Title
Bending of a Cracked, Reinforced Viscoelastic Beam

Authors
Sackman, Jerome
Nickell, Robert

Publication Date
1966-12-01

Division

STRUCTURES AND MATERIALS RESEARCH
DEPARTMENT OF CIVIL ENGINEERING

REPORT NO. 66-23

BENDING OF A CRACKED, REINFORCED VISCOELASTIC BEAM

by

J. L. SACKMAN

and

R. E. NICKELL

Interim Technical Report
U.S. Army Research Office (Durham)
Project No. 4547–E

DECEMBER, 1966

STRUCTURAL ENGINEERING LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY CALIFORNIA

Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering
and Structural Mechanics

Report Number 66-23

BENDING OF A CRACKED, REINFORCED VISCOELASTIC BEAM

CORRIGENDA

1. In the fourth line from the bottom on page 10, "t_o" should be "t_i."
2. In equation (9h) on page 15, "$\sigma^*(28,28)$" should be "$\sigma_c^*(28,28)$."
3. In equation (11c) on page 16, "$\varepsilon^*(28,28)$" should be "$\varepsilon_c^*(28,28)$."
4. In equation (11d) on page 16, "$\varepsilon^*(t,\tau)$" should be "$\varepsilon_c^*(t,\tau)$."
5. In the third line on page 17, "σ^*" and "ε^*" should be "σ_c^*" and "ε_c^*," respectively.
6. In equation (15) on page 18, "$\sigma^*(t,\tau)$" should be "$\sigma_c^*(t,\tau)$" in both integrals.
7. In line 9 and in equation (16) on page 20, "$\varepsilon^*(t,t_o)$" should be "$\varepsilon_c^*(t,t_o)$."
8. In equation (17) on page 20, "$\varepsilon^*(t,t)$" and "$\varepsilon^*(t,\tau)$" should be "$\varepsilon_c^*(t,t)$" and "$\varepsilon_c^*(t,\tau)$," respectively.
9. In the second line on page 21, "$\varepsilon^*(t,\tau)$" should be "$\varepsilon_c^*(t,\tau)$."
10. In equation (18) on page 21, "$\sigma_c^{(n)}(\ )$" should be "$\sigma_c^{(n)}(t)$."
11. In equation (19a) on page 21, "$\varepsilon_c^*(t,\tau)$" should be "$\varepsilon_c^*(t,t_o)$."
12. In equation (19b) on page 21, "$\varepsilon^*(t,\tau)$" and "$\varepsilon_c^*(t,\tau)$" should be "$\varepsilon_c^*(t,\tau)$" and "$\varepsilon_c^*(t,t_o)$," respectively.
13. In equation (20) of page 22, "$\varepsilon^*(t,t)$," "$\varepsilon^*(t,t_o)$," "$\sigma^{(n)}(t)$" and "$\sigma(t)$" should be "$\varepsilon_c^*(t,t)$," "$\varepsilon_c^*(t,t_o)$," "$\sigma_c^{(n)}(t)$" and "$\sigma_c(t)$," respectively.
14. In equation (21) on page 22, "$\sigma^*(\tau,\tau)$" and "$\sigma^*(t,\tau)$" should be "$\sigma_c^*(\tau,\tau)$" and "$\sigma_c^*(t,\tau)$," respectively.

15. In line 7 on page 22, "$\sigma^{(n)}(t)$" and "$\sigma(t)$" should be "$\sigma_c^{(n)}(t)$" and "$\sigma_c(t)$," respectively.

16. In equation (22) on page 22, and in equation (23) on page 23, "$\varepsilon^*(t,t)$" and "$\varepsilon^*(t,t_o)$" should be "$\varepsilon_c^*(t,t)$" and "$\varepsilon_c^*(t,t_o)$," respectively.

17. In equation (24c) on page 24, "$\varepsilon^*(t,t_o)$" should be "$\varepsilon_c^*(t,t_o)$."

18. In equation (25d) on page 25, "$\varepsilon^*(t,t)$" and "$\varepsilon^*(t,t_o)$" should be "$\varepsilon_c^*(t,t)$" and "$\varepsilon_c^*(t,t_o)$," respectively.

19. In equation (35b) on page 29, "$\sigma_c^*(\infty)$" should be "$\sigma_c^*(t)$."

20. In the sixth line from the bottom on page 30, "$\varepsilon^*(t,\tau)$" should be "$\varepsilon_c^*(t,\tau)$."

21. In equation (40c) on page 32, "$\varepsilon^*(t_o,t_o)$" should be "$\varepsilon_c^*(t_o,t_o)$."

22. In the eighth line from the bottom on page 32, "$\varepsilon^*(t_o,t_o)$" should be "$\varepsilon_c^*(t_o,t_o)$."

23. In equation (41) of page 32, and in the first line on page 33, "$\varepsilon^*(t,\tau)$" should be "$\varepsilon_c^*(t,\tau)$."

Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering
and Structural Mechanics

Report Number 66-23

BENDING OF A CRACKED, REINFORCED VISCOELASTIC BEAM

by

J. L. Sackman
Professor of Civil Engineering
University of California at Berkeley

and

R. E. Nickell
Graduate Student in Civil Engineering
University of California at Berkeley

Contract Number DA-31-124-ARO-D-460
DA Project No: 20014501B33G
ARO Project No: 4547-E

Structural Engineering Laboratory
University of California
Berkeley, California
December 1966

SUMMARY

Conventional cracked, reinforced concrete beam theory is extended to include general linear viscoelastic behavior of the matrix (concrete) material in compression. An analytical formulation is developed which leads to a single governing equation on the position of the neutral axis. Although it has not been possible to obtain a general closed solution, some important properties of the solution of this nonlinear, Volterra-type integral equation are established. The relationship of this theory to a popular approximate procedure, known as the effective modulus method, is discussed. The nonlinear integral equation is resolved numerically, and results are shown for several cases of interest.

INTRODUCTION

It has been known for many years that even under constant load there is, as time progresses, a change in the stress distribution in reinforced beams composed of materials which exhibit creep. The analysis of this problem is interesting -- and of some technical importance, as reinforced concrete beams are known to behave in this manner [1, 2].

If it is assumed that the material of the beam behaves as a linear viscoelastic solid (with the response in tension the same as the response in compression), and that the reinforcement behaves as a linear elastic (or even linear viscoelastic) material, then the formulation of the problem of beam bending is quite straightforward. A fairly complete analysis of the one-dimensional version of this problem is given in the excellent work of Arutiunian [3]. Some interesting properties of the solution are that, in general, the neutral axis of strain does not coincide with the neutral axis of stress. Further, the positions of these neutral axes, even when the beam is subjected to constant load, change with time.

Such considerations raise no basic difficulties in the analysis of the uncracked beam. But should the matrix material of the beam crack in tension, then basic complexities are introduced. Here, we study the case of a cracked beam, and present a formulation based on suitable extensions of what may be termed "conventional" cracked, reinforced beam theory.

It appears to us that the first attempt to analyze the creep of a cracked, reinforced beam was made in the admirable paper by Faber [1]. There, Faber presented a very simple analysis of reinforced concrete beams based on his introduction of an ad hoc hypothesis now called the "effective" or "reduced" modulus method [2, 4]. The notion of viscoelasticity (or hereditary elasticity, as Volterra's work [5, 6] was termed by Picard [7]) was not well known then. Neither was it known that, within the working stress range, the response of concrete in compression could be approximated quite well by the assumption of linear viscoelastic behavior. Thus Faber could hardly have been expected to supply, at that time, a more sophisticated analysis based on viscoelasticity theory. But the method that Faber introduced is interesting, and still remains of practical importance.

The next analysis concerned with the creep of a cracked, reinforced beam appears to be that reported in the commendable work of Glanville and Thomas [2]. Again the material of interest was reinforced concrete. Not only were the numerical results of an analysis (based on a constant moment history) presented, but they were compared to some experimental results. Two types of analyses were considered: one based on the effective modulus method, and one on a procedure (later [4]) called the "rate of creep" method. The introduction of the rate of creep method (apparently first given by Glanville [8]) was an attempt to bring into the theory a more realistic rate-dependent stress-strain law (for the concrete in compression) than is implied by the effective modulus method. The rate of creep method leads to a rather peculiar stress-strain relationship, which is directly related to the usual hereditary stress-strain relation of linear viscoelasticity only under special circumstances. (This has been commented upon in [9].)

Although the authors presented the numerical results for the creep of a cracked, reinforced beam based on the rate of creep method, they gave no details of that analysis. Their reason for this was that the rate of creep method required a much more complex analysis than the effective modulus method, but the results obtained were not significantly closer to the experimental results than those of the simpler method. It is a pity that the details of this analysis were not given, for we suspect that a proper formulation for that theory might be similar in certain fundamental aspects to the formulation given here. However, the authors did point out that under certain conditions the neutral axes of stress and strain need not coincide and that they shift, with time, towards the (tensile) reinforcement. Faber [1] had previously pointed out this second phenomenon.

In an article by Torroja and Paez [10], we find reference made to the creep of a cracked, reinforced concrete beam. To describe the creep property of the concrete in compression, they depict a complicated rheological model. No details (or even hints) are given as to the procedure of analysis, but qualitative curves illustrating the concrete compressive stress, at various times, through the depth of a beam subjected to constant load are presented. These curves are similar in form to those which would be obtained either by the effective modulus method, or the method to be presented here.

To our knowledge these works constitute the only analytical treatments of this problem available in the literature. We are aware of some related current research by Distéfano and Guinea [11], and by Bresler and Selna [12], but we have not yet been able to secure published reports of those investigations.

In this paper, the analysis presented is based on the usual hypotheses of Bernoulli-Euler beam theory. We also adopt the hypothesis of conventional reinforced concrete beam theory: the material of the beam can sustain compression only, it "cracks" in tension. We further assume, as is conventional, that the reinforcement utilized to sustain tension is linearly elastic and does not slip relative to the immediately surrounding material of the beam. The major difference between our theory and the conventional theory of reinforced concrete beams is that we assume, instead of linearly elastic behavior, linearly viscoelastic behavior of the material of the beam in compression. This viscoelastic behavior is taken to be general - that is, the viscoelastic material is allowed to age with time -- a phenomenon actually exhibited by concrete.

We consider only those cases where the neutral axes monotonically progress, with time, towards the (tensile) reinforcement. Coupled with our assumption of a zero tensile "cracking" stress, this leads to the fact that the neutral axes of strain and of (compressive) stress coincide. This simplifies the formulation of the problem, and it is only that simplified version which we present.

The essential statement of the problem is reduced to a single nonlinear integral equation of the Volterra type. We cannot find a general closed solution of that equation. However, we give some general properties of the solution. Further, we are able to write down some asymptotic results for nonaging materials. Also, we show a relationship between the effective modulus method and our theory.

The governing nonlinear integral equation lends itself well to a numerical analysis. Such numerical results we present for some cases of practical interest -- based on experimentally determined creep properties of concrete.

FORMULATION

The pertinent geometry is shown in Figure 1. We are concerned with a beam of rectangular cross-section having width b, and reinforcement placed at a depth d below the top surface of the beam. The span coordinate is x and the depth coordinate y. Up to some initial time, which is denoted as t_o, the beam is supposed to be in a completely quiescent state. At t_o the beam is subjected to a transverse load which gives rise to a bending moment distribution $M(x,t)$.

We start our analysis by utilizing one of the basic hypotheses of elementary beam theory: plane sections before bending remain plane after bending. This implies that the longitudinal strain, $\varepsilon(x,t)$, is then linear in y:

$$\varepsilon(x,t) = \alpha(x,t) + \frac{y}{d}\,\beta(x,t) \qquad (1)$$

Our major task is the determination of the two fundamental unknown functions of elementary beam theory, α and β. If we neglect the deflection due to shear strain and restrict attention to "small deflection" theory - as is commonly done in elementary theory - then we note that β represents a non-dimensional curvature of the beam.

It is convenient to introduce a function, $k(x,t)$, locating the neutral axis of strain. Clearly, from equation (1), if the neutral axis of strain (where $\varepsilon(x,t) = 0$) is located at a depth $k(x,t)d$, then k is given by

$$k(x,t) = -\,\alpha(x,t)/\beta(x,t) \qquad (2)$$

For our analysis, it turns out that the neutral axis of strain has the property that it separates the zone of compressive stress from the "tensile" zone (i.e., the zone of zero stress) in the matrix material of the beam. This property follows if: (1) attention is restricted to cases where $k(x,t)$ is a monotonically increasing function of t ; (2) it is assumed that the matrix material of the beam cannot sustain <u>any</u> tensile stress (i.e., it "cracks" under tension of any magnitude); (3) conditions are completely quiescent prior to the loading of the beam. Under these restrictions the neutral axis of strain penetrates into a region where the matrix material of the beam has been previously subjected only to zero stress, and therefore the neutral axis of strain and of stress coincide. Such need not be the case if any of the above restrictions are relaxed. (Here we define the neutral axis of stress as that axis in the cross-section of the beam, parallel to the z axis, which separates the zone of compressive stress from the zone of zero stress in the matrix material of the beam.)

We next consider the stress-strain relationships governing the reinforcement and the matrix material. As in the conventional elementary theory for cracked, reinforced concrete beams, it is assumed that the reinforcement behaves as a linearly elastic material. Therefore

$$\sigma_s = E_s \, \varepsilon_s = E_s \, (\alpha + \beta) \qquad (3)$$

where the subscript s refers to the reinforcement, σ represents the normal stress in the x direction and E_s is Young's modulus of the reinforcement. In writing equation $(3)_2$ we have used equation (1) and have tacitly assumed that the strain variation over the reinforcement cross-section is negligible, and that the reinforcement does not slip relative to the matrix material.

At this point we depart from the conventional theory of cracked, reinforced concrete beams by assuming that the compressive behavior of the matrix material of our beam is governed by a general (or aging) linear viscoelastic stress-strain relationship.* If we use the subscript c to refer to the matrix material, then for $t > t_i$, the stress-strain relation for a typical fiber of this material may be written in any of the four forms

$$\varepsilon_c(x,t) = \int_{t_i}^{t} \frac{\partial}{\partial\tau}\sigma_c(x,\tau)\, \varepsilon_c^*(t,\tau)\,d\tau \tag{4a}$$

$$\varepsilon_c(x,t) = -\int_{t_i}^{t} \sigma_c(x,\tau)\frac{\partial}{\partial\tau}\, \varepsilon_c^*(t,\tau)\,d\tau \tag{4b}$$

$$\sigma_c(x,t) = \int_{t_i}^{t} \frac{\partial}{\partial\tau}\varepsilon_c(x,\tau)\, \sigma_c^*(t,\tau)\,d\tau \tag{5a}$$

$$\sigma_c(x,t) = -\int_{t_i}^{t} \varepsilon_c(x,\tau)\frac{\partial}{\partial\tau}\, \sigma_c^*(t,\tau)\,d\tau \tag{5b}$$

Where ε_c and σ_c are the compressive strain and stress, respectively, in the matrix material, and t_i is the time at which the compressive strain or

* On many occasions, such an assumption has been shown to be a decent approximation of the behavior of concrete within working-stress levels. (See reference [9] for a review of the pertinent literature on this point.)

stress is initiated in the fiber, quiescent compressive conditions having been implicitly assumed for $t<t_i$. In the above $\varepsilon_c^*(t,\tau)$ is the specific creep function and $\sigma_c^*(t,\tau)$ the specific relaxation function of the matrix material. Physically, $\varepsilon_c^*(t,\tau)$ represents the strain at time t in our matrix material, occurring in a standard creep test, due to the application of a unit load applied at time τ. (τ is sometimes referred to as the age of the material at the time of loading.) A similar interpretation holds for σ_c^* in a unit relaxation test. Clearly both $\varepsilon_c^*(t,\tau)$ and $\sigma_c^*(t,\tau)$ are identically zero for $t<\tau$.

For convenience of representation, the stress-strain relationship appropriate to each fiber making up the matrix material of the beam may be thought of as being the law governing the mechanical model depicted in Figure 2. The knife edges, A , come apart in tension, and the general (i.e., aging) linear viscoelastic model, B, which is effective in compression, is governed by equation (4) or (5).

