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BY

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TIEBOUT BIAS AND THE DEMAND FOR LOCAL PUBLIC SCHOOLING

Daniel Rubinfeld, Perry Shapiro and Judith Roberts*

February 1986

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 $\label{eq:2} \begin{split} \frac{d\mathbf{r}}{d\mathbf{r}}&= \frac{1}{2}\mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) + \frac{1}{2}\mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) \mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) \mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) \mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) \mathbf{r}_{\mathrm{eff}}\left(\mathbf{r}_{\mathrm{eff}}\right) \mathbf{r}_{\mathrm{eff}}\left$ $\label{eq:2.1} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} \frac{d\mu}{\sqrt{2\pi}} \left(\frac{d\mu}{\mu} \right)^2 \frac{d\mu}{\mu} \left(\frac{d\mu}{\$ $\sim 10^{-1}$ $\mathcal{A}^{\text{max}}_{\text{max}}$

Tiebout Bias and the Demand for Local Public Schooling

ABSTRACT

The estimation of the demand for public goods has long been a concern of public finance economists. Until recently demand estimates were obtained, either with aggregate or micro survey data, using single equation estimation techniques. However, demand estimates may be biased when individuals' choices of communities are dependent upon the quantity and quality of public good provided. This paper spells out the nature of this bias (called Tiebout bias), and suggests an improved maximumlikelihood estimation technique. The technique is applied to a data set involving local public education in Michigan.

 $\hat{\mathcal{L}}$ $\label{eq:2.1} \frac{1}{2}\sum_{i=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\frac{1}{2}\sum_{j=1}^n\$

 \sim

 $\sim 10^{11}$ km $^{-1}$ $\bar{\mathcal{E}}$

 \mathcal{L}_{max}

$1.$ Introduction

The estimation of household demands for public goods has long been a concern of public finance economists. Until relatively recently, demand estimates were obtained primarily using aggregate data and single equation estimation techniques. However, Goldstein and Pauly (1982) have argued that these estimates may be biased if individuals sort themselves into communities in part on the basis of local public sector activity. They illustrate the possibility of such bias -- called Tiebout bias -- using a model of demand in which income is the single explanatory variable. The model suggests that, under reasonable conditions, previous estimates of income elasticities obtained from aggregate data are likely to be biased upward.

A new approach to the estimation of demand for public goods, suggested by Bergstrom, Rubinfeld and Shapiro (1982), uses micro data collected from a survey of individuals. Like its predecessors based on aggregate data, Bergstrom, Rubinfeld and Shapiro's approach employs a single equation technique to estimate demand parameters. In comparison to most aggregate models, however, this approach produces income and price elasticities which are relatively low.

This paper presents an argument similar to that made by Goldstein and Pauly in the context of aggregate models, raising the possibility that micro-based estimates might also be subject to a kind of Tiebout bias. Simply put, the bias arises because single equation models of demand fail to account for the sorting of individuals among communities. We make the argument in the context of Bergstrom, Rubinfeld and Shapiro's model.

The micro approach has substantial promise because the data are rich enough to allow one to test for Tiebout bias, and to the extent that it is present, devise new and better estimation techniques. This paper provides a start in that direction by suggesting the conditions under which Tiebout bias might

occur and by attempting to estimate the extent to which this bias may be present. The paper goes further by suggesting a model structure in which more detailed questions concerning the specification and estimation of the demand for local public goods can be answered. The structure is sufficiently broad to allow for the inclusion of political as well as economic determinations of the demand and supply of local public goods.

Section 2 contains a brief heuristic discussion of the Tiebout bias problem in the context of both micro and aggregate data. The argument is meant to illustrate both the source of the potential bias due to the community selection process as well as the direction of bias. In Section 3 the theoretical econometric discussion of Tiebout bias in the micro model is presented. The theory both suggests the source of bias, and allows us to develop a consistent and efficient estimator of the demand function parameters. The estimator is similar in spirit to the two-stage estimators proposed by Heckman (1978), but substantially different in practice: our estimator involves a least-squares first-stage and an ordinal multinomial probit second stage and the solution is obtained using an iterative procedure. Section 4 contains the empirical analysis of the demand for local public schooling with and without Tiebout bias The results make it clear that Tiebout bias can be an important corrections. problem, but that the estimates of its importance are sensitive to the model of community choice and public goods choice. Some brief concluding comments appear in Section 5.

The Tiebout Bias Problem - A Heuristic Description $2.$

Demand functions for public goods such as education have historically been estimated using aggregate data and single-equation estimation procedures. A

 $\overline{2}$

typical approach involves relating the aggregate outcome in terms of dollars per pupil of school spending to the indicators of the demographic and economic composition of the relevant populations. In order for demand functions to be inferred from such data, a political theory, typically the "median-voter" model is invoked to relate the expenditures of a jurisdiction to the characteristics of its population. The "median-voter" is usually taken to be the individual with the median income, residing in a house with median house value in a community with no renter population.

Goldstein-Pauly (1981) have raised serious questions concerning the application of the median voter concept. Their argument builds upon the notion that the population of each jurisdiction should not be taken as exogenous. Instead individuals will sort themselves out among jurisdictions through migration. To the extent that the sorting is based on the level of public expenditures, rather than exogenous factors, the usual estimates of price and income elasticities will be biased upward.

