad \text{for all} \quad x > 1, \ x \neq x_0, \ y \in \mathbb{R}. \tag{6.5}$$

Since $C \ (E, X) \neq \emptyset$, we have that $E \quad e \ (X)$. We claim that

$$\bar{E} < E \quad \text{implies} \quad C \ (\bar{E}, X) = \emptyset. \tag{6.6}$$

Given this claim, we would have $\bar{E} < e \ (X)$ so that

$$E = \sup\{\bar{E} : \bar{E} < E\} \le e(X),$$

thereby showing that $E = e(X)$, as desired.

Assume $\bar{E} < E$, and let us now proceed to verify (6.6). Write

$$\Phi(x, y; \bar{E}, X) = 4(E - \bar{E})(x^2 - 1)/2x + \Phi(x, y; E, X). \tag{6.7}$$

By (6.5), the equation (6.7) implies that

$$\Phi(x, y; \bar{E}, X) > 0, \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

Thus, by Lemma 6.1, we discover that

$$C(\bar{E}, X) \subseteq \{(1, 0)\}.$$

If $X_1 \neq X_2$, then $\Phi(1, 0; \bar{E}, X) = (X_1 - X_2)^2 > 0$ so that $(1, 0) \notin C(\bar{E}, X)$ and (6.6) holds in this case.

Next, suppose that $X_1 = X_2$. Then

$$\Phi(1, 0; \bar{E}, X) = \Phi(1, 0; E, X) = 0,$$

and by (6.5), we see that

$$\partial_x \Phi(1, 0; E, X) \ge 0.$$

Thus, by (6.7), we find that

$$\partial_x \Phi(1, 0; \bar{E}, X) = 4(E - \bar{E}) + \partial_x \Phi(1, 0; E, X) > 0.$$

By Lemma 6.1, we conclude that $(1, 0) \notin C(\bar{E}, X)$, and again (6.6) holds.

□

For convenience we summarize the relationship between the exceptional point $(1,0)$ and the sets $C(E,X)$.

Corollary 6.4

Fix $X \in \mathbb{R}^2$.

1. $(1,0) \in C(E,X)$ if and only if $X_1 = X_2$ and $E \geq \kappa + \frac{1}{4}X_1^2$.

2. $(1,0) \in C(E,X)$ is a critical point of $\Phi(x,y;E,X)$ if and only if $X_1 = X_2$ and $E = \kappa + \frac{1}{4}X_1^2$.

3. $\{(1,0)\} = C(E,X)$ if and only if $\kappa > 0$, $X_1 = X_2$ and $E = \kappa + \frac{1}{4}X_1^2 = e(X)$.

Proof. The statement (i) was shown in Lemma 6.1, and (ii) follows from Lemma 6.2. In the next result we shall see that $C_0(E,X)$ is either empty or unbounded. Thus, (iii) is just Lemma 6.3. □

Lemma 6.5

At each point $(x,y) \in C(E,X)$ such that $\nabla\Phi(x,y;E,X) \neq 0$, the set $C(E,X)$ is a locally smooth curve.

The sets $C(E,X)$ are closed and connected subsets of $P(D)$.

If $C_0(E,X) \neq \emptyset$, then it is unbounded.

If $\kappa > 0$, then $C(e(X),X)$ is a singleton, and for $E > \bar{E} \geq e(X)$, $C(E,X)$ is a closed curve enclosing $C(\bar{E},X) \setminus \{(1,0)\}$.

Proof. By Lemma 6.1,

$$C(E,X) \setminus \{(1,0)\} \subset \{(x,y) : \Phi(x,y;E,X) = 0,\ x > 1\},$$

so the smoothness of $C(E,X)$ away from critical points of $\Phi(x,y;E,X)$ in the region

$\{x > 1\}$ follows by the implicit function theorem. If $(1,0) \in C_\kappa(E,X)$ is not a critical point of $\Phi_\kappa(x,y;E,X)$, then by Lemma 6.1, $\partial_x \Phi_\kappa(1,0;E,X) < 0$, and the implicit function theorem provides a smooth local parameterization of $C_\kappa(E,X)$ of the form $(x(y),y)$, $|y| \ll 1$, with

$$x(0) = 1, \quad x'(0) = 0, \quad x''(0) = -1/\partial_x \Phi_\kappa(1,0;E,X) > 0,$$

describing a curve contained in $\mathcal{P}(D)$.

To prove the other statements, we consider the cases $\kappa = 0$ and $\kappa > 0$ separately.

Suppose that $\kappa = 0$. If $E = 0$ and $C_0(0,X) \neq \emptyset$, then we have $X = 0$. By definition, $\Phi_0(x,y;0,0) = \frac{1}{2}y^2$, and so by Lemma 6.1,

$$C_0(0,0) = \{(x,0) : x \geq 1\},$$

which is a closed, connected, and unbounded set. If $E > 0$, then $P_0(x;E,X) \to -\infty$, as $|x| \to \infty$. By Lemma 6.1,

$$C_0(E,X) \subset \{(x,y) : \Phi_0(x,y;E,X) = 0,\ x \geq 1\},$$

so $P_0(x;E,X)$ must have real roots $x_1(E,X) \leq x_2(E,X)$. Since

$$P_0(1;E,X) = -(X_1 - X_2)^2 \leq 0,$$

it follows that $x_1(E,X) \leq 1 \leq x_2(E,X)$. If $x_1(E,X) = 1 < x_2(E,X)$, then $\partial_x \Phi_0(1,y;E,X) < 0$, and by Lemma 6.1, $(1,0) \notin C_0(E,X)$ and so

$$C_0(E,X) = \{(x,y) : y^2 = P_0(x;E,X)/x,\ x \geq x_2(E,X)\}.$$

This also holds when $x_1(E,X) < 1 \leq x_2(E,X)$ or when $x_1(E,X) = x_2(E,X) = 1$. Thus,

$C_0(E,X)$ again is a closed, connected, and unbounded set.

Now suppose that $\kappa > 0$. Note that $P_\kappa(x;E,X) \to -\infty$, as $x \to \infty$ and $P_\kappa(1;E,X) = -(X_1 - X_2)^2 \le 0$. So if $C_\kappa(E,X) \neq \emptyset$, then $P_\kappa(x;E,X)$ must have three real roots (counting multiplicity) with

$$x_1(E,X) \le 1 \le x_2(E,X) \le x_3(E,X).$$

By Lemma 6.1, $(1,y) \in C_\kappa(E,X)$ if and only if

$$y = 0 \quad \text{and} \quad \partial_x \Phi_\kappa(1,0;E,X) = -\tfrac{1}{2}P_\kappa'(1;E,X) \le 0.$$

It follows that

$$C_\kappa(E,X) = \{(x,y) : y^2 = P_\kappa(x;E,X)/x,\ x_2(E,X) \le x \le x_3(E,X)\}. \qquad (6.8)$$

Thus, $C_\kappa(E,X)$, $\kappa > 0$, is a simple closed closed curve and a closed, bounded, and connected set.

We note that for $E > \bar{E} \ge e_\kappa(X)$, we have

$$P_\kappa(x;E,X) - P_\kappa(x;\bar{E},X) = 4(E - \bar{E})(x^2 - 1) > 0, \quad x > 1.$$

Thus, the enclosure claim is a consequence of (6.8).

The fact that $C_\kappa(e_\kappa(X),X)$ is a singleton was shown in Lemma 6.3.

□

Observe that (with the exception of Corollary 5.6) the results from Section 5.2 until here are purely algebraic. They have nothing to do with the dynamics of the system (4.7). The next section will make the connection with this reduced Hamiltonian and the actual dynaimcs.

6.2 Reduction to the Phase Plane

If $A \in C^0(\mathbb{R};\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R};\mathbb{M}^2)$, then $(x(t),y(t)) = \mathcal{P}(A(t),A'(t))$ is a C^1 planar curve. We now show that given a solution $A(t)$ of the system (4.7), its phase plane curve $\mathcal{P}(A(t),A'(t))$ satisfies a Hamiltonian system.

Theorem 6.6

Fix $\kappa \geq 0$ and $(E,X) \in [e_\kappa(X),\infty) \times \mathbb{R}^2$. Suppose that $A \in C^0(\mathbb{R},\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R},\mathbb{M}^2)$ is a solution of (4.6), (4.7), (4.8) with initial data in $\mathcal{D}_\kappa(E,X)$. Put

$$(x(t),y(t)) = \mathcal{P}(A(t),A'(t)).$$

Then

$$\begin{aligned} x'(t) &= \partial_y \Phi_\kappa(x(t),y(t);E,X) = y(t) \\ y'(t) &= -\partial_x \Phi_\kappa(x(t),y(t);E,X), \end{aligned} \tag{6.9}$$

and the solution orbit satisfies

$$(x(t),y(t)) \in \mathcal{C}_\kappa(E,X), \tag{6.10}$$

for all $t \in \mathbb{R}$.

Proof. By Theorem 5.1, we have

$(A(t),A'(t)) \in \mathcal{D}_\kappa(E,X)$, for all $t \in \mathbb{R}$, and thus, (6.10) follows by definition of $\mathcal{C}_\kappa(E,X)$.

The first equation of (6.9) holds because

$$x'(t) = (\tfrac{1}{2}|A(t)|^2)' = \langle A(t),A'(t)\rangle = y(t)$$

and $\partial_y \Phi_\kappa(x,y;E,X) = y$.

To verify the second, we compute using (4.7), Lemma 5.5, and the definition of the enery

$$
\begin{aligned}
y^{0}(t) &= x^{00}(t) \\
&= hA^{00}(t), A(t)i + jA^{0}(t)j^{2} \\
&= \Big\langle \quad \kappa A(t) + \Lambda \ \ (A(t), A^{0}(t))A(t) \quad ^{>}, A(t)\Big\rangle + jA^{0}(t)j^{2} \\
&= \quad \kappa jA(t)j^{2} + 2\Lambda \ \ (A(t), A^{0}(t)) + jA^{0}(t)j^{2} \\
&= \quad 2\kappa jA(t)j^{2} + 2E \ \ (A(t), A^{0}(t)) + 2\Lambda \ \ (A(t), A^{0}(t)) \\
&= \quad 4\kappa x(t) + 2E + \frac{2E \quad X_1 X_2}{x(t)^2}.
\end{aligned} \tag{6.11}
$$

A short algebraic manipulation using Defnition 6.1 confirms that

$$
\begin{aligned}
4\kappa x + 2E + \frac{2E \quad X_1 X_2}{x^2} &= \frac{xP^{0}(x; E, X) \quad P \ \ (x; E, X)}{2x^2} \\
&= \quad \partial_x \Phi \ \ (x, y; E, X),
\end{aligned}
$$

for all (x, y) with $x \quad 1$, which completes the verification of (6.9). □

Observe that (6.9) has a Hamiltonian structure. The key result (6.10) will allow us understand the behavior of the orbits $(x(t), y(t))$ of (6.9) corresponding to solutions of (4.7) by studying the sets $C \ \ (E, X)$.

Further, when $\kappa = 0$,

$$
x^{00} = \frac{2E(x \quad 1)^2 + (4Ex \quad X_1 X_2)}{x^2}.
$$

For the existence and uniqueness of (6.9) we start with some $(x(0), y(0))$ *2* $C \ \ (E, X)$. By Lemma 6.1 there is some data (A_0, B_0) *2 D* (E, X) such that

$$
P(A_0, B_0) = (x(0), y(0)).
$$

This data has a corresponding A *2* $C^0(\mathsf{R}, \mathrm{SL}(2, \mathsf{R}))$ \ $C^2(\mathsf{R}, \mathsf{M}^2)$ which is a solution of (4.6), (4.7), (4.8). By Theorem 6.6, $(x(t), y(t)) = P(A(t), A^{0}(t))$ is the desired solution.

Corollary 6.7

Fix $\kappa \quad 0$ and $(E,X) \in [e\ (X), \infty) \quad \mathbb{R}^2$. For any initial data $(x(0),y(0)) \in C\ (E,X)$, the IVP for (6.9) has a unique global solution $(x,y) \in C^1(\mathbb{R},\mathbb{R}^2)$ with $(x(t),y(t)) \in C\ (E,X)$, for all $t \in \mathbb{R}$.

Proof. Given $(x(0),y(0)) \in C\ (E,X)$, use Lemma 6.1 to find data $(A_0,B_0) \in D\ (E,X)$ such that

$$P(A_0,B_0) = (x(0),y(0)).$$

Let $A \in C^0(\mathbb{R},\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R},\mathrm{M}^2)$ be the solution of (4.6), (4.7), (4.8) with this data. By Theorem 6.6, $(x(t),y(t)) = P(A(t),A'(t))$ is the desired solution. □

While the quantity $\Phi\ (x(t),x'(t);E,X)$ is conserved along all solutions of the reduced system (6.9), we emphasize that only the portion of the zero level set in $C\ (E,X)$ corresponds to solutions of the full system (4.7) .

We are now ready to describe the level sets of Φ which correspond to trajectories of (4.7). This characterization will be essentially establish, for the $\kappa > 0$ case, periodicity of the magnitude of our solutions.

Lemma 6.8

Fix $\kappa \quad 0$, $(E,X) \in [e\ (X), \infty) \quad \mathbb{R}^2$, with $(\kappa,E,X) \neq 0$.

If $C\ (E,X)$ does not contain a critical point of $\Phi\ (x,y;E,X)$, then it is a smooth curve in $\mathbb{R}^2$ consisting of a single orbit of (6.9).

If $C\ (E,X)$ contains a single critical point p of $\Phi\ (x,y;E,X)$, then each component of $C\ (E,X) \setminus \{p\}$ is a smooth curve in $\mathbb{R}^2$ consisting of a single orbit of (6.9).

If γ is a nontrivial orbit of (6.9) in $C\ (E,X)$, then either γ is a closed orbit or its alpha- and omega-limit sets are subsets of a critical point $\{p\}$.

Proof. Since $(\kappa,E,X) \neq 0$, $C\ (E,X)$ can contain at most one critical point of $\Phi\ (x,y;E,X)$, Suppose that $C(E,X)$ contains no critical points of $\Phi(x,y;E,X)$. Then the orbit through each point of $C(E,X)$ is open in $C(E,X)$. Since $C(E,X)$ is connected, it can contain only one orbit. If $C(E,X)$ contains a critical point p of $\Phi(x,y;E,X)$, then the same argument is valid on each component of $C(E,X) \setminus \{p\}$. These nontrivial orbits are C^1 curves, by (6.9).

Let γ be a nontrivial orbit in $C(E,X)$. Since $C(E,X)$ is a closed set, it contains $\omega(\gamma)$, the omega-limit set of γ. If $\omega(\gamma) \neq \emptyset$, then it is an invariant set for (6.9). If $\gamma \cap \omega(\gamma) \neq \emptyset$, then $\gamma \subset \omega(\gamma)$. In this case, γ must be a closed orbit. Here's the proof: We can write

$$\gamma = \{\varphi(t) = (x(t), y(t)) : t \in \mathbb{R}\},$$

for some solution $(x(t), y(t))$ of (6.9). $\varphi(0)$ is not a critical point, so by the implicit function theorem, there exists an $\varepsilon > 0$ and a neighborhood N of $\varphi(0)$ such that

$$\{\varphi(t) : t \in (-\varepsilon, \varepsilon)\} = \{(x,y) \in \mathbb{R}^2 : \Phi(x,y;E,X) = 0\} \cap N.$$

Since $\varphi(0) \in \omega(\gamma)$, there exists a sequence $t_j \to \infty$ such that $\varphi(t_j) \to \varphi(0)$. Thus, since $\varphi(t) \in C(E,X)$ for all $t \in \mathbb{R}$, there exists a $t_j > \varepsilon$ such that

$$\varphi(t_j) \in \{(x,y) \in \mathbb{R}^2 : \Phi(x,y;E,X) = 0\} \cap N.$$

It follows that there exists $\tau \in (-\varepsilon, \varepsilon)$ such that $\gamma(t_j) = \gamma(\tau)$. This proves that γ is a closed orbit.

If $q \in C(E,X) \setminus \gamma$ is not a critical point, then its orbit, call it η, is an open subset of $C(E,X)$. This implies that $\eta \cap \omega(\gamma) = \emptyset$, and so $q \notin \omega(\gamma)$. Therefore, we have either $\gamma \cap \omega(\gamma) \neq \emptyset$, in which case γ is closed, or $\omega(\gamma) \cap \gamma = \emptyset$, in which case $\omega(\gamma)$ can only contain critical points of $\Phi(x,y;E,X)$ in $C(E,X)$. The same argument applies for $\alpha(\gamma)$. □

6.3 Special Solutions

With the knowledge of our magnitude's dynamics in hand we describe the behavoir of some simple solutions to (4.7). These act as a sort of boundary on our set of more generic soltuions.

Equilibria

Lemma 6.9

Fix $\kappa \ \ 0$. A solution of (4.6), (4.7), (4.8) is an equilibrium if and only the initial data satisfies $(A_0, B_0) \in \mathcal{D}\ (\kappa, 0)$.

Proof. First, we note that

$$\mathcal{D}_0(0,0) = \{(A,B) \in \mathcal{D} : A \in \mathrm{SL}(2,\mathbb{R}),\ B = 0\},$$

and for $\kappa > 0$,

$$\mathcal{D}\ (\kappa,0) = \{(A,B) \in \mathcal{D} : A \in \mathrm{SO}(2,\mathbb{R}),\ B = 0\}.$$

Moreover, if $A(t) = A_0$ is an equilibrium solution, then $A'(t) = 0 = B_0$.

Suppose first that $\kappa = 0$. If $A(t)$ is an equilibrium solution, then $(A_0, B_0) = (A_0, 0) \in \mathcal{D}_0(0,0)$. Conversely, if $(A_0, B_0) \in \mathcal{D}_0(0,0)$, then $B_0 = 0$ implies that $\Lambda_0(A_0, B_0) = 0$, and so $A(t) = A_0$ is an equilibrium solution of (4.7).

Now suppose that $\kappa > 0$. Then $A(t) = A_0$ is an equilibrium solution of (4.7) if and only if

$$\kappa A_0 = \Lambda\ (A_0, 0)\mathrm{cof}A_0.$$

By Lemma 5.5, this is equivalent to

$$\kappa A_0 = \frac{4E\ (A_0, 0)}{|A_0|^4}\mathrm{cof}A_0 = \frac{2\kappa}{|A_0|^2}\mathrm{cof}A_0.$$

Taking the norm of both sides gives $|A_0|^2 = 2$, so that $A_0 \in \mathrm{SO}(2, \mathbb{R})$. Conversely, by Lemma 3.3 , we see that A_0 is an equilibrium solution

if $A_0 \in \mathrm{SO}(2, \mathbb{R})$. Thus, when $\kappa > 0$, all equilibrium solutions correspond to initial data $(A_0, 0)$ with $A_0 \in \mathrm{SO}(2, \mathbb{R})$, i.e. $(A_0, B_0) \in \mathcal{D}\ (\kappa, 0)$. □

By Lemma 5.7, equilibrium solutions of (4.7) are those which minimize the energy over $\mathcal{D}$. It is interesting to note that the set of equilibrium for Perfect Fluids is identifiable with the entirety of $\mathrm{SL}(2, \mathbb{R})$ wheras any devation from $\mathrm{SO}(2, \mathbb{R})$ will results in non trival dynaicms of the MHD equations.

Rigid motion

Recall the definition of rigid motion given in Definition 4.3. We rephrasae it equivilently here in a way which is more illustrative of its connection with the previous sections.

Definition 6.2. A solution $A(t)$ of the system (4.7) shall be called ***rigid*** if $(x(t), y(t)) = \mathcal{P}(A(t), A'(t))$ is an equilibrium solution of (6.9), or equivalently, if $\mathcal{P}(A(t), A'(t)) = (x, 0)$ for some constant $x \geq 1$.

Equilibrium solutions of (4.7) are also rigid solutions.

If $A(t)$ is rigid with $\frac{1}{2}|A(t)|^2 = x$, for some constant $x \geq 1$, then the fluid domains are ellipses with principal axes of fixed lengths, i.e. $z \mapsto A(t)z$ is a rigid motion.

Lemma 6.10

A solution of the IVP (4.6), (4.7), (4.8)

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

with initial data $(A_0, B_0) \in D\ (E, X)$ is rigid if and only if $P(A_0, B_0)$ is a critical point of $\Phi\ (x, y; E, X)$ in $C\ (E, X)$.

In particular, initial data $(A_0, B_0) \in D\ (e\ (X), X)$, $\kappa > 0$, yields a rigid solution.

Proof. If $A(t)$ is rigid, then

$$P(A(t), A'(t)) = (x_0, 0) = P(A_0, B_0) \in C\ (E, X)$$

is an equilibrium solution of (6.9).Thus, $(x_0, 0)$ is a critical point of Φ .

Next suppose that $(x_0, 0) = P(A_0, B_0)$ is a critical point of Φ in $C\ (E, X)$. Since Φ can only have one critical point on $C\ (E, X)$, $(x(t), y(t)) = (x_0, 0)$ is the unique solution of (6.9) with data $(x_0, 0)$. Let $A(t)$ be the solution of the IVP (4.6), (4.7), (4.8) with initial data (A_0, B_0). By Theorem 6.6, $(x(t), y(t)) = P(A(t), A'(t))$ solves (6.9) with data $(x_0, 0)$. Thus, $P(A(t), A'(t)) = (x_0, 0)$, and so $A(t)$ is rigid.

The final statement is a consequence of Lemma 6.3. □

Next, we consider the special case of rigid motion in $\mathrm{SO}(2, \mathbb{R})$ which will play a special role in what follows.

Lemma 6.11

Fix κ 0. The following statements are equivalent:

1. The function $A(t)$ is a solution of (4.7) in

$$C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

whose initial data satisfies

$$(A_0, B_0) \in D\ (E, X), \quad \text{with} \quad X_1 = X_2, \quad E = \kappa + \tfrac{1}{4}X_1^2,$$

and $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, for some $t_0 \in \mathbb{R}$.

2. The function $A(t)$ is a rigid solution of (4.7) in

$$C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

with $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$ for some $t_0 \in \mathbb{R}$.

3. The function $A(t)$ is a solution of (4.7) in

$$C^0(\mathbb{R}, \mathrm{SO}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2).$$

4. The function $A(t)$ is given by

$$A(t) = U\left(\tfrac{1}{2}X_1 t + \theta\right) = \exp\left[\left(\tfrac{1}{2}X_1 t + \theta\right) Z\right],$$

for some $\theta,\ X_1 \in \mathbb{R}$.

Proof. We shall prove the implications cyclically.

Suppose that (1) holds. The conditions on the invariants (E, X) imply that $P(1; E, X) = P'(1; E, X) = 0$. By Lemma 6.2, $(1, 0)$ is a critical point of $\Phi(x, y; E, X)$ in $C(E, X)$, and so it is an equilibrium solution of (6.9). Since $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, Lemma 6.1says that $\mathsf{P}(A(t_0), A'(t_0)) = (1, 0)$. By Theorem 6.6, $\mathsf{P}(A(t), A'(t))$ is a solution of (6.9), and by uniqueness, it must be equal to the equilibrium solution $(1, 0)$. Thus, $A(t)$ is rigid.

Suppose next that (2) holds. Since $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, $\mathsf{P}(A(t_0), A'(t_0)) = (1, 0)$. Since $A(t)$ is rigid, we have

$$\mathsf{P}(A(t), A'(t)) = \mathsf{P}(A(t_0), A'(t_0)) = (1, 0),$$

for all $t \in \mathbb{R}$. Thus, $A(t) \in SO(2, \mathbb{R})$ for all $t \in \mathbb{R}$.

Suppose next that (3) holds. Differentiating the identity

$$A(t)A(t)^{\top} = I,$$

we find that $A'(t)A(t)^{\top}$ is anti-symmetric. Note that since $A(t) \in SO(2, \mathbb{R})$, we have

$$L(A(t), A'(t)) = A'(t)A(t)^{-1} = A'(t)A(t)^{\top}$$

is anti-symmetric. Thus, using Corollary 5.2 and Lemma 5.1 , we obtain

$$L(A(t), A'(t)) = \tfrac{1}{2}X_2 Z = \tfrac{1}{2}X_1 Z,$$

and so, we see that

$$A'(t) = \tfrac{1}{2}X_1 Z A(t).$$

The explicit solution is

$$A(t) = \exp\left[\tfrac{1}{2}X_1(t - t_0)\, Z\right] A(t_0) = U\left(\tfrac{1}{2}X_1 t\right) U\left(-\tfrac{1}{2}X_1 t_0\right) A(t_0).$$

Since $A(t_0) \in SO(2, \mathbb{R})$, we may use Lemma 3.8to write

$$U\left(-\tfrac{1}{2}X_1 t_0\right) A(t_0) = U(\theta),$$

for some $\theta \in \mathbb{R}$. This leads to the desired formula.

If (4) statement holds, then (1) follows by direct calculation using the explicit formula for $A(t)$. □

Observe that solutions in $SO(2, \mathbb{R})$ are periodic.

Here we again see a fundamental difference between even and odd dimensions. There are no nontrivial solutions of the equation (4.4) in the form $A(t) = \exp(Wt)$ with W anti-symmetric in odd dimensions.

The particular solutions in two dimensions which give rise to these rigid solutions will form an important submanifold of $\mathcal{D}$.

Definition 6.3. For each $X_1 \in \mathbb{R}$, define

$$\mathcal{R}(X_1) = \{(U, \tfrac{1}{2}X_1 ZU) : U \in \mathrm{SO}(2, \mathbb{R})\}.$$

Lemma 6.12

For each $\kappa \quad 0$ and $X_1 \in \mathbb{R}$, $\mathcal{R}(X_1)$ coincides with the orbit of a rigid rotational solution of (4.7).

If $X = (X_1, X_1)$ and $E = \kappa + \frac{1}{4}X_1^2$, then $\mathcal{R}(X_1) \quad \mathcal{D}\ (E, X)$.

Additionally, $\mathcal{R}(X_1) = \mathcal{D}\ (E, X)$ if and only if $E = e\ (X)$.

Proof. Let $\kappa \quad 0$ and $X_1 \in \mathbb{R}$. Set $A(t) = U\left(\frac{1}{2}X_1 t\right)$, $t \in \mathbb{R}$. By Lemma 6.11 (4), $A(t)$ is a rigid rotational solution of (4.7). Its orbit

$$(A(t), A'(t)) = \left(A(t), \tfrac{1}{2}X_1 A(t)\right)$$

is equal to $\mathcal{R}(X_1)$, since, by (3.8), $A(t)$ parameterizes $\mathrm{SO}(2, \mathbb{R})$.

The second statement follows from Lemma 6.11 (1).

Finally, we show that the inclusion is an equality if and only if $E = e\ (X)$.

Note that $\mathcal{P}(\mathcal{R}(X_1)) = \{(1, 0)\}$. Thus, we have

$$\{(1, 0)\} = \mathcal{P}(\mathcal{R}(X_1)) \quad \mathcal{P}(\mathcal{D}\ (E, X)) = C\ (E, X). \tag{6.12}$$

If $E > e\ (X)$, then $C\ (E, X)$ is not a singleton, by Lemma 6.3, and we see that $\mathcal{D}\ (E, X) \setminus \mathcal{R}(X_1) \neq \emptyset$.

If, on the other hand, $E = e(X)$, then $C(E, X)$ is a singleton, and (6.12) implies that $C(E, X) = \{(1, 0)\}$. If $(A, B) \in \mathcal{D}(E, X)$, then $\mathcal{P}(A, B) = (1, 0)$. By (6.1), $A \in \mathrm{SO}(2, \mathbb{R})$, and by Lemma 5.3, $B = \frac{1}{2}X_1 Z A$, since $E = e(X)$. This shows that $(A, B) \in \mathcal{R}(X_1)$, and so $\mathcal{D}(E, X) \subset \mathcal{R}(X_1)$. □

Later, we shall see that the invariant manifolds $\mathcal{R}(X_1)$ are hyperbolic,

Solutions with vanishing pressure

The Lagrangian, L, defined in Remark 4.1 represents a constrained harmonic oscillator, $\kappa > 0$, or geodesics in $\mathrm{SL}(2, \mathbb{R})$, $\kappa = 0$. For solutuions When the Lagrange multiplier vanishes the constraining term in the Lagrangian, and so the pressure, vanishes giving rise to ODEs which can be solved by hand.

Lemma 6.13

Let $(A_0, B_0) \in \mathcal{D}(E, X)$ with $2E - X_1 X_2 = 0$.

If $\kappa = 0$, then

$$A(t) = B_0 t + A_0$$

is the solution of (4.6), (4.7), (4.8) in $C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$.

If $\kappa > 0$, then

$$A(t) = (\cos \sqrt{\kappa}\, t) A_0 + \frac{1}{\sqrt{\kappa}} (\sin \sqrt{\kappa}\, t) B_0$$

is the solution of (4.6), (4.7), (4.8) in $C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$.

Proof. Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be the solution of (4.7) with data in $\mathcal{D}(E, X)$. If $2E - X_1 X_2 = 0$, then by Theorem 5.1and Lemma 5.5, we have that

$$\Lambda(A(t), A'(t)) = 0, \quad \text{for all} \quad t \in \mathbb{R}.$$

The formulas follow directly by solving the IVP for the linear equation resulting from (4.7)

$$A'' + \kappa A = 0.$$

□

If we recall the alternative definition of $\Lambda\ $ givin in Lemma 5.5 we see the condition the condition $2E - X_1X_2 = 0$ is equivalent to assuming $\det B_0 = \kappa$.

It is clear that when $\kappa > 0$, the solution $A(t)$ is $(2\pi/\sqrt{\kappa})$-periodic. A quick computation can give us the norm of our path

$$|A(t)|^2 = \tfrac{1}{2}(1+\cos 2\sqrt{\kappa}t)|A_0|^2 + \tfrac{1}{\sqrt{\kappa}}\sin 2\sqrt{\kappa}t\langle A_0, B_0\rangle + \tfrac{1}{2}(1-\cos 2\sqrt{\kappa}t)|B_0|^2$$

which is $(\pi/\sqrt{\kappa})$-periodic.

The levelset of $\Phi\ $ that gives rise to these motions, $C\ (E,X)$, is an ellipse:

$$y^2 + 4\kappa(x - E/2\kappa)^2 = 4\kappa + E^2/\kappa - X_1^2 - X_2^2.$$

By the Cauchy-Schwarz inequality and the condition $2E - X_1X_2 = 0$, we have $|X_i| \geq E/\sqrt{\kappa} = X_1X_2/(2\sqrt{\kappa})$, so we see that $|X_i| \geq 2\sqrt{k}$, and therefore the right-hand side is nonnegative:

$$\frac{1}{4\kappa}(X_1^2 - 4\kappa)(X_2^2 - 4\kappa) \geq 0.$$

Remark 6.1. When $\kappa = 0$, $A(t)$ is a line in $\mathrm{SL}(2,\mathbb{R})$. $C_0(E,X)$ is a parabola:

$$y^2 - 2X_1X_2x + X_1^2 + X_2^2 = 0.$$

Chapter 7

The Motions

Here we describe the motions for MHD and perfect fluids. We begin by reconstructing the paths in $\mathrm{SL}(2,\mathbb{R})$ from out local coodinate representations.

7.1 Reconstruction

We now show that solutions $A(t)$ of the system (4.7) can be recovered from knowledge of its phase plane curve $\mathcal{P}(A(t), A'(t))$ and its initial data (4.6), using local coordinates. The proof is complicated by the coordinate singularity on $\mathrm{SO}(2,\mathbb{R})$.

In order to avoid repetition, we enforce the following standing assumption throughout this section:

(A) The parameter $\kappa \geq 0$ and the invariants $(E, X) \in [e(X), \infty) \times \mathbb{R}^2$ are fixed, and the initial data satisfies $(A_0, B_0) \in \mathcal{D}(E, X)$.

We summarize some previous results here for convenience.

Lemma 7.1

Suppose that (**A**) holds. If $A_0 \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then there exists

$$s(0) = (s_1(0), s_2(0), s_3(0)) \in \mathbb{R}^3 \times [1, \infty)$$

such that

$$A_0 = \boldsymbol{A}(s(0)) \quad \text{and} \quad \tfrac{1}{2}|A_0|^2 = s_3(0),$$

where $\boldsymbol{A}(s)$ was defined in Lemma 3.12Moreover, there holds

$$B_0 = \frac{X_1 + X_2}{g_{11}(A_0)}\tau_1(A_0) + \frac{X_1 \quad X_2}{g_{22}(A_0)}\tau_2(A_0) + \langle A_0, B_0\rangle\, \tau_3(A_0).$$

If $A_0 \in \mathrm{SO}(2, \mathbb{R})$, then

$$X_1 = X_2, \quad E \quad \kappa + \tfrac{1}{4}X_1^2,$$

and there exists

$$s(0) = (s_1(0), s_2(0), s_3(0)) \in \mathbb{R}^2 \quad [1, \infty)$$

such that

$$A_0 = U(2s_1(0)), \quad \tfrac{1}{2}|A_0|^2 = 1 = s_3(0),$$

and

$$B_0 = \tfrac{1}{2}X_1\, U(2s_1(0))\, Z + \beta\, U(2s_2(0))\, M,$$

with

$$\beta = \left(E \quad \kappa \quad \tfrac{1}{4}X_1^2\right)^{1/2}.$$

Lemma 7.2

Suppose that $(\mathbf{A})$ holds. There exists a unique curve $s = (s_1, s_2, s_3) \in C^2(\mathsf{R}, \mathsf{R}^2 \times [1, \infty))$ such that $(s_3(t), s_3'(t))$ solves (6.9) with initial data $(s_3(0), s_3'(0)) = \mathsf{P}(A_0, B_0)$ and $(s_1(t), s_2(t))$ solves

$$s_1'(t) = \frac{X_1 + X_2}{4(s_3(t) + 1)}$$

$$s_2'(t) = \begin{cases} \dfrac{X_1 - X_2}{4(s_3(t) - 1)}, & \text{if } X_1 \neq X_2 \\ 0, & \text{if } X_1 = X_2, \end{cases} \tag{7.1}$$

with initial data $(s_1(0), s_2(0))$ defined by Lemma 7.1.

If $A \in C^0(\mathsf{R}, \mathrm{SL}(2, \mathsf{R})) \cap C^2(\mathsf{R}, \mathsf{M}^2)$ solves the IVP (4.6), (4.7), (4.8) with initial data (A_0, B_0), then $\mathsf{P}(A(t), A'(t)) = (s_3(t), s_3'(t))$.

Proof. The existence and uniqueness of a solution $(x, y) \in C^0(\mathsf{R}, \mathsf{R}^2)$ to (6.9) with initial data $\mathsf{P}(A_0, B_0)$ is given Corollary 6.7. Since the first equation of (6.9) says that $x'(t) = y(t)$, we can label the solution as (s_3, s_3'). The proof of the corollary also shows that $\mathsf{P}(A(t), A'(t)) = (s_3(t), s_3'(t))$.

We know that $(s_3(t), s_3'(t)) \in C\ (E, X)$, for all $t \in \mathsf{R}$. If $X_1 \neq X_2$, then $s_3(t) > 1$, by Lemma 6.1, so the right-hand sides of (7.1) are well-defined known functions. The solutions s_1 and s_2 are obtained by integration. □

Remark 7.1. The value of $s_3(0)$ is consistent in Lemmas 7.1 and 7.2.

Lemma 7.3

Suppose that $(\mathbf{A})$ holds. Let $s \in C^0(\mathsf{R}, \mathsf{R}^2 \times [1, \infty))$ be the curve constructed in Lemma 7.2. If, on some open interval I, there holds $s_3(t) > 1$, then $\mathsf{A} \circ s(t)$ solves (4.7) on I.

Proof. Define

$$\bar{A}(t) = \mathsf{A} \circ s(t).$$

Since $s \in C^2$ and $s_3(t) > 1$ on I, we see by the definition given in Lemma 3.12that

$$\bar{A} \in C^0(I, \mathrm{SL}(2,\mathbf{R})) \cap C^2(I, \mathbf{M}^2) \quad \text{and} \quad s_3(t) = \tfrac{1}{2}|\bar{A}(t)|^2.$$

Thus, we have that

$$\bar{A}(t) \in \mathrm{SL}(2,\mathbf{R}) \setminus \mathrm{SO}(2,\mathbf{R}), \quad t \in I,$$

by Lemma 3.7It also follows that

$$s_3'(t) = \left\langle \bar{A}(t), \bar{A}'(t) \right\rangle, \quad t \in I,$$

and so by definition, we have

$$P(\bar{A}(t), \bar{A}'(t)) = (s_3(t), s_3'(t)), \quad t \in I.$$

By Corollary 6.7, $(s_3(t), s_3'(t)) \in C\ (E, X)$, and so

$$(\bar{A}(t), \bar{A}'(t)) \in \mathcal{D}\ (E, X) = P^{\ 1}(C\ (E, X)), \quad t \in I. \tag{7.2}$$

In the following calculation, we suppress the dependence of functions upon the independent variable t in order to simplify the formulas. All calculations are valid on the interval I where we have assumed that $s_3 > 1$. Since the metric $g(\bar{A})$ depends only on $s_3 = \frac{1}{2}|\bar{A}|^2$, we shall write $g(s_3)$ for $g(\bar{A}) = g(A\ \ s)$, with abuse of notation.

Using the Christoffel symbols from Lemma 3.15 and the second fundamental form of Definition 3.10, a standard geometric calculation yields

$$\bar{A}'' = \sum_{i=1}^{3} \left[s_i'' + \sum_{j,k=1}^{3} \Gamma^i_{jk}(\bar{A}) s_j' s_k' \right] \tau_i(\bar{A}) + \sum_{j,k=1}^{3} \Pi[\tau_j(\bar{A}), \tau_k(\bar{A})] s_j' s_k' N(\bar{A}).$$

By Definitions 3.8 and (3.11), we have that for any $A \in \mathrm{SL}(2,\mathbf{R}) \setminus \mathrm{SO}(2,\mathbf{R})$

$$A = \frac{1}{g_{33}(A)}\tau_3(A) + \frac{2}{|A|}N(A) \quad \text{and} \quad \mathrm{cof}A = |A|N(A).$$

It follows that

$$\begin{aligned}
\bar{A}'' + \kappa\bar{A} - \Lambda(\bar{A}, \bar{A}')\mathrm{cof}\bar{A} \\
&= \sum_{i=1}^{3}\left[s_i'' + \sum_{j,k=1}^{3}\Gamma^i_{jk}(\bar{A})s_j's_k'\right]\tau_i(\bar{A}) + \frac{\kappa}{g_{33}(\bar{A})}\tau_3(\bar{A}) \\
&\quad + \left[\sum_{j,k=1}^{3}\Pi[\tau_j(\bar{A}), \tau_k(\bar{A})]s_j's_k' + \frac{2\kappa}{|\bar{A}|} - |\bar{A}|\Lambda(\bar{A}, \bar{A}')\right]N(\bar{A}).
\end{aligned}$$

From this we see that $\bar{A}$ satisfies (4.7) on I if and only if the system

$$\begin{aligned}
&s_i'' + \sum_{j,k=1}^{3}\Gamma^i_{jk}(\bar{A})s_j's_k' = 0, \quad i = 1, 2 \\
&s_3'' + \sum_{j,k=1}^{3}\Gamma^3_{jk}(\bar{A})s_j's_k' + \frac{\kappa}{g_{33}(\bar{A})} = 0 \\
&\sum_{j,k=1}^{3}\Pi[\tau_j(\bar{A}), \tau_k(\bar{A})]s_j's_k' + \frac{2\kappa}{|\bar{A}|} - |\bar{A}|\Lambda(\bar{A}, \bar{A}') = 0
\end{aligned}$$

holds on I.

By (7.2), $(\bar{A}, \bar{A}') \in D(E, X)$, so Lemma 5.5 tells us that

$$\Lambda(\bar{A}, \bar{A}') = \frac{2E - X_1X_2}{2s_3^2}.$$

Using Lemmas 3.15 and 3.18, we find that our system is equivalent to

$$s_i'' + \frac{g_{ii}'(s_3)}{g_{ii}(s_3)} s_i' s_3' = 0, \quad i = 1, 2 \tag{7.3}$$

$$s_3'' - \frac{g_{11}'(s_3)}{2g_{33}(s_3)}(s_1')^2 - \frac{g_{22}'(s_3)}{2g_{33}(s_3)}(s_2')^2 + \frac{g_{33}'(s_3)}{2g_{33}(s_3)}(s_3')^2 + \frac{\kappa}{g_{33}(s_3)} = 0 \tag{7.4}$$

$$\frac{g_{11}(s_3)}{2s_3}(s_1')^2 + \frac{g_{22}(s_3)}{2s_3}(s_2')^2 + \frac{g_{33}(s_3)}{2s_3^2}(s_3')^2 + \frac{\kappa}{s_3} - \frac{2E - X_1X_2}{2s_3^2} = 0, \tag{7.5}$$

where as mentioned above $g(s_3) = g(\bar{A}) = g(A \circ s)$ and $g'(s_3)$ indicates the derivative in s_3.

The equations (7.3) hold thanks to the definitions (7.1). Again using (7.1), we find after some computation that (7.5) is equivalent to the equation $\Phi\ (s_3, s_3'; E, X) = 0$, which holds by Lemma 6.1, since $(s_3, s_3') \in C\ (E, X)$ and $s_3 > 1$. Finally, (7.4) is equivalent to the equation

$$s_3'' + \partial_x \Phi\ (s_3, s_3'; E, X) + \frac{g_{33}'(s_3)}{g_{33}(s_3)} \Phi\ (s_3, s_3'; E, X) = 0, \tag{7.6}$$

which holds by (6.9) and the fact that $\Phi\ (s_3, s_3'; E, X) = 0$. Thus, we have verified that $\bar{A} = A \circ s$ solves (4.7) on I. □

Theorem 7.4

Suppose that (**A**) holds, and let

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be the solution of (4.6), (4.7), (4.8) with initial data (A_0, B_0). Let $s \in C^0(\mathbb{R}, \mathbb{R}^2 \times [1, \infty))$ be the curve constructed in Lemma 7.2.