It is to be understood in (4) and (5) that

$$\frac{\partial}{\partial \tau}\sigma_c \ , \ \frac{\partial}{\partial \tau}\varepsilon_c^* \ , \ \frac{\partial}{\partial \tau}\varepsilon_c \ , \ \frac{\partial}{\partial \tau}\sigma_c^*$$

are, in general, distributions. Then, in (4a) and (5a) the integration is interpreted to start at a value slightly less than t_o, whereas in (4b) and (5b) it is interpreted to end at a value slightly greater than t. In this way it is seen that these equations contain the "instantaneous elastic" response of the material, without our having to bother to always explicitly write it out. Equations (4a) and (5a) represent the usual step-function formulation of the superposition integral, whereas equations (4b) and (5b) represent the usual impulse (or delta-function) form.

The creep and relaxation functions are, of course, related. If either of these functions is known, then the other may be obtained by solving any of the four integral equations (for $t \geq t_o$)

$$\int_{t_o}^{t} \frac{\partial}{\partial \tau} \sigma_c^*(\tau, t_o) \varepsilon_c^*(t, \tau) d\tau = H(t - t_o) \tag{6a}$$

$$\int_{t_o}^{t} \sigma_c^*(\tau, t_o) \frac{\partial}{\partial \tau} \varepsilon_c^*(t, \tau) d\tau = -H(t - t_o) \tag{6b}$$

$$\int_{t_o}^{t} \frac{\partial}{\partial \tau} \varepsilon_c^*(\tau, t_o) \sigma_c^*(t, \tau) d\tau = H(t - t_o) \tag{6c}$$

$$\int_{t_o}^{t} \varepsilon_c^*(\tau, t_o) \frac{\partial}{\partial \tau} \sigma_c^*(t, \tau) d\tau = -H(t - t_o) \tag{6d}$$

where H is the unit step function. Therefore, we see from equations (6) that ε_c^* and σ_c^* are, essentially, reciprocal kernels.

Up to this point we have incorporated into our theory the standard assumptions of elementary beam theory, the cracking of the matrix material in tension, and the linear viscoelastic behavior of this material in compression. We must now consider the equilibrium conditions; these are stated in the usual form for beam theory by the equations

$$\iint_A \sigma(x,y,t)\,dA = 0 \tag{7a}$$

$$\iint_A \sigma(x,y,t)\,y\,dA = M(x,t) \tag{7b}$$

where A signifies the region of integration, i.e., the beam cross-section. At a typical cross-section there is a moment $M(x,t)$ acting, but there is no resultant axial load. Our procedure now is the ordinary one -- in equations (6) we substitute for the stress its expression in terms of the strain by utilizing suitable stress-strain laws [either equation (3) or (5)]. Then, using equation (1), we substitute for the strain its expression in terms of α and β. In this way we will arrive at the two simultaneous equations governing the fundamental unknowns of our problem.

The substitution of the strain for the stress requires a breaking up of the integration over the cross-sectional area of the beam, since different stress-strain relations hold in different regions of the beam cross-section. This is considered graphically in Figure 3. At the instant t_o when the load is first applied, there is an immediate strain response, as indicated by the line t_o. Above y_o the matrix is in compression, below y_o the matrix has "cracked" and carries no stress; the reinforcement carries the tensile load. As time progresses the compressive stresses in the matrix relax and the neutral axis penetrates into the initially cracked zone, so that the crack partially closes. This brings into compression fibers in the initially cracked zone which had previously been ineffective. These fibers in the initially cracked zone contribute to the compressive stress only from the time they first become compressed until the current time. The time t_1 at which a fiber located at depth y_1 first goes into compression is given by $k^{-1}(y_1/d)$, where k^{-1} is the function inverse to the neutral axis function k.

Indeed, it is the cracking of the matrix and the subsequent necessity of keeping track of the moving boundary of the compressive zone and of the various times at which matrix fibers first start to undergo compression, which makes the problem nonlinear and somewhat complicated.

Recall, as previously explained, for the cases to which we here restrict attention, the neutral axis of strain coincides with that of stress. We use this fact in a vital way, and incorporate it, along with the notions just discussed, into the equilibrium equations (7). This results in the equations

$$-b \int_{o}^{k(t)d} \left\{ \int_{t^*}^{t} \left[\alpha(\tau) + \frac{y}{d} \beta(\tau) \right] \frac{\partial}{\partial \tau} \sigma_c^*(t,\tau) d\tau \right\} dy$$

$$+ A_s E_s [\alpha(t) + \beta(t)] = 0 \tag{8a}$$

$$-b \int_{o}^{k(t)d} \left\{ \int_{t^*}^{t} \left[\alpha(\tau) + \frac{y}{d} \beta(\tau) \right] \frac{\partial}{\partial \tau} \sigma_c^*(t,\tau) \right\} ydy$$

$$+ A_s E_s d [\alpha(t) + \beta(t)] = M(t) \tag{8b}$$

where

$$t^* = k^{-1} (y/d) \tag{8c}$$

and where here and in most of our subsequent work, for convenience, we have suppressed explicit mention of the span coordinate x . In these equations we see the integration over the currently compressed region of the matrix from $y = 0$ to $y = k(t)d$, and the contribution of each fiber in this region, only from the time t^* that the compression first started in the fiber, up to the current time t . The terms outside of the integral signs represent the contributions of the reinforcement.

Thus we see that the governing equations of our problem -- equations (8a) and (8b) -- are two simultaneous, multiple, highly nonlinear, Volterra-type integral equations in α and β .

The iterated integrals as written in equations (8) are carried out by first integrating along a horizontal strip in the t,y plane, as depicted in Figure 4 and then "adding up" all such strips to cover the region of integration shown. We find it convenient to interchange the order of the iterated integral in equations (8), and to first integrate along a vertical strip, as shown in Figure 4, and then to "add up" all such strips to cover the integration region.

The equations resulting from this process are

$$\int_{t_o}^{t} \left\{ \int_{0}^{k(\tau)} \frac{\partial}{\partial\tau} [s^*(t,\tau)] \, [\alpha(\tau)+\xi\beta(\tau)] d\xi \right\} d\tau$$

$$- pn\,[\alpha(t) + \beta(t)] = 0 \qquad (9a)$$

$$\int_{t_o}^{t} \left\{ \int_{o}^{k(\tau)} \frac{\partial}{\partial\tau} [s^*(t,\tau)] \, [\alpha(\tau)+\xi\beta(\tau)]\xi d\xi \right\} d\tau$$

$$- pn \, [\alpha(t) + \beta(t)] = -\tilde{M}(t) \qquad (9b)$$

where

$$\xi = y/d \qquad (9c)$$

$$E_c = \sigma_c^*(28,28) \qquad (9d)$$

$$s^*(t,\tau) = \sigma_c^*(t,\tau)/E_c \qquad (9e)$$

$$p = A_s/bd \qquad (9f)$$

$$n = E_s/E_c \qquad (9g)$$

$$\tilde{M}(t) = M(t)/bd^2\sigma^*(28,28) \qquad (9h)$$

Here we have suitably non-dimensionalized equations (9). Guided by the conventional notation used in the literature on concrete, we have introduced, for convenience, such parameters as the instantaneous elastic modulus of the matrix material at age 28 days (i.e., $E_c=\sigma_c^*(28,28)=\sigma_c^*(t,\tau)$ for $t = \tau =$ 28 days), the ratio of areas of the reinforcement and the matrix (i.e., p), and the ratio of moduli of the reinforcement and the matrix (i.e., n).

We can now carry out the integration on ξ to obtain from (9a) and (9b)

$$\int_{t_o}^{t} \frac{\partial}{\partial\tau} [s^*(t,\tau)] \, [\alpha(\tau)k(\tau) + \frac{1}{2} \beta(\tau) \, k^2(\tau)] \, d\tau$$

$$- pn \, [\alpha(t) + \beta(t)] = 0 \tag{10a}$$

$$\int_{t_o}^{t} \frac{\partial}{\partial\tau} [s^*(t,\tau)] \, [\frac{1}{2} \alpha(\tau)k^2(\tau) + \frac{1}{3} \beta(\tau)k^3(\tau)] \, d\tau$$

$$- pn \, [\alpha(t) + \beta(t)] = - \mathfrak{M}(t) \tag{10b}$$

At this stage it is useful to note that equations (4b) and (5b) may be viewed as an integral transform pair. With this in mind, we may then take the "inverse transform" of equations (10a) and (10b) to obtain

$$\alpha(t)k(t) + \frac{1}{2} \beta(t)k^2(t) - pn \int_{t_o}^{t} [\alpha(\tau)+\beta(\tau)] \frac{\partial}{\partial\tau} e(t,\tau) = 0 \tag{11a}$$

$$\frac{1}{2} \alpha(t)k^2(t) + \frac{1}{3} \beta(t)k^3(t) - pn \int_{t_o}^{t} [\alpha(\tau)+\beta(\tau)] \frac{\partial}{\partial\tau} e(t,\tau) = -m(t) \tag{11b}$$

where

$$\varepsilon^*(28,28) = 1/E_c \tag{11c}$$

$$e(t,\tau) = E_c \, \varepsilon^*(t,\tau) \tag{11d}$$

$$m(t) = \int_{t_o}^{t} \mathfrak{M}(\tau) \frac{\partial}{\partial\tau} e(t,\tau) \, d\tau \tag{11e}$$

In effect, in passing from equations (10a) and (10b) to equations (11a) and (11b), we have merely made use of the reciprocal properties of the kernels σ^* and ε^* .

By suitable manipulation the two governing simultaneous, nonlinear, integral equations (11a) and (11b) can be reduced to a single governing nonlinear integral equation on the neutral axis function $k(t)$. Subtracting equation (11b) from (11a), and using equation (2) to eliminate β , we obtain

$$\alpha(t) = \frac{6m(t)}{k(t)[3-k(t)]} \tag{12}$$

Substituting this result into equation (11a), and again using (2) to eliminate β we finally arrive at the single nonlinear, Volterra-type integral equation governing our problem

$$\frac{1}{2pn}\left[\frac{m(t)}{3-k(t)}\right] + \int_{t_o}^{t}\left[\frac{m(\tau)}{3-k(\tau)}\right]\left[\frac{1-k(\tau)}{k^2(\tau)}\right]\frac{\partial}{\partial\tau}\, e\,(t,\tau)\, d\tau = 0 \tag{13}$$

If we can resolve this equation to obtain $k(t)$, then equation (12) yields $\alpha(t)$, equation (2) $\beta(t)$, and the problem is essentially solved. Having α , β and k , the stress distribution then follows directly. Equation $(3)_2$ yields the stress in the reinforcement, and the stress in the compressed region of the matrix material is obtained from equations (1) and (5b) as

$$\sigma_c(t) = -\int_{t^*}^{t}\left[\alpha(\tau) + \frac{y}{d}\beta(\tau)\right]\frac{\partial}{\partial\tau}\sigma_c^*\,(t,\tau)\, d\tau \tag{14}$$

where t^* is given by equation (8c). Here we assume that having k, we may effect its inverse to obtain t^*. This we can at least accomplish numerically.

It may be noted that for fibers in the initially compressed region at time t_o, t^* is given by t_o. Then for such fibers we have, from (14),

$$\sigma_c(t) = -\int_{t_o}^{t} \alpha(\tau) \frac{\partial}{\partial\tau} \sigma^*(t,\tau)\, d\tau - \frac{y}{d} \int_{t_o}^{t} \beta(\tau) \frac{\partial}{\partial\tau} \sigma^*(t,\tau)\, d\tau \tag{15}$$

for all $t \geq t_o$ and for all y in the initially compressed zone. Thus it is seen that the stress distribution in the initially compressed zone remains for all times linearly distributed with respect to the depth. For those fibers in the initially cracked zone, which come into compression at some time $t^* > t_o$, the dependence of the stress on y is not only through the linear term appearing within the integral in equation (14), but also through the nonlinear term $t^* = k^{-1}(y/d)$ appearing as the lower limit in this integral. Therefore, it is to be expected that the compressive stress distribution, at time $t > t_o$, in the initially cracked zone will be a nonlinear function of the depth y.

PROPERTIES OF THE SOLUTION

Although we have not been able to construct a closed-form solution of equation (13), we can list some interesting and important properties of the solution.

It may be verified by direct substitution in equations (13), (12) and (2) that if $[\alpha_1(x,t), \beta_1(x,t)]$ is the solution to a problem in which the bending moment distribution is given by $M_1(x,t)$, then the solution for a bending moment of $M_2(x,t) = F(x)M_1(x,t)$ (where F is an arbitrary function) is $[\alpha_2(x,t), \beta_2(x,t)]$ with

$$\alpha_2(x,t) = F(x)\alpha_1(x,t) \quad , \quad \beta_2(x,t) = F(x)\beta_1(x,t) \; .$$

In this sense the beam behaves as a linear system.

However, if $[\alpha_1, \beta_1]$ is the solution corresponding to a moment input of M_1, and $[\alpha_2, \beta_2]$ that corresponding to a moment M_2, then the solution $[\alpha_3, \beta_3]$ corresponding to $M_3 = M_1 + M_2$ is not given by $\alpha_3 = \alpha_1 + \alpha_2$, $\beta_3 = \beta_1 + \beta_2$, unless $M_1(x,t)/M_2(x,t) = F(x)$. Therefore, in general, the beam behaves as a nonlinear system. Here we have a case of a physical system in which the multiplicative property common to linear systems holds, but the additive property, in general, does not.

THE EFFECTIVE MODULUS METHOD

There is an approximate method -- commonly called the "effective" or "reduced" modulus method [1,2,4,8] -- popular in analysis and design of concrete structures. We wish to show a relationship between this approximate method and the exact solution to the cracked, reinforced beam problem as formulated here.

In the treatment of concrete structures by the effective modulus method, it is assumed that at any time t the concrete may be considered as an elastic material having a modulus of elasticity given by $1/\varepsilon^*(t,t_o)$, where t_o indicates the time at which loading of the structure was initiated. Thus the entire analysis, based on the effective modulus method, is simply a series of elastic ones. Of course, such an analysis is in general considerably easier to perform than one based on a linear viscoelastic stress-strain relationship.

The essential feature of the effective modulus method is that it approximates the concrete stress-strain relation, say equation (4b), by the relation

$$\varepsilon_c(t) = \sigma_c(t)\varepsilon^*(t,t_o) \tag{16}$$

We may gage the error involved in such an approximation if we suitably rewrite the stress-strain relation (4b). For convenience we rewrite (4b) explicitly extracting the instantaneous elastic response

$$\varepsilon_c(t) = \sigma_c(t)\varepsilon^*(t,t) - \int_{t_o}^{t} \left[\frac{\partial}{\partial\tau} \varepsilon^*(t,\tau) \right] \sigma(\tau) d\tau \tag{17}$$

In the above, it is to be understood that the integration extends from t_o

to a value slightly smaller than t . In this range of integration we assume that $\frac{\partial}{\partial \tau} \varepsilon^*(t,\tau)$ is a continuous, non-positive function of τ . For all real viscoelastic materials with which we are familiar, such is true.

Assuming that $\sigma_c(\tau)$ can be expanded into a Taylor series about t with a radius of convergence at least as large as $t-t_o$, we have

$$\sigma_c(\tau) = \sum_{n=o}^{\infty} \sigma_c^{(n)}(t)\ (\tau-t)^n/n! \tag{18}$$

where the superscript n indicates the n^{th} time derivative of σ_c . Substituting this into equation (17), extracting the first term of the series, and interchanging the order of integration and summation, we obtain

$$\varepsilon_c(t) = \sigma_c(t)\varepsilon_c^*(t,\tau)\ [1 + \text{Error}] \tag{19a}$$

where

$$\text{Error} = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{\sigma_c^{(n)}(t)}{n!} \int_{t_o}^{t} \left[-\frac{\partial}{\partial \tau}\varepsilon^*(t,\tau)\right] [\tau-t]^n \, d\tau}{\sigma_c(t)\ \varepsilon_c^*(t,\tau)} \tag{19b}$$

Clearly, if the absolute value of the term "Error" is small compared to unity, then (19a), the viscoelastic stress-strain relation, reduces to equation (18), the stress-strain relation used in the effective modulus method. Thus the accuracy of the effective modulus method is gauged by the smallness of the term $|\text{Error}|$.