The argument applies directly to median voter models which utilize aggregate data. Goldstein and Pauly suggest, however, that the use of individual (micro) data could eliminate the Tiebout bias problem. Thus, it would be natural to believe that the micro-based estimates of the demand for public goods of Gramlich and Rubinfeld (1982) and Bergstrom, Rubinfeld and Shapiro (1982) are not Tiebout biased. Ladd and Christopherson (1983) have argued to the contrary in the context of the Gramlich-Rubinfeld paper, while Olmsted (undated) has done the same with repsect to the Bergstrom-Rubinfeld-Shapiro paper.

The Tiebout bias problem in the micro context has nothing to do with the median voter model or with the use of median income as a variable in the demand equation. Rather it involves a direct application of the argument

 $\overline{3}$

that when simultaneous equations bias is a possibility (because of community sorting) that the error term in the demand equation (conditional on the explanatory variables) will no longer have zero expected value.

The nature of the Tiebout bias problem in the context of micro demand equations is illustrated in Figure 1. Assume that income and a random error term are the sole determinants of spending demand and that all individuals have income $x_1 = 2$ or income $x_2 = 6$. Furthermore assume that there are three communities, each supplying a different, fixed level of public spending, A_i . The spending levels are $A_1 = 1$, $A_2 = 3$ and $A_3 = 5$.¹ The desired spending level of each individual is $E_i = \frac{1}{2}x_i + \varepsilon_i$ and $\varepsilon_i \sim N(0,1)$.

Assume that all individuals with income $x_1 = 2$ reside in community 1 in which $A_1 = 1$, and that all those with income $x_2 = 6$ reside in community 2 in which $A_2 = 3$ (community 3 has no population). Then the micro approach to demand estimation yields consistent estimators. A random sample of individuals from the population would not contain any individuals who are getting exactly the level of public service that they desire. However, the errors associated with incorrectly assuming that all individuals desire the level of public service provided in the jurisdiction have expectation zero, and are unrelated to the income of the individuals sampled. A regression of actual spending, A_i , on income x_i for a random sample of individuals would yield a slope estimator with expected value equal to the slope obtained by connecting the points P and Q on the graph.

Now consider what happens if the jurisdictions are open to costless migration. People will sort themselves according to their demand for public goods as well as

 $\mathbf{1}$ The example ignores the problem caused by the budget constraint and nonnegativity conditions. This is done for illustrative purposes only; the addition of these conditions would not alter the qualitative conclusion.

their income. We impose a very simple model of individual choice in which people choose communities with an expenditure level closest to their desired amount. Therefore if $E_i \le 2$ the choice is A_i , if $2 < E_i < 4$ the choice is A_2 and if $E_i \ge 4$ the choice is A_3 . The relation between A and x is

$$
A_i = \frac{1}{2}X_i + \varepsilon_i + v_i
$$

The values of v_i , the difference between actual and desired expenditures, are depicted in Figure 1. From the distributional assumption made about ε_{i} , $E(v_i|x_1) > 0$ and $E(v_i|x_2) = 0$. Therefore a regression of A_i on x_i would violate the fundamental condition of orthogonality between x and the error $\varepsilon_i + v_i$. The resulting slope estimator would be biased. In fact, if A_i and x_i were the logarithms of the expenditures and income, the slope is the income elasticity of demand. In this case, with negative covariance between x_i and $\varepsilon_i + v_i$, the income elasticity, will be underestimated.²

Of course, the story is not complete as shown in Figure 1. For one thing, a similar story could be told about the price elasticity of demand as well as other demand parameters. For another, a more general model would allow for the level of public spending to be endogenous. We consider the more general model in Section 3, while at the same time illustrating how consistent parameter estimates can be obtained by a two-stage process, in which the first stage involves the consistent estimation of the conditional expectation of ε .

The Tiebout bias problem, therefore, can arise when either micro or macro data are being utilized. To illustrate the nature of the problem in more detail we have chosen to focus on the micro approach to the estimation of education demand functions as applied by Bergstrom-Rubinfeld-Shapiro (hereafter BRS).

This is the case discussed by Ladd and Christopherson (1983).

 $\overline{\mathbf{c}}$

We stress, however, that the general approach to the specification of demand models has broader application - to the estimation of the demand for all locally provided public goods (including air pollution and other neighborhood attributes).

Theoretical Analysis 3.

Assume that the demand for public school expenditures is given by:

$$
E_{i} = \beta_{0} + x_{1i} \beta + \varepsilon_{i} \tag{1}
$$

where E_i is the logarithm of individual i's desired per pupil expenditure on public schools; x_{1i} is a k_1x1 vector including socio-economic and demographic characteristics of the individual (including income, tax-price, children in school and race, for example) and school district characteristics β_0 is a constant and β is a k_1x1 vector of demand parameters. The random variable ε is distributed $N(0, \sigma_{\varepsilon}^2)$ and is assumed to be uncorrelated with all personal characteristics.

Not all individuals within a jurisdiction will get to consume the level of school expenditures that is desired, for a host of reasons relating to the fact that the pure model of Tiebout sorting is not descriptive of the real world. We represent the difference between the actual provision of per-pupil spending and desired spending by:

$$
A_{i} - E_{i} = \gamma_{0} + x_{1i}^{*} \gamma + x_{2i}^{*} \tau + u_{i} = v_{i}
$$
 (2)

where A_i is the logarithm of the jurisdiction's per pupil spending on education, x_2 is a k_2 xl vector of variables which relate to the sorting process but do not directly affect demand, γ is a k_1x1 , τ a k_2x1 vector of sorting parameters,

 $\overline{7}$

and u_i is a random disturbance term.

