Title

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$\frac{3}{2}-$Approximation Algorithm for a Generalized, Multiple Depot Hamiltonian Path Problem

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Abstract

We consider a Generalized, Multiple Depot Hamiltonian Path Problem (GMDHPP) and show that it has an algorithm with an approximation ratio of $\frac{3}{2}$ if the costs are symmetric and satisfy the triangle inequality. This improves on the 2-approximation algorithm already available for the same.

Keywords

Approximation algorithms, Hamiltonian path problem, Traveling salesman problem, Vehicle routing.

I. Introduction

Recently, we [20] presented a $\frac{5}{3}-$approximation algorithm[1] for one variant of the multiple depot Hamiltonian Path Problem (HPP). Apart from this result, there are no algorithms in the literature with approximation ratios better than 2 for any variant of the multiple depot Traveling Salesman Problem (TSP) or multiple depot HPP. This paper considers a different, generalized variant of the multiple depot HPP and shows that it has a $\frac{3}{2}$-approximation algorithm. Specifically, this paper addresses the following Generalized Multiple Depot Hamiltonian Path Problem (GMDHPP): Given a set of k distinct depots where a salesman is present at each depot, a positive constant p and a set of n destinations to visit, the objective of GMDHPP is to

- choose at most p salesmen,
- assign paths to the chosen salesmen such that each destination is visited exactly once by one chosen salesman, and,
- the sum of the cost of the paths of all the chosen salesmen is minimized. The cost of a path is the total cost of the edges present in the path.

GMDHPP is a generalization of a single depot, single salesman HPP considered by Hoogeveen [12] and is NP-Hard. In [12], Hoogeveen presented a $\frac{3}{2}-$approximation algorithm for a single depot, single salesman version of the GMDHPP. In general, there are two subproblems when dealing with any multiple depot routing problem. The first subproblem is the partitioning problem which essentially requires finding a subset of destinations for each salesman to visit. Given the subset of vertices for a salesman to visit, the objective of the second subproblem, namely the sequencing problem, is to find an optimal sequence that produces the minimum cost path or tour. With respect to these two problems, consider the following algorithm for GMDHPP:

1. *Solving the partitioning problem:* Find a minimum cost forest with k trees spanning all the depots and the destinations such that there are at most p non-trivial trees (a tree with at least one edge) with no path joining any two depot vertices. Since there are k depot vertices and no path can join any two depot vertices, there must be exactly one depot vertex in each tree of the minimum cost forest. The depot vertices present in the non-trivial trees correspond to the chosen salesmen. There are algorithms available in the literature ([7], [6], [21]) to find such a minimum cost forest in polynomial time.

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[1]An $\alpha-$approximation algorithm for problem P is an algorithm that
- has a polynomial-time running time, and
- returns a solution whose cost is within α times the optimal cost of problem P.

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2. *Solving the sequencing problem:* Double the edges in each nontrivial tree to get an Eulerian graph for each chosen salesman. Shortcut the edges in each Eulerian graph to obtain a path for each chosen salesmen.

It has been shown ([21]) that the above algorithm has an approximation ratio of 2 when the costs satisfy the triangle inequality. Also, using a similar approach, 2-approximation algorithms have been obtained for other variants of the multiple depot TSP or HPP also ([17], [18], [19]). Except for the $\frac{5}{3}-$approximation algorithm for one particular variant of the multiple depot HPP, there are no algorithms in the literature for any multiple depot TSP or HPP that has an approximation ratio better than 2. In the following subsection, we review this variant and its approximation algorithm.

A. Review of the $\frac{5}{3}$-approximation algorithm

The problem considered in [20] was a Multiple Depot, Terminal Hamiltonian Path Problem (MDTHPP) stated as follows: Given k salesmen that start at k distinct depot vertices, k terminal vertices and n $(\geq k)$ destination vertices, the problem is to choose paths for each of the salesmen so that (1) each salesman starts at his respective depot vertex, visits at least one destination vertex and reaches any one of the terminal vertices not visited by other salesmen, (2) each destination vertex is visited exactly once and (3) the cost of the paths is a minimum among all possible paths for the salesmen. The criteria for the cost of paths considered is the total cost of the edges traveled by the entire collection. The single depot, single terminal version with one salesman corresponding to the MDTHPP has a $\frac{5}{3}-$approximation algorithm by Hoogeveen [12]. The $\frac{5}{3}-$approximation algorithm in [20] for MDTHPP used the following approach:
1. *Solving the partitioning problem:* Formulate the MDTHPP as a minimum cost forest problem subject to degree constraints on all the vertices. Penalize the degree constraints to obtain a corresponding Lagrangian dual problem. Solve this Lagrangian dual problem to obtain a minimum cost forest. A destination (or a terminal) is assigned to the depot (or the salesman located at the depot) if it is connected to the depot in the minimum cost forest. This step solves the partitioning problem by assigning a set of destinations and a terminal for each salesman located at the depots.
2. *Solving the sequencing problem:* Use Hoogeveen's algorithm [12] available for the single depot, single terminal HPP on each partition to obtain a path for each salesman.

The $\frac{5}{3}-$approximation ratio could be proved because the partitioning problem was addressed by solving a Lagrangian dual problem of the MDTHPP. The approximation algorithm presented in this paper for GMDHPP is similar to the $\frac{5}{3}-$approximation algorithm presented for the MDTHPP. However, the techniques required to show the approximation ratio are different. In the following subsection, we present the outline of the $\frac{3}{2}-$approximation algorithm for GMDHPP.