We obtain as a crude bound on the absolute value of "Error"

$$\left| \text{Error} \right| \leq \left[1 - \frac{\varepsilon^*(t,t)}{\varepsilon^*(t,t_o)} \right] \sum_{n=1}^{\infty} \left| \frac{\sigma^{(n)}(t)}{\sigma(t)} \right| \frac{(t-t_o)^n}{n!} \tag{20}$$

A cruder bound, but one in a more convenient form, may be obtained from (20). But first it is convenient to introduce some characteristic time which can be used to nondimensionalize our results. For this purpose we introduce an "average relaxation time" , T , defined by

$$T = \lim_{t\to\infty} \left\{ \frac{1}{t-t_o} \int_{t_o}^{t} \frac{\sigma^*(\tau,\tau)}{\left[\frac{\partial}{\partial\tau} \sigma^*(t,\tau) \right]_{t\to\tau^+}} d\tau \right\} \tag{21}$$

Then, if we define $S_M(t)$ as the maximum value of $\left| T^n \sigma^{(n)}(t)/\sigma(t) \right|$ over n (n=1,2,3,....), we obtain from (20)

$$\left| \text{Error} \right| \leq \left[1 - \frac{\varepsilon^*(t,t)}{\varepsilon^*(t,t_o)} \right] \left[\exp\left(\frac{t-t_o}{T} \right) - 1 \right] S_M(t) \tag{22}$$

Although this expression may be difficult to apply quantitatively, because it might be hard to evaluate $S_M(t)$, it leads to useful qualitative information. This expression shows that, in general, the shorter the time interval (t_o, t) over which the effective modulus method is applied, the smaller will be the error associated with this approximation. Secondly, noting that $S_M(t)$ is a measure -- in some sense -- of the fluctuation of the function σ_c , it is seen that, in general, the smaller the fluctuation of σ_c in the interval (t_o, t) (and hence the smaller $S_M(t)$) the smaller will be the error associated with the effective modulus method. These restrictions on the applicability of the reduced modulus method have been known for some time to analysts of concrete structures, and appear to have been arrived at through a considerable amount of experience gained by comparing predictions based on the reduced modulus method to a variety of experiments and to solutions of problems based on viscoelasticity theory.

A simple, but crude, bound on the absolute value of "Error" - and one which is relatively easy to deal with quantitatively - can be obtained directly from equation (17) as

$$|\,\text{Error}\,| \leq \left[1 - \frac{\varepsilon^*(t,t)}{\varepsilon^*(t,t_o)} \right] B(t) \tag{23}$$

where $B(t)$ is defined to be the maximum over τ of $|\,1 - \sigma_c(\tau)/\sigma_c(t)\,|$, with $t_o \leq \tau \leq t$. (It is noted that (23) also predicts a smaller error the smaller the time interval over which the approximation is applied, and the smaller the fluctuation of σ_c over that time interval.) If σ_c is a known function then it is simple to evaluate the bound (23).

In some problems (such as the one we have formulated here) σ_c is not directly known, but is a portion of the solution. Then $B(t)$ can not be directly evaluated unless the solution has been obtained. However, using only the effective modulus method, one may obtain an idea of the applicability of that method to the particular problem being considered. Compute $B(t)$ from the values of σ_c obtained by use of the effective modulus method. Use this in (23) to compute $|\,\text{Error}\,|$. If $|\,\text{Error}\,| << 1$, then the effective modulus solution should closely approximate the viscoelastic solution.

Of particular interest in this investigation is the calculation of the neutral axis function $k(t)$. For our formulation, this requires a resolution of the nonlinear integral equation (13). However, an approximate solution may be obtained by use of the effective modulus method. If we apply this method to the determination of $k(t)$, then the analysis parallels that which occurs in conventional (elastic) cracked, reinforced concrete beam theory. The relationship determining $k(t)$ becomes the simple algebraic equation

$$k^2(t) + 2pn_e(t)[k(t)-1] = 0 \tag{24a}$$

where

$$n_e(t) = E_s/E_e(t) \tag{24b}$$

$$E_e(t) = 1/\epsilon^*(t,t_o) \tag{24c}$$

To obtain an indication of the accuracy of the effective modulus approximation (24a), we may turn to equation (13) governing the viscoelastic formulation, and operate on it in a manner similar to that above. If we do so, then equation (13) may be written as

$$g(t)-f(t)e(t,t_o)\ [1 + \text{Error}] = 0 \tag{25a}$$

where

$$g(t) = \frac{1}{2pn}\left[\frac{m(t)}{3-k(t)}\right] \tag{25b}$$

$$f(t) = \left[\frac{m(t)}{3-k(t)} \right] \left[\frac{1-k(t)}{k^2(t)} \right] \tag{25c}$$

$$\left| \text{Error} \right| \leq \left[1 - \frac{\epsilon^*(t,t)}{\epsilon^*(t,t_o)} \right] C(t) \tag{25d}$$

and $C(t)$ is defined as the maximum value over τ of $\left| 1 - f(\tau)/f(t) \right|$, with $t_o \leq \tau \leq t$. If $\left| \text{Error} \right| << 1$, we can neglect this term compared to 1, and then equation (25a) reduces to (24a). Thus we can use the bound given by equation (25d) to gauge the applicability of the effective modulus method to the calculation of $k(t)$. Similar to our preceding discussion, this bound indicates that, in general, the less fluctuation in the applied moment and the shorter the interval of time over which we utilize the effective modulus method, the better will be this approximation. In particular we see that the reduced modulus method gives the exact answer for $t = t_o^+$, (i.e., immediately after loading).

The difficulty in quantitatively applying (25d) is that we must know the history of the neutral axis function, k, governed by the viscoelastic formulation. However, in a manner similar to that previously described, we may use the easily obtained effective modulus solution to give an indication of the accuracy of that approximate method. In (25c) and (25d) use the values of k predicted by the effective modulus method. If this yields a bound for $\left| \text{Error} \right|$ in equation (25d) which is negligibly small compared to unity, then the effective modulus method should be a good approximation. In a subsequent section we will apply this procedure, and compare its predictions to the results of a numerical solution of equation (13).

FINAL VALUES FOR NONAGING MATERIALS

It is of interest to consider the special case in which the matrix is a nonaging viscoelastic material. Then the creep and stress relaxation functions, $\epsilon_c^*(t,\tau)$ and $\sigma_c^*(t,\tau)$, take on the special forms $\epsilon_c^*(t-\tau)$ and $\sigma_c^*(t-\tau)$, respectively. Thus for the nonaging material the creep and relaxation functions are functions of a single variable, rather than of two variables as is the case for aging materials.

For the nonaging matrix material the stress-strain relation (4b) takes the form

$$\epsilon_c(t) = \int_{t_i}^{t} \mathring{\epsilon}_c^*(t-\tau)\ \sigma_c(\tau)\ d\tau \tag{26}$$

where $\mathring{\epsilon}_c^*(t)$ is the derivative of $\epsilon_c^*(t)$. The integral equation (13) governing $k(t)$ then becomes

$$g(t) - \int_{o}^{t} \mathring{e}(t-\tau) f(\tau) d\tau = 0 \tag{27a}$$

where

$$\mathring{e}(t) = \mathring{\epsilon}_c^*(t)/\epsilon_c^*(o) \tag{27b}$$

and $g(t)$ and $f(t)$ are defined by equations (25b) and (25c), respectively, with

$$m(t) = -\int_{o}^{t} \widetilde{M}(\tau)\mathring{e}(t-\tau)\ d\tau \tag{27c}$$

In the above, without any loss of generality, we have conveniently set $t_o = 0$.

Let us restrict attention to a matrix material exhibiting bounded creep, and to a moment input which asymptotically attains a final value, i.e.

$$\lim_{t\to\infty} \varepsilon_c^*(t) = \varepsilon_{c\infty}^* < \infty \tag{28a}$$

$$\lim_{t\to\infty} \tilde{M}(t) = \tilde{M}_\infty < \infty \tag{28b}$$

Then it is easy to establish that

$$\lim_{t\to\infty} m(t) = - \tilde{M}_\infty \varepsilon_{c\infty}^* = m_\infty \tag{28c}$$

In this case an exact expression for the value of $k(t)$ as $t\to\infty$ may be determined by use of the Laplace transform and the associated final-value theorem.

Taking the Laplace transform of equation (27a) we obtain

$$\bar{g}(p) - p\bar{e}(p)\bar{f}(p) = 0 \tag{29}$$

where p is the transform parameter and a superposed bar indicates the transform of a function. Multiply equation (29) by p and consider the limit of this expression as $p\to 0$

$$\lim_{p\to 0} [p\bar{g}(p)] - \lim_{p\to 0} [p\bar{e}(p)] \lim_{p\to 0} [p\bar{f}(p)] = 0 \tag{30}$$

Using the final-value theorem, we obtain

$$g(\infty) - e(\infty)f(\infty) = 0 \tag{31}$$

Now, recalling equations (25b), (25c), (27b) and (28), we obtain

$$k^2(\infty) + 2pn_\infty\, [k(\infty)-1] = 0 \tag{32a}$$

where

$$n_\infty = E_s/E_{c\infty} \tag{32b}$$

$$E_{c\infty} = 1/\varepsilon^*_{c\infty} \tag{32c}$$

This expression, from which the exact value of $k(t)$ (as $t\to\infty$) may be obtained, is identical to the result we would be led to by use of the reduced modulus method for $t\to\infty$. Thus for the nonaging matrix material exhibiting bounded creep, the reduced modulus method yields the exact result immediately upon loading, and as $t\to\infty$. For intermediate times, the reduced modulus method will be in error.

Having $k(\infty)$, all other quantities of interest for $t\to\infty$ follow directly. From equations (12), (28c) and (2) we obtain

$$\alpha(\infty) = -\frac{6\tilde{M}_\infty\, \varepsilon^*_{c\infty}}{k(\infty)\,[3-k(\infty)]} \tag{33a}$$

$$\beta(\infty) = -\alpha(\infty)/k(\infty) \tag{33b}$$

The reinforcement stress then follows from equation (3)

$$\sigma_s(\infty) = E_s[\alpha(\infty) + \beta(\infty)]\ , \tag{34}$$

and the stress in the matrix follows from equation (26). Applying the final value theorem to equation (14) (written for the nonaging matrix material) we obtain

$$\sigma_c(\infty) = \left[\alpha(\infty) + \frac{y}{d}\,\beta(\infty)\right] \sigma^*_{c\infty} \tag{35a}$$

where

$$\sigma^*_{c\infty} = \lim_{t\to\infty} \sigma^*_c(\infty) \tag{35b}$$

It is easy to establish that

$$\sigma^*_{c\infty} = 1/\varepsilon^*_{c\infty} \tag{36}$$

For the aging material, results similar to those given above cannot be established.

NUMERICAL ANALYSIS

Although we are not able to obtain a closed solution of the governing nonlinear integral equation (13), numerical solutions are readily effected.

Suppose we know the values of $k(t)$ up until some time t_{i-1} , and we seek its value at t_i . Then we can rewrite (13) as

$$\frac{1}{2pn}\left[\frac{m_i}{3-k_i}\right] + \int_{t_o}^{t_{i-1}} f(\tau)\dot{e}(t_i,\tau)\, d\tau$$

$$+ \int_{t_{i-1}}^{t_i^+} \left[\frac{m(\tau)}{3-k(\tau)}\right]\left[\frac{1-k(\tau)}{k^2(\tau)}\right]\dot{e}(t_i,\tau)\, d\tau = 0 \tag{37a}$$

where

$$\dot{e}(t,\tau) = \frac{\partial}{\partial\tau} e(t,\tau) \tag{37b}$$

and where the subscript i associated with a function indicates the value of that function at time t_i . Since we assume we are given the geometry of the beam, the moment history $M(t)$ and the material properties (i.e., the creep function $\varepsilon^*(t,\tau)$, and E_s) , then $m(t)$ (and hence m_i) may be considered a known function. We can numerically obtain it on a computer using the definitions (11e) and (9h). Recalling the definition of $f(t)$, equation (25c), and that we are assuming $k(t)$ known in the interval (t_o, t_{i-1}) , then the first integral appearing in equation (37a) may also be directly computed.

Let us evaluate the second integral appearing in (37a) by using the trapezoidal rule

$$\int_{t_{i-1}}^{t_i^+} \left[\frac{m(\tau)}{3-k(\tau)}\right]\left[\frac{1-k(\tau)}{k^2(\tau)}\right] \dot{e}(t_i,\tau)\, d\tau \doteq$$

$$\left[h_{i-1} - e(t_i,t_i)\right]\left[\left(\frac{m_i}{3-k_i}\right)\left(\frac{1-k_i}{k_i^2}\right)\right] - h_{i-1}f_{i-1} \tag{38a}$$

where we have taken account of the singular (i.e., delta-function) behavior of $\dot{e}(t,\tau)$ at $t = \tau$, and where

$$h_{i-1} = \frac{e(t_i,t_i) - e(t_i,t_{i-1})}{2} \tag{38b}$$

In the above we have approximated $\dot{e}(t_i,t_i)$ with a backward difference and $\dot{e}(t_i,t_{i-1})$ with a forward difference.

Substituting (38a) into (37a), and performing some elementary manipulations, we obtain

$$A_i\ k_i^3 + B_i\ k_i^2 + C_i\ k_i - C_i = 0 \tag{39a}$$

where

$$A_i = - h_{i-1}f_{i-1} - V_{i-1} \tag{39b}$$

$$B_i = 3\ V_{i-1} + 3h_{i-1}f_{i-1} + \frac{m_i}{2pn} \tag{39c}$$

$$C_i = m_i e(t_i,t_i) - h_{i-1}m_i \tag{39d}$$

$$V_{i-1} = \int_{t_o}^{t_{i-1}} f(\tau)\dot{e}(t_i,\tau)\, d\tau \tag{39e}$$

To obtain the proper root, k_i , of the cubic equation (39a), it is convenient to use Newton's method of approximation [13]. In order to ensure rapid convergence of this iterative procedure, it is important to have a good starting approximation for k_i . For this starting approximation we simply use the value of k found at the previous time step, i.e., k_{i-1} .

To initiate the entire procedure, we set $t_i = t_o^+$, and then the cubic equation (39a) reduces to the quadratic equation

$$k_o^2 + 2pn_o (k_o-1) = 0 \tag{40a}$$

where

$$k_o = k(t_o^+) \tag{40b}$$

$$n_o = E_s \varepsilon^*(t_o,t_o) \tag{40c}$$

This equation is the result which would be obtained if our matrix were an elastic material with a compressive modulus of $1/\varepsilon^*(t_o,t_o)$. The proper root of (40a) is easily obtained to any degree of accuracy by use of the quadratic formula. Having k_o we proceed to obtain $k_1 = k(t_1)$ by the method outlined above, and so on for all k_i .

We present some numerical results for a variety of situations based on a creep function for concrete proposed by Hanson [14]

$$\varepsilon^*(t,\tau) = \frac{1}{E(\tau)} + f(\tau)\ln(t-\tau+1) \tag{41}$$

where t and τ are in days. The creep function parameters $E(\tau)$ and

$f(\tau)$ necessary to define $\epsilon^{*}(t,\tau)$ in (41) are shown plotted in Figure 5. They are based on data given in the report by Hanson for Shasta Dam concrete [14].

In the calculation of the compressive stress in the matrix by means of equation (14), the stress relaxation function for the matrix is required. It was obtained from the creep function by solving the linear integral equation (6).

It is of interest to note that we also computed $k(t)$ by means of another numerical procedure. Before we discovered the manipulations which reduced the complicated system of nonlinear equations (8) to the much simpler single, nonlinear equation (13), we dealt directly with equations (8). We constructed an algorithm which allowed us to solve equations (8) for $k(t)$ by numerical means. This algorithm (which we do not bother to record here) is much more complicated than the one required to resolve equation (13). But the results obtained were very good, in that they checked with the results obtained from a numerical analysis of equation (13) to within five significant figures.

DISCUSSION OF RESULTS

In Figures 6 to 19 we show some results for different loading conditions, based on a numerical resolution of equation (13). For all of the figures shown, the creep function given by equation (41) was utilized, and the beam parameters employed throughout were $p = 0.0158$ and $E_s = 30.2 \times 10^6$ psi. Figures 6 to 17 are for cases in which the beam section under consideration is subjected to a constant moment of nondimensional magnitude $\tilde{M} = 0.693 \times 10^{-4}$. This moment is applied when the concrete has attained an age in days of either 10, 30, 60, 120 or 240. Figures 18 and 19 show results for some variable moment histories.

