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### Author

Haldar, Lina Chopra

### Publication Date

2014

Peer reviewed|Thesis/dissertation

Students' Understandings of Arithmetic Generalizations

by

Lina Chopra Haldar

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Education

In the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Geoffrey B. Saxe, chair

Professor Alan Schoenfeld

Professor Silvia Bunge

Spring 2014

Students' Understandings of Arithmetic Generalizations

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by

Lina Chopra Haldar

## Abstract

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Professor Geoffrey Saxe, Chair

This study examines fourth graders' understandings of arithmetic generalizations, the general properties of arithmetic that hold true for all numbers. Its focus is on three types of generalizations: (a) *direction of change* (e.g., addition of positive numbers increases the numerical value, while subtraction of positive numbers decreases the numerical value); (b) *identity* (e.g., the addition or subtraction of 0 to any number leaves its value unchanged); and (c) *relationship between operations* (e.g., addition and subtraction are inverse operations). Using a between subjects design, two interview studies were conducted to investigate the character of children's understandings and to understand how students' production of generalizations varied across different tasks (e.g., *I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?*). Study 1 (n=24) focused on students' additive thinking in the context of addition and subtraction tasks, while Study 2 (n=24) focused on multiplicative thinking in the context of multiplication and division tasks.

In both studies, qualitative analyses of the video data revealed four levels in student thinking, levels that show a spectrum of increasing generality with which students treat arithmetic operations. At Levels 1 and 2, students rely on specific instances and substitution of values. At Levels 3 and 4 (advanced generalizations), students do not rely on any examples and make generalizations about the arithmetic operations. Further quantitative analyses revealed that student thinking was not always consistent and that students' production of advanced generalizations was affected by generalization type and domain type. In both studies, the *identity* tasks prompted the most advanced generalizations, while the *relationship between operations* tasks elicited the least advanced generalizations from students. Similarly, children were more likely to produce an advanced generalization in the additive domain than in the multiplicative domain. Although the multiplicative tasks may have been more challenging for students, the character of students' thinking and the difficulty of the tasks in relation to one another were similar across both domains. These parallel findings across the studies indicate that additive and multiplicative generalizations may involve similar developmental progressions.

This dissertation provides insight into a developmental model about children's construction of arithmetic generalizations. First, the four levels of generality describe qualitative shifts in student thinking and, second, these findings indicate that the development of students' understandings of arithmetic generalizations may be heterogeneous across a range of generalizations. This work can contribute to teachers' knowledge for teaching and inform the design of a developmentally appropriate instructional sequence.

This is dedicated to my husband, best friend, and biggest cheerleader,  
Prateek Haldar.

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## Chapter 1: Introduction

This dissertation reports on the findings from an interview study conducted with fourth grade students to examine their understandings of arithmetic generalizations. Arithmetic generalizations concern the properties of arithmetic operations that hold true for all numbers, and researchers and educators have highlighted the importance of arithmetic generalizations in the development of algebraic thinking in the early grades (Carpenter, Levi, Franke, & Zeringue, 2005; Carraher & Schliemann, 2007; Kaput, 2008). Examples of arithmetic generalizations include arithmetic identities (e.g., for any  $N$ ,  $N + 0 = N$ ) and inverse relations (e.g., for any  $N$ ,  $N + a - a = N$ ). This dissertation investigates how children in urban fourth grade classrooms reason with additive and multiplicative tasks that require arithmetic generalizations. It also represents an important step of a broader research agenda aimed at fostering the development of algebraic thinking in the elementary grades.

Research has demonstrated that young students can in fact engage with ideas of arithmetic generalizations in the elementary grades, but additional research is needed to understand how children's understandings of arithmetic generalizations develop. This prior work has primarily involved case studies and classroom episodes that highlight the effectiveness of particular instructional approaches that create contexts to formulate and justify arithmetic generalizations (Bastable and Schifter, 2008; Carpenter, Franke, & Levi, 2003; Dougherty, 2008). This dissertation extends this work by conducting a more fine-grained analysis of student thinking to systematically examine and compare students' understandings across a range of arithmetic generalizations.

In this dissertation, I report on an interview study that investigates fourth graders' developing understandings of arithmetic generalizations in the context of addition, subtraction, multiplication, and division tasks. This study reveals that there are different ways students treat arithmetic operations, as illustrated in the vignette below. I include this sampling of student reasoning because it introduces the ideas I engage with in this dissertation.

*Students in Ms. Grisham's fourth grade class were asked the following question in one-on-one interviews: "I am thinking about a number. If I subtract 7 from that number first and then add 5, what will happen to my number?" There was a wide range of responses. Leila explained that "it all depends on what the number is" and there is no way to know how the number changes. Jose, on the other hand, explained, "If the number was 9, if you subtract 7, it'll be 2. And then add 5, it'll be 7. So I think the number will be smaller." Kumar used a different type of reasoning and argued that the number will get smaller "because 7 is bigger than 5", and I subtract more than I add. Mary said that the number would "get less by 2" because the difference between 5 and 7 is 2. While most students in Ms. Grisham's class generated seemingly correct solutions, the mathematical sense-making they demonstrated varied significantly. [interviews conducted by Lina C. Haldar in a public elementary school in the San Francisco Bay Area]*

In the observation, while students are all engaged in reasoning about my query, their solutions reveal varied understandings. I argue that a systematic analysis of these understandings and of the variation in these understandings across a range of tasks can provide insight into a developmental model about children's construction of arithmetic generalizations. Further, such an analysis has the potential to inform elementary math curricula with an appropriate instructional sequence, one that builds on students' intuitions.

This dissertation provides a detailed examination of student thinking and reasoning across a range of arithmetic generalization tasks. In my methods and analyses, I focus on two

principal research questions. First, can fourth graders make arithmetic generalizations in the context of additive arithmetic operations (addition and subtraction) and in the context of multiplicative arithmetic operations (multiplication and division)? Further, what is the quality of their reasoning when they engage with arithmetic generalizations? Second, how does students' production of generalizations vary across different arithmetic domains (e.g., additive versus multiplicative) and across the different types of generalizations within each domain (e.g., identity versus inverse operations)? In this chapter, I discuss the motivation behind these research questions, and I explain how these questions build on prior research and represent an important next step of scholarly work. I conclude this chapter with an overview of my dissertation that outlines the remaining chapters.

## Review of the Literature

Arithmetic generalizations are an important building block for the development of algebraic thinking. In the first part of this section, I discuss the importance of arithmetic generalizations in the elementary grades and efforts to integrate arithmetic generalization activities into the elementary mathematics classroom. This prior work converges on the idea that children can in fact engage with arithmetic generalizations, but it does not address questions regarding the cognition or developmental processes that are involved. To consider these questions, in the second section, I draw on Piaget's work on arithmetic generalizations when reasoning about physical quantities, which I build on and extend to when children are asked to reason about generalizations in my dissertation without physical quantities.

**The Role of Arithmetic Generalizations in Elementary Mathematics.** In this section, I first explain how arithmetic generalizations are related to algebraic thinking and I then review work that has focused on arithmetic generalizations in the elementary grades. A review of this literature highlights the role and importance of arithmetic generalizations in the elementary curriculum as a resource for students' developing algebraic understandings, and it establishes why it is important that we understand how students construct generalizations. This prior work has been compelling and insightful, and has opened avenues for research like my dissertation work.

Educators and researchers have explored how to support the development of algebraic thinking in elementary school. Their work and this dissertation is grounded in the notion that there is not a linear progression from arithmetic to algebraic thinking, and that instead, the two strands of thinking can develop simultaneously and inform one another (Carraher & Schliemann, 2007; Kaput, 2008; Vygotsky, 1986). Arithmetic thinking involves a focus on specific numbers and computation, while algebraic thinking deals with the *relations* between numbers and ideas of mathematical generalization, specifically generalized arithmetic (Carraher & Schliemann, 2007; Gelman & Gallistel, 1978; Kaput, 2008; Kieran, 2004; Sfard, 1995). Both strands of thinking are important in the elementary mathematics curriculum. Vygotsky argued specifically that algebraic thinking involves making generalizations about number, providing children with a broader perspective on relations between numbers. Similarly, Carraher and Schliemann (2007) argue, "Arithmetic is a *part* of algebra, namely, that part that deals with number systems, the number line, simple functions and so on. Arithmetic deals with the part of algebra in which particular numbers and measures are treated as instances of more general examples" (p. 698). For example,  $10 - 5 + 5 = 10$  is one of an infinite number of ways to express the number 10, and it is an

instance of the more general concept  $N - a + a = N$ , or the addition and subtraction of the same number has no effect on a number. This is an example of an arithmetic generalization, which focuses on relations between numbers and operations.

There are many reasons for making arithmetic generalizations a focus in elementary mathematics. First, arithmetic generalizations play an important role in middle and high school mathematics, and students' understanding of arithmetic generalizations can provide a foothold into understanding algebraic equations. Unfortunately, most high school students do not realize that the procedures they use to solve equations and simplify algebraic expressions are based on arithmetic generalizations (Carpenter et al., 2005a; Kieran, 1989). Kieran (1989), for example, found that 12-year-olds often judged  $x + 37 = 150$  to have the same solution as  $x = 37 + 150$ . Additionally, she found that students often attempt to balance equations by adding a number to one side and subtracting that same number from the other. Similarly, to solve a problem like  $3x - 6 = 9$ , my high school students would typically say "you need to add 6 to both sides" or "you need to do the opposite on both sides," but most of them could not explain why, and often had difficulty using this strategy with more complex problems. An understanding of inverse relationships, or  $N - a + a = N$ , would help students make sense of algebraic equations, but research indicates that students typically use procedural understandings rather than a conceptual understanding of the properties of arithmetic to solve algebraic equations (Boulton-Lewis, 2001; Demana, 1988; MacGregor, 1996).

Second, arithmetic generalizations should be a focus in elementary mathematics because researchers have produced evidence suggesting that engaging with arithmetic generalizations can support greater understanding of the equal sign and variables, both of which have been documented as major learning hurdles for students in algebra (Dougherty, 2008; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kaput, 2008; Schoenfeld & Arcavi, 1988). Design experiments with small groups of students as well as large-scale studies support this argument. Dougherty (2008), for example, has used continuous measures (e.g., length) in her *Measure Up* curriculum to draw on the composition and decomposition of number to motivate students' understandings of arithmetic generalizations, and she has found that this instructional approach helps students learn to appropriately use letters as variables. For example, a length,  $e$ , can be decomposed into two parts,  $c$  and  $x$ , and students construct mathematical expressions, like  $e = x + c$  and  $e - x = c$  to describe the relationships between the lengths. In this manner, they use letters as variables, and engage with the arithmetic generalizations of commutativity and inverse relations. Similarly, Jacobs et al. (2007) discuss the use of true or false number sentences to encourage students to discuss arithmetic generalizations (e.g., 'True or false:  $78 - 49 = 78$ ' to explore students' understandings of the identity principle for subtraction). In their classroom studies, they found that this approach encouraged students to formulate arithmetic generalizations, which helped students learn to treat the equal sign as a symbol that represents a relation rather than a signal to carry out a computation. To measure the effectiveness of this approach on a larger scale, Jacobs et al. (2007) compared 89 participating classrooms (classrooms whose teachers engaged in professional development focused on ideas of arithmetic generalizations), with 92 nonparticipating classrooms. Students from the participating classrooms performed significantly better on a written assessment aimed at measuring their understanding of the equal sign<sup>1</sup>. A focus on arithmetic generalizations in the elementary grades can therefore play an important role in addressing some of the difficulties students encounter in algebra in later years.

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<sup>1</sup> This assessment included four tasks designed to assess whether students believed the equal sign was a signal to carry out a calculation or if they understood the equal sign as symbol that represents a relationship of equivalence. In

Third, arithmetic generalizations can support a deeper understanding of numerical relations and operations, which allows students to develop more flexible and efficient strategies that are grounded in conceptual understandings. Carpenter et al. (2005b), for example, provide some case studies of interviews with students who were drawn from the participating classrooms in the Jacobs et al. study. In one case, for instance, a third grader was presented with a series of true or false number sentence tasks:  $3 \times 7 = 7 + 7 + 7$ ,  $3 \times 7 = 14 + 7$ , and  $4 \times 6 = 12 + 12$ . From the student's responses to these tasks, it was clear that she had some understanding of the relationship between addition and multiplication, as well as the distributive property. The student was then asked to solve  $4 \times 7 = \underline{\quad}$ , which she solved quickly by explaining that she could simply add 14 to 14. This example illustrates how this third grader's understanding of the distributive property and the general relationship between addition and multiplication supported her learning of number facts and multiplication. The study showed that students who demonstrated an understanding of arithmetic generalizations, like the distributive property, were more likely to solve arithmetic tasks with strategies that highlighted their conceptual understandings, rather than procedural knowledge.

Fourth, a focus on arithmetic generalizations is in line with the Common Core State Standards for Mathematics (2012), which includes 'Operations and Algebraic Thinking' as one of the five primary domains for the elementary mathematics standards. Grade 2 standards, for example, include, "Understand and apply properties of operations and the relationship between addition and subtraction." The Grade 3 standards include, "Understand properties of multiplication and the relationship between multiplication and division." As I will discuss later in this chapter and in Chapter 2, these ideas are central to arithmetic generalizations and my dissertation work.

Finally, a focus on arithmetic generalizations is consistent with a critical strand in math education reform that identifies justification as a central mathematical practice (Common Core State Standards Initiative, 2012; National Research Council, 2001). The Common Core State Standards Initiative (2012) states, "One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from". Arithmetic generalizations can be integrated into the classroom by providing students the opportunity to justify the observations they make about the properties of number and operations. Bastable and Schifter's (2008) work in classrooms has highlighted this. I focus next on Bastable and Schifter's work because they have concentrated primarily on arithmetic generalizations in their data from Teaching to the Big Ideas (TBI), a four-year teacher enhancement project (Schifter, 1999; Bastable & Schifter, 2008). I conclude this section with this work because it lays some of the groundwork for my dissertation. First, it demonstrates that young students can in fact formulate and justify arithmetic generalizations. Second, it illustrates that students exhibit different types of reasoning about these arithmetic generalizations, particularly with respect to the level of generality with which they treat arithmetic operations and justify their observations.

A primary focus of the TBI project was to help teachers create a context for students to construct understandings of arithmetic operations on their own. In Schifter's (1999) discussion of teachers' descriptions of classroom episodes, she shows that instead of explicitly showing students how to add/subtract/multiply/divide, teachers had students approach problems on their own with minimal guidance. For example, an episode that Schifter discusses involved the following problem that was presented to first graders: *Sabrina and Yvonne have 14 stickers when they put their stickers together. Yvonne has 6 stickers. How many stickers does Sabrina have?*

The problem was simply given to students without any discussion of it being a subtraction problem, so students had to decide how to approach it. Some students built a tower with 14 counters and another tower with 6 counters, and then they counted the difference between the two towers. Other students took a counting approach where they counted up from 6 until they arrived at 14. Some students wrote out  $6 + \underline{\quad} = 14$  and used their memorized number facts to conclude that it should be 8. A few students even wrote out  $6 + \underline{\quad} = 14$  and then concluded that because  $14 - 8 = 6$ , the solution must be 8.

Schifter argues that when children have experiences like this with arithmetic, they develop deeper understandings of operations, which leads them to make observations about arithmetic generalizations (e.g., commutativity). She describes a classroom episode in which second graders referred to the idea of commutativity with addition as ‘turnarounds’. In other words, students observed that the order of the addends did not affect the sum, or that they could ‘turn around’ the expression and arrive at the same solution (e.g.,  $4 + 6 = 10$  and  $6 + 4 = 10$ ). These students were then asked to justify their observations, which most were able to do with varying levels of abstraction. Some students justified their observation by referring to only one specific instance (e.g.,  $35 + 70 = 105$  and  $70 + 35 = 105$ ). Other students used base ten blocks to explain that a number like 105 can be configured in many ways and any two of its parts can be added in either order. Still other students’ justifications did not rely on a specific number like 105 and, instead, they generalized that any number of blocks can be divided into two parts and the order in which you combine them does not matter. Thus, Schifter supports the idea that providing children with opportunities to explore arithmetic problems with limited instruction allows them to understand operations at a deeper level, which ultimately leads to the discovery and understanding of arithmetic generalizations.

The research on arithmetic generalizations opens up important directions for elementary mathematics education. This work has established that young students can engage with ideas of arithmetic generalizations in the elementary grades given appropriate supports, and it illustrates supportive contexts that can lead students to formulate and justify arithmetic generalizations in a classroom setting (e.g., True/False number sentences). The findings also suggest that integrating arithmetic generalizations into the elementary curriculum is a promising approach for supporting students’ deeper understanding of number and operation.

The implications of the findings, however, are limited in two important ways. First, the research has been primarily descriptive and exploratory in nature, reviewing classroom episodes that involve a small number of students engaged with particular mathematical tasks supported by a strong teacher. Thus, the evidence is limited with regard to children’s capabilities in producing arithmetic generalizations when contexts are not as supportive and contexts for generalizations differ from the ones observed. Second, this research does not provide insight into the processes whereby students construct arithmetic generalizations. For example, in the classroom episodes that Bastable and Schifter (2008) describe, students make observations about the properties of arithmetic operations. However, there is no discussion of whether student observations related to generalizations were more or less likely to occur across different domains (e.g., additive versus multiplicative) or types of arithmetic generalizations (e.g., identity versus inverse relations); to produce such data would require a different kind of method, one that makes use of a controlled study in which students are presented with tasks that vary systematically in arithmetic domain and generalization type. My dissertation is an important next step in scholarly research as it examines student thinking and reasoning about arithmetic generalizations across two domains

(additive and multiplicative relations) and three generalization types (*direction of change*, *identity*, and *relationship between operations*).

**Understanding Arithmetic Generalizations through a Piagetian Lens.** I draw on Piaget's work in this section because he offers a useful developmental framework that has implications for a treatment of arithmetic generalizations. My research is informed by Piaget's treatment of three constructs: (a) number conservation, particularly with regard to his argument that conservation emerges as a result of the child's coordination of relations; (b) logical necessity, particularly with regard to the transition from empirical judgments to judgments that invoke logical necessity; and (c) the additive composition of number, specifically to understand the distinction between additive and multiplicative reasoning. The coordination of relations is a key idea that is important across these three constructs and informed the conceptual framework I describe in the next section. In turn, the conceptual framework informed the research design that I employed to explore qualitative differences in students' production of generalizations and it motivated the use of particular variables that come into play in the analyses in forthcoming chapters.

**Number Conservation.** Piaget (1965) argued that conservation of disquantities, the idea that any discontinuous quantity remains numerically invariant regardless of spatial changes in arrangement (spreading apart or crowding together), is fundamental to numerical understanding. In an illustrative study, Piaget put the same number of beads one by one into two containers that were the same size, and then poured the beads into two containers that were different sizes. Before and after the transformation, he asked children if the containers held the same number of beads. Piaget found that at Stage 1 (approximately age four), children agreed upon the initial equality but asserted that the quantity of beads changes with the transfer to containers of different dimensions. These children rely on perceptual features to make judgments about quantity. At Stage 2 (approximately age five), children can conserve in some situations guided by an emerging capability to coordinate relations between spatial extension and crowdedness, but when they do make conservation judgments, conservation is an empirical observation (e.g., when reminded of the one-to-one correspondence that was initially established, a child then conserves and coordinates the differences between the dimensions of the two containers). Finally, at Stage 3 (as early as age six), children conserve regardless of spatial arrangement and their ability to conserve relies on an understanding of one-to-one correspondence and reversibility (i.e., objects can be transformed and returned to their original state). Noteworthy in Piaget's conservation studies is that children's explanations about conservation at Stage 3 show that their judgments are not limited to specific numbers or values. Rather, they are about discontinuous quantities in general: Children explain that number is conserved because one-to-one correspondence implies lasting equivalence, an analysis applicable to any discontinuous quantity. One-to-one correspondence requires the coordination of relations, and the production of arithmetic generalizations through the coordination of relations is core to the framework that I make use of in the design of my studies.

Of course, there are hosts of differences between Piaget's treatment of the coordination of relations that lead to conservation judgments and the studies to follow. Most importantly, unlike Piaget's conservation studies, I focus on the coordination of relations when physical quantities are not present. In the interview tasks that I use, number is represented in arithmetic language (e.g., "*I am thinking about a number. If I add 5 to that number and then subtract 5, what happens*").

*to my number?*”). Specifically, examples of arithmetic generalizations that this dissertation explores include identities (e.g.,  $5 + 0 = 5$ , or for any  $N$ ,  $N + 0 = N$ ) and inverse relations (e.g.,  $2 + 5 - 5 = 2$ , or for any  $N$ ,  $N + a - a = N$ ). When adding 5, for example, the number increases by 5, and when subtracting 5, the number decreases by 5 and returns to its original value. In these tasks, children need to coordinate their understandings of arithmetic operations and the magnitudes of operands.

**Logical Necessity.** At Stage 3 in Piaget’s conservation studies, children no longer rely on empirical observation, but rather what Piaget calls ‘logical necessity’; children understand conservation as a necessity rather than something that needs to be empirically observed or proven. The idea of logical necessity is central to Piaget’s theory of development, and it plays an important role in this dissertation research as well. Piaget explored the notion of logical necessity with physical quantities, while I explore it in the context of problems of arithmetic generalizations when quantities are not present. Like conservation, an arithmetic generalization is a property that holds true for all numbers. Consider the following task: “I am thinking about a number. If I subtract 7 from that number first and then add 5, what will happen to my number?” As discussed earlier, children approach this task in a variety of ways. Some children need to consider specific examples and use substitution for the unknown number. Other children understand the relationship between the operations and recognize that the number will always decrease by two; it is a logical necessity for these children.

The transition from empirical judgments to judgments that invoke logical necessity through the coordination of relations is another important idea that I considered in my dissertation. In Piaget’s conservation tasks, for example, children coordinate the differences in the heights and cross sections of the containers (e.g., one container may be taller but narrower, while another may be shorter but wider). The tasks in this dissertation require coordinating relations as well, and I expected to find differences in student thinking across the different types of arithmetic generalizations as they require different coordination of relations. For example, I hypothesized that tasks that require the coordination of more than one operation (e.g., “*I am thinking about a number. If I subtract 7 from that number first and then add 5, what will happen to my number?*”) may present more challenges to students than those that involve only one operation (e.g., “*I am thinking about a number. If I add 0 to that number, what will happen to my number?*”).

**Additive Composition of Number.** An important component in the development of children’s arithmetic thinking involves the additive composition of number. Piaget (1965) explains that the additive composition of number involves the recognition that a whole, or number, remains constant irrespective of the additive composition of its parts, or addends. For example, it requires the understanding of 10 as a totality and the ability to group various combinations of objects in additive compositions to create 10 (e.g.,  $5 + 5 = 1 + 9 = 2 + 8 = 3 + 7$ ). With respect to numbers or unknowns like  $N + 5$ , the additive composition of number becomes important because students must recognize that  $N + 5 = N + 1 + 1 + 1 + 1 + 1$ . In other words, students must understand that  $N + 5$  is a number that is 5 units greater than  $N$ . Piaget found that the concrete operational child, or a child between the age of 7 and 11 who can reason logically with concrete objects and events, can also reason with discrete objects in this manner; he can construct various groupings of counters with the understanding that adding counters to one set means subtracting counters from another set, so the total number of counters remains

constant. In this construction lies an implicit understanding of the additive composition of number as well as the inverse relationship between addition and subtraction. Both are key in understanding arithmetic generalizations and my dissertation extends this work by exploring similar ideas without physical quantities and with arithmetic language instead (e.g., “*I am thinking about a number. If I add 5 to that number and then subtract 5, what happens to my number?*”).

Steffe (1992) built on Piaget’s work and this notion of a composite unit to draw a clear distinction between additive and multiplicative reasoning. Like Piaget, he argued that additive reasoning involves a treatment of number as a composite unit that highlights the composition and decomposition of number. Multiplicative reasoning then involves a coordination of composite units. “For a situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit” (p. 305). First, consider the task of adding 2 markers to a bag that already has 5 markers. The composite units of 2 and 5 are combined and the referent unit of 1 marker is preserved ( $2 = 1+1$ ,  $5 = 1+1+1+1+1$ ,  $2+5=1+1+1+1+1+1$ ). The sum, or total markers in the bag, is 7 markers. Further, subtracting or taking away 2 markers from the bag with 7 markers also preserves the referent unit of 1 marker and highlights the inverse relationship between addition and subtraction ( $5 + 2 - 2 = 5$ ).

Next, consider the task of placing 2 markers on each of 5 desks and finding the total number of markers. The composite units of 2 and 5 cannot simply be combined and neither referent unit (markers per desk or number of desks) is preserved. The product, or total number of markers is 10. Ten is the result of coordinating two composite units, 2 and 5, while finding the total of 1s. In other words, 1 desk needs 2 markers, 2 desks need 4 markers, 3 desks need 6 markers, 4 desks need 8 markers, and 5 desks need 10 markers. Further, dividing the 10 markers onto 5 desks equally also involves coordinating two composite units resulting in 2 markers per desk and highlighting the inverse relationship between multiplication and division ( $2 \times 5 \div 5 = 2$ ).

Thus, additive reasoning involves composite units with addition and subtraction as inverse operations, while multiplicative reasoning involves a coordination of composite units with multiplication and division as inverse operations. This distinction between additive and multiplicative reasoning informed the conceptual framework I discuss next and why I focused on comparing students’ thinking across the additive and multiplicative domains.

## Conceptual Framework

To explore students’ developing abilities to produce arithmetic generalizations, I differentiated arithmetic generalizations into those that involve judgments about additive as distinct from multiplicative relations (see Figure 1). Within each domain, I focus on the three types of generalizations: (a) *direction of change* (e.g., addition of positive numbers increases the numerical value, while subtraction of positive numbers decreases the numerical value); (b) *identity* (e.g., the addition or subtraction of 0 to any number leaves its value unchanged); and (c) *relationship between operations* (e.g., addition and subtraction are inverse operations). A core hypothesis is that a child’s capability to produce arithmetic generalizations in one domain or generalization type may not develop in concert with her capability to produce generalizations in another domain or generalization type. The relations children need to coordinate to arrive at different arithmetic generalizations may present different challenges, so the development of

students' understandings of arithmetic generalizations may be heterogeneous across a range of generalizations.

I selected these specific types of generalizations for three main reasons. First, they require students to consider relations of greater than, less than, and equal to, which Carpenter et al. (2003) identify as an important entry point for building students' understandings of arithmetic generalizations in the elementary grades. Second, for the purposes of this study and the planned analyses that focus on comparisons across the different generalizations, I limited the types of generalizations to those that are relevant in both the additive and multiplicative context for the four operations of addition, subtraction, multiplication, and division. The commutative property, for example, only holds true for addition and multiplication, so it was not included in the study. Lastly, these types of generalizations align with the expectations of the elementary standards. As noted earlier, the Common Core State Standards (2012) highlight the importance of the arithmetic properties and the relationships between the different operations. By the end of third grade, for instance, students are expected to understand properties of both additive and multiplicative operations, as well as the relationships between addition and subtraction, and between multiplication and division. I selected arithmetic generalizations that are highlighted in the Common Core State Standards in order to produce relevant findings that can inform curriculum development.