Note that the variables that explain sorting can include some of the demand-determining variables - e.g., income can affect demand and mobility as well. However, not all demand variables need appear in (2), since some of the x_{1i} can be constrained to have zero coefficients. We assume in our theoretical discussion that education is a pure public good so that A does not vary across individuals within a given jurisdiction. This assumption can be relaxed in the empirical work without much difficulty. To keep the analysis as clear as possible we have chosen to omit jurisdiction subscripts.

We assume that v is distributed $N(0, \sigma_v^2)$. The relationship between $(A_i - E_i)$ and the variables x_1 and x_2 is a complex one that might be related to the process by which individuals locate themselves among communities and the process by which A is politically determined. For example, we might expect $(A_i - E_i)$ to be low in absolute value for recent movers who had some selection among public service bundles in making their move. It is the fact that y may be nonzero as well as the fact that ε and v might be correlated that creates what we will call the Tiebout bias problem.

Of course, $(A_i - E_i)$ is not directly observable. However, BRS utilized the responses to a survey of 2001 Michigan voters conducted by Courant, Gramlich and Rubinfeld (1980) to obtain information about (A_i-E_i) . Each survey respondent was asked whether he or she wanted more, about the same, or less expenditures on public education, as well as his or her individual characteristics. A response of "more" was assumed to be made if the level of educational expenditure was sufficiently smaller than the desired level.³ "Same" was the response if the

A "more" response was recorded if and only if the respondent said more and then answered "yes" to a follow-up question: Your taxes will go up if there are larger expenditures; do you still want more?

actual level was sufficiently close to the desired amount, while "less" was the answer if expenditures were substantially larger than the desired level. These qualitative ranges can be specified in terms of the random variable v and a threshold level δ by the following three equations:

More if
$$
v \leftarrow \delta
$$
 (3)

$$
Same if $\delta \langle v \langle \delta \rangle$ (4)
$$

$$
Less if v > \delta
$$
 (5)

Substituting from (1) and (2) the conditions then become:

More if
$$
E_i > A_i + \delta
$$
 or $\epsilon_i > A_i - \beta_0 - x_{1i} + \delta$ (6)

$$
\text{Same if } A_i - \delta \leq E_i \leq A_i + \delta \qquad \text{ or } A_i - \beta_0 - x_{1i}^{\prime} \beta - \delta \leq \epsilon_i \leq A_i - \beta_0 - x_{1i}^{\prime} \beta + \delta \qquad (7)
$$

$$
\text{Less if } E_i < A_i - \delta \qquad \text{or } \varepsilon_i < A_i - \beta_0 - x_{1i} + \delta \qquad (8)
$$

Maximum-likelihood estimators of the parameters of the demand function can now be obtained if we make some additional assumptions about the distributions of the relevant variables. To allow for Tiebout sorting and its effects, we continue to assume that ε and x are uncorrelated. Likewise we assume that u is uncorrelated with both x_1 and x_2 . However, we require x_1 and x_2 to be normally distributed and for A and ε (and therefore v and ε) to be correlated. The complete set of assumptions is given as follows, with $x = (1, x_1, x_2)$ a kxl vector and $k = k_1 + k_2 + 1$.

 (x, A, ε) is distributed $N((x, \bar{A}, 0), \Sigma)$, where⁴:

$$
\Sigma = \begin{pmatrix} \Sigma_{\mathbf{x}} & \Sigma_{\mathbf{x}A} & 0 \\ \Sigma_{\mathbf{x}A} & \sigma_A^2 & \sigma_{A\varepsilon} \\ 0 & \sigma_{A\varepsilon} & \sigma_{\varepsilon}^2 \end{pmatrix}
$$

Here Σ _y is a kxk matrix, Σ _{xA} is kxl and 0 is kxl.

Using the normality of ε it follows directly that:

$$
\text{Prob (less)} = \int_{-\infty}^{L} f(\eta) d\eta \qquad \qquad \text{Prob (same)} = \int_{L}^{M} f(\eta) d\eta
$$

Prob (more) = $\int_{M} f(\eta) d\eta$, where $f(\cdot)$ is a standard normal density function

$$
L = [As - \beta0 - x1t, \beta - \delta - E(\epsilon[x, A)]/\sigmas
$$
 (9)

$$
M = [Ai - \beta0 - x1i2 \beta + \delta - E(\epsilon[x, A)]/\sigma\epsilon
$$
 (10)

where $E(\epsilon[x,A)$ represents the expectation of ϵ conditional on x and A, and the variance of ε is assumed to be independent of both x and A .

The maximum likelihood procedure in the BRS paper implicitly assumed that E was independent of A as well as x. In this special case the likelihood function to be maximized is given below (F is the cumulative normal distribution function):

4 Some of the variables in the vector x are discrete, rather than continuous so that the normality assumption is not appropriate. It would not be difficult, however, to expand the model to incorporate a vector z of discrete demand determinants. The individual z's would then be interpreted as mean-shifting variables, so that x_1 would become $x_1(z)$, a vector of demand variables whose mean was conditional upon the values taken by the elements of the z vector. We have not included this expanded model in the text solely to simplify the presentation.

$$
\mathcal{L} = \Pi \quad F(\theta_0^L + \theta_1 A_i + \theta_2 x_i) \quad \Pi \quad [F(\theta_0^M + \theta_1 A_i + \theta_2 x_i) - F(\theta_0^L + \theta_1 A_i + \theta_2 x_i)]
$$
\n
$$
\Pi \quad [1 - F(\theta_0^M + \theta_1 A_i + \theta_2 x_i)]
$$
\n
$$
i \in \text{More}
$$