For different ages of loading with a constant moment, the neutral axis history is summarized in Figure 6, the curvature history in Figure 7, the stress history of the reinforcement in Figure 8, and the maximum compressive stress history (at $y=0$) in the matrix material in Figure 9. Strain and stress distribution histories are shown in Figures 10 to 11 and Figures 12 to 17, respectively. These results show a relaxation of the maximum compressive and a build up of the reinforcement stress as time progresses. This is in accord with experience and experiment [2,15]. It may be seen that the largest changes occur soon after loading. For the matrix material considered (concrete), the earlier the loading is applied, the more dramatic are these changes.

The nonlinear distribution of the compressive stress in the initially cracked region of the matrix material is clearly exhibited in Figure 12. Recall that in a previous section we noted that such a behavior was to be expected from our formulation.

In Figure 18 we compare the neutral axis histories produced by three different moment histories, each initiated when the concrete attains an age of 10 days. The variable moment histories for cases A and B are depicted directly on Figure 18. On this same figure a comparison is given between our viscoelastic theory and the results of the effective modulus method. It is seen that the value of $k(t)$ determined from the effective modulus method (and which is independent of the moment history) closely approximates the results obtained from viscoelastic theory for a constant moment history, but poorly approximates the results for a variable moment history.

This agreement, or lack of agreement, of the effective modulus method with our viscoelastic theory could have been predicted on the basis of our previous discussion concerning the applicability of the effective modulus method. For example, if, on the basis of $k(t)$ obtained from the effective modulus method, we compute the term denoted as Error appearing in equations (25), then we obtain for two cases considered in Figure 18 [in the time range 0 to 60 days after loading]: $|\text{Error}| \leq 0.015$ for the case of constant moment, and $|\text{Error}| \leq 0.21$ for case A. Thus for the case of a constant moment history we obtain an error term negligibly small compared to unity, and hence we would expect the effective modulus method to be an accurate approximation. However for the variable moment history treated in case A, the error term is not negligibly small compared to unity, and hence the effective modulus method would not be expected to yield an accurate approximation.

In Figure 19, the stress distribution history in the matrix material for the variable moment history of case A is displayed. Once again we note the marked nonlinearity of this distribution with depth, within the initially cracked zone.

CLOSURE

We have developed an elementary theory to account for the creep of a cracked, reinforced viscoelastic beam. Our procedure has been to take the conventional theory used for the analysis of cracked, reinforced concrete beams, and to extend it by allowing the matrix material to behave as a linear viscoelastic solid in compression. The moment histories which we can admit in our theory are limited to those which give rise to a monotonic shift of the neutral axis towards the (tensile) reinforcement.

Surprisingly, the simple hypotheses employed in our theory lead to a relatively complex governing equation.

It is noted that it is a trivial matter to extend the analytical formulation presented here to include compressive reinforcement, and also the possibility of linear viscoelastic behavior of both tensile and compressive reinforcement. Although we have obtained these results, we do not bother to record them.

Extensions of our analytical formulation for the cracked beam to include the possibility of a non-monotonic behavior of the motions of the neutral axes, and the possibility of a noncoincidence of the neutral axes of stress and strain, have not been given our serious consideration. These appear to be nontrivial topics. Extensions to include shrinkage and thermal strains, a nonzero tensile cracking stress, and slip of the reinforcement also have not been seriously studied by us. At first view, these too appear to be nontrivial extensions. If at the outset of the study, instead of utilizing a strictly analytical formulation (as we have attempted to do), recourse is immediately made to an approximate formulation of the finite-element type, then it would appear that many of the effects mentioned directly above could be included without any great increase in complexity.

REFERENCES

1. O. Faber, "Plastic Yield, Shrinkage, and Other Problems of Concrete, and their Effect on Design," Minutes of Proc. of Inst. of Civil Eng., London, Vol. 225, Part I, 1927, p. 27.

2. W. H. Glanville and F. G. Thomas, "Studies in Reinforced Concrete, IV. Further Investigations on the Creep or Flow of Concrete under Load," Dept. of Scientific and Industrial Research (London), Building Research, Technical Paper No. 21, 1939.

3. N. Kh. Arutiunian, Applications de la Théorie du Fluage, Editions Eyrolles, Paris, 1957 (also, Some Problems in the Theory of Creep in Concrete Structures, Pergamon Press, New York, 1966.)

4. A. D. Ross, "Creep of Concrete under Variable Stress," Journal of the American Concrete Institute, Vol. 29, No. 9, Mar. 1958, p. 739.

5. V. Volterra, "Sulle equazioni integro-differenziali della teoria dell' elasticità," R. C. Accad. Lincei (5), Vol. 18, 1909, p. 296.

6. V. Volterra, "Equazioni integro-differenziali della elasticità nel caso della isotropia," R. C. Accad. Lincei (5), Vol. 18, 1909, p. 577.

7. E. Whittaker, "Biography of Vito Volterra," in Theory of Functionals and Integro-Differential Equations, by V. Volterra, Dover Pub., New York, 1959, p. 5.

8. W. H. Glanville, "Studies in Reinforced Concrete, III. The Creep or Flow of Concrete under Load," Dept. of Scientific and Industrial Research (London), Building Research, Technical Paper No. 12, 1930.

9. J. L. Sackman, "Creep of Concrete and Concrete Structures," Proc. of the Princeton Univ. Conf. on Solid Mechanics, Princeton, N. J., 1965, p. 15.

10. E. Torroja and A. Paez, "Set and Reinforced Concrete," Chap. VIII of Building Materials, Their Elasticity and Inelasticity, edited by M. Reiner, North-Holland Pub., Amsterdam, 1954.

11. J. N. Distéfano and J. Guinea, "Redistribucion de Tensiones en Vigas Viscoelasticas Armadas," presented at the X Jornadas de Ingenieria, August 1965, Buenos Aires, Argentina.

12. B. Bresler and L. Selna, Work Now in Progress in the Dept. of Civil Eng., Univ. of Calif., Berkeley, 1966.

13. Fr. A. Willers, Practical Analysis, Dover Pub., New York, 1948.

14. J. A. Hanson, "A Ten-Year Study of Creep Properties of Concrete," Concrete Lab. Report No. SP-38, U. S. Bureau of Reclamation, 1953.

15. G. W. Washa and P. G. Fluck, "Effect of Compressive Reinforcement on the Plastic Flow of Reinforced Concrete Beams," Journal of the American Concrete Institute, Vol. 24, No. 2, Oct. 1952, p. 89.

LIST OF FIGURES

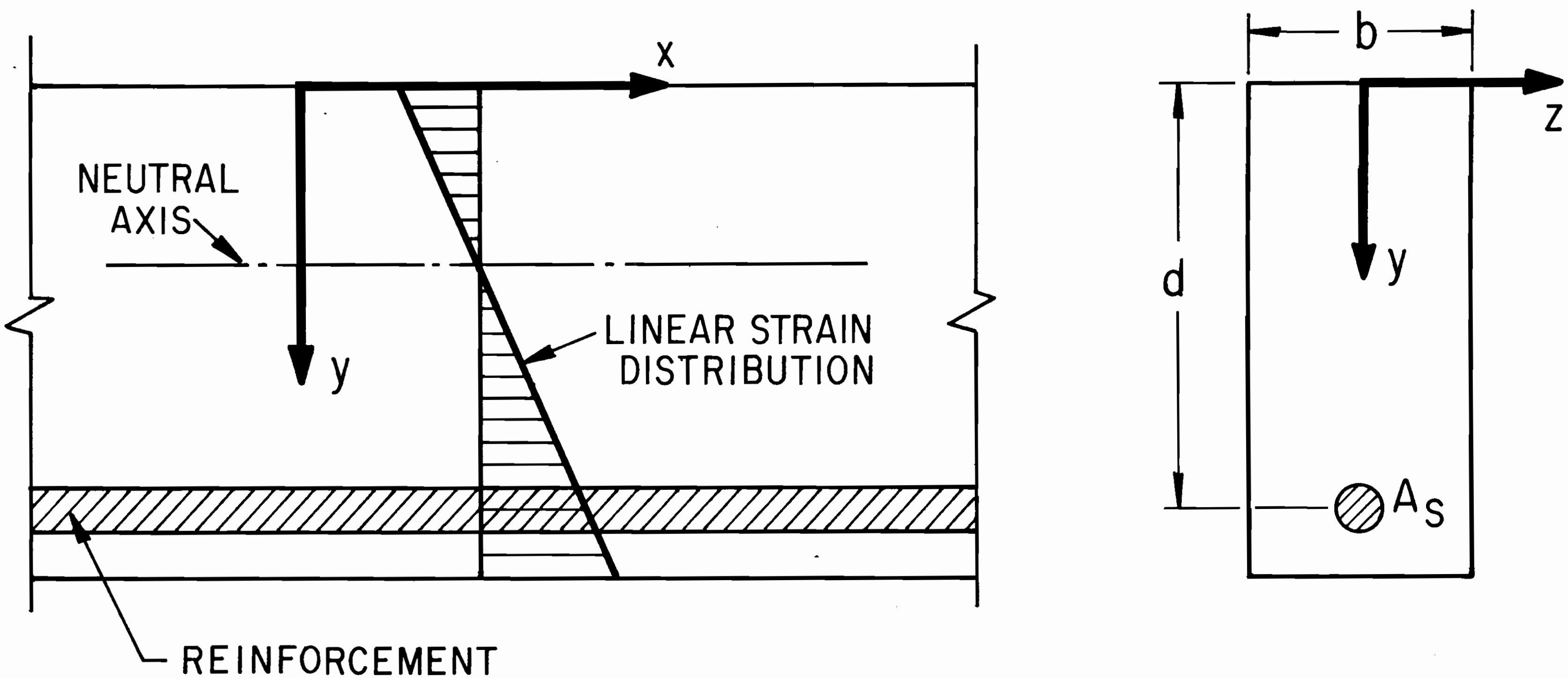

x
y
NEUTRAL
AXIS
LINEAR STRAIN
DISTRIBUTION
REINFORCEMENT
b
z
y
d
A_s

Figure 1

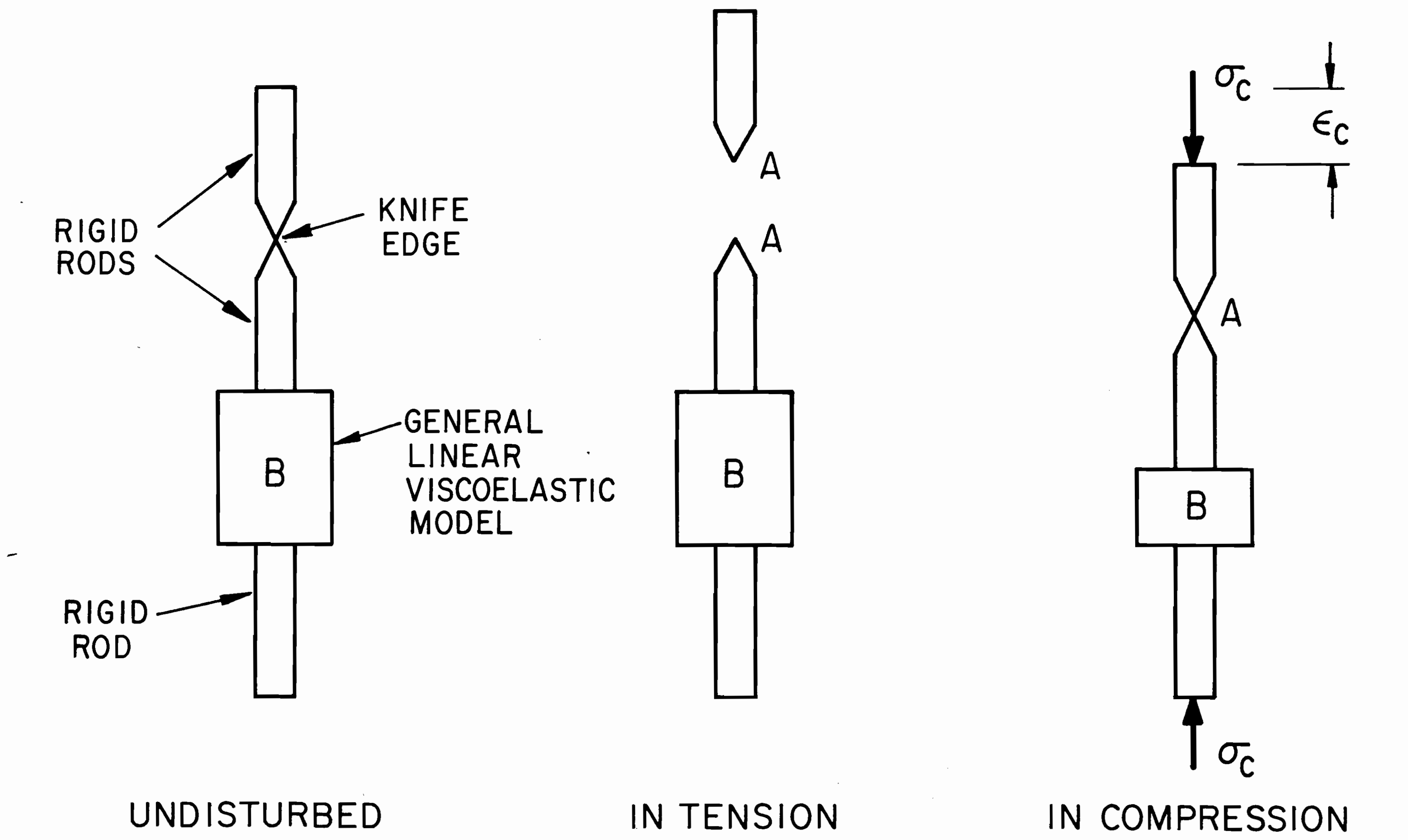

RIGID
RODS
KNIFE
EDGE
GENERAL
LINEAR
VISCOELASTIC
MODEL
RIGID
ROD
B
A
A
B
σ_c
ϵ_c
A
B
σ_c
UNDISTURBED
IN TENSION
IN COMPRESSION

Figure 2

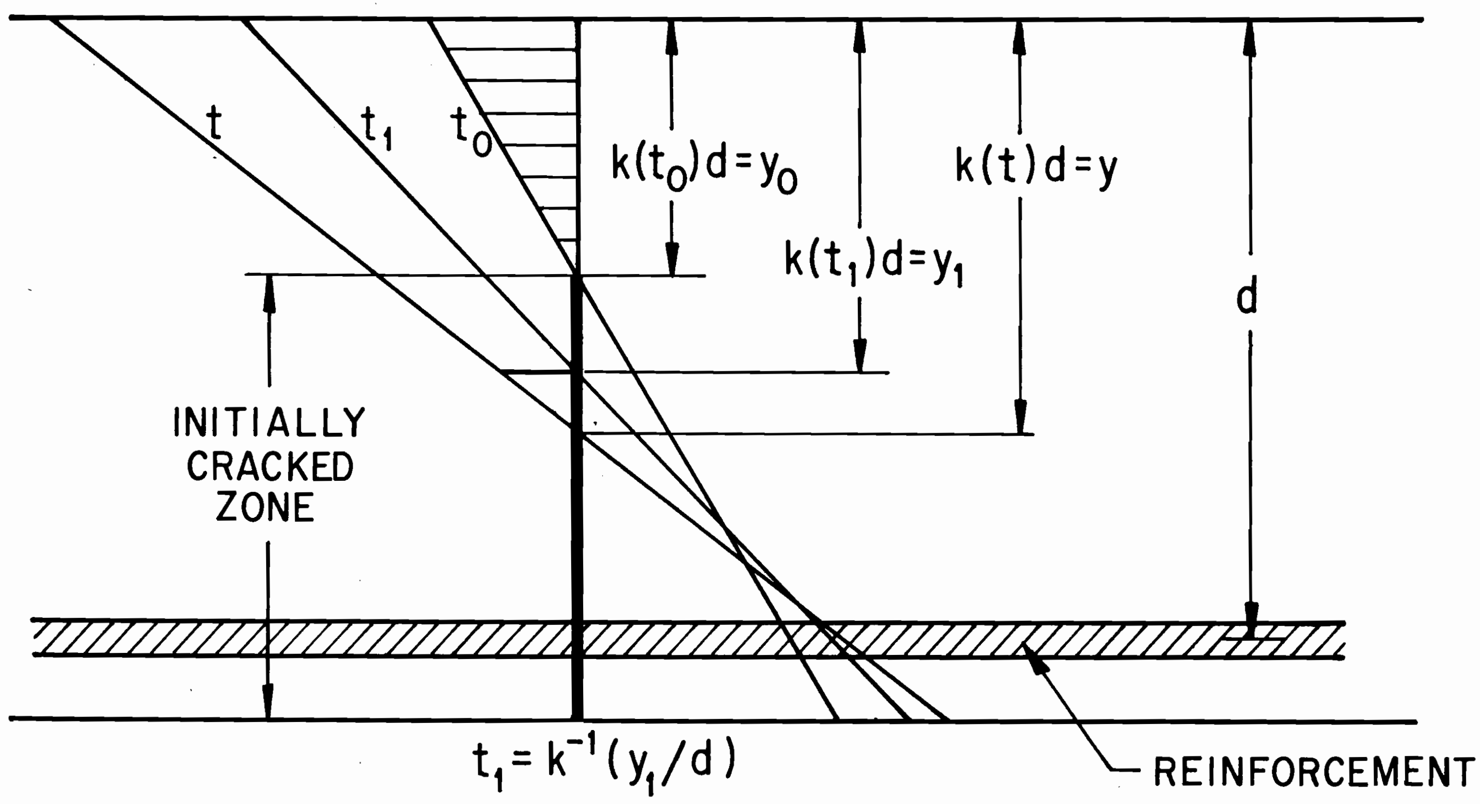

t
t1
t0
k(t0)d=y0
k(t1)d=y1
k(t)d=y
d
INITIALLY
CRACKED
ZONE
t1=k-1(y1/d)
REINFORCEMENT