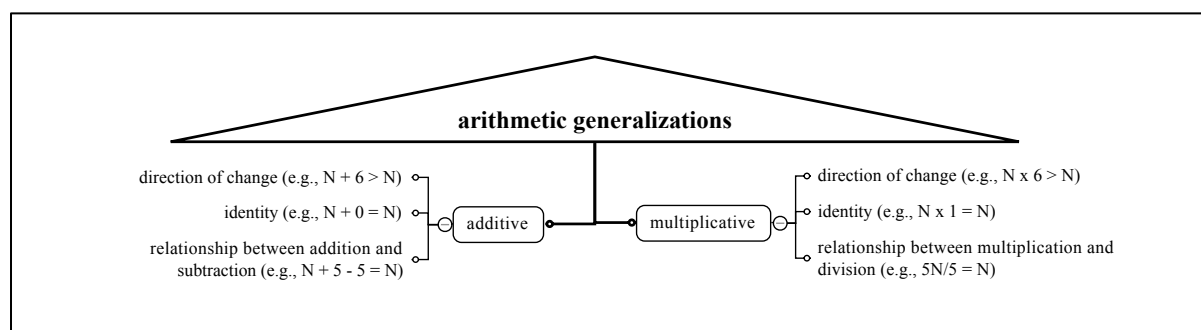


Figure 1. Conceptual framework of mathematical territory.

## Dissertation Overview

To study students' understandings of arithmetic generalizations, I probed student thinking in an interview study that involved problems that students typically do not see in standard mathematics curricula (e.g., *I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?*). All interview tasks were designed to examine students' understandings of core ideas of addition, subtraction, multiplication and division with positive numbers (negative numbers are typically not introduced in the elementary grades and the Common Core State Standards introduce them in sixth grade). Study 1 focused on students' additive thinking in the context of addition and subtraction tasks, while Study 2 focused on multiplicative thinking in the context of multiplication and division tasks.

Problems were open-ended and did not include formal mathematical notation. I used words rather than mathematical expressions to present each mathematical task. I avoided letter notation, which is a known source of confusion for students (Goldenberg & Shteingold, 2008; MacGregor & Stacey, 1997; Christou et al., 2007; Küchemann, 1981; Collis, 1975). The tasks

were open-ended in nature so I could listen to and probe student thinking, and so that students' different ideas could surface.

The interview data from this dissertation illuminate different types of understandings that students demonstrate and the varying levels of generality with which they treat arithmetic operations. The data also provide some insight into what types of generalizations may develop earlier or be more intuitive for students. These findings can inform a model of the development in children's thinking and reasoning about arithmetic generalizations, and I am optimistic that this dissertation will provide promising avenues for researchers to pursue.

In subsequent chapters, I discuss the study design and results of my dissertation study. Chapter 2 describes the research design and methodology of the dissertation study. Chapters 3 and 4 review the findings from Study 1 and Study 2 respectively, including both qualitative and quantitative analyses, focusing on the patterns in student thinking and variations across the different tasks. Chapter 4 concludes with cross-study analyses between Study 1 and Study 2. I conclude with Chapter 5, a discussion of the implications of this dissertation and how future research can build on it further.

## Chapter 2: Methods

This chapter details the methods for Study 1 on additive relations and Study 2 on multiplicative relations. I limit this chapter to overarching structural features related to study design and procedures that are the same across both studies. I provide additional study-specific details in Chapter 3 and Chapter 4, the chapters that report results for the respective studies on arithmetic generalizations in the context of additive and multiplicative operations.

### Participants

Participants in the study were 4<sup>th</sup> graders from the San Francisco Bay Area (n=48), with 24 students (15 boys, 9 girls) participating in Study 1 and 24 students (14 boys, 10 girls) in Study 2. All participants were recruited from a single mid-size school in an urban district. The school had a diverse student population, both ethnically<sup>2</sup> and socioeconomically<sup>3</sup>, and students from four different classrooms were invited to participate. Students who provided the appropriate parent permission forms and were *not* identified as ‘English Language Learners’ (ELL) by the school district were included in the study. Although English was not the first language for some participants, I excluded ELL students because I wanted to make sure all participants understood the interview tasks. I obtained all participants’ math scores from their Grade 3 California Standardized Testing and Reporting (STAR) test. The STAR scale scores for each grade and subject area range from 150 to 600. The mean math score for participants in the study was 443, with a range from 292 to 600 (SD = 83). California uses five performance levels to report student achievement, and Table 1 breaks down participants’ scores by performance level. Students were matched on their math scores from the STAR exam and then randomly assigned to either Study 1 (M = 440; SD = 83) or Study 2 (M = 447; SD = 85). The majority of students in both studies performed in the proficient or advanced range.

Table 1: Breakdown of Students’ Proficiency Levels on STAR Test

	Far Below Basic (150 to 258)	Below Basic (259 to 299)	Basic (300 to 349)	Proficient (350 to 401)	Advanced (402 to 600)
Study 1: Additive Relations	0	1	1	7	15
Study 2: Multiplicative Relations	0	0	2	6	16

I selected students in fourth grade for three reasons. First, by fourth grade, the operations of addition, subtraction, multiplication, and division had been covered in the school I conducted interviews at and in most curricula in general. Second, the Common Core State Standards Initiative (2012) recommend that students understand the general properties of addition, subtraction, multiplication, and division, including the relationship between the operations, by the end of third grade. It is therefore important to examine to what extent students can actually demonstrate these types of understandings, so that educators have a better idea of how to support students in meeting these new standards. Lastly, the Common Core State Standards Initiative (2012) recommend that equations with unknowns be introduced to students in fourth grade, and

<sup>2</sup> Asian (39%), White (35%), Latino (12%), African American (7%), other or multiethnic (7%); English learners (45%)

<sup>3</sup> 17% from socioeconomically disadvantaged backgrounds

the arithmetic generalizations I examine are necessary for a deep, conceptual understanding of solving equations.

## Design and Procedures

Both studies made use of a semi-structured clinical interview format. Ginsburg (1997) explains why the clinical interview is often the best approach to access and understand student thinking, and why standardized approaches are not always effective. He explains that standardized methods are not suitable for complex thinking and that children may not interpret the same questions in the same ways. Furthermore, as demonstrated by the vignette at the beginning of this paper, student thinking can be quite nuanced in the domain of mathematical reasoning I explore in this dissertation, and a clinical interview allowed me to gain access to different types of thinking. Ginsburg (1997) argues that “complex thinking generally does not take place in a short period of time in response to a narrowly focused test item” (p. 17). A clinical interview allowed me to create a more accurate model that describes students’ nuanced, leveled understandings, along with qualitative descriptions of what these levels look like. The interview was, however, semi-structured and there were some probe questions that were asked in all interviews. A semi-structured protocol provided some level of consistency across interviews, which was important for direct comparison across students and tasks.

Study 1 focused on additive thinking in the context of addition and subtraction tasks, while Study 2 focused on multiplicative thinking in the context of multiplication and division tasks. The interview procedures and study design for Study 1 and Study 2 were identical, but the interview tasks were different across the studies. Each task was an opportunity for a conversation that would allow me to examine students’ understandings of arithmetic generalizations in two domains (additive and multiplicative generalizations), focusing on three types of generalizations (generalizations related to *identity*, *direction of change*, and *relationship between operations*). All interviews were videotaped and the videos were used for coding and analyses. Any written work was also scanned and used for analyses. In this section, I describe the different tasks and the methods for administering the tasks.

**Interview Tasks and Procedures.** Students were presented with two baseline tasks and 10 focal tasks, which were the focus of analyses. The purpose of the baseline tasks was to ensure that students had some minimal fluency with arithmetic, and to make sure they understood the wording of the tasks. One task involved a computation using one operation, and the other involved a computation using two operations (see Table 2). The baseline tasks were similar in wording and structure to the interview tasks. All students included in the study were successful on both baseline tasks.

Table 2: Baseline Tasks

	Study 1: Additive Relations	Study 2: Multiplicative Relations
Baseline Task 1: One Operation	I am thinking about the number 4. If I add 6 to that number, what will happen to my number?	I am thinking about the number 2. If I multiply that number by 6, what will happen to my number?
Baseline Task 2: Two Operations	I am thinking about the number 2. If I add 5 to that number, and then subtract 5, what will happen to my number?	I am thinking about the number 6. If I multiply that number by 5 and then divide by 5, what will happen to my number?

The focal tasks are detailed in Figure 2 and Figure 3 for the additive and multiplicative relations studies, respectively. Before beginning the focal tasks, I explained to students that the

number I was thinking about was positive for every task. For each task, the tables include the arithmetic generalization that the task was designed to explore; I describe the generalization with both letter notation (e.g.,  $N + 6 > N$ ;  $N \times 6 > N$ ) and natural language (e.g., addition and multiplication makes numbers greater).

Every task from the additive relations study was matched with a task from the multiplicative relations study, and Figure 2 and Figure 3 convey the parallelism of the two studies. The *addition identity/A-ID1*<sup>4</sup> task in Study 1 and *multiplication identity/M-ID1* task in Study 2, for example, examined students' reasoning about identity for addition and identity for multiplication respectively. Within each study, consecutive tasks were also paired, or designed in a particular manner to allow for comparison. The first two tasks listed in Figure 2 for Study 1, for example, were paired in the sense that *adding/A-D1* examined students' understandings of direction of change for addition, while *subtracting/A-D2* examined students' understandings of direction of change for subtraction.

Additionally, for the *direction of change* and *relationship between addition/subtraction* tasks, there were two task subtypes. For the *direction of change* tasks, two tasks involved only one operation and two tasks involved two operations and comparing the sums or differences. For the *relationship between addition/subtraction* tasks, two tasks were symmetrical and had equal addends and subtrahends, while two tasks were asymmetrical and involved unequal addends and subtrahends. The focal tasks were designed in this fashion to allow analyses of task types and subtypes across and within studies to gain insight into the development of students' understandings of arithmetic generalizations. The rationale behind this was based in my hypothesis that particular arithmetic generalizations (e.g., *identity*) develop earlier than others (e.g., *relationship between addition and subtraction*). In subsequent chapters, I elaborate further on the similarities and differences between the different tasks and task subtypes for each study.

I printed each task on a piece of paper and read tasks aloud to students as they progressed through the interview. Students had a marker to make any marks on the paper that they needed. Follow up probe questions included the following:

1. Do you think the number will get larger? Smaller? Stay the same? (to be included only if student does not include this in initial response)
2. Can you explain your thinking?
3. How can you know for sure if you don't know what number(s) we begin with?
4. Will this be true no matter what number I was thinking about in the beginning?
5. I noticed you [wrote these numbers / operations down]. Can you explain how that helped you?

All students received these standard probe questions to allow for comparisons across students. Depending on students' responses, I typically asked many additional questions to better understand their thinking.

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<sup>4</sup> This task notation will be used throughout this dissertation. A = additive; M = multiplicative; D = direction of change; ID = identity; R = relationship between operations. The number denotes the specific task as it is displayed in Figure 2 or Figure 3.

Generalization Type	Task	Arithmetic Generalization <sup>5</sup> (where N represents a positive integer)	Problem Subtype
Direction of Change	Adding/A-D1 I am thinking about a number. If I add 6 to that number, what will happen to my number?	$N + 6 > N$ <i>Addition makes numbers greater.</i>	Not comparing
	Subtracting/A-D2 I am thinking about a number. If I subtract 3 from that number, what will happen to my number?	$N - 3 < N$ <i>Subtraction makes numbers lesser.</i>	
	Adding/comparing/A-D3 Your teacher and I are both thinking about the same number. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?	$N + 8 > N + 6$ <i>Adding a greater number will yield a greater sum.</i> <i>Adding a lesser number will yield a lesser sum.</i>	Comparing
	Subtracting/comparing/A-D4 Your teacher and I are both thinking about the same number. I subtract 5 from my number. Your teacher subtracts 7 from her number. Who will get a larger number after subtracting? Me or your teacher?	$N - 5 > N - 7$ <i>Subtracting a greater number will yield a lesser difference.</i> <i>Subtracting a lesser number will yield a greater difference.</i>	
Identity	Addition identity/A-ID1 I am thinking about a number. If I add 0 to that number, what will happen to my number?	$N + 0 = N$ <i>0 is the additive identity.</i> <i>If you add 0 to any number, the number will remain the same.</i>	
	Subtraction identity/A-ID2 I am thinking about a number. If I subtract 0 from that number, what will happen to my number?	$N - 0 = N$ <i>0 is the additive identity.</i> <i>If you subtract 0 from any number, the number will remain the same.</i>	
Relationship between Addition/Subtraction	Addend before subtrahend/addend equal to subtrahend/A-R1 I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?	$N + 5 - 5 = N$ <i>Addition and subtraction are inverse operations.</i> <i>If you add and subtract the same number, your initial number will remain the same.</i>	Symmetric
	Subtrahend before addend/subtrahend equal to addend/A-R2 I am thinking about a number. If I subtract 6 from that number and then add 6, what will happen to my number?	$N - 6 + 6 = N$ <i>Addition and subtraction are inverse operations.</i> <i>If you add and subtract the same number, your initial number will remain the same.</i>	
	Addend before subtrahend/addend greater than subtrahend/A-R3 I am thinking about a number. If I add 5 to that number and then subtract 3, what will happen to my number?	$N + 5 - 3 = N + 2$ <i>Addition and subtraction are inverse operations.</i> <i>Adding more than what is subtracted yields a number that is greater than the initial number.</i> <i>Subtracting less than what is added yields a number that is greater than the initial number.</i>	Asymmetric
	Subtrahend before addend/subtrahend greater than addend/A-R4 I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?	$N - 7 + 5 = N - 2$ <i>Addition and subtraction are inverse operations.</i> <i>Subtracting more than what is added yields a number that is less than the initial number.</i> <i>Adding less than what is subtracted yields a number that is less than the initial number.</i>	

Figure 2. Study 1 – Additive relations: Focal tasks.

<sup>5</sup> All arithmetic generalizations apply only for operations on positive integers.

Generalization Type	Task	Arithmetic Generalization <sup>6</sup> (where N represents a positive integer)	Problem Subtype
Direction of Change	Multiplying/M-D1 I am thinking about a number. If I multiply that number by 6, what will happen to my number?	$6N > N$ <i>Multiplication makes numbers greater.</i>	Not comparing
	Dividing/M-D2 I am thinking about a number. If I divide that number by 3, what will happen to my number?	$N/3 < N$ <i>Division makes numbers lesser.</i>	
	Multiplying/comparing/M-D3 Your teacher and I are both thinking about the same number. I multiply the number by 6. Your teacher multiplies the number by 8. Who will get a larger number after multiplying? Me or your teacher?	$8N > 6N$ <i>Multiplying by a greater number will yield a greater product.</i> <i>Multiplying by a lesser number will yield a lesser product.</i>	Comparing
	Dividing/comparing/M-D4 Your teacher and I are both thinking about the same number. I divide the number by 5. Your teacher divides the number by 7. Who will get a larger number after dividing? Me or your teacher?	$N/5 > N/7$ <i>Dividing by a greater number will yield a lesser quotient.</i> <i>Dividing by a lesser number will yield a greater quotient.</i>	
Identity	Multiplication identity/M-ID1 I am thinking about a number. If I multiply that number by 1, what will happen to my number?	$N \times 1 = N$ <i>1 is the multiplicative identity.</i> <i>If you multiply any number by 1, the number will remain the same.</i>	
	Division identity/M-ID2 I am thinking about a number. If I divide that number by 1, what will happen to my number?	$N/1 = N$ <i>1 is the multiplicative identity.</i> <i>If you divide any number by 1, the number will remain the same.</i>	
Relationship between Multiplication/Division	Multiplier before divisor/multiplier equal to divisor/M-R1 I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?	$5N/5 = N$ <i>Multiplication and division are inverse operations.</i> <i>If you multiply and divide by the same number, your initial number will remain the same.</i>	Symmetric
	Divisor before multiplier/divisor equal to multiplier/M-R2 I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number?	$6N/6 = N$ <i>Multiplication and division are inverse operations.</i> <i>If you multiply and divide by the same number, your initial number will remain the same.</i>	
	Multiplier before divisor/multiplier greater than divisor/M-R3 I am thinking about a number. If I multiply that number by 5 and then divide by 3, what will happen to my number?	$5N/3 > N$ <i>Multiplication and division are inverse operations.</i> <i>Multiplying by more than you divide by yields a number that is greater than the initial number.</i> <i>Dividing by less than you multiply by yields a number that is greater than the initial number.</i>	Asymmetric
	Divisor before multiplier/divisor greater than multiplier/M-R4 I am thinking about a number. If I divide that number by 7 and then multiply by 5, what will happen to my number?	$5N/7 < N$ <i>Multiplication and division are inverse operations.</i> <i>Dividing by more than you multiply by yields a number that is less than the initial number.</i> <i>Multiplying by less than you divide by yields a number that is less than the initial number.</i>	

Figure 3. Study 2 – Multiplicative relations: Focal tasks.

<sup>6</sup> All arithmetic generalizations apply only for operations on positive integers.

**Piloting.** I finalized the interview tasks by piloting with fourth and fifth graders, which involved interviewing 32 students and administering a paper and pencil fixed response assessment (see Appendix) with 100 students. The pilot interviews were videotaped. These pilot data were helpful in refining the methods that I used for my two studies reported in Chapters 3 and 4. First, watching and analyzing the videos of interviews allowed me to better understand what types of questions would be effective in engaging students with the mathematical ideas I was interested in exploring. Second, reviewing the interview and assessment data helped me decide what types of problems and arithmetic generalizations to focus on for this dissertation. Third, preliminary qualitative and quantitative analyses allowed me to begin to understand the different types of thinking that students exhibit and to confirm that thinking does in fact vary across different types of arithmetic generalizations. I designed the dissertation study to explore these ideas further and more systematically.

**Counterbalancing.** Tasks were administered in counterbalanced order to control for practice effects and to make sure order and task type were not confounded. I created 8 different orders for each study, and three students received each order within each study (see Figure 4).

	Ordinal Position of Task									
	1	2	3	4	5	6	7	8	9	10
Order 1	ID1	D1	R1	D2	R2	ID2	D3	R3	D4	R4
Order 2	R2	ID1	D2	R1	D1	R4	ID2	D4	R3	D3
Order 3	D3	R3	ID1	D4	R4	D1	R1	ID2	D2	R2
Order 4	R4	D4	R3	ID1	D3	R2	D2	R1	ID2	D1
Order 5	ID2	D1	R1	D2	R2	ID1	D3	R3	D4	R4
Order 6	R2	ID2	D2	R1	D1	R4	ID1	D4	R3	D3
Order 7	D3	R3	ID2	D4	R4	D1	R1	ID1	D2	R2
Order 8	R4	D4	R3	ID2	D3	R2	D2	R1	ID1	D1

Figure 4. Different orders in which tasks were administered. D = direction of change (dark blue); ID = identity (red); R = relationship between operations (light blue). The numbers, 1, 2, 3, and 4 refer to the specific task as detailed in Figures 2 and 3.

I used the Latin square approach to obtain a limited set of different orders (see Figure 5). For each type of arithmetic generalization, I created an array, and each row represented a different order. The tasks were arranged so that every row and column of the array included each task from the set exactly once. These orders were then integrated to obtain the 8 different orders detailed in Figure 4.

	Direction of Change				Identity		Relationship between Addition/Subtraction and Multiplication/Division			
Order 1	D1	D2	D3	D4	ID1	ID2	R1	R2	R3	R4
Order 2	D2	D1	D4	D3	ID2	ID1	R2	R1	R4	R3
Order 3	D3	D4	D1	D2			R3	R4	R1	R2
Order 4	D4	D3	D2	D1			R4	R3	R2	R1

Figure 5. Different orders for each set of tasks of each type of arithmetic generalization. D = direction of change; ID = identity; R = relationship between operations.

### Procedures for Data Analysis

The interview tasks produced student responses and reasoning, which were the primary data source for the analyses I discuss in the next three chapters. A fine-grained analysis of students' responses allowed me to identify patterns and commonalities in student thinking, as well as recognize how student thinking shifted across tasks. In this section, I review the procedures I used for data analysis, including the development of the coding scheme and the coding process. The analyses were aimed at addressing the research questions I described Chapter 1:

1. Can fourth graders make arithmetic generalizations in the context of additive arithmetic operations (addition and subtraction) and in the context of multiplicative arithmetic operations (multiplication and division)? Further, what is the quality of their reasoning when they engage with arithmetic generalizations?
2. How does students' production of generalizations vary across different arithmetic domains (e.g., additive versus multiplicative) and across the different types of generalizations within each domain (e.g., identity versus inverse operations)?

**Research Question 1: Character of Students' Understanding.** I completed a fine-grained analysis of students' responses to the interview tasks to address the first research question, focusing on the extent to which fourth graders can make arithmetic generalizations and the character of their understandings. I identified patterns in student thinking across tasks to conceptualize different types of thinking and levels of generality with which students treated the operations. The data revealed four distinct levels of thinking that I elaborate on in the next two chapters.

The four identified levels of thinking became the basis of the coding scheme, with which I coded students' reasoning with a Level 1, Level 2, Level 3, or Level 4 code. Level 4 represented the most generalized thinking that students demonstrated, while Level 1 was the least generalized level of thinking. Level 1 codes were accompanied by an *a*, *b*, *c*, *d*, or *e* code, which provided further insight into the character of student thinking and which I will elaborate on in the next chapter. Student interviews were videotaped. Responses for each task were assigned a code (i.e., 1a, 1b, 1c, 1d, 1e, 2, 3, or 4). Another doctoral student in mathematics education was trained to use the coding scheme and served as a second coder for reliability purposes. In the next two chapters, I elaborate on the coding process and provide excerpts from transcripts that provide the reader with a better understanding of how I differentiated between the levels of thinking and how I made coding decisions.

**Research Question 2: Variations in Student Thinking Across Tasks.** I designed the Study 1 and Study 2 tasks in a manner that allowed me to analyze differences across arithmetic generalization types within and across the additive and multiplicative domains. Figure 6 details the analytical framework I used to pair and group students' responses within and across studies to complete these analyses. I will discuss the Study 1 analyses in more detail to explain this figure. The *adding/A-D1* and *subtracting/A-D2* tasks, for instance, are paired tasks. Student responses to the *adding/A-D1* and *subtracting/A-D2* tasks were compared to determine if the operations of addition and subtraction elicited different types of responses from students. As detailed in Figure 6, there are a total of 5 sets of paired tasks in Study 1. Furthermore, the *adding/A-D1* and *subtracting/A-D2* tasks did not involve comparing sums or differences, while the *adding/comparing/A-D3* and *subtracting/comparing/A-D4* tasks did involve comparing; student responses across these two sets of tasks were analyzed to determine if comparing played a role in students' thinking. *Not comparing/comparing* and *symmetric/asymmetric* were the task subtypes. Finally, there were three types of arithmetic generalizations that I examined: *direction of change*, *identity*, and *relationship between addition and subtraction*.

STUDY 1 ANALYSES (Chapter 3)			STUDY 2 ANALYSES (Chapter 4)		
Paired Tasks	Task Subtypes	Type of Arithmetic Generalization	Paired Tasks	Task Subtypes	Type of Arithmetic Generalization
[ A-D1 ] [ A-D2 ]	[ A-D1 ] [ A-D2 ]	A-D1 A-D2	[ M-D1 ] [ M-D2 ]	[ M-D1 ] [ M-D2 ]	M-D1 M-D2
[ A-D3 ] [ A-D4 ]	[ A-D3 ] [ A-D4 ]	A-D3 A-D4	[ M-D3 ] [ M-D4 ]	[ M-D3 ] [ M-D4 ]	M-D3 M-D4
[ A-ID1 ] [ A-ID2 ]	A-ID1 A-ID2	A-ID1 A-ID2	[ M-ID1 ] [ M-ID2 ]	M-ID1 M-ID2	M-ID1 M-ID2
[ A-R1 ] [ A-R2 ]	[ A-R1 ] [ A-IR2 ]	A-R1 A-R2	[ M-R1 ] [ M-R2 ]	[ M-R1 ] [ M-R2 ]	M-R1 M-R2
[ A-R3 ] [ A-R4 ]	[ A-R3 ] [ A-R4 ]	A-R3 A-R4	[ M-R3 ] [ M-R4 ]	[ M-R3 ] [ M-R4 ]	M-R3 M-R4
	not comparing / comparing	direction of change		not comparing / comparing	direction of change
	symmetric / asymmetric	identity		symmetric / asymmetric	identity
		relationship between addition/subtraction			relationship between multiplication/division
CROSS STUDY ANALYSES (Chapter 4)					
Paired Tasks Across Studies	Task Subtypes	Type of Arithmetic Generalization			
[ A-D1 M-D1 ] [ A-D2 M-D2 ] [ A-D3 M-D3 ] [ A-D4 M-D4 ]	[ A-D1 M-D1 ] [ A-D2 M-D2 ] [ A-D3 M-D3 ] [ A-D4 M-D4 ]	A-D1 M-D1 A-D2 M-D2 A-D3 M-D3 A-D4 M-D4	[ A-ID1 M-ID1 ] [ A-ID2 M-ID2 ]	A-ID1 M-ID1 A-ID2 M-ID2	A-ID1 M-ID1 A-ID2 M-ID2
[ A-R1 M-R1 ] [ A-R2 M-R2 ] [ A-R3 M-R3 ] [ A-R4 M-R4 ]	[ A-R1 M-R1 ] [ A-R2 M-R2 ] [ A-R3 M-R3 ] [ A-R4 M-R4 ]	A-R1 M-R1 A-R2 M-R2 A-R3 M-R3 A-R4 M-R4			
	not comparing / comparing	direction of change			
	symmetric / asymmetric	identity			
		relationship between inverse operations			

Figure 6. Analytical frame. A = additive; M = multiplicative; D = direction of change; ID = identity; R = relationship between operations.

### Chapter 3: Students' Understandings of Arithmetic Generalizations in the Context of Addition and Subtraction Tasks

This chapter examines fourth graders' understandings of arithmetic generalizations in the context of addition and subtraction tasks. I first consider students' thinking and performance across tasks to understand and identify patterns in the character of children's reasoning. I then examine students' thinking and performance across generalization types (*direction of change*, *identity*, and *relationship between addition and subtraction*) and task subtypes to identify similarities and differences in students' generalizations across different tasks.

#### Overview of Tasks

This section orients the reader to the additive tasks in the study, highlighting similarities and differences between the different types of tasks and task subtypes. The purpose of this section is to help the reader understand how the tasks were designed in relation to one another, which is important in making sense of the forthcoming analyses.

Each interview began with a set of two baseline tasks. The baseline tasks were followed by 10 focal tasks. As discussed in Chapter 2, a set of probe questions followed students' initial responses and those probe questions were the same across all tasks.

**Baseline Tasks.** Two baseline tasks, similar in wording and structure to the focal tasks, were presented to students to determine whether they had some minimal fluency with addition and subtraction, and to make sure that they understood the wording of the focal tasks that followed. These tasks were important for purposes of validity, because the findings from the study cannot be attributed to either students' inability to add and subtract or their understanding of the task. If students did not demonstrate proficiency on the baseline tasks, then difficulties in producing the arithmetic generalizations might be attributable to the inability to add or subtract or to understand the wording of the task, rather than their ability to coordinate arithmetic relations to produce a generalization. The first baseline task involved one operation, while the second baseline task involved two operations (see Table 3).