B. Our approach

We formulate GMDHPP as a minimum cost constrained forest problem with added degree constraints on the vertices (section (II)). By dualizing the degree constraints, we obtain a Lagrangian dual for the GMDHPP (equation (14)). We first show that this Lagrangian dual problem can be solved in polynomial time using the Ellipsoid method (Proposition II.2). By solving the Lagrangian dual for the MDTHPP, we find a subset of destination vertices for each salesman to visit. For each $i = 1, .., k$, we use Hoogeveen's algorithm [12] to find a path for all the salesmen who have at least one destination vertex to visit as follows:
1. Find the minimum cost spanning tree (T_i) corresponding to the vertices of the i^{th} salesman.
2. Find the minimum cost perfect matching (M_i) on the wrong degree vertices present in T_i. A vertex has a wrong degree if

- it is a destination vertex and its degree is odd.
- it is a depot vertex and its degree is even.

3. Add the edges of M_i to T_i to get a new graph for the i^{th} salesman. Shortcut the edges in the new graph to get a path for the i^{th} salesman.

First, the total cost of all the minimum spanning trees found in step (1) of the Hoogeveen's algorithm is shown to be bounded by the optimal cost of GMDHPP (Proposition IV.1). Another crucial part of obtaining a $\frac{3}{2}-$approximation algorithm is due to Proposition IV.5. Proposition IV.5 upper bounds the total cost of matching by half the optimal cost of GMDHPP if the costs satisfy the triangle inequality. This helps us prove that the algorithm presented above for GMDHPP has a $\frac{3}{2}-$approximation ratio. To show the upper bound of the total cost of matching, we also had to prove the following result:

- For a single salesman, Single Depot HPP (SDHPP), the cost of matching using the Hoogeveen's algorithm is at most half the optimal LP relaxation cost of the SDHPP if the costs satisfy the triangle inequality (Proposition IV.4). A similar result has already been shown for the single TSP by Wolsey [24], Shmoys and Williamson [23]. In this paper, we adapt the proof of Shmoys and Williamson [23] to prove a similar result for the SDHPP.

II. Problem formulation

Let $D = \{1, 2, 3, \cdots, k\}$ be the set of vertices representing all the depots. There is one salesman located at each depot. Let $U = \{k+1, k+2, k+3, \cdots, k+n\}$ be the set of vertices denoting n destinations. Let $V := D \bigcup U$. Since we are considering a multiple depot problem, let $k \geq 2$. We also assume there are at least three destination vertices ($n \geq 3$) to eliminate trivial cases. Let p denote the maximum number of salesmen that could chosen for visiting all the destinations. The edge joining vertices i and j has a cost $C_{ij} \in \mathbb{Q}^+$ associated with it where $\mathbb{Q}^+$ is the set of all positive rational numbers. Assume that all costs are positive and symmetric, i.e., $C_{ij} > 0$ and $C_{ij} = C_{ji}$ for all $i, j \in V$. Let $C_{min} = \min_{i,j\in V, i<j} C_{ij}$ and $C_{max} = \max_{i,j\in V, i<j} C_{ij}$. $C_{min} > 0$ by assumption. To simplify the notation in the later stages, the GMDHPP is defined for a set of vertices, $S \subseteq V$. Let D_S and U_S denote the set of all the depots and destinations present in S respectively (i.e. $D_S = S \bigcap \{1, 2, \cdots, k\}$, $U_S = S \bigcap \{k+1, k+2, \cdots, n\}$). Decision variable x_{ij} is used to represent the choice of the edge between vertex i and j for all $i, j \in S$. $x_{ij} = 1$ implies the edge joining vertex i and vertex j is chosen and $x_{ij} = 0$ otherwise. Let the variables x_{ij} $\forall i, j \in S, i < j$ be tersely denoted as x. For a given $S \subseteq V$ with $|D_S| \geq 1$, the GMDHPP is formulated as follows:

$$C_S^{opt} = \min_x \sum_{i\in S, j\in S, i<j} C_{ij} x_{ij} \tag{1}$$

$$\begin{aligned}
\sum_{j\in U_S} x_{ij} &\leq 1 \text{ for all } i \in D_S, & (2)\\
\sum_{j\in S, i<j} x_{ij} + \sum_{j\in S, j<i} x_{ji} &\leq 2 \text{ for all } i \in U_S, & (3)\\
\sum_{i\in D_S j\in U_S, i<j} x_{ij} &\leq p, & (4)\\
\sum_{i\in S, j\in S, i<j} x_{ij} &= |U_S|, & (5)\\
\sum_{i\in R, j\in R, i<j} x_{ij} &\leq |R| - 1, \text{ for all } R \subseteq S, & (6)\\
\sum_{i\in R, j\in R, i<j} x_{ij} &\leq |R| - 2, \text{ for all } R \text{ such that } R \subseteq S \text{ and } |R \textstyle\bigcap D_S| = 2, & (7)\\
x_{ij} &\in \{0, 1\}, \text{ for all } i, j \in S, i < j. & (8)
\end{aligned}$$

C_S^{opt} denotes the optimal cost of the GMDHPP corresponding to the set of vertices $S \subseteq V$. Equation (2) states that the degree of each depot vertex must be at most 1. Equation (3) requires that the degree of each destination vertex must be at most 2. Equation (4) states that at most p salesmen can be used for visiting all the destinations. Equation (5) states that the total number of edges in any feasible solution to the GMDHPP must be equal to $|U_S|$ (i.e there are $|D_S|$ trees). Equation (6) eliminates the presence of a cycle in any feasible solution. Equation (7) eliminates the possibility of a path joining any two depot vertices in D_S.

Let equations (2),(3) be written as $A_S^1 x \leq B_S^1$ and (4)-(7) be written as $A_S^2 x \leq B_S^2$. Define $P(S) := \{x : A_S^1 x \leq B_S^1, A_S^2 x \leq B_S^2, x \geq 0\}$. y is a feasible solution to the GMDHPP if y is present in $\{x : x \in P(S), x \text{ is an integer}\}$. The GMDHPP can now be restated as

$$C_S^{opt} = \min_x \{C_S(x) : x \in P(S), x \text{ is an integer}\}, \tag{9}$$

where $C_S(x) = \sum_{i \in S, j \in S, i<j} C_{ij} x_{ij}$. The LP relaxation of this problem is:

$$C_S^{lp} = \min_x \{C_S(x) : x \in P(S)\}. \tag{10}$$

In the above equation, C_S^{lp} denotes the optimal LP relaxation cost of the GMDHPP defined for the set of vertices $S \subseteq V$.