If $(1, 0) \notin C\ (E, X)$ or if $(1, 0) \in C\ (E, X)$ is a critical point of $\Phi\ (x, y; E, X)$, then

$$A(t) = A \quad s(t), \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof. By Corollary 6.7, we have $(s_3(t), s_3'(t)) \in C\ (E, X)$, for all $t \in \mathbb{R}$. If $(1,0) \notin C\ (E,X)$, then $s_3(t) > 1$, for all $t \in \mathbb{R}$.

If $(1,0) \in C\ (E,X)$ is a critical point of $\Phi\ (x,y;E,X)$, then $(1,0)$ is an equilibrium solution of (6.9). Thus, if $(s_3(t_0), s_3'(t_0)) = (1,0)$, at a single time t_0, then $s_3(t) = 1$, for all $t \in \mathbb{R}$. Otherwise, $(s_3(t), s_3'(t)) \in C\ (E,X) \setminus \{(1,0)\}$, and we obtain $s_3(t) > 1$, for all $t \in \mathbb{R}$.

If $s_3(t) = 1$, for all $t \in \mathbb{R}$, then $A(t) \in \mathrm{SO}(2, \mathbb{R})$, for all $t \in \mathbb{R}$, so $X_1 = X_2$, by (5.1). By Lemma 6.11, we have that

$$A(t) = U(\tfrac{1}{2}X_1 t + \theta),$$

for some $\theta \in \mathbb{R}$. Since

$$U(\theta) = A(0) = U(2s_1(0)),$$

we may take $\theta = 2s_1(0)$.

On the other hand, we can calculate the function $s_1(t)$ directly from (7.1), and we find

$$s_1(t) = \tfrac{1}{4}X_1 t + s_1(0).$$

Since $s_3(t) = 1$, the formula in Lemma 3.12reduces to

$$A \quad s(t) = U(s_1(t) + s_2(t))\ I\ U(s_1(t) \quad s_2(t)) = U(2s_1(t)).$$

This shows that $A(t) = A \quad s(t)$, for all $t \in \mathbb{R}$, when $s_3(t) = 1$.

Now let us assume that $s_3(t) > 1$, for all $t \in \mathbb{R}$. Define

$$\bar{A}(t) = \mathcal{A} \circ s(t).$$

By applying Lemma 7.3 on the interval $I = \mathbb{R}$, we see that

$$\bar{A} \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

is a solution of (4.7).

We now check the initial data of $\bar{A}$. By Lemma 7.1 we have

$$\bar{A}(0) = \mathcal{A} \circ s(0) = A_0.$$

By Lemmas 3.12and 7.1, we have

$$\bar{A}'(0) = (\mathcal{A} \circ s)'(0) = \sum_{i=1}^{3} s_i'(0)\tau_i(\mathcal{A} \circ s(0)) = \sum_{i=1}^{3} s_i'(0)\tau_i(A_0).$$

From (7.1), we see that

$$s_1'(0) = \frac{X_1 + X_2}{g_{11}(A_0)} \quad \text{and} \quad s_2'(0) = \frac{X_1 - X_2}{g_{22}(A_0)}.$$

Moreover, $s_3'(0) = \langle A_0, B_0 \rangle$, by definition. Thus, from Lemma 7.1 we find that $\bar{A}'(0) = B_0$

Having shown that $\bar{A}$ solves (4.7) with the same initial data as A, we conclude that $\bar{A} = \mathcal{A} \circ s = A$, by uniqueness of solutions to the IVP. $\square$

Lemma 7.5

Suppose that (**A**) holds. Suppose that $(1, 0) \in C(E, X)$ and $(1, 0)$ is not a critical point of $\Phi(x, y; E, X)$. Let

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be a solution of (4.6), (4.7), (4.8) with initial data (A_0, B_0).

If $\kappa = 0$, then there exists a unique $t_0 \in \mathbb{R}$ such that

$$P(A(t_0), A'(t_0)) = (1, 0).$$

If $\kappa > 0$, then $P(A(t), A'(t))$ is periodic with minimum period $T > 0$. Morevoer, there exists a unique $t_0 \in \mathbb{R}$ such that $0 \in [t_0, t_0 + T)$ and

$$\{t \in \mathbb{R} : P(A(t), A'(t)) = (1, 0)\} = \{t_j = t_0 + jT : j \in \mathbb{Z}\}.$$

Proof. By Lemma 6.8, $C\ (E, X)$ consists of a single smooth orbit

$$(x(t), y(t)) = P(A(t), A'(t)).$$

Thus, by Lemma 6.5, there exists $t_0 \in \mathbb{R}$ such that $(x(t_0), y(t_0)) = (1, 0)$. If $\kappa = 0$, this orbit is unbounded, so it is not closed, and t_0 is the unique time with this property. If $\kappa > 0$, then the orbit is closed and therefore periodic with minimal period $T > 0$. Since T is a minimal period, we have that the set $\{t_0 + jT\}$, $j \in \mathbb{Z}$, coincides with the set of times t where $(x(t), y(t)) = (1, 0)$. We can redefine t_0, if necessary, so that $0 \in [t_0, t_0 + T)$. □

Theorem 7.6

Suppose that (**A**) holds, and let

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be the solution of (4.6), (4.7), (4.8) with initial data (A_0, B_0).

Let $s(t) \in C^0(\mathbb{R}, \mathbb{R}^2 \times [1, \infty))$ be the curve constructed in Lemma 7.2.

If $(1,0) \in C(E,X)$ and $(1,0)$ is not a critical point of $\Phi(x,y;E,X)$, then

$$A(t) = (\mathrm{cof})^{n(t)} \mathcal{A}(s(t)),$$

where $n(t) = 0, 1$ is the piece-wise constant right continuous function with $n(0) = 0$ and jump discontinuities on the set $\{t_j\}$ from Lemma 7.5.

Proof. Define $\bar{A}(t) = (\mathrm{cof})^{n(t)} \mathcal{A}(s(t))$. The goal is to prove that $A = \bar{A}$, using the same uniqueness argument as in the proof of Theorem 7.4.

Note that $s \in C^2(\mathbb{R})$, and so $\mathcal{A} \circ s \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$. Since the cofactor map leaves $\mathrm{SL}(2,\mathbb{R})$ invariant, we see that $\mathrm{cof}\mathcal{A} \circ s \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$, as well. Thus, $\bar{A}$ is continuous, except possibly at the points $\{t_j\}$. However, at the points $\{t_j\}$, we have $s_3 = 1$, and so by Lemma 3.7, $\mathcal{A} \circ s(t_j) \in \mathrm{SO}(2,\mathbb{R})$. By Lemma 3.3, $\mathcal{A} \circ s(t_j) = \mathrm{cof}\mathcal{A} \circ s(t_j)$, so we see that $\bar{A} \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$.

Examining the definition of $\mathcal{A} \circ s$, we see that this function could fail to be differentiable at the times t_j when $s_3(t_j) = 1$, because of the term $\sqrt{s_3(t) - 1}$.

Let us suppose first that $\kappa = 0$. Then by Lemma 7.5, there exists a single time $t_0 \in \mathbb{R}$ such that $s(t_0) = 1$. Assume that $0 \in [t_0, \infty)$ so that

$$\bar{A}(t) = \begin{cases} \mathrm{cof}\mathcal{A} \circ s(t), & t < t_0 \\ \mathcal{A} \circ s(t), & t \geq t_0. \end{cases}$$

(If $0 \in (-\infty, t_0)$, then the cofactor would be applied on the other interval.) Now $s_3(t_0) = 1$ is a minimum value for s_3, so $s_3'(t_0) = 0$ and $s_3''(t_0) \geq 0$. Since $(1,0)$ is not a critical point of $\Phi(x,y;E,X)$, $P(x;E,X)$ has a simple root at $x = 1$, by Lemma 6.2. Since $(s_3(t), s_3'(t))$ satisfies (6.9), we have $s_3''(t_0) = -P'(1;E,X)/2 \neq 0$. Thus, $s_3''(t_0) > 0$, and we can write

$$s_3(t) - 1 = \alpha(t)(t - t_0)^2, \quad \alpha \in C^2, \quad \alpha(t_0) = \tfrac{1}{2} s''(t_0) > 0. \tag{7.7}$$

Thus, $\alpha(t)$ is strictly positive in a neighborhood of $t = t_0$. From this we see that the function

$$\sqrt{\alpha(t)}\,(t - t_0) = \begin{cases} -\sqrt{s_3(t) - 1}, & \text{if } t < t_0 \\ \sqrt{s_3(t) - 1}, & \text{if } t \geq t_0 \end{cases}$$

is C^2 in a neighborhood of $t = t_0$. (If $t_0 \in (-\infty, 0)$, then the signs of the two terms above would be reversed.) This shows that

$$\begin{aligned} &\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{s_3(t)+1} & \sqrt{\alpha(t)}\,(t - t_0) \\ \sqrt{\alpha(t)}\,(t - t_0) & \sqrt{s_3(t)+1} \end{bmatrix} \\ &\quad = \begin{cases} \mathrm{cof} H(s_3(t)), & \text{if } t < t_0 \\ H(s_3(t)), & \text{if } t \geq t_0 \end{cases} \\ &\quad = (\mathrm{cof})^{n(t)} H(s_3(t)). \end{aligned} \tag{7.8}$$

This function belongs to C^2 for t near t_0. Finally, we conclude that

$$\bar{A}(t) = U(s_1(t) + s_2(t))\,(\mathrm{cof})^{n(t)} H(s_3(t))\,U(s_1(t) - s_2(t))$$

belongs to $C^2(\mathsf{R}, \mathsf{M}^2)$.

If $\kappa > 0$, then the set $\{t_j\}$ is countable and a repetition of the argument of the previous paragraph near each t_j again shows that $\bar{A} \in C^2(\mathsf{R}, \mathsf{M}^2)$.

Next, we show that $\bar{A}$ solves (4.7).

Suppose that $s_3(t) > 1$ on some open interval I. Then by Lemma 7.3, $A \circ s \in C^0(I; \mathrm{SL}(2, \mathsf{R})) \cap C^2(I, \mathsf{M}^2)$ solves (4.7) on I. Since the cofactor map leaves solutions of (4.7) invariant, we see that $\mathrm{cof} A \circ s$ also solves (4.7) on I. Therefore, $\bar{A}$ solves (4.7) except on the at most countable set of isolated points $\{t_j\}$. Having shown that $\bar{A} \in C^2(\mathsf{R}, \mathsf{M}^2)$, it follows that $\bar{A}$ solves (4.7) on R.

It remains to verify that A and $\bar{A}$ share the same initial data.

If $A_0 \notin \mathrm{SO}(2, \mathsf{R})$, then according to Lemma 7.1, our choice $s(0)$ gives $\bar{A}(0) = A \circ s(0) = A_0$. Also, by Lemmas 7.1 and 7.2, we have

$$\bar{A}'(0) = (\mathcal{A} \circ s)'(0) = \sum_{i=1}^{3} s_i'(0)\tau_i(A_0) = B_0.$$

If $A_0 \in \mathrm{SO}(2,\mathbf{R})$, then $\mathcal{P}(A_0, B_0) = (s_3(0), s_3'(0)) = (1, 0)$. By Lemma 7.1, we have

$$A_0 = U(2s_1(0))$$

and

$$B_0 = (\tfrac{1}{2}X_1)\, U(2s_1(0))\, Z + \beta\, U(2s_2(0))\, M,$$

with

$$\beta = \left(E - \kappa - \tfrac{1}{4}X_1^2\right)^{1/2}.$$

Since $s_3(0) = 1$, we have $t_0 = 0$, and so

$$\bar{A}(0) = \mathcal{A} \circ s(0) = U(2s_1(0)) = A_0.$$

Going back to the formula (7.8), we have

$$\bar{A}(t) = \tfrac{1}{\sqrt{2}} U(s_1(t) + s_2(t)) \begin{bmatrix} \sqrt{s_3(t) + 1} & -\sqrt{\alpha(t)}\; t \\ \sqrt{\alpha(t)}\; t & \sqrt{s_3(t) + 1} \end{bmatrix} U(s_1(t) - s_2(t)),$$

for $0 \leq t < T$, where by (7.7), (7.6)

$$\alpha(0) = \tfrac{1}{2}s_1''(0) = -\tfrac{1}{2}\partial_x\Phi\;(1, 0; E, X) = \tfrac{1}{4}P'(1; E, X) = 2\beta^2.$$

Since $X_1 = X_2$, we have $s_2(t) = s_2(0)$, by (7.1). As in Lemma 3.14, this can be written as

$$\bar{A}(t) = \left(\frac{s_3(t)+1}{2}\right)^{1/2} U(2s_1(t)) + t\left(\frac{\alpha(t)}{2}\right)^{1/2} U(2s_2(0))\, M,$$

$0 \le t < T$. Since $\bar{A}$ is C^2, it is enough to compute its right derivative at $t = 0$:

$$\bar{A}'(0) = U(2s_1(0))\, Z\, 2s_1'(0) + \beta\, U(2s_2(0))\, M = B_0,$$

by (7.1), as desired. □

Corollary 7.7

A solution $A(t)$ of (4.7) is symmetric if and only if $s_1(t) = j\pi/2$, for some $j \in \mathbb{Z}$.

7.2 MHD

In this section, we focus on the case where $\kappa > 0$. The next result summarizes the properties of the orbits $C(E, X)$ of (6.9) when $\kappa > 0$. Recall that these orbits are contained in the set

$$P(D) = \{(x, y) : x > 1\} \cup \{(1, 0)\}.$$

Lemma 7.8

Fix $\kappa > 0$.

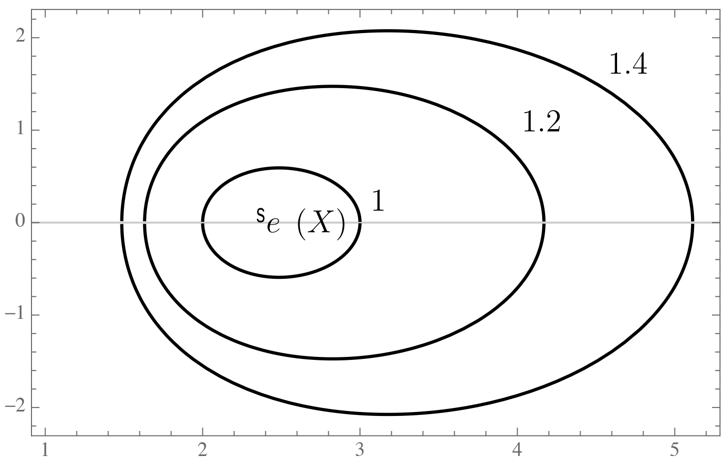
1. If $X_1 \neq X_2$, then:

   (a) $C(e(X), X) = \{(x_0, 0)\}$ where $x_0 > 1$ and $(x_0, 0)$ is a critical point of the Hamiltonian $\Phi(x, y; e(X), X)$, and

   (b) for all $E > \bar{E} \quad e(X)$, $C(E, X)$ is a nontrivial closed orbit of the system (6.9) in $P(D) \setminus \{(1, 0)\}$ enclosing $C(\bar{E}, X)$. (See Figure 7.1)

2. If $X_1 = X_2$ and $\frac{1}{8}X_1^2 \quad \kappa$, then:

Figure 7.1: Level curves $C\ (E, X)$ in the case $X_1 \neq X_2$, with $\kappa = 1/4$, $X_1 = \quad X_2 = 1$, $E = e\ (X)$, 1, 1.2, 1.4.



(a) $e\ (X) = \kappa + \frac{1}{4}X_1^2$,

(b) $C\ (e\ (X), X) = \{(1, 0)\}$ and $(1, 0)$ is a critical point of the Hamiltonian $\Phi\ (x, y; e\ (X), X)$,

(c) for all $E > \bar{E} > e\ (X)$, $C\ (E, X)$ is a nontrivial closed orbit of the system (6.9) in $P(D)$ containing $\{(1, 0)\}$ and enclosing $C\ (\bar{E}, X) \setminus \{(1, 0)\}$. (See Figure 7.2)

3. If $X_1 = X_2$ and $\frac{1}{8}X_1^2 > \kappa$, then:

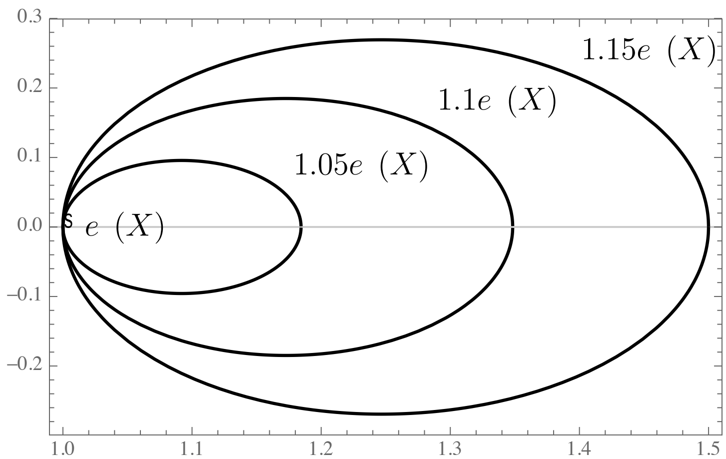
(a) $3\kappa \quad e\ (X) = (2\kappa)^{1/2}|X_1| \quad \kappa < E \quad \kappa + \frac{1}{4}X_1^2$,

(b) $C\ (e\ (X), X) = \{(x_0, 0)\}$ where $x_0 > 1$ and $(x_0, 0)$ is a critical point of $\Phi\ (x, y; e\ (X), X)$,

(c) for all $E \ > E > \bar{E} \quad e\ (X)$, $C\ (E, X)$ is a is a nontrivial closed orbit of (6.9) in $P(D) \setminus \{(1, 0)\}$ enclosing $C\ (\bar{E}, X)$,

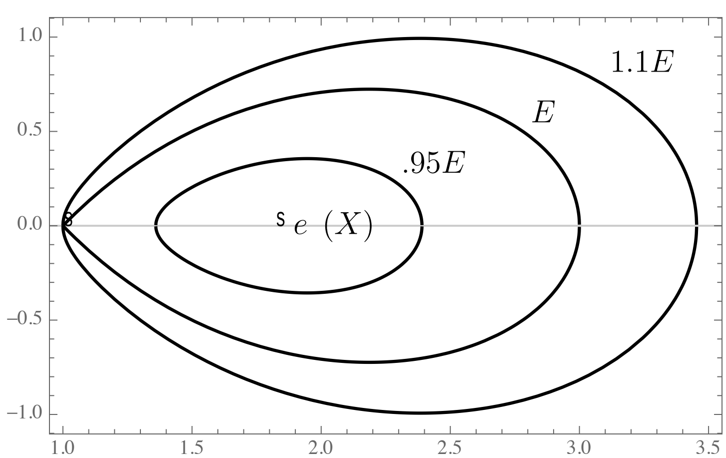
(d) $C\ (E\ , X)$ is a nontrivial closed curve in $P(D)$ containing $\{(1, 0)\}$ and enclosing $C\ (\bar{E}, X) \setminus \{(1, 0)\}$ for $E \ > \bar{E} \quad e\ (X)$, $(1, 0)$ is a critical point of the Hamiltonian $\Phi\ (x, y; E\ , X)$, and $C\ (E\ , X) \setminus \{(1, 0)\}$ is a homoclinic orbit,

Figure 7.2: Level curves $C\ (E, X)$ in the case $X_1 = X_2$, $\kappa \quad \frac{1}{8}X_1^2$, with $\kappa = 1/4$, $X_1 = X_2 = 1$, $E = e\ (X)$, $1.05\,e\ (X)$, $1.1\,e\ (X)$, $1.15\,e\ (X)$.



(e) for all $E > E\ $, $E > \bar{E} \quad e\ (X)$, $C\ (E, X)$ is a nontrivial closed orbit in $P(D)$ containing $f(1,0)g$ and enclosing $C\ (\bar{E}, X)\ n\ f(1,0)g$. (See Figure 7.3)

Figure 7.3: Level curves $C\ (E, X)$ in the case $X_1 = X_2$, $\kappa < \frac{1}{8}X_1^2$, for the values $\kappa = 1/4$, $X_1 = X_2 = 2$, $E = e\ (X)$, $.95E\ $, $E\ $, $1.1E\ $, $E\ \ = \kappa + \frac{1}{4}X_1^2$.



Proof. This is an application of Lemmas 5.8, 6.3, and 6.5. □

Remark 7.2. Cases (2) and (3) of Lemma 7.8 can also be characterized by $X_1 = X_2$ and

$E > \kappa + \frac{1}{4}X_1^2$ or $E = e\ (X)$, respectively, by Lemma 5.8.

Theorem 7.9

Let $\kappa > 0$ and $(E, X) \in [e\ (X), \infty)\ \ \mathbb{R}^2$. Let

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be a solution of (4.6), (4.7), (4.8) with initial data $(A_0, B_0) \in D\ (E, X)$.

1. The solution A is constant if and only if $X = 0$ and $E = \kappa$.
2. The solution A is non-constant and rigid if and only if $X \neq 0$ and either $E = e\ (X)$ or $A_0 \in \mathrm{SO}(2, \mathbb{R})$ and $E = \kappa + \frac{1}{4}X_1^2 > e\ (X)$.

Proof. This is an application of Lemmas 6.9, 6.10, and 6.11. □

The next results concern the homoclinic orbit in case (3d) of Lemma 7.8 .

Theorem 7.10

Let $\kappa > 0$. Suppose that $(E, X) \in [e\ (X), \infty)\ \ \mathbb{R}^2$ satisfies

$$X_1 = X_2 \quad \text{and} \quad E = \kappa + \tfrac{1}{4}X_1^2 > e\ (X).$$

Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a solution of the IVP (4.6), (4.7), (4.8) with initial data $(A_0, B_0) \in D\ (E, X) \setminus R(X_1)$, (cf. Lemma 6.3). Then there exist phases θ such that for $0 < \lambda < \frac{1}{2}(X_1^2\ \ 8\kappa)^{1/2}$, the solution satisfies

$$\lim_{t \to \infty} e^{\ |t|} \left| \frac{d^j}{dt^j} \left[A(t)\ \ U\left(\tfrac{1}{2}X_1 t + \theta\ \right)\right] \right| = 0, \quad j = 0, 1.$$

Proof. The assumptions on the parameters put us in case (3d) of Lemma 7.8, and in particular, we have $\kappa < \frac{1}{8}X_1^2$. The Hamiltonian $\Phi\ (x, y; E, X)$ has a critical point at $(1, 0)$, the set $C\ (E, X) \setminus \{(1, 0)\}$ is a nontrivial homoclinic orbit, and since $A_0 \notin \mathrm{SO}(2, \mathbb{R})$,

$$\{P(A(t), A'(t)) : t \in \mathbb{R}\} = C\ (E, X) \setminus \{(1, 0)\}.$$

Set $(x(t), y(t)) = \boldsymbol{P}(A(t), A'(t))$. Then we have

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \nearrow 0, \quad \text{as} \quad t \to \infty,$$

and

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \searrow 0, \quad \text{as} \quad t \to -\infty.$$

We shall prove the result for $t \to \infty$, the other case being nearly the same.

Using Definition 6.1, (6.9), and the condition on (E, X), we find that

$$\begin{aligned} x'(t) = y(t) &= -\left(\frac{P(x(t); E, X)}{x(t)}\right)^{1/2} \\ &= -\left(\frac{X_1^2 - 4\kappa(x(t)+1)}{x(t)}\right)^{1/2} (x(t) - 1), \quad t \geq 1. \end{aligned} \tag{7.9}$$

Fix $0 < \lambda < \frac{1}{2}(X_1^2 - 8\kappa)^{1/2}$ and choose $t_0 \geq 1$ such that

$$\left(\frac{X_1^2 - 4\kappa(x(t)+1)}{x(t)}\right)^{1/2} > 2\lambda, \quad t \geq t_0.$$

Then

$$x'(t) \leq -2\lambda(x(t) - 1), \quad t \geq t_0,$$

and we obtain the estimate

$$0 < x(t) - 1 \leq (x(t_0) - 1)\exp[-2\lambda(t - t_0)], \quad t \geq t_0.$$

Applying this in (7.9) yields

$$|x'(t)| \lesssim \exp[-2\lambda(t - t_0)], \quad t \geq t_0.$$

Using the notation from Lemma 3.12, it follows that

$$\left|\frac{d^j}{dt^j}[H(x(t)) - I]\right| = \left|\frac{d^j}{dt^j}\begin{bmatrix} \sqrt{\frac{x(t)+1}{2}} - 1 & \sqrt{\frac{x(t)-1}{2}} \\ -\sqrt{\frac{x(t)-1}{2}} & \sqrt{\frac{x(t)+1}{2}} - 1 \end{bmatrix}\right| \lesssim \exp(-\lambda t),$$

for $t \quad t_0$, $j = 0, 1$.

We now use Theorem 7.4 to reconstruct $A(t)$. Define $(s_1(t), s_2(t))$ according to Lemma 7.2. Then since $X_1 = X_2$, we have $s_2(t) = s_2(0)$ and

$$\begin{aligned} s_1(t) &= s_0(0) + \int_0^t \frac{X_1}{2(x(\sigma)+1)} d\sigma \\ &= \tfrac{1}{4} X_1 t + s_0(0) + \tfrac{1}{4} X_1 \int_0^t \frac{x(\sigma) + 1}{x(\sigma)+1} d\sigma \\ &= \tfrac{1}{4} X_1 t + \tfrac{1}{2}\theta_+ + \tfrac{1}{4} X_1 \int_t^1 \frac{x(\sigma) \quad 1}{x(\sigma)+1} d\sigma, \end{aligned}$$

where

$$\tfrac{1}{2}\theta_+ = s_0(0) + \tfrac{1}{4} X_1 \int_0^1 \frac{x(\sigma) + 1}{x(\sigma)+1} d\sigma.$$

Thus, we have

$$\left| \frac{d^j}{dt^j} \left[s_1(t) \quad (\tfrac{1}{4} X_1 t + \tfrac{1}{2}\theta_+) \right] \right| \ . \ \exp(\quad \lambda t), \quad t \quad t_0, \quad j = 0, 1.$$

It follows that

$$\begin{aligned} &\left| \frac{d^j}{dt^j} \left[U(2s_1(t)) \quad U(\tfrac{1}{2} X_1 t + \theta_+) \right] \right| \\ &\qquad = \left| \frac{d^j}{dt^j} \left[U(2s_1(t) \quad \tfrac{1}{2} X_1 t \quad \theta_+) \quad I \right] U(\tfrac{1}{2} X_1 t + \theta_+) \right| \\ &\qquad\qquad . \ \exp(\quad \lambda t), \quad \text{for} \quad t \quad t_0, \quad j = 0, 1. \end{aligned}$$

By Theorem 7.4, we obtain

$$A(t) = U(s_1(t) + s_2(0))\, H(x(t))\, U(s_1(t) \quad s_2(0)).$$

The desired estimates follow after writing

$$\begin{aligned} A(t) \quad U(\tfrac{1}{2} X_1 t + \theta_+) &= U(s_1(t) + s_2(0))\, [H(x(t)) \quad I]\, U(s_1(t) \quad s_2(0)) \\ &\qquad + U(2s_1(t)) \quad U(\tfrac{1}{2} X_1 t + \theta_+). \end{aligned}$$

□

Remark 7.3. The total phase shift is given by the expression

$$\theta_+ \quad \theta \quad = \quad \tfrac{1}{4}X_1 \int_{\infty}^{\infty} \frac{x(\sigma) \quad 1}{x(\sigma)+1} d\sigma.$$

Corollary 7.11

Let $\kappa > 0$. Suppose that

$$X_1 = X_2 \quad \text{and} \quad E = \kappa + \tfrac{1}{4}X_1^2 > e\ (X).$$

Then $\mathcal{R}(X_1) \quad \mathcal{D}\ (E, X)$ corresponds to the orbit of the rigid rotation $U\left(\tfrac{1}{2}X_1 t\right)$. The set $\mathcal{D}\ (E, X) \setminus \mathcal{R}(X_1)$ is a stable and unstable manifold for $\mathcal{R}(X_1)$. Every solution orbit (A, A') in $\mathcal{D}\ (E, X) \setminus \mathcal{R}(X_1)$ is homoclinic to $\mathcal{R}(X_1)$, that is,

$$\lim_{|t| \to \infty} e^{\ |t|} \operatorname{dist}[(A(t), A'(t)), \mathcal{R}(X_1)] = 0,$$

for some $\lambda > 0$.

Proof. This follows from Lemma 6.12 and Theorem 7.10. □

Remark 7.4. $\bigcup_{X_1 \in \mathbb{R}} \mathcal{R}(X_1)$ is a normally hyperbolic invariant manifold.

Theorem 7.12

If $A(t)$ is a solution of the IVP (4.6), (4.7), (4.8) such that the quantity $\frac{1}{2}|A(t)|^2$ is T-periodic for some $T > 0$, then the solution has the form

$$A(t) = U(\omega_1 t)\hat{A}(t)U(\omega_2 t),$$

where $\hat{A}(t)$ is is T-periodic if $(1,0) \notin C\ (E, X)$ and $2T$-periodic if $(1,0) \in C\ (E, X)$. The frequencies are defined by

$$\omega_1 + \omega_2 = \frac{2}{T} \int_0^T \frac{X_1 + X_2}{g_{11}(A(t))} dt$$

and

$$\omega_1 - \omega_2 = \begin{cases} 0, & X_1 = X_2 \\ \dfrac{2}{T}\displaystyle\int_0^T \frac{X_1 - X_2}{g_{22}(A(t))}dt, & X_1 \neq X_2. \end{cases}$$

Proof. Define $(s_1(t), s_2(t))$ as in Lemma 7.2 and ω_1, ω_2 as above. Since $g_{ii}(A(t))$ is T-periodic, $i = 1, 2$, the functions

$$s_1'(t) + s_2'(t) - \omega_1 \quad \text{and} \quad s_1'(t) - s_2'(t) - \omega_2$$

are T-periodic and have mean zero over the interval $[0, T]$. Hence, their antidervatives

$$s_1(t) + s_2(t) - \omega_1 t \quad \text{and} \quad s_1(t) - s_2(t) - \omega_2 t$$

are T-periodic. It follows that

$$U(s_1(t) + s_2(t))\, U(-\omega_1 t) \quad \text{and} \quad U(s_1(t) - s_2(t))\, U(-\omega_2 t)$$

are T-periodic.

Now going back to Theorems 7.4 and 7.6, we find that

$$\hat{A}(t) = U(-\omega_1 t)A(t)U(-\omega_2 t)$$

is T-periodic if $(1, 0) \notin C\,(E, X)$ and $2T$-periodic if $(1, 0) \in C\,(E, X)$. □

Remark 7.5. The result shows that there is monodromy when the solution $A(t)$ passes through $\mathrm{SL}(2, \mathbb{R})$.

Remark 7.6. Note that the result holds for rigid solutions. In this case, the quantity $|A(t)|$ is constant and thus T-periodic for all $T \geq 0$. Any value of $T > 0$ can be used in computing the frequencies.

Theorem 7.13

Let $A(t)$ be a solution of the IVP (4.6), (4.7), (4.8) such that the quantity $|A(t)|$ is T-periodic for some $T > 0$.

For every $N \in \mathbb{N}$, there exists $\ell(N) \in \{1, \dots, N^2\}$ such that

$$|A(2\ell(N)T + t) - A(t)| \le 8\pi |A(t)|/N, \quad \text{for all} \quad t \in \mathbb{R}.$$

If $A(t)$ is rigid, then either $A(t)$ is periodic or the range of $A(t)$ is dense in the sphere of radius $|A_0|$ in $\mathrm{SL}(2, \mathbb{R})$.

Proof. By Theorem 7.12, we may write

$$A(t) = U(\omega_1 t)\hat{A}(t)U(\omega_2 t),$$

in which $\hat{A}(t)$ is $2T$-periodic. (If $A(t)$ does not pass through $\mathrm{SO}(2, \mathbb{R})$, then we know that $\hat{A}(t)$ is T-periodic.)

For every $x \in \mathbb{R}$, there is a unique $k \in \mathbb{Z}$ such that

$$\{x\} = x - 2\pi k \in [0, 2\pi).$$

Consider the set of $N^2 + 1$ ordered pairs

$$\left\{ \, (\{\omega_1 2jT\}, \{\omega_2 2jT\}) : j = 0, 1, \dots, N^2 \, \right\}$$

contained in the square $[0, 2\pi) \times [0, 2\pi)$. Partition this square into N^2 congruent subsquares of side $2\pi/N$. By the pigeonhole principle, two of these ordered pairs belong to the same subsquare. It follows that there exist $k, \ell(N) \in \mathbb{Z}$ such that $0 \le k < k + \ell(N) \le N^2$ and

$$|\{\omega_i 2kT\} - \{\omega_i 2(k + \ell(N))T\}| \le 2\pi/N, \quad i = 1, 2.$$

Thus, there exist $m_i \in \mathbb{Z}$ such that

$$|\omega_i 2\ell(N)T + 2\pi m_i| \le 2\pi/N, \quad i = 1, 2.$$

Define

$$\tau_i = \omega_i 2\ell(N)T + 2\pi m_i.$$

For $i = 1, 2$ and $t \in \mathbb{R}$, we have using Definition 3.6 and the mean value theorem

$$
\begin{aligned}
|U(\omega_i 2\ell(N)T + t) - U(t)| &= |U(\omega_i 2\ell(N)T) - I| \\
&= |U(\tau_i) - I| \\
&= \sqrt{2}\,[(\cos\tau_i - 1)^2 + \sin^2\tau_i]^{1/2} \\
&= 2(1 - \cos\tau_i)^{1/2} \\
&\leq 2|\tau_i| \\
&\leq 4\pi/N.
\end{aligned}
$$

For any $t \in \mathbb{R}$, we have

$$
\begin{aligned}
A(2\ell(N)T + t) &= U(\omega_1(2\ell(N)T + t))\hat{A}(2\ell(N)T + t)U(\omega_2(2\ell(N)T + t)) \\
&= U(\omega_1 2\ell(N)T)U(\omega_1 t)\hat{A}(t)U(\omega_2 t)U(\omega_2 2\ell(N)T) \\
&= U(\tau_1)A(t)U(\tau_2).
\end{aligned}
$$

We now estimate as follows

$$
\begin{aligned}
|A(2\ell(N)T + t) - A(t)| &= |U(\tau_1)A(t)U(\tau_2) - A(t)| \\
&= |[U(\tau_1) - I]A(t)U(\tau_2) + A(t)[U(\tau_2) - I]| \\
&\leq |U(\tau_1) - I||A(t)||U(\tau_2)| + |A(t)||U(\tau_2) - I| \\
&\leq 2(4\pi/N)|A(t)|.
\end{aligned}
$$

This proves the first statement.

If $A(t)$ is rigid, then $|A(t)| = |A_0|$, and so by Theorem 7.12

$$A(t) = U(\omega_1 t)A_0 U(\omega_2 t) = U(\{\omega_1 t\})A_0 U(\{\omega_2 t\}).$$

If ω_1 and ω_2 are rationally dependent, then $A(t)$ is periodic. The curve

$$t \mapsto (\{\omega_1 t\}, \{\omega_2 t\})$$

represents linear flow on the torus. If ω_1 and ω_2 are rationally independent, then it is well-known that the image of the curve is dense in the square $[0, 2\pi) \times [0, 2\pi)$. By Lemma

3.12, the set

$$\{UA_0V : U, V \in \mathrm{SO}(2,\mathbb{R})\}$$

coincides with the sphere of radius $|A_0|$ in $\mathrm{SL}(2,\mathbb{R})$. Thus, the range of $A(t)$ is dense in this sphere. □

Remark 7.7. The only solutions $A(t)$ for which $|A(t)|$ is not periodic are those which are homoclinic to a rigid rotation. Thus, the result shows that, generically, solutions are recurrent.

Remark 7.8. Since

$$|A(t)| \leq \left[2E\ (A(t), A'(t))\right]^{1/2}$$

and the energy is conserved, Theorem 7.13 shows that

$$|A(2\ell(N)T + t) - A(t)| \lesssim 1/N, \quad \text{for all} \quad t \in \mathbb{R}.$$

7.3 Perfect Fluids

Lemma 7.14

Fix $\kappa = 0$ and $(E, X) \in (0, \infty) \times \mathbb{R}^2$. Let

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be a solution of the IVP (4.6), (4.7), (4.8) with initial data in $D_0(E, X)$. The quantity $x(t) = \frac{1}{2}|A(t)|^2$ satisfies

$$x''(t) = \frac{2E(x(t) - 1)^2 + 4Ex(t) - X_1X_2}{x(t)^2}. \tag{7.10}$$

Moreover, $x''(t) \geq 0$, for all $t \in \mathbb{R}$, and if there exists $t_0 \in \mathbb{R}$ such that $x''(t_0) = 0$, then $x(t) = 1$ for all $t \in \mathbb{R}$.

Proof. Equation (7.10) is just a restatement of (6.11) in the case $\kappa = 0$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|X_1X_2| = |\, X_1(A(t),A'(t))\; X_2(A(t),A'(t))\, | &\le |A(t)|^2|A'(t)|^2 \\ &= 4x(t)E_0(A(t),A'(t)) = 4Ex(t).\end{aligned}$$

From this we see that $x''(t) \geq 0$, for all $t \in \mathbb{R}$.

If $x''(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then $E(x(t_0) - 1)^2 = 0$. Since $E > 0$, we have $x(t_0) = 1$. This implies that $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, so we must have $X_1 = X_2$ by equation (5.1).But then $x''(t_0) = 0$ implies that $4E = X_1^2$. By Lemma 6.11, $A(t)$ is a rigid solution in $\mathrm{SO}(2,\mathbb{R})$ for all $t \in \mathbb{R}$, and therefore, $x(t) \equiv 1$. □

Lemma 7.15

Fix $\kappa = 0$ and $(E,X) \in [0,\infty) \times \mathbb{R}^2$.

We have $C_0(0,0) = \{(x,0) : 1 \leq x < \infty\}$, and each point $(x,0) \in C_0(0,0)$ corresponds to an equilibrium solution of (6.9).

If $X_1 = X_2 \neq 0$ and $E = \frac{1}{4}X_1^2$, then $C_0(E,X)$ is the union of an equilibrium solution $\{(1,0)\}$ of (6.9) and two semi-bounded orbits. (See Figure 7.4)

In all other cases, $C_0(E,X)$ is a single orbit which is unbounded as $t \to \pm\infty$. (See Figures 7.4 and 7.5)

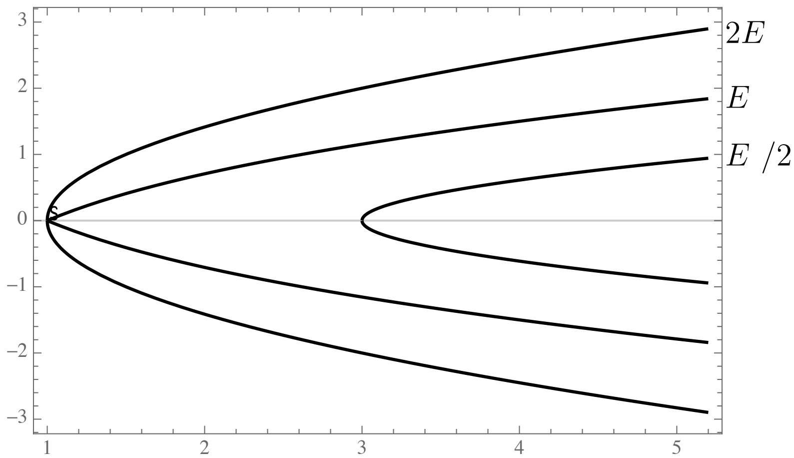
The point $(1,0)$ belongs to $C_0(E,X)$ if and only if $X_1 = X_2$ and $E \geq \frac{1}{4}X_1^2$.

Proof. As already shown in Lemma 6.5, the sets $C_0(E,X)$ are unbounded, and the set $C_0(E,X)$ consists of a single orbit unless it contains a critical point of $\Phi_0(x,y;E,X)$. This occurs when $(E,X) = (0,0)$ and when $X_1 = X_2$, $E = \frac{1}{4}X_1^2$, by Lemma 6.2. Lemma 6.1 gives the condition for $(1,0) \in C_0(E,X)$. □

Theorem 7.16

Let $A \in C^0(\mathbb{R},\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R},\mathbb{M}^2)$ be a solution of the IVP (4.6), (4.7), (4.8) with

Figure 7.4: Level curves $C_0(E, X)$ in the case $X_1 = X_2 = 1$, $E = E\ /2$, E , $2E$, $E \ = \frac{1}{4}X_1^2$.



initial data $(A_0, B_0) \in D_0(E, X)$. If $\sup_{t>0} |A(t)|^2 = \infty$, then there exist A_∞, $B_\infty \in M^2$ such that for $t > 0$, $j = 0, 1, 2$,

$$\left| \frac{d^j}{dt^j} \left[A(t) - (B_\infty t + A_\infty) \right] \right| \lesssim (1 + t)^{-1-j}. \tag{7.11}$$

If $\bar{A}_\infty$, $\bar{B}_\infty \in M^2$ is any pair such that

$$\lim_{t \to \infty} |A(t) - (\bar{B}_\infty t + \bar{A}_\infty)| = 0, \tag{7.12}$$

then $(\bar{A}_\infty, \bar{B}_\infty) = (A_\infty, B_\infty)$.

