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Symplectic approaches in geometric representation theory

by

Xin Jin

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

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Xin Jin

Abstract

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Professor David Nadler, Chair

We study various topics lying in the crossroads of symplectic topology and geometric representation theory, with an emphasis on understanding central objects in geometric representation theory via approaches using Lagrangian branes and symplectomorphism groups.

The first part of the dissertation focuses on a natural link between perverse sheaves and holomorphic Lagrangian branes. For a compact complex manifold X, let $D^b_c(X)$ be the bounded derived category of constructible sheaves on X, and $Fuk(T^*X)$ be the Fukaya category of T^*X. A Lagrangian brane in $Fuk(T^*X)$ is holomorphic if the underlying Lagrangian submanifold is complex analytic in $T^*X_{\mathbb{C}}$, the holomorphic cotangent bundle of X. We prove that under the quasi-equivalence between $D^b_c(X)$ and $DFuk(T^*X)$ established by Nadler and Zaslow, holomorphic Lagrangian branes with appropriate grading correspond to perverse sheaves.

The second part is motivated from general features of the braid group actions on derived category of constructible sheaves. For a semisimple Lie group $G_{\mathbb{C}}$ over $\mathbb{C}$, we study the homotopy type of the symplectomorphism group of the cotangent bundle of the flag variety and its relation to the braid group. We prove a homotopy equivalence between the two groups in the case of $G_{\mathbb{C}} = SL_3(\mathbb{C})$, under the $SU(3)$-equivariancy condition on symplectomorphisms.

Dedicated to my parents.

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Chapter 0

Introduction

Symplectic geometry is the study of invariants and structures of a $2n$-dimensional real manifold equipped with a closed nondegenerate 2-form. The field was originally motivated by classical mechanics and its intriguing nature of softness versus rigidity has stimulated numerous interesting and active research directions in recent decades. One important driving force for the recent fast development of the field is the (homological) mirror symmetry conjecture posted by physicists and Kontsevich (for the homological version) in the 90's. The homological mirror symmetry conjecture predicts some intriguing connections between algebraic geometry and symplectic geometry. On the symplectic side, it focuses on the Fukaya category of a symplectic manifold, which is a categorical invariant consisting of Lagrangian submanifolds as its objects and Lagrangian intersections as its morphisms. The category has its origin from Floer theory and possesses a very rich composition structure (the A_∞-structure) defined by counting J-holomorphic discs with Lagrangian boundary conditions. The very general algebraic set-up of the Fukaya category is still an ongoing project, and this together with the studies of Fukaya categories of examples of symplectic manifolds motivated from mirror symmetry occupy a significant portion of current research topics in symplectic geometry. In what follows, we will mostly restrict ourselves to the Fukaya category of cotangent bundles, where the algebraic framework is well defined and the category is relatively well understood using Morse theory.

Among many others, one significant impact of the introduction of the Fukaya category is on the studies of symplectomorphism groups. Every symplectomorphism naturally acts on the Fukaya category, and it is not isotopic to the identity if the induced action is nontrivial, therefore the Fukaya category gives rise to a categorical representation of the symplectic mapping class group, i.e. the π_0 of the symplectomorphism group. A basic construction of a symplectomorphism that is not isotopic to the identity is the Dehn twist about a Lagrangian sphere, first introduced by Seidel. There are generalizations of this construction for singular symplectic fibrations, namely the symplectic monodromy around a loop (avoiding the singular values) will induce a symplectomorphism on the fiber. In this way, one can usually get an embedding from the fundamental group of the base to the symplectic mapping class group. Our studies of the symplectomorphism group of the cotangent bundles of flag

varieties below are partly motivated from this point of view.