In the following discussion, we show how the GMDHPP formulated in equations (1-8) can be viewed as a constrained forest problem with degree constraints on all the vertices. In this paper, a constrained forest for a given set of vertices S with $|D_S| \geq 1$ is defined as a forest with the following constraints:

- there are exactly $|D_S|$ trees such that no two depot vertices are connected, and
- there are at most p edges joining any vertex in D_S to any vertex in U_S.

Equations (4-8) describe all the constraints mentioned above. Since the forest has exactly $|D_S|$ trees and no two depot vertices must be connected, each tree must contain exactly one depot vertex. The constrained forest problem, denoted by CF, is:

$$C_S^f = \min_x \{C_S(x) : x \in F(S), x \text{ is an integer}\}, \tag{11}$$

where,

$$F(S) = \{x : A_S^2 x \leq B_S^2, x \geq 0\}. \tag{12}$$

In the above equations, C_S^f denotes the optimal cost of the constrained forest problem defined for the set of vertices $S \subseteq V$. This constrained forest problem can be solved in polynomial time using the algorithms in [7], [6] or [21]. The GMDHPP is actually CF with the additional degree constraints present in $A_S^1 x \leq B_S^1$. Before we present the approximation algorithm for GMDHPP in the next section, we formulate a Lagrangian dual problem corresponding to GMDHPP and show its related results that are crucial in proving the approximation ratio.

Given a constrained forest x, let $d_i(x, S)$ denote the degree of vertex i in S. That is,

$$d_i(x, S) = \begin{cases} \sum_{j \in U_S} x_{ij} \text{ for all } i \in D_S, \\ \sum_{j \in S, i<j} x_{ij} + \sum_{j \in S, j<i} x_{ji} \text{ for all } i \in U_S. \end{cases} \tag{13}$$

By dualizing the constraints in $A_S^1 x \leq B_S^1$, we can obtain a Lagrangian dual to the GMDHPP. This Lagrangian dual problem for a given set $S \subseteq V$ can be formulated as $\max_{\pi \geq 0} w(\pi, S)$ where

$$w(\pi, S) \quad = \min_{x \in F(S), x \text{ is an integer}} [C_S(x) + \sum_{i \in D_S} \pi_i (d_i(x,S) - 1) + \sum_{i \in U_S} \pi_i (d_i(x,S) - 2)].$$

In the above equation, π_i is the penalizing variable corresponding to the degree constraint of the i^{th} vertex. Also, let π indicate the penalizing variables π_i for all $i \in S$. By letting

$$v_i(x, S) \quad = \begin{cases} d_i(x,S) - 1 \text{ if } i \in D_S, \\ d_i(x,S) - 2 \text{ if } i \in U_S, \end{cases}$$

we restate the Lagrangian dual problem for a given $S \subseteq V$ as

$$\max_{\pi \geq 0} w(\pi, S)$$

where,

$$w(\pi, S) \quad = \min_{x \in F(S), x \text{ is an integer}} [C_S(x) + \sum_{i \in S} \pi_i v_i(x, S)]. \tag{14}$$

We first state a result that relates C_S^{lp} (the optimal LP relaxation cost), $\max_{\pi \geq 0} w(\pi, S)$ (the optimal Lagrangian dual cost) and C_S^{opt} (the optimal integer programming cost).

Proposition II.1: For any $S \subseteq V$ with $|D_S| \geq 1$,

$$C_S^{lp} \leq \max_{\pi \geq 0} w(\pi, S) \leq C_S^{opt}. \tag{15}$$

Proof: This follows from a known result for integer programs. Refer to page 13 in Fisher [5] or page 330 in Nemhauser and Wolsey [15]. In general, for integer programs with minimization objective, the optimal Lagrangian dual cost can at most be equal to the optimal integer programming cost. Also, the optimal LP relaxation cost can at most be equal to the optimal Lagrangian dual cost.

■

Proposition II.2: The Lagrangian dual problem, $\max_{\pi \geq 0} w(\pi, V)$, can be solved in time polynomial in $n + k$ and $\log nC_{max}$ using the Ellipsoid method.

Proof: This theorem is proven using the results in Grotschel et al. [8]. The Lagrangian dual problem, $\max_{\pi \geq 0} w(\pi, V)$, can also be written as a linear program as follows:

$$\max_{t, \pi \geq 0} \ t$$

$$t \leq C_V(x) + \sum_{i \in V} \pi_i v_i(x, V), \quad \forall \ x \text{ where } x \text{ is a constrained forest}. \tag{16}$$

Using Lemma V.1 proved in the appendix, adding the constraints $0 \leq t \leq nC_{max}$ and $0 \leq \pi_i \leq nC_{max} \ \ \forall i \in V$ to the linear program above does not change its optimal cost. Now, define

$$\begin{aligned} \mathcal{P} := \{t, \pi \quad : \quad & t \leq C_V(x) + \sum_{i \in V} \pi_i v_i(x, V), \forall \ x \text{ where } x \text{ is a constrained forest}, \\ & 0 \leq t \leq nC_{max}, \\ & 0 \leq \pi_i \leq nC_{max} \quad \forall i \in V\}. \end{aligned} \tag{17}$$

To show that the linear program (16) is solvable using the Ellipsoid method, we need to show the following:

- The following separation problem is solvable in polynomial time:

Given any $t^* \in \mathbb{Q}$ and $\pi^* \in \mathbb{Q}^{|V|}$, decide whether $t^*, \pi^* \in \mathcal{P}$ and if not find a violated constraint.

