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On Mean Curvature Diffusion in Nonlinear Image Filtering

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Abstract

Mean curvature diffusion is shown to be a position vector diffusion, tending to scalar diffusion as a flat image region is approached, and providing noise removal by steepest descent surface minimization. At edges, it switches to a nondiffusion state due to two factors: the Laplacian of position vanishes and the magnitude of the surface normal attains a local maximum.

Keywords: image filtering, inhomogeneous diffusion, nonlinear

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1 Introduction

We have introduced an inhomogeneous diffusion that evolves the image surface at a rate proportional to its mean curvature (El-Fallah & Ford, 1993; El-Fallah & Ford, 1994a; El-Fallah & Ford, 1994b). This mean curvature diffusion (MCD) preserves edges while reducing the surface area (thereby removing noise) and imposing regularity.

In this paper, it is shown that MCD can be cast as a position vector diffusion tending to scalar (homogeneous) diffusion as a flat image region is approached. This characterization clearly indicates the preservation of edges under MCD. The first variation of surface area and Schwartz's inequality are then used to show that MCD removes noise by steepest descent surface area minimization.

We briefly review MCD in section 2, and then relate MCD to the position vector diffusion in section 3, In section 4, we explain the properties of MCD in removing noise. We conclude in section 5 by stressing the geometric interpretation of an image and the corresponding position vector diffusion.

2 Review of Mean Curvature Diffusion

Mean curvature diffusion is defined on the three-dimensional Euclidean space $\mathbf{E}^3$ and is interpreted geometrically by characterizing the image $I(x_1, x_2)$ as a surface $\mathcal{S}$ on $\mathbf{E}^2$

$$\mathcal{S}: \; g(\mathbf{x}) = x_3 - I(x_1, x_2) = 0 \tag{1}$$

where g is a differentiable real-valued function and $\mathbf{x} = (x_1, x_2, x_3)$ is the natural coordinate vector function of $\mathbf{E}^3$ defined such that for each point $\mathbf{p} = (p_1, p_2, p_3)$ in $\mathbf{E}^3$: $x_i(\mathbf{p}) = p_i$. The gradient vector field ∇g for the surface $\mathcal{S}$

$$\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} \mathrm{U}_i \,, \tag{2}$$

where U_i is the unit vector field in the positive x_i direction, can be shown to be a non-vanishing *normal* vector field on the entire surface (see (El-Fallah & Ford, 1994b)). From (1) and (2) the magnitude of the surface normal can be expressed in terms of the image gradient

$$|\nabla g| = \sqrt{|\nabla I|^2 + 1} \,, \tag{3}$$

and the *unit normal vector field* on a neighborhood of $\mathbf{p}$ in $\mathcal{S}$ is

$$\mathcal{N} \triangleq \frac{\nabla g}{|\nabla g|} \tag{4}$$

The diffusion of g is modeled by

$$\frac{\partial g}{\partial t} = \nabla \bullet (C \, \nabla g) \tag{5}$$

and our interest is in the properties of inhomogeneous diffusion in which the diffusion coefficient is the inverse of the surface gradient magnitude

$$C = \frac{1}{|\nabla g|} \tag{6}$$

Rewriting (5) by substituting (4) and (6)

$$\frac{\partial g}{\partial t} = \nabla \bullet \mathcal{N} = \frac{\partial \mathcal{N}_1}{\partial x_1} + \frac{\partial \mathcal{N}_2}{\partial x_2} \tag{7}$$

It can be shown that the terms on the right are the normal curvatures in the x_1 and x_2 directions. The *mean curvature* H is the average value of the normal curvature in *any* two orthogonal directions, and since the directions x_1 and x_2 are orthogonal, we obtain

$$\frac{\partial g}{\partial t} = 2\mathrm{H} \tag{8}$$

Thus, the surface under this diffusion evolves at a rate twice the mean curvature of the image. Application of MCD to an isolated noisy edge (El-Fallah & Ford, 1994b) shows that the evolution of the surface results in surface area reduction (noise removal), arriving at a minimal surface at convergence (complete noise removal) with edge enhancement and an intact edge location.

The surface view point discussed in this section has recently been adopted in (Kimmel *et al.*, 1997) where it was also extended to color. This work was based on the more extensive work reported in (Sochen *et al.*, 1996). Mean curvature flow has also been reported in (Malladi & Sethian, 1996) together with min/max curvature flow. Other recent studies of MCD properties include (Fischl, 1997; Fischl & Schwartz, 1997).