Geometric representation theory lies in the crossroads of a number of different fields, including representation theory, algebraic geometry, number theory and symplectic geometry. The field grew from a sequence of significant discoveries of connections among $\mathcal{D}$-modules, constructible sheaves, and representation theory, including the microlocal studies of $\mathcal{D}$-modules, the Riemann-Hilbert correspondence, the theory of perverse sheaves, and the Beilinson-Bernstein localization theorem. One of the most important applications of these results is the resolution of the Kazhdan-Lusztig conjecture, which opens up totally new perspectives on questions related to representation theory. In recent years, one of the leading projects in the field is the Geometric Langlands program, which has its origin from number theory and has been motivated largely from the S-duality in physics. The statement of the Geometric Langlands conjecture can be interpreted as a version of mirror symmetry, whose"symplectic" side is given by the category of $\mathcal{D}$-modules. There have been some understandings of the relation between $\mathcal{D}$-modules and Lagrangian branes from the physical point of view, and more recently, the mathematical understanding of this relation is given by the Nadler-Zaslow correspondence composing with the Riemann-Hilbert correspondence. One of the main results presented below is an understanding of perverse sheaves through Lagrangian branes, based on the Nadler-Zaslow correspondence in the holomorphic symplectic setting.

The following is a brief description of the contents of the dissertation. Chapter 1 contains background materials for the subsequent chapters, especially for Chapter 2. Then Chapter 2 and 3 contain results in different directions lying in the intersections of symplectic geometry and geometric representation theory, with motivations and applications in both fields and the approaches being mostly symplecto-geometric.

Chapter 2 focuses on the close relation between holomorphic Lagrangian branes and perverse sheaves. Let's first have a quick overview of constructible sheaves and perverse sheaves. A constructible sheaf on a manifold X is a cochain complex of sheaves of $\mathbb{C}$-vector spaces, whose cohomology sheaves are locally constant with respect to some stratification of X. The notion of constructible sheaves first showed its importance in the studies of holonomic $\mathcal{D}$-modules over a complex manifold or a smooth algebraic variety over $\mathbb{C}$. $\mathcal{D}$-modules are basically algebraic systems of linear differential equations, and holonomic $\mathcal{D}$-modules are a subclass of these, whose solution spaces are finite dimensional. Taking the local solution spaces of a holonomic $\mathcal{D}$-module naturally gives a constructible sheaf, and this functor gives the Riemann-Hilbert correspondence, which is a (contravariant) equivalence between the derived category of (regular) holonomic D-modules and the derived category of constructible sheaves. On an open dense subset of the manifold, every constructible sheaf thus obtained restricts to a local system, and the complement to this where the sheaf fails to be a local system can be thought as the singularities of the $\mathcal{D}$-module. There are much refined studies of such singularities via the microlocal approach, described in terms of the *microlocal support* or *singular support*, which are conical Lagrangian subvarieties in the cotangent bundle of X. Perverse sheaves is a special kind of constructible sheaves, which is characterized as the heart of the interesting t-structure on the derived category of constructible sheaves (different

from the trivial t-structure!) induced from the obvious t-structure on the derived category of holonomic $\mathcal{D}$-modules by the Riemann-Hilbert correspondence. So one can simply say that a perverse sheaf represents a single holonomic $\mathcal{D}$-module.

The Nadler-Zaslow correspondence gives a natural symplecto-geometric interpretation of constructible sheaves. Roughly speaking, for every real analytic manifold X, the correspondence gives a *microlocal functor* μ_X from the derived category of constructible sheaves on X to the derived Fukaya category of T^*X, by sending each constructible sheaf $\mathcal{F}$ to a (complex of) Lagrangian brane(s)[1] in T^*X which is asymptotic to the singular support of $\mathcal{F}$ near the infinity of T^*X. It is worth to note that though one knows the image of every sheaf in terms of iterated cones of standard branes, which are generators in the Fukaya category, it is always interesting to find the most geometric object(s) representing a fixed sheaf, e.g. a single Lagrangian brane. This question is in general not easy, neither is the converse question of finding the sheaf corresponding to a given brane. To get a geometric understanding of perverse sheaves in this manner, we go to the complex setting, where X is a complex manifold and we view T^*X as a real symplectic manifold having a complex structure (not compatible with the symplectic form) induced from that of X. Now there is a natural subcategory of holomorphic Lagrangian branes in the Fukaya category, and our main theorem in Chapter 1 says that they give rise to perverse sheaves.