- Let $B(y,\rho)$ for $y \in \mathbb{R}^{|V|+1}, \rho \in \mathbb{R}^+$ be a ball of radius ρ centered at y where $\mathbb{R}$ denotes the set of all real numbers. There exists $y_1, y_2 \in \mathbb{R}^{|V|+1}$ and $\rho_1, \rho_2 \in \mathbb{R}^+$ where $\log \rho_1, \log \rho_2$ are polynomial functions of the input such that $B(y_1,\rho_1) \subseteq \mathcal{P} \subseteq B(y_2,\rho_2)$.

Since the constrained forest problem is solvable in polynomial time ([7], [6], [21]), we know that the separation problem is solvable in polynomial time. To show that a ball is contained in $\mathcal{P}$ choose $t_o = \frac{C_{min}}{n+k}$, $\pi_i^o = \frac{C_{min}}{n+k}$ $\forall i \in V$ and $\rho_1 = \frac{C_{min}}{n+k}$. $C_{min} > 0$ by assumption. Let the center of the ball be $y_1 = (t_o, \pi_1^o, \cdots, \pi_{n+k}^o)$. Lemma V.2 in the appendix proves that $B(y_1,\rho_1) \subseteq \mathcal{P}$. As in the definition of $\mathcal{P}$, all the variables, $(t, \pi_1, \cdots, \pi_{n+k})$, are positive and upper bounded by nC_{max}. Therefore by choosing $\rho_2 = nC_{max}$, one can conclude that $\mathcal{P} \subseteq B(0,\rho_2)$.

Since $\mathcal{P}$ is full dimensional, bounded and the separation problem is solvable in polynomial time, one can use the results in Grotschel et al. [8] to conclude that the linear program formulated in (16) is solvable in time polynomial in the number of vertices (i.e. $n+k$) and $\log \rho_2$ (i.e. $\log nC_{max}$) using the Ellipsoid method. ■

We now state a result regarding the decomposition of the optimal Lagrangian dual cost of the GMDHPP. A similar result was also shown for a different multiple depot, terminal HPP in [20].

Proposition II.3: Let (π^*, x^*) solve the Lagrangian dual problem, $\max_{\pi \geq 0} w(\pi, V)$, where

$$w(\pi, V) = \min_{x \in F(V), x \text{ is an integer}} [C_V(x) + \sum_{i \in V} \pi_i v_i(x, V)].$$

Let $P_i(x^*)$ be the set of all vertices present in the i^{th} tree of the optimal solution x^*. Then,

$$\max_{\pi \geq 0} w(\pi, V) = \sum_{m=1,\cdots,k} \max_{\pi^m \geq 0} w(\pi^m, P_m(x^*)),$$

where π^m represents π_i for all $i \in P_m(x^*)$.

Proof: Refer to the appendix. ■

III. Approximation algorithm for GMDHPP

We generate a feasible solution, x^{approx}, for the GMDHPP using the following algorithm called *approx*:

- Solve the Lagrangian dual problem,

$$\max_{\pi \geq 0} \min_{x \in F(V), x \text{ is an integer}} [C_V(x) + \sum_{i \in V} \pi_i v_i(x, V)]$$

using the Ellipsoid method. Let (π^*, x^*) be the corresponding optimal solution. x^* is a disjoint forest with k trees such that a depot vertex is contained in each tree. Let $P_i(x^*)$ be the set of vertices present in the i^{th} tree. Note that $P_i(x^*) \bigcap P_j(x^*) = \emptyset$ for $i \neq j$ and $\bigcup_{i \in 1,\cdots,k} P_i(x^*) = V$. Let the depot vertex present in $P_i(x^*)$ be denoted as r_i. We use Hoogeveen's approximation algorithm [12] to construct a path for each salesman $i \in \{1, \cdots, k\}$ as follows:

1. If $|P_i(x^*)| = 1$, then $P_i(x^*)$ is a trivial component and contains only the depot vertex r_i. Hence, we need not proceed further for this component.
2. Find the minimum spanning tree T_i corresponding to the vertices in $P_i(x^*)$.

3. Find the set of odd degree vertices S_i of T_i. Note that the number of odd degree vertices $|S_i|$ is even.
4. If the depot vertex r_i present in T_i has even degree, then let $S_i := S_i \bigcup \{r_i\}$ else let $S_i := S_i \backslash \{r_i\}$.
5. Find the minimum cost matching M_i on S_i of cardinality $\frac{|S_i|-1}{2}$.
6. Now consider the graph $G_i = (P_i(x^*), E_i)$ where E_i is the union of all the edges present in T_i and M_i. G_i is connected and has either 0 or 2 odd degree vertices.
– If there are 2 odd degree vertices, then one of the 2 odd degree vertices must be the depot vertex r_i.
– If there are 0 odd degree vertices, then remove an edge e_i joining the depot vertex r_i to any other vertex in $P_i(x^*)$. Then the resulting graph $G_i = (P_i(x^*), E_i \backslash \{e_i\})$ would have two odd degree vertices with the depot vertex being one of them.
7. Construct an Eulerian path that traverses each edge in G_i exactly once. The Eulerian path will have the two odd degree vertices as its end points. This construction can be done using the algorithms given in [12],[25].
8. Shortcut this Eulerian path to produce a Hamiltonian path starting at depot vertex r_i and visiting all the vertices present in $P_i(x^*)$. If the costs satisfy the triangle inequality, the cost of this Hamiltonian path will be less than or equal to the cost of the Eulerian path.

The following is the main result of this paper:

Theorem III.1: Let the costs be symmetric and satisfy the triangle inequality. Algorithm *approx* provides a feasible solution, x^{approx}, with an approximation ratio of $\frac{3}{2}$ for GMDHPP and is solvable in time polynomial in $n+k$ and $\log nC_{max}$.