The θ 's are parameters, with θ_0^L and θ_0^M reflecting the fact that the constant will vary by response category. Since,

$$
\theta_0^L = -(\beta_0 + \delta)/\sigma_{\varepsilon} \qquad \theta_0^M = -(\beta_0 - \delta)/\sigma_{\varepsilon}
$$

$$
\theta_1 = 1/\sigma_{\varepsilon} \qquad \theta_2 = -\beta/\sigma_{\varepsilon}
$$

consistent estimators of the demand parameters are given by:

$$
\hat{\beta}_0 = -(\hat{\theta}_0^L + \hat{\theta}_0^M)/2\hat{\theta}_1
$$

\n
$$
\hat{\delta} = -(\hat{\theta}_0^L + \hat{\theta}_0^M)/2\hat{\theta}_1
$$

\n
$$
\hat{\delta} = -(\hat{\theta}_0^L + \hat{\theta}_0^M)/2\hat{\theta}_1
$$

In the more general case in which Tiebout bias is a possibility $E(\epsilon[x,A)$ is no longer identically zero and, in particular, no longer independent of x and A. This possibility is illustrated by our previous example in Figure 1, where the values of ε _i, as well as v_i , are drawn. The following relationships can be seen to hold directly from Figure 1:

$$
E(\epsilon[x_1, A_1) < 0
$$
\n
$$
E(\epsilon[x_1, A_2) > 0
$$
\n
$$
E(\epsilon[x_1, A_2) > 0
$$
\n
$$
E(\epsilon[x_1, A_3) > 0
$$
\n
$$
E(\epsilon[x_2, A_2) = 0
$$
\n
$$
E(\epsilon[x_2, A_3) > 0
$$
\n
$$
E(\epsilon[x_2, A_3) > 0
$$
\n
$$
E(\epsilon[x_1, A_1) > E(\epsilon[x_2, A_1)
$$
\n
$$
E(\epsilon[x_1, A_3) > E(\epsilon[x_1, A_3) > E(\epsilon[x_2, A_3])
$$

In this case the maximum-likelihood estimation procedure becomes substantially more complex. The correct procedure can be developed if we first evaluate $E(\epsilon[x,A)$ in the general case, as is done in the lemmas which follow.

Lemma 1.
$$
E(\epsilon[x,A) = \lambda[(A-\overline{A}) - \Sigma_{xA}^* \Sigma_x^{-1}(x-\overline{x})]
$$

where:

$$
\lambda = \sigma_{A\epsilon} \frac{\det(\Sigma_{\mathbf{x}})}{\det\begin{pmatrix} \Sigma_{\mathbf{x}} & \Sigma_{\mathbf{x}A} \\ \Sigma_{\mathbf{x}A} & \sigma_A^2 \end{pmatrix}}
$$

The proof follows from a standard result from multivariate statis-Proof. tics for the conditional expectation of a normal variable.

Lemma 2.
$$
E(\epsilon[x,A) = \lambda[(A-\overline{A}) - (x_1^{\dagger} - \overline{x}_1^{\dagger})(\beta + \gamma) - (x_2^{\dagger} - \overline{x}_2^{\dagger})\tau]
$$

Proof. From (1) and (2),
$$
A = \beta_0 + \gamma_0 + x_1^*(\beta + \gamma) + x_2^* \tau + (u + \epsilon)
$$

Therefore, Σ_{xA}^{-2} Σ_x^{-1} = $(\beta_0 + \gamma_0, \beta + \gamma, \tau)$ and the result follows (recall that x_1 and x_2 are uncorrelated with both u and ε). In this case, the bounds of the integrals which go into the likelihood function (equations (9) and (10)) change in the following manner (we have substituted for $E(\epsilon[x,A)$ using lemma $2):$

$$
\mathbf{L}^* = \{ \mathbf{A}_i (1-\lambda) - \mathbf{B}_0 (1-\lambda) - \delta - \mathbf{x}_{1i} [\beta(1-\lambda) - \lambda \gamma] + \mathbf{x}_{2i}^* \mathbf{L} \lambda + \lambda (\bar{\mathbf{A}} - \mathbf{B}_0 - (\beta + \gamma) \bar{\mathbf{x}}_1 - \mathbf{t} \bar{\mathbf{x}}_2) \} / \sigma_e \tag{11}
$$

$$
M^* = \left\{ A_{i}(1-\lambda) - \beta_0(1-\lambda) + \delta - x_{1i} \left[\beta(1-\lambda) - \lambda \gamma \right] + x_{2i} \tau \lambda + \lambda (\bar{A} - \beta_0 - (\beta + \gamma)\bar{x}_1 - \tau \bar{x}_2) \right\} / \sigma_e \tag{12}
$$

In this formulation, σ_e^2 is the variance of ε conditional on x and A. Then it follows that

$$
\text{plim } \hat{\beta}_0 = \beta_0 + \frac{\lambda}{1 - \lambda} \tag{13}
$$

$$
\rho \lim_{\delta} \delta = \delta / (1 - \lambda) \tag{14}
$$

$$
\text{plim } \hat{\beta} = \beta + \frac{\lambda}{1 - \lambda} \gamma \tag{15}
$$

where $\hat{\beta}_0$, $\hat{\delta}$ and $\hat{\beta}$ represent the BRS ordered probit estimator applied to the demand-determining characteristic vector x_1 .