Figure 3

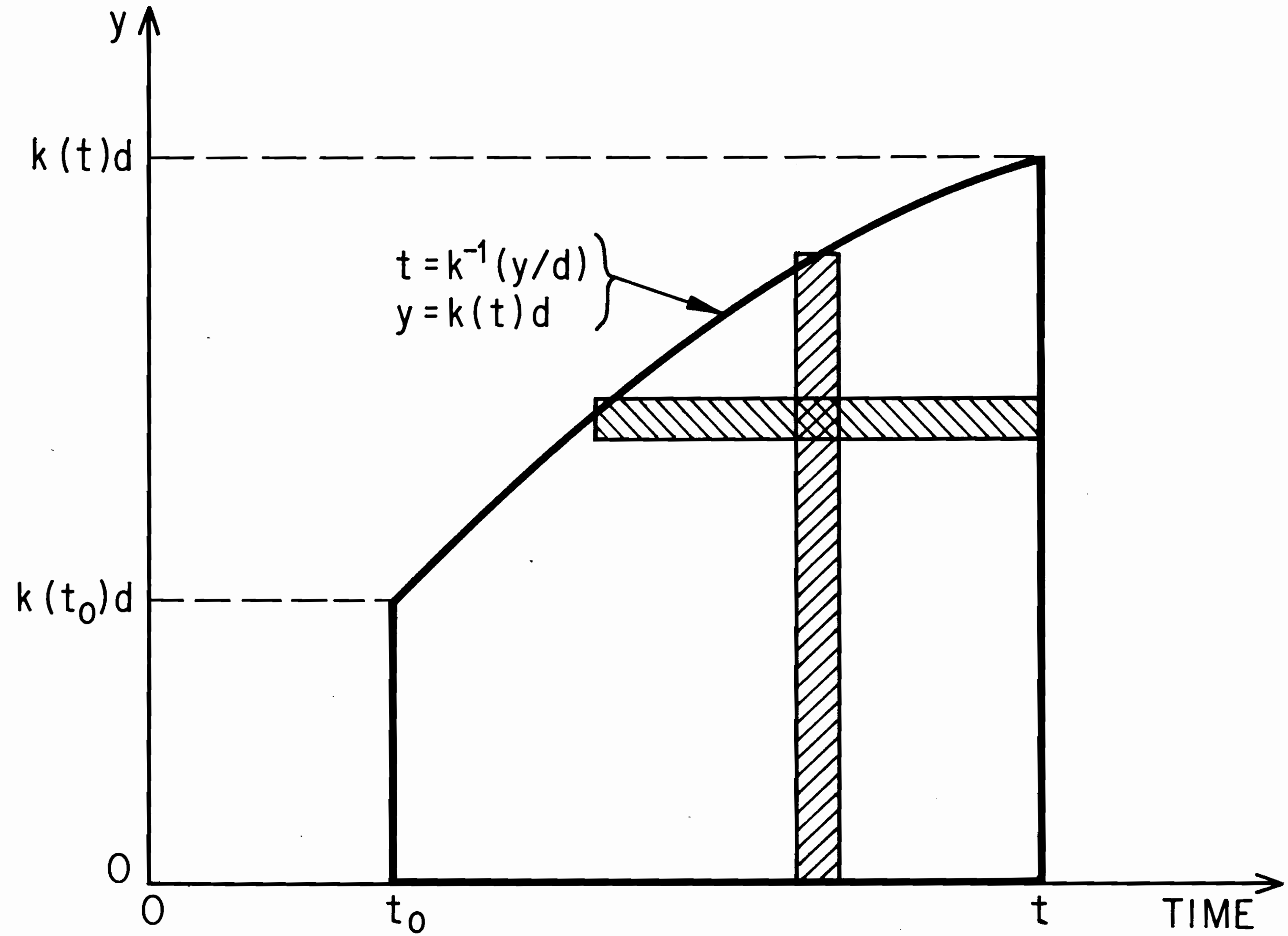


Figure 4

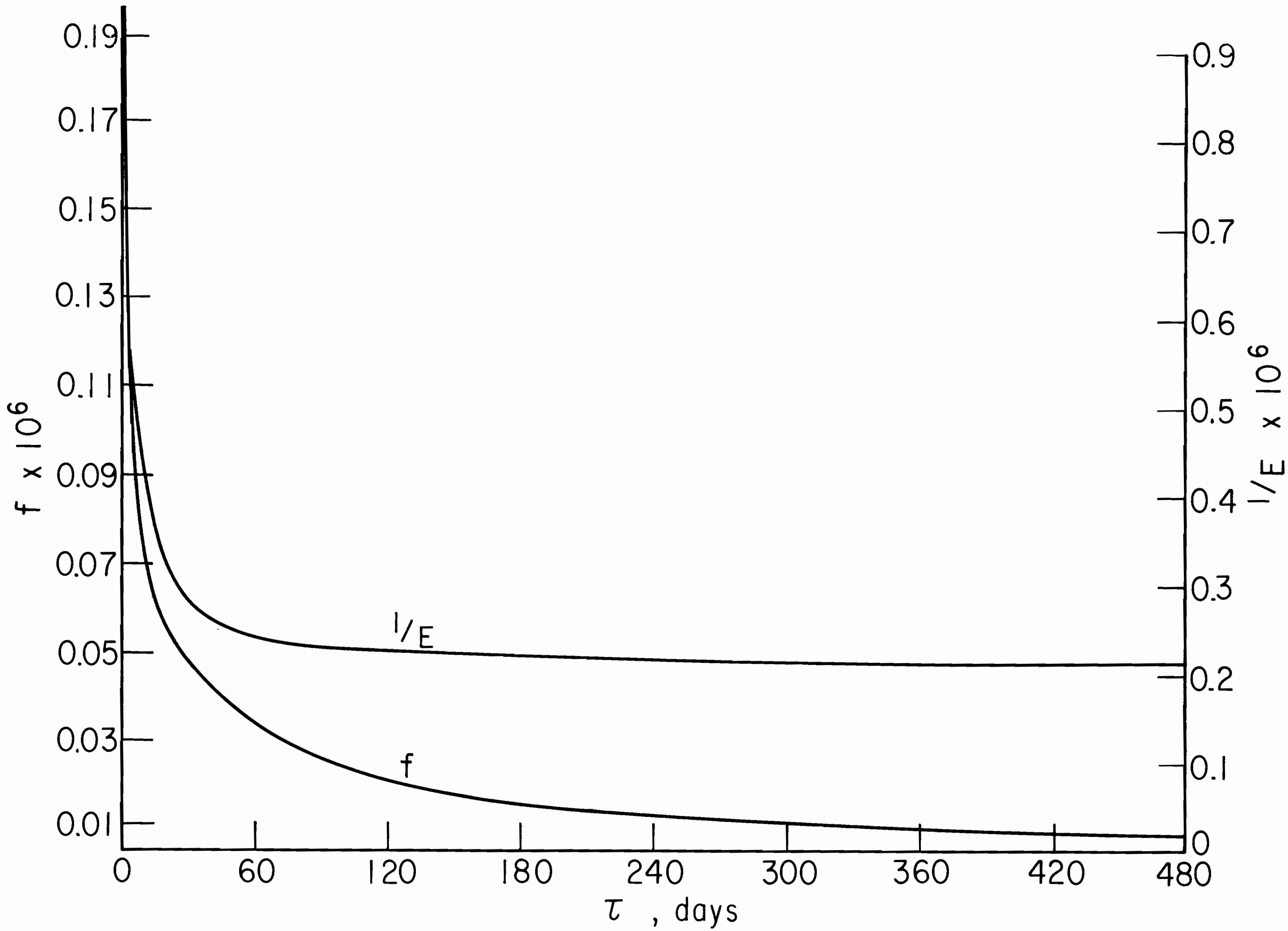

0.19
0.17
0.15
0.13
0.11
0.09
0.07
0.05
0.03
0.01
f x 10^6
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1/E x 10^6
1/E
f
0
60
120
180
240
300
360
420
480
τ , days

Figure 5

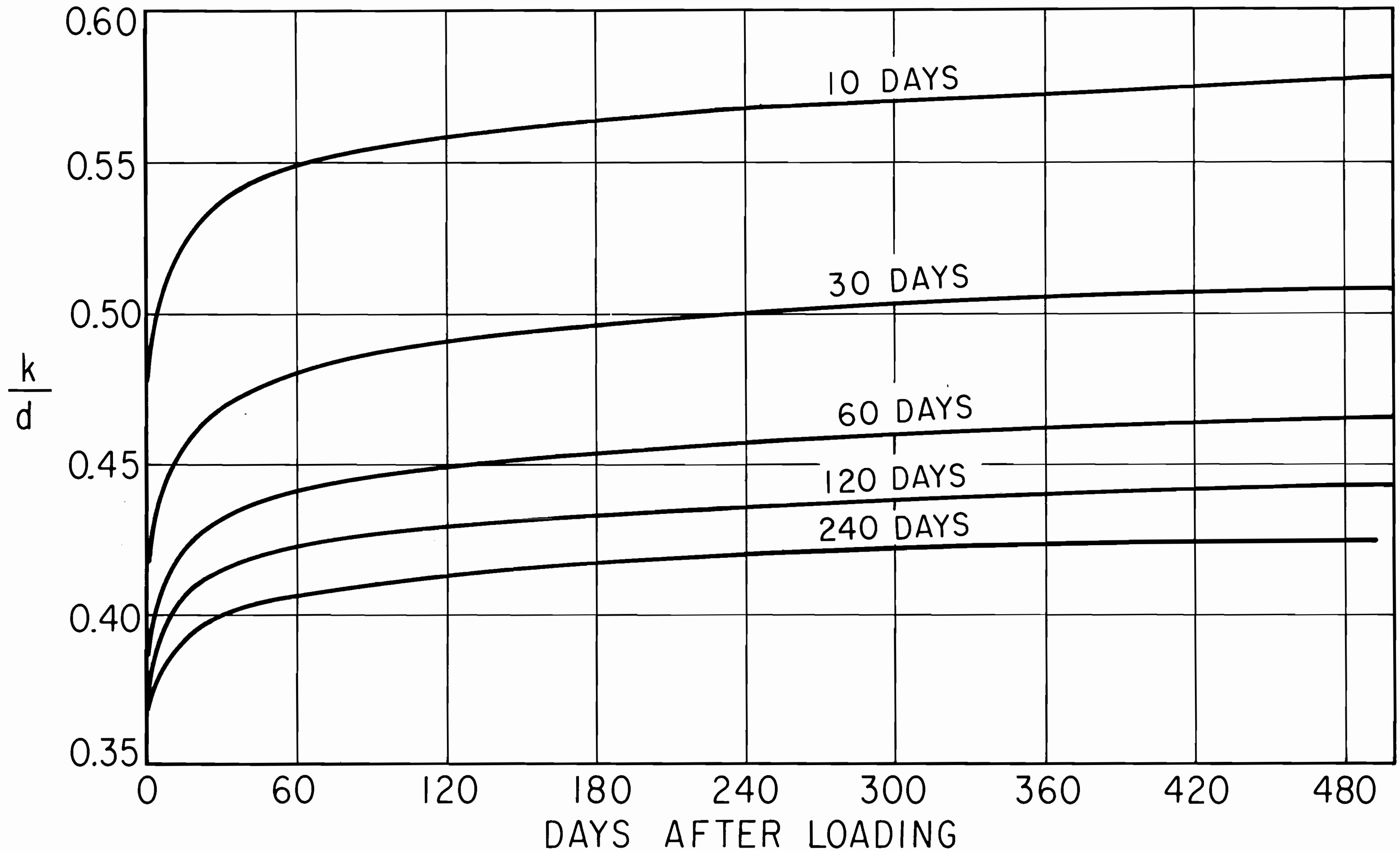
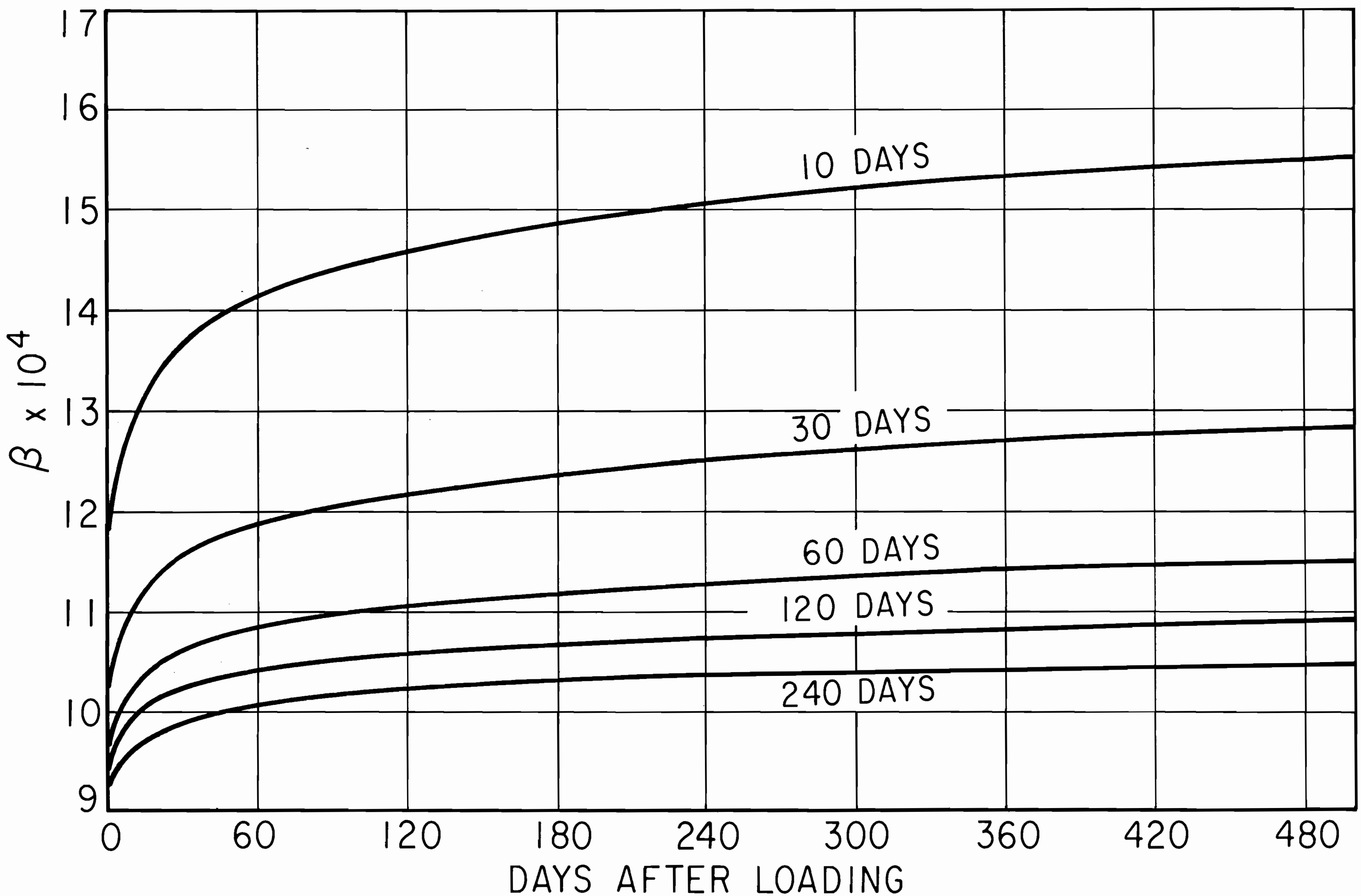