The vectors A_∞, B_∞ satisfy

$$E_0(A_\infty, B_\infty) = \tfrac{1}{2}|B_\infty|^2 = E > 0, \quad X(A_\infty, B_\infty) = X,$$

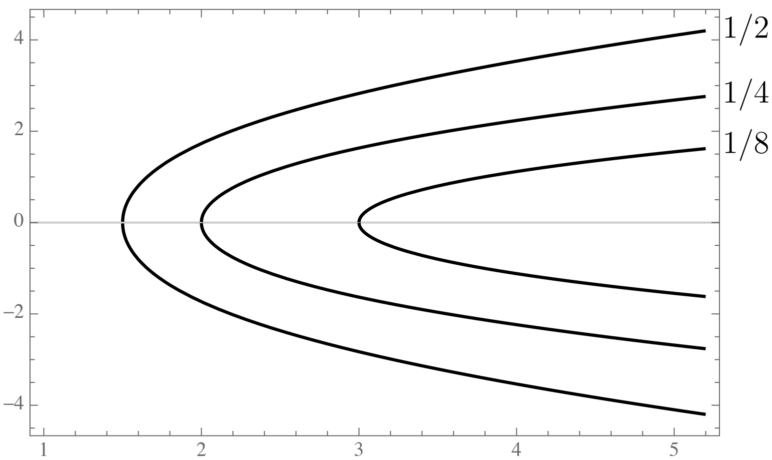
and

$$\langle B_\infty, \mathrm{cof} A_\infty \rangle = \det B_\infty = 0, \quad \det A_\infty = \frac{X_1 X_2}{2E}.$$

If $\det B_0 = 0$, then $(A_\infty, B_\infty) = (A_0, B_0) \in D(E, X)$ and

$$A(t) = B_0 t + A_0.$$

Figure 7.5: Level curves $C_0(E,X)$ in the case $X_1 = -X_2 = 1$, $E = 1/8,\ 1/4,\ 1/2$.



Proof. Suppose that A is a solution in $\mathcal{D}_-(E,X)$ with $\sup_{t>0}|A(t)|^2 = \infty$. Set $(x(t),y(t)) = \mathcal{P}(A(t),A'(t))$. Then $\sup_{t>0} x(t) = \infty$, and so there exists $t_0 > 0$ such that $x'(t_0) > 0$. Since $x(t)$ is not identically equal to 1, Lemma 7.14 implies $x''(t) > 0$, for all $t \in \mathbb{R}$. It follows that $x'(t) \geq x'(t_0) > 0$, for $t \geq t_0$, and consequently, $x(t) \to \infty$, as $t \to \infty$. From (7.10), there exists $t_1 > 0$ such that

$$x''(t) \geq E > 0, \quad t \geq t_1.$$

After integration, this leads to the lower bound

$$x(t) \geq \tfrac{1}{2}E(t-t_1)^2 + y(0)(t-t_1) + x(0), \quad t \geq t_1,$$

and thus,

$$|A(t)|^2 \gtrsim (1+t)^2, \quad t \geq 0.$$

Since $A(t)$ solves (4.7), we obtain from Lemma 5.5 that

$$|A''(t)| \lesssim |A(t)|^{-3} \lesssim (1+t)^{-3}, \quad t \geq 0.$$

Thus, by Lemma 6 of [37], we can write

$$A(t) = B_\infty t + A_\infty + A_1(t),$$

with

$$\begin{aligned}
B_\infty &= B_0 + \int_0^\infty A''(s)ds, \\
A_\infty &= A_0 - \int_0^\infty \int_s^\infty A''(\sigma)d\sigma ds, \\
A_1(t) &= \int_t^\infty \int_s^\infty A''(\sigma)d\sigma ds.
\end{aligned}$$

Note that our estimate for $|A''(t)|$ implies that

$$\left|\frac{d^j}{dt^j} A_1(t)\right| \lesssim (1+t)^{-1-j}, \quad t \geq 0, \quad j = 0, 1, 2,$$

thereby proving (7.11).

If (7.12) holds, then using (7.11), we find that

$$\lim_{t \to \infty} |(B_\infty - \bar{B}_\infty)t + (A_\infty - \bar{A}_\infty)| = 0,$$

and uniqueness of the states (A_∞, B_∞) follows from this.

Applying (7.11), we find

$$E = \tfrac{1}{2}|A'(t)|^2 = \tfrac{1}{2}|B_\infty + A_1'(t)|^2 = \tfrac{1}{2}|B_\infty|^2 + O(t^{-1}), \quad t > 0.$$

Sending $t \to \infty$ shows that $E = \frac{1}{2}|B_\infty|^2$.

For the other invariants, we have

$$\begin{aligned}
X = X(A(t), A'(t)) &= X(B_\infty t + A_\infty + A_1(t), B_\infty + A_1'(t)) \\
&= tX(B_\infty, B_\infty) + X(A_\infty, B_\infty) + O(t^{-1}).
\end{aligned}$$

By equation (3.1), we see that $X(B_\infty, B_\infty) = 0$, and so letting $t \to \infty$ we obtain $X = X(A_\infty, B_\infty)$.

Since $A(t) \in \mathrm{SL}(2, \mathbb{R})$, we get from Lemma 3.3

$$
\begin{aligned}
2 &= 2\det A(t) \\
&= \langle A(t), \operatorname{cof} A(t)\rangle \\
&= t^2 \langle B_\infty, \operatorname{cof} B_\infty\rangle + 2t\langle A_\infty, \operatorname{cof} B_\infty\rangle \\
&\quad + \langle A_\infty, \operatorname{cof} A_\infty\rangle + 2t\langle B_\infty, \operatorname{cof} A_1(t)\rangle + O(t^{-1}) \\
&= 2t^2 \det B_\infty + 2t\langle A_\infty, \operatorname{cof} B_\infty\rangle \\
&\quad + 2\det A_\infty + 2t\langle B_\infty, \operatorname{cof} A_1(t)\rangle + O(t^{-1}).
\end{aligned}
$$

This implies that

$$
2\det B_\infty = \langle B_\infty, \operatorname{cof} B_\infty\rangle = 0, \quad \langle A_\infty, \operatorname{cof} B_\infty\rangle = 0,
$$

and

$$
2\det A_\infty + \lim_{t\to\infty} 2t\langle B_\infty, \operatorname{cof} A_1(t)\rangle = 2.
$$

Using the formula for $A_1(t)$, l'Hôpital's rule, (4.7), Lemma 5.5, and (7.11), we find that

$$
\lim_{t\to\infty} tA_1(t) = \lim_{t\to\infty} \tfrac{1}{2}t^3 A_1''(t) = \lim_{t\to\infty} \tfrac{1}{2}t^3 A''(t) = (2E - X_1X_2)\frac{\operatorname{cof} B_\infty}{|B_\infty|^4}.
$$

From this follows

$$
\det A_\infty = 1 - \frac{2E - X_1X_2}{|B_\infty|^2} = \frac{X_1X_2}{2E}.
$$

If $\det B_0 = 0$, then by Lemma 5.5, we get $2E - X_1X_2 = 0$. Lemma 5.5 then says that $\Lambda_0(A(t), A'(t)) = 0$, for all $t \in \mathbb{R}$. So the equations of motion simplify dramatically to $A''(t) = 0$, and from this we see that $A(t)$ must be linear in t. □

Remark 7.9. If $(A_0, B_0) \in \mathcal{D}$ and $\det B_0 = 0$, then $A(t) = B_0 t + A_0$ is a geodesic line in $\mathrm{SL}(2,\mathbb{R})$, by Lemma 6.13.

Remark 7.10. In Theorem 7.12, if $\det B_0 \neq 0$, then $A_\infty \notin \mathrm{SL}(2,\mathbb{R})$, and hence $(A_\infty, B_\infty) \notin \mathcal{D}$.

Remark 7.11. An analogous result holds when $\sup_{t<0} |A(t)|^2 = \infty$.

Theorem 7.17

Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a ***non-rigid*** solution of the IVP (4.6), (4.7), (4.8) in $\mathcal{D}_0(E, X)$.

If $\sup_{t>0} |A(t)|^2 < \infty$, then $X_1 = X_2 \neq 0$, $E = \frac{1}{4}X_1^2$, the orbit $(A(t), A'(t))$ belongs to the set

$$W^s(X_1) = \{(A, B) \in \mathcal{D}_0(E, X) : \langle A, B\rangle < 0\},$$

and there exists a phase θ_+ such that for every $0 < \lambda < \frac{1}{2}|X_1|$,

$$\left| \frac{d^j}{dt^j} \left[A(t) - U\left(\tfrac{1}{2}X_1 t + \theta_+\right)\right] \right| \lesssim \exp(-\lambda t),$$

for all $t \geq 0$, $j = 0, 1$.

Proof. Since the solution A is non-rigid and semi-bounded, Lemma 7.15 implies that $X_1 = X_2 \neq 0$ and $E = \frac{1}{4}X_1^2$. By Lemma 7.14, we have

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \nearrow 0, \quad \text{as} \quad t \to \infty.$$

Thus, $y(t) = \langle A(t), A'(t)\rangle < 0$, $t \in \mathbb{R}$, and so the solution orbit $(A(t), A'(t))$ lies in $W^s(X_1)$. Since the phase plane orbit $(x(t), y(t))$ lies in $C_0(E, X)$, we have $\Phi_0(x(t), y(t); E, X) = 0$, $t \in \mathbb{R}$. Using Definition 6.1, (6.9), and the condition on (E, X), we find that

$$x'(t) = y(t) = -|X_1|\, x(t)^{-1/2}(x(t) - 1), \quad t \in \mathbb{R}.$$

Given $0 < \lambda < |X_1|/2$, choose t_0 large enough so that

$$|X_1|\, x(t)^{-1/2} \geq 2\lambda, \quad t \geq t_0.$$

Then

$$x'(t) \leq -2\lambda(x(t) - 1), \quad t \geq t_0.$$

From this we obtain the estimates

$$0 < x(t) \le 1 - (x(t_0) - 1)\exp[-2\lambda(t - t_0)],$$

and

$$|x'(t)| \lesssim \exp[-2\lambda(t - t_0)],$$

for $t \le t_0$. The rest of the proof proceeds exactly as in Theorem 7.10.

□

Remark 7.12. There is an obvious companion result in the case when $\sup_{t<0} |A(t)|^2 < \infty$ for the set

$$W^u(X_1) = \{(A, B) \in \mathcal{D}(E, X) : \langle A, B\rangle > 0\},$$

with $X_1 = X_2$ and $E = \frac{1}{4}X_1^2$.

Corollary 7.18
For $0 \neq X_1 \in \mathbb{R}$ and $E = \frac{1}{4}X_1^2$, the sets $W^s(X_1)$ and $W^u(X_1)$ are stable and unstable manifolds for $R(X_1)$.

7.4 The Picture in $T_A\mathrm{SL}(2, \mathbb{R})$

Several special situations have emerged: the existence of stable and unstable manifolds for $\mathrm{SO}(2, \mathbb{R})$, the existence of solutions with vanishing pressure, and the existence of rigid solutions. Here we shall attempt to visualize the corresponding tangent directions in $T_A\mathrm{SL}(2, \mathbb{R})$ for a fixed point $A \in \mathrm{SL}(2, \mathbb{R})$.

Let us first assume that $A \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R})$. By Lemma 3.12, we can represent an element $B \in T_A\mathrm{SL}(2, \mathbb{R})$ using the normalized frame $\{\hat{\tau}_i(A)\}$ from Definition 3.8 as

$$B = \sum_i c_i\, \hat{\tau}_i(A),$$

in which

$$c_1 = \frac{X_1 + X_2}{\sqrt{g_{11}}}, \quad c_2 = \frac{X_1 - X_2}{\sqrt{g_{22}}}, \quad \text{with} \quad X_i = X_i(A,B), \quad g_{ii} = g_{ii}(A).$$

The metric g was given in Lemma 3.9Thus, we have

$$X_1 = \tfrac{1}{2}(\sqrt{g_{11}}\, c_1 + \sqrt{g_{22}}\, c_2) \quad \text{and} \quad X_2 = \tfrac{1}{2}(\sqrt{g_{11}}\, c_1 - \sqrt{g_{22}}\, c_2). \tag{7.13}$$

We also have

$$E = E(A,B) = \tfrac{1}{2}|B|^2 + \tfrac{\;}{2}|A|^2 = \tfrac{1}{2}\sum_i c_i^2 + \tfrac{\;}{2}|A|^2. \tag{7.14}$$

By Lemma 5.5, solutions with vanishing pressure are characterized by the condition $E = \frac{1}{2}X_1X_2$. Using expressions (7.13), we find that

$$E = \tfrac{1}{8}(g_{11}\, c_1^2 - g_{22}\, c_2^2).$$

From (7.14), this leads to the relation

$$c_1^2 - c_2^2 - \frac{2}{|A|^2}c_3^2 = 2\kappa.$$

Thus, the set

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : E(A,B) = \tfrac{1}{2}X_1(A,B)X_2(A,B)\}$$

is a two-sheeted hyperboloid when $\kappa > 0$, and a cone when $\kappa = 0$. The region of positive pressure is connected, and the region of negative pressure has two connected components.

The critical point $(1,0)$ for (6.9) corresponds to the family of rotating solutions. The homoclinic orbits produce a stable/unstable manifold characterized by the conditions $X_1 = X_2$ and $E = \kappa + \frac{1}{4}X_1^2$. Here, we have $c_2 = 0$, and so

$$E = \kappa + \tfrac{1}{16}g_{11}\, c_1^2.$$

Thus, in local coordinates, the set

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : E(A,B) = \kappa + \tfrac{1}{4}X_1(A,B)^2,\ X_1(A,B) = X_2(A,B)\}$$

is given by

$$\tfrac{1}{8}g_{22}\, c_1^2 \quad c_3^2 = \tfrac{}{2} g_{22}.$$

This describes a hyperbola in the $\hat{\tau}_1$, $\hat{\tau}_3$ plane with two branches, each contained within one of the components of negative pressure. The limiting solution is a rotation of the form $U(\frac{1}{2}X_1 t + \theta)$, by Lemma 6.11. Since X_1 and c_1 have the same sign, we see that the branch with $c_1 > 0$ corresponds to counterclockwise rotation in the limit. When $\kappa = 0$, the hyperbola degenerates to a pair of lines through the origin. Parameter values $c_3 < 0$ along these lines correspond to stable directions while values $c_3 > 0$ correspond to unstable directions.

By Lemmas 6.2 and 6.10, the set

$$\{B \in T_A \mathrm{SL}(2, \mathbf{R}) : (A, B) \text{ is initial data for a rigid solution of (4.7)}\}$$

is equal to

$$\{B \in T_A \mathrm{SL}(2, \mathbf{R}) : (x, y) = \boldsymbol{P}(A, B) \text{ satisfies } y = 0,\ P\ (x; E, X) = 0,\ P'(x; E, X) = 0\}.$$

The condition $P\ (x; E, X) = 0$ is the same as (7.14). Now $y = c_3 = 0$, so we have

$$E = \tfrac{1}{2}(c_1^2 + c_2^2) + \kappa x.$$

The condition $P'(x; E, X) = 0$ is equivalent to

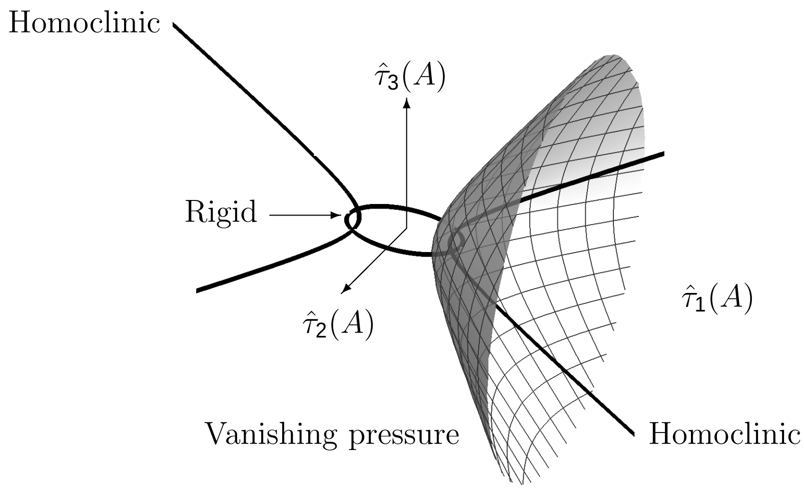
$$8Ex = 4\kappa(3x^2 \quad 1) + \tfrac{1}{2}g_{11}c_1^2 + \tfrac{1}{2}g_{22}c_2^2.$$

We find that the local coordinates $(c_1, c_2, 0)$ of B must lie on the ellipse

$$\frac{c_1^2}{g_{11}} + \frac{c_2^2}{g_{22}} = \kappa/2.$$

This intersects the hyperboloid of data with vanishing pressure at four points. The ellipse shrinks to the origin as $\kappa \to 0$. See Figures 7.6 and 7.7.

Figure 7.6: Distinguished directions in $T_A\mathrm{SL}(2,\mathbb{R})$ for a fixed $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $\kappa = 1/2$. The branch of pressureless directions in the half space $c_1 < 0$ is not shown.



When $A \in \mathrm{SO}(2,\mathbb{R})$, we have

$$B = \sum_{i=1}^{3} c_i \hat\tau_i(A) \quad \text{with} \quad c_i = \langle B, \hat\tau_i(A) \rangle .$$

This yields

$$c_1 = \frac{1}{\sqrt{2}} X_1, \quad c_2 = -\frac{1}{\sqrt{2}} \langle B, U(2s_2)K \rangle , \quad c_3 = \frac{1}{\sqrt{2}} \langle B, U(2s_2)M \rangle ,$$

for an arbitrary $s_2 \in \mathbb{R}$. We have

$$E = \kappa + \tfrac{1}{2} \sum_{i=1}^{3} c_i^2 = \kappa + \tfrac{1}{4} X_1^2 + \tfrac{1}{2} \sum_{i=2}^{3} c_i^2 .$$

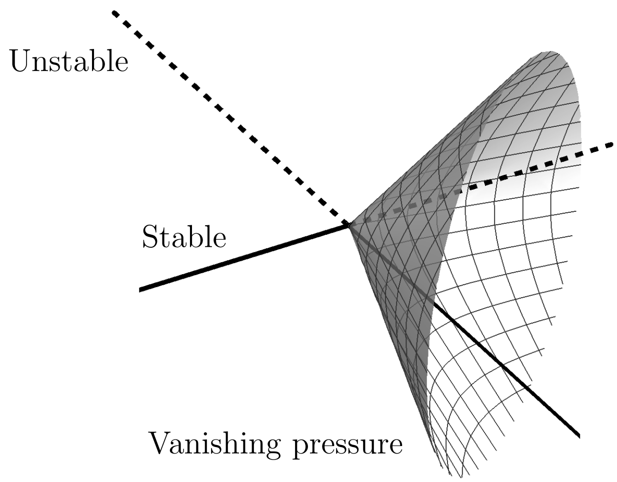
The pressureless solutions are described by the equation

$$c_1^2 - c_2^2 - c_3^2 = 2\kappa,$$

which is consistent with taking the limit as $A \to \mathrm{SO}(2,\mathbb{R})$.

The rigid solutions are given by $E = \kappa + \frac{1}{4}X_1^2$, or equivalently $c_2 = c_3 = 0$. The segment $|c_1| \leq 4\kappa$ along the $\hat{\tau}_1(A)$ axis arises as the limit $A \to \mathrm{SO}(2,\mathbb{R})$. The portion $|c_1| > 4\kappa$ corresponds to the limit set of the homoclinic orbits.

Figure 7.7: Distinguished directions in $T_A\mathrm{SL}(2,\mathbb{R})$ for a fixed $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $\kappa = 0$. The cone of pressureless directions in the half space $c_1 < 0$ is not shown.



Chapter 8

Full Hamiltonian

This chapter is a slight aside to explore the Hamiltonian structure of (4.4). By full Hamiltonian we mean the actual Hamiltonian of the system which is gotten by the energy. Even though it is "full" it will still reduce to a two dimensional phase diagram. The analysis for trajectories with invariants $X_1 = X_2$ is much richer and requires different coordinates than we have been using thus far and so we separate the two cases.

8.1 When $X_1 \neq X_2$

By Lemma 5.1 having $X_1 \neq X_2$ is sufficient to keep us away from $\mathrm{SO}(2, \mathbb{R})$. As a reminder the coordinates we will use are

$$A(s) = U(s_1 + s_2)H(s_3)U(s_1 - s_2)$$

where $U(\sigma)$ and $H(\sigma)$ are rotations and hyperbolic rotations respectively. The phase space for our dynamics will be (s, s') where the s' coordinates are the compononets of a tangent vectors A' to A written in the $\{\tau_i\}$ frame defined in Lemma 3.9.

Rather than view the Lagrangian as a map on the full matrix space $\mathbb{M}^2$ constrained to $\mathrm{SL}(2, \mathbb{R})$, we express it in local coordinates (s, s'). Here our Lagrangian becomes

$$\begin{aligned} L[s, s'] &= \tfrac{1}{2}|gs'|^2 - \frac{\kappa}{2}|s|^2 \\ &= \tfrac{1}{2}|gs'|^2 - \frac{\kappa}{2}s_3 \end{aligned}$$

where g is the metric on the tangent spaced defined in (3.12) and $| \ |$ is the usual Euclidean three norm.

$$L[s, s'] = \langle s', gs' \rangle - V(s),$$

where $\langle a, b \rangle$ is now denoting the standard Euclidean inner product. The Legendre Transformation

$$(q, p) = (s, gs')$$

will allow us to transform out problem into the Hamiltonian formulation.Our potential term does not depend on s_1 or s_2 so the conjugate momentum associated to these coordinates will invariant under the Hamiltonian flow. Explicitly, the Hamiltonian is

$$H[q, p] = \tfrac{1}{2}\left\langle p, g^{-1}p \right\rangle + \kappa q_3,$$

These allow us to easily recover the angular coordinate description from the Hamiltonian flow

$$q' = \partial_p H \quad p' = -\partial_q H. \tag{8.1}$$

Lemma 8.1

If $X_1 \neq X_2$ then p_1 and p_2 are constant and the dynamics of q_1 and q_2 are completely determined by their initial value and the path $q_3(t)$. Explicitly $p_1 = X_1 + X_2$, $p_2 = X_1 - X_2$, and the angular coordinates are

$$q_i(t) = q_i(0) + (X_1 + (\quad 1)^{i+1} X_2) \int_0^t g_{ii}{}^{1}(q_3(s))ds.$$

Proof. For this proof let $i = 1$ or 2. The Hamiltonian flow (8.1) implies the p_i and q_i satisfy

$$\begin{cases} p_i^0 & = & 0 \\ q_i^0 & = & g_{ii}{}^{1}(q_3)p_i \end{cases} \tag{8.2}$$

Thus the p_i are constant. The Legendre Transform defining the p_i gives us that

$$p_i = g_{ii} s_i^0,$$

which by Lemma 3.9 implies

$$p_i = X_1 + (\quad 1)^{i+1} X_2.$$

Since the p_i are constant our Hamiltonian only depends on q_3 and p_3. Therefore, we can solve the reduced dynamics for this two dimensional Hamiltonian and get $q_3(t)$. Integrating (8.2) gives the desired result for the angular variables.

□

Using Lemma 8.1 we can write the Hamiltonian as

$$H[p,q] = H_{(X_1;X_2)}[p_3,q_3] = \frac{1}{8}\left(\frac{(X_1+X_2)^2}{q_3+1} + \frac{(X_1 \quad X_2)^2}{q_3 \quad 1}\right) + (q_3 \quad \frac{1}{q_3})p_3^2 + \kappa q_3.$$

We will keep the invariants fix and use the abusive of notation $H_{X_1;X_2}[q_3,p_3] = H[q_3,p_3]$. The gradients of which are

$$\partial_{q_3} H \;=\; \frac{1}{8}\left(\frac{(X_1+X_2)^2}{(q_3+1)^2} + \frac{(X_1 \quad X_2)^2}{(q_3 \quad 1)^2}\right) + (1+\frac{1}{q_3^2})p_3^2 + \kappa, \tag{8.3a}$$

$$\partial_{p_3} H \;=\; 2(q_3 \quad \frac{1}{q_3})p_3,$$

and

$$D^2 H = \begin{pmatrix} \frac{1}{4}\left(\frac{(X_1+X_2)^2}{(q_3+1)^3} + \frac{(X_1 \quad X_2)^2}{(q_3 \quad 1)^3}\right) \quad \frac{2p^2}{q_3^3} & 2(1+\frac{1}{q_3^2})p_3 \\ 2(1+\frac{1}{q_3^2})p & 2(q_3 \quad \frac{1}{q_3}) \end{pmatrix}. \tag{8.3b}$$

With these we can fully characterize the (q, p) dynamics.

Theorem 8.2

If $X_1 \neq X_2$ then H restricted to

$$R = \{(q_3, p_3) : q_3 > 1\}$$

has a unique critical point where it's global min, $e \;(X)$, is achieved. Further, all other non empty level sets are closed curves in R which surround this critical point and are the unique curves on which H obtains their energy.

Proof. Notice that (8.3a) implies that all critical points must lie on the line $p = 0$ so

$$\partial_{q_3} H[q_3, 0] = \frac{1}{8}\left(\frac{(X_1+X_2)^2}{(q_3+1)^2} + \frac{(X_1 \quad X_2)^2}{(q_3 \quad 1)^2}\right) + \kappa.$$

This is strictly increasing for $q_3 > 1$. Thus there is a unique $q \;\; > 1$ for which $(q \;\;, 0)$ is a critical point of H. At such a critical point (8.3b) becomes

$$D^2 H(q \;\;, 0) = \begin{pmatrix} \frac{1}{4}\left(\frac{(X_1+X_2)^2}{(q \;\; +1)^3} + \frac{(X_1 \quad X_2)^2}{(q \quad 1)^3}\right) & 0 \\ 0 & 2(q \quad \frac{1}{q}) \end{pmatrix}$$

Its Hessian is positive definite so the critical point is a local minimum, H strictly convex near the critical point, and the critical point is isolated. Let $e(X)$ be this minimal energy and $\bar{E} > e(X)$. Then $\bar{E}$ is a regular value of H so the level set must be a closed one dimensional manifold. By conservation of energy this manifold is compact and thus homeomorphic to a circle.

To see that the level set surrounds the critical point we let Γ be the interior of the region surrounded by $H^{-1}(\{\bar{E}\})$. Then H restricted to the compact set $\bar{\Gamma}$ is C^1 and so it obtains its max and min. Since H is constant on $\partial\Gamma$ and is not constant on all of $\bar{\Gamma}$ it must have a max or min on the interior which implies there should be a critical point which we showed must be $(q, 0)$.

To see that there is only one closed level set we argue similarly. Were there two level sets for $\bar{E}$ then both must surround the critical point and they cannot intersect, since they are manifolds, so one curve must surround the other. Then the annular region who's boundary is the union of both these curves is a compact set where H is constant on the boundary meaning it must contain a critical point which is a contradiction. □

From here we know that trajectories which have $X_1 \neq X_2$ have magnitudes which are constant, rigid motions, or their magnitudes are periodic with some period T.

8.2 When $X_1 = X_2$

To handle the case where $X_1 = X_2$, i.e. when there is the possibility of entering $\mathrm{SO}(2, \mathbb{R})$, we will need new coordinates. Once we have coordinates where our metric is non singular near $\mathrm{SO}(2, \mathbb{R})$ we will follow the argument used in the $X_1 \neq X_2$ case.

Before deriving our coordinates we will derive a coordinate free description of the problematic sets. Notice if a trajectory $A(t)$ has $X_1 = X_2$ then by Lemma 3.9 $\partial_2 A(s) = X_1 - X_2 = 0$ and so $A(s_1(t), s_2(t), s_3(t)) = A(s_1(t), s_2(0), s_3(t))$. Thus these trajectories are constrained to the image of planes of constant s_2 coordinate.

Lemma 8.3

Let $s_2 = \omega/2$, then the image of this plane under the coordinates is

$$\{A \in \mathrm{SL}(2,\mathbb{R}) \mid \langle A, \cos(\omega)K - \sin(\omega)M\rangle = 0\}.$$

Proof. Let $\omega = 2s_2$, by Lemma 3.14

$$A(s) = \mu_+(s_3)U(2s_1)I + \mu_-(s_3)U(\omega)M$$

so

$$\begin{aligned}(A - \mathrm{cof}(A)) &= 2\mu_- U(\omega)M,\\ (A - \mathrm{cof}(A))M &= 2\mu_- U(\omega),\end{aligned}$$

and

$$\begin{aligned}\langle I, (A - \mathrm{cof}(A))M\rangle &= 4\mu_- \cos(\omega),\\ \langle Z, (A - \mathrm{cof}(A))M\rangle &= 4\mu_- \sin(\omega).\end{aligned}$$

By Lemma 3.3

$$\begin{aligned}\langle I, (A - \mathrm{cof}(A))M\rangle &= \langle M, A - \mathrm{cof}(A)\rangle\\ &= \langle M, A\rangle - \langle M, \mathrm{cof}(A)\rangle\\ &= \langle M, A\rangle - \langle \mathrm{cof}(M), A\rangle\\ &= 2\langle M, A\rangle.\end{aligned}$$

Since $ZM = K$ we can argue as above to get

$$\langle Z, A - \operatorname{cof}(A)\rangle = 2\langle K, A\rangle.$$

If $\omega = k\pi/2$ for some $k \in \mathbb{Z}$ then we get the desired result. Otherwise dividing these two inner products gives the desired result. □

Suppose we have a trajectory, $A(t)$, on which $s_2(t) \equiv \omega/2$. Then for a fixed θ the trajectory $U(\theta)A(t)U^{\top}(\theta)$ is also a solution with the same invariants E, X_1, and X_2. Thus a description of the dynamics on the set where $s_2 \equiv 0$ will describe all dynamics for which $X_1 = X_2$. We let

$$\mathcal{C} = \{A \in \mathrm{SL}(2, \mathbb{R}) \mid \langle A, K\rangle = 0\}$$

In the usual basis from Definition (3.2) $\mathcal{C}$ can be written as $[A] = (a_0, a_1, 0, a_2)$. By Lemma 3.1 the condition that this hyper plane intersect $\mathrm{SL}(2, \mathbb{R})$ becomes

$$a_0^2 + a_1^2 - a_2^2 = 1,$$

which we see is a hyperbola in the (a_0, a_2) and (a_1, a_2) plane, and a circle in the remaining (a_0, a_1) plane. This leads to the coordinate choice

$$A(q_1, q_2) = \cosh(q_1)U(q_2) + \sinh(q_1)M. \tag{8.4}$$

It is quick to check that this is indeed a parameterization of $\mathcal{C}$. The following Lemma collects the useful facts about these coordinates.

Lemma 8.4

Let $A = A(q_1, q_2) \in \mathcal{C}$ be written in the (8.4) coordinates, then its norm can be expressed as

$$\tfrac{1}{2}|A|^2 = \sinh(q_1)^2 + \cosh(q_1)^2. \tag{8.5}$$

The tangent vectors to C at $A(q_1, q_2)$ are

$$\begin{aligned} \xi_1 &= \sinh(q_1)U(q_2) + \cosh(q_1)M, \\ \xi_2 &= \cosh(q_1)ZU(q_2), \end{aligned}$$

and the metric on C induced by our usual metric on $\mathrm{SL}(2, \mathsf{R})$ in these coordinates is

$$g(q_1, q_2) = g(q_1) = \mathrm{diag}(2(2\cosh^2(q_1) - 1), 2\cosh^2(q_1)).$$

Finally, if $A' = q_1'\xi_1 + q_2'\xi_2$ then

$$g_{22}(q_1)q_2' = X_1. \tag{8.7}$$

Proof. The norm property (8.5) follows by the fact that span$\{I, Z\}$ is orthogonal to span$\{K, M\}$. The tangent vectors follow by differentiating (8.4) and the characterization of $\mathrm{SO}(2, \mathsf{R})$ in (3.8). Out metric on $\mathrm{SL}(2, \mathsf{R})$ we are using is induced by M_2, so C's induced metric in $\mathrm{SL}(2, \mathsf{R})$ is the metric induced by M_2 as well. By Lemma 3.1 ξ_1 is orthogonal to ξ_2, and their norms are easy to compute. For (8.7) we first note that

$$\begin{aligned} \langle ZA, \xi_1 \rangle &= \langle \cosh(q_1)ZU(q_2) + \sinh(q_1)ZM, \sinh(q_1)U(q_2) + \cosh(q_1)M \rangle \\ &= \langle \cosh(q_1)ZU(q_2), \sinh(q_1)U(q_2) \rangle + \langle \cosh(q_1)ZU(q_2), \cosh(q_1)M \rangle \\ &\quad + \langle \sinh(q_1)ZM, \sinh(q_1)U(q_2) + \cosh(q_1)M \rangle \\ &= 0, \end{aligned}$$

and

$$
\begin{aligned}
\langle ZA, \xi_2 \rangle &= \langle \cosh(q_1) ZU(q_2) + \sinh(q_1) ZM, \cosh(q_1) ZU(q_2) \rangle \\
&= 2\cosh^2(q_1).
\end{aligned}
$$

Since $X_1 = \langle A', ZA \rangle$ if we expand our tangent vector in the $\{\xi_1, \xi_2\}$ basis we have

$$
\begin{aligned}
X_1 &= \langle A', ZA \rangle \\
&= q_1' \langle \xi_1, ZA \rangle + q_2' \langle \xi_2, ZA \rangle \\
&= q_1' 2\cosh^2(q_1) \\
&= q_1' g_{22}(q_1).
\end{aligned}
$$

As required. □

Notice the metric can, again, be written as a function of the magnitude:

$$
g(q_3) = g(\cosh(q_3)) = \operatorname{diag}(|A|^2, \tfrac{1}{2}|A|^2 + 1).
$$

Importantly, the metric is non singular everywhere on $\mathbb{C}$ and in particular on $\mathrm{SO}(2, \mathbb{R})$. We can analyze the dynamics using the Hamiltonian formulation in these coordinates. The Lagrangian in these coordinates is

$$
L[q, q'] = \langle q', g q' \rangle - \frac{\kappa}{2} g_{11}(q_1).
$$

Applying the Legendre Transform $(q, q') \to (q, gq')$ gives us the Hamiltonian

$$
H[q, p] = \tfrac{1}{2} \left\langle p, g^{-1} p \right\rangle + \frac{\kappa}{2} g_{11}(q_1).
$$

A quick computation gives the gradients of H

$$
\partial_q H = \left(-\tfrac{1}{2} \left\langle p, g' g^{-2} p \right\rangle + \frac{\kappa}{2} g_{11}'(q_1), 0 \right) \tag{8.8a}
$$

$$
\partial_p H = g^{-1} p. \tag{8.8b}
$$

From (8.8a) we see that the Hamiltonian Flow for (p_2, q_2) is

$$\begin{aligned} p_2^0 &= 0 \\ g_2^0 &= g_{22}^{\ \ 1}(q_1)p_2. \end{aligned}$$

Thus p_2 is constant and by (8.8b) and Lemma 8.4

$$p_2 = q_2^0 g_{22}(q_1) = X_1.$$

Therefore q_2's dynamics are completely determined by its initial value, the invariant X_1 and the path $q_1(t)$ as was the case in the previous section. This again allows us to reduce the Hamiltonian dynamics to the (q_1, p_1) plane. With abuse of notation we will write

$$\begin{aligned} H_{X_1}[q_1, q_2, p_1, p_2] = H[q_1, p_1] &= \frac{p_2^2}{2g_{22}(q_1)} + \frac{p_1^2}{2g_{11}(q_1)} + \frac{\kappa}{2} g_{11}(q_3) \\ &= \frac{X_1^2}{2g_{22}(q_1)} + \frac{2p_1^2}{2g_{11}(q_1)} + \frac{\kappa}{2} g_{11}(q_3), \end{aligned}$$

making our Hamiltonian

$$H[q_1, p_1] = \frac{X_1^2}{4\cosh^2(q_1)} + \frac{p_1^2}{4(2\cosh^2(q_1) \quad 1)} + \kappa(2\cosh^2(q_1) \quad 1).$$

The Hamiltonian flow is then

$$\begin{aligned} \frac{dq_1}{dt} &= \partial_{p_1} H[q_1, p_1] = \frac{p_1}{2(2\cosh^2(q_1) \quad 1)} \\ \frac{dp_1}{dt} &= \partial_{q_1} H[q_1, p_1] = \sinh(q_1)\left[\frac{X_1}{2\cosh^3(q_1)} \quad \frac{p_1^2 \cosh(q_1)}{(2\cosh^2(q_1) + 1)^2} + 4\kappa\cosh(q_1)\right] \end{aligned}$$

Which gives us the following lemma.

Lemma 8.5

$H[q_1, p_1]$ has the following critical point description.

1. If $X_1^2 > 8\kappa$ then H has three critical points $(0,0)$ and $(0, \quad q\ )$ where $q\ \ > 0$ is a solution to

$$\cosh^4(q\ ) = \frac{X_1^2}{8\kappa}.$$

   In this case the origin is a hyperbolic critical point and the other two are centers.

2. If $X_1^2 \quad 8\kappa$ then H has a unique critical point at the origin. This critical point is an absolute minimum of the Hamiltonian.

Proof. From (8.9) we see that all critical points occur on the $p_1 = 0$ line and $(0,0)$ is always a critical point. The remaining are found by solving

$$\begin{aligned} \frac{X_1^2}{2\cosh^3(q_1)} + 4\kappa\cosh(q_1) &= 0 \\ \frac{X_1^2}{2} + 4\kappa\cosh^4(q_1) &= 0 \\ \cosh^4(q_1) &= \frac{X_1^2}{8\kappa}. \end{aligned}$$

We see this has two solutions precisely when $X_1^2 > 8\kappa$ and no solutions if $X_1^2 \quad 8\kappa$. For the classification we first note that

$$\partial_{q_1 p_1} H[q_1, 0] = 0,$$

and

$$\partial_{p_1 p_1} H[q_1, p_1] = \frac{1}{2(2\cosh^2(q_1) \quad 1)} > 0.$$

The second derivative with respect to q_1

$$\partial_{q_1 q_1} H[q_1, p_1] = \cosh(q_1)\left[\frac{X_1^2}{2\cosh^3(q_1)} \quad \frac{p_1^2 \cosh(q_1)}{(2\cosh^2(q_1)+1)^2} + 4\kappa\cosh(q_1)\right]$$
$$+ \sinh(q_1)\partial_{q_1}\left[\frac{X_1^2}{2\cosh^3(q_1)} \quad \frac{p_1^2 \cosh(q_1)}{(2\cosh^2(q_1)+1)^2} + 4\kappa\cosh(q_1)\right]$$
$$= \cosh(q_1)\left[\frac{X_1^2}{2\cosh^3(q_1)} \quad \frac{p_1^2 \cosh(q_1)}{(2\cosh^2(q_1)+1)^2} + 4\kappa\cosh(q_1)\right]$$
$$+ \sinh(q_1)\left[\frac{3X_1^2 \sinh(q_1)}{2\cosh^4(q_1)} \quad p_1^2 \partial_{q_1}\left[\frac{\cosh(q_1)}{(2\cosh^2(q_1)+1)^2}\right] + 4\kappa\sinh(q_1)\right].$$

So at $(0,0)$

$$D^2 H = \begin{pmatrix} \frac{X_1^2}{2} + 4\kappa & 0 \\ 0 & \frac{1}{6} \end{pmatrix}.$$

When $X_1^2 \quad 8\kappa$ this is a, possibly degenerate, local min. When $X_1^2 > 8\kappa$ the origin is hyperbolic and we now have two other critical points at which

$$\partial_{q_1 q_1} H[\quad q \ , 0] = \sinh(q \ )^2 \left[\frac{3X_1^2}{2\cosh^4(q \ )} + 4\kappa\right] > 0$$

and so both of these are local mins of the Hamiltonian making them centers. □

When $X_1^2 \quad 8\kappa$ then the characterization of the trajectories goes just as in Theorem 8.2. There will be a minimum energy achieved at the origin and all other higher energies will have a unique level set which surrounds the origin. Interestingly, this implies there are two time during an orbit at which $q_1 = 0$ and so if the magnitude has period T then the (q_1, p_1) trajectory has period $2T$. In this way we recover the complicated period doubling of previous sections.

When $X_1^2 > 8\kappa$ we get much richer dynamics. Since the origin is now a hyperbolic critical point there homoclinic orbits.

Definition 8.1. The **critical energy** for the system (8.9) with $X_1 = X_2$ and $X_1^2 > 8\kappa$ is the energy of a trajectory through the origin, explicitly

$$E \ = \frac{X_1^2}{4} + \kappa.$$

Trajectories with $E > E$ and $E < E$ will be called **super critical** and **subcritical**, respectively.

The **shifted Hamiltonian** is

$$H\ [q_1, p_1] = H[q_1, p_1] \quad E\ .$$

The shifted Hamiltonian can be expressed as

$$H\ [q_1, p_1] = \frac{X_1^2}{4}\left[\frac{1}{\cosh^2(q_1)} \quad 1\right] + \frac{p_1^2}{4(2\cosh^2(q_1) \quad 1)} + 2\kappa(\cosh^2(q_1) \quad 1).$$

Theorem 8.6

Let $X_1 = X_2$ satisfy $X_1^2 > 8\kappa$. Then the origin is a hyperbolic critical point with two homoclinic orbits (to itself). These homoclinic loops each surround a global minimum of the Hamiltonian. The min/max values of q_1 on the homoclinic orbits are $\quad q_m$ where $q_m > 0$ and

$$\cosh^2(q_m) = \frac{X_1^2}{8\kappa}.$$

Proof. The level set through the origin is a zero set of the shifted Hamiltonian and so

$$p_1^2 = \frac{4}{2\kappa}\left(2 \quad \frac{1}{\cosh^2(q_1)}\right)\left(\frac{X_1^2}{8\kappa}\left[1 \quad \cosh^2(q_1)\right] + \cosh^4(q_1) \quad \cosh^2(q_1)\right).$$

The solution set is non empty when

$$\frac{X_1^2}{8\kappa}\left[1 \quad \cosh^2(q_1)\right] + \cosh^4(q_1) \quad \cosh^2(q_1) < 0$$

$$\left(\cosh^2(q_1) \quad 1\right)\left(\cosh^2(q_1) \quad \frac{X_1^2}{8\kappa}\right) < 0.$$

We see that the homoclinic orbits form a lemniscate like curve with intersection point at the origin. We also see tha the max q_1 on the curve satisfies

$$\cosh^2(q_1) = \frac{X_1^2}{8\kappa}.$$

Whereas the critical points which are energy minimums satisfy

$$\cosh^2(q_1) = \sqrt{\frac{X_1^2}{8\kappa}} < \frac{X_1^2}{8\kappa}$$

and so $q_m > q\ $. This shows us the homoclinic orbits surround the energy minimums.

