Implicit understanding of functions in quantitative reasoning[1]

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Abstract

We present a theoretical analysis of students' implicit understanding of the concepts of variables and functions, and present a cognitive model of this understanding based on the idea that reasoning involves a successful interaction between psychological agents and the things and other people in a situation. In the first part of the paper, we provide evidence that middle- and high-school students demonstrate implicit understanding of functional relations among quantities when they reason about a physical model of linear functions. Implicit understanding is knowledge of concepts or principles that enables and constrains performance, but is not articulate. In the second part of the paper, we describe several theoretical properties of our computational model: a.) activities are modelled as interactions between a person and a situation; b.) reasoning is modelled as a form of activity that produces new information; and c.) understanding is modelled as attunement to the constraints of conceptual activities.

In most information-processing models, the symbolic expressions of some programming language are assumed to correspond to cognitive representations that are constructed through perception and reasoning. We have developed a model based on the idea that reasoning involves a successful interaction between psychological agents and the things and other people in a situation, in which only some symbolic expressions are interpreted as cognitive representations. We interpret other symbolic expressions as statements that are assumed to be true about the situation, but we do not assume that these expressions correspond to representations constructed by the person whose reasoning the model simulates. We make this distinction in order to capture several important theoretical properties in our model. Firstly, activities are modelled as interactions between a person and a situation. Secondly, reasoning is modelled as a form of activity that produces new information. Finally, understanding is modelled as attunement to the constraints of conceptual activities.

The conceptual domain that we model involves the concepts of variables and functions. The first part of this paper summarizes an empirical investigation of middle- and high-school students' implicit understanding of functional relations in the context of a physical model. In the second part of the paper, we provide a theoretical analysis of students' understanding, and describe how our model functions within this theoretical framework.

Implicit Understanding

We use the term "implicit understanding" of a concept or principle to refer to reasoning in which the concept or principle plays a functional role, but the person does not give evidence of having an explicit representation of the concept or principle. That is, implicit understanding is knowledge of concepts or principles that enables and constrains performance, but is not articulate. The concept of implicit understanding has been used to account for successful reasoning in several domains. For instance, Gelman and Gallistel (1978) inferred from pre-school children's counting performance that they have implicit understanding of the principles of one-to-one correspondence, the order of numerals, and cardinality. Children's implicit understanding also has been studied in the areas of biology (Carey, 1985; Hatano & Inagaki, 1987), physical causality (Bullock, Gelman & Baillargeon, 1982), theory of mind (Wellman & Estes, 1986), ontological categories (Keil, 1979), and physical force and motion (diSessa, 1983; McCloskey, 1983).

This study is an attempt to demonstrate the same kind of understanding by middle- and high-school students of the concepts of variables and functions when they reason about a physical model of linear function. The particular physical model that we used is sketched in Figure 1, and is a variation of a device used by Piaget, Grize, Szeminska, & Bang (1977). The device consists of a board, about a yard long, with two grooves that run lengthwise. Beside each groove is a ruler, marked in inches, and inside each groove is a small metal block with a pointer on its top surface. At the end of each groove an axle with a handle is attached to a wheel so that the axle can be turned. Metal spools of various circumferences can be placed on the two axles, and each spool is attached by a string to a block. As a spool turns, the string winds around the spool and the block moves toward the spool. The two axles can be linked together so the spools turn simultaneously, or separated so that they turn independently.

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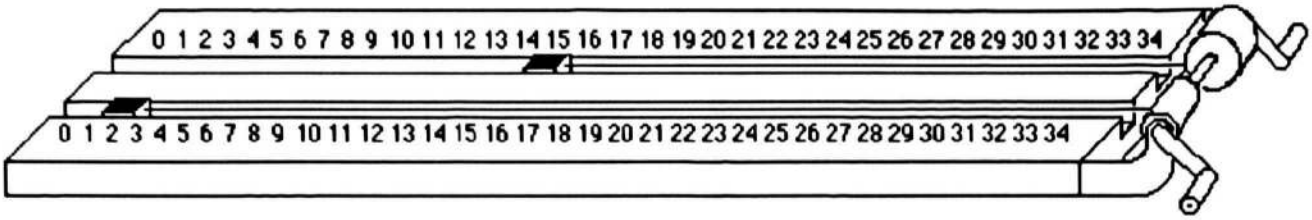

Figure 1. Physical model of linear functions.