Table 3: Study 1 Baseline Tasks

Baseline Task 1: Addition	Baseline Task 2: Addition and Subtraction
I am thinking about the number 4. If I add 6 to that number, what will happen to my number?	I am thinking about the number 2. If I add 5 to that number, and then subtract 5, what will happen to my number?

**Focal Tasks for Analyses.** A total of 10 focal tasks were presented to students and these tasks are the focus of the analyses in this chapter. Students were presented 4 *direction of change* tasks, 2 *identity* tasks, and 4 *relationship between addition and subtraction* tasks. The 10 tasks were all similar in structure and wording, but each task addressed a different arithmetic generalization. I provide additional detail about the different tasks and their role in the analyses in the next three sections.

**Direction of Change Tasks.** The *direction of change* tasks, depicted in Figure 7, examined students' understandings of the directionality of change when an operation is performed (i.e., addition of positive numbers increases the numerical value, while subtraction of

positive numbers decreases the numerical value). As discussed in the previous chapter, the first two tasks in the figure (*adding/A-D1* and *subtracting/A-D2*) were paired and designed to allow me to examine differences between students' understandings of addition and subtraction. The third and fourth tasks (*adding/comparing/A-D3* and *subtracting/comparing/A-D4*) were designed in a similar manner. Additionally, there were two task subtypes, *not comparing* and *comparing*. The first two tasks in Figure 7 are *not comparing* tasks; students were required to consider one operation and evaluate how the number changes. The second two tasks are *comparing* tasks; students were asked to consider two operations and compare the effect of one to the other.

Task	Arithmetic Generalization	Task Subtype
Adding/A-D1 I am thinking about a number. If I add 6 to that number, what will happen to my number?	$N + 6 > N$ <i>Addition makes numbers greater.</i>	not comparing
Subtracting/A-D2 I am thinking about a number. If I subtract 3 from that number, what will happen to my number?	$N - 3 < N$ <i>Subtraction makes numbers lesser.</i>	
Adding/comparing/A-D3 Your teacher and I are both thinking about the same number. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?	$N + 8 > N + 6$ <i>Adding a greater number will yield a greater sum.</i> <i>Adding a lesser number will yield a lesser sum.</i>	comparing
Subtracting/comparing/A-D4 Your teacher and I are both thinking about the same number. I subtract 5 from my number. Your teacher subtracts 7 from her number. Who will get a larger number after subtracting? Me or your teacher?	$N - 5 > N - 7$ <i>Subtracting a greater number will yield a lesser difference.</i> <i>Subtracting a lesser number will yield a greater difference.</i>	

Figure 7. Direction of Change Tasks. A = additive; D = direction of change.

In addition to the standard probe questions students received across all tasks, a few students needed some clarification for the *comparing* tasks. These students began by substituting two different numbers instead of one number in the task. When this occurred, I would say, "I see what you did, but it says here that we are thinking about the *same* number."

**Identity Tasks.** The *identity* tasks, depicted in Figure 8, focused on students' thinking about properties of an identity operation: the addition or subtraction of 0 to any number leaves its value unchanged. The two tasks were paired and designed to allow me to make comparisons between students' understandings of addition and subtraction.

Task	Arithmetic Generalization
Addition identity/A-ID1 I am thinking about a number. If I add 0 to that number, what will happen to my number?	$N + 0 = N$ <i>0 is the additive identity.</i> <i>If you add 0 to any number, the number will remain the same.</i>
Subtraction identity/A-ID2 I am thinking about a number. If I subtract 0 from that number, what will happen to my number?	$N - 0 = N$ <i>0 is the additive identity.</i> <i>If you subtract 0 from any number, the number will remain the same.</i>

Figure 8. Identity Tasks. A = additive; ID = identity

**Relationship between Addition and Subtraction Tasks.** The relationship between addition and subtraction tasks, depicted in Figure 9, explored students' reasoning about the

inverse relationship between addition and subtraction. As discussed in the previous chapter, the first two tasks in the figure (addend before subtrahend/addend equal to subtrahend/A-R1 and subtrahend before addend/subtrahend equal to addend/A-R2) were paired and designed to allow me to examine whether the order of operations played a role in students' understanding about inverse operations. The third and fourth tasks (addend before subtrahend/addend greater than subtrahend/A-R3 and subtrahend before addend/subtrahend greater than addend/A-R4) were designed in a similar manner. Additionally, there were two task subtypes, symmetric and asymmetric. The first two tasks in Figure 9 are symmetric tasks: students need to reason with addends and subtrahends that are equal to one another, which leaves the value of the initial number unchanged. The second two tasks are asymmetric tasks; students need to reason with unequal addends and subtrahends, which requires them to compare the magnitudes of the addend and subtrahend to determine how the initial number changes.

Task	Arithmetic Generalization	Task Subtype
Addend before subtrahend/addend equal to subtrahend/A-R1  I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?	$N + 5 - 5 = N$  <i>Addition and subtraction are inverse operations.</i>  <i>If you add and subtract the same number, your initial number will remain the same.</i>	symmetric
Subtrahend before addend/subtrahend equal to addend/A-R2  I am thinking about a number. If I subtract 6 from that number and then add 6, what will happen to my number?	$N - 6 + 6 = N$  <i>Addition and subtraction are inverse operations.</i>  <i>If you add and subtract the same number, your initial number will remain the same.</i>	
Addend before subtrahend/addend greater than subtrahend/A-R3  I am thinking about a number. If I add 5 to that number and then subtract 3, what will happen to my number?	$N + 5 - 3 = N + 2$  <i>Addition and subtraction are inverse operations.</i>  <i>Adding more than what is subtracted yields a number that is greater than the initial number.</i>  <i>Subtracting less than what is added yields a number that is greater than the initial number.</i>	asymmetric
Subtrahend before addend/subtrahend greater than addend/A-R4  I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?	$N - 7 + 5 = N - 2$  <i>Addition and subtraction are inverse operations.</i>  <i>Subtracting more than what is added yields a number that is less than the initial number.</i>  <i>Adding less than what is subtracted yields a number that is less than the initial number.</i>	

Figure 9. Relationship between Addition and Subtraction Tasks. A = additive; R = relationship between operations.

In addition to the standard probe questions students received across all tasks, a few students required some clarification for these tasks. These students parsed the task into two separate questions, rather than one. For example, for the *subtrahend before addend/subtrahend greater than addend/A-R4*, two students said the number would decrease if you subtract 7 and the number would increase if you add 5. In these instances, I would say something like, “I see what you did, but we are going to subtract 7 and add 5 one after the other. I am thinking about a number. We’re going to subtract 7 from that number. And then whatever number we get after subtracting, we’re going to add 5 to *that*. So what happens after we do *both* of those things?”

## Results

The results are presented in three sections. In the first, I briefly discuss the findings from the baseline tasks. In the second, I focus on the character of students' understandings and consider variations in the level of generalizations students produce when they consider the addition and subtraction tasks. I provide an overview of four levels of student generalizations that I observed and coded for each task, and I then discuss both clear and less clear examples of these levels. In the third, I compare students' levels in the generalizations they produced both across and within the different types of tasks, *direction of change*, *identity*, and *relationship between addition and subtraction*. The findings show that these fourth graders often produced upper levels of generalization (Levels 3 and 4) but their thinking often varied across the different types of tasks.

**Baseline Tasks.** Students' success on the baseline tasks indicates that the findings I will describe in this chapter cannot be explained by a lack of computational fluency or confusion over the wording of the tasks. All 24 students in Study 1 provided correct solutions and appropriate explanations for both baseline tasks. All students responded with the correct solutions of 10 and 2 for *Baseline Task 1: Addition* and *Baseline Task 2: Addition and Subtraction* respectively, and they explained the steps they took to arrive at their solutions.

**Research Question 1: Quality of Students' Understandings.** A fine-grained qualitative analysis involved an iterative process of watching the interviews, observing patterns in student thinking, and identifying similarities and differences across students' explanations. I designed a coding scheme to capture the quality of students' arithmetic generalizations. This scheme consisted of four levels (see Table 4), and students' explanations for each task were coded as Level 1, 2, 3, or 4.

In Table 4, I describe and provide examples of the levels of thinking that I have conceptualized based on the interview data (the sample responses are reflective of the types of thinking found across tasks). Level 1 and Level 2 thinking are more concrete in nature because students' explanations are tied to specific instances or examples, rather than abstractions or generalizations about the arithmetic operations. Level 3 and 4 are more generalized because students do not rely on specific instances or examples, and instead, make generalizations about the arithmetic operations. At Level 1, students do not demonstrate any understanding of the arithmetic generalization. They may, for example, argue that the number can change in a variety of ways and that there is no possible way to know how the number changes without knowing what it is. At Level 2, students use substitution of one or more numbers for the unknown number, and make a generalization based on the instance or instances they examine. At Level 3, students generalize without relying on specific instances, but they have difficulty explaining why the generalization holds true. Level 4 is what I describe as the most generalized level of thinking, at which students generalize properties of addition and subtraction to all numbers without considering specific instances or numbers to make sense of the task, and they can explain the rationale behind the generalization they make. Table 5 provides short excerpts from students' responses for each task at each level, which illustrates further how I categorized and coded students' responses.

A total of 240 responses (24 students x 10 tasks) were coded. Another doctoral student in mathematics education was trained to use the coding scheme and served as a second coder for

reliability purposes. He coded 20% of the data (5 interviews selected at random). Inter-rater reliability was calculated as the percentage of agreements out of the total number of codes. Coder agreement was 98%, and any disagreements were resolved through a discussion.

Table 4: Coding Scheme for Levels of Generality

Level Description	Examples from Subtrahend Greater than Addend/A-R4: <i>I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?</i>
<p><b>Level 1. No Generalization:</b> Students do not demonstrate any understanding of the arithmetic generalization and may do one of the following:</p> <p>(1a/<b>more information required</b>): state that additional information, like a specific number, is necessary to compare and evaluate the numbers</p> <p>(1b/<b>incorrect generalization</b>): make an incorrect mathematical generalization based on the information given (e.g., the order of operations)</p> <p>(1c/<b>specific number</b>): respond with a specific number that they arrived at after performing a computation</p> <p>(1d/<b>combination</b>): more than one of the above (often 1a and 1b)</p> <p>(1e/<b>no solution</b>): say that they do not know or provide unrelated response</p>	<p>“It depends on what the number is. It could get bigger. It could get smaller. We just don’t know.”</p> <p>“The last thing you do is add and adding makes numbers larger. So the number would get larger.”</p> <p>“The number would be 5. I started with 7, then I got 0, and then I added 5 and got 5.”</p>
<p><b>Level 2. Generalization Based on Substitution:</b> Students rely on specific numbers and mathematical instances to make a generalization about the arithmetic operation. They substitute a specific number for the unknown value in the task to make a generalization, and to compare and evaluate numbers. Like at Level 1, students do not demonstrate an understanding of the arithmetic generalization that is independent from the specific context, but they do make an accurate generalization based on the substitution of one or more numbers.</p>	<p>“If the number was 9, if you subtract 7, it’ll be 2. And then add 5, it’ll be 7. So I think the number will be smaller.”</p>
<p><b>Level 3. Generalization that does not Rely on Substitution:</b> Students demonstrate a developing understanding of the arithmetic generalization. They do not rely on the substitution of numbers to compare and evaluate numbers, and they make a correct generalization, but they cannot explain the mathematical reasoning behind the generalization they make. They often refer to their prior math/classroom experience instead. Like at Level 2, students do not demonstrate a deep understanding of the arithmetic generalization, but their understanding is independent of the specific context and they do not rely on substitution.</p>	<p>“I think the number would get smaller because 5 is less than 7, but I’m not exactly sure why. I just know that you need to look at the 5 and 7.”</p> <p>“The number would get smaller because that’s what I’ve seen happen in the classroom.”</p> <p>“The number would always get smaller, but it’s hard to explain why.”</p>
<p><b>Level 4. Generalization Based on Reasoning about Operation(s):</b> Students demonstrate a well-developed understanding of the arithmetic generalization and do not rely on the substitution of numbers to compare and evaluate numbers. Like at Level 3, students do not rely on substitution, but they can also demonstrate a deeper understanding of the arithmetic generalization.</p>	<p>“It would get smaller because 7 is more than 5, and you’re subtracting more than you’re adding. It’s like you’re taking away more than you’re giving back.”</p> <p>“Your number will be 2 less. You’ve only added back 5, not the full 7. And 7 is 2 more than 5.”</p>

Table 5: Sample Student Responses Coded at each Level for each Task

Task	Level 1: No Generalization	Level 2: Generalization Based on Substitution	Level 3: Generalization that does not Rely on Substitution	Level 4: Generalization Based on Reasoning about Operation(s)
<p>Adding/A-D1</p> <p>I am thinking about a number. If I add 6 to that number, what will happen to my number?</p>	<p>"That's another one where I need to know what the number is." (ID 11)</p>	<p>(no occurrences)</p>	<p>(no occurrences)</p>	<p>"It will get bigger. This number [6] will be added to that number to make it bigger....Anything plus a number that's more than zero will make it bigger." (ID 01)</p>
<p>Subtracting/A-D2</p> <p>I am thinking about a number. If I subtract 3 from that number, what will happen to my number?</p>	<p>"I don't know...no [after he was asked whether there is any way to know if/how the number changes]" (ID 14)</p>	<p>(no occurrences)</p>	<p>"If you have a number and minus 3 from it, it will probably be an odd or even number left....It will get smaller." The student did not explain why the number will get smaller. (ID 16)</p>	<p>"It will be 3 numbers less because you subtract 3 from it...subtracting the 3 will make it less because subtracting is to make it less, and we're subtracting 3, which will make it 3 less." (ID 03)</p>
<p>Adding/comparing/A-D3</p> <p>Your teacher and I are both thinking about the same number. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?</p>	<p>"It depends on what number." The student also nodded and said 'yes' after he was asked whether either person can get the larger number depending on what number they begin with. (ID 14)</p>	<p>"So you can never go higher than the teacher...If you had 8 and if you add minus 4 to 8, it'll turn into 4. But then if you do minus 4 at 6, it'll turn into 2. And if you plus 1 at 8, it turns into 9. And if you plus 1 at 6, it's going to be 7." (ID 2)</p>	<p>(no occurrences)</p>	<p>"The teacher because you're adding 6 and she's adding 8. 6 is 2 less than 8." (ID 08)</p>
<p>Subtracting/comparing/A-D4</p> <p>Your teacher and I are both thinking about the same number. I subtract 5 from my number. Your teacher subtracts 7 from her number. Who will get a larger number after subtracting? Me or your teacher?</p>	<p>"My teacher because she has a larger number subtracted with her number...this is a bigger number [pointing to 7] than your number [5]." (ID 18)</p>	<p>"If you subtract 7 [from 17 he wrote on page], then that'll just equal 10. And then if you subtract 5, that'll equal 2, I mean 12. And 12 is bigger than 10. So that would mean that you would get the larger number." [ID 10]</p>	<p>(no occurrences)</p>	<p>"You would because you're subtracting 5, a lower number than 7...yours would be larger by 2 numbers." (ID 07)</p>
<p>Addition identity/A-ID1</p> <p>I am thinking about a number. If I add 0 to that number, what will happen to my number?</p>	<p>(no occurrences)</p>	<p>(no occurrences)</p>	<p>"It would stay the same...because it's true for every number." [no further explanation provided despite follow up questions] (ID 04)</p>	<p>"It will be the same number. If you subtract 0, you get the same number, and if you add 0, you get the same number...because 0 doesn't do anything to that number." (ID 09)</p>
<p>Subtraction identity/A-ID2</p> <p>I am thinking about a number. If I subtract 0 from that number, what will happen to my number?</p>	<p>(no occurrences)</p>	<p>(no occurrences)</p>	<p>"The number, it will stay the same...because you're subtracting 0...I don't know [when asked to explain why]." (ID 14)</p>	<p>"Your number is not going to change at all because you're taking away nothing." (ID 19)</p>

Direction of Change

Identity

Task	Level 1: No Generalization	Level 2: Generalization Based on Substitution	Level 3: Generalization that does not Rely on Substitution	Level 4: Generalization Based on Reasoning about Operations
Addend before subtrahend/addend equal to subtrahend/A-R1 I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?	"The number that's only 0, it's going to stay the same. But other numbers, it's going to get smaller." The student then responded 'yes' when asked whether it depends on what number I am thinking about. (ID 13)	"Let's go for 10. So if you add 5, it would be 15. 15 subtract 5 equals 10. So you're not going to change any number... And if you use a different number, you'll get that number." (ID 04)	(no occurrences)	"It would be the same number no matter what... unless you change one of the 5s.... You're adding 5 to whatever number, it's 5 more. And then you're taking 5 away from whatever number that is. So then it would be the same number." (ID 17)
Subtrahend before addend/subtrahend equal to addend/A-R2 I am thinking about a number. If I subtract 6 from that number and then add 6, what will happen to my number?	"I think it might be 6... I thought the first number was 6. And then you have to subtract 6 and that's 0. And then you add 6 again and it will just be 6." Student then said that the number will not always stay the same and that it could also increase or decrease depending on what number I am thinking about. (ID 05)	"It'll stay 6 because if you minus 6, you get 0 and when you add 6 again, it's going to be 6 still." When asked if this is true for all numbers, the student said yes. (ID 16)	(no occurrences)	"It will be the same thing and... because you're just subtracting it, then you're gonna add it again, so it'll be the same number." (ID 15)
Addend greater than subtrahend/A-R3 I am thinking about a number. If I add 5 to that number and then subtract 3, what will happen to my number?	"It will get smaller because it says it will subtract 3... subtract means lose." (ID 24)	(no occurrences)	(no occurrences)	"It will be 2 more than the number you are thinking about... because if you first add 5 and then subtract 3 from 5, it's 2. But I don't know which number so it's plus 2 to any number." (ID 20)
Subtrahend greater than addend/A-R4 I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?	"It's depending what number you're thinking about." Student then said that the number can increase, decrease, or stay the same. (ID 22)	(no occurrences)	"Because 7 and 5 are different numbers, I would say it would get smaller... I can't think of how to explain this." (ID 04)	"If you subtract 7, you're gonna get a smaller number. If you add 5, you're gonna get a bigger number but it's not going to be as high as the first number." (ID 23)

Relationship between Addition/Subtraction

Overall, students demonstrated robust understandings of the arithmetic generalizations with 187 out of 240, or 78%, of these responses coded as Level 4 and most of the remaining responses coded as Level 1 (18%). As described earlier (see Table 4), there were five different varieties of a Level 1 code (1a/*more information required*, 1b/*incorrect generalization*, 1c/*specific number*, 1d/*combination*, and 1e/*no solution*). Of the 43 Level 1 responses, 28 were coded as 1a. With a Level 1a response, students argued that they needed to know what the number was to know how it changed. Figure 10 shows the breakdown of Level 1 responses, illustrating that 1a explanations were the most common and 1b and 1d were the next most common categories for Level 1 responses.

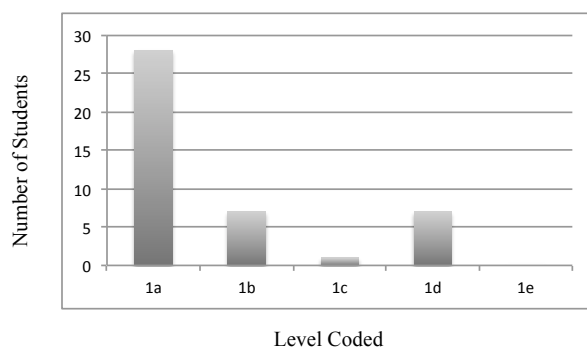


Figure 10. Breakdown of Level 1 Responses

I now turn to a more detailed description of each of the four levels, including excerpts from transcripts. Each level represents my effort to capture student thinking; on some occasions, students' solutions were well captured by my level descriptions, whereas in other cases, levels were a more approximate description of student thinking. For each level of thinking, I describe the nature of the students' solutions, particularly in relation to other levels of thinking, and I provide one clear and one less clear example. These sections also provide the reader with a better understanding of what the interviews with students were like, and what the raw data looked like.

**Level 1: Generalization Based on Substitution.** In a Level 1 response, students do not make the arithmetic generalization that the task was aimed at exploring. They often explain that they need to know more about the number the interviewer was thinking about to know if and how the number changes, or they make an incorrect generalization. Below, I provide some excerpts from interviews to provide the reader with prototypical Level 1 responses, one clear case and one less clear case but coded as Level 1.

*Clear Example of Level 1 Response.* In the interaction below, Student ID 11 responded to the task, "I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?" (*addend before subtrahend/addend equal to subtrahend/A-R1*).

- ID 11: I think I need to know the number.  
 Halдар: Okay, so is it like the other ones you talked about where you said there's no way to know for sure [how the number changes] until you know what number I'm thinking about?  
 ID 11: Yeah.

This student's response was categorized as Level 1a/*more information required* response. In his response, he did not make any sort of generalization about the arithmetic operations or the relationship between addition and subtraction. His explanation was also more concrete in nature as he stated that he needed to know what number I was thinking about to figure out how the number changes.

*Less Clear Example of Level 1 Response.* In the interaction below, Student ID 13 responded to the same task, *addend before subtrahend/addend equal to subtrahend/A-R1*. I categorized her response as 1d/*combination*, but it was not as clear as the previous example that I discussed.

- ID 13: If it was 0, just like for a particular example, if it was 0 and you add 5 to that number, and you subtract 5, it will be 0. But if you're thinking about a number that's greater than 0, then you'll get an answer other than 0.
- Haldar: Okay. Do you know if the number will get larger or smaller, or if there's no way to tell without knowing what number I'm thinking about?
- ID 13: Smaller.
- Haldar: It will get smaller? Okay. How do you know the number will get smaller?
- ID 13: Because well...even though you add 0 to 5, and you subtract it by 5, and the answer will be 0, 0 is no bigger than 5.
- Haldar: Okay, but when you compare it to the first number, the number that I'm thinking about, does it get bigger or smaller?
- ID 13: Well one of the numbers that's only 0, it's going to stay the same. But other numbers, it's going to get smaller.
- Haldar: Okay, so if I was thinking about the number 0, you think the number would stay the same?
- ID13: Yes.
- Haldar: And if I'm thinking about a different number, it may not stay the same?
- ID 13: Yes.
- Haldar: So does it depend on what number I'm thinking about?
- ID 13: Yes.
- Haldar: And it could stay the same *or* get smaller?
- ID 13: Yeah.
- Haldar: Could it get bigger also?
- ID 13: No.
- Haldar: Okay, so it all depends on what number I'm thinking about?
- ID 13: Yeah.

This example is less clear than the previous example because the student first stated that the number will get smaller, but she then shifted her argument and argued that the number will get smaller or stay the same. This student initially tried substitution with 0, but she did not make a clear conclusion or generalization from that substitution. In fact, she stated that the number the interviewer was thinking about can stay the same *or* get smaller after adding 5 and subtracting 5. Since this student made an incorrect generalization that the number will always stay the same or get smaller (1b/*incorrect generalization*) and concluded that how the number changes depends on the initial number the interviewer is thinking about (1a/*more information required*), I categorized her response as Level 1d.

**Level 2: Generalization Based on Substitution.** In a Level 2 response, students make the arithmetic generalization that the task is aimed at exploring, but their reasoning is concrete in

nature: Students rely on specific examples and use substitution to draw conclusions about the numbers in the task. They base their generalizations on one or more examples that they try, and not on any properties of addition or subtraction that they refer to; they plug in a number and make a conclusion about all numbers based on that result. Like at Level 1, students do not demonstrate an understanding of the arithmetic generalization that is grounded in an understanding of the properties of the operations. However, unlike a Level 1 response, a Level 2 response includes an accurate generalization.

To confirm that students did in fact use the number they referred to as a way to figure out the problem, I would ask explicit questions like, “Did you use the number 5 to help you figure out how the number changes?” or “Did using 0 help you figure it out or are you trying to show me what you mean with an example?” This clarification was important because some students with Level 3 and Level 4 responses would also refer to specific instances. All students who made generalizations were also asked a question like, “How can you know for sure if you don’t know what number we are thinking about in the beginning?” Students’ responses to this set of questions helped me sort students’ responses appropriately. Below I provide excerpts from interviews to provide the reader with both a clear and less clear example of a Level 2 response.

*Clear Example of Level 2 Response.* In the interaction below, Student ID 02 responded to the task, “Your teacher and I are both thinking about the same number. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?” (*adding/comparing/A-D3*).

- ID 02: Mmmm...same...no, I’m not sure.  
 Haldar: Okay.  
 ID 02: I’m not really sure because, like last time there’s minus 1 [referring to -1]. So if I minus 4 [referring to -4] at, like the teacher is minus 4. Hmmm. Let me see.  
 Haldar: Okay, take your time. [pause while student seems to be working through problem in her head] Can you think out loud so I can try to understand what you’re thinking?  
 ID 02: Okay, now I get it. So you can never go higher than the teacher.  
 Haldar: Oh, so you’re saying my number will never be larger than the teacher’s?  
 ID 02: Yeah.  
 Haldar: So the teacher will always have a larger number?  
 ID 02: Yeah.  
 Haldar: How do you know the teacher will always have a larger number?  
 ID 02: Because if you have 8 and you add minus 4 to 8, then it’ll turn to 4. But if you do minus 4 at the, add 6, then you’ll turn to 2. So 8 turns more larger.  
 Haldar: Okay.  
 ID 02: And if you plus 1 at 8, and 8 plus 1 turns 9. And if you plus 1 at 6, it turns 7.  
 Haldar: So did you use the numbers -4 and 1 to help you figure it out?  
 ID 02: Yeah!  
 Haldar: Okay, so did those examples help you figure out what happens to the numbers?  
 ID 02: Yeah.  
 Haldar: But how can you know for sure if you don’t know what numbers we’re starting with at the beginning?  
 ID 02: I think...from my thinking...hmmm...from my mind...you can never make higher than the teacher on this number. Because if you have the same number, you can’t even pick and make it higher.  
 Haldar: And will this be true no matter what number we’re both thinking about at the beginning?  
 ID 02: Yes.

Unlike the Level 1 examples I discussed in the previous section, this student made the arithmetic generalization that the task explored. She appeared to arrive at this generalization by plugging in -4 and 1 into the problem and performing the operations on those values. She also confirmed that it was in fact these examples of -4 and 1 that helped her figure out the problem. With my last two questions (“But how can you know for sure if you don’t know what numbers we’re starting with at the beginning?” and “And will this be true no matter what number we’re both thinking about at the beginning?”), I gave this student the opportunity to demonstrate a higher level of thinking to justify her conclusion, but she did not extend her reasoning by talking more generally about the operation of addition or the addends, so her response was categorized as a Level 2.

*Less Clear Example of Level 2 Response.* In the interaction below, Student ID 04 responded to same task, *adding/comparing/A-D3*. His response was also categorized as Level 2, but it was slightly less clear than the previous example that I discussed.

- ID 04: Teacher, because 50...50 plus 8 equals to 58. And um 50 plus 6 is only 56. Which is bigger? 58.
- Haldar: So did you use the number 50 to help you figure this one out?
- ID 04: Yeah, pretty much, yes.
- Haldar: But how can you be sure if you don’t know what number we’re starting with in the beginning?
- ID 04: Because it says your teacher and I are both thinking about the same numbers. So it will be the same, we’ll both be thinking about 50.
- Haldar: Will this be true no matter what number we’re thinking about at the beginning?
- ID 04: Yes.
- Haldar: And why will that always be true?
- ID 04: Because [pauses], I really can’t explain it.