□

The remaining description of the subcritical and super critical energies follow from arguments similar to those in the $X_1 \neq X_2$ case. We state the results here without proof. Recall that $e\ (X)$ is the minimum energy for a fixed (X_1, X_2). If a trajectory reaches this energy Lemma 8.5 implies that this will be a fixed point $(\ \ q\ ,0)$ and so the path is a rigid motion.

Theorem 8.7

Consider a solution of (8.9) with energy $\bar{E}$.

Subcritical: Let $E\ > \bar{E} > e\ (X)$. Then there are two orbits, reflections about $q_1 = 0$ of each other, which are closed loops surrounding a center and contained in a petal of the homoclinic lemniscate.

Super critical: Let $\bar{E} > E\ $. Then there is a unique orbit with energy $\bar{E}$ and it surrounds the homoclinic lemiscate. These orbits pass through the lines $q_1 = 0$ and

$p_1 = 0$ twice each.

Theorem 8.6 and 8.7 give a complete description of the MHD dynamics in the $X_1 = X_2$ case. If we fix this X_1 then κ, the strength of the magnetic field, acts as a bifurcation parameter. Though we don't have a good interpretation of X_1, in this case it is the vorticity of the fluid.

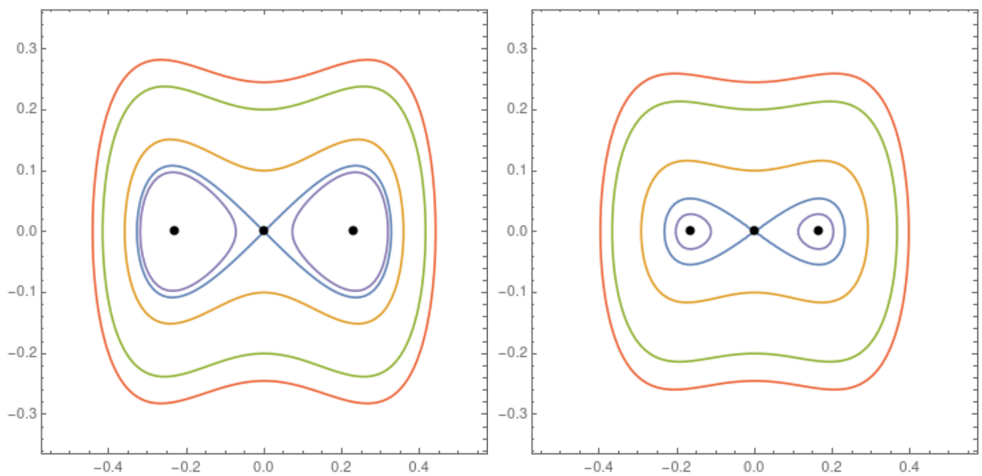


Figure 8.1: Contour plot of H for $X_1^2 > 8\kappa$. κ is increasing left to right.

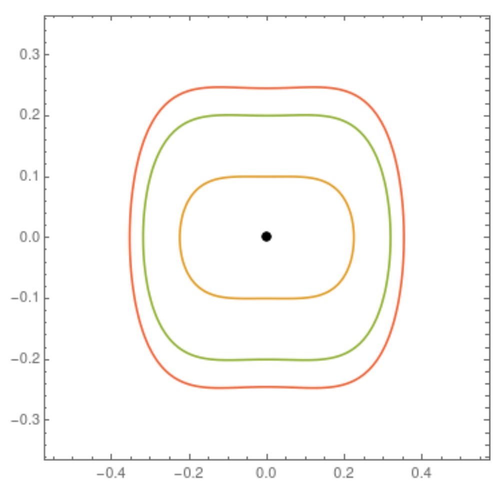


Figure 8.2: Sample level sets for $X_1^2 = 8\kappa$.

Chapter 9

First Order Quadratically Coupled Waves

9.1 Introduction

We will be analyzing the system

$$\begin{cases} \partial_t u + A\partial_x u = N_{B(t)@_x}(u,u) \\ u(x,0) = u_0(x) \end{cases} \tag{9.1}$$

where A is an n n symmetric-hyperbolic matrix. Given a linear operator, X, on a Banach Algebra the bilinear operator, N_X, defined by X will be denoted

$$N_X(u,v) = uXv. \tag{9.2}$$

When viewed as a function of u,v, and X, $N_X(u,v)$ is trilinear. For (9.1) the bilinear operator is defined component-wise by n bilinear operators

$$N_{B(t)@_x}(u,v)_i = u_k B_{ijk}(t)\partial_x v_j. \tag{9.3}$$

For fixed k the matrix $B_{ijk}(t) = B_k(t)$ is symmetric and satisfies the decay conditions

$$t^m \frac{d^m}{dt^m}[B_{ijk}] \qquad C \begin{cases} (1+t)^{\quad 1} & i = j = k \;\theta > 0 \\ (1+t)^{\quad 1} & \text{else} \end{cases},$$

for $m = 0, 1, 2$.

Since A is hyperbolic we can without loss of generality assume it is diagonal with entries λ_i for $1 \quad i \quad n$. We further assume that the wave speeds, λ_i, are distinct; however, without much trouble we can modify our nonlinearity to accommodate the distinct wave packets and get global existence of (9.1).

Chapter 10

Local Theory

In this section we will construct a solution to (9.1) which exists on a finite time interval.First we smooth (9.1), allowing us to treat it as an ODE in a suitable Banach Space. After collecting some commutation results we use a standard Picard argument to get a family of smooth approximations to (9.1), each of which has its own interval of existence. In order to bring all these approximations onto the same existence interval we will use energy estimates on the smoothed equation. These energy estimates provide us a template for showing our approximations are Cauchy and will be inherited by the actual solution to (9.1).

Before working in the modified Sobolev Spaces we collect some properties for our operators of (9.1) in H^r. We define

$$L = A\partial_x$$

to be our ***linear operator*** and

$$\mathsf{L} = \partial_t + L \tag{10.1}$$

to be our ***time dependent linear operator***. The linear operator is an order one operator and the nonlinear operator can be viewed as the map

$$N : H^r \times H^{r+2} \to H^r \quad \text{or} \quad N : H^{r+1} \times H^{r+1} \to H^r.$$

Lemma 10.1

Let $f_0 \in H^r$, $f, g \in H^{r+1}$ and $h \in H^{r+2}$, then

$$|Lf|_r \le c|f|_{r+1} \tag{10.2a}$$
$$|N_{B\partial_x}(f, g)|_r \le c|f|_{r+1}|g|_{r+1} \tag{10.2b}$$
$$|N_{B\partial_x}(f_0, h)|_r \le c|f_0|_r|h|_{r+2}. \tag{10.2c}$$

Proof. For (10.2a) notice:

$$\begin{aligned} |Lf|_r = |A\partial_x f|_r &= \sum_{k=0}^{r} |A\partial_x^k \partial_x f|_0 \\ &\le c|\partial_x f|_r \\ &\le c|f|_{r+1} \end{aligned}$$

For (10.2b) we start with the L_2 norm of the nonlinearity.

$$|N_{B\partial_x}(f, g)|_0 = \left[\int (fB(t)\partial_x g)^2 dx\right]^{1/2} \le c|f|_\infty|\partial_x g|_0 \le c|f|_1|g|_1.$$

For the general case.

$$|N_{B\partial_x}(f, g)|_r = \sum_{k=0}^{r} |\partial_x^k N_{B\partial_x}(f, g)|_0$$

and each term in the sum satisfies

$$
\begin{aligned}
|\partial_x^k N_{B\partial_x}(f,g)|_0 &\le \sum_{m=0}^{k} |N_{B\partial_x}(\partial_x^m f, \partial_x^{k-m} g)|_0 \\
&\le \sum_{m=0}^{k} c|\partial_x^m f|_1 |\partial_x^{k-m} g|_1 \\
&\le c|f|_{r+1}|g|_{r+1}
\end{aligned}
$$

Bound (10.2c) comes from the L_2 bound

$$
|N_{B\partial_x}(f_0,h)|_0 \le c|f|_0|\partial_x h|_{L^\infty} \le c|f|_0|h|_2,
$$

and arguing as in (10.2b). □

Next we derive two integration by parts results. We emphasize the extra regularity required to handle the nonlinearity. This is the primary reason we must smooth (9.1) before performing energy estimates.

Lemma 10.2

Let $f(\cdot,t) \in H^1$ and $g(\cdot,t) \in H^2$ on some time interval, then the time dependent linear operator satisfies

$$
(\mathbf{L} f, f)_2 = \partial_t \left[\tfrac{1}{2}|f(\cdot,t)|_0^2\right]
$$

and

$$
\left(\partial_x g, N_{B(t)\partial_x}(f,\partial_x g)\right)_2 = -\tfrac{1}{2}\left(\partial_x g, N_{B(t)\partial_x}(\partial_x f, g)\right)_2
$$

Proof. For the first

$$
\begin{aligned}
(Lf, f)_2 &= (\partial_t f, f)_2 + (A\partial_x f, f)_2 \\
&= \partial_t \left[\tfrac{1}{2}|f(\cdot, t)|_0^2\right] + (A\partial_x f, f)_2 \,.
\end{aligned}
$$

Since A is a symmetric and commutes with ∂_x, an anti-symmetric operator on H^1, $A\partial_x$ is anti-symmetric. So the last term vanishes giving the desired result.

For the second we expand out and get

$$
\left(\partial_x g, N_{B(t)\partial_x}(f, \partial_x g)\right)_2 = \int \langle \partial_x g, f_i B_i \partial_x \partial_x g \rangle dx.
$$

By symmetry of the B_i

$$
\begin{aligned}
\partial_x \langle \partial_x g, f_i B_i \partial_x g \rangle &= \langle \partial_x g, f_i B_i \partial_x \partial_x g \rangle + \langle \partial_x \partial_x g, f_i B_i \partial_x g \rangle + \langle \partial_x g, \partial_x f_i B_i \partial_x g \rangle \\
&= 2\langle \partial_x g, f_i B_i \partial_x \partial_x g \rangle + \langle \partial_x g, \partial_x f_i B_i \partial_x g \rangle .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int \langle \partial_x g, f_i B_i \partial_x \partial_x g \rangle dx &= \tfrac{1}{2} \int \partial_x \langle \partial_x g, f_i B_i \partial_x g \rangle dx \quad \tfrac{1}{2} \int \langle \partial_x g, \partial_x f_i B_i \partial_x g \rangle dx \\
&= 0 \quad \tfrac{1}{2} \left(\partial_x g, N_{B(t)\partial_x}(\partial_x f, g)\right)_2 .
\end{aligned}
$$

□

10.1 Spaces and Operators

To get global existence of solutions to (9.1) we will take advantage of the commuting vector fields. This was inspired by similar work done in [23, 36]. Specifically we work with the scaling operator, or S-derivative,

$$
S = t\partial_t + x\partial_x .
$$

To build the space we wish to construct our solution in we use several modified Sobolev spaces.

Definition 10.1. We define the **time dependent norm**

$$\|u\|_{r,k} = \|u\|_{r,k}(t) = \sum_{0 \le j \le k} |S^j u(\cdot, t)|_{r-k}, \tag{10.4}$$

$H_T^{r,2}$ by the norm

$$\|u\|_{H_T^{r,2}} = \sup_{t \in [0,T)} \|u\|_{r,2},$$

and the time independent Banach Space

$$|v|_{r,k} = \sum_{0 \le j \le k} |R^j v|_{r-j}.$$

Here R is the 1D radial derivative $R = x\partial_x$.

We now derive analogs of the bounds of Lemma 10.1 into $\tilde{H}^{r,2}$ and $H_T^{r,2}$.

Lemma 10.3

R satisfies the commutation relations

$$[R, \partial_x^n] = \quad n\partial_x^n \tag{10.5a}$$
$$[R^2, \partial_x^n] = \quad 2n\partial_x^n R + n^2\partial_x^n \tag{10.5b}$$

and S satisfies

$$[S, \partial_x^n] = \quad n\partial_x^n \tag{10.6a}$$
$$[S^2, \partial_x^n] = \quad 2n\partial_x^n S + n^2\partial_x^n. \tag{10.6b}$$

Proof. Notice $[R, \partial_x] = \quad \partial_x$ and a quick induction argument gives the general relation (10.5a). We can then use this to build up to (10.5b)

$$\begin{aligned} R^2\partial_x^n &= R(\partial_x^n R + [R, \partial_x^n]) \\ &= R\partial_x^n R \quad nR\partial_x^n \\ &= (\partial_x^n R + [R, \partial_x^n])R \quad n(\partial_x^n R + [R, \partial_x^n]) \\ &= \partial_x^n R^2 \quad 2n\partial_x^n R + n^2\partial_x^n. \end{aligned}$$

For (10.6a) note $[t\partial_t, \partial_x^n] = 0$. So

$$\begin{aligned} [S, \partial_x^n] &= [t\partial_t + R, \partial_x^n] \\ &= [R, \partial_x^n] \end{aligned}$$

(10.5a) gives the result. For the second order term

$$\begin{aligned} [S^2, \partial_x^n] &= [(t\partial_t)^2, \partial_x^n] + [2t\partial_t R, \partial_x^n] + [R^2 m\partial_x^n] \\ &= 2t\partial_t[R, \partial_x^n] + [R^2, \partial_x^n]. \end{aligned}$$

Applying the relations (10.5) gives

$$\begin{aligned} [S^2, \partial_x^n] &= \quad 2nt\partial_t\partial_x^n \quad 2n\partial_x^n R + n^2\partial_x^n \\ &= \quad 2nS\partial_x^n + 2nR\partial_x^n \quad 2n\partial_x^n R + n^2\partial_x^n \\ &= \quad 2n\partial_x^n S \quad 2n[S, \partial_x^n] + 2n[R, \partial_x^n] + n^2\partial_x^n \\ &= \quad 2n\partial_x^n S + n^2\partial_x^n \end{aligned}$$

as required. □

To bound the nonlinearity in $\tilde{H}^{r;2}$ and $H_T^{r;2}$ we will need modified Leibniz Rules for its interaction with R and S derivatives. These commutations produce lower order terms than standard Leibniz Rules. The S-derivatives get additional terms since they hit the coefficients $B(t)$. We let

$$N^{(k)}_{B\partial_x}(f,g) = N_{(t\partial_t)^k[B(t)]\partial_x}(f,g). \tag{10.7}$$

Lemma 10.4

If $f \in \tilde{H}^{2,2}$ and $g \in \tilde{H}^{3,2}$ then R derivatives of the nonlinearity are give by

$$RN_{B\partial_x}(f,g) = N_{B\partial_x}(Rf,g) + N_{B\partial_x}(f,Rg) - N_{B\partial_x}(f,g) \tag{10.8a}$$

$$\begin{aligned} R^2 N_{B\partial_x} = {} & N_{B\partial_x}(R^2 f,g) + 2N_{B\partial_x}(Rf,Rg) + N_{B\partial_x}(f,R^2 g) \\ & - 2N_{B\partial_x}(Rf,g) - 2N_{B\partial_x}(f,Rg) + N_{B\partial_x}(f,g). \end{aligned} \tag{10.8b}$$

If $f \in H^{2,2}_T$ and $g \in H^{3,2}_T$ for some $T > 0$ then the S-derivatives of the nonlinearity satisfy

$$\begin{aligned} SN_{B\partial_x}(f,g) = {} & N_{B\partial_x}(Sf,g) + N_{B\partial_x}(f,Sg) \\ & - N_{B\partial_x}(f,g) + N_{t\partial_t B\partial_x}(f,g) \end{aligned}$$

$$S^2 N_{B\partial_x} = \sum_{\mathsf{m}_2} N^{(m_1)}_{B\partial_x}(S^{m_2} f, S^{m_3} g)$$

where $\mathsf{m}_2 = \{m_i : m_1 + m_2 + m_3 \le 2\}$.

Proof. To begin we use the product rule and trilinearity of $N(\ ,\ )$.

$$\begin{aligned} RN_{B\partial_x}(f,g) = {} & N_{B\partial_x}(Rf,g) + N_{RB\partial_x}(f,g) \\ = {} & N_{B\partial_x}(Rf,g) + N_{B(\partial_x R - \partial_x)}(f,g) \\ = {} & N_{B\partial_x}(Rf,g) + N_{B\partial_x}(f,Rg) - N_{B\partial_x}(f,g). \end{aligned}$$

The second order relation (10.8b) comes from:

$$\begin{aligned} R^2 N_{B\partial_x}(f,g) = {} & N_{B\partial_x}(R^2 f, g) + 2N_{RB\partial_x}(Rf,g) + N_{R^2 B\partial_x}(f,g) \\ = {} & N_{B\partial_x}(R^2 f, g) + 2N_{B\partial_x}(Rf, Rg) - 2N_{B\partial_x}(Rf,g) + N_{R^2 B\partial_x}(f,g). \end{aligned}$$

For the last term we use (10.5b) and trilinearity of $N_{B\partial_x}$.

$$\begin{aligned} N_{R^2 B\partial_x}(f,g) = {} & N_{\partial_x (R - I)^2}(f,g) \\ = {} & N_{B\partial_x}(f, R^2 g) - 2N_{B\partial_x}(f, Rg) + N_{B\partial_x}(f,g), \end{aligned}$$

adding this to the first computation gives the desired result.

For the S-derivatives we proceed similarly. Let $B^{(n)} = (t\partial_t)^n B(t)$ and notice

$$\begin{aligned} SN_{B\partial_x}(f,g) = {} & N_{B\partial_x}(Sf,g) + N_{S[B\partial_x]}(f,g) \\ = {} & N_{B\partial_x}(Sf,g) + N^{(1)}_{B\partial_x}(f,g) + N_{BS\partial_x}(f,g) \\ = {} & N_{B\partial_x}(Sf,g) + N^{(1)}_{B\partial_x}(f,g) + N_{B(\partial_x S - \partial_x)}(f,g) \\ = {} & N_{B\partial_x}(Sf,g) + N_{B^{(1)}\partial_x}(f,g) + N_{B\partial_x}(f,Sg) - N_{B\partial_x}(f,g). \end{aligned}$$

The second order term follows similarly

$$\begin{aligned} S^2 N_{B\partial_x}(f,g) &= N_{B\partial_x}(S^2 f,g) + 2N_{S[B\partial_x]}(Sf,g) + N_{S^2[B\partial_x]}(f,g) \\ &= N_{B\partial_x}(S^2 f,g) + 2N^{(1)}_{B\partial_x}(f,g) + 2N_{B\partial_x}(f,Sg) - 2N_{B\partial_x}(f,g) \\ &\quad + N^{(2)}_{B\partial_x}(f,g) + 2N^{(1)}_{BS\partial_x}(f,g) + N_{BS^2\partial_x}(f,g) \end{aligned}$$

The commutation relations (10.6a) and (10.6b) and the trilinearity of $N(\cdot,\cdot)$ gives the desired result. □

Lemma 10.3 will allow us to use Lemma 10.1 to get bounds on our operators in our modified Sobolev Spaces.

Lemma 10.5

If $f_0 \in \tilde{H}^{r,2}$, $f, g \in \tilde{H}^{r+1,2}$, and $h \in \tilde{H}^{r+2,2}$, then

$$|Lf|_{r,2} \le c|f|_{r+1,2} \tag{10.10a}$$
$$|N_{B\partial_x}(f,g)|_{r,2} \le c|f|_{r+1}|g|_{r+1,2} \tag{10.10b}$$
$$|N_{B\partial_x}(f_0,h)|_{r,2} \le c|f_0|_{r,2}|h|_{r+2,2}. \tag{10.10c}$$

If $f_0 \in H_T^{r,2}$, $f,g \in H_T^{r+1,2}$, and $h \in H_T^{r+2,2}$ for some $T>0$, then

$$\|Lf\|_{r,2}(t) \le c\|f\|_{r+1,2}(t)$$
$$\|N_{B\partial_x}(f,g)\|_{r,2}(t) \le c\|f\|_{r+1,2}(t)\|g\|_{r+1,2}(t) \tag{10.11a}$$
$$\|N_{B\partial_x}(f_0,h)\|_{r,2}(t) \le c\|f_0\|_{r,2}(t)\|h\|_{r+2,2}(t). \tag{10.11b}$$

Proof. For the linear bound (10.10a) we have

$$|Lf|_{r,2} = |Lf|_r + |RLf|_{r-1} + |R^2Lf|_{r-2}.$$

The first term is handled by (10.2a) of Lemma 10.1. For the other two

$$|R^jLf|_{r-j} = \sum_{k=0}^{r-j} |\partial_x^k R^j A\partial_x f|_0$$

for $j=1,2$. For $j=1$ we use the commutation relations in Lemma 10.3,

$$\begin{aligned} |\partial_x^k RA\partial_x f|_0 = {} & |\partial_x^k A(\partial_x R + [R,\partial_x])f|_0 \\ \le {} & c|\partial_x^{k+1} f|_0 + c|\partial_x^{k+1} Rf|_0 \\ \le {} & c|f|_{k+1} + c|Rf|_{k+1} \\ \le {} & c|f|_r + c|f|_{r+1,1} \\ \le {} & c|f|_{r+1,1} \end{aligned}$$

The second order term is similar. For the nonlinearity we begin in L_2 and apply the commutation relation (10.8a) from Lemma 10.3.

$$
\begin{aligned}
jRN_{B@_x}(f,g)j_0 \quad & jN_{B@_x}(Rf,g)j_0 + jN_{RB@_x}(f,g)j_0 \\
& jN_{B@_x}(Rf,g)j_0 + jN_{B@_x}(f,Rg)j_0 + jN_{B@_x}(f,g)j_0 \\
& jRfj_1 jgj_1 + jfj_1 jRgj_1
\end{aligned}
$$

and then

$$
jRN_{B@_x}(f,g)j_{r \quad 1} = \sum_{k=0}^{r \quad 1} j\partial_x^k RN_{B@_x}(f,g)j_0.
$$

The commutation relations in Lemma 10.3 makes the individual terms

$$
\begin{aligned}
j\partial_x^k RN_{B@_x}(f,g)j_0 = \; & j(R\partial_x^k + k\partial_x^k)N_{B@_x}(f,g)j_0 \\
& jR\partial_x^k N_{B@_x}(f,g)j_0 + kj\partial_x^k N_{B@_x}(f,g)j_0 \\
& jR\partial_x^k N_{B@_x}(f,g)j_0 + kjfj_{k+1} jgj_{k+1} \\
& jR\partial_x^k N_{B@_x}(f,g)j_0 + cjfj_{r+1;1} jgj_{r+1;1}.
\end{aligned}
$$

The remaining term is handled as in Lemma 10.1.

$$
\begin{aligned}
jR\partial_x^k N_{B@_x}(f,g)j_0 \quad & \sum_{j=0}^{k} jRN_{B@_x}(\partial_x^j f, \partial_x^{k \quad j} g)j_0 \\
& c\sum_{j=0}^{k} jR\partial_x^j fj_1 j\partial_x^{k \quad j} gj_1 + j\partial_x^j fj_1 jR\partial_x^{k \quad j} gj_1 \\
& jfj_{k+2;1} jgj_{k+2;1} \\
& jfj_{r+1;1} jgj_{r+1;1}
\end{aligned}
$$

The second order terms are next. From Lemma 10.3

$$
\begin{aligned}
|R^2 N_{B\partial_x}|_0 = {} & |N_{B\partial_x}(R^2 f, g)|_0 + 2|N_{B\partial_x}(Rf, Rg)|_0 + |N_{B\partial_x}(f, R^2 g)|_0 \\
& + 2|N_{B\partial_x}(Rf, g)|_0 + 2|N_{B\partial_x}(f, Rg)|_0 + |N_{B\partial_x}(f, g)|_0 \\
\le {} & |R^2 f|_1 |g|_1 + 2||Rf|_1 |Rg|_1 + |f|_1 |R^2 g|_1 \\
& + 2|Rf|_1 |g|_1 + 2|f|_1 |Rg|_1 + |f|_1 |g|_1 \\
\le {} & |f|_{3,2} |g|_{3,2}. \qquad (10.12)
\end{aligned}
$$

Then as with the first order term we use Lemma 10.3.

$$
\begin{aligned}
|R^2 N_{B\partial_x}(f, g)|_{r-2} = {} & \sum_{k=0}^{r-2} |\partial_x^k R^2 N_{B\partial_x}(f, g)|_0 \\
\le {} & \sum_{k=0}^{r-2} |R^2 \partial_x^k N_{B\partial_x}(f, g)|_0 + |[R^2, \partial_x^k] N_{B\partial_x}(f, g)|_0 \\
\le {} & \sum_{k=0}^{r-2} |R^2 \partial_x^k N_{B\partial_x}(f, g)|_0 + n \sum_{j=0}^{k} |N_{B\partial_x}(\partial_x^j f, \partial_x^{k-j} g)|_0 \\
\le {} & c|f|_{k+1} |g|_{k+1} + \sum_{k=0}^{r-2} |R^2 \partial_x^k N_{B\partial_x}(f, g)|_0.
\end{aligned}
$$

To bound the individual terms in the remaining sum we first apply the product rule

$$
|R^2 \partial_x^k N_{B\partial_x}(f, g)|_0 \le \sum_{j=0}^{k} |R^2 N_{B\partial_x}(\partial_x^j f, \partial_x^{k-j} g)|_0,
$$

and then bound these individual terms. Each of these can be expanded according to (10.8b) in Lemma 10.3 as

$$
\begin{aligned}
|R^2 N_{B\partial_x}(\partial_x^j f, \partial_x^{k-j} g)|_0 \le {} & |N_{B\partial_x}(R^2 \partial_x^j f, \partial_x^{k-j} g)|_0 + 2|N_{B\partial_x}(R\partial_x^j f, R\partial_x^{k-j} g)|_0 + |N_{B\partial_x}(\partial_x^j f, R^2 \partial_x^{k-j} g)|_0 \\
& + 2|N_{B\partial_x}(R\partial_x^j f, \partial_x^{k-j} g)|_0 + 2|N_{B\partial_x}(\partial_x^j f, R\partial_x^{k-j} g)|_0 + |N_{B\partial_x}(\partial_x^j f, \partial_x^{k-j} g)|_0.
\end{aligned}
$$

Applying bound (10.12) gives us

$$|R^2 N_{B@x}(\partial_x^j f, \partial_x^{k-j} g)|_0 \leq |\partial_x^j f|_{3;2} |\partial_x^{k-j} g|_{3;2}$$
$$\leq |f|_{r+1;2} |g|_{r+1;2}.$$

Bound (10.10c) follows similarly once we have the bound

$$|R^2 N_{B@x}|_0 = |N_{B@x}(R^2 f_0, h)|_0 + 2|N_{B@x}(Rf_0, Rh)|_0 + |N_{B@x}(f_0, R^2 h)|_0 + 2|N_{B@x}(Rf_0, h)|_0 + 2|N_{B@x}(f_0, \ldots$$

$$\leq |R^2 f_0|_0 |h|_2 + 2||Rf_0|_0 |Rh|_2 + |f_0|_0 |R^2 h|_2$$
$$+ 2|Rf_0|_0 |g|_2 + 2|f_0|_0 |Rh|_2 + |f_0|_0 |h|_2$$

$$\leq |f_0|_{1;2} |h|_{4;2}.$$

and arguing as we did for (10.10b). The S-derivatives follow by the same argument.

□

We are now ready to smooth (9.1). We use convolution with a family of smooth bump functions. The specific family of functions we use are Good Kernels as defined in [39].

Definition 10.2. (Stein)

Let K be an indexed family of L_2 functions on $\mathbb{R}$. If the family satisfies

(i) $\int_{\mathbb{R}} K\, dx = 1$.

(ii) $|K| < A\delta^{-1}$.

(iii) $|K(x)| \leq A\delta/|x|^2$

then we say the convolution operator K is an approximation to the identity.

To construct these let ϕ be a non-negative smooth bump function, supported on $[-1, 1]$, even, and have unit mass. We will show

$$\phi(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right) \tag{10.13}$$

defines an approximation to the identity in our solution spaces.

Definition 10.3. Let ϕ be as in (10.13) above. Then J is the **smoothing operator**

$$J\ [u] = \phi \quad u.$$

and

$$J^{(l)}[u] = (R^l\phi\ ) \quad u$$

We define the smooth linear and nonlinear operators by

$$\begin{aligned} L\ (f) = &\ A\partial_x J\ f \\ &\text{and} \\ N\ (f, g) = &\ N_{B(t)@x}(J\ f, J\ g), \end{aligned}$$

respectively.

The smoothed version of (9.1) is

$$\begin{cases} \partial_t v = \quad J\ L\ v + J\ N\ (v, v) \\ v(0) = J\ u_0 \end{cases} . \tag{10.14}$$

10.2 Smooth Commutators

All of the results rely on the boundedness and convergence properties of $J^{(l)}$. We handle the R-derivatives first and then the S-derivatives.

Lemma 10.6

The $J^{(l)}$ satisfy

$$
\begin{aligned}
[R, J^{(l)}] &= J^{(l+1)} + J^{(l)} \\
[R^2, J^{(l)}] &= [R, J^{(l+1)}] + [R, J^{(l)}] + 2[R, J^{(l)}]R \qquad (10.15a) \\
[L\ , J^{(l)}] &= 0. \qquad (10.15b)
\end{aligned}
$$

Proof. We first note the product rule for convolutions. For simplicity assume that f and g are compactly supported, then

$$x(f \quad g) = (xf) \quad g + f \quad (xg).$$

We combine this with the fact that convolution commutes with differentiation we have

$$
\begin{aligned}
R(f \quad g) &= x\partial_x(f \quad g) \\
&= (xf^0) \quad g + f^0 \quad (xg) \\
&= (Rf) \quad g + f \quad (xg)^0 \\
&= (Rf) \quad g + f \quad g + f \quad (Rg).
\end{aligned}
$$

Using this we get

$$
\begin{aligned}
RJ^{(l)}f &= R(R^l\phi \quad f) \\
&= R^l\phi \quad (Rf) + (RR^l\phi\ ) \quad f + R^l\phi \quad f \\
&= J^{(l)}[Rf] + J^{(l+1)}f + J^{(l)}f.
\end{aligned}
$$

Therefore,

$$[R, J\ ] = J^{(l+1)} + J^{(l)}.$$

For (10.15b) we expand the commutator as

$$[L\ , J\ ] = [A\partial_x J\ , J\ ]$$

and note that convolutions commute with each other as well as differentiation so the commutator vanishes as required.

For (10.15a) we have

$$\begin{aligned}[R^2, J^{(l)}] &= R[R, J^{(l)}] + [R, J^{(l)}]R \\ &= R(J^{(l+1)} + J^{(l)}) + [R, J^{(l)}]R \\ &= [R, J^{(l+1)} + J^{(l)}] + (J^{(l+1)} + J^{(l)})R + [R, J^{(l)}]R \\ &= [R, J^{(l+1)}] + [R, J^{(l)}] + 2[R, J^{(l)}]R\end{aligned}$$

□

Definition 10.4. Given two linear operators X and Y on an algebra A and X's associated billinear form N_X, we define the **bilinear commutator of Y with N_X** by

$$fY, N_X g(u, v) = Y[N_X(u, v)] \quad N_X(Yu, v) \quad N_X(u, Yv) \quad N_{[X;Y]}(u, v)$$

Notice

$$f\partial_x, N_{B@_x} g = 0.$$

The smoothing operator will generate non-trivial bilinear commutators with R, or S in the time dependent setting. The following lemma tracks these.

Lemma 10.7

L satisfies

$$[R, L\ ] = \quad L\ + L[R, J\ ] \qquad (10.16a)$$

and

$$[R, J\ L\ ] = \quad J\ L\ + 2L\ [R, J\ ]. \qquad (10.16b)$$

N satisfies

$$\{R, N\}(u, v) = N_{B\partial_x}([R, J]u, J v) + N_{B\partial_x}(J u, [R, J]v).$$

Along with the modified Leibniz rule

$$\begin{aligned} R^2 N(u, v) = \; & R\{R, N\}(u, v) + \sum_{m_1, m_2} N(R^{m_1}u, R^{m_2}v) \qquad (10.17) \\ & + \sum_{k_1, k_2} \{R, N\}(R^{k_1}u, R^{k_2}v) \end{aligned}$$

where $0 \quad m_1 + m_2 \quad 2$ and $0 \quad k_1 + k_2 \quad 1$.

Proof. For (10.16a) we see

$$\begin{aligned} RL &= RA\partial_x J \\ &= A([R, \partial_x] + \partial_x R)J \\ &= \quad L + A\partial_x([R, J] + J R) \\ &= \quad L + A\partial_x[R, J] + L R \end{aligned}$$

To handle the (10.16b) we apply the identity $[A, BC] = [A, B]C + B[A, C]$. For the first

$$\begin{aligned} [R, J L] &= [R, J]L + J [R, L] \\ &= [R, J]L \quad J L + J A\partial_x[R, J] \\ &= \quad J L + [R, J]L + L [R, J] \end{aligned}$$

The nonlinear commutator term is similar

$$\begin{aligned} RN(u, v) = \; & RN_{B\partial_x}(J u, J v) \\ = \; & N_{B\partial_x}(RJ u, J v) + N_{RB\partial_x}(J u, J v) \\ = \; & N(Ru, v) + N_{B\partial_x}([R, J]u, J v) + N(u, Rv) \\ & + N_{B\partial_x}(J u, [R, J]v) + N_{[R, B\partial_x]}(J u, v), \end{aligned}$$

and so

$$fR, N\, g(u,v) = \; N_{B@x}([R, J\,]u, J\,v) + N_{B@x}(J\,u, [R, J\,]v).$$

For the final term

$$\begin{aligned} R^2 N\,(u,v) = \;& R\left(fR, N\, g(u,v) + N\,(Ru,v) + N\,(u,Rv) + N_{[R;B@x]}(J\,f, J\,g)\right) \\ = \;& R fR, N\, g(u,v) \; + fR, N\, g(Ru,v) + N\,(R^2u,v) \\ & + N\,(Ru,Rv) \quad N\,(Ru,v) + fR, N\, g(u,Rv) \\ & + N\,(Ru,Rv) + N\,(u,R^2v) \quad N\,(u,Rv) \\ & \quad fR, N\, g(u,v) \quad N\,(Ru,v) \quad N\,(u,Rv) + N\,(u,Rv). \end{aligned}$$

Since $N_{[R;B@x]}(f,g) = \quad N_{B@x}(f,g)$ we have the desired result. □

We define the time dependent versions of our smooth linear operator to be

$$L \;\; = \partial_t + J\,L \;\; . \tag{10.18}$$

Many commutators share the term $J\,L \; + [R, J\,L\,]$ and so we abbreviate it

$$H \; = J\,L \; + [R, J\,L\,] = 2[R, J\,]L\; . \tag{10.19}$$

Lemma 10.8

L satisfies the S commutation relations

$$[S, J\,] = [R, J\,], \tag{10.20a}$$

$$[S, L\;] = \quad L \; + H\;, \tag{10.20b}$$

and the S^2 commutation relations

$$[S^2, J\,] = 2[R, J\,]S + [R, J^{(1)}] + [R, J\,], \tag{10.21a}$$
$$[S^2, \boldsymbol{L}\,] = \quad 2S\boldsymbol{L} \;\; + \boldsymbol{L} \;\; + H\,S + SH\,.$$

The smoothed nonlinear operator satisfies the modified Leibniz relations

$$SN \; = N\,(Su, v) + N\,(u, Sv) + N_{[S;B@_x]}(J\,u, J\,v) + \boldsymbol{f}R, N\,\boldsymbol{g}$$

and

$$S^2 N\,(u, v) = \sum_{\mathsf{m}_2} N^{(m_3)}(S^{m_1}u, S^{m_2}v) + \sum_{k_1;k_2} \boldsymbol{f}R, N\,\boldsymbol{g}(S^{k_1}u, S^{k_2}v) \tag{10.22a}$$
$$+ \;\; S\boldsymbol{f}R, N\,\boldsymbol{g}(u, v).$$

where

$$\mathsf{m}_2 = \boldsymbol{f}m_1 + m_2 + m_3 \quad\;\; 2\boldsymbol{g},$$
$$0 \quad\; k_1 + k_2 \quad\; 1,$$

and

$$N^{(n)}(u, v) = N_{(t@_t)^n[B(t)]@_x}(J\,u, J\,v).$$

Proof. Since $[S, \partial_t] = [t\partial_t, \partial_t] = \quad \partial_t$ and $[\partial_t, J\,] = 0$ we have

$$[S, J\,] = [t\partial_t, J\,] + [R, J\,]$$
$$= [R, J\,]$$

which gives (10.20a).

For (10.20b)

$$
\begin{aligned}
[S, \mathsf{L}\ ] &= \ [t\partial_t + R, \partial_t + J\ L\ ] \\
&= \ [t\partial_t, \partial_t] + [R, J\ L\ ] \\
&= \quad \mathsf{L}\ + J\ L\ + [R, J\ L\ ] \\
&= \quad \mathsf{L}\ + H\ .
\end{aligned}
$$

For (10.21)

$$
\begin{aligned}
[S^2, J\ ] &= [S, J\ ]S + S[S, J\ ] \\
&= [R, J\ ]S + S[R, J\ ] \\
&= 2[R, J\ ]S + [S, [R, J\ ]] \\
&= 2[R, J\ ]S + [t\partial_t, [R, J\ ]] + [R, [R, J\ ]] \\
&= 2[R, J\ ]S + [R, [R, J\ ]],
\end{aligned}
$$

and

$$
\begin{aligned}
[S^2, \mathsf{L}\ ] &= \ [S, \mathsf{L}\ ]S + S[S, \mathsf{L}\ ] \\
&= \ (\quad \mathsf{L}\ + H\ )S + S(\quad \mathsf{L}\ + H\ ) \\
&= \quad 2\mathsf{L}\ S + H\ S + SH \\
&= \ \ 2S\mathsf{L}\ + \mathsf{L}\ + H\ S + SH\ .
\end{aligned}
$$

For the nonlinear commutators

$$
\begin{aligned}
SN\ (u, v) &= \ N_{B@_x}(SJ\ u, J\ v) + N_{SB@_x}(J\ u, J\ v) \\
&= \ N\ (Su, v) + N_{B@_x}([R, J\ ]u, J\ v) + N\ (u, Sv) \\
&\qquad + N_{B@_x}(J\ u, [R, J\ ]v) + N_{[S;B@_x]}(J\ u, J\ v) \\
&= \ N\ (Su, v) + N\ (u, Sv) + N_{[S;B@_x]}(J\ u, J\ v) + \mathsf{f}R, N\ \mathsf{g}.
\end{aligned}
$$

The modified Leibniz rule comes from the identity (10.17). Arguing as we did there we get

$$
S^2 N\ (u, v) = \ S\left(\mathsf{f}S, N\ \mathsf{g}(u, v) + N\ (Su, v) + N\ (u, Sv) + N_{[S;B@_x]}(J\ f, J\ g)\right),
$$

but now we need to handle the commutator term differently since

$$[S, B\partial_x] = tB'(t)dx \quad B\partial_x.$$

Plugging this into the bilinear commutator identity gives the desired result. □

Using the above lemma we can track the S-derivatives of solutions to (10.14). Showing convergence of our smooth approximations will require energy ODI's in the difference of approximations so we compute the dynamics of a generalization of (10.14).

Lemma 10.9

Let f, g, h 2 $H_T^{2;2}$ for some $T > 0$ satisfy

$$L \ f = J\,(N\,(g,h) + N\,(g,h)) + F(x,t)$$

then

$$L \ Sf = J \left[\sum_{\mathsf{m}_1} N^{(m_1)}(S^{m_2}g, S^{m_3}h) + N^{(m_1)}(S^{m_2}h, S^{m_3}g) \right] + SF(x,t) + \mu_{\ ;1}(f,g,h), \tag{10.23a}$$

and

$$L \ S^2 f = J \left[\sum_{\mathsf{m}_2} N^{(m_1)}(S^{m_2}g, S^{m_3}h) + N^{(m_1)}(S^{m_2}h, S^{m_3}g) \right] + (S \quad I)^2 F(x,t) + \mu_{\ ;1}(f,g,h) + \mu_{\ ;2}(f,g,h). \tag{10.23b}$$

Where $\mathsf{m}_i = f(m_1, m_2, m_3) : 0 \quad m_1 + m_2 + m_3 \quad i g$ and the $\mu_{\ ;i}$'s are the commutator terms:

$$\mu_{\,;1}(f,g,h) = J\ \left(\{R, N\}(g,h) + \{R, N\}(h,g)\right) \\ + [R, J\,]\,(N\ (g,h) + N\ (h,g)) + H\ f$$

and

$$\mu_{\,;2}(f,g,h) = \sum_{k_1;k_2} \left(\{R, N\}(S^{k_1}g, S^{k_2}h) + \{R, N\}(S^{k_1}h, S^{k_2}g)\right) \\ +\ [S^2, J\,]\,(N\ (g,h) + N\ (h,g)) + H\ Sf + SH\ f \\ + S\,(\{R, N\}(g,h) + \{R, N\}(h,g))$$

where $0 \quad k_1 + k_2 \quad 1$.