In this physical model, the distance that a block moves is directly proportional to the number of turns of the spool, and the position of a block after a given number of turns can be found by adding the initial position to the distance travelled. Because of these properties, the winch device can be thought of as an embodiment of the linear function $y = mx + b$, where y is the position of the block after a certain number of turns, b is the block's starting position, m is the circumference of the spool, and x is the number of turns.

Participants and Procedure

There were three groups of students: four pairs of seventh graders who had no prior algebra instruction, seven pairs of ninth graders who had one year of algebra instruction, and four pairs of 11th graders who had two years of algebra instruction. Sessions of approximately 45 minutes were conducted with pairs of students of the same sex. Students were asked a series of increasingly difficult problems by an interviewer.

Results

Overall, students were quite successful on these tasks. Transcriptions of the interview sessions were used to infer whether students demonstrated implicit understanding of linear functions. Two examples are discussed below to provide a sense of the kind of interactions that we observed.

Multiple-solution Problem. Approximately 20 minutes into the interview, after an introduction to the winch, students were asked "How could you make both blocks get to 24 at the same time?", and then "Can you think of another way?". We then asked for solutions with unequal starting positions, with different spool sizes, and with different numbers of turns. Symbolically, this question required finding conditions such that $m_1x_1 + b_1$, the expression for one track of the winch, equals $m_2x_2 + b_2$, the expression for the second track of the winch, and both equal 24. The easiest solution is to have all the variables equal, which is the solution that every pair of students chose initially. If one of the variables is made unequal, such as making one spool size larger than the other, then another variable has to be made unequal in the opposite direction by an appropriate amount.

The operations that students performed to arrive at their answers were functionally equivalent to algebraic operations; however, the understanding they showed about variables and functions was implicit in their reasoning about quantities in the physical system, rather than being about the symbolic expressions of algebra. We analyzed students' solutions to infer features of the problem spaces in which they reasoned about the winch. Implicit understanding of a concept will be reflected in the problem-solving operators that are part of a person's problem space.

For this problem, we focussed on students' implicit understanding of variables. For example, one 7th-grade pair said:

S2: You could put a bigger one [spool] and put the other one [block] ahead. [...] 24, this one [a 6-spool] will be up there in 4 turns, this one [a 3-spool] in 8.

INT: Okay.

S2: So that's like 2 turns of this [red's spool] to 1 turn of this [blue's spool].

To say how the two blocks can get to 24 at the same time, given that one of the spools is larger than the other, students need to understand: (a) that a specified property functions as a variable, in this example, that spool size is a parameter of the system; (b) the way in which a variable influences the behavior of the system, in this case, that the block with the larger spool will move further on each turn; and (c) how the variables interact, for instance, that a variable other than spool size can be varied to compensate for the effect of the difference in spool sizes. All students were able to give answers in which variables compensated for one another.

Quantitative Inference Problem. The second example occurred approximately 35 minutes into the interview and had a situation with a 3-spool for the red block and a 6-spool for the blue block. The axles were linked and both blocks started at 0. The formulaic representation for this question is $y1 = 3x$ and $y2 = 6x$. The question had five parts, the first two are discussed briefly:

(a) When you turn the handle, will the red block ever get ahead? (When or why not?)

(b) How far ahead will the blue block be after 4 turns?

To answer questions such as these, students needed to make inferences based on functional relations among quantitative properties of the systems. To the extent that they reasoned successfully, we can infer that they had (at least) implicit understanding of those functions.

The first part of this question is about a general property of the two functions - one of the functions is monotonically greater than the other. An 11th grader said that the red block would never get ahead, and when asked "Why?" replied:

S1: Because right off from the start, after the first turn this one [6-spool] has a 3-inch lead, so it can never get ahead. Every turn it'll gain 3 inches.