With this formulation it becomes clear that there are two distinct sources of bias which might arise in the demand estimation process. One possibility is that the covariance between A (or v) and ε is non-zero (and thus λ is non-zero). This might be labelled Tiebout bias, and occurs because the demand for public goods affects the choice of residential communities, causing a non-zero correlation between A and ε . The second possibility is that the sorting equation, which explains why some individuals under- or overconsume the public good, is a function of the demand determining variables, i.e., that $y \neq 0$. The first source of bias causes difficulties for the estimation of all the relevant parameters. However, in the special case in which $y = 0$, β is a consistent estimator of β , but the constant β_0 and threshold parameter $\hat{\delta}$ are inconsistently estimated by $\hat{\beta}_0$ and $\hat{\delta}$. If there is no Tiebout bias, $\lambda = 0$, and all the estimators are consistent.

On the basis of the more general formulation the following important theorems can be summarized:

Theorem 1. When Tiebout bias is present and demand-determining variables also affect community sorting, β_0 , δ and β as estimated using the BRS maximumlikelihood procedure are inconsistent.

Theorem 2. When Tiebout bias is present and the set of sorting variables and demand determining variables are mutually exclusive and orthogonal, i.e., when $y = 0$, then the BRS technique will yield a consistent estimator of β .

Theorem 3. When Tiebout bias is not present, $\hat{\beta}_0$, $\hat{\beta}$ and $\hat{\delta}$ are consistent estimators.

The theorems suggest that the BRS demand parameters are consistently estimated: (1) when $Cov(A, \varepsilon) = 0$, or when the actual level of politically determined per-pupil spending on education is uncorrelated with any omitted demand determining characteristics; or (2) when $y = \tau = 0$, so that sorting and demand will not be confused. One possible assumption under which $Cov(A, \varepsilon) = 0$ might occur is if individuals are randomly assigned to communities and migration is not possible.⁵ A different set of assumptions is given by Theorem 2, in which $y = 0$ and x_1 and x_2 are orthogonal. This assumption holds when the extent to which individuals demands deviate from the actual public service provision is uncorrelated with demand characteristics.⁶

This technical discussion of the problem caused by endogenous community choice can be understood in a non-technical way. Consider the coefficient of A in the BRS maximum-likelihood procedure. The larger is θ_1 , the greater the

⁵ We should note that in the case of perfect Tiebout sorting, $y = \tau = 0$, and there is no Tiebout bias. However, the BRS technique is not applicable in this situation.

This is a special case of a more general result concerning the consistency of slope estimators in qualitative choice models. See Ruud (1983).

increase in the probability of a less response resulting from the increase in A (as we survey in communities with larger A's), holding x constant. However, if people select communities according to their tastes for public goods, the change in the probability of a less response to a change in A is smaller than if A's and x's were matched at random (that is, if community choice were not influenced by public good preferences). In fact, if there is sorting by public good preferences, there is a tendency towards a same response. This means that Tiebout sorting will systematically bias downward the absolute value of θ_1 . In other words, without correcting for changes in the conditional expectation of ε , the effect of actual expenditure levels on response probabilities is systematically underestimated.

A similar argument implies that the absolute value of the estimated coefficients of x, the θ_2 vector, will be downward biased. However, the maximum likelihood estimates of the demand parameters involve the ratio of θ_2 to θ_1 we cannot say a priori whether the BRS demand parameters would be biased upward or downward.

Consistent Maximum-Likelihood Estimation

The Tiebout bias problem can be solved by recognizing that the demand and community matching function represent a pair of simultaneous equations. In terms of equations (1) and (2)

$$
E_{i} = \beta_{0} + x_{1i}^{*} \beta + \varepsilon_{i}
$$

\n
$$
A_{i} = (\beta_{0} + \gamma_{0}) + x_{1i}^{*} (\beta + \gamma) + x_{2i}^{*} \tau + \omega_{i}
$$

where $w_i = u_i + \varepsilon_i$. If u_i is normally distributed, the random variables ε_i and w_i are bivariate normal with correlation ρ .⁷ In this case the log likelihood

Thus, $\sigma_{\varepsilon}^2 = \sigma_{w}^2 (1 - \rho^2)$.

function for the observed pattern of survey responses is

$$
\mathcal{L} = \sum_{i \in \text{Less}} \log F(L^*) + \sum_{i \in \text{Same}} \log [F(M^*) - F(L^*)]
$$

+
$$
\Sigma
$$
 log[1-F(M*)] - $\frac{1}{2}\Sigma \log 2\pi\sigma_w^2$
iéMore

$$
-\frac{1}{2\sigma_{w}^{2}}\Sigma\{\mathbf{A}_{i}[(\beta_{0}+\gamma_{0}) + \mathbf{x}_{1i}^{'}(\beta+\gamma) + \mathbf{x}_{2i}^{'}\tau]\}^{2}
$$
 (16)

where L* and M* are defined in equations (11) and (12). In this case $\lambda = \rho(\sigma_g/\sigma_m)$. Maximization of (16) with respect to the parameters of the demand and matching functions yields consistent and efficient estimators.⁸

4. Empirical Analysis

As in the original BRS paper, the data involved a subsample of 945 homeowners who responded to the question of whether they would like more, the same or less spending on public education. The definitions of all variables utilized in the estimation procedure are given in Table 1. (BRS contains a more detailed description of the data.)