Proof. Using the commutation relations in Lemma 10.8

$$\begin{aligned} L\ Sf =&\ SL\ f + [S, L\ ]f \\ =&\ SJ\ (N\ (g,h) + N\ (h,g)) + SF \quad L\ f + J\ L\ f + [R, J\ L\ ]f \\ =&\ J\ S\,(N\ (g,h) + N\ (h,g)) + J\ (N\ (g,h) + N\ (h,g)) \\ &+ [R, J\,]\,(N\ (g,h) + N\ (h,g)) + H\ f + SF(x,t) + F(x,t). \end{aligned}$$

Again by Lemma 10.8

$$\begin{aligned} SN\ (g,h) =&\ N\ (Sg,h) + N\ (g,Sh) + N^0(g,h) \\ &\ N\ (g,h) + \{R, N\}(g,h), \end{aligned}$$

and similarly for $SN\ (h,g)$. Thus

$$\begin{aligned} L\ Sf =&\ \sum N^{(m_1)}(S^{m_2}g, S^{m_3}h) + N^{(m_1)}(S^{m_2}h, S^{m_3}g) \\ &+ \{R, N\}(g,h)\ + [R, J\,]\,(N\ (g,h) + N\ (h,g)) \\ &+ H\ f + SF(x,t) + F(x,t), \end{aligned}$$

where $m_1 + m_2 + m_3 \quad 1$.

For (10.23b) we use Lemma 10.8

$$\begin{aligned}
\mathsf{L}\ S^2 f = &\ S^2 \mathsf{L}\ f + [S^2, \mathsf{L}\ ] f \\
= &\ S^2 J\ (N\ (g,h) + N\ (h,g)) + [S^2, \mathsf{L}\ ] f + S^2 F(x,t) \\
= &\ S^2 J\ (N\ (g,h) + N\ (h,g)) \quad 2\mathsf{L}\ Sf \\
&\ + H\ Sf + SH\ f + S^2 F(x,t),
\end{aligned}$$

$\mathsf{L}\ Sf$ is handled by (10.23a). For the higher order term notice

$$\begin{aligned}
S^2 J\ (N\ (g,h) + N\ (h,g)) = &\ J\ S^2\ (N\ (g,h) + N\ (h,g)) \\
&\ + [S^2, J\ ]\ (N\ (g,h) + N\ (h,g))\,.
\end{aligned}$$

By (10.22a) in Lemma 10.8

$$\begin{aligned}
S^2 N\ (g,h) = &\sum_{\mathsf{m}_2} N^{(m_3)}(S^{m_1} g, S^{m_2} h) + \\
&\sum_{k_1;k_2} \big(\mathsf{f} R, N\ \mathsf{g}(S^{k_1} g, S^{k_2} h) + \mathsf{f} R, N\ \mathsf{g}(S^{k_1} h, S^{k_2} g) \big) \\
&+\ S\ (\mathsf{f} R, N\ \mathsf{g}(g,h) + \mathsf{f} R, N\ \mathsf{g}(h,g)) + H\ Sf + SH\ f,
\end{aligned}$$

where $\mathsf{m}_2 = \mathsf{f} m_1 + m_2 + m_3 \quad 2\mathsf{g}$, and $0 \quad k_1 + k_2 \quad 1$. This holds for $S^2 N\ (h,g)$ giving us

$$\begin{aligned}
\mathsf{L}\ S^2 f = &\quad 2\mathsf{L}\ Sf +\ J\ \left[\sum_{\mathsf{m}_2} N^{(m_3)}(S^{m_1} g, S^{m_2} h) + N^{(m_3)}(S^{m_1} h, S^{m_2} g) \right] \\
&+ \sum_{k_1;k_2} \big(\mathsf{f} R, N\ \mathsf{g}(S^{k_1} g, S^{k_2} h) + \mathsf{f} R, N\ \mathsf{g}(S^{k_1} h, S^{k_2} g) \big) \\
&+\ S\ (\mathsf{f} R, N\ \mathsf{g}(g,h) + \mathsf{f} R, N\ \mathsf{g}(h,g)) + S^2 F(x,t).
\end{aligned}$$

Subbing in for $S\mathsf{L}\ f$ gives the desired result. □

10.3 Smoothed Operators

Existence to a solution of (10.14) for a fixed $\epsilon > 0$ will follow from boundedness properties of $J^{(l)}$ in $\tilde{H}^{r;2}$. However, for global existence of (9.1) we will need decay results for the commutators of Section 10.2 in the time dependent space $H_T^{s;2}$. These decay estimates will be inherited from the decay in $\tilde{H}^{r;2}$ and so we collect them here alongside the boundedness results for $\tilde{H}^{r;2}$.

To begin, we note the reason we use the name approximation to the identity as in [39] is because for any $f \in L_2$

$$\lim_{\to 0^+} |J f \quad f|_0 \to 0$$

the result is extended in [41] for good kernels that satisfy the moment condition

$$\int x\phi(x)dx = 0. \tag{10.25}$$

We note that (10.25) is satisfied by our kernel since ϕ is even.

Lemma 10.10

(Bona and Smith) If $f \in H^r$ then

$$|J f|_{r+} \qquad c\epsilon \quad |f|_r$$
$$|f \quad J f|_r \qquad c\epsilon \; |f|_r$$

for all $\alpha > 0$ and $\beta \in [0, r]$

Notice in Lemma 10.6 that the smooth commutator terms are sums of $J^{(l)} + J^{(l+1)}$. The following lemma shows that these operators are indeed approximating "minus" each other in the H^r sense.

Lemma 10.11

$(\ \ 1)^l J^{(l)}$ is an approximation of the identity in the H^r sense. Specifically, for all $f \in H^r$

$$
\begin{aligned}
|J^{(l)} f|_{r+} &\quad c\epsilon \quad |f|_r \\
|f \quad J^{(l)} f|_r &\quad c\epsilon\ |f|_r .
\end{aligned}
$$

Proof. We begin with the unit mass condition (i) of Definition 10.2. For any smooth compactly supported ϕ

$$\int R\phi dx = \int x\partial_x \phi dx = \quad \int \phi dx.$$

Inductively we see $\int R^l \phi dx = (\ \ 1)^l \int \phi dx$. Thus for our unit mass ϕ

$$\int R^l \phi \ dx = (\ \ 1)^l$$

and so $(\ \ 1)^l R^l \phi$ has unit mass.

For (ii) notice that since $\phi \in C^\infty$ for all k there is a c_k so that $|\phi^{(k)}| < c_k$. Then

$$|\phi\ (x)| = \frac{1}{\epsilon}|\phi(x/\epsilon)| \quad c_0/\epsilon.$$

Then for $R\phi$ we have

$$|R\phi\ | = |\frac{x^2}{\epsilon^2}\phi^0(x/\epsilon)|,$$

so when $x < \epsilon$

$$|R\phi\ | < \frac{c_1}{\epsilon}$$

When $x \quad \epsilon$ then $\phi^0 = 0$. So we have the result for $R\phi$. The general result follows by noticing

$$R^l = \sum_{0 \quad i \quad l} x^i \partial_x^i,$$

and using induction. Recall that the sum potentially supresses constants.

$$\begin{aligned} |R^l \phi| &\quad \sum_{0 \quad i \quad l} |x^i \partial_x^i \phi^{(i)}(x/\epsilon)| \\ &\quad \frac{1}{\epsilon} \sum_{0 \quad i \quad l} c_i \end{aligned}$$

and so (ii) is satisfied.

Condition (iii) follows by a similar splitting and bounding but we use the fact that

$$\frac{1}{\epsilon^2} < \frac{1}{|x|^2}$$

when $|x| < \epsilon$.

□

Noticing that

$$\begin{aligned} |[R, J^{(l)}] \quad f|_r &= |J^{(l+1)} f + J^{(l)} f + f \quad f|_r \\ &\quad |J^{(l+1)} f + f|_r \quad + |J^{(l)} f \quad f|_r \quad , \end{aligned}$$

we can use (10.27d) of Lemma 10.13 to get the following corollary.

Corollary 10.12

Let $f \in H^r$ then the as ϵ goes to zero

$$|[R, J^{(l)}] f|_r \to 0.$$

Further, if $\beta \in [0, r)$, then the commutator satisfies the decay estimates

$$\left|[R, J^{(l)}]f\right|_r \qquad c\epsilon \ jfj_r, \tag{10.26a}$$

and J is symmetric on L_2.

Proof. Without loss of generality assume l is even. From Lemma 10.6 we get

$$j[R, J^{(l)}]fj_r \quad jJ^{(l)} \quad fj_r + jJ^{(l+1)} + fj_r.$$

Lemma 10.11 gives the convergence.

For (10.26a) assume f and β are as above and notice

$$\begin{aligned} j[R, J^{(l)}]\ fj_r \quad &= jJ^{(l+1)}f + J^{(l)}f + f \quad fj_r \\ &\quad jJ^{(l+1)}f + fj_r \quad + jJ^{(l)}f \quad fj_r \quad . \end{aligned}$$

Lemma 10.11 gives the desired decay result.

The symmetry property comes from the fact that convolution against a smooth bump function is symmetric on L_2. □

We are now ready to work in our actual spaces of interest $\tilde{H}^{r;2}$ and $H_T^{r;2}$.

Lemma 10.13

$(\ 1)^l J^{(l)}$ is an approximation to the identity in both $\tilde{H}^{r;2}$ and $H_T^{r;2}$. Specifically let $\alpha > 0$ and β 2 $[0, r)$, then if f 2 $\tilde{H}^{r;2}$

$$jJ\ fj_{r+\ ;2} \qquad c\epsilon \quad jfj_{r;2} \tag{10.27a}$$

$$jf \quad J\ fj_{r\ \ ;2} \qquad c\epsilon\ jfj_{r;2} \tag{10.27b}$$

and if f 2 $H_T^{r;2}$ then for all t 2 $[0, T)$

$$\|J f\|_{r+\ ;2}(t) \quad c\epsilon \quad \|f\|_{r;2}(t) \tag{10.27c}$$

$$\|f \quad J f\|_{r \ \ ;2}(t) \quad c\epsilon \ \|f\|_{r;2}(t) \tag{10.27d}$$

Proof. We begin with (10.27a). Let $\alpha > 0$, expanding the norm and commuting terms gives us

$$\begin{aligned} |J f|_{r+\ ;2} = |J f|_{r+} \ + |RJ f|_{r+ \ \ 1} + |R^2 J f|_{r+ \ \ 2} \\ |J f|_{r+} \ + |J Rf|_{r+ \ \ 1} + |J R^2 f|_{r+ \ \ 2} \\ + |[R, J] f|_{r+ \ \ 1} + |[R^2, J] f|_{r+ \ \ 2}. \end{aligned}$$

(10.15a) of Lemma 10.6 gives

$$\begin{aligned} |J f|_{r+\ ;2} \quad |J f|_{r+} \ + |J Rf|_{r+ \ \ 1} + |J R^2 f|_{r+ \ \ 2} + |[R, J] f|_{r+ \ \ 1} \\ + |[R, J^{(1)}] f|_{r+ \ \ 2} + 2|[R, J] Rf|_{r+ \ \ 2} + |[R, J] f|_{r+ \ \ 2}. \end{aligned}$$

Since $R^k f \ 2 \ H^{r \ \ k}$ Lemma 10.10 gives us

$$\begin{aligned} |J f|_{r+\ ;2} \quad c\epsilon \quad |f|_{r+\ ;2} + |[R, J] f|_{r+ \ \ 1} \\ + |[R, J^{(1)}] f|_{r+ \ \ 2} + 2|[R, J] Rf|_{r+ \ \ 2} + |[R, J] f|_{r+ \ \ 2} \end{aligned}$$

(10.26a) in Lemma 10.12 gives us

$$\begin{aligned} |J f|_{r+\ ;2} \quad & c\epsilon \quad |f|_{r;2} + 2c\epsilon^1 \quad |f|_r + +2c\epsilon^2 \quad |f|_r + c\epsilon^1 \quad |f|_{r;1} \\ & c\epsilon \quad |f|_{r;2}. \end{aligned}$$

For (10.27c) we argue similarly but use the commutation relations (10.20a) and (10.21a) of Lemma 10.8. We drop the explicit time dependence to condense notation:

$$
\begin{aligned}
kJ\ fk_{r+\ \ ;2} &= jJ\ fj_{r+}\ \ + jSJ\ fj_{r+\ \ 1} + jS^2J\ fj_{r+\ \ 2} \\
& jJ\ fj_{r+}\ \ + jJ\ Sfj_{r+\ \ 1} + jJ\ S^2 fj_{r+\ \ 2} \\
& \quad + j[R, J\ ]fj_{r+\ \ 1} + j[S^2, J\ ]fj_{r+\ \ 2}. \\
& c\epsilon\ \ \ kfk_{r;2} + j[R, J\ ]fj_{r+\ \ 1} \\
& \quad + j[R, J^{(1)}]fj_{r+\ \ 2} + 2j[R, J\ ]Sfj_{r+\ \ 2} + j[R, J\ ]fj_{r+\ \ 2} \\
& c\epsilon\ \ \ kfk_{r;2} + c\epsilon^1\ \ \ kfk_{r;0} + 2c\epsilon^2\ \ \ kfk_{r;0} + 2c\epsilon^2\ \ \ kfk_{r;1} \\
& c\epsilon\ \ \ kfk_{r;2:}
\end{aligned}
$$

(10.27b) and (10.27d) are handled similarly.

□

Bound (10.27a) allows us to control the smoothed operators on $\tilde{H}^{r;2}$.

Lemma 10.14

Let $f, g\ 2\ \tilde{H}^{r;2}$, with $r\ \ \ \ 2$, then $L\ \ $ satisfies

$$
\begin{aligned}
jJ\ L\ \ fj_{r;2} &\quad c\epsilon\ \ ^1 jfj_{r;2} \\
jJ\ L\ \ fj_{r\ \ k;2} &\quad cjfj_r
\end{aligned}
$$

for all $k\ 2\ [1, r]$, and $N\ \ $ satisfies

$$
jJ\ N\ (f, g)j_{r;2} \quad c\epsilon\ \ ^2 jfj_{r;2} jgj_{r;2}
$$

and

$$
jJ\ N\ (f, g)j_{r\ \ k;2} \quad cjfj_{r;2} jgj_{r;2}
$$

Proof. Since $f\ 2\ \tilde{H}^{r;2}$ by Lemma 10.13 $J\ f$ is as well. In particular

$$
jJ\ L\ \ fj_{r;2} \quad cjL\ \ fj_{r;1} = \ \ cjLJ\ fj_{r;1}.
$$

By Lemmas 10.5 and 10.13

$$\begin{aligned} |LJ f|_{r,2} &\le c|J f|_{r+1,2} \\ &\le c\epsilon^{-1}|f|_{r,2}, \end{aligned}$$

putting this together with our first bound gives the desired result, and similarly for the $r - k$ norm.

The N bounds are gotten by first applying Lemma 10.13, expanding the definition of N, and using Lemma 10.1:

$$\begin{aligned} |J N(f,g)|_{r,2} &\le c|N_{B\partial_x}(J f, J g)|_{r,2} \\ |N_{B\partial_x}(J f, J g)|_{r,2} &\le c|J f|_{r+1,2}|J g|_{r+1,2} \\ &\le c\epsilon^{-2}|f|_{r,2}|g|_{r,2}. \end{aligned}$$

As required. The $r - k$ norm follows similarly.

□

Combining these decay estimates with the commutator relations in Lemma 10.7 will be our primary tool in showing local existence of smoothed solutions. The following lemma will give us decay of the "unexpected" commutator terms.

Lemma 10.15

Let f and g be in $\tilde{H}^{r,2}$ then for any $\beta \in [0, r)$ and $k = 0, 1, 2$

$$|[R, J L]f + J L f|_{r-1,k} \le c|[R, J]f|_{r,k} \qquad (10.28a)$$

$$\begin{aligned} |\{R, N\}(f,g)|_{r-1,k} &\le c|[R, J]f|_{r-1,k}|g|_{r,k} \qquad (10.28b) \\ &+ c|[R, J]g|_{r,k}|f|_{r-1,k} \end{aligned}$$

Further ,

$$
\begin{aligned}
|[R, L\ ]f + L\ f|_{r-1;k} &\to 0 \\
|\{R, N\ \}(f, g)|_{r-1;k} &\to 0
\end{aligned}
$$

Proof. We begin with (10.28a). Using the commutation identity (10.15b) from Lemma 10.7 we see

$$|[R, J\ L\ ]f + J\ L\ f|_{r-1;k} = 2|L\ [R, J\ ]f|_{r-1;k}$$

By the commutation relation in Lemma 10.12 and the fact that $J^{(l)}$ is a convolution operator we get

$$[\partial_x, [R, J^{(l)}]] = 0,$$

and so

$$
\begin{aligned}
|L[R, J\ ]f|_{r-1;k} = {} & |A[R, J\ ]\partial_x f|_{r-1;k} \\
& |A[R, J\ ]f|_{r;k} \\
& c|[R, J\ ]f|_{r;k}.
\end{aligned}
$$

For the bilinear commutator term we use Lemma 10.7:

$$
\begin{aligned}
|\{R, N\ \}(f, g)|_{r-1;k} \quad & |N_{B@_x}([R, J\ ]f, J\ g)|_{r-1;k} + |N_{B@_x}(J\ f, [R, J\ ]g)|_{r-1;k} \\
& c\left(|[R, J\ ]f|_{r-1;k}|J\ g|_{r;k} + |J\ f|_{r-1;k}|[R, J\ ]g|_{r;k}\right).
\end{aligned}
$$

The convergence result follows from bounds (10.28a) and (10.28b) together with Lemma 10.13. □

Combining Lemmas 10.15 and 10.13 gives the following corollary.

Corollary 10.16

Let f and g be in H^r then

$$
\begin{aligned}
|[R, J L]f + J L f|_r &\to 0 \\
|\{R, N\}(f,g)|_r &\to 0
\end{aligned}
$$

as $\epsilon \to 0$. Further, for any $\beta \in [0, r)$

$$
\begin{aligned}
|[R, J L]f + J L f|_{r-\beta-1} &\leq c\epsilon\, |f|_r \\
|\{R, N\}(f,g)|_{r-\beta-1} &\leq c\epsilon\, |f|_r |g|_r
\end{aligned}
$$

These decay properties of $J^{(\epsilon)}$ in $\tilde{H}^{r,2}$ can now be pulled up to the space $H_T^{r,2}$.

Lemma 10.17

Let $f \in H_T^{r,2}$, then as $\epsilon \to 0$

$$\|J^{(\epsilon)} f - f\|_{H_T^{r,2}} \to 0$$

and for every $\beta \in [0, r)$

$$\|J^{(\epsilon)} f - f\|_{H_T^{r-\beta,2}} \leq c\epsilon\, \|f\|_{H_T^{r,2}}.$$

Proof. We surpress the time dependence of the $\| \ \|_{r,2}(t)$ norm for convenience. Expanding the norm gives us

$$\|J^{(\epsilon)} f - f\|_{r-\beta,2} \leq |J^{(\epsilon)} f - f|_{r-\beta} + |S(J^{(\epsilon)} f - f)|_{r-\beta-1} + |S^2(J^{(\epsilon)} f - f)|_{r-\beta-2}.$$

Using the commutation relations in Lemma 10.8 we see

$$
\begin{aligned}
|S(J^{(l)}f \quad f)|_{r \quad 1} &= |J^{(l)}Sf \quad Sf + [R, J^{(l)}]f|_{r \quad 1} \\
& |(J^{(l)}Sf \quad Sf|_{r \quad 1} + |[R, J^{(l)}]f|_{r \quad 1}
\end{aligned}
$$

and

$$
\begin{aligned}
|S^2(J^{(l)}f \quad f)|_{r \quad 2} &= |J^{(l)}S^2 f \quad S^2 f + [S^2, J^{(l)}]f|_{r \quad 2} \\
& |J^{(l)}S^2 f \quad S^2 f|_{r \quad 2} + |[S^2, J^{(l)}]f|_{r \quad 2}.
\end{aligned}
$$

The second order commutator term is handled by

$$
\begin{aligned}
|[S^2, J^{(l)}]f|_{r \quad 2} & |[R, J^{(l)}]Sf|_{r \quad 2} + |S[R, J^{(l)}]f|_{r \quad 2} \\
& 2|[R, J^{(l)}]Sf|_{r \quad 2} + |[S, [R, J^{(l)}]]f|_{r \quad 2},
\end{aligned}
$$

and by Lemma 10.6

$$
|[S, [R, J^{(l)}]]f|_{r \quad 2} \quad |[S, J^{(l)}]f|_{r \quad 2} + |[S, J^{(l+1)}]f|_{r \quad 2}.
$$

Combining all these and applying Lemma 10.11 gives the desired result.

□

Corollary 10.18

Let f and g be in $H_T^{r,2}$ then as $\epsilon \to 0$

$$
\begin{aligned}
\|[R, J \; L \;]f + J \; L \; f\|_{r \quad 1} &\to 0, \\
\|\{R, N \; \}(f, g)\|_{r \quad 1} &\to 0,
\end{aligned}
$$

and for any $\beta \in [0, r)$

$$\begin{aligned} \|[R, J\,L\,]f + J\,L\,f\|_{r-1} &\leq c\epsilon\, \|f\|_{r,2} \\ \|fR, N\,g(f,g)\|_{r-1} &\leq c\epsilon\, \|f\|_{r,2}\|g\|_{r,2} \end{aligned}$$

for all $t \in [0,T)$.

10.4 Existence

With the bounds from the previous section we are now able to construct local in time solutions to the smoothed equation (10.14) through Picard Iteration. We construct our solution in a slightly different space.

Definition 10.5. Our temporary solution space

$$X(r,j,T) = C^j\left([0,T); \tilde{H}^{r,2}\right)$$

is defined by the norm

$$\|v\|_{X(r,j,T)} = \sum_{0 \leq i \leq j} \sup_{t \in [0,T)} |\partial_t^i v|_{r,2}.$$

Before showing existence we verify that $X(r,1,T)$ is sufficient to get existence in our desired space $H_T^{r,2}$.

Lemma 10.19

Let $r \geq 2$ and $T > 0$ then $X(r,2,T) \subset H_T^{r,2}$.

Proof. Let $f \in X(r,2,T)$. Notice it suffices to show the scaling derivatives are controlled.

$$\begin{aligned} |Sf|_{r-1} &\leq |t\partial_t f|_{r-1} + |Rf|_{r-1} \\ &\leq T|\partial_t f|_{r-1} + \|f\|_{X(r,2,T)} \\ &\leq (T+1)\|f\|_{X(r,2,T)}. \end{aligned}$$

For the second order terms

$$
\begin{aligned}
|S^2 f|_{r,2} &\leq |t^2\partial_t^2 f|_{r,2} + 2|t\partial_t R f|_{r,2} + |R^2 f|_{r,2} \\
&\leq T^2\|f\|_{X(r,2,T)} + 2T\|f\|_{X(r,2,T)} + \|f\|_{X(2,T,j)} \\
&\leq (T^2 + 2T + 1)\|f\|_{X(r,2,T)}.
\end{aligned}
$$

So $\|f\|_{H_T^{r,2}} \leq C\|f\|_{X(r,2,T}$. $\square$

Our target space for energy estimates is $X(r+1, 2, T)$, but here we show existence of higher regularity solutions.

Theorem 10.20

Let $u_0 \in \tilde{H}^{r,2}$ then for every $\delta, \epsilon > 0$ there exists a $T(\delta, \epsilon)$ so that the smoothed equation (10.14) has a solution

$$v(t) \in X(r+k, j, T(\epsilon))$$

for any $k, j \in \mathbb{N}$ in some ball about u_0.

Proof. **Initial Solutions:** We begin by constructing our solution in $X(r, 1, T(\epsilon))$. Let $\rho > 0$ and B be the $\tilde{H}^{r,2}$ ball about initial data u_0. For this proof we set the right hand side of (10.14) to

$$F(f) = J L f + J N(f, f).$$

Let $f \in B$ then by Lemma 10.14

$$
\begin{aligned}
|F(f)|_{r,2} &\leq c|L f|_{r,2} + c|N(f,f)|_{r,2} \\
&\leq c(\epsilon^{-1}|f|_{r,2} + \epsilon^{-2}|f|_{r,2}^2).
\end{aligned}
$$

Thus

$$\begin{aligned} |F(f)|_{r,2} &\le c(\epsilon^{-1}(\rho + |J u_0|_{r,2}) + \epsilon^{-2}(\rho + |J u_0|_{r,2})^2_{r,2}) \\ &\le c(\epsilon^{-1}(\rho + |u_0|_{r,2}) + \epsilon^{-2}(\rho + |u_0|_{r,2})^2_{r,2}). \end{aligned}$$

So F is bounded on B.

To see that it is locally Lipschitz first note that any bounded linear operator is locally Lipschitz. For the non-linear piece let $v_1, v_2 \in B$ then by bilinearity of N we have

$$|N(v_1, v_1) - N(v_2, v_2)|_{r,2} \le |N(v_1 - v_2, v_1)|_{r,2} + |N(v_2, v_1 - v_2)|_{r,2}$$

and by Lemma 10.14

$$\begin{aligned} |N(v_1, v_1) - N(v_2, v_2)|_{r,2} &\le c\epsilon^{-2}|v_1 - v_2|_{r,2}(|v_1|_{r,2} + |v_2|_{r,2}) \\ &\le c\epsilon^{-2}(\rho + |J u_0|_{r,2})|v_1 - v_2|_{r,2} \\ &\le c\epsilon^{-2}(\rho + |u_0|_{r,2})|v_1 - v_2|_{r,2}, \end{aligned}$$

putting this together

$$|F(v_1) - F(v_2)|_{r,2} \le c(\epsilon^{-1} + \epsilon^{-2}(\rho + |u_0|_{r,2}))|v_1 - v_2|_{r,2}.$$

Thus F is locally Lipschitz on B.

Therefore by the Picard Existence Theorem there is a unique solution to (10.14) in $X(r, 1, T(\epsilon))$.

Extending to spatial derivatives: Let $v(t)$ be such a constructed solution to (10.14) then $\partial_x v$ satisfies

$$\begin{cases} \partial_t \partial_x v = \partial_x F(v) \\ \partial_x v(0) = \partial_x J u_0 \end{cases}.$$

We write the right hand side of the equation as

$$F^0(v) = \epsilon \ {}^1(\phi^0) \quad (L \ v + N \ (v, v)).$$

Then $\partial_x v$ satisfies the integral equation

$$\partial_x v(t) = \partial_x v(0) + \int_0^{T(\ )} F^0(v(t))dt$$

which can be bounded as

$$j\partial_x v(t) j_{r;2} \quad j\partial_x J \ \ u_0 j_{r;2} + \int_0^{T(\ )} jF^0(v(t))j_{r;2} dt.$$

Since $v(t)$ 2 $B \ (u_0)$ Lemma 10.13 gives us

$$j\partial_x v(t) j_{r;2} \quad c\delta \ {}^1 j u_0 j_{r;2} + \epsilon \ {}^1 \int_0^{T(\ )} jL \ v j_{r;2} + jN \ (v, v) j_{r;2} dt.$$

Then by Lemma 10.5

$$j\partial_x v(t) j_{r;2} \quad c\delta \ {}^1 j u_0 j_{r;2} + \epsilon \ {}^1 T(\epsilon) c\epsilon \ {}^1 j v j_{r;2} + c\epsilon \ {}^2 j v j^2_{r;2}.$$

Our solution is in $X(r, 1, T(\epsilon))$ so the right hand side is uniformly bounded on $[0, T(\epsilon))$, making $\partial_x v$ 2 $X(r, 1, T(\epsilon, \delta))$ and so v 2 $X(r + 1, 1, T(\epsilon, \delta))$. The higher spatial derivatives follow similarly.

Extending to Time Derivatives: From our initial Picard Iteration any solution of (10.14) has $\partial_t v(t)$ 2 $\tilde{H}^{r;2}$. Differentiating the smoothed equation gives us

$$\partial_t^2 v = \ \partial_t F \ (v) = \ L \ \partial_t v + N \ (\partial_t v, v) + N \ (v, \partial_t v) + N^0(v, v),$$

where

$$N^0(v, v) = J \ N_{B^0(t)@_x}(J \ v, J \ v).$$

The $\tilde{H}^{r,2}$ norm is then

$$
\begin{aligned}
|\partial_t^2 v|_{r,2} &\leq |L\, \partial_t v|_{r,2} + |N(\partial_t v, v)|_{r,2} + |N(v, \partial_t v)|_{r,2} + |N'(v,v)|_{r,2} \\
&\leq c\epsilon^{-1}|v|_{r,2} + 2c\epsilon^{-2}(|\partial_t v|_{r,2}|v|_{r,2} + |v|_{r,2}^2) < \infty,
\end{aligned}
$$

so our solution is in $X(r, 2, T(\epsilon))$. The higher time derivatives follow similarly. Repeating this argument with the spatial derivatives gives the desired result. □

10.5 Energy Estimates

Currently our solution's existence interval depends on both ϵ and δ. To bring all our approximations into $H_T^{s,2}$ we will use energy estimates. In order to get a local in time exiestence result for (9.1) we could work simply in $H_T^{2,0}$. However, in order to prove global existence we will need more.

Definition 10.6. The scaling energies of a function $u \in H_T^{s,k}$ are

$$E_{s,k}[v(\cdot, t)] = \tfrac{1}{2}\|v\|_{s,k}^2(t)$$

Before deriving our energy dynamics we need an integration by parts result for our smoothed bilinear operator.

Lemma 10.21
Let $f \in H_T^{2,0}$, $g \in H_T^{1,0}$ for some $T > 0$, and $|f(\cdot,t)|_0^2 \in C^1([0,T))$ then the smoothed time dependent linear operator and smoothed nonlinear operator satisfy

$$
\begin{aligned}
(f, L\, f)_2 &= -\partial_t\left[\frac{1}{2}|f(\cdot,t)|_0^2\right], \\
(\partial_x f, J\, N(g, \partial_x f))_2 &= -\frac{1}{2}(\partial_x f, J\, N(g, f))_2.
\end{aligned}
$$

Proof. Recall J is symmetric on L_2 and so

$$\begin{aligned}(f, \mathsf{L}\ f)_2 = &\ (f, \partial_t f)_2 + (f, J\ L\ f)_2 \\ = &\ \partial_t \left[\frac{1}{2}|f(\ ,t)|_0^2\right] + (f, J\ L\ f)_2 \\ = &\ \partial_t \left[\frac{1}{2}|f(\ ,t)|_0^2\right] + (J\ f, A\partial_x J\ f)_2.\end{aligned}$$

Since $A\partial_x$ is anti-symmetric on H^1 $J\ A\partial_x$ is as well so the remaining term vanishes. For the nonlinear term we have by the symmetry of $B_j(t)$

$$\begin{aligned}(\partial_x f, J\ N\ (g, \partial_x f))_2 &= (\partial_x J\ f, N\ (g, \partial_x f))_2 \\ &= \int \langle \partial_x J\ f, B_j(t)\partial_x \partial_x f \rangle J\ g dx \\ &= \frac{1}{2}\int \partial_x \left[\langle \partial_x J\ f, g B_j \partial_x J\ f \rangle\right] dx \quad \frac{1}{2}\int \langle \partial_x f, J\ \partial_x g B_j \partial_x f \rangle dx \\ &= \quad \frac{1}{2}(J\ f, N\ (\partial_x g, f))_2\end{aligned}$$

as required. □

We now derive the dynamics of the high energy. Recall that $\sum$ denotes a linear combination of terms with some coefficients potentially suppressed.

In order to simplify our notation and emphasize the terms which come from commuting S over J we define the product terms $P_{s;k;}\ (f, g, h)$ to be the "expected" terms in the energy estimates.

$$P_{s;k;}\ (f, g, h) = \sum_{I_k(s)} \left(\partial_x^{n_1\ \ k} S^k f, N^{(m_1)}(\partial_x^{n_2} S^{m_2} g, \partial_x^{n_3} S^{m_3} h)\right)_2. \qquad (10.32)$$

The index set is all allowed powers of the $\partial_x^{n_i} S^{m_i}$ that appear after the Leibniz Rules in Lemma 10.8 and taking into consider integration by parts in Lemma 10.21.

$$\mathsf{I}_k(s) = \left\{ (\mathbf{m}, \mathbf{n}) \; 2 \; \mathsf{N}^3 \quad \mathsf{N}^3 : \begin{array}{rl} k \quad n_1 & \quad s \\ m_1 + m_2 + m_3 & \quad k \\ n_2 + n_3 & \quad n_1 \quad k \\ m_3 = k \quad) & \quad n_2 < n_1 \quad k \end{array} \right\}. \tag{10.33}$$

Notice that $P_{s;0;}(v, v, v)$ is the usual term appearing in energy estimates. The "unexpected" terms that come commuting S past J are the $\mu_{;i}$ defined in Lemma 10.9. These will be shown to decay.

Lemma 10.22

Let v be an $H_T^{s;2}$ solution of (10.14) for some $T > 0$ and $s \quad 2$. The dynamics of the corresponding energies are

$$E^0_{s;0} = P_{s;0;}(v\ , v\ , v\ ), \tag{10.34a}$$

$$\begin{aligned} E^0_{s;1} = {} & E^0_{s;0} + P_{s;1;}(v\ , v\ , v\ ) \\ & + \sum_{n_1=1}^{s} \left(J\, \partial_x^{n_1 \quad 1} S v\ ,\ \partial_x^{n_1 \quad 1} \mu_{;1}(v\ , v\ , v\ ) \right) \end{aligned} \tag{10.34b}$$

and

$$\begin{aligned} E^0_{s;2} = {} & E^0_{s;1} + P_{s;2;}(v\ , v\ , v\ ) \\ & + \sum_{n_1=2}^{s} \left(J\, \partial_x^{n_1 \quad 2} S^2 v\ ,\ \partial_x^{n_1 \quad 2} \mu_{;2}(v\ , v\ , v\ ) \right). \end{aligned}$$

Proof. Since the energy level will be s and the smoothing factor ϵ will be the same throughout this proof We let $\mathsf{I}_k(s) = \mathsf{I}_k$, abbreviate $v\ = v$, $\mu_{;i}(v, v, v) = \mu_{;i}$, and $P_{s;k;}(v, v, v) = P_{s;k}(v)$.

Pure ∂_x dynamics: Let $0 \quad n_1 \quad s$, then differentiating (10.14) with resepct to x n_1 times gives us

$$L\ \partial_x^{n_1} v \ = \ J\, N\,(v, \partial_x^{n_1} v) \ + \sum J\, N\,(\partial_x^{n_2} v, \partial_x^{n_3} v) \qquad \begin{cases} n_2 + n_3 = n_1 \\ n_3 < n_1 \end{cases} .$$

Taking the in L_2 inner product of both sides with $\partial_x^{n_1} v$ gives

$$(\partial_x^{n_1} v, L\ \partial_x^{n_1} v)_2 = \ (\partial_x^{n_1} v, J\, N\,(v, \partial_x^{n_1} v) + \sum (\partial_x^{n_1} v, J\, N\,(\partial_x^{n_2} v, \partial_x^{n_3} v))_2.$$

By Theorem 10.20 $\partial_t v\ 2\ H_T^{s;2}$. Therefore $jv(\ ,t)j_0^2\ 2\ C^1([0,T))$ and so by the integration by parts result of Lemma 10.21 and symmetry of J from Lemma 10.13 we have

$$\begin{aligned} \partial_t[\frac{1}{2} j\partial_x^{n_1} v j_0^2] &= \ (J\, \partial_x^{n_1} v, N\,(v, \partial_x^{n_1} v)_2 + \sum (J\, \partial_x^{n_1} v, N\,(\partial_x^{n_1} v, \partial_x^{n_2} v))_2 \\ &= \quad (J\, \partial_x^{n_1} v, N\,(\partial_x v, \partial_x^{n_1\ \ 1} v)_2 + \sum (J\, \partial_x^{n} v, N\,(\partial_x^{n_1} v, \partial_x^{n_2} v))_2. \end{aligned}$$

We absorb the first term into the sum and sum over $0 \quad n_1 \quad s$ which gives

$$\sum_{n_1 \ \ s} \partial_t[\frac{1}{2} j\partial_x^{n} v j_0^2] = \ \sum_{\text{I}_0} (J\, \partial_x^{n_1} v, N\,(\partial_x^{n_2} v, \partial_x^{n_3} v))_2. \ = P_{s;0}(v). \tag{10.35}$$

The left is $E_{s;0}^{0}$ which gives (10.34a).

One S derivative: Letting $f, g, h = v$, $N\,(g,h) = \frac{1}{2} N\,(v,v)$, and $F(x,t) = 0$ in Lemma 10.9 and differentiating $n_1 \quad 1$, for $1 \quad n_1$, times with respect to x gives

$$L\ \partial_x^{n_1\ \ 1} S v = \ N\,(v, \partial_x^{n_1\ \ 1} S v) + \sum N^{(m_1)}(\partial_x^{n_2} S^{m_2} v, \partial_x^{n_3} S^{m_3} v) + \ \partial_x^{n_1\ \ 1}\,[\mu\ _{;1}]$$

where

$$\begin{cases} n_2 + n_3 = n_1 \quad 1 \\ n_3 < n_1 \quad 1 \end{cases} .$$

Taking the inner product with $\partial_x^{n_1\ \ 1} S v$, applying Lemma 10.21, and summing over $1 \quad n_1 \quad s$ gives us

$$\partial_t \left[\sum_{n_1 \quad s} \frac{1}{2} j \partial_x^{n \quad 1} S v j_0^2 \right] = \sum_{\text{I}_1} (\partial_x^{n_1 \quad 1} S v, N^{(m_1)}(\partial_x^{n_2} S^{m_2} v, \partial_x^{n_3} S^{m_3} v))_2$$

$$+ \sum_{n_1=1}^{s} (\partial_x^{n_1 \quad 1} S v, \partial_x^{n_1 \quad 1} \mu_{\ ;1})_2$$

$$= P_{s;1}(v) + \sum_{n_1=1}^{s} (\partial_x^{n_1 \quad 1} S v, \partial_x^{n_1 \quad 1} \mu_{\ ;1})_2 \tag{10.36}$$

Adding (10.35) and (10.36) gives us the first dynamics equation (10.34b).

Two S derivatives: Using Lemma 10.9 with the same substitutions as before and differentiating $n_1 \quad 2$ times with respect to x gives

$$L \ \partial_x^{n_1 \quad 2} S^2 v = \ J \ N \ (v, \partial_x^{n_1 \quad 2} S^2 v) + \partial_x^{n_1 \quad 2} \left[J \sum_{\text{m}_2} N^{(m_1)}(S^{m_2} v, S^{m_3} v)) + \mu_{\ ;1} + \mu_{\ ;2} \right].$$

Taking the inner product with $\partial_x^{n_1 \quad 2} S^2 v$ and summing over $2 \quad n_1 \quad s$ gives us

$$\begin{aligned}
\partial_t \left[j \partial_x^{n_1 \quad 2} S^2 v j_0^2 \right] = \ & (\partial_x^{n_1 \quad 2} S^2 v, N \ (v, \partial_x^{n_1 \quad 2} S^2 v))_2 \\
& + \sum_{\text{I}_2} (\partial_x^{n_1 \quad 2} S^2 v \ , \ N^{(m_1)}(S^{m_2} \partial_x^{n_2} v, \partial_x^{n_3} S^{m_3} v))_2 \\
& + \sum_{n_1} (\partial_x^{n_1 \quad 2} S^2 v, \partial_x^{n_1} [\mu_{\ ;1} + \mu_{\ ;2}])_2 \\
= \ & (\partial_x^{n_1 \quad 2} S^2 v, N \ (v, \partial_x^{n_1 \quad 2} S^2 v))_2 + P_{s;2}(v) \\
& + \sum_{n_1} (\partial_x^{n_1 \quad 2} S^2 v, \partial_x^{n_1} [\mu_{\ ;1} + \mu_{\ ;2}])_2
\end{aligned}$$

The integration by parts in Lemma 10.21 gives the desired result

□

Note the $\mu_{\ ;i}$ always have one less S-derivative than the energy equation they appear in. To get the precise decay it will be easier to bound these terms by commutators whose decay we know.

When the explicit forms of constants is not important we will write

$$f \lesssim g \quad , \quad f \quad cg. \tag{10.37}$$

For some $c > 0$.

Lemma 10.23

Let $\mu_{;1}$ and $\mu_{;2}$ be as in Lemma 10.22 and $f, g, h \in H_T^{s;1}$ for some $T > 0$. Then for all $t \in [0, T)$ we have

$$\begin{aligned}\|\mu_{;1}(f,g,h)\|_{s\ 1;0}(t) \quad & c\left(\|h\|_{s;0}\|[R,J]g\|_{s;0} + \|[R,J]h\|_{s;0}\|g\|_{s;0}\right) \\ & + c\left(\|[R,J]\left(N(g,h) + N(h,g)\right)\|_{s\ 1;0} + \|[R,J]f\|_{s;0}\right)\end{aligned}$$

and

$$\begin{aligned}\|\mu_{;2}(f,g,h)\|_{s\ 2;0} \quad & c\left(\|h\|_{s;1}\|[R,J]g\|_{s;1} + \|[R,J]h\|_{s;1}\|g\|_{s;1}\right) \\ & c\left(\|[S^2,J]\left(N(g,h) + N(h,g)\right)\|_{s\ 2;0} + \|[R,J]f\|_{s;1}\right).\end{aligned}$$

Proof. For this proof we use the abbreviation $\mu_{;i} = \mu_{;i}(f,g,h)$. We start with $\mu_{;1}$:

$$\begin{aligned}\|\mu_{;1}\|_{s\ 1;0} \quad & \|J\left(\{R,N\}(g,h) + \{R,N\}(h,g)\right)\|_{s\ 1;0} \\ & + \|[R,J]\left(N(g,h) + N(h,g)\right)\|_{s\ 1;0} + \|H f\|_{s\ 1;0}.\end{aligned}$$

All of these are handled by an application of Lemma 10.15. Explicitly

$$\begin{aligned}\|\{R,N\}(g,h)\|_{s\ 1;0} & \lesssim \|[R,J]g\|_{s\ 1;0}\|h\|_{s;0} + \|[R,J]h\|_{s}\|g\|_{s\ 1;0} \\ & \lesssim \|[R,J]g\|_{s;0}\|h\|_{s;0} + \|[R,J]h\|_{s}\|g\|_{s;0},\end{aligned}$$

and similarly for $\{R,N\}(h,g)$.