This example is less clear than the previous example. From this student’s initial response, it was unclear whether he used 50 to substitute into the problem and draw a conclusion, or if he was referring to 50 as an illustrative example. However, when I explicitly asked him if he used 50 to help him figure it out, he said yes. And when asked how he can be sure if he doesn’t know what number we begin with, he again stated “we’ll both be thinking of 50.” Again, with my last two questions, I gave this student the opportunity to demonstrate a higher level of thinking to explain the conclusion he arrived at, but he did not do so, and his response was therefore categorized as a Level 2.

***Level 3: Generalization that does not rely on Substitution.*** In a Level 3 response, students make the arithmetic generalization that the task explores, and they do not rely on a concrete example to arrive at the generalization. However, they have trouble justifying the arithmetic generalization in their explanation; they often say they do not know how to explain it or they simply restate the generalization and that it’s true for all numbers. Like at Level 2, students do not demonstrate a deep understanding of the arithmetic generalization, but unlike the prior level, students show a new level of generalization in their reasoning that is independent of the specific context or the substitution of numbers.

*Clear Example of Level 3 Response.* In the interaction below, Student ID 14 responded to the task, “I am thinking about a number. If I subtract 0 from that number, what will happen to my number?” (*subtraction identity/A-ID2*).

ID 14: The number will stay the same.  
 Haldar: Can you explain why it will stay the same?  
 ID 14: Because you're subtracting 0.  
 Haldar: Okay, and how can you know for sure if you don't know what number I'm actually thinking about?  
 ID 14: Well...I don't know.  
 Haldar: Will the number always stay the same no matter what number I'm thinking about at the beginning?  
 ID:14 [nods head yes]

Unlike the Level 2 examples I discussed in the previous section, this student did not rely on the substitution of numbers or concrete values to make the appropriate arithmetic generalization. He clearly stated that the number will always stay the same when you subtract 0 and that it holds true for all numbers, but to justify his reasoning, he simply restated the problem ("Because you're subtracting 0.") He did not make any reference to the properties of subtraction or 0 as a number.

*Less Clear Example of Level 3 Response.* In the interaction below, Student ID 04 responded to the task, "I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?" (*subtrahend greater than addend/A-R4*).

ID 04: So 10 subtracting 7 equals 3. And then 5 adding 3 equals 8.  
 Haldar: Okay...so did my number get smaller? Bigger?  
 ID 04: Smaller. Two numbers smaller.  
 Haldar: Will the number always get smaller no matter what number I'm thinking about, when I subtract 7 and then add 5?  
 ID 04: Yeah. Yes.  
 Haldar: How do you know?  
 ID 04: Since 7 is bigger than 5, right? So it will be smaller.  
 Haldar: And why are you comparing the 7 and 5.  
 ID 04: Since 7 and 5 are like different numbers. I would say it would get smaller.  
 Haldar: And why smaller? What about the 7 and the 5 tell you that?  
 ID 04: I can't think of how to explain this. [he then refers to his initial example of 10, and another example of 50 to illustrate that the number will get smaller]

This example is less clear than the previous example because the student used substitution, but also reasoned about the addend and subtrahend independently. If this student's justification was based solely on the example of 10, this response would have been categorized as Level 2. Though he began and ended his explanation by referring to concrete examples, he also compared the subtrahend, 7, and addend 5, which is an important part of a Level 4 response for this task. However, based on his utterances in our exchange, it was difficult to conclude to what extent he actually understood the relationship between the 5 and 7 or the relationship between addition and subtraction. He was not explicit and when I probed his thinking more about this relationship, he returned to the concrete example. This response was therefore categorized as Level 3 because he could not explain how he used the 5 and 7 to make sense of the problem.

**Level 4: Generalization Based on Reasoning about Operations.** In a Level 4 response, students make the arithmetic generalization that the task is aimed at exploring, and they do not rely on a concrete example to arrive at the generalization. Like at Level 3, students do not rely on

substitution, but unlike a Level 3 response, a Level 4 response includes appropriate mathematical reasoning or justification behind the generalization. This reasoning usually involves a discussion of the general properties of the operations involved in the task.

*Clear Example of Level 4 Response.* In the interaction below, Student ID 03 responded to the task, “I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?” (*subtrahend greater than addend/A-R4*).

- ID 03: You will have 2 less than the number you started off with because you subtracted 7 from it, but you only got 5 back. And 5 plus 2 equals 7, which means you only have 2 less than the number you started with.
- Haldar: And, again, how do you know for sure if you don't know what number I'm starting off with?
- ID 03: I know for sure because 7 is 2 more than 5. But you subtracted 7 and then added 5, which means you only have 2 less than the number you started off with.
- Haldar: Will this be true no matter what number I'm thinking about in the beginning?
- ID 03: Yes.

This student did not rely on the substitution of values to make sense of the problem or to determine how the number changes. He recognized the inverse relationship between addition and subtraction, and he demonstrated an understanding of the asymmetrical relationship between the addend and subtrahend.

*Less Clear Example of Level 4 Response.* In the interaction below, Student ID 20 responded to the same task, *subtrahend greater than addend/A-R4*, but his explanation is not as clear.

- ID 20: First it gets smaller, then it gets bigger [pauses]...well, it's still smaller. It's still smaller by 2. The number is 2 smaller than that number.
- Haldar: Will the number always get smaller by 2?
- ID 20: Yeah.
- Haldar: And how did you figure that out? Can you explain why?
- ID 20: Because if you minus 7 from the number and you add 5, you get 2 because there is 2 remaining when you add 5 again to the 7 [that is taken away].
- Haldar: Okay, and will that be true no matter what number I'm thinking about?
- ID 20: Yeah, yeah.

This example is less clear because the language that this student used is difficult to follow. However, it is important to note that English was not the first language for this student. His response was categorized as Level 4 because he made the appropriate generalization that the task is aimed at exploring and he justified this generalization by talking about how the number first decreases, and then increases, but still experiences a net decrease. He also talked about the relationship between the addend and subtrahend, though not necessarily as articulately as the student in the last example.

**Research Question 2: Variations in Student Thinking Across Tasks.** In this section, I compare students' thinking across the three generalization types: *direction of change*, *identity*, and *relationship between addition and subtraction*. In my comparative analysis, I also consider task subtypes. For example, I compare generalizations that involve symmetric relations with

generalizations that involve asymmetric relations. I show that although the 4<sup>th</sup> graders in the study demonstrated competency with arithmetic generalizations, most students showed inconsistency in their thinking across tasks, with some types of generalizations appearing more difficult than others.

In much of the forthcoming analyses, I group Level 1 and Level 2 responses together, and I group Level 3 and Level 4 responses together; Levels 1 and 2 are both more concrete in nature as students rely on specific numbers, while Levels 3 and 4 involve generalizing that does not require students to deal with specific instances. Combining scores in this manner was helpful because there were a limited number of Level 2 and Level 3 responses and because it made trends in the data more visible. For each task, students received either a ‘generalization score’ of 0 for Level 1 and Level 2 responses, or they received a generalization score of 1 for Level 3 and Level 4 responses.

The proportion of students who received advanced generalization scores (a generalization score of 1) for each task is detailed in Table 6. In this table, I also provide the breakdown of students by level for each task. Note that Level 1 and Level 4 responses occurred the most frequently across tasks, while Level 2 and Level 3 explanations did not occur as frequently. There was also the most variability in student thinking for the *relationship between addition and subtraction* tasks. This table provides an overview of some of the patterns I observed in the data, illustrating which tasks elicited the most generalized levels of thinking from students, and how student thinking varied for each task. I analyze these patterns further in the following sections.

Table 6: Proportion of Students with Advanced Generalization Score and Breakdown of Student Responses by Level for Each Task

Task		Proportion of Students with Advanced Generalization Score (Level 3 or 4)	Frequency Breakdown of Students' Responses by Level			
			Level 1	Level 2	Level 3	Level 4
<b>Direction of Change</b>	Adding/A-D1	.917	2	0	0	22
	Subtracting/A-D2	.917	2	0	1	21
	Adding/comparing/A-D3	.875	1	2	0	21
	Subtracting/comparing/A-D4	.750	4	2	0	18
<b>Identity</b>	Addition identity/A-ID1	1.000	0	0	1	23
	Subtraction identity/A-ID2	1.000	0	0	1	23
<b>Relationship between Addition/Subtraction</b>	Addend before subtrahend/addend greater than subtrahend/A-R1	.708	6	1	0	17
	Subtrahend before addend/subtrahend greater than addend/A-R2	.708	6	1	0	17
	Addend greater than subtrahend/A-R3	.542	11	0	0	13
	Subtrahend greater than addend/A-R4	.542	11	0	1	12

To compare students' understandings across the three types of arithmetic generalizations, I calculated a mean proportion of advanced generalization scores for each type of generalization. For example, if a student's responses were coded with 4, 3, 1, and 1 for the *direction of change* tasks, the proportion of tasks she received an advanced generalization score for was .5. These proportions were averaged across each type of arithmetic generalization and these mean scores were .864, 1.000, and .625 for the *direction of change*, *identity*, and *relationship between*

*addition/subtraction* tasks respectively. A Friedman test<sup>7</sup> showed that there was a statistically significant effect of generalization type on students' production of advanced generalizations ( $\chi^2(2) = 22.082, p = 0.000$ ). Post hoc analysis with Wilcoxon signed-rank tests was conducted with a Bonferroni correction applied, resulting in a significance level set at  $p < 0.017$ , and there were statistically significant differences across all types of generalizations: (a) *direction of change* and *identity* tasks ( $Z = -2.410, p = .016$ ), with medium effect size of  $r = .347$ ; (b) *direction of change* and *relationship between addition/subtraction* tasks ( $Z = -3.007, p = .003$ ) with medium/large effect size of  $r = .434$ ; (c) *identity* and *relationship between addition/subtraction* tasks ( $Z = -3.332, p = .001$ ) with medium/large effect size of  $r = .481$ . These results indicate that students provided the most generalized levels of responses for the *identity* tasks and the least generalized responses to the *relationship between addition/subtraction* tasks, and the differences between all three types of generalizations were statistically significant.

To understand how student thinking varied across tasks, I next analyzed each students' across-task profile, and observed that while some students used the same level of generality across all tasks, most did not. As displayed in Figure 11, 9 of the 24 students demonstrated a consistent level of thinking across all tasks, responding with Level 4 explanations across all tasks. Fifteen out of the 24 students demonstrated *at least* two different levels of thinking over the course of the interview. Of the 11 students who demonstrated two levels of thinking, 10 of them responded with only Level 1 and Level 4 explanations across all tasks, demonstrating again that Level 1 and Level 4 responses were the most common among students and Level 2 and Level 3 responses did not occur as frequently.

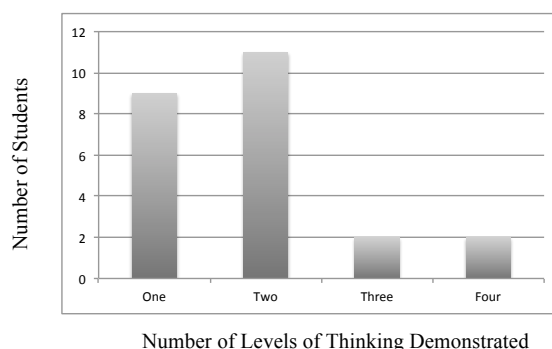


Figure 11. Number of levels of thinking demonstrated by students across tasks.

These data illustrate that even the fourth graders who demonstrated the most generalized thinking for some tasks, reverted to less generalized ways of thinking for particular tasks. This finding warranted further analyses, which I will discuss in detail in the following sections.

In the remaining sections, I examine student thinking within each type of arithmetic generalization more closely to understand these findings. To do so, I first provide an overview of student performance for each type of generalization. I then analyze how student thinking varied across particular variables, like the type of operation and order of operations, and how individual students' responses varied across tasks.

<sup>7</sup> A Friedman test was used because the data was ordinal and non-normally distributed.

**Direction of Change Tasks.** Figure 12 presents the frequency with which students used the four generalization levels for each of the *direction of change* tasks. Level 4 responses occurred the most frequently across tasks and most students received an advanced generalization score for each *direction of change* task. Of the 24 participants, 17 students responded with a Level 4 explanation across all four tasks, demonstrating that the fourth graders had a robust understanding of this type of generalization. Next, I analyze the effect of the two different operations and task subtypes, and the consistency of students' responses across the *direction of change tasks*.

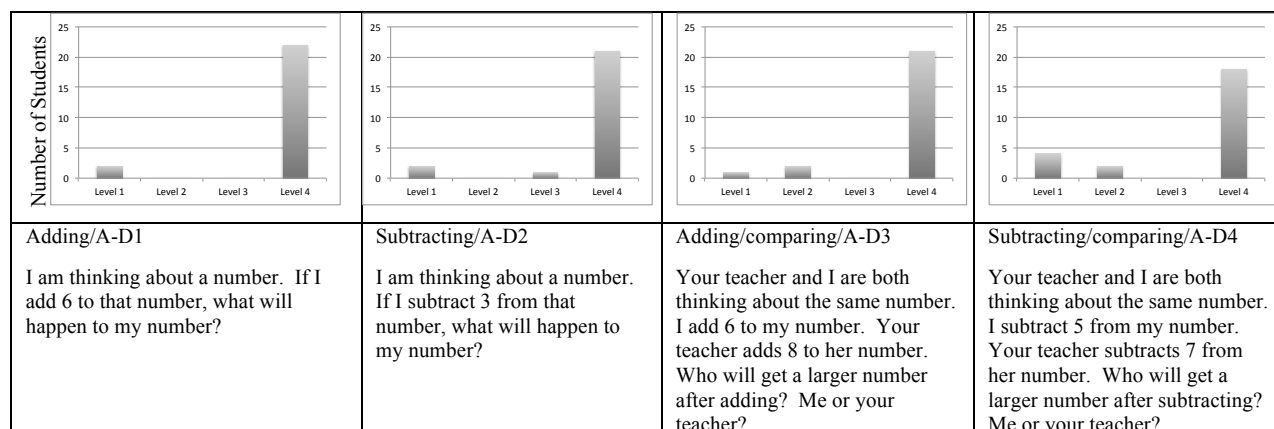


Figure 12. Breakdown of student responses for *direction of change* tasks.

**Effect of Different Variables.** As discussed in Chapter 2, tasks were paired in their design to evaluate the effects of the (a) addition versus subtraction problem contexts, and (b) comparing versus not comparing problem contexts. For example, *adding/A-D1* examined students' understandings of direction of change for addition, while *subtracting/A-D2* examined students' understandings of direction of change for subtraction. Similarly, *adding and comparing/A-D3* and *subtracting and comparing/A-D4* involved comparing sums and differences, while *adding/A-D1* and *subtracting/A-D2* did not involve comparing. These analyses were important in identifying variables that influenced student thinking.

The addition versus subtraction problem context did not have a statistically significant effect on students' production of advanced generalizations. As illustrated in Figure 13, 22 students (91.7%) had an advanced generalization score for both *adding/A-D1* and *subtracting/A-D2*, so there did not appear to be any differences across students' thinking for these tasks. Similarly, 21 students (87.5%) received an advanced generalization score for *adding and comparing/A-D3*, while 18 students (75%) students received an advanced generalization score on *subtracting and comparing/A-D4*, and a McNemar's Chi-square test<sup>8</sup> revealed that this difference was not statistically significant ( $\chi^2(1, N = 24) = 3, p = .250$ ).

<sup>8</sup> A McNemar's Chi-square test was used because of the within-subjects design and because the data was non-normally distributed.

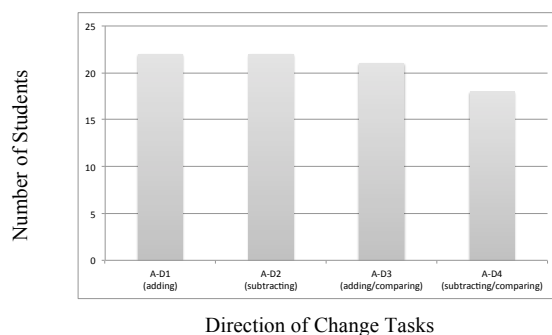


Figure 13. Students with advanced generalization scores for direction of change tasks.

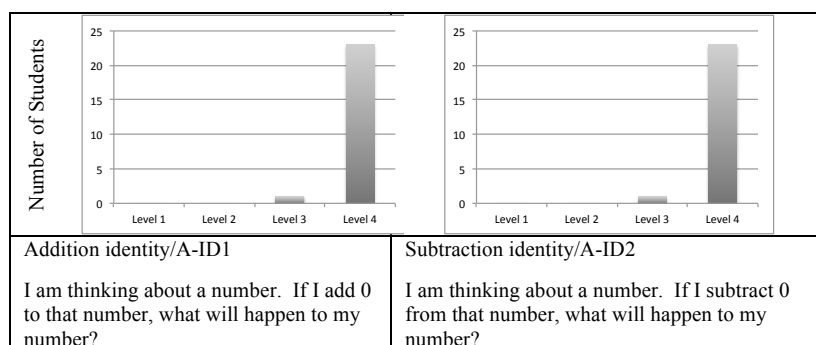
Similarly, the comparing versus not comparing problem context did not have a statistically significant effect on students' production of advanced generalizations. I added students' generalization scores of 0 or 1 for the two tasks that did not involve comparing and the two tasks that did involve comparing, for subtotal scores that could range from 0 to 2. The means for students' subtotals across these types of tasks were very similar at 1.83 (SD=.48) and 1.62 (SD=.71) respectively, indicating that student thinking did not vary much when comparison was and was not involved in the *direction of change* tasks. A Wilcoxon signed-rank test confirmed that the difference in students' responses for these tasks was not statistically significant ( $Z = -1.667$ ,  $p = .096$ ). Thus, the fourth graders demonstrated similar levels of understanding for the *direction of change* tasks across both the addition and subtraction tasks, and across the two task subtypes that were included.

*Consistency Across Student Responses.* Individual students were not always consistent with respect to the level of generality with which they treated the operations in these tasks. Seven students responded inconsistently across the *direction of change* tasks, and demonstrated some range in the level of generality with which they treated the operations of addition and subtraction. Table 7 illustrates how two students varied in their thinking across the tasks. Student ID 14, for instance, provided a Level 4 response for the task that involved simple addition and no comparing, but he provided Level 1 responses for the remaining *direction of change* tasks. Student ID 04 provided Level 4 explanations for the tasks that did not involve comparing, and he seemed to be able to generalize that addition makes numbers greater in value, while subtraction makes numbers lesser in value. However, he did not demonstrate an understanding of the generalization that adding a greater number yields a greater sum, and subtracting a smaller number yields a greater difference. Note that I include Student ID 04 in the next two sections as well, so the reader can see how a sample student responded across *all* interview tasks.

Table 7: Sample Range of Thinking for Direction of Change Tasks

<i>DIRECTION OF CHANGE TASKS</i>				
Adding/A-D1	Subtracting/A-D2	Adding/comparing/A-D3	Subtracting/comparing/A-D4	
I am thinking about a number. If I add 6 to that number, what will happen to my number?	I am thinking about a number. If I subtract 3 from that number, what will happen to my number?	Your teacher and I are both thinking about the same number. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?	Your teacher and I are both thinking about the same number. I subtract 5 from my number. Your teacher subtracts 7 from her number. Who will get a larger number after subtracting? Me or your teacher?	
<b>ID 04</b>	Level 4 Response: The student explained that the number will get larger “since it’s adding, not subtracting. . . Subtracting is like cutting something and adding is like adding more water to a bottle.”	Level 4 Response: The student explained that the number will get smaller “since it’s subtracting” and it will get smaller no matter what number I am thinking about at the beginning.	Level 2 Response: The student explained that the teacher will get the larger number “because like 50 . . . 50 plus 8 equals to 58 and 50 plus 6 is only 56 . . . which is bigger? 58.” When I asked how he can know for sure if he doesn’t know what number we are thinking about at the beginning, he said he couldn’t explain it.	Level 2 Response: “For example 10. If you subtract 7 from 10, it would be 3. And if you subtract 5 from 10, it would be 5. So 5 is bigger than 3.” When asked how he can know for sure if he doesn’t know what number we are thinking about at the beginning, he tried it again with 20 to arrive at the same conclusion.
<b>ID 14</b>	Level 4 Response: The student explained that the number will get bigger “because you’re adding” and it will get bigger no matter what number I am thinking about at the beginning.	Level 1 Response: “I don’t know.” When asked if there was any way to know if/how the number changes, he said no. When asked if he could tell me if/how the number changes if he knew what number I was thinking about, he said yes.	Level 1 Response: “It depends on what number.” The student also nodded and said ‘yes’ after he was asked whether either person can get the larger number depending on what number we begin with.	Level 1 Response: “The teacher because his number is more bigger.” When asked if the teacher will always get the larger number, the student said that it would depend on what number we are thinking about at the beginning and the teacher would not always get the larger number.

**Identity Tasks.** Figure 14 presents the frequency with which students used the four generalization levels for each of the *identity* tasks. Level 4 responses occurred the most frequently across tasks and *all* students provided at least a Level 3 response for each of the two identity tasks, *addition identity/A-ID1* and *subtraction identity/A-ID2*. Of the 24 participants, 22 students responded with a Level 4 explanation across both tasks, demonstrating that the fourth graders had a robust understanding of this type of generalization. Next, I analyze the effect of the two different operations and the consistency of students’ responses across the *identity* tasks.

Figure 14. Breakdown of Student Responses for *Identity* Tasks

**Effect of Different Variables.** The *identity* tasks were paired in their design to evaluate the effects of the addition versus subtraction problem contexts. *Addition identity/A-ID1* probed

students' thinking about the identity element for addition, while *subtraction identity/A-ID2* examined students' understandings of the identity element for subtraction. As revealed in Figure 14, 24 students (100%) responded with advanced generalization scores for both tasks, so there was no difference in students' thinking across these two tasks. Thus, the fourth graders demonstrated similar levels of understanding for the *identity* tasks for both addition and subtraction.

*Consistency Across Student Responses.* Students were fairly consistent with respect to the level of generality with which they treated the operations of addition and subtraction. Only two students responded inconsistently across the *identity* tasks, and demonstrated some range in thinking. Table 8 illustrates how Student ID 04 varied in his thinking across the tasks. Student ID 04 provided a Level 3 explanation for *addition identity/A-ID1*, but provided a Level 4 response for *subtraction identity/A-ID2*.

Table 8: Sample Range of Thinking for Identity Tasks

<i>IDENTITY TASKS</i>	
Addition identity/A-ID1	Subtraction identity/A-ID2
I am thinking about a number. If I add 0 to that number, what will happen to my number?	I am thinking about a number. If I subtract 0 from that number, what will happen to my number?
<b>ID 04</b>	
Level 3 Response: "It would stay the same...because it's true for every number." [no further explanation provided despite follow up questions]	Level 4 Response: "It'll stay the same. Zero is nothing...it has nothing in it. It'll always be the same."

*Relationship between Addition and Subtraction Tasks.* Figure 15 presents the frequency with which students used the four generalization levels for each of the *relationship between addition and subtraction* tasks. Level 1 and Level 4 responses occurred the most frequently across tasks (a minimum of 23 out of 24 responses), while Level 2 and Level 3 responses did not occur often (a maximum of 1 out of 24 responses). The majority of students provided a Level 3 or Level 4 response for each task. Of the 24 participants, 10 students responded with a Level 4 explanation across all four tasks. In comparison with the *direction of change* and *identity* tasks, the *relationship between addition and subtraction* tasks elicited far more Level 1 and Level 2 explanations as well as more variability in student thinking. In fact, 5 students responded with Level 1 explanations across all four tasks, and there were no instances of this for the other two types of arithmetic generalizations. Next, I analyze the effect of order and symmetry, and the consistency of students' responses across the *relationship between addition/subtraction tasks*.

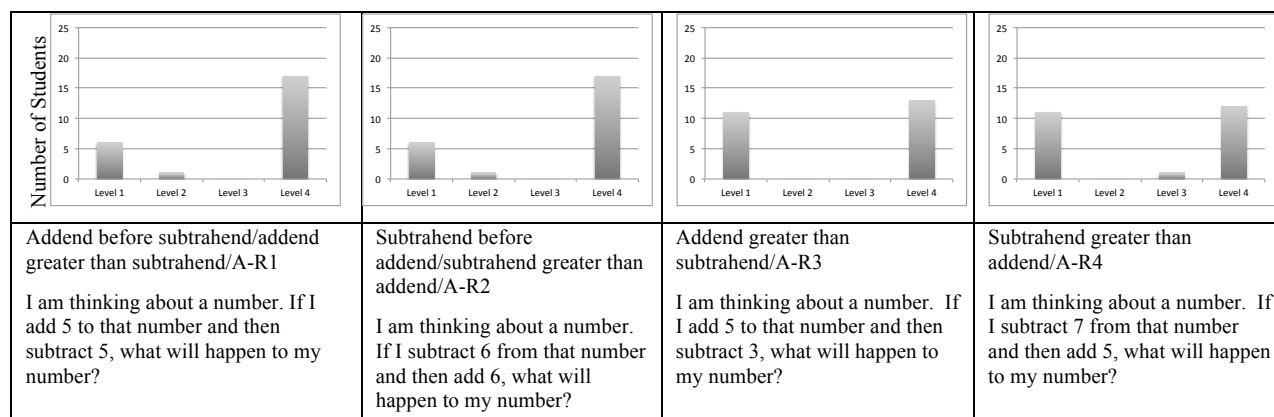
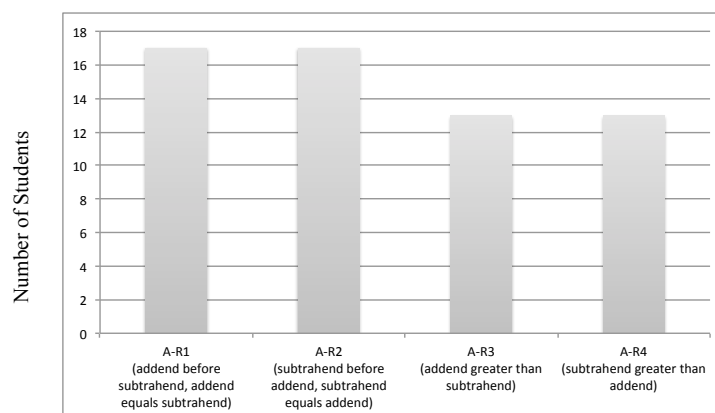


Figure 15. Breakdown of student responses for *relationship between addition/subtraction* tasks.

*Effect of Different Variables.* Tasks were paired in their design to evaluate the effects of the (a) order of operations, and (b) symmetry. For example, *addend before subtrahend/addend equal to subtrahend/A-R1* and *subtrahend before addend/subtrahend equal than addend/A-R2* were similar in structure, but the order of the operations varied across the tasks. Similarly, *addend before subtrahend/addend equal to subtrahend/A-R1* and *subtrahend before addend/subtrahend equal than addend/A-R2* were symmetric tasks (i.e., addend and subtrahend equal), while *addend greater than subtrahend/A-R3* and *subtrahend greater than addend/A-R4* were asymmetric (i.e., addend and subtrahend not equal).