Theorem 0.0.1. *Let X be a compact complex manifold. Then for any holomorphic Lagrangian brane L in the Fukaya category of T^*X, we have $\mu_X^{-1}(L)$ is a perverse sheaf (up to a grading shift).*

One would hope that there should be an equivalence between perverse sheaves and holomorphic Lagrangian branes. Since there are global obstructions to the existence of holomorphic branes, it is not true in general that every perverse sheaf can be represented by a single brane. However, this might be true locally, and is an interesting question to be explored. This theorem also leads to many other interesting questions both in symplectic geometry and geometric representation theory. For example, the theorem implies there is a natural t-structure on the Fukaya category of the cotangent bundle of every complex manifold, and it suggests the same holds for the Fukaya category of more general holomorphic symplectic manifold. Among many others, the conical symplectic resolutions, including the cotangent bundles of partial flag varieties and ALE spaces, etc., are important objects in geometric representation theory since the modules over their quantizations (a notion similar to $\mathcal{D}$-modules) give representations of certain algebras via a generalized Beilinson-Bernstein localization formalism. In such general cases, one expects that the Riemann-Hilbert correspondence is between the category of modules over the quantizations and the Fukaya category on the manifold, and the expected t-structure inherited from the former to the Fukaya category should be the t-structure induced from the holomorphic symplectic structure.

[1]The word brane means some extra structures are decorated on the Lagrangian in order to obtain a well-defined category structure.

Chapter 3 contains studies of the symplectomorphism group of the cotangent bundle of the flag variety $T^*\mathcal{B}$ for a semisimple Lie group $G_{\mathbb{C}}$, where $\mathcal{B} = G_{\mathbb{C}}/B$ for a Borel subgroup B in $G_{\mathbb{C}}$. The motivation is twofold. On one hand, there is a well-known braid group action on the derived constructible category of sheaves on $\mathcal{B}$, which is $G_{\mathbb{C}}$-equivariant. As we have seen above, the derived constructible category of sheaves on $\mathcal{B}$ is equivalent to the derived Fukaya category of $T^*\mathcal{B}$, so this gives rise to a $G_{\mathbb{C}}$-equivariant braid group action on the Fukaya category as well. Since every (reasonable) symplectomorphism of $T^*\mathcal{B}$ induces an automorphism of the Fukaya category, it is natural to ask whether the braid group action is coming from symplectomorphisms of the space. Moreover, one could ask the following stronger question: is the group of "$G_{\mathbb{C}}$-equivariant" symplectomorphisms of $T^*\mathcal{B}$ homotopy equivalent to the braid group? On the other hand, there have been constructions of symplectomorphisms by means of symplectic monodromies of a symplectic fibration as being mentioned above, which are usually given by (family) Dehn twists. There is a natural symplectic fibration with generic fiber symplectomorphic to $T^*\mathcal{B}$ from the adjoint quotient map of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and the fundamental group of the base is just the braid group associated to $G_{\mathbb{C}}$. In this way, we get an embedding from the braid group to the symplectic mapping class group of $T^*\mathcal{B}$.

In Chapter 3, we first formulate in an appropriate way of the group of "$G_{\mathbb{C}}$-equivariant" symplectomorphisms of $T^*\mathcal{B}$ that we are interested in, since the notion is not standard in the literature. Along the way, we make a natural connection to the *Steinberg variety*, which plays a key role in geometric representation theory. Then we make a conjecture about the homotopy equivalence relation between the "$G_{\mathbb{C}}$-equivariant" symplectomorphism group and the braid group. Our main theorem is the following (see Theorem 3.1.4 for the precise statement).