3 The Vector Heat Equation

The surface representation in (1) describes the surface by following a set of points in 3-space. In this section we study the motion of the surface by tracking the point $(\mathbf{x}, t)$ *on* the surface. For small $\Delta\mathbf{x}$ and small Δt the tracked point evolves to $(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t)$ on the new surface, and we have from the representation in (1)

$$g\left(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t\right) = g\left(\mathbf{x}, t\right) = 0\,. \tag{9}$$

Expanding to the first order, we have the approximation

$$g\left(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t\right) \approx g\left(\mathbf{x}, t\right) + \nabla g \bullet \Delta\mathbf{x} + \frac{\partial g}{\partial t}\,\Delta t \tag{10}$$

Using (9) and (10) and dividing by $|\nabla g|$

$$\frac{\nabla g}{|\nabla g|} \bullet \Delta\mathbf{x} = -\frac{1}{|\nabla g|}\,\frac{\partial g}{\partial t}\,\Delta t\,. \tag{11}$$

Using (4), (6), and the result in (8), we have

$$\mathcal{N} \bullet \Delta\mathbf{x} = -\,C\left(2\mathrm{H}\right)\Delta t \tag{12}$$

With $\mathbf{n}$ denoting the unit surface normal $\mathcal{N}\left(\mathbf{p}\right)$, as Δt tends to zero

$$\mathbf{n} \bullet \frac{\partial \mathbf{x}}{\partial t} = -\,C\left(2\mathrm{H}\right) \tag{13}$$

which shows that the surface moves with a normal velocity of magnitude $[-\,C\left(2\mathrm{H}\right)]$. Thus the motion up to tangential diffeomorphism is equivalent to

$$\frac{\partial \mathbf{x}}{\partial t} = -\,C\left(2\mathrm{H}\right)\,\mathbf{n} = C\left(-2\mathrm{H}\,\mathbf{n}\right) \tag{14}$$

where the mean curvature vector ($-2\mathrm{H}\,\mathbf{n}$) is the Laplacian of position $\nabla^2\,\mathbf{x}$ (cf. (Laugwitz, 1965), p. 131). Thus, the motion is described by

$$\frac{\partial \mathbf{x}}{\partial t} = C\ \nabla^2\,\mathbf{x} \tag{15}$$

which is the position vector equivalent of the scalar heat equation

$$\frac{\partial I}{\partial t} = c\,\nabla^2\,I \tag{16}$$

where c is a constant conduction coefficient independent of space-time.

Rewriting (15) using (6) we obtain

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{1}{|\nabla g|}\ \nabla^2\,\mathbf{x} \tag{17}$$

which characterizes MCD and shows that its rate is inversely proportional to the magnitude of the surface normal and directly proportional to the Laplacian of position. As an edge is approached, the magnitude of the surface normal approaches its maximum value, as indicated by (3), and the Laplacian of position (mean curvature vector) tends to zero, driving the diffusion rate to zero and preserving the edge. Thus there are two contributing factors halting the diffusion at edges: The Laplacian of position and the magnitude of the surface normal.

When diffusion or averaging occurs according to (17), it is an averaging of a position vector which is qualitatively different than that characterized by the scalar heat equation in (16). The former averages the position vector consisting of the three coordinates of space. This simultaneous averaging of the three coordinates results in surface averaging which we term *free* averaging, as it does not involve eroding the surface. The latter averages only the scalar intensity with disregard to the other two coordinates of space (x_1 and x_2) and results in eroding the surface, destroying structural information. It is also clear from (17) that convergence corresponds to the vanishing of the Laplacian of position. Thus at convergence the Laplacian of each coordinate function vanishes rendering all three functions harmonic, leading to regularity everywhere.

In noninformative flat regions the magnitude of the surface normal tends to the scalar value of one from (3) and the diffusion becomes homogeneous. The diffusion of (17) automatically switches

from a non-diffusion state at an edge to a homogeneous diffusion state in flat regions. This can be easily shown mathematically by taking limits as both I_{x_1} and I_{x_2} tend to zero in a flat region. Taking this limit in (4) results in $\mathbf{n}$ tending to the unit vector (0,0,1), and applying the same limit in (8) results in 2H approaching the Laplacian: $I_{x_1x_1} + I_{x_2x_2} = \nabla^2 I$. This results in (15) approaching

$$\frac{\partial \mathbf{x}}{\partial t} = \left(\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial I}{\partial t}\right) = (1)\,\nabla^2 I\,(0,0,1) \tag{18}$$

which is the scalar heat equation (16) with $c = 1$, and with $\frac{\partial x_1}{\partial t} = \frac{\partial x_2}{\partial t} = 0$ indicating that the two coordinates x_1 and x_2 remain unchanged, unlike the above mentioned free averaging of the three coordinates.