The order of operations did not have a statistically significant effect on students' production of advanced generalizations. As depicted in Figure 16, 17 students (70.8%) had an advanced generalization score for *addend before subtrahend/addend equal to subtrahend/A-R1* task and *subtrahend before addend/subtrahend equal than addend/A-R2*, so there did not appear to be any differences across students' thinking for these tasks. Similarly, 13 students (54.2%) had advanced generalization scores for the *addend greater than subtrahend/A-R3* task and *subtrahend greater than addend/A-R4* task, again indicating no difference across students' thinking for these tasks.



Relationship between Addition/Subtraction Tasks

Figure 16. Students with advanced generalization scores for *relationship between addition/subtraction* tasks

Similarly, the symmetry of the tasks did not have a statistically significant effect on students' production of advanced generalizations. I calculated students' subtotal scores for the two symmetrical tasks and the two asymmetrical tasks by assigning students a score of either 1 for Levels 3 and 4, or a score of 0 for Levels 1 and 2, for a total maximum score of 2 for each subtotal. The mean score for symmetrical tasks was 1.42 (SD=.88), while the mean score for asymmetrical tasks was 1.08 (SD=.93). A Wilcoxon signed-rank test showed that the difference between students' responses for these different problem types was not statistically significant ( $Z = -1.651, p = .099$ ). Thus, neither the order of operations nor the symmetry of the tasks appeared to prompt differences in the types of thinking students demonstrated.

*Consistency Across Student Responses.* Many students responded inconsistently with respect to the level of generality with which they treated the arithmetic operations. Nine students responded inconsistently across the *relationship between addition/subtraction* tasks, and demonstrated some range in thinking. Table 9 illustrates how two students varied in their thinking across the tasks. Student ID 12, for example, responded with Level 4 explanations to symmetric tasks, but she responded with Level 1 explanations for asymmetric tasks.

Table 9: Sample Range of Thinking for Relationship between Addition/Subtraction Tasks

<i>RELATIONSHIP BETWEEN ADDITION/SUBTRACTION TASKS</i>				
	Addend before subtrahend/addend greater than subtrahend/A-R1	Subtrahend before addend/subtrahend greater than addend/A-R2	Addend greater than subtrahend/A-R3	Subtrahend greater than addend/A-R4
	I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?	I am thinking about a number. If I subtract 6 from that number and then add 6, what will happen to my number?	I am thinking about a number. If I add 5 to that number and then subtract 3, what will happen to my number?	I am thinking about a number. If I subtract 7 from that number and then add 5, what will happen to my number?
<b>ID 04</b>	Level 2 Response: "Let's go for 10. So if you add 5, it would be 15. 15 subtract 5 equals 10. So you're not going to change any number...And if you use a different number, you'll get that number."	Level 1 Response: The student tries to subtract 6 from 0, but then states that that's not possible. Then he concludes that there's no way to know what happens to the number "since you don't really have a number to work with...you would have to know what number you start with."	Level 4 Response: "It would always get bigger by 2 numbers...[after student is asked to explain why]...by subtracting the 3 from 5."	Level 3 Response: The student first states that the number will get smaller after he tries it with 10. Then he explains that the number will always get larger "because 7 is bigger than 5...because 7 and 5 are different numbers, I would say it would get smaller." He then says it can't explain why the number will get smaller.
<b>ID 12</b>	Level 4 Response: "It will stay the same...it's basically like subtracting the number and then adding the number to get back."	Level 4 Response: "If you subtract 6 from a number, it would go smaller. And then once you have that number and add 6 more, it would go back to the same number." She then states that that would happen with all numbers.	Level 1 Response: "It depends basically what number."	Level 1 Response: The student stated that "there is no way to know." When asked, "Do you need to know what number I'm thinking about to figure it out?" she said "yes".

## Summary and Discussion

In this chapter, I analyzed students' construction of arithmetic generalizations in the context of addition and subtraction tasks. The 10 focal tasks were all similar in structure and wording, but each task addressed a different arithmetic generalization. My focus was on three types of generalizations: *direction of change*, *identity*, and *relationship between addition and*

*subtraction*. The analysis of student interviews focused first on identifying patterns in and understanding the character of student thinking, and second on analyzing if and how children's thinking varied across the different tasks.

A fine-grained qualitative analysis involved an iterative process of watching the interviews, observing patterns in student thinking, and identifying similarities and differences across students' explanations. This analyses revealed four levels in student thinking for each generalization task, levels that reveal a spectrum of increasing generality with which students approach these tasks. These levels became the basis of a coding scheme designed to capture the quality of students' arithmetic generalizations. This scheme consisted of four levels and students' explanations for each task were coded as Level 1, 2, 3, or 4. At Levels 1 and 2, students rely on specific instances and substitution of values. At Levels 3 and 4, students do not rely on any examples and make generalizations about the arithmetic operations.

Overall, students demonstrated robust understandings of the arithmetic generalizations with 187 out of 240, or 78%, of these responses coded as Level 4 and most of the remaining responses coded as Level 1 (18%). Level 4 and Level 1 codes therefore captured the most common types of thinking among students, while Level 2 and Level 3 codes did not occur as frequently. The relative absence of Level 2 and Level 3 codes may be explained by the small sample size of twenty-four and the limited number and type of arithmetic generalizations in this study; with a larger sample size and additional tasks, I would expect to see more Level 2 and Level 3 responses.

Though students demonstrated robust understandings overall, student thinking was not always consistent across tasks. To investigate this further, I compared student thinking across different variables, including generalization type and task subtypes (operation, comparing, order of operations, and symmetry) and I examined how individual students' responses varied across tasks. The findings indicate that generalization type did have a statistically significant effect on students' production of advanced generalizations, while task subtypes did not. Students provided the most generalized levels of responses for the *identity* tasks and the least generalized responses to the *relationship between addition/subtraction* tasks, and the differences between all three types of generalizations were statistically significant. Many students demonstrated a range of thinking within each type of generalization as well and the across task profiles illustrated this finding. I included Student ID 04 in each of these across task profiles, so the reader has a sense of how one particular student's responses varied across all 10 tasks.

The findings from this study provide insight into a developmental model about children's construction of arithmetic generalizations. First, the four levels of generality that I found do not necessarily indicate a fixed trajectory, but they do represent different ways students approach these types of tasks and help us describe qualitative shifts in their thinking. It also appears that students can 'skip' levels, move back and forth between levels, and may exhibit different levels of thinking across different types of tasks. Second, as I hypothesized in Chapter 1, these findings indicate that a child's capability to produce arithmetic generalizations across the generalization types may develop differently. Specifically, students' production of advanced generalizations decreased from *identity*, to *direction of change*, to *relationship between addition and subtraction*. Furthermore, this order of difficulty of the generalization types parallels the increasing need to coordinate relations in these tasks. In other words, as children were required to coordinate more relations, the likelihood of producing an advanced generalization decreased. All the *relationship between addition and subtraction* tasks required students to coordinate their understandings of addition and subtraction. Similarly, two of the *direction of change* tasks required coordinating

two different operands and the sums or differences. The *identity* tasks, though, did not require this kind of coordination and therefore may have been more intuitive for children.

## Chapter 4: Students' Understandings of Arithmetic Generalizations in the Context of Multiplication and Division Tasks

This chapter examines fourth graders' understandings of arithmetic generalizations in the context of multiplication and division tasks. The reader will find that the structure of this chapter on multiplicative relations mirrors the prior chapter on additive relations. I first consider students' thinking and performance across tasks to understand and identify patterns in the character of children's reasoning. I then examine students' thinking and performance across generalization types (*direction of change*, *identity*, and *relationship between multiplication and division*) and task subtypes to identify similarities and differences in students' generalizations across different tasks. I conclude this chapter with a cross-study analysis that examines the findings from the multiplicative study in relation to the additive study discussed in Chapter 3.

### Overview of Tasks

The first part of this chapter orients the reader with the multiplicative tasks in the study, highlighting similarities and differences between the different types of tasks and task subtypes. Each interview began with a set of two baseline tasks. Like the procedures used in the prior chapter, I began each interview with two baseline tasks, the purpose of which I detail below. The baseline tasks were followed by 10 focal tasks that probed students' production of different types of multiplicative generalizations. As discussed in Chapter 2, a set of probe questions followed students' initial responses and those probe questions were the same across all tasks.

**Baseline Tasks.** Two baseline tasks, similar in wording and structure to the focal tasks, were presented to students to determine whether they had some minimal fluency with addition and subtraction, and to make sure that they understood the wording of the focal tasks that followed. These tasks were important for purposes of validity, because the findings from the study cannot be attributed to either students' inability to add and subtract or their understanding of the task. If students did not demonstrate proficiency on the baseline tasks, then difficulties in producing the arithmetic generalizations might be attributable to the inability to multiply or divide or to understand the wording of the task, rather than their ability to coordinate arithmetic relations to produce a generalization. The first baseline task involved one operation, while the second baseline task involved two operations (see Table 10).

Table 10: Study 2 Baseline Tasks

Baseline Task 1: Multiplication	Baseline Task 2: Multiplication and Division
I am thinking about the number 2. If I multiply that number by 6, what will happen to my number?	I am thinking about the number 6. If I multiply that number by 5 and then divide by 5, what will happen to my number?

**Focal Tasks for Analyses.** A total of 10 focal tasks were presented to students. The tasks included 4 *direction of change* tasks, 4 *identity* tasks, and 4 *relationship between multiplication and division* tasks. The 10 tasks were all similar in structure and wording, but each task focused on a different arithmetic generalization. I provide additional detail about the different tasks and their role in the analyses in the next three sections.

**Direction of Change Tasks.** The *direction of change* tasks, depicted in Figure 17, examined students' understandings of the directionality of change when an operation is performed (i.e., multiplication of positive integers increases the numerical value, while division of positive integers decreases the numerical value). Figure 17 lists each of the *direction of change* tasks. The first two tasks in the figure (*multiplying/M-D1* and *dividing/M-D2*) were designed to allow me to examine differences between students' understandings of multiplication and division. The third and fourth tasks (*multiplying/comparing/M-D3* and *dividing/comparing/M-D4*) were designed in a similar manner that allowed me to compare student thinking across the multiplication and division context. Additionally, there were two task subtypes, *not comparing* and *comparing*. The first two tasks in Figure 17 are *not comparing* tasks; students are required to consider one operation and evaluate how the number changes. The second two tasks are *comparing* tasks; students are asked to consider two operations and compare the effect of one to the other.

Task	Arithmetic Generalization	Problem Subtype
Multiplying/M-D1 I am thinking about a number. If I multiply that number by 6, what will happen to my number?	$6N > N$ <i>Multiplication makes numbers greater.</i>	not comparing
Dividing/M-D2 I am thinking about a number. If I divide that number by 3, what will happen to my number?	$N/3 < N$ <i>Division makes numbers lesser.</i>	
Multiplying/comparing/M-D3 Your teacher and I are both thinking about the same number. I multiply the number by 6. Your teacher multiplies the number by 8. Who will get a larger number after multiplying? Me or your teacher?	$8N > 6N$ <i>Multiplying by a greater number will yield a greater product.</i> <i>Multiplying by a lesser number will yield a lesser product.</i>	comparing
Dividing/comparing/M-D4 Your teacher and I are both thinking about the same number. I divide the number by 5. Your teacher divides the number by 7. Who will get a larger number after dividing? Me or your teacher?	$N/5 > N/7$ <i>Dividing by a greater number will yield a lesser quotient.</i> <i>Dividing by a lesser number will yield a greater quotient.</i>	

Figure 17. Direction of change tasks. M= multiplicative; D = direction of change.

In addition to the standard probe questions students were asked across all tasks, a few students needed some clarification for the *comparing* tasks. These students began by substituting two different numbers for the numbers that the teacher and I are thinking about in the task. When this occurred, I would say, "I see what you did, but it says here that we are thinking about the *same* number."

**Identity Tasks.** The *identity* tasks focused on students' thinking about properties of identity (i.e., the multiplication or division by 1 with any number leaves its value unchanged). Figure 18 lists each of the *identity* tasks. The two *identity* tasks were designed to allow me to make comparisons between students' understandings of multiplication and division.

Task	Arithmetic Generalization
Multiplication identity/M-ID1 I am thinking about a number. If I multiply that number by 1, what will happen to my number?	$N \times 1 = N$ <i>1 is the multiplicative identity.</i> <i>If you multiply any number by 1, the number will remain the same.</i>
Division identity/M-ID2 I am thinking about a number. If I divide that number by 1, what will happen to my number?	$N/1 = N$ <i>1 is the multiplicative identity.</i> <i>If you divide any number by 1, the number will remain the same.</i>

Figure 18. Identity tasks. M = multiplicative; ID = identity.

***Relationship between Multiplication and Division Tasks.*** The *relationship between multiplication and division* tasks explored students' reasoning about the inverse relationship between multiplication and division. Figure 19 lists each of the *relationship between multiplication and division* tasks. The first two tasks in the figure (*multiplier before divisor/multiplier equal to divisor/M-R1* and *divisor before multiplier/divisor equal to multiplier/M-R2*) were designed to allow me to examine whether the order of operations played a role in students' understanding of inverse operations. The third and fourth tasks (*multiplier before divisor/multiplier greater than divisor/M-R3* and *divisor before multiplier/divisor greater than multiplier/M-R4*) were designed in a similar manner, to understand the role of both the order and asymmetry of the operations in student thinking. Additionally, there were two task subtypes, *symmetric* and *asymmetric*. The first two tasks in Figure 19 are *symmetric* tasks; students need to reason with equal multipliers and divisors, which leaves the value of the initial number unchanged. The second two tasks are *asymmetric* tasks; students need to reason with unequal multipliers and divisors, which requires them to compare the magnitudes of the multiplier and divisor to determine how the initial number changes. As noted earlier, the *asymmetric* tasks present an additional challenge, as they require children to coordinate their understanding of both the operations as well as the magnitude of the operands.

Task	Arithmetic Generalization	Problem Subtype
Multiplier before divisor/multiplier equal to divisor/M-R1  I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?	$5N/5 = N$  <i>Multiplication and division are inverse operations.</i>  <i>If you multiply and divide by the same number, your initial number will remain the same.</i>	symmetric
Divisor before multiplier/divisor equal to multiplier/M-R2  I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number?	$6N/6 = N$  <i>Multiplication and division are inverse operations.</i>  <i>If you multiply and divide by the same number, your initial number will remain the same.</i>	
Multiplier before divisor/multiplier greater than divisor/M-R3  I am thinking about a number. If I multiply that number by 5 and then divide by 3, what will happen to my number?	$5N/3 > N$  <i>Multiplication and division are inverse operations.</i>  <i>Multiplying by more than you divide by yields a number that is greater than the initial number.</i>  <i>Dividing by less than you multiply by yields a number that is greater than the initial number.</i>	asymmetric
Divisor before multiplier/divisor greater than multiplier/M-R4  I am thinking about a number. If I divide that number by 7 and then multiply by 5, what will happen to my number?	$5N/7 < N$  <i>Multiplication and division are inverse operations.</i>  <i>Dividing by more than you multiply by yields a number that is less than the initial number.</i>  <i>Multiplying by less than you divide by yields a number that is less than the initial number.</i>	

Figure 19. Relationship between multiplication and division tasks. M = additive; R = relationship between multiplication and division.

In addition to the standard probe questions students received across all tasks, a few students required some clarification for these tasks. These students parsed the task into two separate questions, rather than one. For example, for *divisor before multiplier/divisor greater than multiplier/M-R4*, three students said the number would decrease if you divide by 7 and the number would increase if you multiply by 5. In these instances, I would say something like, “I see what you did, but we are going to divide by 7 and multiply by 5 one after the other. I am thinking about a number. We’re going to divide that number by 7. And then whatever number we get after dividing, we’re going to multiply *that* number by 5. What happens after we do both those things?”

## Results

The results are presented in three sections. In the first, I briefly discuss the findings from the baseline tasks. In the second, I focus on the character of students’ understandings and consider variations in the levels of generalizations students’ produce when they consider the multiplication and division tasks. I provide an overview of four levels of student generalizations that I observed and coded for each task, and I then discuss both clear and less clear examples of these levels. In the third, I compare students’ levels in the generalizations they produced both across and within the different types of tasks, *direction of change*, *identity*, and *relationship between multiplication and division*. The findings show that these fourth graders demonstrated

competency in making generalizations, but their thinking often varied across the different types of tasks.

**Baseline Tasks.** Students' overall success on these tasks indicates that the findings I discuss in this chapter based on the focal tasks cannot be explained by a lack of computational fluency or confusion over the wording of the tasks. All 24 students in Study 2 responded with the correct solution of 12 and an appropriate explanation for *Baseline Task 1: Multiplication*. For *Baseline Task 2: Multiplication and Division*, which involved two consecutive operations, 20 students responded with the correct solution of 6 and explained the steps they took to arrive at the solution. Four students, however, did not respond with a solution of 6. These students seemed to rely primarily on memorized number facts to approach the task. For example, three students said that  $6 \times 5 = 30$  without any sort of calculation or explanation. Two of these three students went on to state that  $30 \div 5 = 5$ , again without any calculation or explanation, indicating that they were relying on memorized number facts. The other student said he wasn't sure what 30 divided by 5 is. The fourth student also relied on number facts and she incorrectly stated that  $6 \times 5 = 24$ . When she couldn't divide 24 by 5, she did  $24 - 5 = 19$  and explained that division is similar to subtraction. The 4 students who responded with incorrect solutions did not demonstrate any confusion about the wording of the tasks, and it is difficult to draw any conclusions about their computational fluency because they relied on memorized number facts, which were not accompanied by much explanation.

**Research Question 1: Quality of Students' Understandings.** As discussed in the previous chapter, a fine-grained qualitative analysis, which involved watching the interviews and observing patterns in student thinking, led to the development of a coding scheme. The scheme was designed to capture the quality of students' multiplicative generalizations. Like the additive scheme described in Chapter 3, the multiplicative scheme consisted of 4 levels.

Table 11 contains a description and illustrations of each of the four levels of thinking that I have conceptualized based on the interview data (the sample responses are reflective of the types of thinking found across tasks). As discussed in the previous chapter, Level 1 and Level 2 thinking are more concrete in nature because students' explanation are tied to specific instances or examples, rather than abstractions or generalizations about the arithmetic operations. Level 3 and Level 4 are more generalized because students do not rely on specific instances or examples, and instead, make generalizations about the arithmetic operations. At Level 1, students do not demonstrate any understanding of the arithmetic generalization. They may, for example, argue that the number can change in a variety of ways and that there is no possible way to know how the number changes without knowing what it is. At Level 2, students use substitution of one or more numbers for the unknown number, and make a generalization based on the instance or instances they examine. At Level 3, students generalize without relying on specific instances, but they have difficulty explaining why the generalization holds true. Level 4 is what I describe as the most generalized level of thinking, at which students generalize properties of multiplication and division to all numbers without considering specific instances or numbers to make sense of the task, and they can explain the rationale behind the generalization they make. Table 12 provides short excerpts from students' responses for each task at each level, which illustrates further how I categorized and coded students' responses.

There were a total of 240 (24 students x 10 tasks) responses that were coded. Another doctoral student in mathematics education was trained to use the coding scheme and served as a

second coder for reliability purposes. He coded 20% of the data (5 interviews selected at random). Inter-rater reliability was calculated as the percentage of agreements out of the total number of codes. Coder agreement was 98%, and any disagreements were resolved through a discussion.

Table 11: Coding Scheme for Levels of Generality

Level Description	Examples from divisor before multiplier/divisor greater than multiplier/M-R4 task
<p><b>(1) No Generalization:</b> Students do not demonstrate any understanding of the arithmetic generalization and may do one of the following:</p> <p>(1a/<b>more information required</b>): state that additional information, like a specific number, is necessary to compare and evaluate the numbers</p> <p>(1b/<b>incorrect generalization</b>): make an incorrect mathematical generalization based on the information given (e.g., the order of operations)</p> <p>(1c/<b>specific number</b>): respond with a specific number that they arrived at after performing a computation</p> <p>(1d/<b>combination</b>): more than one of the above (often 1a and 1b)</p> <p>(1e/<b>no solution</b>): say that they do not know or provide unrelated response</p>	<p><i>I am thinking about a number. If I divide that number by 7 and then multiply by 5, what will happen to my number?</i></p> <p>“It depends on what the number is. It could get bigger. It could get smaller. We just don’t know.”</p> <p>“The last thing you do is multiply and multiplying makes numbers larger. So the number would get larger.”</p> <p>“The number would be 5. I started with 7, then I got 1, and then I multiplied by 5 and got 5.”</p>
<p><b>(2) Generalization Based on Substitution:</b> Students rely on specific numbers and mathematical instances to make a generalization about the arithmetic operation. They substitute a specific number for the unknown value in the task to make a generalization, and to compare and evaluate numbers. Like at Level 1, students do not demonstrate an understanding of the arithmetic generalization that is independent from the specific context, but they do make an accurate generalization based on the substitution of one or more numbers.</p>	<p>“If the number was 14, if you divide by 7, it’ll be 2. And then multiply by 5, it’ll be 10. So I think the number will be smaller.”</p>
<p><b>(3) Generalization that does not Rely on Substitution:</b> Students demonstrate a developing understanding of the arithmetic generalization. They do not rely on the substitution of numbers to compare and evaluate numbers, and they make a correct generalization, but they cannot explain the mathematical reasoning behind the generalization they make. They often refer to their prior math/classroom experience instead. Like at Level 2, students do not demonstrate a deep understanding of the arithmetic generalization, but their understanding is independent of the specific context and they do not rely on substitution.</p>	<p>“I think the number would get smaller because 5 is less than 7, but I’m not exactly sure why. I just know that you need to look at the 5 and 7.”</p> <p>“The number would get smaller because that’s what I’ve seen happen in the classroom.”</p> <p>“The number would always get smaller, but it’s hard to explain why.”</p>
<p><b>(4) Generalization Based on Reasoning about Operation(s):</b> Students demonstrate a well-developed understanding of the arithmetic generalization and do not rely on the substitution of numbers to compare and evaluate numbers. Like at Level 3, students do not rely on substitution, but they can also demonstrate a deeper understanding of the arithmetic generalization.</p>	<p>“It would get smaller because 7 is more than 5, and you’re dividing by more than you’re multiplying by. It’s like you’re taking away more than you’re giving back.”</p>

Table 12: Sample Student Responses Coded at each Level for each Task

Task	Level 1: No Generalization	Level 2: Generalization Based on Substitution	Level 3: Generalization that does not Rely on Substitution	Level 4: Generalization Based on Reasoning about Operation(s)
<p>Multiplying/M-D1</p> <p>I am thinking about a number. If I multiply that number by 6, what will happen to my number?</p>	<p>"It'd still depend on what number." (ID 28)</p>	<p>(no occurrences)</p>	<p>"It'll get bigger...because if you think about a number like 6 times 6, it'll be like 36... [it will be true for] every number!" Student lists other examples it is true for. (ID 47)</p>	<p>"It'll get larger... because when you're doing times, you'll get the same answer as adding 6 in a row." (ID 27)</p>
	<p>"I need to know the number." (ID 40)</p>	<p>Student pauses to think for a long time. "It'll get smaller... 6 divided by 3 equals 2." Student then states that he used the number 6 to help him figure it out. (ID 47)</p>	<p>(no occurrences)</p>	<p>"It would get smaller because it's kind of like subtracting... because it like divides it by parts." (ID 25)</p>
	<p>"It depends on what number you're thinking about... [but] the teacher will get the bigger number most of the time." (ID 45)</p>	<p>"...if it's what 1 times, multiplied, it would be 8. Big number! And 2 times 8, it would still be the bigger number. So no matter what, my teacher will win." (ID 29)</p>	<p>"Our teacher's gonna get it bigger because she has the bigger number [pointing to multiplier of 8]." Student does not talk about any other ideas related to multiplication. (ID 48)</p>	<p>"My teacher because, well 8 is greater than the number 6 and they are both multiplying. And then when you multiply something by 8, it will be greater than if you multiply something by 6." (ID 32)</p>
<p>Dividing/M-D2</p> <p>I am thinking about a number. If I divide that number by 3, what will happen to my number?</p>	<p>"You because you divided it by a smaller number and my teacher divided by a bigger number...you get less if you divide by the bigger number...Well you guys may get the same amount too." After asked if it depends on the number we are thinking about, student said yes. (ID 36)</p>	<p>"If you start small you go big, and if you start big you go small. Well that's what I heard of, so I think you'll get the bigger number. So like if you do 5 plus 5, you'll get 10...and another number with another number, subtraction, it would get lower." This student's thinking was not very clear, but she was the only student who was coded as Level 2 for this task. (ID 29)</p>	<p>(no occurrences)</p>	<p>"You because you're dividing it in 5 and you don't have to divide it in so many parts. But when you do it divide it by 7, you need to divide it into smaller parts because you need more space." (ID 37)</p>
	<p>"I don't know what number and I'm not so sure." (ID 42)</p>	<p>The student uses the number 7 to approach the task. "The number that you put in turns out because 'I'm 7! I'm going in! 1 times 7 equals 7.'" (ID 40)</p>	<p>"It'll stay the same because 1 times 5 equals 5...I'm giving you an example. Because anything times 1 will stay the same." (ID 34)</p>	<p>"When you multiply something by 1, it will always be the number you started out with... 6 times 2 would be 12 because it's 6 and then another 6, but 6 times 1, 1 mean it would be just 6 because it's only one 6." (ID 35)</p>
<p>Multiplication identity/M-ID1</p> <p>I am thinking about a number. If I multiply that number by 1, what will happen to my number?</p>	<p>Student first explains that 7 divided by 1 is 1. "No matter what the number is, you're still gonna get 1." (ID 38)</p>	<p>"If you get 1 and then divide it by 1, it equals 1. If you divide 2 by that, you get 2. Oh I got it! Because it's 1, even if you use 100, it'll come back as 100." (ID 40)</p>	<p>"It would be the same thing...[explains that 5 divided by 5 is 1 and that multiplication is like division] I know because my teacher did a lot of division when I was in third grade." (ID 31)</p>	<p>"It'll stay the same because times 1 and divided by 1 are the same. So if you times 1 by something, it ends up being the same... Dividing is basically putting stuff in groups, so if you put it in one group, it'll stay the same." (ID 44)</p>
	<p>Division identity/M-ID2</p> <p>I am thinking about a number. If I divide that number by 1, what will happen to my number?</p>	<p>(no occurrences)</p>	<p>(no occurrences)</p>	<p>(no occurrences)</p>

Direction of Change

Task	Level 1: No Generalization	Level 2: Generalization Based on Substitution	Level 3: Generalization that does not Rely on Substitution	Level 4: Generalization Based on Reasoning about Operations
Multiplier before divisor/multiplier equal to divisor/M-R1 I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?	"It will either get smaller or stay the same... maybe it could [get bigger] but probably not." (ID 43)	"7 times 5 equals 35. 35 divided in 5... [pauses] actually, it would be any number! Like 3 times 5 goes 15. 5 divided into 15 is 3. It turned out the number that you're multiplying." When asked if this is true for all numbers, the student said yes. (ID 34)	"You'll get the same number that you multiplied with 5." When asked if this is true for all numbers, the student said yes, but she could not explain why. (ID 37)	"It will stay the same because if you multiply and then you divide, multiply and divide are opposites. So if you multiply, dividing would be like taking away the number you multiplied, so it would be the same number." (ID 39)
Divisor before multiplier/divisor equal to multiplier/M-R2 I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number?	"I think it will get bigger." Then student substitutes with 42, and concludes that the number will stay the same, but says that she is "not sure" if it will always stay the same and that it will depend on the number. (ID 37)	"18 divided by 6 equals 3, times 6... it will become the same number." Student tries it again with 36 and states that the number will always stay the same. (ID 47)	"I think it might actually stay the same... because if you divide it by 6 and then multiply it by 6, division and multiplication are opposites, so say the number is like 30, and then you divide it by 6, it will just become 5. But then if you multiply by 6 again, it will become 30 again." (ID 35)	"It will turn into a number bigger than it, and then it will turn into a number smaller than the number it just turned into and then it will be the exact same... it's kind of like realigning it, it will just turn back, because divide and multiply and pretty much the exact opposite things." (ID 41)
Multiplier before divisor/multiplier greater than divisor/M-R3 I am thinking about a number. If I multiply that number by 5 and then divide by 3, what will happen to my number?	"It will have a smaller number... if you divided, you will always get a smaller number." (ID 30)	(no occurrences)	(no occurrences)	"It will get bigger. Well if you multiply by 5 and then divide by 3, 5 is bigger than 3, and multiply is the opposite of divide. So it will end up a little bit bigger." (ID 33)
Divisor before multiplier/divisor greater than multiplier/M-R4 I am thinking about a number. If I divide that number by 7 and then multiply by 5, what will happen to my number?	Student first tries substitution with 30 and says the number will get smaller, but then says it "it depends on what number you're thinking about at the beginning", and that the number could also get bigger or stay the same. (ID 48)	(no occurrences)	(no occurrences)	"It'll get a little smaller because 7 is bigger than 5, and once you divide something by a bigger number than you multiply it, then it will get smaller." (ID 46)

Overall, students varied in their levels of arithmetic generalizations with somewhat robust understandings of the arithmetic generalizations, with 114, or 47.5%, of these responses coded as Level 4, and 89, or 37%, of the responses that were coded as Level 1. The remaining 15.5% of responses were coded as Level 2 or Level 3. As detailed in Figure 20, there were five different ways a student's explanation could be coded as a 1 (*1a/more information required, 1b/incorrect generalization, 1c/specific number, 1d/combination, and 1e/no solution*). Of the 89 Level 1 responses, 58 were coded as 1a. With a Level 1a response, students argued that they needed to know what the number was to know how it changed. Figure 20 shows the breakdown of Level 1 responses, illustrating that 1a explanations were the most common and 1b, 1d, and 1e were all equally common.