0.60
0.55
0.50
0.45
0.40
0.35
k/d
10 DAYS
30 DAYS
60 DAYS
120 DAYS
240 DAYS
0
60
120
180
240
300
360
420
480
DAYS AFTER LOADING

Figure 6



17
16
15
14
13
12
11
10
9
$\beta \times 10^4$
10 DAYS
30 DAYS
60 DAYS
120 DAYS
240 DAYS
0
60
120
180
240
300
360
420
480
DAYS AFTER LOADING

Figure 7

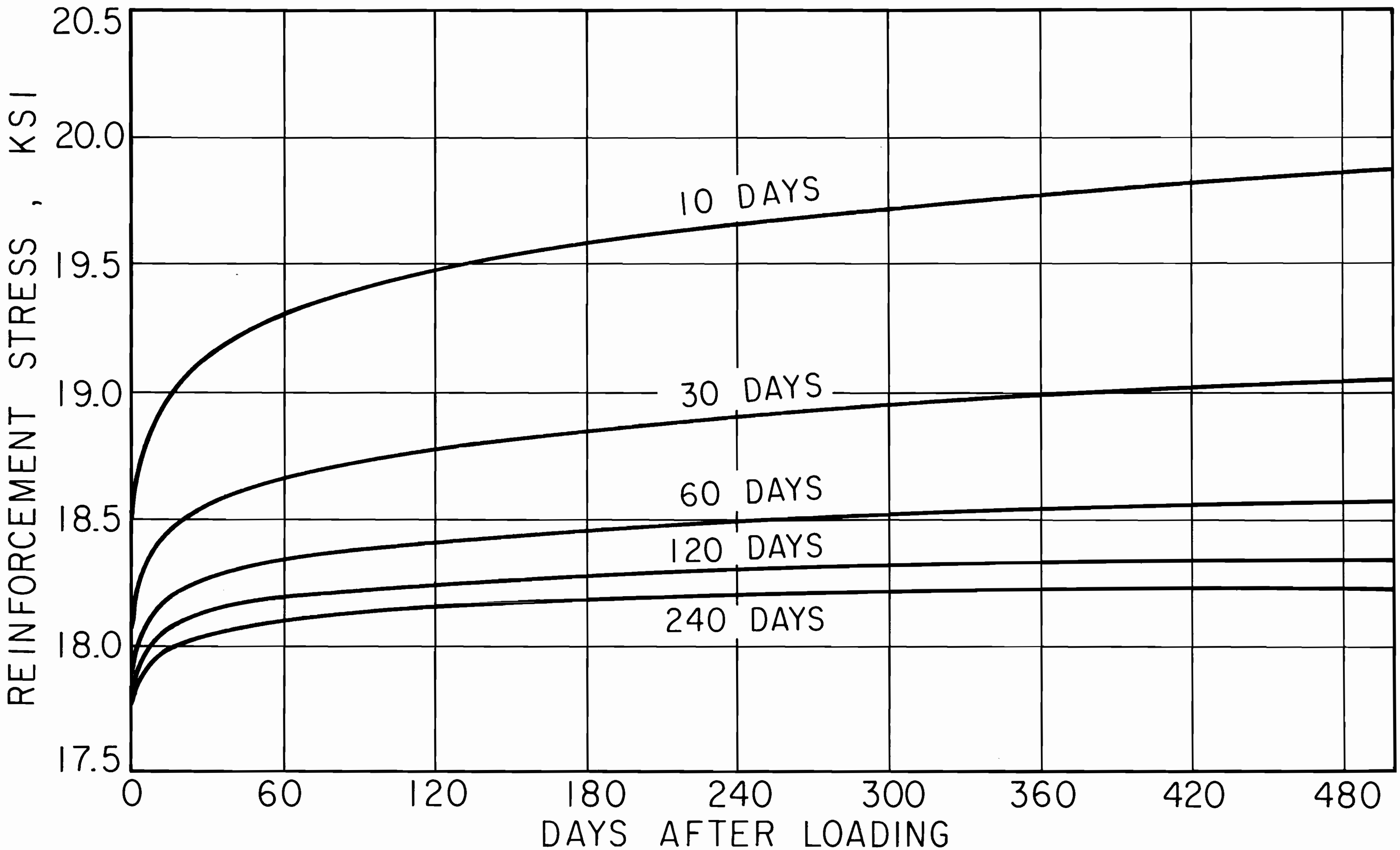

REINFORCEMENT STRESS , KSI
20.5
20.0
19.5
19.0
18.5
18.0
17.5
10 DAYS
30 DAYS
60 DAYS
120 DAYS
240 DAYS
0
60
120
180
240
300
360
420
480
DAYS AFTER LOADING

Figure 8

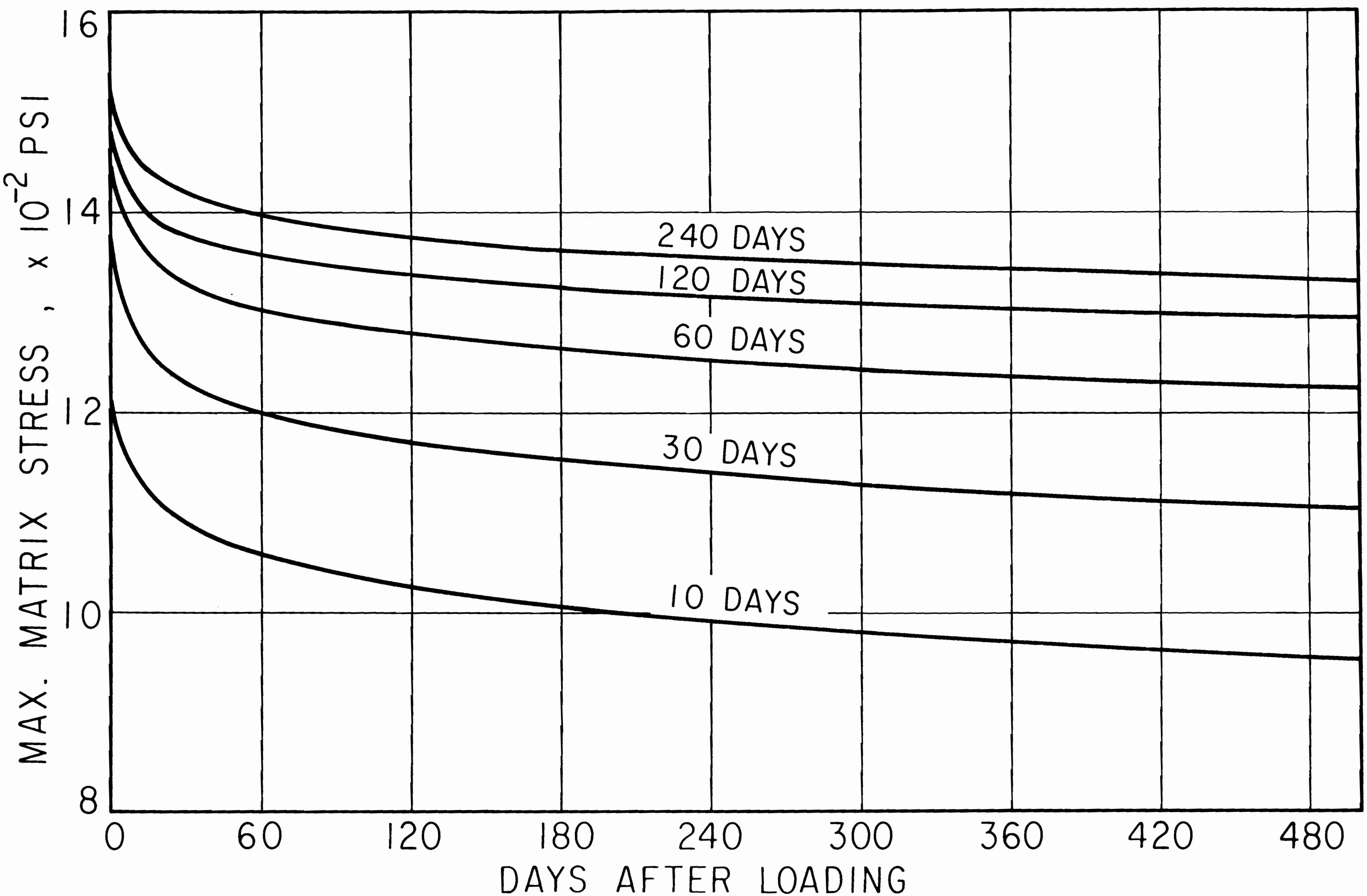
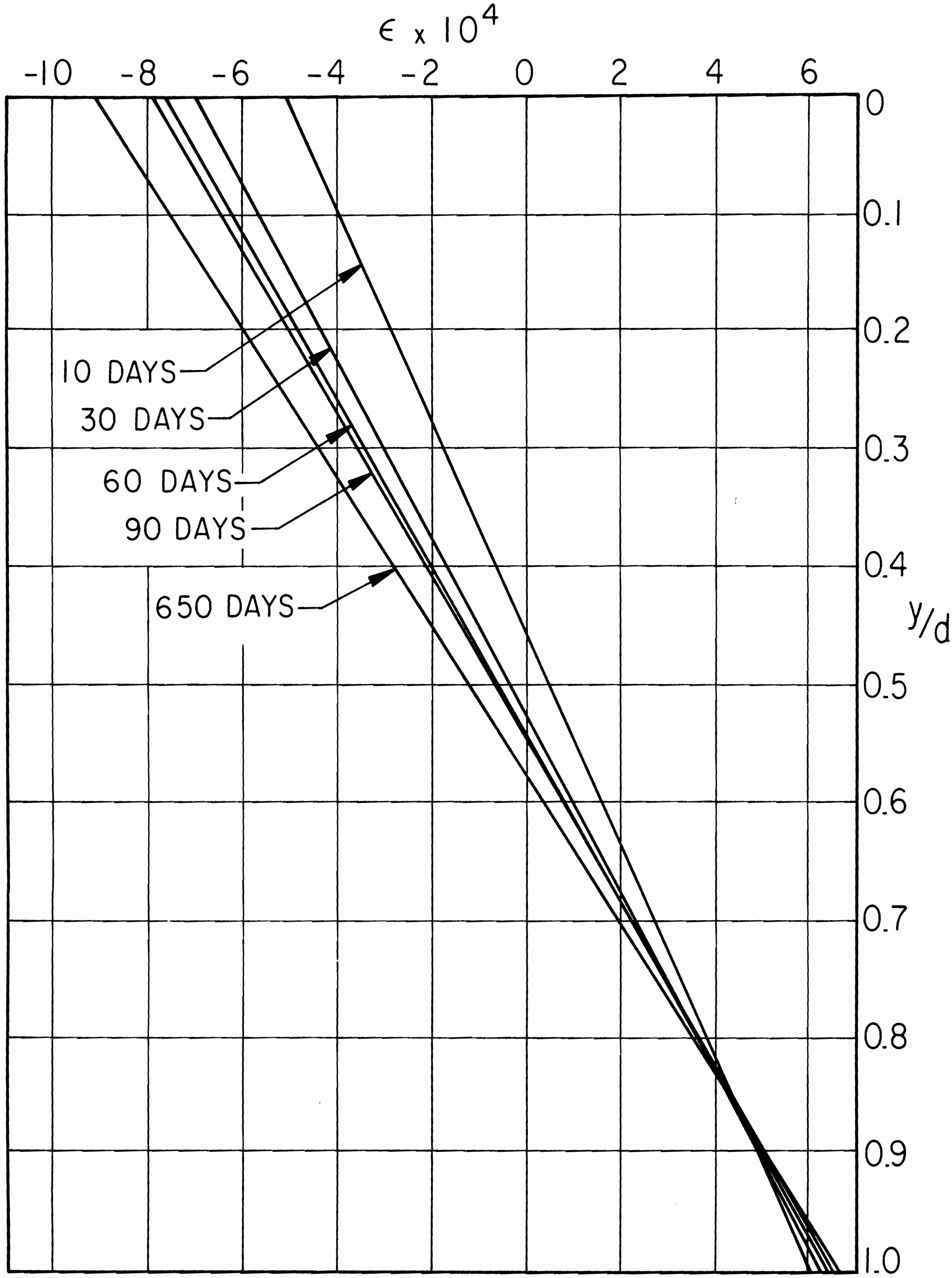
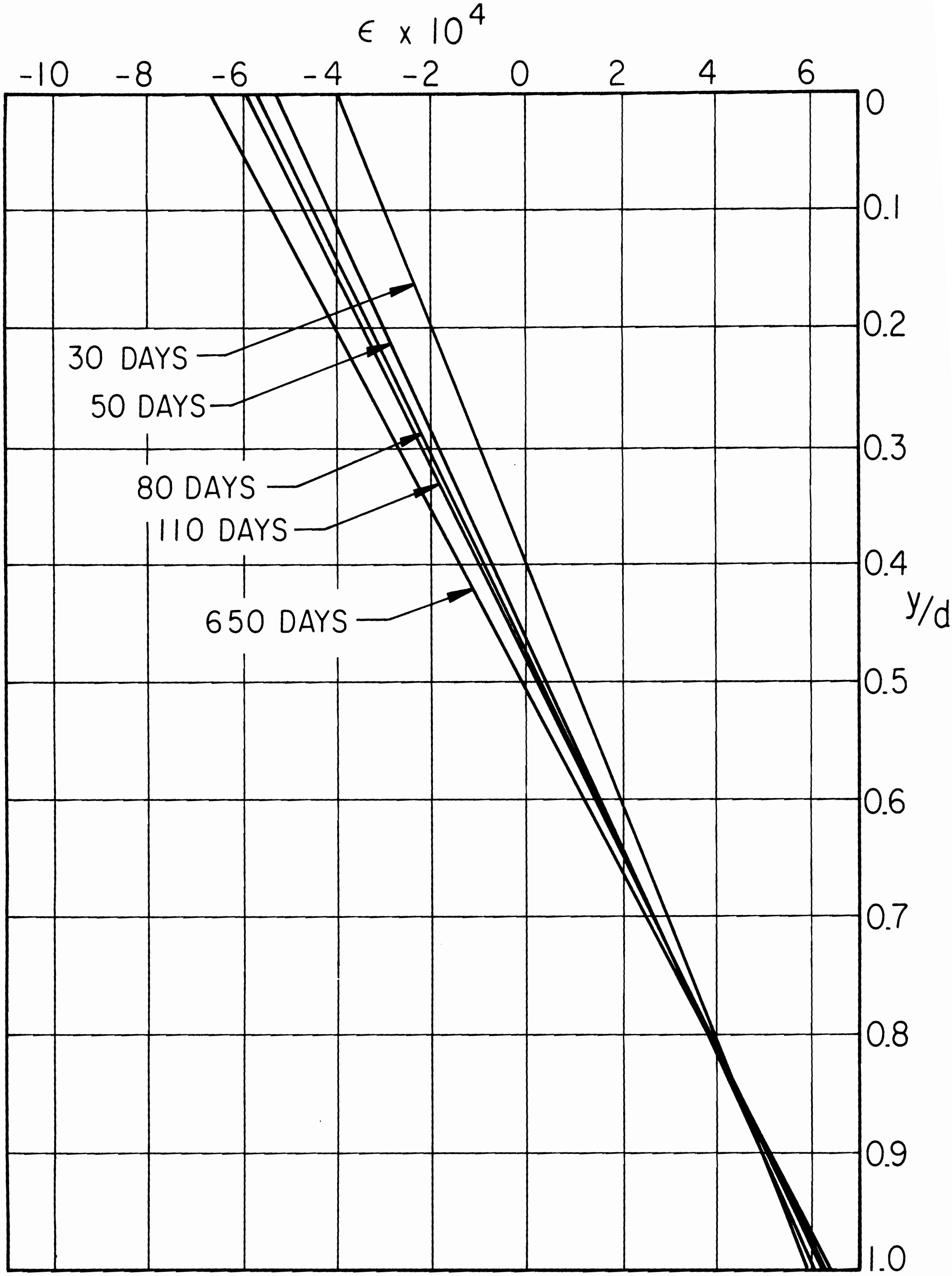