4 The Evolving Surface

In Section 2 mean curvature was expressed in two forms: the average of two normal curvatures in orthogonal directions, and the divergence of the unit surface normal. In this section, we develop the relationship between mean curvature and the first variation of surface area. We first show that the curvature vector of a planar curve determines the change in length as the curve is deformed. This result is then extended to a 2-D surface to show that mean curvature determines the change in surface area as a surface is deformed. This result is then used to show the properties of MCD in noise removal.

4.1 The Evolving Curve

We consider the deformation of a planar curve, shown in Fig. 1, where an infinitesimal piece of planar curve $\mathrm{d}\ell$ is pushed a distance $\mathrm{d}\alpha$ in the direction of its curvature vector $\boldsymbol{\kappa}$, which is the rate of change of the unit tangent vector with respect to arc length. The original arc lies on a circle of radius $\frac{1}{\kappa}$, and the evolved arc is on a circle of radius

$$\frac{1}{\kappa} - \mathrm{d}\alpha = \frac{1}{\kappa}\left(1 - \kappa \mathrm{d}\alpha\right). \tag{19}$$

The length of the arc changes from $\mathrm{d}\ell$ to $(1 - \kappa \mathrm{d}\alpha)\,\mathrm{d}\ell$, producing a decrease in length of $\kappa \mathrm{d}\alpha\,\mathrm{d}\ell$.

More generally, if the displacement is a vector $\mathrm{d}\boldsymbol{\alpha}$ not necessarily in the direction of $\boldsymbol{\kappa}$, the

incremental decrease in length is $\boldsymbol{\kappa} \bullet \mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\ell$. The decrease of length for a planar curve is thus $\int \boldsymbol{\kappa} \bullet \mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\ell$. If the curve moves with initial velocity $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}$, then the initial rate of change in length is given by

$$-\int \boldsymbol{\kappa} \bullet \mathbf{v}\,\mathrm{d}\ell \tag{20}$$

4.2 First Variation of Surface Area

For a surface moving at a speed $\mathbf{v}$ and with $\mathcal{A}$ denoting the surface area, the first variation of surface area is defined as

$$\delta^1(\mathcal{S}) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\mathcal{S} + t\,\mathbf{v})\Big|_{t=0}, \tag{21}$$

and it is a linear operator on smooth vector fields $\mathbf{v}$ on 3-space, and is the initial rate of area of $\mathcal{S}$ as it is pushed by the vector field $\mathbf{v}$. The surface is moving with a velocity vector $\mathbf{v}$ which can be decomposed into two components, tangential to the surface and normal to the surface. Since this equation is linear in $\mathbf{v}$ we may consider tangential and normal variations in surface area separately. Tangential variations correspond to sliding the surface along itself, with $\delta^1(\mathcal{S}) = 0$. Let $\mathrm{v}\boldsymbol{n}$ be a small normal variation, and consider an infinitesimal area $\mathrm{d}x_1\,\mathrm{d}x_2$ at $\mathbf{p}$. Since any surface will have a maximum and minimum normal curvatures κ_1 and κ_2 called principal curvatures occurring in two orthogonal directions called principal directions we may assume that the principal directions point along the axes, and applying (20) on both directions with κ_1 and κ_2 being the two principal curvatures, the new infinitesimal area is

$$(1 - \mathrm{v}\kappa_1)\,\mathrm{d}x_1\,(1 - \mathrm{v}\kappa_2)\,\mathrm{d}x_2 = \mathrm{d}x_1\mathrm{d}x_2 - \mathrm{v}\,(\kappa_1 + \kappa_2)\,\mathrm{d}x_1\mathrm{d}x_2 + \mathrm{v}^2\kappa_1\kappa_2\mathrm{d}x_1\mathrm{d}x_2 \tag{22}$$