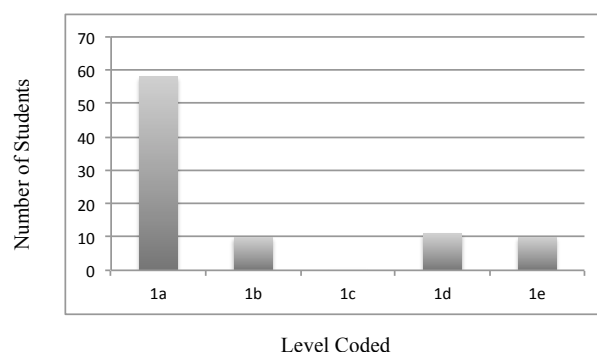


Figure 20. Breakdown of Level 1 responses.

The next four sections include excerpts from transcripts and an overview of some decision rules to help the reader understand how I conceptualized these levels and how student responses were coded. For each level of thinking, I describe the nature of the students' solutions, particularly in relation to other levels of thinking, and I provide one clear and one less clear example from the transcripts. These sections also provide the reader with a better understanding of what the interviews with students were like, and what the raw data looked like.

**Level 1: No Generalization.** In a Level 1 response, students do not make the arithmetic generalization that the task was aimed at exploring. They often explain that they need to know more about the number the interviewer was thinking about to know if and how the number changes, or they make an incorrect generalization. Below, I provide some excerpts from interviews to provide the reader with both a clear and less clear example of a Level 1 response.

*Clear Example of Level 1 Response.* In the interaction below, Student ID 13 responds to the task, "I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?" (*multiplier before divisor/multiplier equal to divisor/M-R1*).

- ID 13: It'll get smaller.  
 Halдар: Okay, and can you explain why it will get smaller?  
 ID 13: Because if you multiply, it will get a big number and then if you divide it, it will get a smaller number.  
 Halдар: Okay, and how can you be sure if you don't know what number I'm thinking about in the beginning?  
 ID 13: Because if you divide, it's probably going to be a smaller number. And if you multiply, it's probably going to be a bigger number. If you do multiply first and then

do divide, it will get a smaller number. And if you do division first, and then multiply second, it'll be a bigger number.

Haldar: Ohhh, so it has something to do with the order?  
ID 13: [nods her head]

This student's response was categorized as a Level 1b/*incorrect generalization* response. Though she demonstrates knowledge of the effects of multiplication and division, she does not say anything that indicates an understanding of their inverse relationship with one another. Further, she clearly makes an inappropriate generalization about the relationship between multiplication and division that is based on the order of the operations.

*Less Clear Example of Level 1 Response.* In the interaction below, Student ID 43 responds to the same task, *multiplier before divisor/multiplier equal to divisor/M-R1*. I coded her response as *1a*, but it was not as clear as the previous example.

ID 43: It will go back to, [pauses] it will be the same number.  
Haldar: The number will stay the same?  
ID 43: Yeah.  
Haldar: Can you explain why the number will stay the same?  
ID 43: Well, because I used the number 5. So 5 times 5 equals 25. And then 25 divided by 5 equals 5.  
Haldar: So you used the number 5 to help you figure it out?  
ID 43: Yeah, and it goes back to 5.  
Haldar: Okay. So you used the number 5. You did 5 times 5 equals 25?  
ID 43: [nods]  
Haldar: And then you did 25 divided by 5 and you got 5 again?  
ID 43: [nods]  
Haldar: How can you be sure if you don't know what number I'm thinking about? I may not be thinking about the number 5 right?  
ID 43: It will get the same number because if any number times 5 has either a 0 or a 5 at the end, so it will either be like [pauses], and then you divide it by 5, it will get smaller.  
Haldar: And how do you know that you'll get the same number again? How do you know the number I'm thinking about will stay the same?  
ID 43: It'll either get smaller or stay the same.  
Haldar: Okay, it'll get smaller or stay the same. And why can it only get smaller? Why can't it get any bigger?  
ID 43: Well, maybe it could, but probably not.  
Haldar: Does it depend on what number I'm thinking about?  
ID 43: Yeah.  
Haldar: So if you knew what number I was thinking about, you could know for sure?  
ID 43: Yeah.  
Haldar: But you think that it could bigger, or smaller, *or* stay the same depending on what number?  
ID 43: Yeah [nods her head].

This example is less clear than the previous example because it seems like a clear Level 2 response for the first half of her explanation. The student first states that the number will stay the same and that she used substitution with the number 5 to figure it out. With the probe questions, however, it becomes clear that she is not making a generalization about all numbers and that the number could also increase or decrease. Though this student did use substitution to make an initial conclusion, she did not generalize to all numbers. Her response was therefore coded as Level 1d/*combination* because over the course of her explanation she made an incorrect

generalization and she also said that how the number changes depends on what number we begin with.

**Level 2: Generalization Based on Substitution.** In a Level 2 response, students make the arithmetic generalization that the task is aimed at exploring, but their reasoning is concrete in nature. In other words, students rely on specific examples and use substitution to draw conclusions about the numbers in the task. They typically base their conclusions, or generalizations, on one or more examples that they try, and not on any properties of multiplication or division that they refer to; they plug in a number and make a conclusion about all numbers based on that substitution. Like at Level 1, students do not demonstrate an understanding of the arithmetic generalization that is independent from the specific context. However, unlike a Level 1 response, a Level 2 response includes an accurate generalization.

As I discussed in the previous chapter, I asked students explicit questions to confirm that they did in fact use the number they referred to as a way to figure out the problem. For example, I would ask questions like, “Did you use the number 5 to help you figure out how the number changes?” or “Did using 0 help you figure it out or are you trying to show me what you mean with an example?” All students who made generalizations were also asked a question like, “How can you know for sure if you don’t know what number we are thinking about in the beginning?” Students’ responses to this set of questions helped me classify students’ responses using the four-level scheme. Below I provide the reader with both a clear and less clear example of a Level 2 response.

*Clear Example of Level 2 Response.* In the interaction below, Student ID 47 responds to the task, “I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number? (*divisor before multiplier/divisor equal to multiplier/M-R2*)

- ID 47: 18 divided by 6 equals 3, times 6, [pauses] it will become the same number.  
 Haldar: So it sounded like you used 18 to figure that one out?  
 ID 47: Yeah.  
 Haldar: How can you know for sure if you don’t know what number I’m thinking about. I may not be thinking about the number 18 right?  
 ID 47: Like uh 2 divided by 6, uh I need to use a big number. 36 divided by 6 equals 6, 6 times 6 equals 36. So it’s like the same.  
 Haldar: Okay. Will that be true no matter what number I’m thinking about?  
 ID 47: Yeah.

Unlike the Level 1 examples discussed in the previous section, this student makes an appropriate arithmetic generalization and he arrives at this generalization with the substitution of the values 18 and 36. He also confirms that it was in fact the example of 18 that helped him figure out the problem. This student clearly used substitution and then used the substitution to make a generalization, and he did not reason about the operations of multiplication or division generally, so his response was coded as Level 2.

*Less Clear Example of Level 2 Response.* In the interaction below, Student ID 34 responds to same task, *divisor before multiplier/divisor equal to multiplier/M-R2*. His response was also coded as Level 2, but it was slightly less clear than the previous example that I discussed.

- ID 34: It would stay the same.  
 Haldar: Can you explain why you think it will stay the same?  
 ID 34: 'Cause like 3 times 6 equals 18. 18 divided into 6 will be 3. 3 times 6 equals 18.  
 Haldar: Oh, so you can go back and forth like that?  
 ID 34: Mmhmm [nods head].  
 Haldar: I see. What if I wasn't thinking about that number? How do you know for sure?  
 ID 34: It'd stay the same 'cause the example I did.  
 Haldar: Did you use the number 3 to help you figure it out?  
 ID 34: [nods] I just picked it randomly.  
 Haldar: Okay, but will that work for any number?  
 ID 34: Yeah.  
 Haldar: No matter what number I'm thinking about, will that number always stay the same after I divide by 6 and multiply by 6?  
 ID 34: Mmhmm [nods].  
 Haldar: Do you know why?  
 ID 34: 'Cause 5 divided into 1 will be 5. So 5 times 1 equals 5. It'll stay the same.

This example is less clear than the previous example because it was unclear whether he used 18 to substitute into the problem and draw a conclusion, or if he was referring to 18 as an illustrative example to demonstrate his understanding of how the number would stay the same. I coded his response as a Level 2 for two main reasons. First, when I explicitly asked him if he used 3 to help him figure it out, he said yes. Second, when I asked him why the number would always stay the same, he again refers to another example. With my last questions, I gave this student the opportunity to demonstrate a higher level of thinking to explain the conclusion he arrived at, but he was unable to do so, and his response was therefore categorized as a Level 2.

***Level 3: Generalization that does not rely on Substitution.*** In a Level 3 response, students make the arithmetic generalization that the task is aimed at exploring, and they do not rely on a concrete example to arrive at the generalization. However, they have trouble justifying the arithmetic generalization in their explanation; they often say they do not know how to explain it or they simply restate the generalization and that it's true for all numbers. Like at Level 2, students do not demonstrate a deep understanding of the arithmetic generalization, but unlike the prior level, students show a new level of generalization of their reasoning that is independent of the specific context or the substitution of numbers.

*Clear Example of Level 3 Response.* In the interaction below, Student ID 28 responds to the task, "I am thinking about a number. If I divide that number by 1, what will happen to my number?" (*division identity/M-ID2*)

- ID 28: Oh. That. If we divided by like a million, it'd still be like a million. You can choose any number, it would be the same answer, no matter what.  
 Haldar: How do you know?  
 ID 28: It's kind of the same for every single rule. If you like minus, well *kind of* for every single rule except minus. If you minus, then if it's like 2 minus 1, then it would be 1. Unless you do 0. So, but if you do it for like and plus...  
 Haldar: Oh are you talking about rules about the number 1?  
 ID 28: Mmhmm [nods head]. So if you're multiplication, it would be like, for an example, like 100 times 1, it would still be 100. If it's like, for division, 89 divided by 1, still it would just, I mean that's the rule for multiplication and division.  
 Haldar: Do you know why that's the rule?  
 ID 28: No [shakes head].

Haldar: Will this be true no matter what number I'm thinking about at the beginning?  
 ID 28: [nods head] If you're thinking about a million, it doesn't matter.

Unlike the Level 2 examples I discussed in the previous section, this student does not rely on the substitution of numbers to make the appropriate arithmetic generalization. He clearly states that the number will always stay the same when you divide by 1 and that it holds true for all numbers, but to justify his reasoning, he simply refers to 'the rules'. I asked him if he knew why the rule works to determine whether he could explain it using the ideas of division, but he did not do so, which is why his response was coded as Level 3.

*Less Clear Example of Level 3 Response.* In the interaction below, Student ID 31 responds to the same task, *division identity/M-ID2*. His explanation was coded as Level 3, but it was less clear than the previous example.

ID 31: It'll be the same thing.  
 Haldar: Okay. Can you explain why?  
 ID 31: Because if you did 5 divided by 5, 5 goes into 5 one time. And that would be pretty much it.  
 Haldar: Did you use the number 5 to help you figure it out?  
 ID 31: Mhmm [nods head].  
 Haldar: And how can you know for sure if you don't know what number I'm thinking about? I may not be thinking about the number 5 right?  
 ID 31: Mhmm.  
 Haldar: So how do you know the number will stay the same if you don't know what number I'm thinking about?  
 ID 31: Because it's almost like multiplication, but you just divide it. So whatever number you're thinking about, it'll be the same thing, because multiplication and division are almost the same thing.  
 Haldar: Okay. Can you tell me more about multiplication and division? What do you mean they are almost the same?  
 ID 31: Because you have to divide the number to get the number right here [pointing to quotient of 1 on his paper from finding 5 divided by 5].  
 Haldar: Oh, okay. And what does that have to do with multiplication?  
 ID 31: Um.  
 Haldar: I'm just trying to understand what you were saying about multiplication and division.  
 ID 31: You said, how would it be the same?  
 Haldar: Well you said multiplication is like division. What were you saying about multiplication and division?  
 ID 31: That they're almost the same because you need to multiply it to get the number right there [pointing to his paper again, but it was unclear what he was referring to].  
 Haldar: Oh, okay, and what do you have to multiply by what? What are you multiplying?  
 ID 31: ...It's just I know because our teacher did a lot of division when I was in third grade. So and she, the only thing she told us was that multiplication was almost like division and she taught us division. And we started getting better at it, and a lot of people came up and they said what is, like, what is 6 divided by 1. And she was helping us, and we all ended up with the same number.  
 Haldar: Okay, so you've seen this kind of problem a lot, so that's how you figured it out?  
 ID 31: Mhmm [nods head].  
 Haldar: Will this be true no matter what number I'm thinking about at the beginning? Will you always get the same number?  
 ID 31: Mhmm [nods head].

This example is less clear than the previous example because the student used substitution initially, but he also speaks generally about the relationship between multiplication and division. Though he begins and ends his explanation by referring to the concrete examples of 5 and 6, he also talks about both the relationship between multiplication and division, as well as his prior experience with division and how that informed his thinking. He initially approaches the task by dividing 5 by 5, instead of dividing 5 by 1, and speaks somewhat generally about the relationship between multiplication and division. However, based on his utterances in our exchange, it was difficult to conclude to what extent he actually understood the relationship between multiplication and division, or how that relationship was relevant to understanding the task. He was not explicit or clear, and when I probed his thinking more about this relationship, he returned to the concrete example of 6 divided by 1. If this student's explanation was based solely on the examples of 5 and 6, this response would have been categorized as Level 2. His explanation was therefore categorized as Level 3 because he did not clearly demonstrate an understanding of the relationship between multiplication and division or how it was relevant for this task, but he also did not seem to rely on substitution.

**Level 4: Generalization Based on Reasoning about Operations.** In a Level 4 response, students make the arithmetic generalization that the task is aimed at exploring, and they do not rely on a concrete example to arrive at the generalization. Like at Level 3, students do not rely on substitution, but unlike a Level 3 response, a Level 4 response includes appropriate mathematical reasoning or justification behind the generalization. Students' explanations at Level 4 involve some discussion about the general properties of the operations.

*Clear Example of Level 4 Response.* In the interaction below, Student ID 44 responds to the task, "I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number?" (*divisor before multiplier/divisor equal to multiplier/M-R2*)

- ID 44: It will turn into a number bigger than it, and then it will turn into a number smaller than the number it just turned into, and then it'll be the exact same.
- Haldar: It'll be the exact same? Okay, can you explain why it'll stay the same?
- ID 44: Because if you multiply it by 6 and whatever number it turns into, when you divide by 6 it's kind of like rewinding it. It will just turn back because divide and multiply are pretty much the exact opposite things.
- Haldar: So how can you know for sure that the number will stay the same if you don't know what number I'm thinking about?
- ID 44: Because it happens with any number. Like if it's 3 times 6, that's 18, divided by and then 18 divided by 6 is 3. Same with all the other numbers.
- Haldar: Oh, so this will be true no matter what number I'm thinking about?
- ID 44: Yeah.

This student does not rely on the substitution of values to make sense of the problem or to determine that the number will stay the same. He recognizes the inverse relationship between multiplication and division, and explains this relationship with the analogy of 'rewinding' as well as with an example.

*Less Clear Example of Level 4 Response.* In the interaction below, Student ID 25 responds to the same task, *divisor before multiplier/divisor equal to multiplier/M-R2*, but his explanation is not as clear.

- ID 25: It's kind of like plus and minus. If you uh 1 plus 6 equals 6, then you divide it by 6, it equals 1. Because like you're kind of like adding it and subtracting.
- Haldar: So you're telling me that the number will turn into 1?
- ID 25: Well if you're thinking of the number 1 because it's just like plus and minus. If you have 19 plus 1 equals 20, then minus 1 equals 19. So you would get the number you are starting by.
- Haldar: Okay, so you're talking about adding and then subtracting, but here we're dividing and then multiplying.
- ID 25: Yeah, it's kind of like a fact family. Like multiplying is kind of like the same as adding, and dividing is kind of the same as subtracting.
- Haldar: Okay. So what you've told me so far is if we start with the number 1, we'll get the number 1 again?
- ID 25: Yeah.
- Haldar: So will the number always stay the same?
- ID 25: The number will always stay the same if like the number you're multiplying by is the same and the number you're dividing by is the same.
- Haldar: So in this problem, will it always stay the same?
- ID 25: Yeah.
- Haldar: How can you be sure if you don't know what number we're starting with?
- ID 25: It doesn't matter what number you're starting with. It's just the numbers with like, if you have 5 and 5 [uses fingers and appears to be pointing to the divisor and multiplier in the task], it will always end up in the same number you're starting by. Like if you do 5 multiplied by 5, then divided by 5, it will also equal 5. Same way with 6, 7, 8, ...
- Haldar: So this will be true no matter what number I'm thinking about in the beginning?
- ID 25: Yeah.

This example is less clear because the logic and language that this student uses is difficult to follow. In his initial statement, for instance, this student adds 6 to 1, and then divides by 6. The task, however, is about division and multiplication. Later, he cites another example and uses multiplication and division, though he multiplies first and then divides. This could, however, indicate an understanding that the order of the operations does not affect the result, and I should have responded with additional follow up questions. His response was categorized as Level 4 because he makes the appropriate generalization that the task is aimed at exploring and he justifies this generalization by comparing the relationship between multiplication and division to the relationship between addition and subtraction. He also talks about the relationship between the multiplier and divisor and why their equivalence is important in understanding why the number will always stay the same.

**Research Question 2: Variations in Student Thinking Across Tasks.** In this section, I compare students' thinking across the three generalization types: *direction of change*, *identity*, and *relationship between multiplication and division*. In my comparative analysis, I also consider task subtypes, like symmetric/non-symmetric and comparing/not comparing, like I did in Chapter 3. I show that although the 4<sup>th</sup> graders in the study demonstrated competency with arithmetic generalizations, most students showed inconsistency in their thinking across tasks, with some types of generalizations appearing more difficult than others.

In much of the forthcoming analyses, I group Level 1 and Level 2 responses together, and I group Level 3 and Level 4 responses together, as I did in Chapter 3. I structured the analyses in this manner because Levels 1 and 2 are both more concrete in nature as students rely on specific numbers, while Levels 3 and 4 require generalizing that does not require students to deal with

specific numbers. Combining scores in this manner makes trends in the data more visible. For each task, students received either a ‘generalization score’ of 0 for Level 1 and Level 2 responses, or they received a ‘generalization score’ of 1 for Level 3 and Level 4 responses.

The proportion of students who received an advanced generalization score (generalization score of 1) for each task is detailed in Table 13. In this table, I also provide the breakdown of students by level for each task. Note that Level 1 and Level 4 responses occurred the most frequently across tasks, while Level 2 and Level 3 explanations did not occur as frequently. This table provides an overview of some of the patterns I observed in the data, illustrating which tasks elicited the most generalized levels of thinking from students, and how student thinking varied for each task. I analyze these patterns further in the following sections.

Table 13: Proportion of Students with Advanced Generalization Score and Breakdown of Student Responses by Level for Each Task

	Task	Proportion of Students with Advanced Generalization Score (Level 3 or 4)	Frequency Breakdown of Students' Responses by Level			
			Level 1	Level 2	Level 3	Level 4
<b>Direction of Change</b>	Multiplying/M-D1	.750	2	0	0	22
	Dividing/M-D2	.542	2	0	1	21
	Multiplying/comparing/M-D3	.833	1	2	0	21
	Dividing/comparing/M-D4	.583	4	2	0	18
<b>Identity</b>	Multiplication identity/M-ID1	.917	0	0	1	23
	Division identity/M-ID2	.792	0	0	1	23
<b>Relationship between Multiplication/Division</b>	Multiplier before divisor/multiplier greater than divisor/M-R1	.500	6	1	0	17
	Divisor before multiplier/divisor greater than multiplier/M-R2	.375	6	1	0	17
	Multiplier greater than divisor/M-R3	.333	11	0	0	13
	Divisor greater than multiplier/M-R4	.333	11	0	1	12

To compare students' understandings across the three types of arithmetic generalizations, I calculated a mean proportion of advanced generalization scores for each type of generalization. For example, as I discussed in Chapter 3, if a student's responses were coded with 4, 3, 1, and 1 for the *direction of change* tasks, the proportion of tasks she received an advanced generalization score for was .5. These proportions were averaged across each type of arithmetic generalization and these mean scores were .678, .855, and .385 for the *direction of change*, *identity*, and *relationship between multiplication/division* tasks respectively. A Friedman test<sup>9</sup> showed that there was a statistically significant effect of generalization type on students' production of advanced generalizations ( $\chi^2(2) = 20.246, p = 0.000$ ). Post hoc analysis with Wilcoxon signed-rank tests was conducted with a Bonferroni correction applied, resulting in a significance level set at  $p < 0.017$ , and there were statistically significant differences between the following types of generalizations: (a) *direction of change* and *relationship between multiplication/division* tasks ( $Z = -3.220, p = .001$ ) with medium/large effect size of  $r = .465$ ; and (b) *identity* and *relationship between multiplication/division* tasks ( $Z = -3.358, p = .001$ ) with medium/large effect size of  $r = .485$ . The difference between *direction of change* and *identity* tasks was not statistically significant. These results indicate that students provided the most generalized levels of responses

<sup>9</sup> A Friedman test was used because the data was ordinal and non-normally distributed.

for the *direction of change* and *identity* tasks and the least generalized responses to the *relationship between multiplication/division* tasks.

To understand how student thinking varied across tasks, I next analyzed each students' across-task profile, and I observed that while some students used the same level of generality across all tasks, most did not. As displayed in Figure 21, 18 of the 24 students demonstrated *at least* two different levels of thinking over the course of the interview. Six students demonstrated a consistent level of thinking across all tasks. Of these 6 students, 5 students responded with Level 4 explanations across all tasks, and 1 student responded with Level 1 explanations across all tasks.

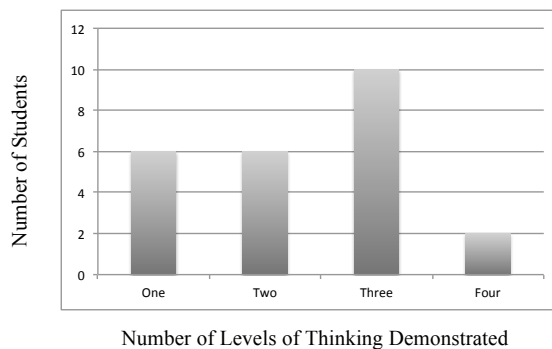


Figure 21. Number of levels of thinking demonstrated by students across tasks.

These data illustrate that even the fourth graders who demonstrated the most generalized thinking for some tasks, reverted to less generalized ways of thinking for particular tasks. This finding warranted further analyses, which I will discuss in detail in the following sections.

In the remaining sections, I examine student thinking within each type of arithmetic generalization more closely to understand these findings. To do so, I first provide an overview of student performance for each type of generalization. I then analyze how student thinking varied across particular variables, like the type of operation and order of operations, and how individual students' responses varied across tasks.

***Direction of Change Tasks.*** Figure 22 presents the frequency with which students used the four generalization levels for each of the *direction of change* tasks. Level 4 responses occurred the most frequently across tasks. Of the 24 participants, 12 students responded with a Level 4 explanation across all four tasks, demonstrating that the fourth graders had a robust understanding of this type of generalization. Next, I analyze the effect of the two different operations and task subtypes, and the consistency of students' responses across the *direction of change tasks*.

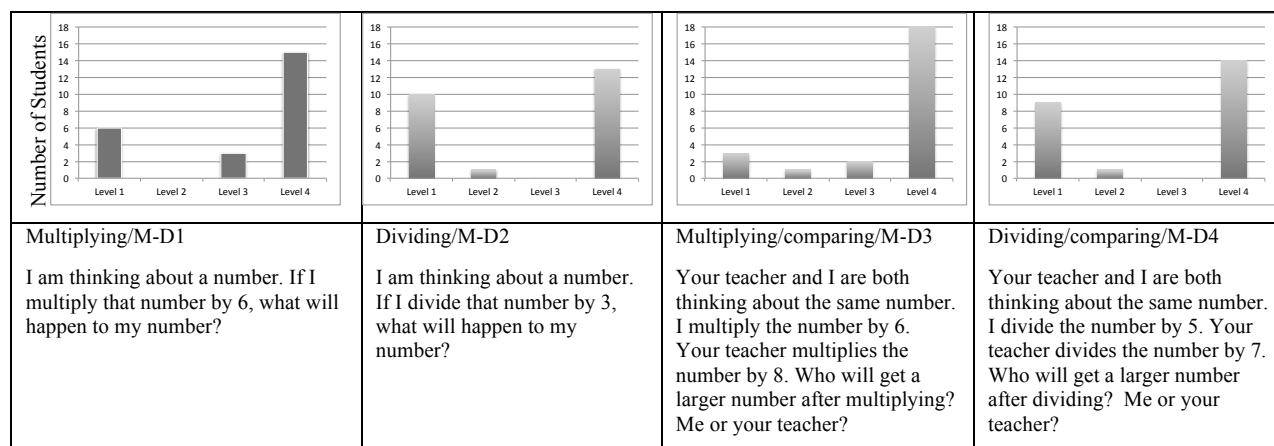


Figure 22. Breakdown of Student Responses for *Direction of Change* Tasks.