The Tiebout bias question revolves around two central issues. The first is whether community choice is systematically affected by preferences for public In terms of the model presented here the test for this bias is simple goods. whether or not the parameter λ is significantly different from zero. The

⁸ In order to find the maximum likelihood estimates we used an iterative, two-stage procedure which is the sequential solution to a probit and regression problem. This procedure, which is of some inherent interest because of its broader applicability, is described in the appendix so as not to divert the reader's attention from the substantive issues at hand.

TABLE 1 - DEFINITION OF VARIABLES

second is the extent to which the variables that explain the matching of individuals to their preferred public expenditures are the same variables that explain their demand. In order to test for these two sources of bias there must be some measurable set of variables, x_2 's that explain the matching of preferences with communities, but do not affect demand.

A proper empirical analysis of the Tiebout bias question would involve a complete theoretical specification of a model of community choice which would suggest the proper identifying restrictions. This specification, which is beyond the scope of this paper, would have to incorporate variables which arose from the modelling of the politics of local public goods supply as well as the socioeconomics of migration. We offer instead a set of x_2 variables which could reasonably be expected to affect the degree of preference-community mismatch, v, but not to affect underlying preferences.

In general we would expect that individuals who are most likely to have low values of v in absolute value are those a) who have recently moved, b) who live in a metropolitan area in which there is substantial choice among public sector bundles, and/or c) who have tastes that are reasonably similar to others with equivalent incomes. If we were explaining the absolute value of v, therefore, a number of explanatory variables would come immediately to mind. These would include a) a dummy reflecting a recent move (not available in the Courant, Gramlich, Rubinfeld survey), and b) an indicator of the extent to which community choices are available (e.g. a dummy indicating presence in a suburban district in a metropolitan area). However, our concern is somewhat different. We are looking for instruments that might be correlated with v, not its absolute value, and which do not appear in the demand equation. Whether the same variables are relevant is a question for which we do not have a confident response at this time.

In any case, we have tried three variables as instruments in our tests for Tiebout bias. First, is the variable PCEXP, which measures the percentage change in per pupil expenditures in the school district over the previous year (1976-77 to 1977-78). Because moving is costly, households will often choose to remain in a community even though local public spending (or average community demand) changes at a different rate than household demand. A relatively large value of PCEXP would, if unexpected, reflect a greater likelihood that individuals are consuming more than their desired levels of public spending. 9 Second, is the variable SMSA, an SMSA dummy meant to reflect the availability of community choice in the metropolitan area. Finally, the third variable was CCITY, a dummy variable equal to one when the school district was located in the central city. The central city location was assumed to reflect a limited public education selection despite the fact that the individual resided in a metropolitan area in which there was substantial choice.

The results of the Tiebout bias analysis are given in Tables 2 and 3. The first results in column (1) of Table 2 are the single equation probit coefficients, equivalent to the ones computed in BRS. If there were no Tiebout bias, these would be estimates of the demand parameters divided by the conditional standard deviation of ε . The resulting implied demand parameters are given in column (2) of Table 3. These results are discussed, in detail, in the BRS paper. The striking result is that both the price and income elasticities of demand are smaller than those found in previous median voter studies. However, if there is residential sorting by preferences for education, these original estimates could be inconsistent.

See Roberts (1985) for a complete discussion of the conditions under which PCEXP will be correlated with v, but not with demand.

TABLE 2

Maximum Likelihood Coefficients

TABLE 2 (continued)

TABLE 3
Demand (β 's) and Community Matching (γ 's) Parameters

Table 3 (continued)

 x_i :

Column (2) of Table 2 gives the equivalent results if the possibility of Tiebout sorting is allowed. (Column (3) represents the sum of the demand and sorting coefficients.) The straightforward test for Tiebout bias is the test of the hypothesis that the parameter λ is significantly different from zero. The results from Table 2 are convincing - an asymptotic t-ratio of 2.36 (on the coefficient λ/σ_{ρ}) suggests that the hypothesis that $\lambda = 0$ can be rejected at the 5% significance level. In fact, $\hat{\lambda}$ turns out to be very close to 1 for the results described in Table 2. From a practical perspective, this suggests strongly that individuals may sort themselves based upon public goods preferences. We have chosen to present this particular specification, less because of the attraction of the underlying theoretical model, but more because it illustrates the real possibility of Tiebout bias.

As suggested before, failure to correct for Tiebout bias will cause the effects of changes in actual level of expenditures, A, on the response probabilities to be understated. This is exactly what we observe. The coefficient on A (LNEXP), the estimated value of $\frac{1}{\sigma_{\rho}}$, is 0.43 without the sorting correction. When the effect of Tiebout sorting is accounted for, the coefficient increases by a factor of 4 to 1.74. All the remaining coefficients relating to personal characteristics are relatively unchanged under the Tiebout bias correction. Only the coefficient on the variables that relate to the school district - LNENRL, LNPUB, DETRT, LNCTCH, LNCY, LNCW - change substantially after the correction, and most of these coefficients have low t-ratios.

Each of the variables that is used to explain community matching, but not demand, appears to be significant. The coefficient on PCEXP is large and over 12 times its standard error. Since these variables explain the size of the matching error, v_i , it appears that the larger is the percent change in expenditures the greater the probability that actual expenditures exceed desired expenditures.

Table 3 reports the demand parameters estimated in three different ways. The first is a regression of actual expenditures on various independent vari-The second is a single equation probit, equivalent to the one reported ables. The third uses a full information maximum likelihood estimator which in BRS. accounts for the Tiebout bias. The t-ratios are reported for the regression estimates. In the case of the bias-corrected procedure, the community matching parameters, y and t, are reported as well (column 4). These are calculated as the difference between the values of $\beta + \gamma$ calculated from column 3 of Table 2 and the estimated demand parameters, β 's, reported in column 3 of Table 3.