The second part of this question, which called for a calculation, was "How far ahead will the blue block be after four turns?" or, in symbolic terms, evaluate $6x - 3x$ for $x = 4$. One of the 7th-grade pairs said:

S2: Let's see [measures intervals with fingers from 0 on blue side] 1, 2, 3, so 24. And then this one will be — what was that, 3? — so 3 times 4 is 12. So it'll be —
S1: 12 inches — double the amount.
S2: So, however many turns we turn together, this one [blue] will always be double the amount, so like if it was 30, then this one would be at 15.
INT: Ok, and how far ahead would the blue block be after 4 turns?
S1&2: 12.

This pair found the blue block's position by counting increments on the winch, and inferred the red block's position by multiplication. They then stated a general property of the two functions: that the position of the blue block will always be twice the position of the red. On this problem, all students recognized general properties of the relationship between the two functions, and were able to use the functional relations to calculate specific answers.

Overall, our results indicate that children have the capacity for reasoning about functional relations among quantities in the context of a physical model. We are currently exploring the hypothesis that this implicit understanding could be used as a foundation for learning about symbolic representations of algebra, and thus provide an environment for developing conceptual understanding of the mathematical symbols that denote variables and functions.

Theoretical Analysis

In this section, we discuss the theoretical framework that we are using to develop a computational model of the reasoning described above. We discuss implicit understanding by developing the idea that reasoning involves a successful interaction between a person and a situation, which includes both objects and people (Brown, Collins & Duguid, 1989; Greeno, 1989; Jordan, 1987; Laboratory of Comparative Human Cognition, 1983; Lave, 1988; McCabe & Balzano, 1986; Shaw, Turvey, & Mace, 1982; Suchman, 1987). Within our analysis representations play a significant role in reasoning; however, we also assume some features of the environment can support cognitive activity without being mentally represented. Gibson's (1979) concepts of direct perception of affordances are used to describe this support. Our analysis extends Gibson's concept of affordances to include affordances for reasoning since we consider reasoning to be a form of activity that produces new information.

Action Schemata

Figure 2 presents a taxonomy of some of the elements of our model. Each branch of the tree will be discussed in turn. A *state of affairs*, a term from situation theory (Barwise, 1989; Devlin, 1988), is represented by a true statement about a situation, such as "Emmy is eating yogurt". A state of affairs is a relation between objects that holds in the world. We characterize the space in which the model operates as a set of states of affairs, and will later distinguish between types of states of affairs and how they are used in reasoning.

An action schema is a description of an agent-situation interaction. This concept of a schema is different from that developed in cognitive science during the 1970's, where it was assumed to consist of a representation of abstract concepts which could be used to construct a representation of the current situation. In contrast, we assume that a schema is a pattern of interaction that a person has learned to participate in. This is consistent with Bartlett's use of the term: "'Schema' refers to an active organisation of past reactions, or of past experiences, which must always be supposed to be operating in any well-adapted organic response" (Bartlett, 1932, p. 201).

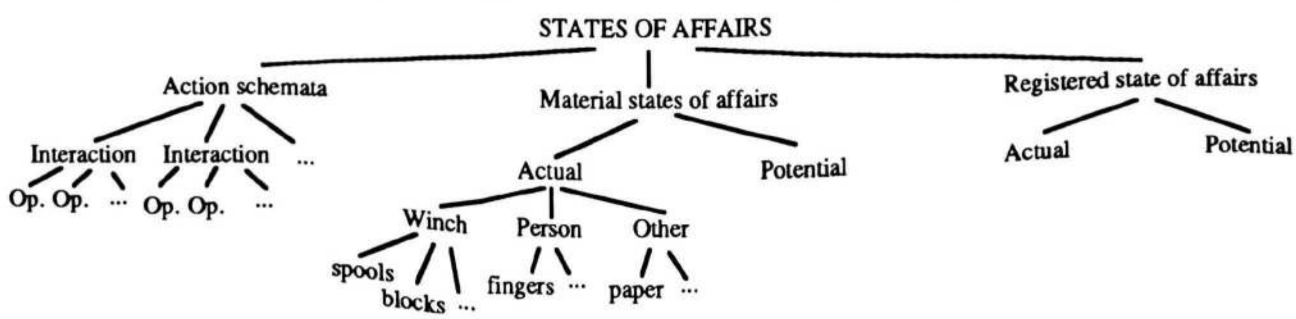

Figure 2. Taxonomy of model.