*Effect of Different Variables.* As discussed in Chapter 2, tasks were paired in their design to evaluate the effects of the (a) multiplication versus division problem contexts, and (b) comparing versus not comparing problem contexts. For example, *multiplying/M-D1* examined students' understandings of direction of change for multiplication, while *dividing/M-D2* examined students' understandings of direction of change for division. Similarly, *multiplying and comparing/M-D3* and *dividing and comparing/M-D4* involved comparing products and quotients, while *multiplying/M-D1* and *dividing/M-D2* did not involve comparing. These analyses were important in identifying variables that influenced student thinking.

The multiplication versus division problem context did have a statistically significant effect on students' production of advanced generalizations for the comparing set of tasks, but not for the tasks that did not involve comparing. As illustrated in Figure 23, 18 students (75%) had an advanced generalization score for *multiplying/M-D1*, while 13 students (54.2%) had an advanced generalization score for *dividing/M-D2*. A McNemar's Chi-square test, however, revealed that this difference was not statistically significant ( $\chi^2(1, N = 24) = 5, p = .063$ ). Responses to *multiplying/comparing/M-D3* and *M-dividing/comparing/M-D4* were also compared, and 20 students (83.3%) and 14 students (58.3%) received advanced generalization scores for each respective task. A McNemar's Chi-square test showed that this difference was statistically significant ( $\chi^2(1, N = 24) = 6, p = .031$ ) with a medium effect size ( $\Phi = .354$ ), indicating that comparing quotients was more challenging than comparing products for these students.

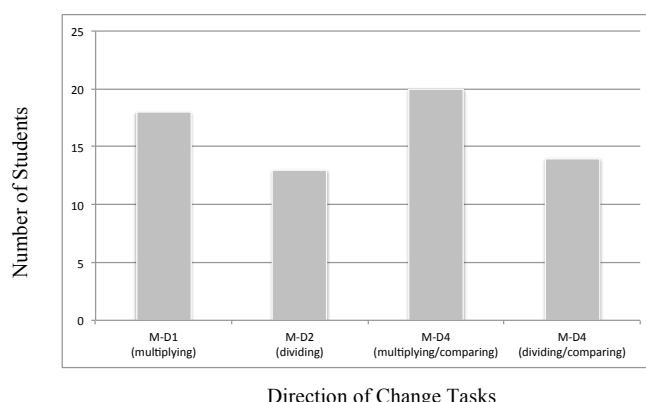


Figure 23. Students with advanced generalization score for *direction of change* tasks

The comparing versus not comparing problem context did not have a statistically significant effect on students' production of advanced generalizations. I calculated students' subtotal scores for the two tasks that did not involve comparing and the two tasks that did involve comparing to determine if there was a difference across these types of tasks. As noted earlier, the subtotal score was calculated by assigning students a score of either 1 for Levels 3 and 4, or a score of 0 for Levels 1 and 2, for a total maximum subtotal score of 2. The means for students' subtotals across these types of tasks were similar at 1.29 and 1.42 respectively, indicating that student thinking did not vary much when comparison was involved in the *direction of change* tasks. A Wilcoxon signed-rank test confirmed that this difference was not statistically significant ( $Z = -1.342$ ,  $p = .180$ ).

*Consistency Across Student Responses.* Individual students were not always consistent with respect to the level of generality with which they treated the operations in the tasks. Nine students responded inconsistently across the *direction of change* tasks, and demonstrated some range in the level of generality with which they treated the operations of multiplication and division. Table 14 illustrates how two students varied in their thinking across the tasks. Student ID 47, for instance, demonstrated 4 different levels of thinking across the *direction of change* tasks. Student ID 48, provided Level 3 explanations for the tasks that involved multiplication and Level 1 explanations for the division tasks. Note that I include Student ID 47 in the next two sections as well, so the reader can see how a sample student responded across *all* interview tasks.

Table 14: Sample Range of Thinking for Direction of Change Tasks

<i>DIRECTION OF CHANGE TASKS</i>			
Multiplying/M-D1	Dividing/M-D2	Multiplying/comparing/M-D3	Dividing/comparing/M-D4
I am thinking about a number. If I multiply that number by 6, what will happen to my number?	I am thinking about a number. If I divide that number by 3, what will happen to my number?	Your teacher and I are both thinking about the same number. I multiply the number by 6. Your teacher multiplies the number by 8. Who will get a larger number after multiplying? Me or your teacher?	Your teacher and I are both thinking about the same number. I divide the number by 5. Your teacher divides the number by 7. Who will get a larger number after dividing? Me or your teacher?
<b>ID 47</b>	Level 3 Response: “It will get bigger because if you think about a number like 6 times 6, it will be like 36. And 5 times 6 equals 30. And 4 times 6 equals 24.” When asked if he used those numbers to help him figure it out, the student shook his head and said, “For every number,” but he did not explain why.	Level 2 Response: “It will get smaller...like 6 divided by 3 is 2.” When asked if he used the number 6 to help him figure it out, he said yes. When asked how he can know for sure, he replied with “Uh, 3 divided by 3 is 1.”	Level 4 Response: The student first states that the teacher will always get the larger number and refers to the example with the number 9. When asked how he can know for sure since we may not be thinking about the number 9, the student says, “because this number [pointing to 8] is bigger than the other number [pointing to 6].”
<b>ID 48</b>	Level 3 Response: “It would pretty much always get bigger. If you multiply any number by a higher number, it will always be higher.”	Level 1 Response: “It depends. If this number right here [pointing to the ‘I am thinking about a number’] was 27, then that answer would be 9. And if it was 24, it’d be 8. And if it was 21, it’d be 7.” When asked if the number will always get smaller when dividing by 3, the student explained that “it depends on what number you start with...it may get bigger, but it may also stay the same.”	Level 1 Response: The student first tries substitution with 35 and concludes that I would get the larger number. When asked if I will always get the larger number even if we start with a number other than 35, the student says, “That’s kind of hard to explain.” She then revisits the example of 35 and explains that if it was a different number, “kids couldn’t do it, only college students could do it.” When asked if it would depend on what number we are thinking about, she said yes.

**Identity Tasks.** Figure 24 presents the frequency with which students used the four generalization levels for each of the *identity* tasks. Level 3 and Level 4 responses occurred the most frequently across both tasks. Of the 24 participants, 19 students produced advanced generalizations across both tasks, demonstrating that a majority of the fourth graders had an understanding of generalizations related to multiplicative identity on the tasks used. Next, I analyze the effect of the two different operations and the consistency of students’ responses across the *identity* tasks.

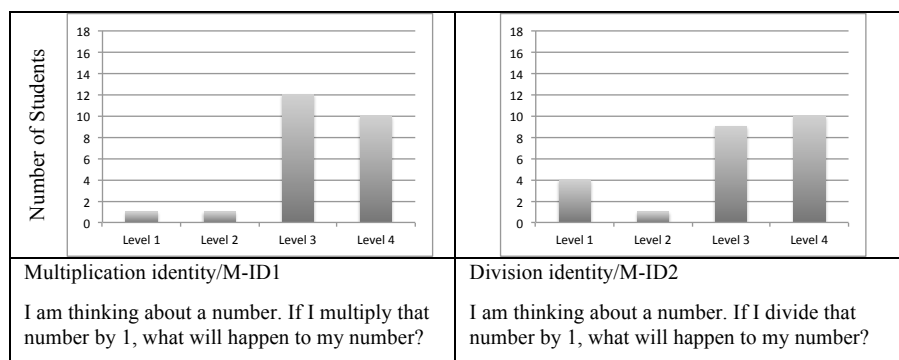


Figure 24. Breakdown of student responses for *identity* tasks.

*Effect of Different Variables.* The *identity* tasks were paired in their design to evaluate the effects of the multiplication versus division problem contexts. *Multiplication identity/M-ID1* probed students' thinking about the identity element for multiplication, while *division identity/M-ID2* examined students' understandings of the identity element for division. As illustrated in Figure 25, 22 students (91.7%) had an advanced generalization score, or Level 3 or Level 4 response, for the multiplication task, while 19 students (79.2%) had an advanced generalization score for the division task. A McNemar's Chi-square test, however, revealed that this difference was not statistically significant ( $\chi^2(1, N = 24) = 3, p = .250$ ). Thus, the fourth graders demonstrated similar levels of understanding of the *identity* tasks for both multiplication and division.

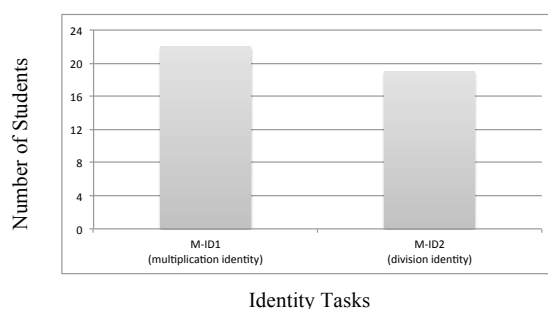


Figure 25. Students with advanced generalization score for *identity* tasks.

*Consistency Across Student Responses.* Students were fairly consistent with respect to the level of generality with which they treated the operations of multiplication and division. Seven students responded inconsistently across the *identity* tasks, and demonstrated some range in thinking, and only 2 of these 7 students received an advanced generalization score. Table 15 illustrates how these two students responded with Level 3 responses for the multiplication task and Level 1 responses for the division task.

Table 15: Sample Range of Thinking for Identity Tasks

<i>IDENTITY TASKS</i>	
(M-ID1/multiplication identity)	(M-ID2/division identity)
I am thinking about a number. If I multiply that number by 1, what will happen to my number?	I am thinking about a number. If I divide that number by 1, what will happen to my number?
<b>ID 47</b> Level 3 Response: “It will be the same number as what number you’re thinking of...because if you write a big number and you write times 1, it will be the same.”	Level 1 Response: First, the student stated, “It will get smaller because we’re only dividing.” Then the student seems to change his mind and said that “if you divide by 1, the number always becomes 1.” The student changes his mind again and says the number can get bigger or smaller depending on what number we begin with.
<b>ID 38</b> Level 3 Response: “It will stay the same because if you multiply any number by 1, it always stays the same.” When asked why that is true, she said, “I don’t know.”	Level 1 Response: “It will be 1 because that you divided it by 1.” She then refers to the example of 7 and how 7 divided by 1 is 1, and states, “No matter what the number is, you’re still going to get 1.”

***Relationship between Multiplication and Division Tasks.*** Figure 26 presents the frequency with which students used the four generalization levels for each of the *relationship between multiplication and division* tasks. Level 1 and Level 4 responses occurred the most frequently across tasks, while Level 2 and Level 3 responses did not occur often. Of the 24 participants, 7 students responded with a Level 4 explanation across all four tasks, and 11 students responded with Level 1 explanations across all tasks. Next, I analyze the effect of order and symmetry, and the consistency of students’ responses across the *relationship between multiplication/division tasks*.

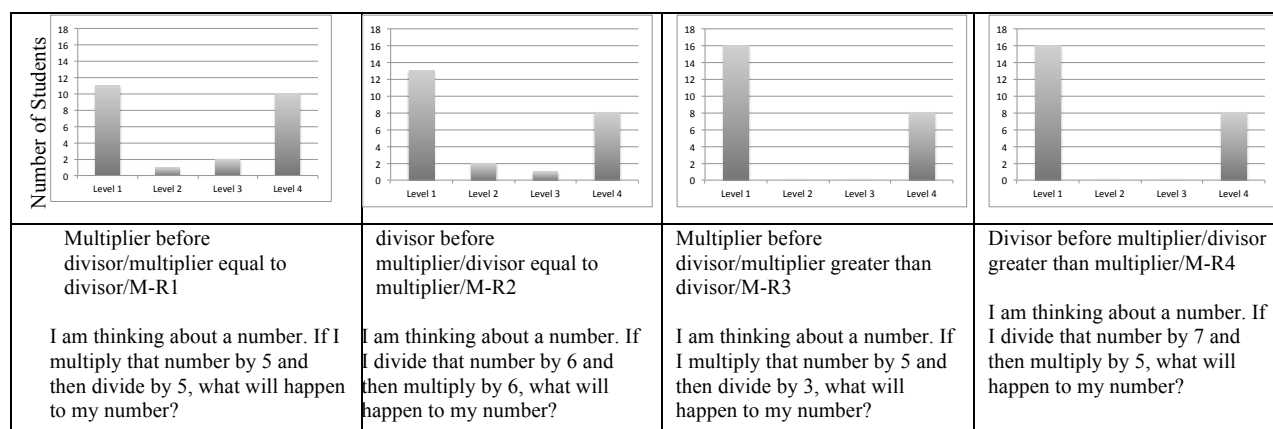


Figure 26. Breakdown of student responses for *relationship between multiplication/division* tasks.

***Effect of Different Variables.*** Tasks were paired in their design to evaluate the effects of the (a) order of operations, and (b) symmetry. For example, *multiplier before divisor/multiplier equal to divisor/M-R1* and *divisor before multiplier/divisor equal to multiplier/M-R2* were similar in structure, but the order of the operations varied across the tasks. Similarly, *multiplier before divisor/multiplier equal to divisor/M-R1* and *divisor before multiplier/divisor equal to multiplier/M-R2* were symmetric tasks (i.e., multiplier and divisor equal), while *multiplier greater than divisor/M-R3* and *divisor greater than multiplier/M-R4* were asymmetric (i.e., multiplier and divisor not equal).

The order of operations did not have a statistically significant effect on students’ production of advanced generalizations. As illustrated in Figure 27, 12 students (50.0%) and 9 students (37.5%) received an advanced generalization score for *multiplier before*

*divisor/multiplier equal to divisor*/M-R1 and *divisor before multiplier/divisor equal to multiplier*/M-R2 respectively. A McNemar's Chi-square test, however, revealed that this difference was not statistically significant ( $\chi^2(1, N = 24) = 3, p = .250$ ). Similarly, 8 students (33.3%) students received advanced generalization scores for both *multiplier greater than divisor*/M-R3 and *divisor greater than multiplier*/M-R4 task, again indicating no difference across students' thinking for these tasks. These analyses produced no evidence that order of operations affects performance on the tasks.

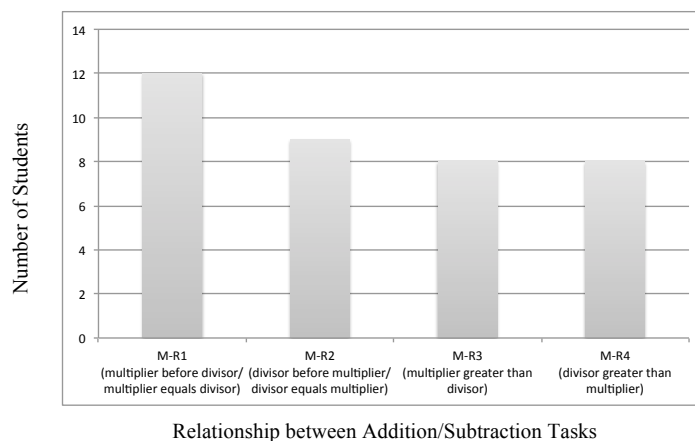


Figure 27. Students with advanced generalization score for *relationship between multiplication/division* tasks.

Similarly, the symmetry of the task did not have a statistically significant effect on students' production of advanced generalizations. I calculated students' subtotal scores for the two symmetrical tasks (i.e., multiplier and divisor equal) and two asymmetrical tasks (i.e., multiplier and divisor not equal). As noted earlier, this was calculated by assigning students a score of either 1 for Levels 3 and 4, or a score of 0 for Levels 1 and 2, for a total maximum subtotal score of 2. The mean score for symmetrical tasks was .88 (SD=.95), while the mean score for asymmetrical tasks was .67 (SD=.96), and a Wilcoxon signed-rank test showed that this difference between students' responses for these different task types was not statistically significant ( $Z = -1.518, p = .129$ ). Thus, neither the order of operations nor the symmetry of the tasks appeared to prompt significant differences in the types of thinking students demonstrated.

*Consistency Across Student Responses.* Many students responded inconsistently with respect to the level of generality with which they treated the arithmetic operations. Six students responded inconsistently across the *relationship between multiplication/division* tasks, and demonstrated some range in thinking. Table 16 illustrates how two students varied in their thinking across the tasks. Student ID 47, for example, responded with a Level 4 explanation for the symmetric task that involved multiplication first. For the symmetric task that required division first, however, he provided a Level 2 response, indicating that the order of the operations may have influenced his thinking. For the asymmetric tasks, he responded with Level 1 explanations for both tasks, which may mean that the symmetry of the tasks also influenced his thinking. Similarly, Student ID 25, provided Level 4 explanations for the symmetric tasks and Level 1 explanations for the asymmetric tasks, indicating that symmetry may have affected his thinking as well.

Table 16: Sample Range of Thinking for Relationship between Multiplication/Division Tasks

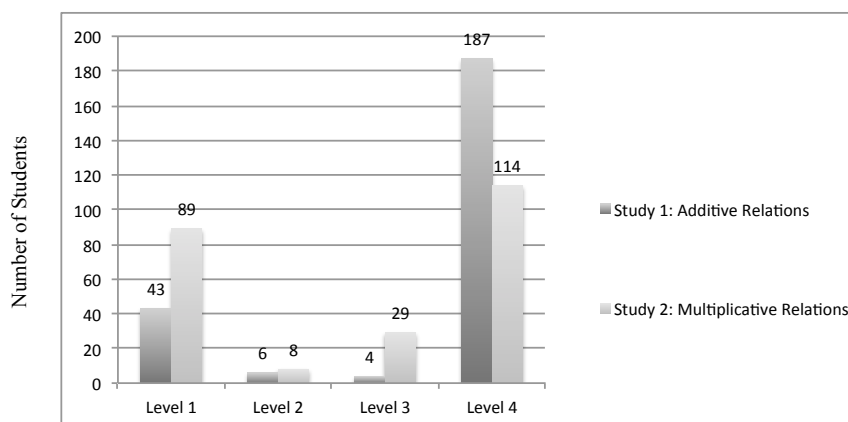
<i>RELATIONSHIP BETWEEN MULTIPLICATION/DIVISION TASKS</i>				
	Multiplier before divisor/multiplier equal to divisor/M-R1	Divisor before multiplier/divisor equal to multiplier/M-R2	Multiplier before divisor/multiplier greater than divisor/M-R3	Divisor before multiplier/divisor greater than multiplier/M-R4
	I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?	I am thinking about a number. If I divide that number by 6 and then multiply by 6, what will happen to my number?	I am thinking about a number. If I multiply that number by 5 and then divide by 3, what will happen to my number?	I am thinking about a number. If I divide that number by 7 and then multiply by 5, what will happen to my number?
<b>ID 47</b>	Level 4 Response: “It depends on the number you are thinking about and then it would stay the same.” The student then demonstrates with the example of 5 and states that “the divide is the opposite of the multiply, and the multiply is the opposite of the divide.”	Level 2 Response: “18 divided by 6 equals 3, times...[pauses] It will become the same number.” When asked how he can know for sure and that I may not be thinking about the number 18, the student used substitution with 36 and arrived at same conclusion, and he states that the number will always stay the same.	Level 1 Response: The student first said that he just is not sure. When asked if it depends on what number we begin with, he said yes.	Level 1 Response: “It gets bigger because of the last part [gesturing to the ‘multiplying by 5’ in written task]. The student also said that he was not sure if the number always gets bigger.
<b>ID 25</b>	Level 4: “Your number will become the exact same number as your start. Like if you say 1 times 5 equals 5, then if you divide it by 5, it would equal 1.” The student then drew a comparison to adding and then subtracting, and referred to another example of 10. When asked how he can know for sure without knowing what number we start with, he explained, “It doesn’t matter the number. It just matters what you’re dividing by and multiplying by... If you multiply one number and then you divide the same number as you were multiplying by, it would equal the same number that you are starting with.”	Level 4: “It’s kind of like plus and minus...you will get the number you are starting by.” When asked about multiplication and division, he explained, “It’s kind of like a fact family. Multiplying is kind of the same as adding, and dividing is kind of the same as subtracting.” When asked if the number will always stay the same, the student explained, “The number will always stay the same if the number you’re multiplying by is the same and the number you’re dividing by is the same.”	Level 1: The student first tried substitution with 2, and arrived at 3 ½. He then stated, “You will always get a smaller number.” As he explained, he stated, “When you’re dividing, your number’s always going to get smaller. When you’re multiplying, it depends on the number.” In additional probing about this specific problem, he explained, “The number will get smaller because there’s a division thing.”	Level 1: “It would get bigger.” When asked how he knew, the student changed his mind and explained, “Well it depends on the number you’re dividing and multiplying by.” When asked if he was referring to the 7 and 5, or the number we start with, he said it would depend on the number we begin with.

**Comparison of Findings from Additive and Multiplicative Domains.** This section discusses the findings from Chapter 3 and this chapter in relation to one another to examine how students’ understandings of additive and multiplicative relations are similar and different. The first part of this chapter focuses on patterns in thinking that were similar across the two studies, while the second part of this chapter highlights differences.

**Similarities in Findings Across Additive and Multiplicative Domains.** There were many parallels in the findings from the additive and multiplicative studies, suggesting that within each domain, student thinking develops in similar ways. The following findings were apparent across both studies: (a) students demonstrated 4 levels of thinking that varied with respect to the generality with which they treated the arithmetic operations; (b) Level 1 and Level 4 responses were the most common, while Level 2 and Level 3 responses occurred less frequently; (c) *direction of change* and *identity* tasks elicited the most generalized levels of thinking, while the *relationship between operations* tasks elicited the least generalized thinking; (d) the majority of students exhibited at least two levels of thinking across tasks; and (e) while generalization type

did effect the production of students' advanced generalization in both studies, most task subtypes did not (the exception was in the multiplicative study, where I found that comparing two quotients may be more challenging for students than comparing two products).

Figure 28 details the breakdown of student responses for each study by level and Table 17 displays the proportion of students with advanced generalization scores of 1 for each task. Additionally, Table 18 provides sample student responses to illustrate what the different levels of thinking looked like for two parallel additive and multiplicative tasks. These figures and tables highlight the similarities in the findings across both studies.



Frequency of Each Level Response for each Study

Figure 28. Breakdown of student responses for each study

Table 17: Proportion of Students with Advanced Generalization Score for Each Task

Study 1: Additive Relations		Proportion of Students with Advanced Generalization Score (Level 3 or 4)	Study 2: Multiplicative Relations	
Task			Task	Proportion of Students with Advanced Generalization Score (Level 3 or 4)
<b>Direction of Change</b>	Adding/A-D1	.917	Multiplying/M-D1	.750
	Subtracting/A-D2	.917	Dividing/M-D2	.542
	Adding/comparing/A-D3	.875	Multiplying/comparing/M-D3	.833
	Subtracting/comparing/A-D4	.750	Dividing/comparing/M-D4	.583
<b>Identity</b>	Addition identity/A-ID1	1.000	Multiplication identity/M-ID1	.917
	Subtraction identity/A-ID2	1.000	Division identity/M-ID2	.792
<b>Relationship between Operations</b>	Addend before subtrahend/addend greater than subtrahend/A-R1	.708	Multiplier before divisor/multiplier greater than divisor/M-R1	.500
	Subtrahend before addend/subtrahend greater than addend/A-R2	.708	Divisor before multiplier/divisor greater than multiplier/M-R2	.375
	Addend greater than subtrahend/A-R3	.542	Multiplier greater than divisor/M-R3	.333
	Subtrahend greater than addend/A-R4	.542	Divisor greater than multiplier/M-R4	.333

Table 18: Sample Student Responses from Parallel Tasks from Each Study

Task	Level 1: No Generalization	Level 2: Generalization Based on Substitution	Level 3: Generalization that does not Rely on Substitution	Level 4: Generalization Based on Reasoning about Operations
<b>Study 1: Additive Relations</b> Addend before subtrahend/addend equal to subtrahend/A-R1 I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?	“The number that’s only 0, it’s going to stay the same. But other numbers, it’s going to get smaller.” The student then responded ‘yes’ when asked whether it depends on what number I am thinking about. (ID 13)	“Let’s go for 10. So if you add 5, it would be 15. 15 subtract 5 equals 10. So you’re not going to change any number...And if you use a different number, you’ll get that number.” (ID 04)	(no occurrences)	“It would be the same number no matter what... unless you change <i>one</i> of the 5s.... You’re adding 5 to whatever number, it’s 5 more. And then you’re taking 5 away from whatever number that is. So then it would be the same number.” (ID 17)
	<b>Study 2: Multiplicative Relations</b> Multiplier before divisor/multiplier equal to divisor/M-R1 I am thinking about a number. If I multiply that number by 5 and then divide by 5, what will happen to my number?	“It will either get smaller or stay the same...maybe it could [get bigger] but probably not.” (ID 43)	“7 times 5 equals 35. 35 divided in 5...[pauses] actually, it would be any number! Like 3 times 5 goes 15. 5 divided into 15 is 3. It turned out the number that you’re multiplying.” When asked if this is true for all numbers, the student said yes. (ID 34)	“You’ll get the same number that you multiplied with 5.” When asked if this is true for all numbers, the student said yes, but she could not explain why. (ID 37)

**Differences between Additive and Multiplicative Domains.** Despite the similarities in the findings across the additive and multiplicative domains, the differences across the two studies indicate that overall, students produced more advanced generalizations for the tasks that involved addition and subtraction tasks. A binary logistic regression<sup>10</sup> was conducted to analyze the effect of domain type and generalization type on students' generalization score for each task. There was not a statistically significant interaction between domain type and the *direction of change* tasks ( $p = .960$ ), domain type and the *identity* tasks ( $p = .774$ ), or domain type and the *relationship between operations* tasks ( $p = .997$ ). These results indicate that the effect of generalization type was the same across both the additive and multiplicative domains, as illustrated by the similar shape of the graphs in Figure 29. The main effects analysis showed that the additive domain was associated with an increase in the odds of producing an advanced generalization, with an odds ratio of 2.66 (Wald  $\chi^2(1) = 10.805$ ,  $p = .001$ ). I completed post-hoc analyses to explore these findings further and identify the source of these effects.

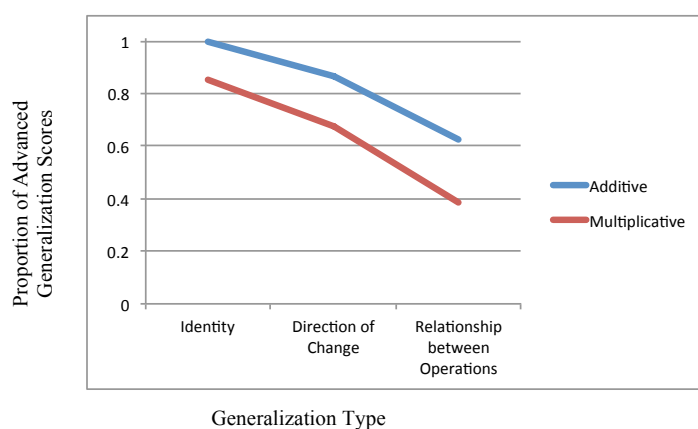


Figure 29. Proportion of advanced generalization scores for each study by generalization type.