IV. Proof of theorem III.1

The computational complexity of algorithm *approx* is dominated by the Ellipsoid method which is solvable in time polynomial in $n+k$ and $\log nC_{max}$ by Proposition II.2. The rest of the section constructs the necessary results to prove the following:

If the costs satisfy the triangle inequality, then

$$C_V^{opt} \leq \sum_{i,j \in V, i<j} C_{ij} x_{ij}^{approx} \leq \frac{3}{2} C_V^{lp}.$$

The total cost of the feasible solution, x^{approx}, is upper bounded by

$$\sum_{i \in 1,\cdots,k} (C(T_i) + C(M_i)) \tag{18}$$

where $C(T_i)$ and $C(M_i)$ represent the total cost of the edges present in T_i and M_i respectively.

Proposition IV.1: (**Bound on the cost of spanning trees**)

$$\sum_{i=1,\cdots,k} C(T_i) \leq C_V^{opt}. \tag{19}$$

Proof: $P_i(x^*)$ has one depot vertex. Therefore the path elimination constraints present in $F(P_i(x^*))$ between any two depot vertices are trivially satisfied. Remaining constraints in $F(P_i(x^*))$ describe the spanning tree constraints corresponding to $P_i(x^*)$ with a degree constraint on the depot vertex r_i. Since T_i is the minimum spanning tree corresponding to the vertices in $P_i(x^*)$ with no degree constraints,

$$C(T_i) \quad \leq \quad \min_x \{C_{P_i(x^*)}(x) : x \in F(P_i(x^*)), x \text{ is an integer}\}. \tag{20}$$

Let π^i represent π_j for all $j \in P_i(x^*)$. If all the penalizing variables are zero (*i.e.* $\pi^i = 0$), then note that $w(0, P_i(x^*)) = \min_x \{C_{P_i(x^*)}(x) : x \in F(P_i(x^*)), x \text{ is an integer}\}$ by definition. Therefore,

$$\begin{aligned} C(T_i) &\leq \min_x \{C_{P_i(x^*)}(x) : x \in F(P_i(x^*)), x \text{ is an integer}\} \\ &= w(0, P_i(x^*)) \\ &\leq \max_{\pi^i \geq 0} w(\pi^i, P_i(x^*)). \end{aligned} \tag{21}$$

From equation (21) and Propositions II.3, II.1,

$$\begin{aligned} \sum_{i=1,\cdots,k} C(T_i) &\leq \sum_{i=1,\cdots,k} \max_{\pi^i \geq 0} w(\pi^i, P_i(x^*)) \\ &= \max_{\pi \geq 0} w(\pi, V) \\ &\leq C_V^{opt}. \end{aligned} \tag{22}$$

■

In the following subsection we show that $\sum_{i=1,\cdots,k} C(M_i)) \leq \frac{1}{2} C_V^{opt}$.

A. Bound on the cost of matching

We first show that for a single salesman HPP the cost of matching is upper bounded by half the optimal LP relaxation cost of the HPP. Consider a matching problem on a set of vertices denoted by $\bar{V}$. Assuming that $|\bar{V}|$ is odd, the objective of the minimum cost matching problem is to find a matching M with $\frac{|\bar{V}|-1}{2}$ edges that has minimum cost. Due to Edmonds (1965), this matching problem can be formulated as a linear program as follows:

$$C(M) := \min \sum_{i \in \bar{V}, j \in \bar{V}, i<j} C_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j \in \bar{V}, i<j} x_{ij} + \sum_{j \in \bar{V}, j<i} x_{ji} &\leq 1 \text{ for all } i \in \bar{V}, \\ \sum_{i \in \bar{V}, j \in \bar{V}, i<j} x_{ij} &= \frac{|\bar{V}|-1}{2}, \\ \sum_{i \in R, j \in R, i<j} x_{ij} &\leq \frac{|R|-1}{2}, \text{ for all } R \subset \bar{V}, |R| \geq 3, |R| \text{ odd}, \\ 0 \leq x_{ij} &\leq 1 \text{ for all } i, j \in \bar{V}, i<j. \end{aligned} \tag{23}$$

Now, consider the Hamiltonian path problem of finding a minimum cost path that visits each vertex in $\bar{V}$ exactly once. In this path problem note that the start or the end vertex of the path is not specified. A integer programming formulation of this Hamiltonian path problem is

$$\min \sum_{i \in \bar{V}, j \in \bar{V}, i<j} C_{ij} x_{ij} \tag{24}$$

$$\sum_{j\in\bar{V},i<j} x_{ij} + \sum_{j\in\bar{V},j<i} x_{ji} \;\leq\; 2 \text{ for all } i \in \bar{V}, \tag{25}$$

$$\sum_{i\in\bar{V},j\in\bar{V},i<j} x_{ij} \;=\; |\bar{V}| - 1, \tag{26}$$

$$\sum_{i\in R,j\in R,i<j} x_{ij}, \;\leq\; |R| - 1, \text{ for all } R \subseteq \bar{V}, \tag{27}$$

$$x_{ij} \;\in\; \{0,1\} \text{ for all } i,j \in \bar{V}, i<j. \tag{28}$$

$$\tag{29}$$

Consider a LP relaxation of the above problem where the constraints $x_{ij} \in \{0,1\}\ \forall i,j \in \bar{V}, i<j$ are replaced with $x_{ij} \geq 0\ \ \forall i,j \in \bar{V}, i<j$. Let $C^{HPP}_{\bar{V}}$ be the optimal cost of this LP relaxation.

Proposition IV.2: $C(M) \leq \frac{1}{2}C^{HPP}_{\bar{V}}$.