Expanding the form of H and using the commutation relations in Lemma 10.7 gives us

$$kH\ fk_{s\ \ 1;0} = k[R, J\ L\ ]f + J\ L\ fk_{s\ \ 1;0}$$
$$.\quad k[R, J\ ]fk_{s;0}.$$

For $\mu\ _{;2}$ we first notice

$$k\mu\ _{;2}k_{s\ \ 2;0}\ .\ \sum_{k_1;k_2} k\left(fR, N\ g(S^{k_1}g, S^{k_2}h) + fR, N\ g(S^{k_1}h, S^{k_2}g)\right) k_{s\ \ 2;0}$$
$$+\ \ k[S^2, J\ ]\left(N\ (g,h) + N\ (h,g)\right) k_{s\ \ 2;0} + kH\ fk_{s\ \ 2;1}$$
$$+\, k\left(fR, N\ g(g,h) + fR, N\ g(h,g)\right) k_{s\ \ 2;1}$$

where 0 $\quad k_1 + k_2 \quad$ 1. Again using (10.28b) from Lemma 10.15 we see

$$kfR, N\ g(S^{k_1}g, S^{k_2}h)k_{s\ \ 2;0}\ .\ k[R, J\ ]S^{k_1}gk_{s\ \ 2;0}kS^{k_2}hk_{s\ \ 1;0}$$
$$+\, kS^{k_1}gk_{s\ \ 2;0}k[R, J\ ]S^{k_2}hk_{s\ \ 1;0}$$
$$.\ k[R, J\ ]gk_{s\ \ 1;1}khk_{s;1} + kgk_{s\ \ 1;1}k[R, J\ ]hk_{s;1}.$$

and $fR, N\ g(S^{k_1}h, S^{k_2}g)$ satisfies an analogous bound.

This gives us

$$\sum_{k_1;k_2} kfR, N\ g(S^{k_1}g, S^{k_2}h) + fR, N\ g(S^{k_1}h, S^{k_2}g)k_{s\ \ 2;0}$$
$$.\ khk_{s;1}k[R, J\ ]gk_{s;1} + k[R, J\ ]hk_{s;1}kgk_{s;1},$$

Lemma 10.15 handles the remaining term.

□

The $\mu\ _{;i}$ control the convergence of our smoothed energies to the energies of a solution to (9.1). Their decay in low energies will be used to get a uniform time interval of existence for our smoothed approximations, their boundedness for our high energies will allow us to get polynomial bounds for the high energies, and both of these will be used to show our approximations form a Cauchy sequence. Finally, once we have convergent sequence we will see the $\mu_{;i}$ on this sequence will decay even at high energies allowing solutions to (10.14) to pass on their energy estimates to solutions of (9.1). To meet all these needs we collect various applications of Lemma 10.23 and the decay results of Lemma 10.18 in the following Theorem.

Theorem 10.24

Let $\mu_{;i}$ be as in (10.24).

If $f, g, h \in H_T^{s;i}$ then

$$\|\mu_{;i}(f,g,h)\|_{s-i;0} \le c\left(\|f\|_{s;i-1} + \|g\|_{s;i-1}\|h\|_{s;i-1}\right), \tag{10.39a}$$

If $f, g, h \in H_T^{s+1;i}$ then

$$\|\mu_{;i}(f,g,h)\|_{s-i;0} \le c\epsilon\left(\|f\|_{s+1;i-1} + \|g\|_{s+1;i-1}\|h\|_{s+1;i-1}\right), \tag{10.39b}$$

and if there are $f, g, h \in H_T^{s;i-1}$ so that f, g, and h converges to f, g, and h respectively, then

$$\|\mu_{;i}(f,g,h)\|_{H_T^{s-i;0}} = \sup_{t\in[0;T)} \|\mu_{;i}(f,g,h)\|_{s-i;0} \to 0 \tag{10.39c}$$

as $\epsilon \to 0$.

Proof. For (10.39a) and (10.39b) we use Lemma 10.23 and bound the nonlinear terms with Lemma 10.14 and the commutator term with Lemma 10.12.

$$\begin{aligned}
\|\mu_{;1}(f,g,h)\|_{s-1} &\lesssim \|h\|_s\|[R,J]g\|_s + \|[R,J]h\|_s\|g\|_s \\
&\quad + \|[R,J]\left(N(g,h) + N(h,g)\right)\|_{s-1} + \|[R,J]f\|_s \\
&\lesssim \epsilon\left(\|h\|_{s+;0}\|g\|_{s+;0} + \|N(g,h)\|_{s-1+} + \|N(h,g)\|_{s-1+} + \|f\|_{s+;0}\right) \\
&\lesssim \epsilon\left(\|f\|_{s+;0} + \|g\|_{s+;0}\|h\|_{s+;0}\right),
\end{aligned}$$

and similarly for $\mu_{;2}$. If we let $\beta = 0$ or 1 we recover (10.39a) and (10.39b) respectively.

For the convergence result (10.39c) we first handle the linear commutator terms. Notice

$$
\begin{aligned}
\|[R, J]f\|_{s;i\ 1} &\quad \|[R, J]f\ \|_{s;i\ 1} + \|[R, J](f \quad f\ )\|_{s;i\ 1} \\
&\quad \|[R, J]f\ \|_{s;i\ 1} + c\|f \quad f\ \|_{s;i\ 1}
\end{aligned}
$$

By Lemma 10.12 since $f \ \in H_T^{s;i\ 1}$

$$\|[R, J]f\|_{s;i\ 1} \to 0$$

with ϵ and by assumption $\|f \quad f\ \|_{s;i\ 1} \to 0$ with ϵ. The same holds for g and h. Notice

$$N(g, h) = N(g \quad g, h \quad h) + N(g, h \quad h) + N(g \quad g, h) + N(g, h).$$

So by Lemma 10.14 and 10.12

$$
\begin{aligned}
\|[R, J^{(k)}]N(g, h)\|_{s\ \ i;i\ 1} &\quad c\,(\|g \quad g\ \|_{s;i\ 1}\|h \quad h\ \|_{s\ \ i;i\ 1} + c\|g\ \|_{s;i\ 1}\|h \quad h\ \|_{s;i\ 1}) \\
&\quad + c\,(\|g \quad g\ \|_{s;i\ 1}\|h\ \|_{s\ \ i;i\ 1} + \|[R, J_k]N(g, h)\|\|_{s\ \ i;i\ 1}).
\end{aligned}
$$

Letting $i = 1$ and $k = 0$ sets us up for the $\mu_{;1}$ bound. In this case since $g, h \in H_T^{s;0}$ Lemma 10.14 implies $N(h, g) \in H^{s;0}$ and so $[R, J]N(g, h) \to 0$ in this space. The remaining difference terms go to zero by assumption.

The only difference in argument for the $\mu_{;2}$ is the new commutator $[S^2, J]$ which by (10.21a) in Lemma 10.8

$$[S^2, J] = 2[R, J]S + [R, J_1] + [R, J],$$

pluigging this into the above bounds for $i = 2$ and $k \quad 1$ gives the desired result.

$\square$

Corollary 10.25

If v is a soltuion to (10.14) then the commutators $\mu_{;i}$ defined in (10.24) satisfies: If $v \in H_T^{s,i}$ then

$$|\mu_{;i}(v, v, v)|_{s \quad i} \quad c \left(E^{1/2}_{s,i \quad 1} + E_{s,i \quad 1}\right),$$

If $v \in H_T^{s+1,i}$ then

$$|\mu_{;i}(v, v, v)|_{s \quad i} \quad c\epsilon \left(E^{1/2}_{s+1,i \quad 1} + E_{s+1,i \quad 1}\right), \tag{10.40a}$$

and if there is a $v \in H_T^{s,i \quad 1}$ so that v converges to v in this space

$$\sup_{t \in [0,T)} |\mu_{;i}(v, v, v)|_{s \quad i} \to 0$$

as $\epsilon \to 0$.

With the commutator terms under control we can now focus on the main contributors to the dynamics. The following integration by parts result will be our primary tool.

Lemma 10.26

Let $f \in L_2$, $g \in H^1$, and $h \in H^2$ then

$$|(J f, N^{(l)}(g, h))_2| \quad \frac{c}{1+t}|f|_0|g|_1|h|_1.$$

If $f, g \in L_2$ and $h \in H^2$ then

$$|(J f, N(g, h))_2| \quad \frac{c}{1+t}|f|_0|g|_0|h|_2$$

Proof. First recall by our assumptions on the nonlinearity

$$|(t\partial_t)^l B_{ijk}(t)| \quad \frac{C}{1+t}$$

for all i, j, kExpand out the inner product as

$$|(J\ f, N\ (g,h))_2| = \ \left| \int J\ f_i\ (t\partial_t)^l B_{ijk}(t)\ J\ g_j \partial_x\ J\ h_k dx \right|$$
$$\frac{C}{1+t} \int |J\ f_i\ \ J\ g_j\ \ \partial_x J\ h_k| dx.$$

By Cauchy Schwartz and the Sobolev Embedding

$$\int |J\ f_i J\ g_j \partial_x J\ h_k| dx \qquad |J\ f|_0 |J\ g|_1 |\partial_x J\ h|_0 \quad \text{or} \quad |J\ f|_0 |J\ g|_0 |J\ h|_2.$$

The result follows from the boundedness properties of J given in Lemma 10.13. □

The "expected" product terms are are handled in an energy splitting fashion similar to [34]. We split the product terms here before deriving our full energy ODI.

Lemma 10.27

Let $f, g, h \in H_T^{s;k}$ with $1 \quad k \quad s$ and choose

$$\frac{s+2}{2} \quad \sigma \quad s$$

then the inner product terms satisfy

$$|P_{s;0;}\ (f,g,h)| \qquad \frac{1}{1+t} \|f\|_s (\|g\|_{\ ;0} \|h\|_s + \|g\|_s \|h\|_{\ ;0}) \tag{10.41}$$

and

$$|P_{s;k;}\ (f,g,h)| \qquad \frac{c}{1+t} \|f\|_{s;k} \left[\begin{array}{l} \ \ \|g\|_{\ ;0} \|h\|_{s;k} + \|g\|_{s;k} \|h\|_{\ ;0} \\ + \|g\|_{\ ;k} \|h\|_{s\ \ k+1;0} + \|g\|_{s\ \ 1;k\ \ 1} \|h\|_{\ ;k\ \ 1} \\ + \|g\|_{\ ;k\ \ 1} \|h\|_{s;k\ \ 1} \end{array} \right].$$

Proof. Firstly

$$|P_{s;k;}(f,g,h)| \quad \sum_{I_k} \left| \left(\partial_x^{n_1} \; {}^k S^k f, N\left(\partial_x^{n_2} S^{m_2} g, \partial_x^{n_2} S^{m_2} h\right)\right) \right|.$$

Starting with $P_{s;0;}$ we notice

$$|P_{s;0;}(f,g,h)| \quad \frac{c}{1+t}\|f\|_{n_1;0}\left[\|g\|_{n_2;0}\|h\|_{n_3+2;0} + \|g\|_{n_2+1;0}\|h\|_{n_3+1;0}\right],$$

and then break the n_i's sum into pieces. Break where $n_3 + 2 \quad \sigma$ to

$$|P_{s;0;}(f,g,h)| \quad \frac{c}{1+t}\|f\|_s\|g\|_s\|h\|_{\;;0}.$$

The compliment of these indicies is $n_3 \quad \sigma \quad 1$ which implies $n_2 \quad \sigma$ and by assumption $n_3 < n_1 \quad 1$ so we have

$$|P_{s;0;}(f,g,h)| \quad \|f\|_s\|g\|_{\;;0}\|h\|_s,$$

adding these gives us (10.41).

For $k \quad 1$ we need to bound the inner product terms of the form

$$I(k,\mathbf{m},\mathbf{n}) = \left| \left(\partial_x^{n_1} \; {}^k S^k f, N\left(\partial_x^{n_2} S^{m_2} g, \partial_x^{n_2} S^{m_2} h\right)\right) \right|,$$

where

$$\mathbf{m} = (m_1, m_2, m_3) \quad \mathbf{n} = (n_1, n_2, n_3).$$

By Lemma 10.26

$$I(k,\mathbf{m},\mathbf{n}) \quad \frac{c}{1+t}\|f\|_{n_1;k}\|g\|_{n_2+m_2;m_2}\|h\|_{n_3+m_3+2;m_3}$$

or

$$I(k, \mathbf{m}, \mathbf{n}) \quad \frac{c}{1+t} kfk_{n_1;k} kgk_{n_2+m_2+1;m_2} khk_{n_3+m_3+1;m_3}.$$

Assume that $m_3 = k$. This implies $m_1 = m_2 = 0$ and $n_3 < n_1 \quad k$. Thus

$$I(k, (0,0,k), \mathbf{n}) \quad \frac{c}{1+t} kfk_{n_1;k} \left[\ kgk_{n_2;0} khk_{n_3+k+2;k} \quad or \quad kgk_{n_2+1;0} khk_{n_3+k+1;k} \ \right].$$

We split the terms first into those which satisfy $n_3 + k + 2 \quad \sigma$. For these we have

$$I(k, (0,0,k), \mathbf{n}) \quad \frac{c}{1+t} kfk_{s;k} kgk_{s \quad k;0} khk_{\quad ;k}$$

and in the compliment $n_3 + k + 2 \quad \sigma$ we get $n_2 + 1 \quad s + 2 \quad \sigma \quad \sigma$. Thus

$$\begin{aligned} I(k, (0,0,k), \mathbf{n}) \quad & \frac{c}{1+t} kfk_{s;k} kgk_{n_2+1;0} khk_{n_3+k+1;k} \\ & \frac{c}{1+t} kfk_{s;k} kgk_{\quad ;0} khk_{s;k} \end{aligned}$$

Therefore when $m_3 = k$ we have

$$I(k, (0,0,k), \mathbf{n}) \quad \frac{c}{1+t} kfk_{s;k} \left[\ kgk_{s \quad k;0} khk_{\quad ;k} + kgk_{\quad ;0} khk_{s;k} \ \right]. \tag{10.42}$$

Now assume $m_2 = k$ so $m_1 = m_3 = 0$ and

$$I(k, (0,k,0), \mathbf{n}) \quad \frac{c}{1+t} kfk_{n_1;k} \left[\ kgk_{n_2+k;k} khk_{n_3+2;0} \quad or \quad kgk_{n_2+k+1;k} khk_{n_3+1;0} \ \right].$$

Then we split again but this time at $n_3 + 2 \quad \sigma$ giving us

$$I(k, (0,k,0), \mathbf{n}) \quad \frac{c}{1+t} kfk_{s;k} kgk_{s;k} khk_{\quad ;0}.$$

Then on the compliment $n_2 + k + 1 \quad \sigma$ and we get

$$I(k,(0,k,0),\mathbf{n}) \quad \frac{c}{1+t}\|f\|_{s;k}\|g\|_{\;;k}\|h\|_{s\quad k+1;0}$$

which gives us

$$I(k,(0,k,0),\mathbf{n}) \quad \frac{c}{1+t}\|f\|_{s;k}\left[\|g\|_{s;k}\|h\|_{\;;0}+\|g\|_{\;;k}\|h\|_{s\quad k+1;0}\right]. \tag{10.43}$$

The remaining terms now all satisfy $m_2, m_3 < k$. Here we split at $n_3+2+k \quad 1 \quad \sigma$ to get

$$I(k,\mathbf{m},\mathbf{n}) \quad \frac{c}{1+t}\|f\|_{s;k}\|g\|_{s\quad 1;k\quad 1}\|h\|_{\;;k\quad 1}.$$

The compliment of these indicies is $n_2+1+k \quad 1 \quad \sigma$ and so

$$I(k,\mathbf{m},\mathbf{n}) \quad \frac{c}{1+t}\|f\|_{s;k}\|g\|_{\;;k\quad 1}\|h\|_{s;k\quad 1},$$

summing these gives us

$$\begin{aligned} I(k,\mathbf{m},\mathbf{n}) \quad & \frac{c}{1+t}\|f\|_{s;k}[\|g\|_{s\quad 1;k\quad 1}\|h\|_{\;;k\quad 1}+\|g\|_{\;;k\quad 1}\|h\|_{s;k\quad 1}] \\ & \frac{c}{1+t}\|f\|_{s;k}[\|g\|_{s\quad 1;k\quad 1}\|h\|_{\;;k\quad 1}+\|g\|_{\;;k\quad 1}\|h\|_{s;k\quad 1}] \end{aligned} \tag{10.44}$$

The result follows by adding (10.42), (10.43), and (10.44).

□

We now derive a general system of energy ODIs. This will allow us to split at whatever energy levels we like base on convenience.

Theorem 10.28

Let v be an $H_T^{s;2}$ solution to the smoothed equation (10.14) on some interval $[0,T)$, with $s \quad 2$ and let

$$\frac{s+2}{2} \quad \sigma \quad s,$$

then the solution's energies satisfy

$$
\begin{aligned}
E'_{s;0} &\quad \frac{c}{1+t} E^{1/2}_{\;;0} E_{s;0}, \\
E'_{s;1} &\quad \frac{c}{1+t} E^{1/2}_{\;;0} E_{s;1} + \frac{c}{1+t} E^{1/2}_{s;1} E^{1/2}_{s;0} [E^{1/2}_{\;;1} + E^{1/2}_{\;;0}] \\
&\qquad + E^{1/2}_{s;1} |\mu_{\;;1}(v)|_{s\;\;1},
\end{aligned}
\tag{10.45a}
$$

and

$$
\begin{aligned}
E'_{s;2} &\quad \frac{c}{1+t} E^{1/2}_{\;;0} E_{s;2} + \frac{c}{1+t} E^{1/2}_{s;2} [E^{1/2}_{s\;\;1;0} E_{\;;2} + E^{1/2}_{s;1}, E^{1/2}_{\;;1}] \\
&\qquad + E^{1/2}_{s;2} (|\mu_{\;;1}(v)|_{s\;\;1} + |\mu_{\;;2}(v)|_{s\;\;2}),
\end{aligned}
$$

for all $t \in [0,T)$. Here we have made the abbreviation $\mu_{\;;i}(v,v,v) = \mu_{\;;i}(v)$.

Proof. By Lemma 10.22 we have

$$E'_{s;0} \quad |P_{s;0}(v)|$$

where for this proof we've abbreviated $P_{s;k;}(v,v,v) = P_{s;k}(v)$ from (10.32). By Lemma 10.27 we see

$$|P_{s;k;}(v,v,v)| \quad \frac{c}{1+t} \|v\|_{s;k} \left[\begin{array}{c} \|v\|_{\;;0} \|v\|_{s;k} + \|v\|_{\;;k} \|v\|_{s\;\;k+1;0} \\ + \|v\|_{s\;\;1;k\;\;1} \|v\|_{\;;k\;\;1} + \|v\|_{\;;k\;\;1} \|v\|_{s;k\;\;1} \end{array} \right].$$

In particular

$$
\begin{aligned}
|P_{s;0}(v)| &\quad \frac{c}{1+t} \|v\|_{s;0} \left[\quad \|v\|_{\;;0} \|v\|_{s;0} + \|v\|_{s;0} \|v\|_{\;;0} \right] \\
&\quad \frac{c}{1+t} E^{1/2}_{\;;0} E_{s;0}.
\end{aligned}
$$

Then we have

$$\begin{aligned} E^{0}_{\ell;1} \quad & jE^{0}_{s;0}j + jP_{s;1}(v)j + \sum_{n_1=1}^{s} j(\partial_x^{n_1 \ \ 1} Sv, \mu_{\ ;1}(v))_2 j \\ & jP_{s;0}(v)j + jP_{s;1}(v)j + E^{1=2}_{s;1} j\mu_{\ ;1}(v)j_0 \end{aligned}$$

and again by Lemma 10.27

$$jP_{s;1}(v)j \quad \frac{c}{1+t} kvk_{s;1} \left[\begin{array}{c} kvk_{\ ;0} kvk_{s;1} + kvk_{\ ;1} kvk_{s;0} \\ + kvk_{s \ \ 1} kvk_{\ ;0} + kvk_{\ ;0} kvk_{s;0} \end{array} \right]$$

and so

$$\begin{aligned} jP_{s;1}(v)j \quad & \frac{c}{1+t} kvk^2_{s;1} kvk_{\ ;0} + \frac{c}{1+t} + kvk_{s;1} kvk_{s;0} [kvk_{\ ;1} + kvk_{\ ;0}] \\ & \frac{c}{1+t} E_{s;1} E^{1=2}_{\ ;0} + \frac{c}{1+t} E^{1=2}_{s;1} E^{1=2}_{s;0} \left[E^{1=2}_{\ ;1} + E^{1=2}_{\ ;0} \right], \end{aligned}$$

combining this with the $P_{s;0}(v)$ bound gives the desired result.

Finally we have

$$\begin{aligned} E^{0}_{s;2} \quad & jE^{0}_{s;0}j + jE^{0}_{s;1}j + jP_{s;2}(v)j + \sum_{n_1=2}^{s} j(\partial_x^{n_1 \ \ 2} S^2 v, \partial_x^{n_1 \ \ 2} \mu_{\ ;2}(v,v))j_0 \\ & jE^{0}_{s;0}j + jE^{0}_{s;1}j + jP_{s;2}(v)j + E^{1=2}_{s;2} j\mu_{\ ;2}(v,v)j_{s \ \ 2}. \end{aligned}$$

Again we use Lemma 10.27

$$jP_{s;2}(v)j \quad \frac{c}{1+t} kvk_{s;2} \left[\begin{array}{c} kvk_{\ ;0} kvk_{s;2} + kvk_{\ ;2} kvk_{s \ \ 1} \\ + kvk_{s \ \ 1;1} kvk_{\ ;1} + kvk_{\ ;1} kvk_{s;1} \end{array} \right]$$

and so

$$
\begin{aligned}
|P_{s;2}(v)| &\le \frac{c}{1+t}\|v\|^2_{s;2}\|v\|_{\sigma;0} + \frac{c}{1+t}\|v\|_{s;2}\|v\|_{s-1}\|v\|_{\sigma;2} + \frac{c}{1+t}\|v\|_{s;2}\|v\|_{s;1}\left[\|v\|_{\sigma;1} + \|v\|_{\sigma;0}\right] \\
&\le \frac{c}{1+t}E_{s;2}E^{1/2}_{\sigma;0} + \frac{c}{1+t}E^{1/2}_{s;2}\left[E^{1/2}_{s-1;0}E^{1/2}_{\sigma;2} + E^{1/2}_{s;1}E^{1/2}_{\sigma;1} + E^{1/2}_{s;1}E^{1/2}_{\sigma;0}\right],
\end{aligned}
$$

Combining with the previous results gives the bound. □

We now switch over to the actual energy levels of interest ℓ and h.

Theorem 10.29

Let v and $H_T^{h;2}$ solution to (10.14) for some fixed $\epsilon, T > 0$. Then if $2 \le h$ and if

$$(h+2)/2 \le l \le h - 2,$$

v's associated energies satisfy the following differential inequalities. For the low energies we have

$$E'_{\ell;0} \le \frac{c}{1+t}E^{3/2}_{\ell;0} \tag{10.46a}$$

$$E'_{\ell;1} \le \frac{c}{1+t}E^{1/2}_{\ell;0}E_{\ell;1} + c\epsilon\, E^{1/2}_{\ell;1}E^{1/2}_{h;0}(1 + E^{1/2}_{h;0}) \tag{10.46b}$$

$$E'_{\ell;2} \le \frac{c}{1+t}E^{1/2}_{\ell;1}E_{\ell;2} + c\epsilon\, E^{1/2}_{\ell;2}E^{1/2}_{h;1}(1 + E^{1/2}_{h;1}) \tag{10.46c}$$

and the high energies satisfy

$$
\begin{aligned}
E'_{h;0} &\le \frac{c}{1+t}E^{1/2}_{\ell;0}E^{1/2}_{h;0} \\
E'_{h;1} &\le \frac{c}{1+t}E^{1/2}_{\ell;1}E^{1/2}_{h;1} + E^{1/2}_{h;1}|\mu_{;1}(v)|_{h-1} && (10.47a) \\
E'_{h;2} &\le \frac{c}{1+t}E^{1/2}_{\ell;1}E_{h;2} + \frac{c}{1+t}E^{1/2}_{h;2}E^{1/2}_{h;0}E^{1/2}_{\ell;2} + E^{1/2}_{h;2}(|\mu_{;1}(v)|_{h-1} + |\mu_{;2}(v)|_{h-2}) && (10.47b)
\end{aligned}
$$

Proof. Bound (10.46a) follows immediately from (10.45a) in Theorem 10.28 with $s = \sigma = l$.

For (10.46b) and (10.46c) we use Theorem 10.28 with $\sigma = \ell$ and $s = h$ and so

$$
\begin{aligned}
E^{0}_{s;1} \quad & \frac{c}{1+t}E^{1/2}_{\ell;0}E_{\ell;1} + \frac{c}{1+t}E^{1/2}_{\ell;1}E^{1/2}_{\ell;0}[E^{1/2}_{\ell;1} + E^{1/2}_{\ell;0}] + E^{1/2}_{\ell;1}|\mu_{\ ;1}(v,v)|_{\ell \quad 1} \\
& \frac{c}{1+t}E^{1/2}_{\ell;0}E_{\ell;1} + E^{1/2}_{\ell;1}|\mu_{\ ;1}(v,v) + H\ v|_{s \quad 1}.
\end{aligned}
$$

Recall that $v \in H^{h;2}_{T}$ and so (10.40a) in Corollary 10.25 gives

$$
\begin{aligned}
|\mu_{\ ;1}(v)|_{\ell \quad 1} \quad & c_2\epsilon\ E^{1/2}_{\ell+1;0}(1 + E^{1/2}_{\ell+1;0}) \\
& c_2\epsilon\ E^{1/2}_{h;0}(1 + E^{1/2}_{h;0})
\end{aligned}
$$

and similarly

$$
|\mu_{\ ;2}(v,v)|_{\ell \quad 1;1} \quad c_1\epsilon\ E^{1/2}_{h;1}(E^{1/2}_{h;1} + 1).
$$

The first bound finishes (10.47a) and to finish the low energy bounds we have

$$
\begin{aligned}
E^{0}_{s;2} \quad & \frac{c}{1+t}E^{1/2}_{\ell;0}E_{\ell;2} + \frac{c}{1+t}E^{1/2}_{\ell;2}[E^{1/2}_{\ell \quad 1;0}E^{1/2}_{\ell;2} + E_{\ell;1}] \\
& \qquad + (|\mu_{\ ;1}(v,v)|_{\ell \quad 1;1} + |\mu_{\ ;2}(v)|_{\ell \quad 1;1})E^{1/2}_{\ell;2} \\
& \frac{c}{1+t}E^{1/2}_{\ell;1}E_{\ell;2} + (|\mu_{;}\ (v,v)|_{\ell \quad 1} + |\mu_{\ ;2}(v)|_{\ell \quad 1;1})E^{1/2}_{\ell;2}.
\end{aligned}
$$

Using the same commutator bounds as above gives the desired result.

The high energy results are obtained by letting $s = h$ and $\sigma = \ell$ in Theorem 10.28 and leaving the commutator terms alone. □

For most of the local theory equation (10.47a) can be simplified giving us a more compact way to write the ODIs.

Corollary 10.30

Let v and $H^{h;2}_{T}$ solution to (10.14) for some fixed $\epsilon, T > 0$. If $2 \quad h$ and

$$
(h+2)/2 \quad l \quad h \quad 2,
$$

then v's associated energies satisfy the following differential inequalities.

$$E'_{\ell,i} \le \frac{c}{1+t} E^{1/2}_{\ell,i-1} E_{\ell,i} + c\epsilon E^{1/2}_{\ell,i} E^{1/2}_{h,i-1}\left(1 + E^{1/2}_{h,i-1}\right), \tag{10.48}$$

$$E'_{h,i} \le \frac{c}{1+t} E^{1/2}_{\ell,i} E^{1/2}_{h,i} + E^{1/2}_{h,i} \sum_{k=1}^{i} |\mu_{\,,k}|_{h-k}. \tag{10.49}$$

For $i = 1, 2$.

Here we collect the raw energy estimates needed to put all of our solutions into the same $H^{h,2}_T$. Since our final solution will inherit the high energy estimates we take care to follow how bounds on the lower energies control the growth rate of higher energies.

Theorem 10.31

Let v be a solution to (10.14) in $H^{h,2}_T$ and assume on some interval $[0, T)$ that:

$E^{1/2}_{\ell,i}(t) \le \Lambda_i$ for $i = 0, 1, 2$, then

$$E^{1/2}_{h,i}(t) \le (1+t)^{c\,\,_i/2}\left[E^{1/2}_{h,i}(0) + t\sum_{k=0}^{i} M_{\,,i,h}\right] \tag{10.50a}$$

Where $M_{\,,0,h} = 0$, and

$$M_{\,,i,s}(T) = \sup_{t\in[0,T)} |\mu_{\,,i}(v)|_{s-i}(t).$$

If

$$E^{1/2}_{\ell,j-1}(t) \le \Lambda_{j-1} \quad \text{and} \quad E^{1/2}_{h,j-1}(t) \le M_{j-1} \quad j = 1, 2$$

then there exists and $\epsilon_j(T, M_{j-1}, E_{\ell,j-1}(0))$ so that for all $\epsilon < \epsilon_j$ we have

$$E^{1/2}_{\ell,j}(t) \le 2E^{1/2}_{\ell,j}(0)(1+t)^{\frac{c}{2}\,\,_{j-1}}. \tag{10.50b}$$

.

If

$$E^{1=2}_{\ell;j}(t) \quad \Lambda_j \quad \text{and} \quad E^{1=2}_{h;0}(t) \quad M_0$$

then

$$E^{1=2}_{h;2}(t) \quad (1+t)^{c \quad 1=2}\left[E^{1=2}_{h;2}(0) + t\left(\frac{c}{2}M_0\Lambda_2 + \sum_{k=0}^{2} M_{\;;i;h}\right)\right]. \tag{10.50c}$$

Proof. To condense notation we will suppress the inputs of the ϵ_i's, $M_{\;;i;s}$'s, and the $\mu_{\;;i}$'s. For (10.50a) we set $M_{\;;0;h} =$ then the ODI's (10.49) of Corollary 10.30 gives us

$$\begin{aligned} E'_{h;i} \quad & \frac{c}{1+t}E^{1=2}_{\ell;i}E^{1=2}_{h;i} + E^{1=2}_{h;i}\sum_{k=0}^{i} j\mu_{\;;k}j_h \quad k \\ & \frac{c\Lambda_i}{1+t}E^{1=2}_{h;i} + E^{1=2}_{h;i}\sum_{k=0}^{i} M_{\;;k;h}(T). \end{aligned}$$

Lemma 11.10 implies

$$E^{1=2}_{h;i}(t) \quad (1+t)^{c \quad i=2}\left[E^{1=2}_{h;i}(0) + t\sum_{k=0}^{i} M_{\;;k;h}(T)\right]$$

as required.

For (10.50b) let $j = 1, 2$, $E^{1=2}_{\ell;j \quad 1} \quad \Lambda_{j \quad 1}$, and $E_{h;j \quad 1} \quad M_{j \quad 1}$. Then the ODIs (10.48) become

$$E'_{\ell;j} \quad \frac{c\Lambda^{1=2}_{j \quad 1}}{1+t}E_{\ell;j} + c\epsilon E^{1=2}_{\ell;j}M^{1=2}_{j \quad 1}\left(1 + M^{1=2}_{j \quad 1}\right).$$

If we let

$$\epsilon_j \quad \frac{E^{1/2}_{h,j}(0)}{cT\left(M^{1/2}_{j-1} + M_{j-1}\right)}$$

then $\epsilon < \epsilon_j$ implies

$$E'_{\ell,j} \quad \frac{c\Lambda^{1/2}_{j-1}}{1+t} E_{\ell,j} + \frac{E^{1/2}_{h,j}(0)}{T} E^{1/2}_{\ell,j}.$$

Again Lemma 11.10 gives the desired result.

The bound (10.50c) follows similarly by using ODI (10.47b) of Theorem 10.29.

□

Manipulating the ODIs in Theorem 10.28 to make them amenable with the ODI results in Section 11.4 will give us our energy estimates. Importantly, all of these estimates will hold on a uniform time interval that depends only on the size of the intial data in $\tilde{H}^{l,2}$.

Theorem 10.32

Let $|u_0|_{\ell,1} \quad M$ and define

$$T_\ell = \quad 1 + 2^{-\frac{2}{cM}}.$$

For all ϵ sufficiently small, any v which is an $H^{h,2}_{T_\ell}$ solution to (10.14) satisfies the following polynomial energy estimates on $[0, T_\ell)$.

For the low energies

$$E^{1/2}_{\ell,i}(t) \quad 2E^{1/2}_{\ell,i}(0)(1+t)^{cM/2} \quad 4E^{1/2}_{\ell,i}(0), \tag{10.51a}$$

and for the high energy

$$E_{h;i}^{1/2}(t) \leq (1+t)^{cM/2}\left[E_{h;i}^{1/2}(0) + t\sum_{j=0}^{i} M_{;i;h}(T_\ell)\right]. \tag{10.51b}$$

Here $i = 0, 1, 2$, and $M_{;i;h}$ are as defined in Theorem 10.31.

Proof. Let $[0, T(\epsilon))$ be v 's interval of existence. Since the energy of our solutions is continuous there is some $0 < \tau < T(\epsilon)$ so that

$$E_{\ell;0}^{1/2}(t) \leq 2E_{\ell;0}^{1/2}(0) \leq 2M \quad t \in [0, \tau)$$

Let $T_{0;}$ be the largest such τ. Then by Theorem 10.29 we see that

$$E_{\ell;0}' \leq \frac{2M}{1+t} E_{\ell;0}$$

for all $t \in [0, T_{0;})$. By Gronwall we have

$$E_{\ell;0}(t) \leq E_{\ell;0}(0)(1+t)^{cM}.$$

If we consider $t < T_\ell$ then

$$E_{\ell;0}(t) \leq 2E_{\ell;0}(0).$$

By maxamality of $T_{0;}$ necessarily $T_\ell < T_{0;}$ and thus (10.51a) holds for $i = 0$ on $[0, T_\ell)$ for all sufficiently small $\epsilon > 0$. Thus for all $t \in [0, T_\ell)$

$$E_{\ell;0}^{1/2}(t) \leq 2^{1/2} E_{\ell;0}^{1/2}(0) \leq 2M.$$

The rest of the argument is a cascading application of the bounds in Theorem 10.31. For (10.51b) with $i = 0$ we use our bound $E_{\ell;0}^{1/2} \leq 2M$ on $[0, T_\ell)$, (10.50a) and the definition of $T_`$:

$$
\begin{aligned}
E_{h;0}^{1=2}(t) \quad & E_{h;0}^{1=2}(0)(1+t)^{cM} \\
& 2E_{h;0}^{1=2}(0).
\end{aligned}
$$

Recall $M_{\ ;0;h} = 0$. Now using our bounds on $E_{`;0}$ and $E_{h;0}$ on $[0, T_`)$ we can apply (10.50b) and our definition of $T_`$

$$
\begin{aligned}
E_{`;1}^{1=2}(t) \quad & 2E_{`;1}^{1=2}(0)(1+t)^{cM} \\
& 4M.
\end{aligned}
$$

The $E_{h;1}$, $E_{h;2}$, and $E_{`;2}$ follow similarly.

□

Definition 10.7. Let $T(\epsilon)$ is the maximal time interval for the existence of a smoothed solution v then Theorem 10.32 guarentees us that for sufficiently some fixed small ϵ_0

$$
\inf_{< \ 0} T(\epsilon) = T \ \ > T_`.
$$

We call $[0, T \ )$ our uniform interval of existence.

Throughout our high energy estimates we made use of the fact that our local existence result in Theorem 10.20 gave us solutions which were $H_T^{h+1;2}$ without much cost. In order to show the v are Cauchy we will need to be more careful.

Theorem 10.33

ϵ be as in Theorem 10.32 and T as in Definition 10.7. If v is the corresponding solution to (10.14) in $H_T^{h+1;2}$, then v satisfies the energy estimates

$$
E_{h+1;0}(t) \quad 2\delta^{\ \ 2} E_{h;0}(0),
$$

$$E^{1=2}_{h+1;1}(t) \quad c(T\ )\delta\ ^1E^{1=2}_{h;1}(0)(1+\delta\ ^1E^{1=2}_{h;1}),$$

and

$$E^{1=2}_{h+1;2}(t) \quad c(T\ )\delta\ ^1E^{1=2}_{h;2}(0)(1+\delta\ ^1E^{1=2}_{h;2})$$

Where $c(T\ ) \quad 0$.

Proof. By Theorem 10.20 our approximations are $H^{h+1;2}_{T}$ for some T. Notice Theorem 10.32 holds for all h and so in particular it holds for $h+1$. Specifically, (10.51b) holds on $[0,T\ )$

$$E_{h+1;0}(t) \quad 2E_{h+1;0}(0).$$

Notice $E_{s;k}(0) = jJ\ u_0j^2_{s;k}$ and so by Lemma 10.13

$$E_{h+1;0}(t) \quad 2\delta\ ^2E_{h;0}(0).$$

For $E_{h+1;1}$ we first use eqrefeq:Ehi bounds poly smooth to get

$$\begin{aligned} E^{1=2}_{h+1;1}(t) \quad & (1+t)^{cE^{1=2}_{h;0}(0)}\left[E^{1=2}_{h+1;1}(0)+tM_{\ ;1;h+1}\right] \\ & (1+T\ )^{cE^{1=2}_{h;0}(0)}\left[E^{1=2}_{h+1;1}(0)+T\ M_{\ ;1;h+1}\right]. \end{aligned}$$

By equation (10.39a) of Corollary 10.25

$$M_{\ ;1;h+1} \quad cE^{1=2}_{h+1;0}(1+E^{1=2}_{h+1;0}).$$

Using this and the argument above gives

$$
\begin{aligned}
E^{1=2}_{h+1;1}(t) \quad & (1+T\ )^{cE^{1=2}_{h;0}(0)}\left[E^{1=2}_{h+1;1}(0)+T\ cE^{1=2}_{h+1;0}(1+E^{1=2}_{h+1;0})\right]. \\
& (1+T\ )^{cE^{1=2}_{h;0}(0)}\left[\delta\ ^{1}E^{1=2}_{h;1}(0)+\delta\ ^{1}T\ cE^{1=2}_{h;0}(1+\delta\ ^{1}E^{1=2}_{h;0})\right] \\
& c(T\ )\delta\ ^{1}E^{1=2}_{h;1}(1+\delta\ ^{1}E^{1=2}_{h;1}).
\end{aligned}
$$

The $E_{h+1;2}$ term follows similarly.

□

10.6 Convergence

We are now ready to construct out solution to (9.1). We will show that for sufficiently small ϵ_0 the solutions to (10.14) form a Cauchy sequence in $H^{h;2}_{T}$. To begin let $\epsilon_1 < \epsilon_2 < \epsilon_0$ where ϵ_0 is chosen as in Theorem 10.32. Let v_1 and v_2 be the solutions to respective smoothed equations. The dynamics of $w = v_1 \quad v_2$ are then given by

$$
\begin{aligned}
\mathsf{L}\ _1 w = \ & J\ _1 N\ _1(w,w) + J\ _1 N\ _1(w,v_2) + J\ _1 N\ _1(v_2,w) \\
& + (\mathsf{L}\ _2 \quad \mathsf{L}\ _1)v_2 + J\ _1 N\ _1(v_2,v_2) \quad J\ _2 N\ _2(v_2,v_2)
\end{aligned}
\tag{10.53}
$$

with zero initial data. We wrap up the forcing terms.

Definition 10.8. We call

$$
D(\epsilon, v_2) = (\mathsf{L}\ _2 \quad \mathsf{L}\ _1)v_2 + J\ _1 N\ _1(v_2,v_2) \quad J\ _2 N\ _2(v_2,v_2)
$$

the difference operator.

To distinguish energies of w from v_2 we let

$$
\mathsf{E}_{s;k} = E_{s;k}[w(\ ,t)] \quad and \quad E_{s;k} = E_{s;k}[v(\ ,t)]. \tag{10.54}
$$

To get the dynamics of w's S-derivatives we use Lemma 10.9 with the nonlinearity

$$N(f,g) = \frac{1}{2}\left(N\left(f,f\right) + N\left(f,g\right) + N\left(g,f\right)\right)$$

and forcing term $F(x,t) = D(\epsilon, v_2)$.