Ignoring the second order variation v^2 and using the definition of mean curvature $\left(\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}\right)$, the new infinitesimal area becomes $\mathrm{d}x_1\mathrm{d}x_2 - \mathrm{v}\,(2\mathrm{H})\,\mathrm{d}x_1\mathrm{d}x_2$. The decrease in surface area is thus

$$\mathrm{v}\,(2\mathrm{H})\,\mathrm{d}x_1\mathrm{d}x_2 = \mathbf{v} \bullet (2\mathrm{H})\,\boldsymbol{n} \tag{23}$$

To find the total decrease we integrate over the surface with $\mathbf{v}$ being the initial velocity and the initial rate of decrease of the surface area, or the first variation of surface area

$$\delta^1(\mathcal{S}) = \int [\mathbf{v} \bullet (2\mathrm{H})\, \boldsymbol{n}]\, \mathrm{d}\mathcal{S} \tag{24}$$

4.3 Steepest Surface Area Descent

Schwarz's inequality states that for any two vector fields $\mathbf{F}$ and $\mathbf{G}$

$$\left| \int \mathbf{F} \bullet \mathbf{G} \right| \leq \left(\int |\mathbf{F}|^2 \right)^{\frac{1}{2}} \left(\int |\mathbf{G}|^2 \right)^{\frac{1}{2}} \tag{25}$$

with equality if and only if $\mathbf{F}$ is a scalar multiple of $\mathbf{G}$, here a scalar is a scalar field which can be any real valued function on $\mathbf{E}^3$ assigning a scalar value to a point $\mathbf{p}$. Vector fields $\mathbf{F}$ and $\mathbf{G}$ assign vectors to $\mathbf{p}$. Combining this inequality with the result in (24) we see that the change in surface area proceeds in the direction of steepest surface descent when $\mathbf{v}$ is a scalar field multiple of the mean curvature vector field. For MCD, $\mathbf{v}$ is $\dfrac{-2\mathrm{H}\,\mathbf{n}}{|\nabla g|}$ which is a scalar field multiple of the mean curvature vector field. We thus conclude that among all vector fields on $\mathcal{S}$ with square integral over $\mathcal{S}$ equal to that of $\dfrac{-2\mathrm{H}\,\mathbf{n}}{|\nabla g|}$ $\left[= \int_{\mathbf{X} \epsilon \mathcal{S}} \left| \dfrac{-2\mathrm{H}\,\mathbf{n}}{|\nabla g|} \right|^2 \mathrm{d}\mathcal{S} \right]$, MCD with $\mathbf{v}_{\mathrm{mcd}} = \dfrac{-2\mathrm{H}\,\mathbf{n}}{|\nabla g|}$ provides the fastest rate of surface area reduction (noise removal).

5 Conclusion

We interpreted the image coordinates and gray level as a *geometric object*, a position vector in 3-space specifying a point on the image surface. By choosing an inhomogeneous diffusion coefficient equal to the inverse of the magnitude of the surface normal we arrived at the position vector heat equation. This equation expresses the diffusion of the position vector of a point on the image surface and indicates clearly the preservation of edges. It also reduces to the homogeneous diffusion equation as a flat region is approached. The first variation of surface area and the position vector heat equation were used to show that the proposed diffusion is the fastest at removing noise, by providing steepest descent surface minimization.

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List of Figures

1 Deformation of an infinitesimal planar curve moving in the direction of $\boldsymbol{\kappa}$ 11

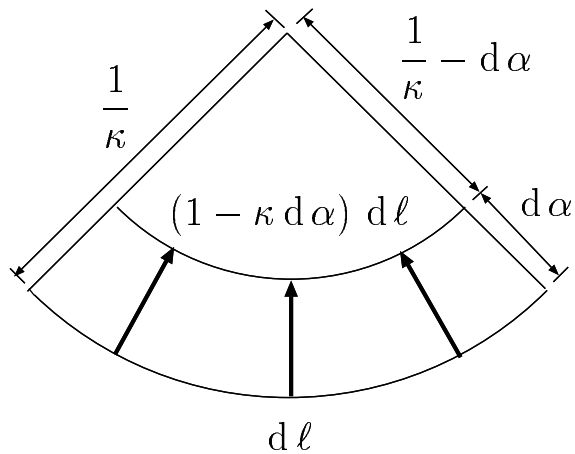

Figure 1: Deformation of an infinitesimal planar curve moving in the direction of $\boldsymbol{\kappa}$