A number of conclusions seem quite striking. First, a comparison of columns (2) and (3) shows that the correction for Tiebout bias leads to lower price and income elasticities of demand. This is especially interesting since the micro price and income elasticities of BRS were substantially lower than the macro elasticities obtained by most other demand studies. It confirms our view that the income and price elasticities of demand for education are quite low. Second, the correction for Tiebout bias substantially lowers the values of a number of other demand coefficients such as BLACK which we found to be unusally high in the BRS paper. Finally, the estimated threshold parameter (Table 3) is substantially lower than in the BRS paper. Its value of 1.65 suggests that people do not discern differences in per pupil expenditures that are within 65% of their ideal level. This value is quite high, but considerably more reasonable than the $649%$ value suggested by the uncorrected single equation probit.

Column (4) includes the consistent estimates of the γ' 's and the τ' 's, which allows us to get a sense of the magnitude of the omitted variable bias. The relevant comparison involves the magnitude of each of the y's to the corresponding β 's that were obtained from the probit equation. In this comparison the y's tell us (roughly) the extent to which demand parameters would be biased were

the demand functions to be estimated without taking simultaneity into account. For example, the price coefficient of .07 should be compared to the price elasticity of demand of -. 11. This suggests that failure to account for simultaneity in the BRS estimation caused the price elasticity of demand to be overestimated by more than 50%. Repeated another way, the Tiebout bias correction leads us to the conclusion that the price elasticity of demand is even lower than the BRS results suggest. A similar conclusion is reached about the size of the income elasticity of demand, since the relevant γ is equal to -.08, while the β from Table 3 is equal to .10.

Another demand variable of particular interest is the BLACK variable. The BRS results suggested a very large differential between black and nonblack demands for public school. Our analysis suggests that the correct black coefficient is roughly one-quarter the size of the BRS coefficient which also incorporated the effect of race-related differences in mobility. A similar comparison would apply to many of the coefficients of the demand variables. To the extent that one believes that the demand variables also determine the migration-community choice decision, then the BRS estimates are likely to overstate the magnitude of the demand parameters.

The regression derived demand parameters in column (1) of Table 3 are consistent with our understanding of the effect of omitted variable bias. As we suggested earlier, if the degree of community matching is correlated with the demand variables x_1 , then a regression of A_i on x_{1i} will produce biased parameter estimates. It is easy to show that this bias is equal to value of the community matching parameters, γ , reported in column (4) of Table 3. This suggests that the differences between the x_1 parameters in column (1) and the equivalent parameters in column (4) should produce unbiased estimates of the true demand parameters. The similarity between these differences and the consistent estimates of column 3 is striking.

$5.$ Conclusion and Additional Comment

This paper represents an assessment of our own earlier work in the light of potential selectivity biases. Bergstrom, Rubinfeld and Shapiro use survey data to estimate the demand function for public education. The price and income elasticities of demand for public education were found to be considerably smaller than those found using aggregate data to estimate median voter models. Further thought about the econometric issues regarding the estimation problem convinced us that the estimates reported might be biased because of selectivity effects similar to those studied by Heckman, Amemiya and others.

The possibility of selectivity induced (Tiebout) biases arises if people select communities on the basis of their individual preferences for public goods. The practical problem in the BRS framework is that strong assumptions are made about the distributions of random variables. To the extent that this error specification is incorrect, the estimated demand parameters will be biased. A complete formulation yields a simultaneous equation model and a more complicated likelihood function than originally specified by BRS.

The mere possibility of bias is not sufficient in itself to justify a reassessment of previous results. However, estimating the model with the complete specification suggests that the Tiebout bias can be important. How important depends heavily on the choice of instruments used to correct for selectivity bias. Thus, a final resolution of this issue awaits more elaborate and thorough models, as well as empirical testing. Our results might be different, for example, if one were to argue that a number of discrete variables appearing in the demand equation ought to be removed and placed in the community choice equation instead.

In any case, we remain quite confident about the relative magnitudes of our original estimates of price and income elasticities of demand for education.

We believe that the price and income elasticities are quite small, substantially smaller than has been suggested by most studies of aggregated data. We look forward to further discussion and analysis of this result since it has important policy implications.

We feel that an additional comment about the low income elasticity of demand is called for because our estimated value is much smaller than values found with community median income estimators. Although we do not have a fully developed model to explain the low value, the finding led us to consider possible explanations.

Suppose education were considered an investment in human capital, the output of which is measured as changes in the wealth, or permanent income, of the student. The marginal product of that investment is the change in permanent income due to a small change in education expenditures. If the marginal product is independent of family income, the optimal investment should not vary substantially between income groups.

One might suspect that both the taste for and productivity of education varies directly with the education of the parents. Our results indicate that the demand for education is between 12% (with the selectivity corrected estimate) and 34% (with the uncorrected estimate) higher for college graduates than for nongraduates.

One finds that community expenditures on education increase substantially with community income. Our result suggest that this may be due to factors which are highly collinear with income, such as education, rather than to income itself.