Proof: If x is a feasible solution to the LP relaxation of the Hamiltonian Path Problem (24-28), then $\frac{x}{2}$ is also a feasible solution to the matching problem. Hence, $C(M) \leq \frac{1}{2}C^{HPP}_{\bar{V}}$.

■

Proposition IV.3: If the costs satisfy triangle inequality, then for any $\bar{V} \subseteq V'$, $C^{HPP}_{\bar{V}} \leq C^{HPP}_{V'}$.

Proof: A Lagrangian dual to the HPP (24-28) is

$$\max_{\pi\geq 0} LD(\pi,\bar{V}), \tag{30}$$

where,

$$LD(\pi,\bar{V}) = \min \sum_{i\in\bar{V},j\in\bar{V},i<j} (C_{ij} + \pi_i + \pi_j)x_{ij} - 2\sum_{i\in\bar{V}} \pi_i$$

$$\begin{aligned}\sum_{i\in\bar{V},j\in\bar{V},i<j} x_{ij} &= |\bar{V}| - 1,\\ \sum_{i\in R,j\in R,i<j} x_{ij} &\leq |R| - 1, \text{ for all } R \subseteq \bar{V},\\ x_{ij} &\in \{0,1\} \text{ for all } i,j \in \bar{V}, i<j.\end{aligned} \tag{31}$$

Also, consider the integer program of a minimum spanning tree problem with the objective formulated in equation (24) and the constraints formulated in (26-28). It is well known (Lawler [14]) that the extreme points of the LP relaxation of this integer program is in one to one correspondence with the set of trees spanning all the vertices in $\bar{V}$. Hence, solving the LP relaxation itself produces the optimal spanning tree. Because of this integrality property, using corollary (6.6) in Nemhauser and Wolsey [15] or the results in Fisher [5], it follows that

$$\max_{\pi\geq 0} LD(\pi,\bar{V}) = C^{HPP}_{\bar{V}}. \tag{32}$$

Now, consider the HPP on the set of vertices $V' := \bar{V}\bigcup\{b\}$. The aim is to show that $C^{HPP}_{\bar{V}} \leq C^{HPP}_{V'}$ if the costs satisfy the triangle inequality. By equation (32), we essentially want to prove that $\max_{\pi\geq 0} LD(\pi,\bar{V}) \leq \max_{\pi'\geq 0} LD(\pi',V')$. Let us prove this by contradiction. Let us assume that $\max_{\pi\geq 0} LD(\pi,\bar{V}) > \max_{\pi'\geq 0} LD(\pi',V')$. Let π^* solve $\max_{\pi\geq 0} LD(\pi,\bar{V})$. Let $\pi_b \geq 0$ be the weight on vertex b. Let π' be such that $\pi'_i = \pi^*_i$ for $i \in \bar{V}$ and $\pi'_b = \pi_b$. For any arbitrary π_b, $\max_{\pi\geq 0} LD(\pi,\bar{V}) > LD(\pi',V')$ by assumption. Given π^*, let a optimal spanning tree corresponding to the minimization problem in $LD(\pi',V')$ be denoted by $T(\pi_b)$. Let us consider the following two cases:

- *Let $\pi_b = 0$. There exists no optimal spanning tree, $T(0)$, with the degree of vertex b greater than 1:*

In this case, vertex b is a leaf in the tree $T(0)$. Let b be connected to vertex q in $T(0)$. Removing the edge joining vertex b and q will result in a tree, $\bar{T} = T(0)\backslash(b,q)$, spanning the vertices in $\bar{V}$. By definition,

$$\begin{aligned} LD(\pi',V') &= \sum_{(i,j)\in T(0)} (C_{ij} + \pi'_i + \pi'_j) - 2\sum_{i\in V'} \pi'_i \\ &= C_{bq} + \pi_b + \pi_q + \sum_{(i,j)\in \bar{T}} (C_{ij} + \pi'_i + \pi'_j) - 2\sum_{i\in \bar{V}} \pi'_i - 2\pi_b \end{aligned} \tag{33}$$

Since, $\pi_q \geq 0$ and $\pi_b = 0$, we get,

$$\begin{aligned} LD(\pi',V') &\geq \sum_{(i,j)\in \bar{T}} (C_{ij} + \pi'_i + \pi'_j) - 2\sum_{i\in \bar{V}} \pi'_i \\ &= \sum_{(i,j)\in \bar{T}} (C_{ij} + \pi^*_i + \pi^*_j) - 2\sum_{i\in \bar{V}} \pi^*_i \\ &\geq \max_{\pi\geq 0} LD(\pi,\bar{V}). \end{aligned} \tag{34}$$

- *Let $\pi_b = 0$. The degree of vertex b in every optimal spanning tree, $T(0)$, is at least 2:*

By suitably increasing π_b, there exists a $\pi_b = \delta \geq 0$, such that there is at least one optimal spanning tree, $T(\delta)$, where the degree of vertex b is exactly equal to 2 (Refer to Lemma 3 of Shmoys and Williamson [23]). Consider such an optimal tree and let p and q be the two vertices connected to vertex b in the same. Let $\bar{T} = T(\delta) \bigcup (p,q)\backslash\{(b,p),(b,q)\}$ denote the tree spanning vertices in $\bar{V}$.
By definition,

$$\begin{aligned} LD(\pi',V') &= \sum_{(i,j)\in T(\delta)} (C_{ij} + \pi'_i + \pi'_j) - 2\sum_{i\in V'} \pi'_i \\ &= C_{pb} + C_{qb} + \pi^*_p + \pi^*_q + 2\delta + \sum_{(i,j)\in T(\delta)\backslash\{(b,p),(b,q)\}} (C_{ij} + \pi^*_i + \pi^*_j) \\ &\quad -2\sum_{i\in \bar{V}} \pi^*_i - 2\delta. \end{aligned} \tag{35}$$