We have analyzed some of the activities that students engaged in with the winch into a network of interaction-operation units. The term *interaction* will be used to emphasize that activity is jointly constituted by an agent and the setting in which the activity occurs. In this view we follow other theorists, including Dewey and Bentley (1949), Lave (1988), McCabe and Balzano (1986), and Riegel and Meacham (1978). The term *operation* will be used to refer to a means or method of an interaction, following a distinction made by Leont'ev (1981) in an analysis of activity. The description of an interaction will focus on the function of activity, while the description of an operation will focus on the ways in which an activity is performed.

For example, imagine a situation in which the block starts at the ten-inch mark, a 4-spool is placed on the winch and the student is asked "Where will the block be after three turns?". An interaction that might occur in this situation could be described as "determining an end position after some number of turns from a starting position". We observed students use at least four different operations for this interaction: (a) turning the handle of the winch, and then attending to the resulting position of the block; (b) marking intervals of distance on the ruler to indicate where the block(s) would be after each turn of the handle; (c) making a table of values to indicate the numerical position of the block after each turn; and (d) multiplying the number of turns times the spool size, and adding that number to the starting position of the block.

The distinction between interactions and operations is relative to the level of description. That is, our analyses are nested, with units that are interactions at one level of description and operations at another. The description at any level distinguishes between an interaction (what is accomplished) and an operation (how it is accomplished). The distinction provides a convenient expression of relations between alternative methods for achieving a functional result.

What does it mean for an activity to be jointly constituted by an agent and the situation? The possibility of a person interacting in a certain way with a physical and/or conceptual system depends on the affordances provided by the situation and the abilities of the person. The term *affordances* (Gibson, 1979) refers to features of a situation that support certain activities - these are specified relative to an agent acting in that situation. *Abilities* are those activities that an agent can perform - these are specified relative to a situation. (We prefer the term *abilities* to the term *effectivities* that Shaw, Turvey, and Mace (1982) coined, but we believe that we mean the same thing.) For example, in order for a student to turn the handle of the winch, the handle must be "graspable" and "turnable", that is, it must be a certain size and shape, and must be connected to the rest of the device in a way that allows it to be rotated. However, a handle is only graspable and turnable by an agent with an appropriately shaped hand and the ability to turn something in a circular motion. An affordance and an ability are two sides of the same possibility: affordances for activities depend on the characteristics of an agent, and abilities for activities depend on characteristics of the situation.

Reasoning as a Form of Activity

Figure 2 indicates the relationships between several kinds of states of affairs. Material states of affairs are those that hold in the physical world. Before distinguishing how different kinds of material states of affairs are used in reasoning, it is necessary to define some terms. A situation type is a parameterized state of affairs, that is, a partially specified state of affairs that applies in many situations. For instance, "Emmy is eating yogurt" is a state of affairs that may or may not hold in a given situation, while "someone is eating yogurt" is a situation type that holds in the class of situations in which a person is eating yogurt.

In any situation, various states of affairs hold. Because of affordances for activity, there are also *potential* states of affairs that hold. A state of affairs that might occur, such as "Emmy might eat yogurt" is a potential state of affairs. These depend on the affordances for an activity being available, as well as an agent with the appropriate abilities. Potential states of affairs designate ways in which an agent or something in the situation can contribute to an interaction.

A *constraint* is a relation between situation types such as "If someone is eating yogurt then that person might drop the spoon". It is this relation between situation types that supports inferences by a person who is attuned to the constraint. For example, if someone is eating yogurt with a spoon, then it can be inferred that the spoon might be dropped.

Actual states of affairs are those that are true about the material aspects of a situation. Potential states of affairs are generated from an actual state of affairs through a set of constraints. Figure 2 shows a further breakdown of kinds of actual states of affairs that we use in modelling interactions with the winch. The same distinctions could be made for the potential states of affairs.