Theorem 0.0.2. *(1) There is a natural surjective group homomorphism from the "$G_{\mathbb{C}}$-equivariant" symplectomorphism group of $T^*\mathcal{B}$ to the braid group, for $G_{\mathbb{C}} = SL_n(\mathbb{C})$.*
(2) The surjective homomorphism in (1) is a homotopy equivalence for $G_{\mathbb{C}} = SL_2(\mathbb{C}), SL_3(\mathbb{C})$.

One of the evidence for our conjecture is the known case for $G_{\mathbb{C}} = SL_2(\mathbb{C})$, where $\mathcal{B} = \mathbb{P}^1$ and the notion of "$G_{\mathbb{C}}$-equivariant" symplectomorphisms is equivalent to that of compactly supported symplectomorphisms. Seidel [27] proved that the compactly supported symplectomorphism group of $T^*\mathbb{P}^1$ is (weakly) homotopy equivalent to $\mathbb{Z}$, which is the braid group for $G_{\mathbb{C}} = SL_2(\mathbb{C})$. We are interested in extending the result to semisimple Lie groups of all types.

Chapter 1

Background

This chapter collects background materials needed for the subsequent chapters, especially for Chapter 2. We first collect basic material on analytic-geometric categories, since this is a reasonable setting for stratification theory (hence for constructible sheaves) and for Lagrangian branes. Then we give a short account of A_∞-categories, which is the algebra basics for Fukaya category. Lastly, we give an overview of the definition of infinitesimal Fukaya categories, with some specific account for $Fuk(T^*X)$ to supplement the main content.

The reader could skip the chapter first and then return to this when necessary.

1.1 Analytic-Geometric Categories.

Analytic-Geometric Categories provide a setting on subsets of manifolds and maps between manifolds, where one can always expect reasonable geometry to happen after standard operations. A typical example is if a C^1-function $f : X \to \mathbb{R}$ is in an analytic-geometric category $\mathcal{C}$ and it is proper, then its critical values form a discrete set in $\mathbb{R}$. For more general and precise statement, see Lemma 1.1.5. This tells us that the map $f = x^2 \sin(\frac{1}{x}) : \mathbb{R} \to \mathbb{R}$ does not belong to any $\mathcal{C}$, and gives us a sense that certain pathological behavior of arbitrary functions and subsets of manifolds are ruled out by the analytic-geometric setting.

The following is a brief recollection of background results from [32]. All manifolds here are assumed to be real analytic, unless otherwise specified.

Definition

An *analytic-geometric category* $\mathcal{C}$ assigns every analytic manifold M a collection of subsets in M, denoted as $\mathcal{C}(M)$, satisfying the following axioms:
(1) $M \in \mathcal{C}(M)$ and $\mathcal{C}(M)$ is a Boolean algebra, namely, it is closed under the standard operations $\cap, \cup, (-)^c$ (taking complement);
(2) If $A \in \mathcal{C}(M)$, then $A \times \mathbb{R} \in \mathcal{C}(M \times \mathbb{R})$;
(3) For any proper analytic map $f : M \to N$, $f(A) \in \mathcal{C}(N)$ for all $A \in \mathcal{C}(M)$;

(4) If $\{U_i\}_{i\in I}$ is an open covering of M, then $A \in \mathcal{C}(M)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$ for all $i \in I$;
(5) Any bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

It is easy to construct a category $\mathcal{C}$ from these data. Namely, define objects as all pairs (A, M) with $A \in \mathcal{C}(M)$, and a morphism $f : (A, M) \to (B, N)$ to be a continuous map $f : A \to B$, such that the graph $\Gamma_f \subset A \times B$ is lying in $\mathcal{C}(M \times N)$. We will always omit the ambient manifolds, and will call A a $\mathcal{C}$-set and $f : A \to B$ a $\mathcal{C}$-map.

The smallest analytic-geometric category is the *subanalytic category* $\mathcal{C}_{an}$ consisting of subanalytic subsets and continuous subanalytic maps. It is enough to assume that $\mathcal{C} = \mathcal{C}_{an}$ throughout the dissertation, but we work in more generality.