Since the costs satisfy the triangle inequality, $C_{pb} + C_{qb} \geq C_{pq}$. Therefore,

$$
\begin{aligned}
LD(\pi', V') &\geq C_{pq} + \pi_p^* + \pi_q^* + \sum_{(i,j)\in T(\delta)\setminus\{(b,p),(b,q)\}} (C_{ij} + \pi_i^* + \pi_j^*) - 2\sum_{i\in\bar{V}} \pi_i^* \\
&= \sum_{(i,j)\in\bar{T}} (C_{ij} + \pi_i^* + \pi_j^*) - 2\sum_{i\in\bar{V}} \pi_i^* \\
&\geq \max_{\pi\geq 0} LD(\pi, \bar{V}).
\end{aligned}
\tag{36}
$$

In the above argument we have shown that there exists a $\pi_b \geq 0$ where $LD(\pi', V') \geq \max_{\pi\geq 0} LD(\pi, \bar{V})$. Hence the assumption must be false. Therefore $\max_{\pi\geq 0} LD(\pi, \bar{V}) \leq \max_{\pi'\geq 0} LD(\pi', V')$.

■

Proposition IV.4: Suppose $\bar{V} \subseteq V'$ and $|\bar{V}|$ is odd. Let M be the minimum cost matching on $\bar{V}$ with cardinality $\frac{\bar{V}-1}{2}$. If the costs satisfy the triangle inequality, then $C(M) \leq \frac{1}{2} C_{V'}^{HPP}$.

Proof: Follows from Propositions IV.2 and IV.3.

Proposition IV.5: (**Bound on the cost of matching**) Suppose $S_i \subseteq P_i(x^*)$ and $|S_i|$ is odd. Let M_i be the minimum cost matching on S_i with cardinality $\frac{S_i-1}{2}$. If the costs satisfy the triangle inequality, then $\sum_{i=1,\cdots,k} C(M_i) \leq \frac{1}{2} C_V^{opt}$.

Proof: Note that the Hamiltonian path problem considered in equations (24-28) does not require the path to start at any depot vertex. Enforcing a constraint that that the path should start at the depot vertex can only increase the optimal LP relaxation cost. Therefore, $C_{P_i(x^*)}^{HPP} \leq C_{P_i(x^*)}^{lp}$. Therefore,

$$
\begin{aligned}
\sum_{i=1,\cdots,k} C(M_i) &\leq \frac{1}{2} \sum_{i=1,\cdots,k} C_{P_i(x^*)}^{HPP} \quad \text{(from Proposition IV.4)} \\
&\leq \frac{1}{2} \sum_{i=1,\cdots,k} C_{P_i(x^*)}^{lp} \\
&\leq \frac{1}{2} \sum_{i=1,\cdots,k} \max_{\pi^i\geq 0} w(\pi^i, P_i(x^*)) \quad \text{(from Proposition II.1)} \\
&= \frac{1}{2} \max_{\pi\geq 0} (\pi, V) \quad \text{(from Proposition II.3)} \\
&\leq \frac{1}{2} C_V^{opt}. \quad \text{(from Proposition II.1)}
\end{aligned}
\tag{37}
$$

■

Combining Propositions IV.5 and IV.1 proves theorem III.1.

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V. Appendix

Lemma V.1: The optimal solution (t^*, π^*) to the linear program formulated in (16) satisfies the following constraints:

$$0 \leq t^* \leq nC_{max} \text{ and } 0 \leq \pi_i^* \leq nC_{max} \ \forall i \in V.$$

Proof: The optimal cost of the Lagrangian dual problem is always upper bounded by the optimal cost of the GMDHPP. Therefore, $t^* \leq C_V^{opt} \leq nC_{max}$. Also since the cost of all the edges are positive, $t^* \geq 0$.

Since $k \geq 2$, one can always choose a constrained forest, x_f, such that

- all depot vertices in x_f have degree equal to 0 except one depot vertex, s, that has degree 1, and
- all destination vertices in x_f have degree equal to 2 except one destination vertex, q, that has degree 1.

For such a constrained forest, $v_i(x_f, V) = 0, \ \forall \ i \in \{s\} \bigcup \{k+1, k+2, \cdots, n\} \backslash \{q\}$ and $v_i(x_f, V) = -1, \ \forall \ i \in \{q\} \bigcup \{1, 2, \cdots, k\} \backslash \{s\}$. Therefore we have, using equation (16),

$$t^* \leq C_V(x_f) - \sum_{i \in \{1, \cdots, k\} \backslash \{s\}} \pi_i^* - \pi_q^*.$$

Since $t^* \geq 0$, and $C_V(x_f) \leq nC_{max}$,

$$\sum_{i \in \{1, \cdots, k\} \backslash \{s\}} \pi_i^* + \pi_q^* \leq C_V(x_f) - t^* \leq nC_{max}. \quad (38)$$

Using the above equation, and by suitably choosing the depot vertex s and destination vertex q, we can conclude that each π_i^* can be upper bounded by nC_{max}. ■

Lemma V.2: Let $t_o = \frac{C_{min}}{n+k}$, $\pi_i^o = \frac{C_{min}}{n+k}$ $\forall i \in V$ and $\rho_1 = \frac{C_{min}}{n+k}$. Let the center of the ball be $y_1 = (t_o, \pi_1^o, \cdots, \pi_{n+k}^o)$. Then $B(y_1, \rho_1) \subseteq \mathcal{P}$.