MAX. MATRIX STRESS, x 10^-2 PSI
16
14
12
10
8
240 DAYS
120 DAYS
60 DAYS
30 DAYS
10 DAYS
0
60
120
180
240
300
360
420
480
DAYS AFTER LOADING

Figure 9



ϵ x 10^4
-10
-8
-6
-4
-2
0
2
4
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
y/d
10 DAYS
30 DAYS
60 DAYS
90 DAYS
650 DAYS

Figure 10



ϵ x 10^4
-10
-8
-6
-4
-2
0
2
4
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
y/d
30 DAYS
50 DAYS
80 DAYS
110 DAYS
650 DAYS

Figure 11

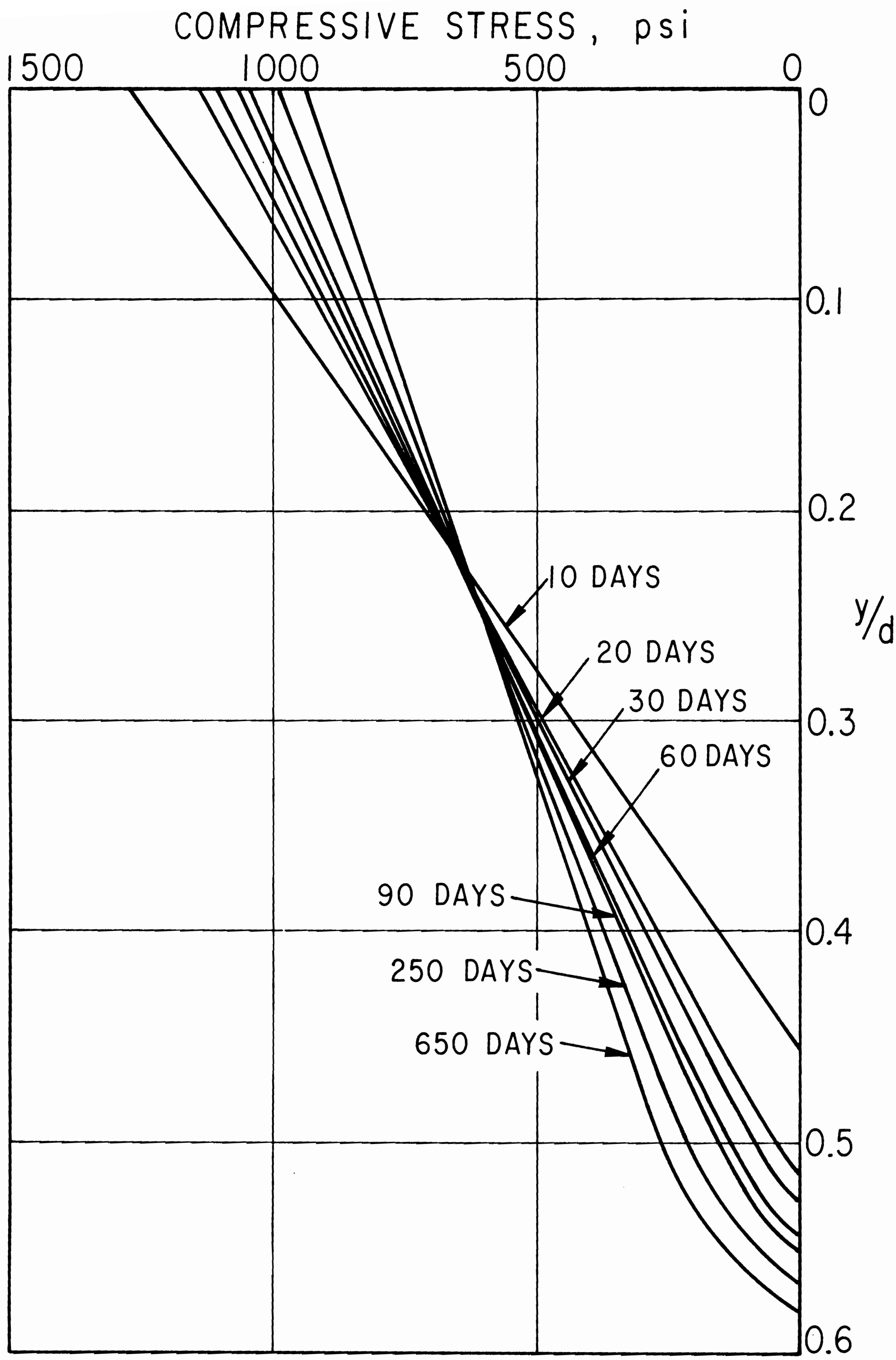

COMPRESSIVE STRESS, psi
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
10 DAYS
20 DAYS
30 DAYS
60 DAYS
90 DAYS
250 DAYS
650 DAYS

Figure 12

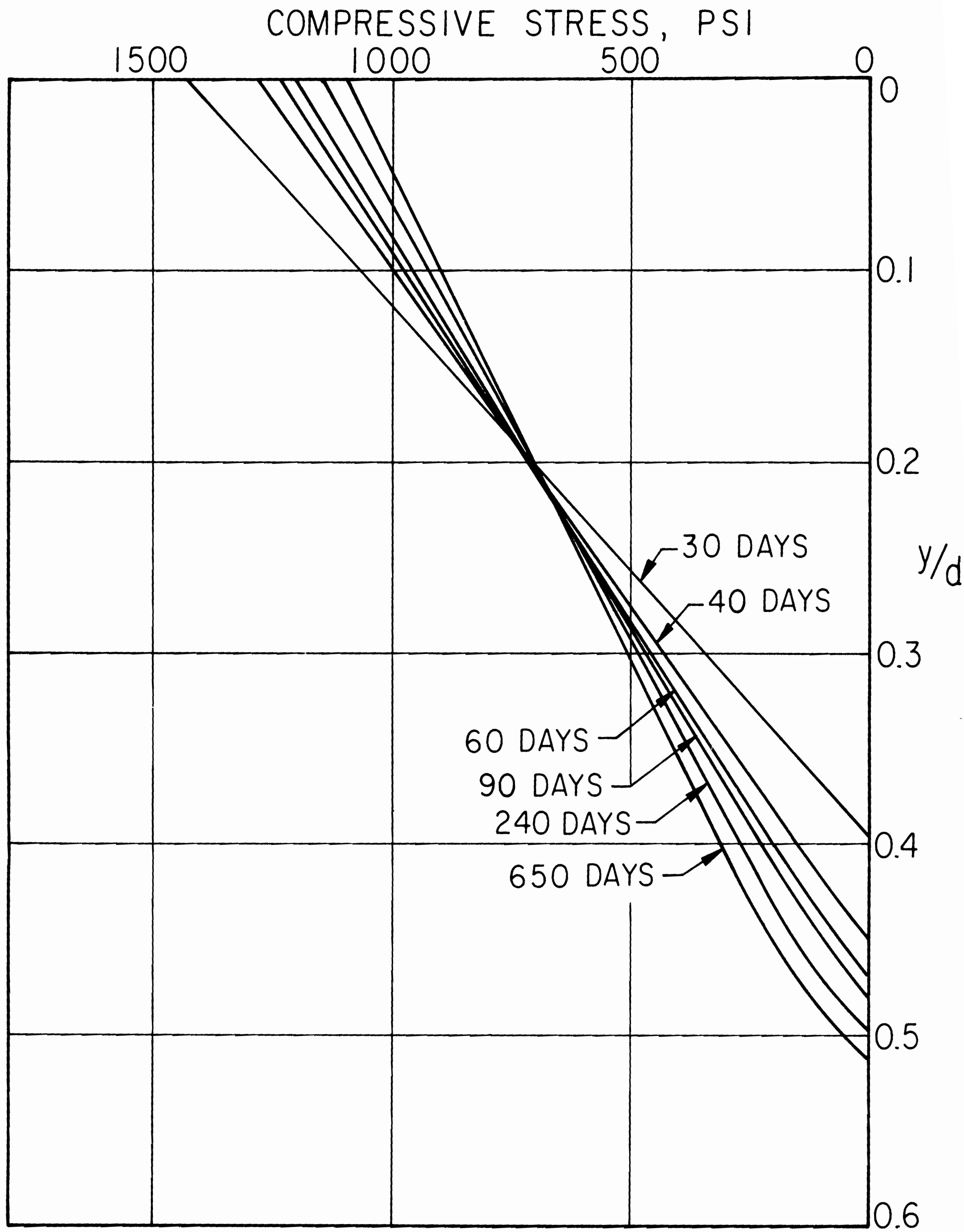

COMPRESSIVE STRESS, PSI
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
30 DAYS
40 DAYS
60 DAYS
90 DAYS
240 DAYS
650 DAYS

Figure 13

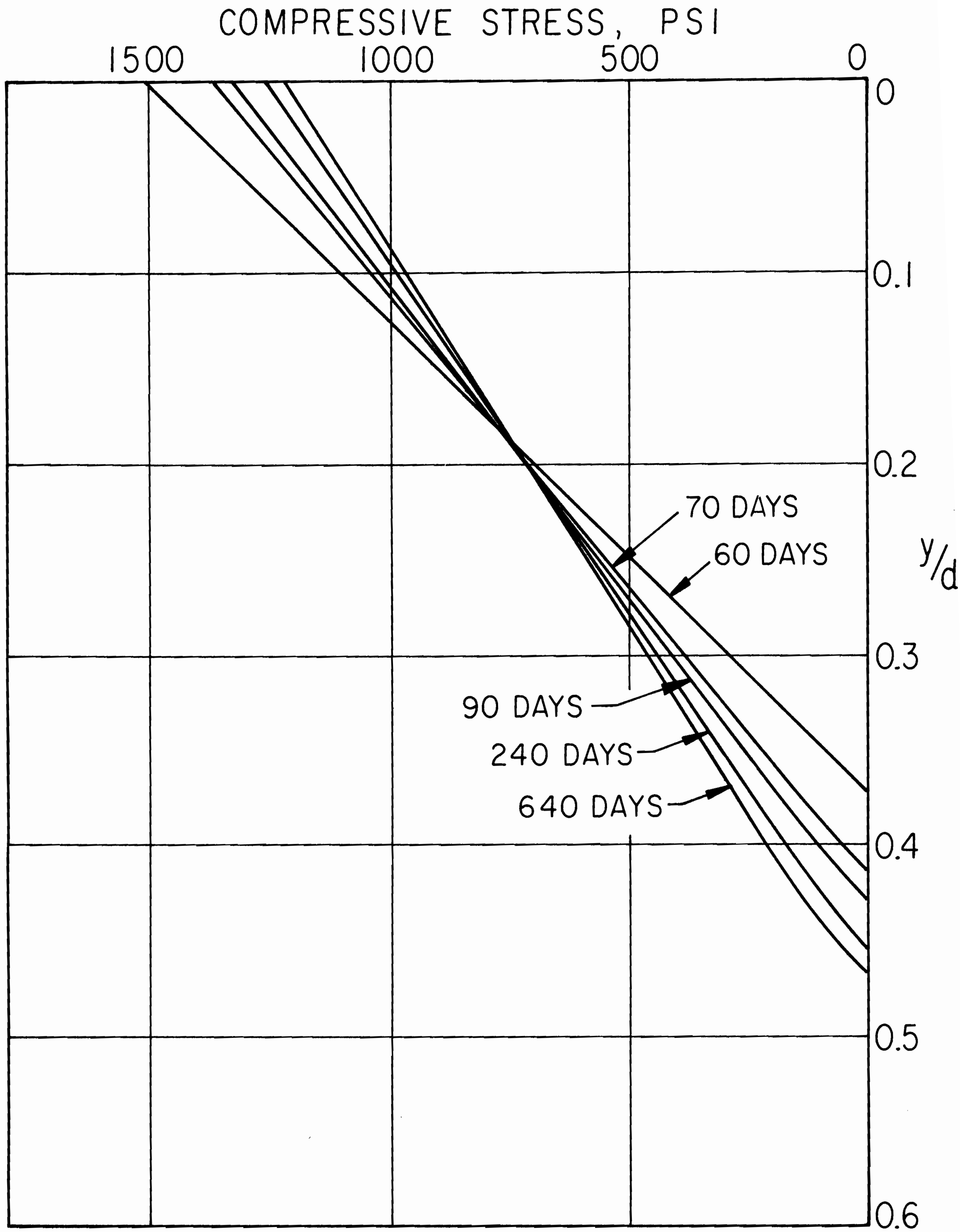

COMPRESSIVE STRESS, PSI
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
70 DAYS
60 DAYS
90 DAYS
240 DAYS
640 DAYS

Figure 14

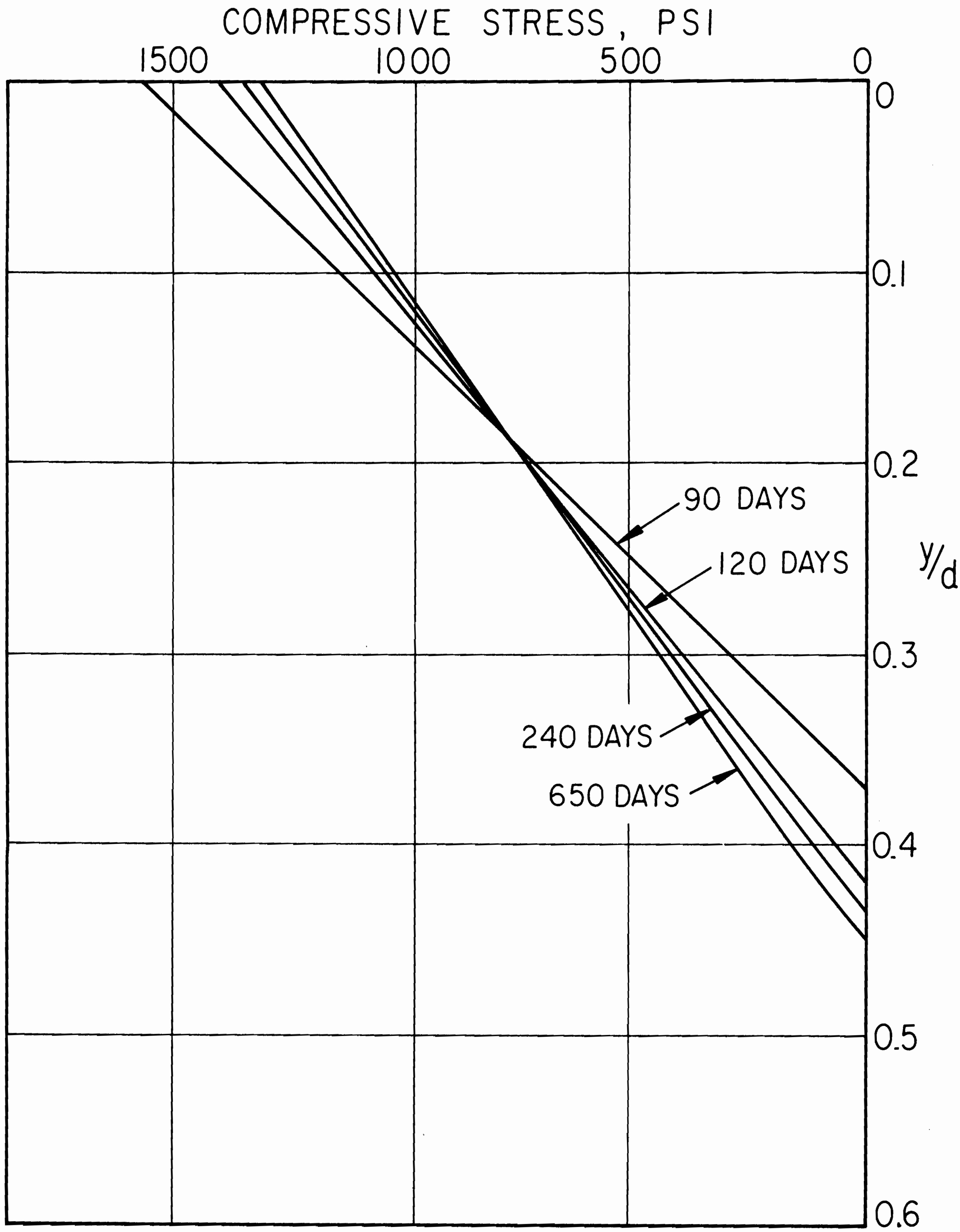
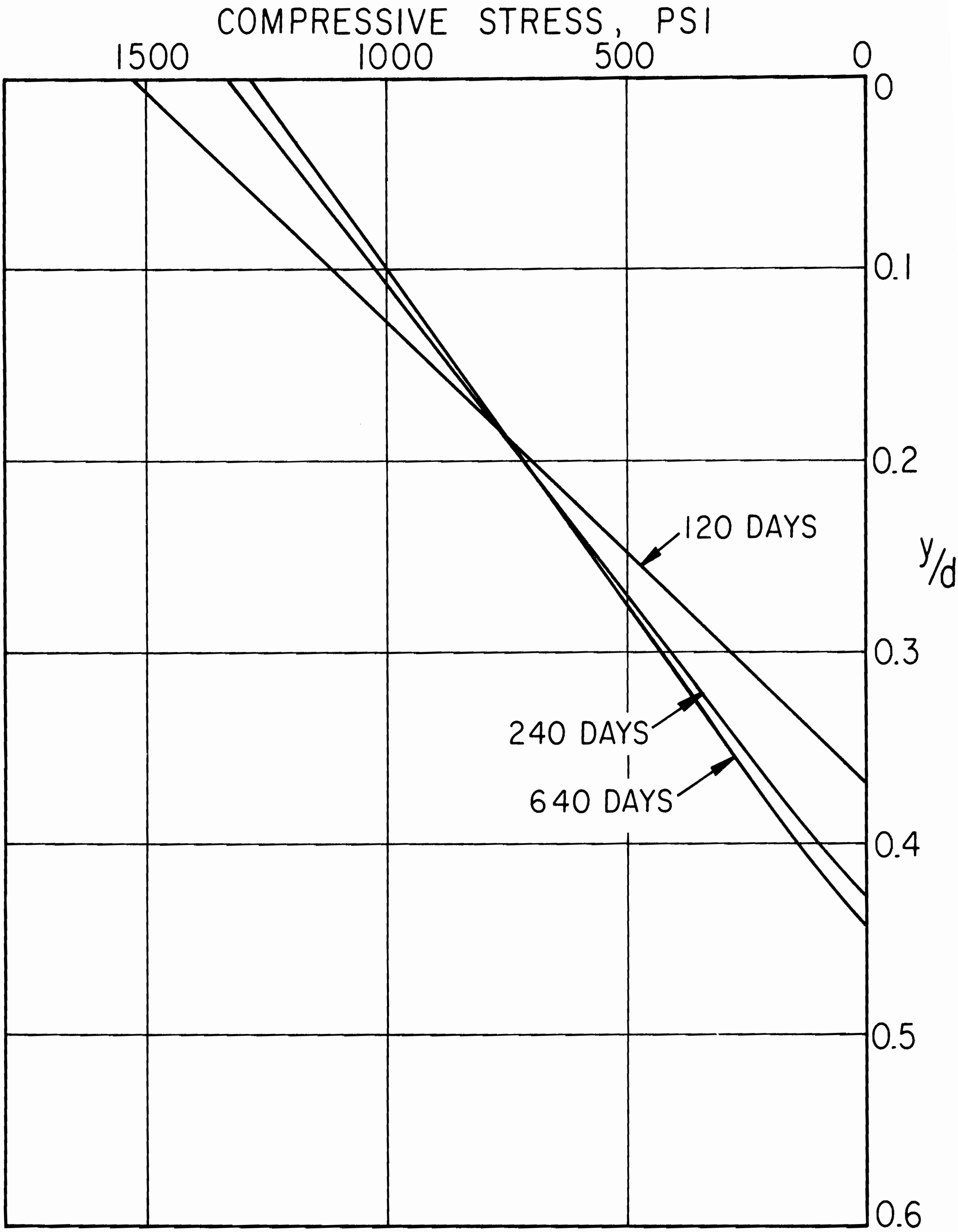