To get the higher derivatives we use the same argument as in Theorem 10.28 but we cannot use integration by parts to reduce the highest differentiation on v_2. The energy dynamics of the difference equation are then

$$\begin{aligned} E^{0}_{s;0} = {} & P_{s;0;\ 1}(w,w,w) + P_{s;0;\ 1}(w,v_2,w) + P_{s;0;\ 1}(w,w,v_2) \\ & + \sum_{n_1=0}^{s} (\partial_x^{n_1} w\ ,\ J_{\ 1} N_{\ 1}(w, \partial_x^{n_1} v_2))_2 + \partial_x^{n_1} D(\epsilon, v_2), \end{aligned}$$

$$\begin{aligned} E^{0}_{s;1} = E^{0}_{s;0} & + P_{s;0;1;\ 1}(w,w,w) + P_{s;0;1;\ 1}(w,v_2,w) + P_{s;0;1;\ 1}(w,w,v_2) \\ & + \sum_{n_1=1}^{s} \left(\partial_x^{n_1\ 1} S w\ ,\ J_{\ 1} N_{\ 1}(w, \partial_x^{n_1\ 1} S v_2)\right)_2 + \partial_x^{n_1\ 1} S D(\epsilon, v_2) \\ & + \left(\partial_x^{n_1\ 1} S w, \partial_x^{n_1\ 1}\left[\mu_{\ 1;1}(w,w,w) + \mu_{\ 1;1}(w,w,v_2) + \mu_{\ 1;1}(w,v_2,w)\right]\right)_2 \end{aligned}$$

$$\begin{aligned} E^{0}_{s;2} = E^{0}_{s;1} & + P_{s;0;2;\ 1}(w,w,w) + P_{s;0;2;\ 1}(w,v_2,w) + P_{s;0;2;\ 1}(w,w,v_2) \\ & + \sum_{n_1=2}^{s} \left(\partial_x^{n_1\ 2} S^2 w\ ,\ J_{\ 1} N_{\ 1}(w, \partial_x^{n_1\ 2} S^2 v_2)\right)_2 + \partial_x^{n_1\ 2} S^2 D(\epsilon, v_2) \\ & \quad + \left(\partial_x^{n_1\ 2} S^2 w\ ,\ \partial_x^{n_1\ 2}\left[\mu_{\ 1;2}(w,w,w)\right]\right)_2 \\ & \quad + \left(\partial_x^{n_1\ 2} S^2 w\ ,\ \partial_x^{n_1\ 2}\left[\mu_{\ 1;2}(w,w,v_2) + \mu_{\ 1;2}(w,v_2,w)\right]\right)_2 . \end{aligned}$$

We note that the decay in time of the coefficients is necessary only for global existence. We carried it along in the energy estimates in Theorem 10.28 so that they may be passed along into the non smoothed setting. For the convergence result we will ignore it and just use the fact that the coefficients are $C^2(\mathsf{R})$ in time. First we bound the difference term.

Lemma 10.34

Let f 2 $H_T^{r+1;k}$ for some $0 < T$ then the difference operator $D(\epsilon, f)$ satisfies

$$jD(\epsilon, f)j_{r \quad ;k} \quad c\epsilon \; jfj_{r+1;k}(1 + jfj_{r+1;k}).$$

Proof. We handle the linear and non linear pieces separately. For the former

$$\begin{aligned} j(L_{\,1} \quad L_{\,2})fj_{r \quad ;k} \quad & jJ_{\,1}A\partial_x J_{\,1}f \quad A\partial_x fj_{r \quad ;k} + jJ_{\,2}A\partial_x J_{\,2}f \quad A\partial_x fj_{r \quad ;k} \\ &= jJ_{\,1}J_{\,1}A\partial_x f \quad A\partial_x fj_{r \quad ;k} + jJ_{\,2}J_{\,2}A\partial_x f \quad A\partial_x fj_{r \quad ;k} \end{aligned}$$

and then by Lemma 10.13 we get

$$\begin{aligned} j(L_{\,1} \quad L_{\,2})fj_{r \quad ;k} \quad & c\epsilon \; j\partial_x fj_{r;k} \\ & c\epsilon \; jfj_{r+1;k}. \end{aligned}$$

We attack the nonlinear part similarly.

$$\begin{aligned} jJ_{\,1}N_{\,1}(f,f) \quad J_{\,2}N_{\,2}(f,f)j_{r \quad ;k} \quad & jJ_{\,1}N_{\,1}(f,f) \quad N_{B@_x}(f,f)j_{r \quad ;k} \\ & + jJ_{\,2}N_{\,2}(f,f) \quad N_{B@_x}(f,f)j_{r \quad ;k}. \end{aligned}$$

Each is handled the same so we consider a general ϵ. Notice

$$\begin{aligned} J\;N\;(f,f) \quad N_{B@_x}(f,f) = {} & J\;[N_{B@_x}(J\;f \quad f, J\;f \quad f) + N_{B@_x}(J\;f \quad f, f)] \\ & + J\;[N_{B@_x}(f, J\;f \quad f) + N_{B@_x}(f,f)] \quad N_{B@_x}(f,f). \end{aligned}$$

Therefore we have

$$\begin{aligned} jJ\;N\;(f,f) \quad N_{B@_x}(f,f)j_{r \quad ;k} \; . \; & jN_{B@_x}(J\;f \quad f, J\;f \quad f)j_{r \quad ;k} + jN_{B@_x}(J\;f \quad f, f)j_{r \quad ;k} \\ & + jN_{B@_x}(f, J\;f \quad f)jj_{r \quad ;k} + jJ\;[N_{B@_x}(f,f)] \\ & \quad N_{B@_x}(f,f)j_{r \quad ;k}. \end{aligned}$$

and so

$$
\begin{aligned}
|J\,N(f,f) \quad N_{B\partial_x}(f,f)|_{r\ ;k} \lesssim{}& |J\,f \quad f|^2_{r\ +1;k} + |J\,f \quad f|_{r\ +1;k}|f|_{r\ +1;k} \\
&+ \epsilon\ |N_{B\partial_x}(f,f)|_{r;k} \\
\lesssim{}& \epsilon^2\ |f|^2_{r+1;k} + \epsilon\ |f|_{r+1;k}|f|_{r\ +1;k} + \epsilon\ |f|^2_{r+1;k}.
\end{aligned}
$$

Summing the bounds gives the desired result.

□

Lemma 10.35

Let $s \quad 2$ and

$$\frac{s+2}{2} \quad \sigma \quad s$$

Then the energies of w satisfy

$$
\begin{aligned}
E'_{s;0} \quad & \left(E^{1/2}_{\ ;0} + E^{1/2}_{\ ;0}\right) E_{s;0} + E^{1/2}_{\ ;0} E^{1/2}_{s;0} E^{1/2}_{s;0} \\
& + E^{1/2}_{s;0} E^{1/2}_{\ ;0} E^{1/2}_{s+1;0} + c\epsilon E^{1/2}_{s;0} E^{1/2}_{s+1;0}(1 + E^{1/2}_{s+1;0}),
\end{aligned}
\tag{10.56a}
$$

$$
\begin{aligned}
E'_{s;1} \quad & \left[E^{1/2}_{\ ;0} + E^{1/2}_{\ ;0}\right] E_{s;1} \\
& + E^{1/2}_{s;1} \left[E^{1/2}_{s;0} E^{1/2}_{\ ;1} + E^{1/2}_{\ ;1} E^{1/2}_{s;0} + E^{1/2}_{s;1} E^{1/2}_{\ ;0}\right] \\
& + E^{1/2}_{s;1} E^{1/2}_{\ ;0} E^{1/2}_{s+1;1} + c\epsilon E^{1/2}_{s;1} E^{1/2}_{s+1;1}(1 + E^{1/2}_{s+1;1}) \\
& + E^{1/2}_{s;1} \left[|\mu_{\ ;1}(w,w,w)|_{s\ 1} + |\mu_{\ 1;1}(w,w,v_2)|_{s\ 1} + |\mu_{\ 1;1}(w,v_2,w)|_{s\ 1}\right],
\end{aligned}
\tag{10.56b}
$$

and

$$
\begin{aligned}
E'_{s;2} \quad & \left[E^{1/2}_{\ ;0} + E^{1/2}_{\ ;0}\right] E_{s;2} \\
& + E^{1/2}_{s;2} \left[E^{1/2}_{s;0} E_{\ ;2} + E^{1/2}_{s;1} E^{1/2}_{\ ;1} + E^{1/2}_{s;2} E^{1/2}_{\ ;0} + E^{1/2}_{\ ;2} E^{1/2}_{s;0} + E^{1/2}_{s;1} E^{1/2}_{\ ;1} + E^{1/2}_{\ ;1} E^{1/2}_{s;1}\right] \\
& + E^{1/2}_{s;2} E^{1/2}_{\ ;0} E^{1/2}_{s+1;2} + c\epsilon E^{1/2}_{s;2} E^{1/2}_{s+1;0}(1 + E^{1/2}_{s+1;0}) \\
& E^{1/2}_{s;2} \left[|\mu_{\ 1;2}(w,w,w)|_{s\ 2} + |\mu_{\ 1;2}(w,w,v_2)|_{s\ 2} + |\mu_{\ 1;2}(w,v_2,w)|_{s\ 2}\right].
\end{aligned}
\tag{10.56c}
$$

Proof. Each energy have high order terms in v_2 and difference terms that will bounded in the same way so we bound them first here. By Lemma 10.22

$$
\begin{aligned}
|D(\epsilon, v_2)|_{s;i} &\leq c\epsilon |v_2|_{s+1;i}(1 + |v_2|_{s;k}) \\
&\leq c\epsilon E_{s+1;i}^{1/2}(1 + E_{s+1;i}^{1/2})
\end{aligned} \tag{10.57a}
$$

and

$$
\begin{aligned}
\left| \sum_{n_1=i}^{s} \left(\partial_x^{n_1-i} w\,,\; J_{-1} N_{-1}(w, \partial_x^{n_1-i} S^i v_2)\right)_2 \right| &\leq \frac{c}{1+t} E_{s;i}^{1/2} |w|_{2;i} |\partial_x S^i v_2|_s \\
&\leq c E_{s;i}^{1/2} E_{2;i}^{1/2} E_{s+1;i}^{1/2} \\
&\leq c E_{s;i}^{1/2} E_{;i}^{1/2} E_{s+1;i}^{1/2} .
\end{aligned} \tag{10.57b}
$$

Pure ∂_x: For (10.56a) we first bound the product terms with (10.41) of Lemma 10.27. Explicitly

$$
\begin{aligned}
|P_{s;0;-1}(w, w, w)| &\leq c E_{;0}^{1/2} E_{s;0}, \\
|P_{s;0;-1}(w, w, v_2) + P_{s;0;-1}(w, w, v_2)| &\leq c E_{;0}^{1/2} \left[E_{s;0}^{1/2} E_{;0}^{1/2} + E_{s;0}^{1/2} E_{;0}^{1/2} \right].
\end{aligned}
$$

The higher order v_2 terms are bounded by (10.57b) with $i = 0$ and the difference terms by 10.57a.

One S: For (10.56b) we note that the pure w terms are bounded in the same way we bounded $E_{s;1}^{0}$ in Theorem 10.28 and so

$$
\begin{aligned}
&\left| P_{s;1;-1}(w, w, w) + \sum_{n_1=1}^{s} \left(\partial_x^{n_1-1} S w, \partial_x^{n_1-1} [\mu_{-1;1}(w, w, w)]\right)_2 \right| \\
&\qquad \leq E_{;0}^{1/2} E_{s;1} + E_{s;1}^{1/2} E_{s;0}^{1/2} E_{;1}^{1/2} + E_{s;1}^{1/2} |\mu_{;1}(w, w, w)|_{s-1}.
\end{aligned}
$$

For the mixed terms we have by Lemma 10.27 for $k \geq 1$

$$|P_{s,k,-1}(w,v_2,w)| \le \frac{c}{1+t}|w|_{s,k}\begin{bmatrix} |v_2|_{\infty,0}|w|_{s,k} + |v_2|_{s,k}|w|_{\infty,0} \\ +|v_2|_{\infty,k}|w|_{s} + |v_2|_{s,k-1}|w|_{\infty,k-1} \\ +|v_2|_{\infty,k-1}|w|_{s,k-1} \end{bmatrix}. \qquad (10.58a)$$

Simplifying (10.58a) for $k = 1$ gives us

$$\begin{aligned} P_{s,1,-1}(w,v_2,w) &\le \frac{c}{1+t}|w|_{s,1}\left[|v|_{\infty,0}|w|_{s,1} + |v|_{\infty,1}|w|_{s} + |v|_{s,1}|w|_{\infty,0}\right] \\ &\lesssim E_{s,1}^{1/2}\left[E_{\infty,0}^{1/2}E_{s,1}^{1/2} + E_{\infty,1}^{1/2}E_{s,0}^{1/2} + E_{s,1}^{1/2}E_{\infty,0}^{1/2}\right]. \end{aligned}$$

We note that the same bound holds for $P_{s,0,1,-1}(w,w,v_2)$.

The mixed commutator terms are handled with a quick Cauchy Schwartz inequality inequality

$$|\left(\partial_x^{n_1-1}Sw\ ,\ \partial_x^{n_1-1}[\mu_{-1,1}(w,w,v_2) + \mu_{-1,1}(w,v_2,w)]\right)_2| \lesssim E_{s,1}^{1/2}(|\mu_{-1,1}(w,w,v_2)|_{s-1} + |\mu_{-1,1}(w,v_2,w)|_{s-1}).$$

For the higher order v_2 terms we use (10.57b) with $i = 1$ and (10.57a) for the difference term and adding these gives the desired result.

Two S's: For (10.56c) we proceed similarly. The terms with only a w are bounded as $E_{s,2}^{0}$ was in Theorem 10.28. In particular the product terms satisfy

$$\begin{aligned} |P_{s,2,-1}(w,w,w)| &\le \frac{c}{1+t}E_{\infty,0}^{1/2}E_{s,2} + \frac{c}{1+t}E_{s,2}^{1/2}[E_{s-1,0}^{1/2}E_{\infty,2} + E_{s,1}^{1/2}E_{\infty,1}^{1/2}] \\ &\lesssim E_{\infty,0}^{1/2}E_{s,2} + E_{s,2}^{1/2}[E_{s,0}^{1/2}E_{\infty,2} + E_{s,1}^{1/2}E_{\infty,1}^{1/2}], \end{aligned}$$

and the pure commutator term can be broken into

$$\begin{aligned} &|\sum_{n_1=2}^{s}\left(\partial_x^{n_1-2}S^2w\ ,\ \partial_x^{n_1-2}\left[\mu_{-1,2}(w,w,w)\right]\right)_2| \\ &\qquad \lesssim E_{s,2}^{1/2}\left[|\mu_{-1,2}(w,w,w)|_{s-1}\right]. \end{aligned}$$

For the mixed terms we again use (10.58a) with $k = 2$ to get

$$|P_{s;2;\ 1}(w, v_2, w)| \quad |w|^2_{s;2}|v_2|_{\ ;0} + |w|_{s;2}|v_2|_{s;2}|w|_{\ ;0} + |w|_{s;2}|v_2|_{\ ;2}|w|_{s} + |w|_{s;2}|v_2|_{s;1}|w|_{\ ;1} + |w|_{s;2}|v_2|_{\ ;1}|w|_{s;1}$$

$$E_{s;2}E^{1/2}_{\ ;0} + E^{1/2}_{s;2}E^{1/2}_{s;2}E^{1/2}_{\ ;0} + E^{1/2}_{s;2}E^{1/2}_{\ ;2}E^{1/2}_{s;0} + E^{1/2}_{s;2}E^{1/2}_{s;1}E^{1/2}_{\ ;1} + E^{1/2}_{s;2}E^{1/2}_{\ ;1}E^{1/2}_{s;1}. \qquad (10.59a)$$

A quick check shows that $P_{s;2;\ 1}(w, w, v_2)$ is also bounded by (10.59a). The mixed commutators are bounded by

$$\sum_{n_1=2}^{s} \left| \left(\partial_x^{n_1\ \ 2} S^2 w\ ,\ \partial_x^{n_1\ \ 2}\left[\mu_{\ 1;2}(w, w, v_2) + \mu_{\ 1;2}(w, v_2, w)\right]\right)_2 \right|$$

$$E^{1/2}_{s;2}\left(|\mu_{\ 1;2}(w, w, v_2)|_{s\ \ 2} + |\mu_{\ 1;2}(w, v_2, w)|_{s\ \ 2}\right)$$

The higher order v_2 terms are again handled with (10.57b) and the difference term by (10.57a). Adding these and the bounds of (10.59) gives the desired result.

□

The ODI's are rather involved so we stress the important pieces. Firstly, they are closed in w but lose regularity in v_2. However, these lost regularity terms are always multiplied by either the lowest w energy, which we expect to be small, or ϵ. Finally, the coefficients of the highest power of the top derivative E present only depends on the low energies of w and v_2.

With the ODI's for the difference of solutions in hand we can now show the sequence is Cauchy. As has been our strategy we work up from the lowest energies to the highest in stages.

Theorem 10.36

Let T and ϵ be as in Theorem 10.32 then

$$E_{\ell;i}^{1/2}(t) \leq c(T)\epsilon,$$

for all $t \in [0, T)$ and $i = 0, 1, 2$.

Proof. Letting $s = \sigma = \ell$ in Lemma 10.35 we get

$$\begin{aligned} E'_{\ell;0} &\leq \left(E_{\ell;0}^{1/2} + 2E_{\ell;0}^{1/2}\right) E_{\ell;0} + \\ &\quad + E_{\ell;0} E_{h;0}^{1/2} + c\epsilon E_{\ell;0}^{1/2} E_{h;0}^{1/2} (1 + E_{h;0}^{1/2}). \end{aligned}$$

Working on $[0, T)$ allows us to use Theorem 10.32 to get

$$E_{\ell;0} \leq E_{h;0} \leq c,$$

absorbing these constants simplifies the ODI to

$$E'_{\ell;0} \leq cE_{\ell;0} + c\epsilon E_{\ell;0}^{1/2}.$$

Corollary 11.11 together with the fact that $E_{\ell;0}(0) = 0$ implies

$$\begin{aligned} E_{\ell;0}^{1/2}(t) &\leq \frac{c\epsilon}{2} e^{ct/2} \\ &\leq c_1(T)\epsilon. \end{aligned}$$

For $E_{\ell;1}$ we argue similarly and use Theorem 10.32 to get uniform bounds on

$$E_{\ell;i} \leq E_{h;i} \leq c$$

for $i = 0, 1$ on the same time interval as above. Absorbing these constants and plugging our new estimate of $E_{\ell;0}$ into the ODI (10.56b) of Lemma 10.35 gives us

$$
\begin{aligned}
E'_{\ell,1} \quad & cE_{\ell,1} + E^{1/2}_{\ell,1}\left(\epsilon E^{1/2}_{\ell,1} + cE^{1/2}_{\ell,1} + c\epsilon\right) + c\epsilon E^{1/2}_{\ell,1} \\
& \quad + E^{1/2}_{\ell,1}\left(|\mu\ _{,1}(w,w,w)|_{l\ \ 1,0} + |\mu\ _{,1}(w,w,v_2)|_{l\ \ 1,0} + |\mu\ _{,1}(w,v_2,w)|_{l\ \ 1,0}\right) \\
& cE_{\ell,1} + c\epsilon E^{1/2}_{\ell,1} \\
& \quad + E^{1/2}_{\ell,1}\left(|\mu\ _{,1}(w,w,w)|_{l\ \ 1,0} + |\mu\ _{,1}(w,w,v_2)|_{l\ \ 1,0} + |\mu\ _{,1}(w,v_2,w)|_{l\ \ 1,0}\right).
\end{aligned}
$$

The first commutator is handled with Corollary 10.25

$$
\begin{aligned}
|\mu\ _{,1}(w,w,w)|_{l\ \ 1,0} \quad & c\left(E^{1/2}_{\ell,0} + E_{\ell,0}\right) \\
& c\epsilon(1+\epsilon) \\
& c\epsilon.
\end{aligned}
$$

Theorem 10.24 bounds the mixed terms by

$$
\begin{aligned}
|\mu\ _{,1}(w,w,v_2)|_{l\ \ 1,0} \quad & c\left(E^{1/2}_{\ell,0} + E^{1/2}_{\ell,0}E^{1/2}_{\ell,0}\right) \\
& c\epsilon
\end{aligned}
$$

which gives the simplified ODI

$$
E'_{\ell,1} \quad cE_{\ell,1} + c\epsilon E_{\ell,1}.
$$

Arguing as we did with $E_{\ell,0}$ gives the desired result. $E_{\ell,2}$ follows by the same argument as for $E_{\ell,1}$. □

Thus our sequence v is Cauchy in $H^{\ell,2}_T$. We now improve this to the higher energies.

Theorem 10.37

Let T be as in Theorem 10.33 and choose our initial smoothing parameter $\delta = \epsilon^p$ for $p < 1/2$, then for sufficiently small $\epsilon > 0$ we have

$$E_{h,i}^{1/2}(t) \le c(T^*)\epsilon^{1-2p},$$

for all $t \in [0, T^*)$ and $i = 0, 1, 2$.

Proof. Working on $[0, T^*)$ allows us to use Theorem 10.32 we get the uniform bounds

$$E_{\ell,i} \le E_{h,i} \le c. \tag{10.60a}$$

By Theorem 10.33 we have

$$E_{h+1,i} \le c\delta^{-2} E_{h,i}(0)$$

for $i = 0, 1, 2$.

We begin with $E_{h,0}$. Letting $s = h$ and $\sigma = \ell$ in (10.56b) of Lemma 10.35 we can use the bounds (10.60a) and the decay from Theorem 10.36 to get the simplified energy ODI

$$\begin{aligned} E'_{h,0} &\le cE_{h,0} + c\epsilon E_{h,0}^{1/2} + \epsilon E_{h,0}^{1/2} E_{h+1,0}^{1/2} + c\epsilon E_{h,0}^{1/2} E_{h+1,0}^{1/2}(1 + E_{h+1,0}^{1/2}) \\ &\le cE_{h,0} + c\epsilon E_{h,0}^{1/2} + \epsilon\delta^{-1} E_{h,0}^{1/2} + c\epsilon\delta^{-2} E_{h,0}^{1/2} \\ &\le cE_{h,0} + c\epsilon^{1-2p} E_{h,0}^{1/2}. \end{aligned}$$

Using Corollary 11.11 as we did in Theorem 10.36 we get

$$E_{h,0}^{1/2}(t) \le c_0(T^*)\epsilon^{1-2p}.$$

We use the same bounds to simplify the ODI (10.56b) of Lemma 10.35

$$
\begin{aligned}
E^{0}_{h;1} \quad & cE_{h;1} + cE^{1=2}_{h;1}\left[\epsilon^{2 \quad 2p} + \epsilon\right] + \epsilon E^{1=2}_{h;1} E^{1=2}_{h+1;1} + c\epsilon E^{1=2}_{h;1} E^{1=2}_{h+1;1}(1 + E^{1=2}_{h+1;1}) \\
& \quad + E^{1=2}_{s;1}\left[j\mu_{\ ;1}(w, w, w)j_{h \quad 1} + j\mu_{\ 1;1}(w, w, v_2)j_{h \quad 1} + j\mu_{\ 1;1}(w, v_2, w)j_{h \quad 1}\right] \\
& cE_{h;1} + c\epsilon^{1 \quad 2p} E^{1=2}_{h;1} \\
& \quad + E^{1=2}_{s;1}\left[j\mu_{\ ;1}(w, w, w)j_{h \quad 1} + j\mu_{\ 1;1}(w, w, v_2)j_{h \quad 1} + j\mu_{\ 1;1}(w, v_2, w)j_{h \quad 1}\right].
\end{aligned}
$$

The remaining commutator terms can be handled by Corollary 10.25 and Theorem 10.24.For the pure terms

$$
\begin{aligned}
j\mu_{\ ;1}(w, w, w)j_{h \quad 1} \quad & c\left(E^{1=2}_{h;0} + E_{h;0}\right) \\
& c\left(\epsilon^{1 \quad 2p} + \epsilon^{2 \quad 4p}\right) \\
& c\epsilon^{1 \quad 2p},
\end{aligned}
$$

and the mixed terms are bounded by

$$
\begin{aligned}
j\mu_{\ 1;1}(w, w, v_2)j_{h \quad 1} \quad & c\left(E^{1=2}_{h;0} + E^{1=2}_{h;0} E^{1=2}_{h;0}\right) \\
& c\left(\epsilon^{1 \quad 2p} + c\epsilon^{1 \quad 2p}\right) \\
& c\epsilon^{1 \quad 2p}.
\end{aligned}
$$

These make our energy ODI

$$
E^{0}_{h;1} \quad cE_{h;1} + c\epsilon^{1 \quad 2p} E^{1=2}_{h;1}
$$

which gives the desired result. The $E_{h;2}$ result follows by the same argument.

□

Now that we have convergence in $H^{h;2}_{T}$ we can use Corollary 10.25 to improve the high energy bounds of Theorem 10.32 and remove their ϵ dependence. We collect this result and a simplification of the bounds in the following Corollary.

Corollary 10.38

If $E^{1/2}_{\ell,1}(t) \quad \Lambda_1$ on $[0, T)$, then for a sufficiently small $\epsilon < \epsilon_0(\Lambda_1, T\ )$ the corresponding smooth approximation v to (10.14) satisfy the polynomial energy estimates

$$E^{1/2}_{p,k}(t) \quad 2E^{1/2}_{p,k}(0)(1+t)^{c\ {}_1^{1/2}}$$

for $p = \ell, h$ and $i = 0, 1$. Further if $E^{1/2}_{\ell,2}(t) \quad \Lambda_2$ then

$$E^{1/2}_{h,2}(t) \quad (1+t)^{c\ {}_1}\left[2E^{1/2}_{h,2}(0) + \frac{c}{2} M_0 \Lambda_2\right]$$

or

$$E^{1/2}_{h,2}(t) \quad 2E^{1/2}_{h,2}(0)(1+t)^{c\ {}_2}$$

Chapter 11

Global Existence

We begin with a continuation condition which will motivate the future energy estimates. Then we finally reap the rewards for carrying through the scaling operators, S, in the local theory in the form of the Scaling Inequality. This will allow us to control weighted energies in terms of higher non-weighted ones. These weightings will give better decay in the dynamics of $E^{0}_{;1}$ which we derive and use to show global existence. To simplify notation in this section we will abbreviate $N_{B(t)\partial_x}(u,v)$ by $N(u,v)$.

11.1 Continutation Condition

Here we will give a basic continuation condition on our local solution to make it global. It relies on uniquesness to solutions of (9.1) which we prove now.

Theorem 11.1

If $u_1, u_2 \in H_T^{r;2}$ for $r \geq 2$ are both solutions to (9.1) on some interval $[0,T)$, then $u_1 \equiv u_2$.

Proof. By Theorem 10.37 we know for $u_0 \in \tilde{H}^{h;2}$ equation (9.1) has a solution in $H_T^{h;2}$ whose energy up to a time dependent on the size of the initial data is bounded by some $M > 0$. Assume u_1 and u_2 are two such solutions. Let $w = u_1 - u_2$ then the dynamics of w are given by

$$Lw = N(w,w) + N(w,u_2) + N(u_2,w).$$

Let $E = \frac{1}{2}|w|_0^2$. Then taking the inner product of the dynamics of w with w gives

$$E' = (w, N(w,w))_2 + (w, N(w,u_2))_2 + (w, N(u_2,w))_2$$

Before bounding the terms we apply an integration by parts to the final term

$$\begin{aligned}(w, N(u_2,w))_2 &= (w, u_{2i}B_i(t)\partial_x w)_2 \\ &= -\tfrac{1}{2}(w, \partial_x u_{2i}B_i(t)w)\end{aligned}$$

A quick Cauchy Schwartz and bounds (10.11a) and (10.11b) of Lemma 10.5 in all basic the energy ODI of the difference

$$\begin{aligned}E' &\le |w|_0^2|w|_2 + |w|_0^2|u_2|_2 + |w|_0^2|u_2|_2 \\ &\le 3ME\end{aligned}$$

Since $E(0) = 0$ Gromwall implies $E = 0$ and so $w = 0$ in L_2. Since $w \in H^h \subset H^2$ this implies that w is actually zero in H^h and so is zero in $H_T^{h,2}$. □

Our continuation principle will be for solutions to (9.1) in $H_T^{h,2}$; however, we will show that boundedness in $H_T^{\ell,1}$ will be sufficient.

Theorem 11.2

Let $u \in H_T^{h,2}$ be the solution to (9.1) for the maximal time T^*. Then either $T^* = \infty$ or

$$\lim_{t \to (T^*)^-} \|u\|_{h,2}(t) = \infty.$$

Proof. Suppose that $T^* < \infty$ and there is some M so that $\|u\|_{H_T^{h,2}} < M$. Then let

$T_0 = T \quad \tau$ and consider the shifted in time problem

$$\begin{aligned} Lu &= N(u,u) \\ u(0,x) &= u(T_0,x). \end{aligned}$$

We could approximate a solution to the shifted equation and get a solution $\tilde{u} \in H_T^{h;2}$, where T depends only on the size of the initial data, which in this case is bounded by M. Letting $\tau = T \; /2$ we get a solution on the interval $[T_0, T \; + T \; /2)$. By Theorem 11.1 the two solutions must agree on $[0, T \;)$ and so $\tilde{u}$ is a continuation of u to a larger time interval violating maximallity of T .

□

11.2 Scaling Inequality

We can now introduce our weights.

Definition 11.1. We call

$$\tilde{W}(x,t) = xI \quad tA$$

the partial weighting matrix and

$$W(x,t) = \mathrm{diag}(\langle x \quad \lambda_i t \rangle),$$

where $\lambda_i = A_{ii}$, the (full) weight matrix. We define the weighted energy of our solution to be

$$X_{\ell;k} = \frac{1}{2} \sum_{j \quad k} |W \partial_x S^k u|^2_{\ell \quad 1 \quad k}$$

The basic relationship of interest is

$$Sf \quad \mathsf{L} f = \tilde{W}\partial_x f. \tag{11.1}$$

Lemma 11.3

Let u be a solution to (9.1) in $H_T^{h;2}$. Then if $2 \quad s \quad h$ we have

$$t^2 j\mathsf{L}\partial_x^s u j_0^2 \quad c\left[E_{s;0}^2 + E_{2;0}E_{s+1;0}\right] \tag{11.2a}$$

and if $3 \quad s \quad h$ we have

$$t^2 j\mathsf{L}\partial_x^{s \quad 1} S u j_0^2 \quad c\left[E_{s;0}E_{s+1;1} + E_{s;1}^2\right] \tag{11.2b}$$

Proof. The first is a striaght forward application of the product rule, decay properties of $B_{ijk}(t)$, and the bounds (10.2c) and (10.2b)

$$\begin{aligned}
t^2 j\mathsf{L}\partial_x^s u j_0^2 = j\partial_x^s \mathsf{L} u j_0^2 = j\partial_x^s N(u,u) j_0^2 & \\
& t^2 \sum_{s_1+s_2=s} jN(\partial_x^{s_1}u, \partial_x^{s_2}u)j_0^2 \\
& t^2 \sum_{s_1 \quad 1} \quad + \quad t^2 \sum_{s_2 \quad s \quad 2} \\
& c\sum_{s_1 \quad 1} juj_{s_1}^2 juj_{s_2+2}^2 + c\sum_{s_2 \quad s \quad 2} juj_{s_1+1}^2 juj_{s_2+1}^2 \\
& c\left[E_{s;0}^2 + E_{2;0}E_{s+1;0}\right].
\end{aligned}$$

For (11.2b) first notice $[S, \mathsf{L}] = \quad \mathsf{L}$ and so

$$t^2 j\mathsf{L}\partial_x^{s \quad 1} S u j_0^2 \quad t^2 j\partial_x^{s \quad 1} S N(u,u) j_0^2 + t^2 j\partial_x^{s \quad 1} N(u,u) j_0^2.$$

The second term is handled as in (11.2a)

$$
\begin{aligned}
t^2|\partial_x^{s-1}N(u,u)|_0^2 &\le c\left[E_{s-1;0}^2 + E_{2;0}E_{s;0}\right] \\
&\le cE_{s;0}^2.
\end{aligned}
$$

Using the product rule of S breaks the first into three pieces

$$
\begin{aligned}
t^2|\partial_x^{s-1}SN(u,u)|_0^2 \le\ & t^2\sum_{s_1+s_2=s-1}|N(\partial_x^{s_1}Su,\partial_x^{s_2}u)|_0^2 + t^2\sum_{s_1+s_2=s-1}|N(\partial_x^{s_1}u,\partial_x^{s_2}Su)|_0^2 \\
&+ t^2\sum_{s_1+s_2=s-1}|N_{tB'(t)}(\partial_x^{s_1}u,\partial_x^{s_2}u)|_0^2.
\end{aligned}
$$

The decay properties of $B(t)$ makes the third term bounded as in (11.2a). For the first we again split and apply (10.2c) and (10.2b)

$$
\begin{aligned}
t^2\sum_{s_1+s_2=s-1}|N(\partial_x^{s_1}Su,\partial_x^{s_2}u)|_0^2 &= t^2\sum_{s_2\le s-2} + t^2\sum_{s_1\ge 1} \\
&\le c\sum_{s_2\le s-2}|Su|_{s_1}^2|u|_{s_2+2}^2 + c\sum_{s_1\ge 1}|Su|_{s_1+1}^2|u|_{s_2+1}^2 \\
&\le c\left[E_{s;1}E_{s;0} + E_{3;0}E_{s;0}\right] \\
&\le c\left[E_{s;1}E_{s;0} + E_{s;0}^2\right] \\
&\le cE_{s;1}^2.
\end{aligned}
$$

The second term is handled similarly but with a slightly more drastic split

$$
\begin{aligned}
t^2\sum_{s_1+s_2=s-1}|N(\partial_x^{s_1}u,\partial_x^{s_2}Su)|_0^2 &= t^2\sum_{s_2\le s-3} + t^2\sum_{s_1\ge 2} \\
&\le c\sum_{s_2\le s-3}|u|_{s_1}^2|Su|_{s_2+2}^2 + c\sum_{s_1\ge 2}|u|_{s_1+1}^2|Su|_{s_2+1}^2 \\
&\le c\left[E_{s-1;0}E_{s;1} + E_{3;0}E_{s+1;1}\right] \\
&\le c\left[E_{s;1}^2 + E_{s;0}E_{s+1;1}\right].
\end{aligned}
$$

Putting all these together gives the desired result.

$\square$

With the dynamics bounded we now bound the weighted energies in terms of the non-weighted ones.

Lemma 11.4

Let u be a solution to (9.1) in $H_T^{h,2}$. If $3 \le \ell \le h-2$ then u satisfies

$$|\tilde{W}\partial_x\partial_x^{\ell-1}u|_0^2 \le c\left[E_{\ell,1}+E_{\ell,0}^2\right]. \tag{11.3a}$$

and if $4 \le \ell \le h-2$ the u satisfies

$$|\tilde{W}\partial_x S\partial_x^{\ell-2}u|_0^2 \le c\left[E_{\ell,2}+E_{\ell,1}^2\right]. \tag{11.3b}$$

Moreover the full weighted energy for such ℓ satisfies

$$\mathcal{X}_{\ell,i} \le E_{\ell,i+1}+cE_{\ell,i}^2. \tag{11.4}$$

Proof. For (11.3a) we use (11.1) and triangle inequality

$$\begin{aligned}
|\tilde{W}\partial_x\partial_x^{\ell-1}u|_0^2 &= |S\partial_x^{\ell-1}u - tL\partial_x^{\ell-1}u|_0^2 \\
&\le |S\partial_x^{\ell-1}u|_0^2 + t^2|L\partial_x^{\ell-1}u|_0^2 \\
&\le E_{\ell,1} + t^2|L\partial_x^{\ell-1}u|_0^2.
\end{aligned}$$

Using bound (11.2a) of Lemma 11.3 with $s=\ell-1$ gives

$$\begin{aligned}
t^2|L\partial_x^{\ell-1}u|_0^2 &\le c\left[E_{\ell-1,0}^2 + E_{2,0}E_{\ell,0}\right] \\
&\le cE_{\ell,0}^2.
\end{aligned}$$

Bound (11.3b) will follow similarly.

$$
\begin{aligned}
|\tilde{W}\partial_x\partial_x^{\ell-2}Su|_0^2 &\le |S\partial_x^{\ell-2}Su|_0^2 + t^2|L\partial_x^{\ell-2}Su|_0^2\\
&\le |\partial_x^{\ell-2}S^2u|_0^2 + (\ell-2)|\partial_x^{\ell-2}Su|_0^2 + t^2|L\partial_x^{\ell-2}Su|_0^2\\
&\le E_{\ell,2} + (\ell-1)E_{\ell,1} + t^2|L\partial_x^{\ell-2}Su|_0^2.
\end{aligned}
$$

Then applying bound (11.2b) of Lemma 11.3 with $s = \ell - 1$ gives us

$$
\begin{aligned}
|\tilde{W}\partial_x\partial_x^{\ell-2}Su|_0^2 &\le c\left[E_{\ell,2} + E_{\ell,1} + E_{\ell-1,0}E_{\ell,1} + E^2_{\ell-1,1}\right]\\
&\le c\left[E_{\ell,2} + E^2_{\ell,1}\right].
\end{aligned}
$$

For the full bounds (11.4) we notice

$$
|W\partial_x u|_0^2 = |\partial_x u|_0^2 + |\tilde{W}u|_0^2.
$$

□

11.3 Energy Estimates

These are our final energy estimates needed for global existence. First we provide a lemma adding weighting to our dynamics and derive the energy ODI. Since our solution is at best in $H_T^{h,2}$ we cannot directly derive high energy estimates and so these must be inherited from our smooth approximations. The low energy dynamics are derived and weighting is added to the ODI which will give us a uniform bound on the low energies for small initial data.

Lemma 11.5

Let $g_i = \langle x - \lambda_i t\rangle$. Then if $f \in H^1$

$$|g_i^{1/2} f_i|_{L^\infty_x} \le c|f|_0 + |W\partial_x f|_0^{1/2}|f|_0^{1/2}$$

. Moreover, if u is a solution to (9.1) then

$$|g_i^{1/2}\partial_x^s u_i|_{L^\infty_x} \le c\left[E_{s;0}^{1/2} + E_{s;0}^{1/4}X_{s+1;0}^{1/4}\right]$$

and

$$|g_i^{1/2}\partial_x^{s-1} S u_i|_{L^\infty_x} \le c\left[E_{s;1}^{1/2} + E_{s;1}^{1/4}X_{s+1;1}^{1/4}\right]$$

Proof. Notice

$$|g_i f_i^2|(y) = \int_{-\infty}^{y} \partial_x(|g_i^{1/2} f_i|^2)dx = \int_{-\infty}^{y} \partial_x[g_i]|f_i|^2 dx + \int_{-\infty}^{y} \langle f_i g_i, \partial_x f_i\rangle dx$$

$\partial_x g_i$ is bounded so we have

$$\begin{aligned} |g_i f_i|^2(y) &\le \int |\partial_x[g_i] f_i^2| dx + \int |\langle f_i g_i, \partial_x f_i\rangle| dx \\ &\le c|f|_0^2 + |f|_0 |g_i \partial_x f_i|_0 \\ &\le c|f|_0^2 + |f|_0 |W\partial_x f|_0 \end{aligned}$$

as required. The bound for $\partial_x^s u$ and $S\partial_x^{s-1}$ are direct substitutions. ☐

Now we recover an analog of the polynomial growth of the high energies in Theorem 10.32.

Theorem 11.6

Let v be the smooth approximation of our solution, u, to (9.1) in $H_T^{h;2}$. If u's corresponding energy satisfies $E_{\ell;1}^{1/2}(t) \le M$ on some subinterval $[0, T)$ then

$$E^{1/2}_{h,2}(t) \leq c_2(1+t)^{cM}\left[E^{1/2}_{h,2}(0) + E_{h,2}(0)\right]$$

where $c_2 = c_2(T, M)$.

Proof. We let $E_{\ell,i,\epsilon} = E_{\ell,i}[v_\epsilon(\cdot,t)]$ and $E_{h,i,\epsilon} = E_{h,i}[v_\epsilon(\cdot,t)]$. Then since $v_\epsilon \to u$ there is an ϵ small enough so that $E^{1/2}_{\ell,1,\epsilon}(t) \leq 2M$ and so by Theorem 10.31

$$\begin{aligned} E^{1/2}_{h,1,\epsilon}(t) &\leq 2E^{1/2}_{h,1,\epsilon}(0)(1+t)^{cM} \\ &\leq c_1(T,M)E^{1/2}_{h,1}(0). \end{aligned}$$

Then using (10.50b)

$$\begin{aligned} E^{1/2}_{\ell,2,\epsilon}(t) &\leq 2E^{1/2}_{\ell,2,\epsilon}(0)(1+t)^{cM} \\ &\leq c_1(T,M)E^{1/2}_{\ell,2}(0), \end{aligned}$$

which by Corollary 10.38 gives us

$$\begin{aligned} E^{1/2}_{h,2,\epsilon}(t) &\leq (1+t)^{cM}\left[2E^{1/2}_{h,2}(0) + c_1(T,M)^2 E^{1/2}_{\ell,2}(0)E^{1/2}_{h,1}(0)\right] \\ &\leq c_2(T,M)(1+t)^{cM}\left[E^{1/2}_{h,2}(0) + E_{h,2}(0)\right] \end{aligned}$$

□

To begin the low energy estimates in ernest we differentiate (9.1) and integrate against the appropriate derivatives as in Lemma 10.28, except now the process is simpler since we no longer have terms coming from commuting with the smoothing operator.

$$E'_{\ell,1} = \sum_{I_0}(\partial_x^{n_1}u, N(\partial_x^{n_2}u, \partial_x^{n_3}u))_2 \tag{11.5a}$$

$$+ \sum_{I_1}(\partial_x^{n_1-1}Su, N^{(m_1)}(\partial_x^{n_2}S^{m_2}u, \partial_x^{n_3}S^{m_3}u))_2. \tag{11.5b}$$

Where $\Lambda_i = \Lambda_i(\ell)$ as defined in (10.33).

Importantly we have used integration by parts from Lemma 10.2 so that in (11.5a) $n_2 < n_1$ and in (11.5b) whenever $m_3 = 1$ $n_2 < n_1 \quad 1$.