Basic facts

Derivatives

Let A be a $(C^1, \mathcal{C})$-submanifold of M. If $A \in \mathcal{C}(M)$, then its tangent bundle TA is a $\mathcal{C}$-set of TM, and its conormal bundle $T^*_A M$ is a $\mathcal{C}$-set in T^*M.

Curve Selection Lemma.

Lemma 1.1.1. *Let $A \in \mathcal{C}(M)$. For any $x \in \overline{A} - A$ and $p \in \mathbb{Z}_{>0}$, there is a $\mathcal{C}$-curve, i.e. a $\mathcal{C}$-map $\rho : [0, 1) \to \overline{A}$, of class C^p with $\rho(0) = x$ and $\rho((0, 1)) \subset A$.*

Defining functions

For any closed set A in M, a *defining function* for A is a function $f : M \to \mathbb{R}$ satisfying $A = \{f = 0\}$.

Proposition 1.1.2. *For any closed $\mathcal{C}$-set A and any positive integer p, there exists a $(C^p, \mathcal{C})$-defining function for A.*

Remark 1.1.3. In Chapter 2, we frequently use the notion of a function f satisfying $\{f > 0\} = V$ for a given open $\mathcal{C}$-set V, and we will call f a *semi-defining function* of V.

Whitney stratifications.

(1) Let $M = \mathbb{R}^N$. A pair of C^p submanifolds (X, Y) in M ($\dim X = n, \dim Y = m$) is said to satisfy the *Whitney property* if
(a) (Whitney property A) For any point $y \in Y$ and any sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ approaching y, if $\lim_{k\to\infty} T_{x_k} X$ exists and equal to τ in $Gr_n(\mathbb{R}^N)$, then $T_y Y \subset \tau$;
(b) (Whitney property B) In addition to the assumptions in (a), let $\{y_k\}_{k\in\mathbb{N}} \subset Y$ be any sequence approaching y. If the limit of the secant lines $\lim_{k\to\infty} \overline{x_k y_k}$ exists and equal to ℓ, then $\ell \subset \tau$.

It is easy to see that Whitney property B implies Whitney property A. The Whitney property obviously extends for any manifold M, just by covering M with local charts.

(2) A C^p *stratification* of a closed subset P is a locally finite partition by C^p-submanifolds $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ satisfying

$$S_\alpha \cap \overline{S}_\beta \neq \emptyset, \alpha \neq \beta \Rightarrow S_\alpha \subset \overline{S}_\beta - S_\beta.$$

A *Whitney stratification* of P in class C^p is a C^p stratification $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ such that every pair (S_α, S_β) satisfies the Whitney property.

We will also need the following notions:

(i) We say that a collection of subsets in M, $\mathcal{A}$, is *compatible* with another collection of subsets $\mathcal{B}$, if for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have either $A \cap B = \emptyset$ or $A \subset B$.

(ii) Two stratifications $\mathcal{S}$ and $\mathcal{T}$ are said to be *transverse*, if for any $S_\alpha \in \mathcal{S}$ and $T_\beta \in \mathcal{T}$, we have $S_\alpha \pitchfork T_\beta$. It is clear that

$$\mathcal{S} \cap \mathcal{T} := \{S_\alpha \cap T_\beta : S_\alpha \in \mathcal{S}, T_\beta \in \mathcal{T}\}$$

is also a stratification.

(iii) Let $f : P \to N$ be a C^1-map, and $\mathcal{S}, \mathcal{T}$ be C^p-stratifications of P and N respectively. The pair $(\mathcal{S}, \mathcal{T})$ is called a *C^p-stratification of f* if $f(S_\alpha) \in \mathcal{T}$ for all $S_\alpha \in \mathcal{S}$, and the map $S_\alpha \to f(S_\alpha)$ is a submersion.

Now assume $\mathcal{C} = \mathcal{C}_{\text{an}}$, $\mathcal{C}^{\mathbb{R}}_{\text{an}}$ or $\mathcal{C}_{\text{an,exp}}$ (see the definitions in [32]).