In the additive study, 79.6% of all coded responses received an advanced generalization score, while 59.6% of responses in the multiplicative study had an advanced generalization score. A Mann-Whitney U test<sup>11</sup> indicated that this difference was statistically significant ( $U = 190$ ,  $p = .04$ ). To explore these differences further, I completed Mann-Whitney U tests for each of the parallel additive and multiplicative tasks and found statistical differences for the tasks detailed in Table 19. For each of these tasks, students produced more advanced generalization scores for the additive task than the corresponding multiplicative task. It is difficult to explain why the differences between some parallel tasks were statistically significant, while others were not. A larger sample size may have resulted in different findings, and this is something I would like to explore further in my future research.

<sup>10</sup> A binary logistic regression was used because generalization score was binary, there were two independent variables (domain and generalization type), and the data was non-normally distributed.

<sup>11</sup> A Mann-Whitney U test was used because the additive and multiplicative groups were independent, and the data was ordinal and non-normally distributed.

Table 19: Mann-Whitney U Test Analyses Across Parallel Additive and Multiplicative Tasks

TASKS		U	Z	p	r
Additive	Multiplicative				
adding/A-D1	multiplying/M-D2	207	-2.278	0.023	0.33 (medium effect size)
subtracting/A-D2	dividing/M-D2	187.5	-2.612	0.009	.38 (medium effect size)
addition identity/A-ID1	multiplication identity/M-ID1	131	-3.999	0.000	.58 (large effect size)
subtraction identity/A-ID2	division identity/M-ID2	129.5	-4.006	0.000	.58 (large effect size)
subtrahend before addend/subtrahend equal to addend/A-R2	divisor before multiplier/divisor equal to multiplier/M-R2	183	-2.425	0.015	.35 (medium effect size)

The previous chapters also examined whether there were differences in student thinking across particular task subtypes, and these results varied across the additive and multiplicative domains. The *multiplying/comparing/M-D3* and *dividing/comparing/M-D4* tasks were compared, and 20 students (83.3%) and 14 students (58.3%) had an advanced generalization score on each respective task. A McNemar's Chi-square test showed that this difference was statistically significant, indicating that comparing quotients was more challenging than comparing products. A similar analysis was completed for *adding/comparing/A-D3* and *subtracting/comparing/A-D4*, but there was not a statistical difference between these tasks. Thus, comparing sums versus differences did not appear to elicit differences in students' thinking, but comparing products versus quotients did. So although there may be similar developmental progressions through the additive and multiplicative domains, an understanding of the relationship between sums and differences may develop earlier for children than the relationship between products and quotients.

The previous chapters also examined differences in student thinking across the three types of arithmetic generalizations. In the additive study, the differences between all three types of arithmetic generalizations were statistically significant, with *identity* tasks eliciting the highest level responses from students and *relationship between addition/subtraction* tasks eliciting the lowest level responses. For the multiplicative study, the *relationship between multiplication/division* tasks elicited the highest level responses, but the difference between the *identity* and *direction of change* tasks was not statistically significant, indicating that perhaps those two types of generalizations develop in concert with one another in the multiplicative domain.

## Summary and Discussion

In this chapter, I analyzed students' construction of arithmetic generalizations in the context of multiplication and division tasks. The 10 focal tasks were all similar in structure and wording, but each task addressed a different arithmetic generalization. My focus was on three

types of generalizations: *direction of change*, *identity*, and *relationship between multiplication and division*. As discussed in the previous chapter, the analysis of student interviews focused first on identifying patterns in and understanding the character of student thinking, and second on analyzing if and how children's thinking varied across the different tasks. I concluded this chapter with a cross-study analysis to compare students' thinking across the additive and multiplicative domains.

A fine-grained qualitative analysis involved an iterative process of watching the interviews, observing patterns in student thinking, and identifying similarities and differences across students' explanations. This analysis revealed the same four levels of thinking discussed in the additive study, levels that reveal a spectrum of increasing generality with which students approach these tasks. These levels became the basis of a coding scheme designed to capture the quality of students' arithmetic generalizations. This scheme consisted of four levels and students' explanations for each task were coded as Level 1, 2, 3, or 4. At Levels 1 and 2, students rely on specific instances and substitution of values. At Levels 3 and 4, students do not rely on any examples and make generalizations about the arithmetic operations.

Overall, students demonstrated somewhat robust understandings of the arithmetic generalizations, with 114 out of 240 responses (47.5%) coded as Level 4, and 89 responses (37%) coded as Level 1. The remaining 15.5% of responses were coded as Level 2 or Level 3. Level 4 and Level 1 codes therefore captured the most common types of thinking among students, while Level 2 and Level 3 codes did not occur as frequently.

Though students demonstrated robust understandings overall, student thinking was not always consistent across tasks. To investigate this further, I compared student thinking across different variables, including generalization type and task subtypes (operation, comparing, order of operations, and symmetry) and I examined how individual students' responses varied across tasks. The findings indicate that generalization type did have a statistically significant effect on students' production of advanced generalizations, while only one task subtype had a significant effect. Students provided the most generalized levels of responses for the *identity* and *direction of change* tasks and the least generalized responses to the *relationship between multiplication/division* tasks. Additionally, for the two comparing tasks, students were more likely to produce an advanced generalization when comparing products versus quotients, indicating that students may have a deeper understanding of products than quotients at this age. Many students demonstrated a range of thinking within each type of generalization as well and the across task profiles illustrated this finding. I included Student ID 47 in each of these across task profiles, so the reader has a sense of how one particular student's responses varied across all 10 tasks.

The cross-study analyses indicate that arithmetic generalizations about additive relations may develop before those about multiplicative relations, but both types of generalizations may involve similar developmental progressions. The additive domain was associated with an increase in the odds of producing an advanced generalization; children relied on substitution and specific examples more with the multiplication and division tasks than they did with the addition and subtraction tasks. However, within each study, the findings were parallel. The character of students' thinking and the difficulty of the tasks in relation to one another were both similar.

The findings from this study provide further insight into a developmental model about children's construction of arithmetic generalizations. First, the four levels of generality represent different ways students approach these types of tasks and help us understand qualitative shifts in their thinking. Second, as I hypothesized in Chapter 1 and as I found in the additive study as

well, these findings indicate that a child's capability to produce arithmetic generalizations across the different tasks may develop at different rates. Specifically, students' production of advanced generalizations was affected by both generalization type and domain type, and I argue that as children were asked to coordinate more relations (e.g., two operations and two operands in relation to an unknown number), the likelihood of producing an advanced generalization decreased.

## Chapter 5: Discussion

The findings from this dissertation provide insight into how students reason about arithmetic generalizations, contributing to the mathematics education research on the development of children's mathematical thinking. In this chapter, I review the key findings from the dissertation, describe the limitations of the study, and discuss the implications and next steps for this line of research.

### Summary and Discussion of Findings

This dissertation reports on two interview studies that were designed in parallel; *Study 1: Additive Relations* examined fourth graders' understandings of arithmetic generalizations that involved addition and subtraction, while *Study 2: Multiplicative Relations* focused on arithmetic generalizations about multiplication and division. To examine how children reason with arithmetic generalizations and how their thinking develops, I discuss two strands of analyses.

First, I examined the character of students' understandings and considered variations in the generality with which students treat the arithmetic operations. I provide an overview of four levels of student thinking that I observed, and these levels were the basis of a coding scheme that I used to code and analyze students' solutions for each task. Students demonstrated robust understandings of the arithmetic generalizations, with primarily Level 3 and Level 4 responses for all the additive tasks and some of the multiplicative tasks. However, students' thinking often varied across the different types of arithmetic generalizations and students demonstrated a range in the level of thinking demonstrated.

Second, I compared students' generalizations across the two studies and within the different types of tasks, *direction of change*, *identity*, and *relationship between operations*. Students' production of advanced generalizations was affected by both generalization type and domain type. The additive tasks elicited more advanced generalizations (Level 3 and Level 4 responses) from students than the multiplicative tasks, indicating that the arithmetic generalizations involving addition and subtraction may be more intuitive than those about multiplication and division. Furthermore, in both studies, students responded with the highest levels of generality for the *identity* tasks and demonstrated less developed understandings of the *relationship between operations* generalizations. Thus, although the multiplicative tasks may have been more challenging for students, children demonstrated the same types of thinking in both studies and the difficulty of the tasks in relation to one another was similar across the additive and multiplicative domains.

As discussed in Chapter 1, this dissertation draws on Piaget's work on generalizations with physical quantities and my findings align with some of his conclusions. First, he argued that children move from relying on empirical observation to 'logical necessity', and my findings suggest a similar qualitative shift in children's thinking from Level 1 to Level 4 responses. At Levels 1 and 2, students rely on specific examples and substitution, while at Levels 3 and 4, students' judgments involve logical necessity. Second, Piaget argued that judgments that invoke logical necessity often involve the coordination of relations, and my findings indicate that the coordination involved in the *relationship between operations* tasks may present more challenges for students than the coordination involved in other tasks. The *relationship between operations* tasks require students to coordinate their understandings of two operations and two operands in relation to an unknown number, and these tasks were the most challenging for students and

elicited the least Level 4 responses in both studies. On the other hand, the *identity* tasks require students to coordinate their understandings of one operation and one operand in relation to an unknown number, and these tasks elicited the most Level 4 responses. Similarly, the multiplicative tasks were more challenging than the additive tasks, and the multiplicative tasks require the coordination of composite units. Recall from Chapter 1 that addition and subtraction involve composite units, while multiplication and division involve the coordination of composite units (e.g., 2 markers + 2 markers = 4 markers and the composite unit is 1 marker for the operands and sum; 4 markers ÷ 2 children = 2 markers per child and the composite units for the operands are 1 marker and 1 child, while the composite unit for the quotient is markers per child). Thus, the developmental rate of arithmetic generalizations across domains and generalization types may differ, and this may be related to the types of coordination students need to make. Third, Piaget's work provides insight into the important distinctions between additive and multiplicative reasoning, and my findings indicate that student reasoning does in fact vary significantly across the additive and multiplicative domains for arithmetic generalizations, though both types of reasoning likely develop gradually and with parallel progressions. Thus, my dissertation extends some of Piaget's work to the sign form, focusing on arithmetic generalizations, and my findings are similar.

## Limitations

This dissertation makes an important contribution to the research on arithmetic generalizations; it extends prior work by conducting a more fine-grained analysis of student thinking to systematically examine and compare students' understandings across a range of generalizations. It is important, though, to note the limitations of this dissertation.

First, the research participants and scope of this dissertation limit the claims that can be made from the findings. Forty-four of the total 48 participants had STAR math scores at the 'proficient' or 'advanced' proficiency levels. This study therefore lacked a diverse student population with respect to math proficiency. Similarly, the scope of this dissertation was limited to only a select number of arithmetic generalizations and there are many more that can and should be explored (e.g., commutativity) in future research. Thus, a larger study that examines additional arithmetic generalizations and includes students with a range in math proficiency is necessary to generalize the findings from this dissertation to the student population as a whole.

Second, in some instances, the probe question, "Will this be true no matter what number I was thinking about in the beginning?" may have led students down a particular path of thinking that they may have not otherwise explored. I included the question to understand if a child was referring to a particular instance or making a generalization, and in most cases I believe the question helped clarify this. It is difficult to say, though, when this question was used for clarifying purposes and when it may have guided student thinking, and this could be a threat to validity.

Third, there may have been a ceiling effect for some of the tasks. A ceiling effect first raises the question of whether children in earlier grades would demonstrate competency on these tasks and whether fourth graders were the appropriate participants for the two studies. Additionally, a ceiling effect makes it difficult to identify differences in students' thinking across particular variables. Both the *identity* tasks in the additive study, for example, prompted only Level 3 and Level 4 responses from students. As a result, it was difficult to identify differences in students' thinking between the two different operations, addition and subtraction.

## Conclusion and Implications

Arithmetic generalizations have been highlighted as a key entry point for ‘early algebra’ to support the development of algebraic thinking in the elementary grades, and my research advances our understandings of how students reason about arithmetic generalizations. The findings from this work can inform both curricula and teachers’ knowledge for teaching.

First, this dissertation provides tasks that are suitable for classroom instruction. Teachers can use these tasks to both encourage students to reason about arithmetic generalizations and to evaluate and understand their students’ knowledge of generalizations. Teachers can also use these tasks alongside some of the early algebra tasks I discussed in Chapter 1, to strengthen students’ understandings and improve student performance. Jacobs et al. (2007), for example, highlight the use of true or false number sentences to encourage students to discuss arithmetic generalizations. To explore students’ understandings of the identity principle, for instance, they include the following task: ‘True or false:  $78 - 49 = 78$ ’. The *identity* tasks from this dissertation can help build students’ knowledge of the identity principle further by extending their understanding to generalized numbers, rather than specific values like 78. Thus, teachers can use the tasks from this dissertation independently or alongside other early algebra tasks to strengthen students’ understandings of arithmetic generalizations.

Second, the tasks from this dissertation can be used as a model for other arithmetic generalizations that were not included in these studies. The tasks and probe questions were successful in engaging students in a conversation about the arithmetic generalizations. Furthermore, the wording and structure of the tasks were accessible to children and can easily be adapted for other arithmetic generalizations. I focused on a limited number of arithmetic generalizations in this dissertation, but teachers can modify these tasks to have productive conversations with their students about many other arithmetic generalizations.

Third, instruction should build on students’ understandings, and these findings suggest that there may be a particular progression that could guide and scaffold instruction. Teachers could, for example, try to build on students’ understandings of properties of identity to develop their understandings of other types of arithmetic generalizations. This can be done with the tasks from this dissertation as well as tasks from other early algebra work. For instance, the task ‘True or false:  $78 - 49 = 78$ ’ may support students’ understandings of a task that involves inverse operations, like ‘True or false:  $78 - 49 + 48 = 78$ ’.

Finally, the levels I have identified can serve as entry points for teachers as they try to make sense of how to guide instruction for elementary students. Like Carpenter, Fennema, and Franke (1996) articulated in their work in the Cognitively Guided Instruction (CGI) curriculum, it is important to help teachers understand children’s thinking and its development in well defined domains (e.g., addition and subtraction). I see this dissertation as a first step in doing just that with arithmetic generalizations.

This dissertation explored how children reason about arithmetic generalizations involving addition, subtraction, multiplication, and division, and it is a first step a broader research agenda aimed at fostering the development of algebraic thinking in the elementary grades. The findings from these interview studies provide insight into children’s construction of arithmetic generalizations across domains and the challenges generalizations present to children. However, further research must be done to learn more about student thinking and how best to integrate arithmetic generalizations into the elementary mathematics classroom. Next steps of my research program will likely include the following: (a) interview studies that examine the same arithmetic

generalizations, but with different types of tasks; (b) interview studies focused on additional types of arithmetic generalizations; and (c) exploratory tutorial studies that examine instructional approaches to support an understanding of arithmetic generalizations.

Though there has been an increasing focus on early algebra in mathematics education research, arithmetic generalizations remain understudied and additional contributions to the literature need to be made to move forward. To date, the majority of the research on arithmetic generalizations has been primarily descriptive in nature, reviewing classroom episodes that highlight instructional contexts that support students' engagement with arithmetic generalizations. My dissertation is an important next step in scholarly research because it makes use of clinical interviews to gain insight into the processes whereby students construct arithmetic generalizations. It is important that future empirical work continues to include interviews and fine-grained analyses of students' thinking to inform our understandings of the processes of cognitive development and to provide detailed knowledge of how children reason about arithmetic generalizations.

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**Appendix: Fixed Response Assessment from Spring 2010 Pilot Data**

Name \_\_\_\_\_ Today's Date \_\_\_\_\_

School \_\_\_\_\_ Date of Birth (month/day/year) \_\_\_\_\_

What grade are you in? \_\_\_\_\_ Gender (circle one): Male Female

**Directions:** Read each question below and think about it carefully (you may use the scrap paper to help you if you like). For each question, there are four examples of students' answers and explanations. Read *all* of the choices and circle the one that most matches how you first thought about the problem. Please be sure to circle *only one* answer. And remember, I want to know about *your* thinking about the problem, not what you think the right answer is. There is no right or wrong answer!!

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**1) I am thinking about a number. If I subtract 0 from that number, what will happen to my number?**

- A. "I need to try it with a number. Let's say the number is fifteen. Fifteen minus zero is fifteen. The number will just stay the same."
- B. "I need to know what the number is to figure it out."
- C. "Subtraction makes numbers smaller and whenever you subtract something from a number, your number will get smaller. So, the number you are thinking about will get smaller."
- D. "The number can't get bigger or smaller, so it must stay the same. If you subtract 0, it's like subtracting nothing."

**2) Your teacher and I are both thinking about the same number. It's bigger than 0. I multiply the number by 6. Your teacher multiplies the number by 8. Who will get a larger number after multiplying? Me or your teacher?**

- A. "Well let's say the number is five. Five times six is thirty, and five times eight is forty. Forty is bigger than thirty, so my teacher gets the larger number."
- B. "Multiplication makes numbers larger, so both of your numbers will become larger. But it's hard to say whose number will be bigger."
- C. "My teacher will get a larger number because she is multiplying by a larger number than you."
- D. "Depending on what the number is, either you or my teacher can have the larger number."

**3) I am thinking about a number. If I add 7 to that number first and then subtract 9, what will happen to my number?**

- A. "I think the number will decrease because you are subtracting more than you are adding, and subtraction makes numbers smaller."
- B. "Let's try a number. Ten plus seven is seventeen, and seventeen minus nine is eight. I started at ten and ended eight. So the number must decrease."
- C. "You need to know what the number that you're adding to and subtracting from is to figure it out."
- D. "The last thing you do is subtract and subtraction makes numbers smaller. So the number must decrease."

**4) I am thinking about a number. If I subtract 7 from that number first and then add 5, what will happen to my number?**

- A. "I can't tell you without knowing what the number is."
- B. "I think I need to try a number and will choose ten. Ten minus seven is three, and three plus five is eight. So the number must get smaller."
- C. "The number must get smaller because you are subtracting more than you are adding, and subtraction makes numbers smaller."
- D. "Adding makes numbers larger. Since adding is the last step, your number must get larger."

**5) I am thinking about a number. If I divide that number by 3 and then multiply by 3, what will happen to my number?**

- A. "Multiplication makes numbers bigger and multiplication is the last thing you do. So the number will get bigger."
- B. "It depends on what the number you are thinking about is."
- C. "I'll have to try it with a number. Nine divided by three is three. And three multiplied by three is nine. It looks like the number stays the same."
- D. "The number will stay the same because you are multiplying by the same number you are dividing by and multiplication and division are like opposites."

**6) I am thinking about a number. It's bigger than 0. If I divide that number by 5 first and then multiply by 4, what will happen to my number?**

- A. "It will get bigger or smaller, but I'm not quite sure because I don't know what the number is."
- B. "You have to look at the last thing you do, which is multiplication here. Since multiplication makes numbers larger, the number will get larger."
- C. "Let's use the number ten. Ten divided by five is two. Two times four is eight. Eight is smaller than ten, so the number gets smaller."
- D. "I think the number will decrease because you are dividing by more than you are multiplying by, and division makes number smaller."

**7) I am thinking about a number and I multiply that number by 2. Your teacher is thinking about a *different* number and multiplies that number by 10. Who will get a larger number after we multiply? Me or your teacher?**

- A. "My teacher but I'm not really sure why."
- B. "Let's say your number is five. Five times two is ten. And let's say my teacher is thinking about the number six. Six times ten is sixty. My teacher will get a larger number because sixty is bigger than ten."
- C. "My teacher will get the larger number because she is multiplying by a larger number."
- D. "It all depends on what numbers you are each thinking about. Since you are both starting with different numbers, I just don't have enough information to figure it out."

**8) I am thinking about a number. If I multiply that number by 4 and then divide by 4, what will happen to my number?**

- A. "I can't tell you unless you tell me what the number is."
- B. "The number will stay the same because you are multiplying by the same number that you are dividing by, and multiplication and division do the opposite to numbers."
- C. "You are dividing at the end and division makes numbers smaller. So your number will get smaller."
- D. "I'll try it with the number two. Well two times four is eight, and eight divided by four is two. So the number must stay the same."

**9) I am thinking about a number. If I multiply that number by 1, what will happen to my number?**

- A. "The number will stay the same because whenever you multiply a number by one, you will get the same number."
- B. "The number will get bigger because multiplication makes numbers larger."
- C. "I don't know. I think I need more information."
- D. "I'll try it with five. Five times one is five, so the number must stay the same."

**10) I am thinking about a number. It's bigger than 0. If I divide that number by 3 first and then multiply by 4, what will happen to my number?**

- A. "For some numbers, it will get bigger. For other numbers, it will get smaller. So the number will get bigger or smaller. It all depends on the number you start with."
- B. "I think the number will increase because you are multiplying by more than you are dividing by, and multiplication makes number bigger."

C. “The last thing you do is multiply and multiplication makes numbers larger. So the number must get larger.”

D. “I’m going to try it with twelve. Twelve divided by three is four, and four times four is sixteen. So the number must increase.”

**11) I am thinking about a number. If I divide that number by 1, what will happen to my number?**

A. “I need to try it with a number, like ten. Ten divided by one is ten. The number stays the same.”

B. “I need to know what the number you’re thinking about is to answer.”

C. “Division always makes numbers smaller, so the number will get smaller.”

D. “The number will stay the same because whenever you divide by one, your number remains the same.”

**12) Your teacher and I are both thinking about the same number. It’s bigger than 0. I add 6 to my number. Your teacher adds 8 to her number. Who will get a larger number after adding? Me or your teacher?**

A. “Addition makes numbers larger, so both of your numbers will become larger. But you can’t really say whose number will be bigger.”

B. “Depending on what the number is, either you or my teacher can have the larger number.”

C. “Well let’s say the number is five. Five plus six is eleven, and five plus eight is thirteen. Thirteen is bigger than eleven, so my teacher gets the larger number.”

D. “My teacher will get a larger number because she is adding a larger number than you.”

**13) Your teacher and I are both thinking about the same number. It’s bigger than 0. I divide the number by 5. Your teacher divides the number by 7. Who will get a larger number after dividing? Me or your teacher?**

A. “I think division makes numbers smaller, so both of your numbers will get smaller. But you can’t really say whose number will be bigger.”

B. “I’ll pretend you’re thinking about the number thirty-five. Thirty-five divided by five is seven. Thirty-five divided by seven is five. You got seven and my teacher got five. So you must get a larger number.”

C. “You will get a larger number because you are dividing by a smaller number.”

D. “My teacher will because seven is bigger than five.”

**14) I am thinking about a number. It’s bigger than 0. If I multiply that number by 4 first and then divide by 3, what will happen to my number?**

- A. "It could get larger, but it could also get smaller if you are thinking about other numbers."
- B. "Three times four is twelve, and twelve divided by three is four. So I think the number increases."
- C. "Division makes numbers smaller. Since you're dividing last, your number will get smaller."
- D. "I think the number will increase because you are multiplying by more than you are dividing by, and multiplication makes numbers bigger."

**15) I am thinking about a number. If I subtract 5 from that number first and then add 7, what will happen to my number?**

- A. "I think the number will increase because you are adding more than you are subtracting, and addition makes numbers larger."
- B. "I think I need to try a number and will choose eight. Eight minus five is three, and three plus seven is ten. So the number must increase."
- C. "You need to know what the number is to answer that question."
- D. "Adding makes numbers larger. Since adding is the last step, your number must get larger. It's important to look at the last step."

**16) I am thinking about a number. If I subtract 7 from that number and then add 7, what will happen to my number?**

- A. "Your number will get larger because the last thing you do is add. Adding makes numbers larger."
- B. "The number will stay the same because you are adding the same number that you are subtracting, and addition and subtraction undo each other."
- C. "Let me try it with a number. Well eight minus seven is one, and one plus seven is eight. So the number must stay the same."
- D. "There is no way to know exactly. I need to know what the number is. Different things could happen."

**17) I am thinking about a number and I divide that number by 3. Your teacher is thinking about a *different* number and divides that number by 8. Who will get a larger number after we divide? Me or your teacher?**

- A. "You will get the larger number and my teacher will get the smaller number. The smaller the number you divide by, the larger the number you get."
- B. "Well if you are thinking about the number thirty, you will get ten because thirty divided by three is ten. And if my teacher is thinking about the number thirty-two, she will get a four because thirty-two divided by eight is four. So you must get the larger number and my teacher must get the smaller number because ten is bigger than four."
- C. "My teacher does because eight is bigger than three."

D. "It all depends on what numbers you are each thinking about. Since you are both starting with different numbers, I just don't have enough information to figure it out."

**18) I am thinking about a number. If I add 8 to that number first and then subtract 6, what will happen to my number?**

A. "It depends on what number you are adding eight to at the beginning."

B. "The number will get larger. Addition makes numbers larger and you are adding more than you are subtracting."

C. "Subtracting makes numbers smaller. The last thing you do is subtract, so the number must get smaller."

D. "Let me try doing it with a number. I'll try two. Two plus eight is ten, and ten minus six is four. So I think the number will get larger."

**19) I am thinking about a number. It's not 0. If I divide that number by itself, what will happen to my number?**

A. "Nobody knows because we don't know the number."

B. "Division makes numbers smaller, so your number will get smaller."

C. "I need to use a number. I'll use four. Four divided by four is one, so I think you will get one."

D. "You will get one because whenever you divide a number by itself, you will get one. Any number goes into itself exactly one time."

**20) I am thinking about a number. It's bigger than 0. If I multiply that number by 4 first and then divide by 5, what will happen to my number?**

A. "You are dividing last and division makes numbers smaller. So your number will get smaller."

B. "Say the number is ten. Ten times four is forty, and forty divided by five is eight. So I think the number will get smaller because eight is smaller than ten."

C. "I think the number will decrease because you are dividing by more than you are multiplying by, and division makes numbers smaller."

D. "Nobody can tell unless they have more information about the number."

**21) Your teacher and I are both thinking about the same number. It's bigger than 0. I subtract 5 from my number. Your teacher subtracts 7 from her number. Who will get a larger number after subtracting? Me or your teacher?**

A. "It all depends on what the number is, but for most numbers, my teacher will get the larger number because seven is bigger than five."

- B. "My teacher will get a smaller number because she is subtracting a larger number than you."
- C. "Well let's say the number is ten. Ten minus five is five, and ten minus seven is three. Five is bigger than three, so I think you will have the larger number and my teacher will have the smaller number."
- D. "Subtraction makes numbers smaller, so both of your numbers will become smaller. But it's hard to say whose will number will be smaller or larger at the end"

**22) I am thinking about a number. If I add 5 to that number and then subtract 5, what will happen to my number?**

- A. "The number will stay the same because you are subtracting the same number that you are adding, and addition and subtraction do the opposite things to numbers."
- B. "I'll use a number to figure it out. I'll use four. Four plus five is nine, and nine minus five is four. So the number must stay the same."
- C. "The last thing you do is subtract and subtracting always makes numbers smaller. So your number will get smaller."
- D. "I don't know because I don't know what the number is."