The extra decay we obtain comes from the fact that for $i \neq j$ the waves do not interact for long times. This is made explicit through

$$\frac{1}{|\langle x \quad \lambda_i t\rangle\langle x \quad \lambda_j t\rangle|_{1\,x}} \quad \frac{c}{1+t}.$$

Theorem 11.7

Let u be a solution of (9.1) with X_ℓ, E_ℓ, and E_h its weighted low, low, and higher energies resepectively . If $\ell \quad 3$ then

$$\begin{aligned} E^{0}_{\ell;1} \quad & \frac{c}{(1+t)^{3/2}}\left[\left(E^{1/2}_{\ell;0} + E^{1/4}_{\ell;0}X^{1/4}_{\ell;0}\right)X^{1/2}_{\ell;1} + \left(E^{1/2}_{\ell;1} + E^{1/4}_{\ell;1}X^{1/4}_{\ell;1}\right)X^{1/2}_{\ell;0}\right]E^{1/2}_{\ell;1} \qquad (11.6) \\ & + X^{1/2}_{\ell;1}E_{\ell;1} + \frac{c}{(1+t)^{1+}}E^{3/2}_{\ell;1} \end{aligned}$$

Proof. **Pure ∂_x terms:** The (11.5a) term can be bounded as

$$\begin{aligned} \sum_{n_1=0}^{\ell}(\partial_x^{n_1}u, N(\partial_x^{n_2}u, \partial_x^{n_3}u))_2 \quad & \sum_{n=0}^{\ell}|(\partial_x^{n_1}u, N(\partial_x^{n_2}u, \partial_x^{n_3}u))_2| \\ & |B_{jik}(t)||\int \partial_x^{n_1}u_i \; \partial_x^{n_2}u_j \; \partial_x\partial_x^{n_3}u_k dx|, \end{aligned}$$

where importantly $n_2 < n_1$.

We begin away from indices where $i = j = k$. Here we have the weaker coefficient decay

$$|B_{jik}(t)||\int \partial_x^{n_1}u_i \; \partial_x^{n_2}u_j \; \partial_x\partial_x^{n_3}u_k dx| \quad \frac{1}{1+t}|\int \partial_x^{n_1}u_i \; \partial_x^{n_2}u_j \; \partial_x\partial_x^{n_3}u_k dx| \qquad (11.7)$$

$j \neq k$: We can introduce the weighting to get

$$\begin{aligned} \left| \int \partial_x^{n_1} u_i \, \partial_x^{n_2} u_j \, \partial_x \partial_x^{n_3} u_k dx \right| &\leq \frac{1}{\| g_j^{1/2} g_k^{1/2} \|_{L^\infty_x}} \int |\partial_x^{n_1} u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; g_k^{1/2} \partial_x \partial_x^{n_3} u_k | dx \\ &\leq \frac{1}{\sqrt{1+t}} \int |\partial_x^{n_1} u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; g_k^{1/2} \partial_x \partial_x^{n_3} u_k | dx. \end{aligned}$$

If $n_3 \quad n_1 \quad 2$ then we get the bound

$$\begin{aligned} \int |\partial_x^{n_1} u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; g_k^{1/2} \partial_x \partial_x^{n_3} u_k | dx &\leq \| g_k^{1/2} \partial_x \partial_x^{n_3} u_k \|_{L^\infty_x} \|\partial_x^{n_1} u_i\|_0 \| g_j^{1/2} \partial_x \partial_x^{n_2 - 1} u_j \|_0 \\ &\leq \| g_k^{1/2} \partial_x \partial_x^{n_3} u_k \|_{L^\infty_x} \; E_{n_1;0}^{1/2} \, X_{n_2;0}^{1/2} \\ &\leq \| g_k^{1/2} \partial_x \partial_x^{n_3} u_k \|_{L^\infty_x} \; E_{\ell;0}^{1/2} \, X_{\ell;0}^{1/2}. \end{aligned} \tag{11.8a}$$

Then by Lemma 11.5

$$\begin{aligned} \| g_k^{1/2} \partial_x \partial_x^{n_3} u_k \|_{L^\infty_x} &\leq c \left[E_{n_3+1;0}^{1/2} + E_{n_3+1;0}^{1/4} X_{n_3+2;0}^{1/4} \right] \\ &\leq c \left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} X_{\ell;0}^{1/4} \right]. \end{aligned} \tag{11.8b}$$

Plugging (11.8b) into (11.8a) and this into (11.7) gives us

$$|B_{jik}(t)| \left| \int \partial_x^{n_1} u_i \, \partial_x^{n_2} u_j \, \partial_x \partial_x^{n_3} u_k dx \right| \leq c \left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} X_{\ell;0}^{1/4} \right] E_{\ell;0}^{1/2} X_{\ell;0}^{1/2}.$$

If $n_3 \quad n_1 \quad 1$ then $n_2 \quad 1$ and using a similar argument as above we get the bounds

$$\begin{aligned} \int |\partial_x^{n_1} u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; g_k^{1/2} \partial_x \partial_x^{n_3} u_k | dx &\leq \| g_j^{1/2} \partial_x^{n_2} u_j \|_{L^\infty_x} \|\partial_x^{n_1} u_i\|_0 \| g_k^{1/2} \partial_x \partial_x^{n_3} u_k \|_0 \\ &\leq \| g_j^{1/2} \partial_x^{n_2} u_j \|_{L^\infty_x} \; E_{n_1;0}^{1/2} \, X_{n_3+1;0}^{1/2} \\ &\leq \| g_j^{1/2} \partial_x^{n_2} u_j \|_{L^\infty_x} \; E_{\ell;0}^{1/2} \, X_{\ell;0}^{1/2} \\ &\leq c \left[E_{n_2;0}^{1/2} + E_{n_2;0}^{1/4} X_{n_2+1;0}^{1/4} \right] E_{\ell;0}^{1/2} \, X_{\ell;0}^{1/2} \\ &\leq c \left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} X_{\ell;0}^{1/4} \right] E_{\ell;0}^{1/2} \, X_{\ell;0}^{1/2}. \end{aligned}$$

j = k ≠ i: We can introduce the weighting and use decay of coefficients as in (11.7)

$$\left|\int \partial_x^{n_1}u_i\ \partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_k\,dx\right| \le \frac{1}{\sqrt{1+t}}\int \left|g_i^{1/2}\partial_x^{n_1}u_i\ g_j^{1/2}\partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_j\right|dx. \tag{11.9a}$$

If $n_1 = 0$ then we have

$$\begin{aligned}\int \left|g_i^{1/2}u_i\ u_j\ g_j^{1/2}\partial_x u_j\right|dx &\le |g_i^{1/2}u_i|_{L^\infty_x}\, E_{0,0}^{1/2}\mathcal{X}_{1,0}^{1/2}\\ &\le c\left[E_{1,0}^{1/2} + E_{1,0}^{1/4}\mathcal{X}_{1+1,0}^{1/4}\right]E_{0,0}^{1/2}\mathcal{X}_{1,0}^{1/2}\\ &\le c\left[E_{\ell,0}^{1/2} + E_{\ell,0}^{1/4}\mathcal{X}_{\ell,0}^{1/4}\right]E_{\ell,0}^{1/2}\mathcal{X}_{\ell,0}^{1/2}.\end{aligned} \tag{11.9b}$$

If $n_3 - n_1 \ge 2$ then we bound (11.9a) as

$$\begin{aligned}\int \left|g_i^{1/2}\partial_x^{n_1}u_i\ g_j^{1/2}\partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_j\right|dx &\le |g_j^{1/2}\partial_x\partial_x^{n_3}u_j|_{L^\infty_x}\, \mathcal{X}_{n_1,0}^{1/2}E_{n_2,0}^{1/2}\\ &\le c\left[E_{n_3+1,0}^{1/2} + E_{n_3+1,0}^{1/4}\mathcal{X}_{n_3+1+1,0}^{1/4}\right]\mathcal{X}_{n_1,0}^{1/2}E_{n_2,0}^{1/2}\\ &\le c\left[E_{\ell,0}^{1/2} + E_{\ell,0}^{1/4}\mathcal{X}_{\ell,0}^{1/4}\right]\mathcal{X}_{\ell,0}^{1/2}E_{\ell,0}^{1/2}\end{aligned} \tag{11.9c}$$

and if $n_3 - n_1 \le 1$ then $n_2 \ge 1$ and so

$$\begin{aligned}\int \left|g_i^{1/2}\partial_x^{n_1}u_i\ g_j^{1/2}\partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_j\right|dx &\le |g_j^{1/2}\partial_x^{n_2}u_j|_{L^\infty_x}\, \mathcal{X}_{n_1,0}^{1/2}E_{n_3,0}^{1/2}\\ &\le c\left[E_{n_2,0}^{1/2} + E_{n_2,0}^{1/4}\mathcal{X}_{n_2+1,0}^{1/4}\right]\mathcal{X}_{n_1,0}^{1/2}E_{n_3+1,0}^{1/2}\\ &\le c\left[E_{1,0}^{1/2} + E_{1,0}^{1/4}\mathcal{X}_{1+1,0}^{1/4}\right]\mathcal{X}_{\ell,0}^{1/2}E_{\ell,0}^{1/2}\\ &\le c\left[E_{\ell,0}^{1/2} + E_{\ell,0}^{1/4}\mathcal{X}_{\ell,0}^{1/4}\right]\mathcal{X}_{\ell,0}^{1/2}E_{\ell,0}^{1/2}.\end{aligned} \tag{11.9d}$$

Adding (11.9b), (11.9c), and (11.9d) and plugging into (11.9a) gives the desired result.

i = j = k: In this case we cannot introduce any weighting and have to rely on the added decay in the coefficients $B_{iii}(t)$ and basic energy estimates. Firstly

$$|B_{iii}(t)|\left|\int \partial_x^{n_1}u_i\ \partial_x^{n_2}u_i\ \partial_x\partial_x^{n_3}u_i dx\right| \quad \frac{1}{(1+t)^{1+}}\left|\int \partial_x^{n_1}u_i\ \partial_x^{n_2}u_i\ \partial_x\partial_x^{n_3}u_i dx\right| \tag{11.10a}$$

again we split where $n_3 \quad n_1 \quad 2$ and get

$$\left|\int \partial_x^{n_1}u_i\ \partial_x^{n_2}u_i\ \partial_x\partial_x^{n_3}u_i dx\right| \quad E^{1/2}_{n_3+2,0}E^{1/2}_{n_1,0}E^{1/2}_{n_2,0}$$
$$E^{3/2}_{\ell,0}$$

and then if $n_3 \quad n_1 \quad 1$ we have

$$\left|\int \partial_x^{n_1}u_i\ \partial_x^{n_2}u_i\ \partial_x\partial_x^{n_3}u_i dx\right| \quad E^{1/2}_{n_2+1,0}E^{1/2}_{n_1,0}E^{1/2}_{n_3+1,0}$$
$$E^{3/2}_{\ell,0}.$$

Plugging these into (11.10a) gives the desired result.

One S: These terms are treated similarly though we first split our sum based on where the scaling operators land. The terms

$$|tB'_{jik}(t)|\left|\int \partial_x^{n_1\ 1}u_i\ \partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_k dx\right| \quad \frac{c}{1+t}\left|\int \partial_x^{n_1\ 1}u_i\ \partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}u_k dx\right|$$

satisfy the same bounds as the Pure ∂_x terms.

$m_3 = 1$: These terms of (11.5b) are of the form

$$B_{jik}(t)\int \partial_x^{n_1\ 1}Su_i\ \partial_x^{n_2}u_j\ \partial_x\partial_x^{n_3}Su_k dx$$

where importantly $n_3 \quad n_1 \quad 2$. We begine away from the self interacting $i = j = k$ terms so we get the same weight decay as (11.7).

$\mathbf{j \neq k}$: We introduce the weighting as

$$\int |\partial_x^{n_1-1} Su_i \ \partial_x^{n_2} u_j \ \partial_x \partial_x^{n_3} Su_k| dx \lesssim \frac{1}{\sqrt{1+t}} \int |\partial_x^{n_1-1} Su_i \ g_j^{1/2} \partial_x^{n_2} u_j \ g_k^{1/2} \partial_x \partial_x^{n_3} Su_k| dx \tag{11.11a}$$

and

$$\begin{aligned} \int |\partial_x^{n_1-1} Su_i \ g_j^{1/2} \partial_x^{n_2} u_j \ g_k^{1/2} \partial_x \partial_x^{n_3} Su_k| &\lesssim |g_j^{1/2} \partial_x^{n_2} u_j|_{L_x^\infty} X_{n_3+2;1}^{1/2} E_{n_1;1}^{1/2} \\ &\lesssim |g_j^{1/2} \partial_x^{n_2} u_j|_{L_x^\infty} X_{n_1;1}^{1/2} E_{n_1;1}^{1/2} \\ &\lesssim c\left[E_{n_2;0}^{1/2} + E_{n_2;0}^{1/4} X_{n_2+1;0}^{1/4}\right] X_{n_1;1}^{1/2} E_{n_1;1}^{1/2} \\ &\lesssim c\left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} X_{\ell;0}^{1/4}\right] X_{\ell;1}^{1/2} E_{\ell;1}^{1/2}. \end{aligned}$$

Plugging these into (11.11a) gives the desired result.

$\mathbf{j = k \neq i}$:

If $n_1 = 1$, that is if we have no pure ∂_x terms, we first perform an integration by parts to get

$$|\int Su_i \ u_j \ \partial_x Su_j dx| \lesssim |\int \partial_x Su_i u_j \ Su_j dx| + |\int Su_i \partial_x u_j Su_j dx|.$$

For the first we have the weighting

$$|\int \partial_x Su_i u_j \ Su_j dx| \lesssim \frac{1}{\sqrt{1+t}} \int |g_i^{1/2} \partial_x Su_i \ g_j u_j \ Su_j| dx$$

and then

$$\begin{aligned}\int |g_i^{1/2}\partial_x S u_i \; g_j u_j \; S u_j| dx &\le |g_j u_j|_{L^\infty_x} \; X_{2,1}^{1/2} \; E_{1,1}^{1/2} \\ &\le c\left[E_{0,0}^{1/2} + E_{0,0}^{1/4} X_{0+1,0}^{1/4}\right] X_{2,1}^{1/2} \; E_{1,1}^{1/2} \\ &\le c\left[E_{\ell,0}^{1/2} + E_{\ell,0}^{1/4} X_{\ell,0}^{1/4}\right] X_{\ell,1}^{1/2} E_{\ell,1}^{1/2}.\end{aligned}$$

For the second term we have the weighing

$$\left|\int S u_i \partial_x u_j S u_j dx\right| \le \frac{1}{\sqrt(1+t)} \int |g_i^{1/2} S u_i \; g_j \partial_x u_j \; S u_j \; | dx$$

and then

$$\begin{aligned}\int |g_i^{1/2} S u_i \; g_j \partial_x u_j \; S u_j \; | dx &\le |g_i^{1/2} S u_i|_{L^\infty_x} \; X_{1,0}^{1/2} \; E_{1,1}^{1/2} \\ &\le c\left[E_{0,1}^{1/2} + E_{0,1}^{1/4} X_{0+1,1}^{1/4}\right] X_{1,0}^{1/2} \; E_{1,1}^{1/2} \\ &\le c\left[E_{\ell,1}^{1/2} + E_{\ell,1}^{1/4} X_{\ell,1}^{1/4}\right] X_{\ell,0}^{1/2} E_{\ell,1}^{1/2}\end{aligned}$$

Now assume $n_1 > 1$. We introduce the weighting

$$\int |\partial_x^{n_1-1} S u_i \; \partial_x^{n_2} u_j \; \partial_x \partial_x^{n_3} S u_j| dx \le \frac{1}{\sqrt{1+t}} \int |g_i^{1/2} \partial_x^{n_1-1} S u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; \partial_x \partial_x^{n_3} S u_j| dx.$$

If we then split on where $n_3 \le n_1 - 3$ and get

$$\begin{aligned}\int |g_i^{1/2} \partial_x^{n_1-1} S u_i \; g_j^{1/2} \partial_x^{n_2} u_j \; \partial_x \partial_x^{n_3} S u_j| dx &\le |\partial_x \partial_x^{n_3} S u_j|_{L^\infty_x} X_{n_1,1}^{1/2} X_{n_2,0}^{1/2} \\ &\le E_{n_3+3,1}^{1/2} X_{n_1,1}^{1/2} X_{n_2,0}^{1/2} \\ &\le X_{\ell,0}^{1/2} X_{\ell,1}^{1/2} E_{\ell,1}^{1/2}\end{aligned}$$

and when $n_3 \ge n_1 - 2$ then $n_2 \le 1$ so we get

$$\int |g_i^{1/2} \partial_x^{n_1 \; 1} Su_i \; g_j^{1/2} \partial_x^{n_2} u_j \; \partial_x \partial_x^{n_3} Su_j| dx \quad |\partial_x^{n_2} u_j|_{L_x^{\infty}} \chi_{n_1;1}^{1/2} E_{n_3+2;1}^{1/2}$$

$$c \left[E_{n_2;0}^{1/2} + E_{n_2;0}^{1/4} \chi_{n_2+1;0}^{1/4} \right] \chi_{\ell;1}^{1/2} E_{\ell;1}^{1/2}$$

$$c \left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} \chi_{\ell;0}^{1/4} \right] \chi_{\ell;1}^{1/2} E_{\ell;1}^{1/2}.$$

i = j = k: In this case we cannot introduce any weighting and have to rely on the added decay in the coefficients $B_{iii}(t)$ and basic energy estimates. Firstly

$$|B_{iii}(t)| \left| \int \partial_x^{n_1 \; 1} Su_i \; \partial_x^{n_2} u_i \; \partial_x \partial_x^{n_3} Su_i dx \right| \quad \frac{1}{(1+t)^{1+}} \left| \int \partial_x^{n_1 \; 1} Su_i \; \partial_x^{n_2} u_i \; \partial_x \partial_x^{n_3} Su_i dx \right|. \tag{11.13a}$$

Again we split where $n_3 \quad n_1 \quad 3$ and get

$$\left| \int \partial_x^{n_1} u_i \; \partial_x^{n_2} u_i \; \partial_x \partial_x^{n_3} u_i dx \right| \quad E_{n_3+3;1}^{1/2} E_{n_1;1}^{1/2} E_{n_2;0}^{1/2}$$

$$E_{\ell;1}^{3/2}$$

and then if $n_3 \quad n_1 \quad 2$ we have

$$\left| \int \partial_x^{n_1 \; 1} Su_i \; \partial_x^{n_2} u_i \; \partial_x \partial_x^{n_3} Su_i dx \right| \quad E_{n_2+1;0}^{1/2} E_{n_1;1}^{1/2} E_{n_3+2;1}^{1/2}$$

$$E_{\ell;1}^{3/2}.$$

Plugging these into (11.13a) gives the desired result.

$m_2 = \mathbf{1}$**:** These terms of (11.5b) are of the form

$$B_{jik}(t) \int \partial_x^{n_1 \; 1} Su_i \; \partial_x^{n_2} Su_j \; \partial_x \partial_x^{n_3} u_k dx$$

We begin away from the self interacting $i = j = k$ terms so we get the same weight decay as (11.7).

$j \neq k$: We can introduce the weighting as

$$\int |\partial_x^{n_1-1} S u_i \, \partial_x^{n_2} S u_j \, \partial_x \partial_x^{n_3} u_k| dx \le \sqrt{\frac{1}{1+t}} \int |\partial_x^{n_1-1} S u_i \, g_j^{1/2} \partial_x^{n_2} S u_j \, g_k^{1/2} \partial_x \partial_x^{n_3} u_k| dx. \tag{11.14a}$$

If $n_3 \le n_1 - 2$ then

$$\begin{aligned}\int |\partial_x^{n_1-1} S u_i \, g_j^{1/2} \partial_x^{n_2} S u_j \, g_k^{1/2} \partial_x \partial_x^{n_3} u_k| dx &\le |g_k^{1/2} \partial_x \partial_x^{n_3} u_k|_{L_x^\infty} \, E_{n_1,1}^{1/2} \, X_{n_2+1,1}^{1/2} \\ &\le c\left[E_{n_3+1,0}^{1/2} + E_{n_3+1,0}^{1/4} X_{n_3+1+1,0}^{1/4}\right] E_{n_1,1}^{1/2} \, X_{n_1,1}^{1/2} \\ &\le c\left[E_{\ell,0}^{1/2} + E_{\ell,0}^{1/4} X_{\ell,0}^{1/4}\right] E_{\ell,1}^{1/2} X_{\ell,1}^{1/2}\end{aligned}$$

and if $n_3 = n_1 - 1$ then $n_2 \le 1$ and so

$$\begin{aligned}\int |\partial_x^{n_1-1} S u_i \, g_j^{1/2} \partial_x^{n_2} S u_j \, g_k^{1/2} \partial_x \partial_x^{n_3} u_k| dx &\le |g_j^{1/2} \partial_x^{n_2} S u_j|_{L_x^\infty} \, E_{n_1,1}^{1/2} X_{n_3+1,0}^{1/2} \\ &\le c\left[E_{n_2+1,1}^{1/2} + E_{n_2+1,1}^{1/4} X_{n_2+1+1,1}^{1/4}\right] X_{\ell,0}^{1/2} E_{\ell,1}^{1/2} \\ &\le c\left[E_{\ell,1}^{1/2} + E_{\ell,1}^{1/4} X_{\ell,1}^{1/4}\right] X_{\ell,0}^{1/2} E_{\ell,1}^{1/2}.\end{aligned}$$

Plugging this into (11.14a) gives the desired result.

$j = k \neq i$: We introduce the weighting as

$$\int |\partial_x^{n_1-1} S u_i \, \partial_x^{n_2} S u_j \, \partial_x \partial_x^{n_3} u_j| dx \le \sqrt{\frac{1}{1+t}} \int |g_i^{1/2} \partial_x^{n_1-1} S u_i \, g_i^{1/2} \partial_x^{n_2} S u_j \, \partial_x \partial_x^{n_3} u_j| dx. \tag{11.15a}$$

If $n_1 = 1$, that is if we have no pure ∂_x terms, we get

$$\int |g_i^{1/2} S u_i \, S u_j \, g_i^{1/2} \partial_x u_j| dx \le |g_i^{1/2} S u_i|_{L_x^\infty} \, E_{1,1}^{1/2} X_{1,0}^{1/2}. \tag{11.15b}$$

Applying Lemma 11.5 gives us

$$
\begin{aligned}
|g_i^{1/2} S u_i|_{L_x^\infty} &\leq c\left[E_{1;1}^{1/2} + E_{1;1}^{1/4} X_{1+1;1}^{1/4}\right] \\
&\leq c\left[E_{\ell;1}^{1/2} + E_{\ell;1}^{1/4} X_{\ell;1}^{1/4}\right].
\end{aligned}
$$

Plugging this into (11.15b) and (11.15a) gives the desired result. Now assume $n_1 > 1$.

If we then split on where $n_3 \leq n_1 - 2$ and get

$$
\begin{aligned}
\int |g_i^{1/2} \partial_x^{n_1-1} S u_i \, \partial_x^{n_2} S u_j \, g_j^{1/2} \partial_x \partial_x^{n_3} u_j| dx &\leq |g_j^{1/2} \partial_x \partial_x^{n_3} u_j|_{L_x^\infty} X_{n_1;1}^{1/2} E_{n_2;1}^{1/2} \\
&\leq c\left[E_{n_3+1;0}^{1/2} + E_{n_3+1;0}^{1/4} X_{n_3+1+1;0}^{1/4}\right] X_{n_1;1}^{1/2} E_{n_2;1}^{1/2} \\
&\leq c\left[E_{\ell;0}^{1/2} + E_{\ell;0}^{1/4} X_{\ell;0}^{1/4}\right] X_{\ell;1}^{1/2} E_{\ell;1}^{1/2}
\end{aligned}
$$

and when $n_3 \geq n_1 - 2$ then $n_2 \leq 1$ so we get

$$
\begin{aligned}
\int |g_i^{1/2} \partial_x^{n_1-1} S u_i \, \partial_x^{n_2} S u_j \, g_j^{1/2} \partial_x \partial_x^{n_3} u_j| dx &\leq |\partial_x^{n_2} S u_j|_{L_x^\infty} X_{n_1;1}^{1/2} X_{n_3+1;0}^{1/2} \\
&\leq E_{n_2+2;1}^{1/2} X_{\ell;1}^{1/2} X_{\ell;0}^{1/2} \\
&\leq E_{3;1}^{1/2} X_{\ell;0}^{1/2} E_{\ell;1}^{1/2}.
\end{aligned}
$$

Since $3 \leq \ell$ we have the desired result.

i = j = k: In this case we cannot introduce any weighting and have to rely on the added decay in the coefficients $B_{iii}(t)$ and basic energy estimates. Firstly

$$
|B_{iii}(t)| \left| \int \partial_x^{n_1-1} S u_i \, \partial_x^{n_2} S u_i \, \partial_x \partial_x^{n_3} u_i dx \right| \leq \frac{1}{(1+t)^{1+}} \left| \int \partial_x^{n_1-1} S u_i \, \partial_x^{n_2} u_i \, \partial_x \partial_x^{n_3} S u_i dx \right|. \tag{11.16a}
$$

Again we split where $n_3 \leq n_1 - 2$ and get

$$\left| \int \partial_x^{n_1} u_i \; \partial_x^{n_2} S u_i \; \partial_x \partial_x^{n_3} u_i dx \right| \le E^{1/2}_{n_3+2,0} E^{1/2}_{n_1,1} E^{1/2}_{n_2,1} \le E^{3/2}_{\ell,1}$$

and then if $n_3 \le n_1 - 1$ we have

$$\left| \int \partial_x^{n_1 - 1} S u_i \; \partial_x^{n_2} S u_i \; \partial_x \partial_x^{n_3} u_i dx \right| \le E^{1/2}_{n_2+1,1} E^{1/2}_{n_1,1} E^{1/2}_{n_3+1,1} \le E^{3/2}_{\ell,1}.$$

Plugging these into (11.16a) gives the desired result.

□

Using can now use the bounds in Lemma 11.4 we can bound some of the more complicated terms in the energy ODI (11.6). Before that though notice by Lemma 11.4

$$\begin{aligned} X_{\ell,0} &\le c\left[E_{\ell,1} + E^2_{\ell,0}\right] \\ &\le cE_{\ell,1}\left[1 + E_{\ell,1}\right] \end{aligned} \tag{11.17}$$

and

$$\begin{aligned} X_{\ell,1} &\le c\left[E_{\ell,2} + E^2_{\ell,1}\right] \\ &\le c\left[E_{h,2} + E^2_{\ell,1}\right] \\ &\le cE_{h,2}\left[1 + E_{\ell,1}\right]. \end{aligned} \tag{11.18}$$

Lemma 11.8

Let u be a solution of (9.1) with X_ℓ, E_ℓ, and E_h its weighted low, low, and higher energies resepectively . If $\ell \ge 3$ then

$$E^{0}_{`;1} \quad \frac{c\left(1+E^{3=4}_{`;1}\right)}{(1+t)^{3=2}} E^{1=2}_{h;2} E_{`;1} + \frac{c}{(1+t)^{1+}} E^{3=2}_{`;1} \tag{11.19}$$

Proof. This is a simple computation bounding the terms

$$\left[\left(E^{1=2}_{`;0} + E^{1=4}_{`;0} X^{1=4}_{`;0}\right) X^{1=2}_{`;1} + \left(E^{1=2}_{`;1} + E^{1=4}_{`;1} X^{1=4}_{`;1}\right) X^{1=2}_{`;0}\right] E^{1=2}_{`;1} + X^{1=2}_{`;1} E_{`;1}$$

Firstly using the bounds (11.17) and (11.18)

$$\begin{aligned}
\left(E^{1=2}_{`;0} + E^{1=4}_{`;0} X^{1=4}_{`;0}\right) X^{1=2}_{`;1} \quad & \left(E^{1=2}_{`;1} + E^{1=4}_{`;1} X^{1=4}_{`;0}\right) X^{1=2}_{`;1} \\
& c\left[1+E^{1=4}_{`;1}\right]\left[1+E^{1=2}_{`;1}\right] E^{1=2}_{h;2} E^{1=2}_{`;1} \\
& c\left[1+E^{3=4}_{`;1}\right] E^{1=2}_{h;2} E^{1=2}_{`;1}.
\end{aligned}$$

Then in a similar argument

$$\begin{aligned}
\left(E^{1=2}_{`;1} + E^{1=4}_{`;1} X^{1=4}_{`;1}\right) X^{1=2}_{`;0} \quad & c\left(E^{1=2}_{`;1} + E^{1=4}_{`;1} E^{1=4}_{h;2}\left[1+E^{1=4}_{`;1}\right]\right) X^{1=2}_{`;0} \\
& c\left(1+E^{1=4}_{`;1}\right) E^{1=2}_{h;2} X^{1=2}_{`;0} \\
& c\left(1+E^{1=4}_{`;1}\right)\left(1+E^{1=2}_{`;1}\right) E^{1=2}_{h;2} E^{1=2}_{`;1} \\
& c\left(1+E^{3=4}_{`;1}\right) E^{1=2}_{h;2} E^{1=2}_{`;1}.
\end{aligned}$$

Finally

$$X^{1=2}_{`;1} E_{`;1} \quad \left(1+E^{1=2}_{`;1}\right) E_{`;1}$$

giving the desired result.

□

We are nwo finally ready to prove global existence for small initial data in the $H^{`;1}_{T}$

sense.

Theorem 11.9

Let $h \quad 6$, $\ell \quad h \quad 2$ and $\ell \quad \min(3, \frac{h+2}{2})$, and u be a solution to (9.1) in $H_T^{h;2}$ where $[0, T\ )$ is the maximal interval of existence.

Fix an $M > 0$ chosen so that $p(M) = \frac{1}{2} \quad 7cM/4 > 0$. If u satisfies the smallness conditions $ju_0j_{`;1} \quad \frac{1}{2}M$ and

$$\left(\frac{E_{`;1}^{1=2}(0)}{M}\right)^{p(M)} e^{c_3 N_1} \quad \frac{e^{\quad c_3 M p(M)=}}{2^{p(M)}} \tag{11.20}$$

where

$$N_1(E_{h;2}(0)) = N_0(E_{h;2}(0))(1 + N_0^{3=4}(E_{h;2}(0))),$$
$$N_0(E_{h;2}(0)) = E_{h;2}^{1=2}(0)\left[1 + E_{h;2}^{1=2}(0)\right],$$

and M is choosen small enough so that $4cM < 1$, c_3 is a positive constant.

Proof. We will supress the inputs of N_i and $p(M)$ for notational simplicity.

Since our energies are continuous in time there is some maximal $T(M) \quad T$ so that

$$E_{`;1}^{1=2}(t) \quad M \tag{11.21}$$

for all t 2 $[t, T(M))$. Then on this interval Theorem 11.6 gives us

$$E_{h;2}^{1=2}(t) \quad c_2(1+t)^{cM}\left[E_{h;2}^{1=2}(0) + E_{h;2}(0)\right]$$
$$c_2 N_0 (1+t)^{cM}$$

This allows us to simplify the ODI (11.19) of Lemma 11.8

$$
\begin{aligned}
E'_{\ell;1} &\le \frac{c_2 N_0 \left(1 + E_{h;2}^{3/4}\right)}{(1+t)^{3/2 - cM}} E_{\ell;1} + \frac{c}{(1+t)^{1+\theta}} E_{\ell;1}^{3/2} \\
&\le \frac{c_2 N_0 \left(1 + c_2^{3/4} N_0^{3/4} (1+t)^{3cM/4}\right)}{(1+t)^{3/2 - cM}} E_{\ell;1} + \frac{c}{(1+t)^{1+\theta}} E_{\ell;1}^{3/2} \\
&\le \frac{c_3 N_1}{(1+t)^{1+p}} E_{\ell;1} + \frac{c}{(1+t)^{1+\theta}} E_{\ell;1}^{3/2} \\
&\le c_4 \left(\frac{N_1}{(1+t)^{1+p}} + \frac{M}{(1+t)^{1+\theta}} \right) E_{\ell;1}
\end{aligned}
$$

By Gronwall

$$
\begin{aligned}
E_{\ell;1}^{1/2}(t) &\le E_{\ell;1}^{1/2}(0) \exp\left[c_4 \int_0^t \frac{N_1}{(1+s)^{1+p}} + \frac{M}{(1+s)^{1+\theta}} ds \right] \\
&\le E_{\ell;1}^{1/2}(0) \exp\left[c_4 \int_0^\infty \frac{N_1}{(1+t)^{1+p}} + \frac{M}{(1+t)^{1+\theta}} dt \right] \\
&\le E_{\ell;1}^{1/2}(0) \exp\left[\frac{c_4 N_1}{p} + \frac{c_4 M}{\theta} \right].
\end{aligned}
$$

To simplify the notation we let $M_0 = E_{\ell;1}^{1/2}(0)$. Notice

$$
M_0 e^{\frac{c_3 N_1}{p}} = \left(M_0^p \, e^{c_4 N_1} \right)^{1/p}
$$

and by our smallness condition (11.20)

$$
M_0^p \, e^{c_4 N_1} \le \left(\frac{M}{2} \right)^p e^{-\frac{c_4 M p}{\theta}}
$$

which implies

$$
E_{\ell;1}^{1/2}(t) \le \tfrac{1}{2} M
$$

on the interval $[0, T(M))$. Thus $T(M)$ cannot be the maximal time on which (11.21) holds. Thereofore the bound holds on all of the existence interval meaning $E_{h;2}(t)$ is also bounded uniformly in time and so by our continuation condition of Theorem 11.2 implies the solution is actually global in time. □

11.4 Supplementary Lemmas

Here are some Lemmas concerning common ODI's in this paper.

Lemma 11.10

Let $y, z \in C^1[0,T)$ and $A, B > 0$ and assume they satisfy

$$\begin{aligned} y'(t) &\leq \frac{A}{1+t} y(t) + B y^{1/2}(t) && (11.22a) \\ z'(t) &\leq A z(t) + B z^{1/2}(t) && (11.22b) \end{aligned}$$

then

$$\begin{aligned} y^{1/2}(t) &\leq (1+t)^{A/2} \left[y_0^{1/2} + \frac{B}{2} t \right] && (11.23a) \\ z^{1/2}(t) &\leq e^{At/2} \left[z_0^{1/2} + \frac{B}{2} t \right] && (11.23b) \end{aligned}$$

for all $t \in [0,T)$. Further

$$\begin{aligned} y^{1/2}(t) &\leq 3 y_0^{1/2} \qquad t \in [0, T_1(y_0, A, B)) && (11.24a) \\ z^{1/2}(t) &\leq 3 z_0^{1/2} \qquad t \in [0, T_2(z_0, A, B)) && (11.24b) \end{aligned}$$

where

$$T_1(y_0, A, B) = \min \left\{ -1 + \left(\frac{3}{2}\right)^{\frac{2}{A}} , \frac{y_0}{B} \right\}$$

$$T_2(z_0, A, B) = \min \left\{ \frac{2}{A} \ln(3/2) , \frac{2}{B} z_0^{1/2} \right\}$$

Proof. Let $w(t) = y^{1/2}(t)$ then (11.22a) becomes

$$
\begin{aligned}
w' &\quad \frac{A}{2(1+t)}w + \frac{1}{2}B \\
w' \quad \frac{A}{2(1+t)}w &\quad \frac{1}{2}B \\
\frac{d}{dt}\left[w(t)(1+t)^{\quad A/2}\right] &\quad (1+t)^{\quad A/2}\frac{1}{2}B.
\end{aligned}
$$

Integrating both sides gives us

$$
\begin{aligned}
w(t)(1+t)^{\quad A/2} \quad w_0 &\quad \frac{1}{2}B\int_0^t (1+s)^{\quad A/2} ds, \\
&\quad \tfrac{1}{2}Bt
\end{aligned}
$$

subbing back in for y and rearranging gives (11.23a) and plugging in for T_1 gives (11.24a). A similar argument for (11.22b) gives (11.23b) and (11.24b). □

Corollary 11.11

Let $y, z \in C^1[0,T)$, $A > 0$, $B \quad > 0$ with $B \quad \to 0$ and assume the functions satisfy

$$
\begin{aligned}
y'(t) &\quad \frac{A}{1+t}y(t) + B\ y^{1/2}(t) \\
z'(t) &\quad Az(t) + B\ z^{1/2}(t)
\end{aligned}
$$

then for sufficiently small $\epsilon(T)$

$$
\begin{aligned}
y^{1/2}(t) &\quad 2y_0^{1/2}(1+t)^{A/2} \\
z^{1/2}(t) &\quad 2z_0^{1/2}e^{At/2}
\end{aligned}
$$

for all $t \in [0,T)$. Further

$$y^{1=2}(t) \quad 4y_0^{1=2} \qquad t \; 2 \; [0, T_1\,(y_0, A))$$
$$z^{1=2}(t) \quad 4z_0^{1=2} \qquad t \; 2 \; [0, T_2\,(z_0, A))$$

where

$$T_1\,(y_0, A) = \quad 1 + (2)^{\frac{2}{A}}$$
$$T_2\,(z_0, A) = \frac{2}{A}\ln(2)$$

Chapter 12

Notation

Symbol	Reference	Description
B	Definition 2.1	The reference domain of the fluid.
$\Omega(t)$	Equation (2.1)	The spatial domain of the fluid.
D_z		The derivative operator with respect to the z variables.
r_z		The gradient operator with respect to the z variables. For vector fields it is equivilent to D_z.
r_z		The divergence operator with respect to the z variables.
∂_n		The derivative in the direction of the outward normal to a (hyper/hypo)-surface.
$v \quad r_x f$		Equivalent to $D_x f v$.
D_t	Definition 2.3	The material time derivative. In spatial coordinates: $\partial_t + (u \quad r_x)$.
$\operatorname{tr} M$		The trace of a matrix M
$A^{\quad >}$		The inverse transpose of A. $(A^{>})^{\quad 1}$.
$f \; . \; g$	Equation (10.37)	f is bounded by a multiple of g. $f \quad cg$.
$\sum f_i$		A sum with coefficients potentialy supressed.
$j \; j_r$		The H^r norm.

Symbol	Reference	Description
$(\cdot,\cdot)_2$	Lemma 10.2	The L^2 inner product.
$N_X(u,v)$	Equation (9.2)	The bilinear operator uXv.
$N_{B(t)\partial_x}(u,v)$	Equation (9.3)	A sum of bilinear operators of the form $N_{B_i(t)\partial_x}(u_i,v)$.
$N(u,v)$		Abbreviation of $N_{B(t)\partial_x}(u,v)$.
$N^{(k)}_{B\partial_x}(f,g)$	Equation (10.7)	$N_{(t\partial_t)^k[B(t)]\partial_x}(f,g)$
L	Equation (10.1)	The linear operator $A\partial_x$.
L	Equation (10.1)	The time dependent linear operator $\partial_t + A\partial_x$.
S		The scaling operator $t\partial_t + x\partial_x$.
$\|u\|_{r,k}(t)$	Equation (10.4)	The time dependent norm $\sum_{0\le j\le k}|S^j u(\cdot,t)|_{r-k}$.
$\|u\|_{H_T^{r,2}}$	Definition 10.1	$\sup_{t\in[0,T)}\|u\|_{r,2}$
R	Definition 10.1	The radial derivative $x\partial_x$.
$|v|_{r,k}$	Definition 10.1	The time independent norm $\sum_{0\le j\le k}|R^j v|_{r-j}$.
ϕ_ϵ	Equation (10.13)	A smooth bump function with unit mass supported on $[-\epsilon,\epsilon]$.
$J_\epsilon[u]$	Definition 10.3	The smoothing operator $\phi_\epsilon * u$.
$J^{(l)}_\epsilon[u]$	Definition 10.3	The smoothing operator $(R^l\phi_\epsilon) * u$.
L_ϵ	Definition 10.3	The smoothed linear operator $A\partial_x J_\epsilon$.
$N_\epsilon(f,g)$	Definition 10.3	The smoothed bilinear operator $N_{B(t)\partial_x}(J_\epsilon f, J_\epsilon g)$
$\{Y, N_X\}(u,v)$	Definition 10.4	The bilinear commutator between a linear operator Y and a bilinear operator N_X. $Y[N_X(u,v)] - N_X(Yu,v) - N_X(u,Yv) - N_{[Y,X]}(u,v)$.
L_ϵ	Equation (10.18)	The time dependent smooth linear operator. $\partial_t + J_\epsilon L_\epsilon$.

Symbol	Reference	Description
$E_{s,k}[v(\cdot,t)]$	Definition 10.6	The time dependent $H_T^{s,k}$ energy. $\frac{1}{2}\|v\|_{s,k}^2(t)$
$P_{s,k,\alpha}$	Equation (10.32)	Product terms for smooth energy estimates. $\sum_{I_k(s)} \left(\partial_x^{n_1} {}^{k}S^{k} f, N^{(m_1)}(\partial_x^{n_2} S^{m_2} g, \partial_x^{n_3} S^{m_3} h)\right)$
$I_k(s)$	Equation (10.33)	Allowed powers of derivatives in the $P_{s,k,\alpha}$. See equation for details.
$[0, T_\epsilon)$	Definition 10.7	The uniform interval of existence for smooth approximations of (9.1).
$D(\epsilon, v_2)$	Definition 10.8	The forcing terms in our difference equation (10.53). $(L_{\epsilon_2} - L_{\epsilon_1})v_2 + J_{\epsilon_1}N_{\epsilon_1}(v_2, v_2) - J_{\epsilon_2}N_{\epsilon_2}(v_2, v_2)$
$\mathcal{E}_{s,k}$	Equation (10.54)	Energy of the difference between smooth approximations. $E_{s,k}[w(\cdot,t)]$.
$\tilde{W}(x,t)$	Definition 11.1	The partial weight matrix. $xI - tA$.
$W(x,t)$	Definition 11.1	The (full) weight matrix. $\mathrm{diag}(\langle x - \lambda_i t\rangle)$.
X_ℓ	Definition 11.1	The weighted energy. $\frac{1}{2}\|W\partial_x\partial_x^\ell u\|_2^2 + \frac{1}{2}\|W\partial_x\partial_x^{\ell-1} S u\|_0^2$.

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