Proposition 1.1.4. *Let P be a closed $\mathcal{C}$-set in M. Let $\mathcal{A}, \mathcal{B}$ be locally finite collections of $\mathcal{C}$-sets in M, N respectively.*
(a) There is a C^p-Whitney stratification $\mathcal{S} \subset \mathcal{C}(M)$ of P that is compatible with $\mathcal{A}$, and has connected and relatively compact strata.
(b) Let $f : P \to N$ be a proper $(C^1, \mathcal{C})$-map. Then there exists a C^p-Whitney stratification $(\mathcal{S}, \mathcal{T}) \subset \mathcal{C}(M) \times \mathcal{C}(N)$ of f such that $\mathcal{S}$ and $\mathcal{T}$ are compatible with $\mathcal{A}$ and $\mathcal{B}$ respectively, and have connected and relatively compact strata.

One can make the strata in (a), (b) to be all cells.

(iv) For any C^p Whitney stratification $\mathcal{S}$ of M, define its associated conormal

$$\Lambda_{\mathcal{S}} := \bigcup_{S_\alpha\in\mathcal{S}} T^*_{S_\alpha} X.$$

Let $f : X \to \mathbb{R}$ be a C^1-map. We say $x \in X$ is a *$\Lambda_{\mathcal{S}}$-critical point of f* if $df_x \in \Lambda_{\mathcal{S}}$. We say $w \in \mathbb{R}$ is a *$\Lambda_{\mathcal{S}}$-critical value of f* if $f^{-1}(w)$ contains a $\Lambda_{\mathcal{S}}$-critical point. More generally, let $\mathbf{f} = (f_1, ..., f_n) : M \to \mathbb{R}^n$ be a proper $(C^1, \mathcal{C})$-map. We say that x is a *critical point* of $\mathbf{f}$ if there is a nontrivial linear combination of $(df_i)_x, i = 1, ..., n$ contained in $\Lambda_{\mathcal{S}}$. Similarly, $\mathbf{w} \in \mathbb{R}^n$ is called a *critical value of* $\mathbf{f}$ if $\mathbf{f}^{-1}(\mathbf{w})$ contains a critical point. Otherwise, $\mathbf{w}$ is called a *regular value of* $\mathbf{f}$.

If in addition $\mathcal{S} \subset \mathcal{C}(M)$ and $f : X \to \mathbb{R}$ is a proper $\mathcal{C}$-map, then we apply Curve Selection Lemma (Lemma 1.1.1) and have

Lemma 1.1.5. *The $\Lambda_{\mathcal{S}}$-critical values of f form a discrete subset of $\mathbb{R}$.*

We will need the following variant of the notion of a fringed set from [7], which is also used in [22].

Definition 1.1.6. A *fringed set* R in $\mathbb{R}^n_+$ is an open subset satisfying the following properties. For $n = 1$, $R = (0, r)$ for some $r > 0$. For $n > 1$, the image of R under the projection $\mathbb{R}^n_+ \to \mathbb{R}^{n-1}_+$ to the first $n-1$ entries is a fringed set in $\mathbb{R}^{n-1}_+$, and if $(r_1, ..., r_{n-1}, r_n) \in R$, then $(r_1, ..., r_{n-1}, r'_n) \in R$ for all $r'_n \in (0, r_n)$.

Corollary 1.1.7. *Let $\mathbf{f} = (f_1, ..., f_n) : M \to \mathbb{R}^n$ be a proper $(C^1, \mathcal{C})$-map. Then there is a fringed set $R \subset \mathbb{R}^n_+$ consisting of $\Lambda_{\mathcal{S}}$-regular values of $\mathbf{f}$.*

Assumptions on X and Lagrangian submanifolds in T^*X

Throughout this chapter and Chapter 2, X is assumed to be a compact real analytic manifold or compact complex manifold. Then T^*X is real analytic. The projectivization $\overline{T^*X} = (T^*X \times \mathbb{R}_{\geq 0} - T^*_X X \times \{0\})/\mathbb{R}^+$ is a semianalytic subset in the manifold $\mathbb{P}_+(T^*X \times \mathbb{R}) = (T^*X \times \mathbb{R} - T^*_X X \times \{0\})/\mathbb{R}^+$.