The final distinction between states of affairs that we make is between material and registered states of affairs, indicated in the first level of Figure 2. We distinguish two aspects of a situation: what is happening - the material situation, and what is recognized about what is happening - the registered situation. Material states of affairs are relations in the world, while registered states of affairs are those states of affairs which have been attended to. Registering a state of affairs often involves saying it and communicating it in a conversational group. In general, it is the set of properties and relations that have symbolic representations. We distinguish registered states of affairs mainly to analyze activities that include inferences and communication. We assume that registration makes the information available for inferences; however, both material and registered states of affairs play a role in supporting certain kinds of activity. We are assuming, therefore, that the role of representation is to provide objects that function in inferences, and we use the notation of registered states of affairs to distinguish symbolic representations of information that are available for inferential reasoning.

We characterize reasoning within this framework as inference of new registered information using an inferential process that involves transformation of a representation. Reasoning about a situation can include generating potential states of affairs and registering those potential states of affairs. Representations of states of affairs afford reasoning activities that infer other states of affairs. In this view, material systems, registered states of affairs, and the person or persons participate interactively in the activity of reasoning.

Understanding Based on Constraints

We hypothesize that activities, such as the interaction-operation units presented earlier, depend on constraints. Turning the handle to reach an end position works because certain relationships exist between states of affairs in the system, such as the handle being connected to the spool and a string being attached to both the spool and the block. Since activities can be conceptual as well as physical, "knowing a concept" in this framework is being attuned to the constraints of conceptual activities. This characterization of knowing a concept includes implicit understanding, as well as more explicit forms of knowing. A reasoner can represent a constraint and reason about it explicitly, as well as simply being able to use a constraint in an activity.

The concept of linear function includes several constraints:

1. *proportional quantities*: a change in one variable causes a proportional change in another variable;
2. *linearly-related measures*: each increment of a variable has the effect of adding a constant number to the previous numerical value of the function;
3. *multiplicative/additive properties*: the cumulative effect of n increments of k each is equal to $n \times k$; and a final quantity measure is equal to an initial measure added to $n \times k$.

In terms of the winch, "proportional quantities" means that each turn of the handle causes a constant change in the position of the block. "Linearly-related measures" means that for each turn of the handle a constant number is added to the numerical position of the block. The "multiplicative/additive" constraint means that the distance a block moves for some number of turns can be calculated by multiplying the spool size times the number of turns, and the final position of the block can be calculated by adding the starting position to the distance moved.

Since we are characterizing activities as being jointly constituted by affordances and abilities, the question arises as to what aspect of a situation is providing a certain constraint. We have analyzed the constraints for the interaction "determining the location of a block after some turns from a starting position". For the operation of turning the handle, all three constraints are embodied in the mechanism of the device. That is, the ending position of the block is determined by the physical constraints of the device. For the operation of measuring increments, the constraint of proportional quantities is provided through the abilities of the student, rather than through the affordances of the winch. The student must explicitly determine the size of the interval to be measured, and simulates the change in the position of the block for each turn. For the operation of creating a table, both the constraints of proportional quantities and linearly related measures are supplied by the abilities of the student. The student explicitly determines a numerical difference and adds that difference repeatedly to calculate where the block would be for each turn of the handle. The number in the last row of the table can then be read as the ending position of the block. For the operation of mental arithmetic, all three constraints have shifted to the abilities of the student. Proportional quantities and linearly related measures are captured as part of the calculations in the multiplicative/additive constraint.

Conclusions

The analysis of this interaction indicates that alternative operations for an interaction differ in having constraints contributed by the situation's affordances or the agent's abilities. Across the four operations discussed above, the interaction is increasingly abstracted from the physical constraints of the device. Having more constraints of a concept embodied in a student's abilities, and reasoning more explicitly about constraints, constitute more understanding of the concept.

We hypothesize that an important aspect of understanding is a kind of correspondence between constraints of interactions. In the handle-turning interaction the proportional quantities constraint is met through the physical constraints of the device. In measuring increments the constraint is met through the size of the interval that the student has created. However, the process of measuring a series of intervals along the ruler only makes sense with respect to the constraints of the physical device. It is through the correspondence of the constraints of the two interactions that measuring increments gains its meaning.

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