Proof: Consider any $t, \pi \in B(y_1, \rho_1)$. Then $0 \leq t \leq \frac{2C_{min}}{n+k}$ and $0 \leq \pi_i \leq \frac{2C_{min}}{n+k}$ for all $i \in V$. Also, for any constrained forest x, $v_i(x, V) \geq -1$ for all $i \in V$. Now consider the right hand side of the constraints in the definition of $\mathcal{P}$ (equation 17):

$$\begin{aligned} C_V(x) + \sum_{i \in V} \pi_i v_i(x, V) &\geq C_V(x) - \sum_{i \in V} \pi_i \\ &\geq nC_{min} - 2C_{min}. \end{aligned}$$

Since $n \geq 3$ and $k \geq 2$,

$$\begin{aligned} C_V(x) + \sum_{i \in V} \pi_i v_i(x, S) &\geq C_{min} \\ &> 2C_{min}/5 \\ &\geq t. \end{aligned}$$

If $t, \pi \in B(y_1, \rho_1)$, it is easy to see that $0 \leq t \leq nC_{max}$ and $0 \leq \pi_i \leq nC_{max}$ $\forall i \in V$. Hence, if t, π $\in B(y_1, \rho_1)$ then $t, \pi \in \mathcal{P}$. This proves the Lemma. ■

Proposition II.3: Let (π^*, x^*) solve the Lagrangian dual problem, $\max_{\pi \geq 0} w(\pi, V)$, where

$$w(\pi, V) = \min_{x \in F(V), x \text{ is an integer}} [C_V(x) + \sum_{i \in V} \pi_i v_i(x, V)].$$

Let $P_i(x^*)$ be the set of all vertices present in the i^{th} tree of the optimal solution x^*. Then,

$$\max_{\pi \geq 0} w(\pi, V) = \sum_{m=1,\cdots,k} \max_{\pi^m \geq 0} w(\pi^m, P_m(x^*)),$$

where π^m represents π_j for all $j \in P_m(x^*)$.

Proof: For $m = 1, \cdots, k$, let r_m denote the depot vertex present in $P_m(x^*)$. We define the following additional constraints on x using $P_m(x^*), m = 1, .., k$ as follows:

$$\begin{aligned} x_{ij} &= 0, \quad \text{for all } i \in P_m(x^*), \quad \text{for all } j \in P_l(x^*), \\ & \quad\ \text{for all } m, l = 1, \cdots, k \text{ and } m \neq l, \\ \sum_{j \in U_{P_m(x^*)}} x_{r_m j} &\leq p, \quad m = 1, \cdots, k, \\ \sum_{i,j \in P_m(x^*), i<j} x_{ij} &= |P_m(x^*)| - 1, \quad m = 1, \cdots, k, \\ \sum_{i,j \in B, i<j} x_{ij} &\leq |B| - 1, \text{ for all } B \subseteq P_m(x^*), \quad m = 1, \cdots, k, \\ x_{ij} &\in \{0, 1\}, \quad \text{for all } i, j \in P_m(x^*),\ i < j,\ m = 1, \cdots, k. \end{aligned} \tag{39}$$

Let all the constraints in (39) be denoted by $A_c x \leq B_c$. Let $F^*(V) = \{x : x \in F(V) \text{ and } A_c x \leq B_c\}$. Adding these constraints based on x^* to the constraints present in $F(V)$ will not change the optimal Lagrangian dual cost, $\max_{\pi \geq 0} w(\pi, V)$. That is,

$$\begin{aligned}\max_{\pi\geq 0} w(\pi, V) &= \max_{\pi\geq 0} \min_{x\in F(V), x \text{ is an integer}} [C_V(x) + \sum_{i\in V} \pi_i v_i(x, V)] \\ &= \max_{\pi\geq 0} \min_{x\in F^*(V), x \text{ is an integer}} [C_V(x) + \sum_{i\in V} \pi_i v_i(x, V)].\end{aligned} \tag{40}$$

We next decompose the objective function and the set of feasible solutions. If $x \in F^*(V)$, then $x_{ij} = 0$ for all $i \in P_m(x^*), j \in P_l(x^*), m, l \in \{1, \cdots, k\}, m \neq l$. Therefore, equation (40) can be written as

$$\max_{\pi\geq 0} w(\pi, V) = \max_{\pi\geq 0} \min_{x\in F^*(V), x \text{ is an integer}} \sum_{m=1,\cdots,k} [C_{P_m(x^*)}(x) + \sum_{i\in P_m(x^*)} \pi_i v_i(x, P_m(x^*))]. \tag{41}$$

Let $F_1 \times F_2$ denote the cartesian product of two sets F_1 and F_2. Then, $x \in F^*(V)$ if and only if $(x^1, x^2, \cdots, x^k) \in \times_{m=1}^{k} F(P_m(x^*))$ where each x^m represents all x_{ij} for $i, j \in P_m(x^*), i < j$. Also let

$$\mathfrak{G}_m(x^m, \pi^m) = C_{P_m(x^*)}(x) + \sum_{i\in P_m(x^*)} \pi_i v_i(x, P_m(x^*)).$$

Hence, equation (41) can be further simplified as follows:

$$\max_{\pi\geq 0} w(\pi, V) = \max_{\pi\geq 0} \min_{(x^1,\cdots,x^k)\in\times_{m=1}^{k} F(P_m(x^*)), x^m \text{ is an integer}} \sum_{m=1,\cdots,k} \mathfrak{G}_m(x^m, \pi^m).$$

This breaks the inner minimization into k independent minimization problems implying

$$\max_{\pi\geq 0} w(\pi, V) = \max_{\pi\geq 0} \sum_{m=1,\cdots,k} \min_{x^m\in F(P_m(x^*)), x^m \text{ is an integer}} \mathfrak{G}_m(x^m, \pi^m).$$

Applying the same reasoning to the maximization problem we get

$$\begin{aligned}\max_{\pi\geq 0} w(\pi, V) &= \sum_{m=1,\cdots,k} \max_{\pi^m\geq 0} \min_{x^m\in F(P_m(x^*)), x^m \text{ is an integer}} \mathfrak{G}_m(x^m, \pi^m) \\ &= \sum_{m=1,\cdots,k} \max_{\pi^m\geq 0} w(\pi^m, P_m(x^*)).\end{aligned}$$

■