Fix an analytic-geometric category $\mathcal{C}$. Define $\mathcal{C}$-sets in $\overline{T^*X}$ to be $\mathcal{C}$-sets in $\mathbb{P}_+(T^*X \times \mathbb{R})$ intersecting $\overline{T^*X}$. All Lagrangian submanifolds L in T^*X are assumed to satisfy $\overline{L} \subset \overline{T^*X}$ a $\mathcal{C}$-set in $\overline{T^*X}$. All subsets of X are assumed to be $\mathcal{C}$-sets unless otherwise specified.

1.2 A_∞-categories

Roughly speaking, A_∞-category is a form of category whose structure is more complicated but more flexible than the classical notion of category: composition of morphisms are not strictly associative but only associative up to higher homotopies, and there are also successive homotopies between homotopies. In this section, we will briefly recall the definition of A_∞-category, left and right A_∞-modules and A_∞-triangulation. The materials are from Chapter 1 [26].

A_∞-categories and A_∞-functors

A *non-unital A_∞-category* $\mathcal{A}$ consists of the following data:
(1) a collection of objects $X \in Ob\mathcal{A}$,
(2) for each pair of objects X, Y, a morphism space $Hom_{\mathcal{A}}(X, Y)$ which is a cochain complex of vector spaces over $\mathbb{C}$,
(3) for each $d \geq 1$ and sequence of objects $X_0, ..., X_d$, a linear morphism

$$m^d_{\mathcal{A}} : Hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes Hom_{\mathcal{A}}(X_0, X_1) \to Hom_{\mathcal{A}}(X_0, X_d)[2-d]$$

satisfying the following identities

$$\sum_{\substack{k+l=d+1,k,l\geq 1\\ 0\leq i\leq d-l}} (-1)^{\dagger_i} m^k_{\mathcal{A}}(a_d,...,a_{i+l+1},m^l_{\mathcal{A}}(a_{i+l},...,a_{i+1}),a_i,...,a_1)=0, \tag{1.1}$$

where $\dagger_i = |a_1|+\cdots+|a_i|-i$ and $a_j \in Hom_{\mathcal{A}}(X_{j-1},X_j)$ for $1\leq j\leq d$.

A special case of an A_∞-category is a dg-category where all the higher compositions $m^d_{\mathcal{A}}, d\geq 3$ vanish.

From the above definition, at the cohomological level for $[a_1]\in H(Hom_{\mathcal{A}}(X_0,X_1),m^1_{\mathcal{A}})$ and $[a_2]\in H(Hom_{\mathcal{A}}(X_1,X_2),m^1_{\mathcal{A}})$, their composition $[a_2]\cdot[a_1] := (-1)^{|a_1|}[m^2_{\mathcal{A}}(a_2,a_1)] \in H(Hom_{\mathcal{A}}(X_0,X_2)),m^1_{\mathcal{A}})$ is well defined, and it is easy to check that the product is associative. We will let $H(\mathcal{A})$ denote the non-unital graded category arising in this way. There is also a subcategory $H^0(\mathcal{A})\subset H(\mathcal{A})$ which only has morphisms in degree 0.

An A_∞-category is called *c-unital* if $H(\mathcal{A})$ is unital. All the A_∞-categories we will encounter are c-unital unless otherwise specified. We will always omit the prefix c-unital at those places. One major benefit of dealing with c-unital A_∞-categories is that one can talk about quasi-equivalence between categories, see below.