Appendix

Due to the difficulty of programming a full maximum likelihood procedure, we estimated the parameters by means of an iterative two-state procedure. The procedure follows from the first order necessary conditions for maximizing the likelihood function (13). It is useful for this case to rewrite the likelihood function

$$
\mathcal{L} = \sum_{i \in \text{less}} \log F[\theta_0^L + \frac{1-\lambda}{\sigma_e} A_i - \frac{\beta - \lambda \theta_1}{\sigma_e} x_{1i} - \frac{\lambda \tau}{\sigma_e} x_{2i}]
$$

$$
+ \sum_{i \in \text{Same}} \{F[\theta_0^M + \frac{1-\lambda}{\sigma_e} A_i - \frac{\beta - \lambda \theta_1}{\sigma_e} x_{1i} - \frac{\lambda \tau}{\sigma_e} x_{2i}]
$$

$$
- F\left[\theta_0^L + \frac{1-\lambda}{\sigma_e} A_i - \frac{\beta - \lambda \theta_1}{\sigma_e} x_{1i} - \frac{\lambda \tau}{\sigma_e} x_{2i}\right]
$$

$$
+ \sum_{i \in \mathsf{More}} \log \{1 - F[\theta_0^M + \frac{1-\lambda}{\sigma_e} A_i - \frac{\beta - \lambda \theta_1}{\sigma_e} x_{1i} - \frac{\lambda \tau}{\sigma_e} x_{2i}] \}
$$

$$
- \frac{1}{2} \sum \log 2\pi \sigma_w^2 - \frac{1}{2\sigma_w^2} \sum (A - \theta_1 x_1 - tx_2)^2
$$

The parameter $\theta_1 = (\beta + \gamma)$.

The necessary conditions for maximizing this likelihood functions are as follows:

$$
\frac{\partial \mathcal{I}}{\partial \hat{\theta}_0^L} = \sum_{i \in \text{less}} \frac{f(L)}{F(L)} - \sum_{i \in \text{Same}} \frac{f(L)}{F(L)} = 0, \tag{A1}
$$

where $f(L)$ and $F(L)$ are the values of the density and cumulative density functions

with values θ_0^L ;

$$
\frac{\partial \mathcal{L}}{\partial \hat{\theta}_0^M} = \sum_{\vec{F}} \frac{f(M)}{F(M)} - \sum_{\vec{F}} \frac{f(M)}{F(M)} = 0
$$
 (A2)

The presentation of the remaining first order conditions is facilitated by defining a vector \wedge , the elements of which, \wedge_{i} , take on the values

$$
\Lambda_{i} = \begin{cases}\n\frac{f(L)}{F(L)} & i \in \text{less} \\
\frac{f(M) - f(L)}{F(M) - F(L)} & i \in \text{Same} \\
\frac{f(M)}{1 - F(M)} & i \in \text{More}\n\end{cases}
$$

Then

$$
\frac{\partial \mathcal{L}}{\partial (1/\sigma_{\mathbf{e}})} = \Lambda' A = 0
$$
 (A3)

$$
\frac{\partial \mathcal{L}}{\partial (\beta/\sigma_{\rho})} = \Lambda' \mathbf{x}_1 = 0
$$
 (A4)

$$
\frac{\partial \mathcal{L}}{\partial (\lambda / \sigma_{\rho})} = -\Lambda' (\mathbf{A} - \hat{\theta}_{1} \mathbf{x}_{1} - \hat{\tau} \mathbf{x}_{2}) = 0
$$
 (A5)

$$
\frac{\partial z}{\partial \hat{\theta}_1} = \Lambda' \frac{\lambda}{\hat{\sigma}_e} x_1 + \frac{1}{\hat{\sigma}_w^2} \Sigma (A - \hat{\theta}_1 x_1 - \hat{\tau} x_2) x_1 = 0
$$
 (A6)

$$
\frac{\partial \mathcal{L}}{\partial \hat{\tau}} = \Lambda' \frac{\lambda}{\hat{\sigma}_e} x_2 + \frac{1}{\hat{\sigma}_w^2} \Sigma (A - \hat{\theta}_1 x_1 - \hat{\tau} x_2) x_2 = 0
$$
 (A7)

$$
\frac{\partial \mathbf{L}}{\partial \hat{\sigma}_{\omega}^2} = -\frac{1}{\hat{\sigma}_{\omega}^2} + \frac{1}{\hat{\sigma}_{\omega}^4} \Sigma (A - \hat{\theta}_1 x_1 - \hat{\tau} x_2)^2 = 0
$$
\n(A8)

The final equation (A8) gives the estimate of σ_{u}^2 as the usual sum of squares of the residuals from a regression of A on x_1 and x_2 .

The two stage iterative procedure starts with the observation that equation (A6) and (A7) would be satisfied in a least squares regression of A + $\frac{\lambda}{\hat{\sigma}}$ $\wedge \hat{\sigma}_{\omega}$ on x_1 and x_2 . A beginning value (in our case $\Lambda = 0$) is chosen and a regression of A on x_1 and x_2 is run to compute initial values of θ_1 , $\hat{\tau}$ and $\hat{\sigma}_{\omega}$. The estimates of θ_1 and t are used to find estimates of θ_0^L , θ_0^M , $1/\sigma_e$, β/σ_e and λ/σ_e that satisfy A1-A5 conditional on the estimated values of θ_1 and τ . With these estimates, a value of $\frac{\lambda}{\sigma_{\rho}}$ $\wedge \hat{\sigma}_{\omega}$ is computed and a regression is run to obtain new estimates of σ_1 , τ and σ_m .

The procedure converges to a set of parameter values that satisfy all the first-order conditions.¹⁰

¹⁰ The procedure just described does not yield independent estimators of σ_{ϵ} and ρ . We have derived maximum-likelihood estimators which do so, but have not presented them here because of the additional detail involved.

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