COMPRESSIVE STRESS, PSI
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
90 DAYS
120 DAYS
240 DAYS
650 DAYS

Figure 15



COMPRESSIVE STRESS, PSI
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
120 DAYS
240 DAYS
640 DAYS

Figure 16

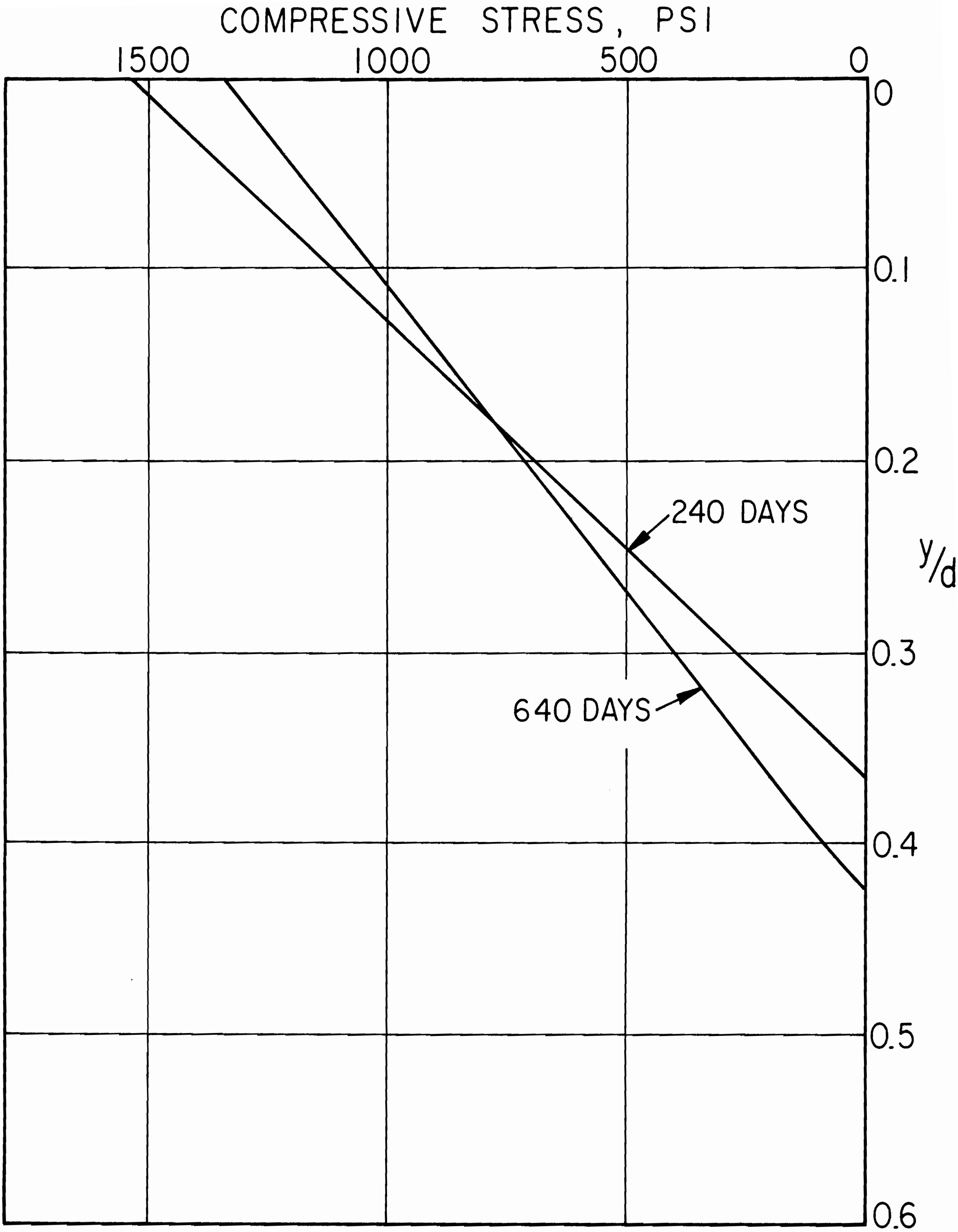

COMPRESSIVE STRESS, PSI
1500
1000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
y/d
240 DAYS
640 DAYS

Figure 17

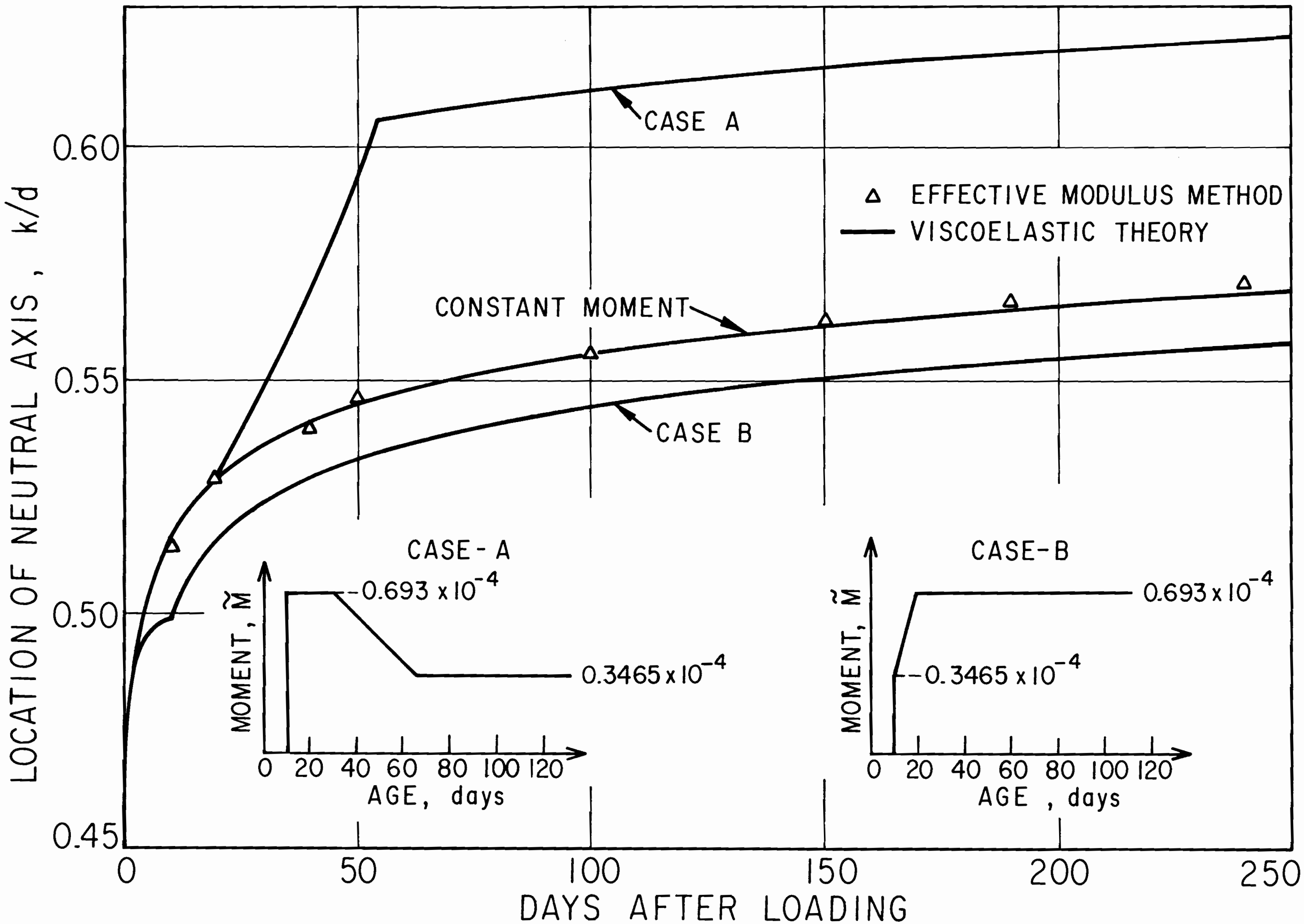

CASE A
EFFECTIVE MODULUS METHOD
VISCOELASTIC THEORY
CONSTANT MOMENT
CASE B
CASE- A
0.693 x 10^-4
0.3465 x 10^-4
MOMENT, M̃
0 20 40 60 80 100 120
AGE, days
CASE-B
0.693 x 10^-4
0.3465 x 10^-4
MOMENT, M̃
0 20 40 60 80 100 120
AGE , days
0.60
0.55
0.50
0.45
LOCATION OF NEUTRAL AXIS , k/d
0
50
100
150
200
250
DAYS AFTER LOADING

Figure 1[illegible]

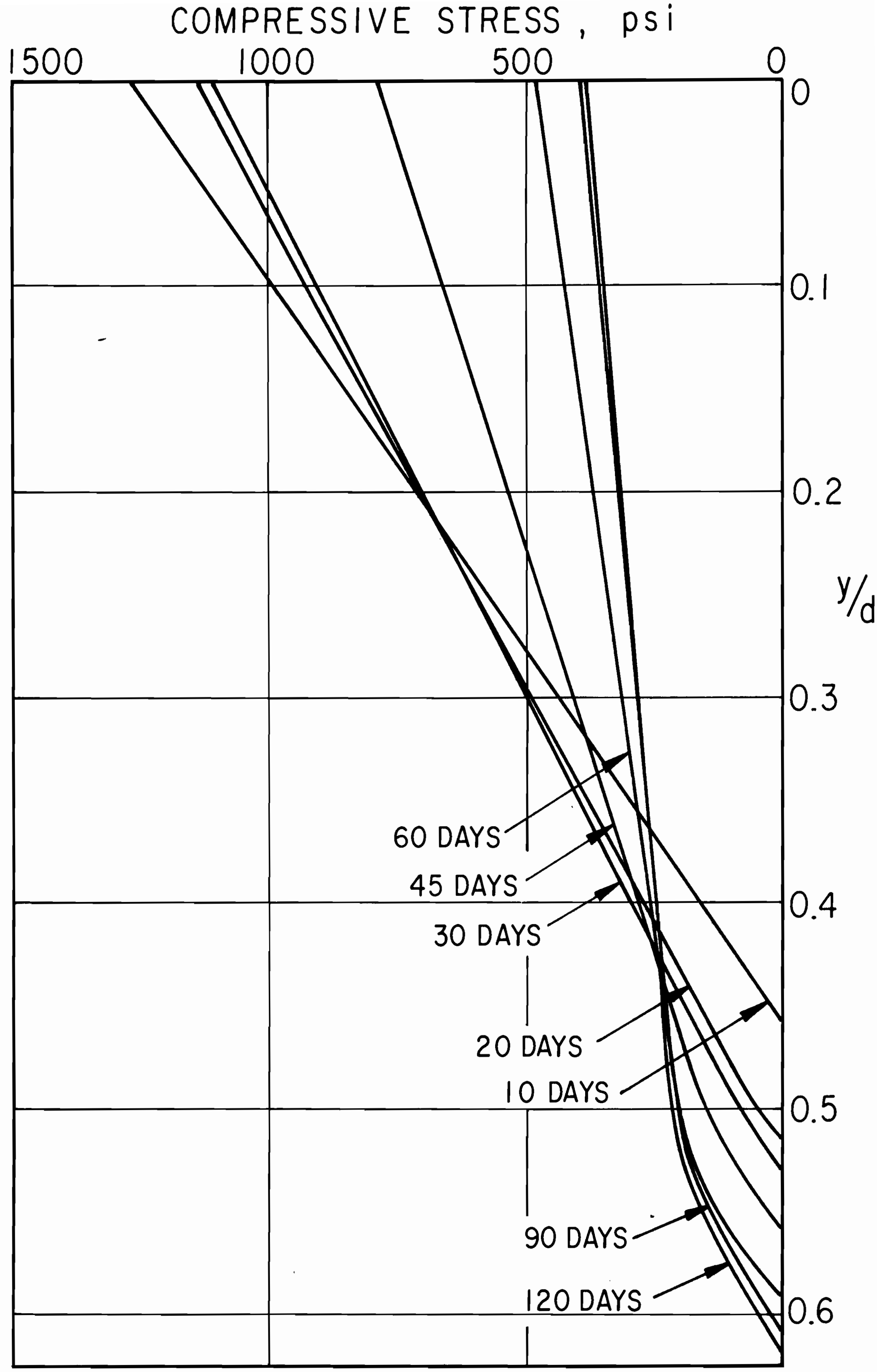

COMPRESSIVE STRESS , psi
1500
1000
500
0
0
0.1
0.2
y/d
0.3
0.4
0.5
0.6
60 DAYS
45 DAYS
30 DAYS
20 DAYS
10 DAYS
90 DAYS
120 DAYS

Figure 19