Given two non-unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, a *non-unital A_∞-functor* $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ assigns each $X\in Ob\mathcal{A}$ an object $\mathcal{F}(X)$ in $\mathcal{B}$, and it consists for every $d\geq 1$ and sequence of objects $X_0,...,X_d\in Ob\mathcal{A}$, of a linear morphism

$$\mathcal{F}^d : Hom_{\mathcal{A}}(X_{d-1},X_d)\otimes\cdots\otimes Hom_{\mathcal{A}}(X_0,X_1)\to Hom_{\mathcal{B}}(\mathcal{F}(X_0),\mathcal{F}(X_d))[1-d],$$

satisfying the identities

$$\begin{aligned}&\sum_{k\geq 1}\sum_{\substack{s_1+...+s_k=d,\\ s_i\geq 1}} m^k_{\mathcal{B}}(\mathcal{F}^{s_k}(a_d,...,a_{d-s_k+1}),...,\mathcal{F}^{s_1}(a_{s_1},...,a_1))\\ =&\sum_{\substack{k+l=d+1,k,l\geq 1\\ 0\leq i\leq d-l}} (-1)^{\dagger_i}\mathcal{F}^k(a_d,...,a_{i+l+1},m^l_{\mathcal{A}}(a_{i+l},...,a_{i+1}),a_i,...,a_1)).\end{aligned}$$

The composition of two A_∞-functors $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ and $\mathcal{G}:\mathcal{B}\to\mathcal{C}$ is defined as

$$(\mathcal{G}\circ\mathcal{F})^d(a_d,...,a_1)=\sum_{k\geq 1}\sum_{\substack{s_1+...+s_k=d,\\ s_i\geq 1}} \mathcal{G}^k(\mathcal{F}^{s_k}(a_d,...,a_{d-s_k+1}),...,\mathcal{F}^{s_1}(a_{s_1},...,a_1)).$$

It is clear that $\mathcal{F}$ descends on the cohomological level to a functor from $H(\mathcal{A})$ to $H(\mathcal{B})$, which we will denote by $H(\mathcal{F})$. One easy example of a functor from $\mathcal{A}$ to itself is the identity functor $id_{\mathcal{A}}$, which is identity on objects and hom spaces and $id^k_{\mathcal{A}}=0$ for $k\geq 2$.

Let $\mathcal{Q}=nu\text{-}fun(\mathcal{A},\mathcal{B})$ be the A_∞-category of non-unital A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$ defined as follows. An element $T=(T^0,T^1,...)$ of degree $|T|=g$, called a *pre-module homomorphism*, in $hom_{\mathcal{Q}}(\mathcal{F},\mathcal{G})$ is a sequence of linear maps

$$T^d : Hom_{\mathcal{A}}(X_{d-1},X_d)\otimes\cdots\otimes Hom_{\mathcal{A}}(X_0,X_1)\to Hom_{\mathcal{B}}(\mathcal{F}(X_0),\mathcal{G}(X_d))[g-d],$$

in particular, T^0 is an element in $Hom_{\mathcal{B}}(\mathcal{F}(X), \mathcal{G}(X))$ of degree g for each X.

We also have the following structures

$$
\begin{aligned}
&(m^1_{\mathcal{Q}}(T))^d(a_d, ..., a_1) \\
&= \sum_{1\le i\le k} \sum_{\substack{s_1+\cdots+s_k=d,\\ s_i\ge 0, s_j\ge 1, j\neq i}} (-1)^{\dagger} m^k_{\mathcal{B}}(\mathcal{G}^{s_k}(a_d, ..., a_{d-s_k+1}), ..., \mathcal{G}^{s_{i+1}}(a_{s_1+\cdots+s_{i+1}}, ..., a_{s_1+\cdots+s_i+1}), \\
&T^{s_i}(a_{s_1+\cdots+s_i}, ..., a_{s_1+\cdots+s_{i-1}+1}), \mathcal{F}^{s_{i-1}}(a_{s_1+\cdots+s_{i-1}}, ..., a_{s_1+\cdots+s_{i-2}+1}), ..., \mathcal{F}^{s_1}(a_{s_1}, \cdots, a_1)) \\
&- \sum_{\substack{r+l=d+1, r, l\ge 1\\ 1\le i\le d-l}} (-1)^{\dagger_i+|T|-1} \sum T^r(a_d, ..., a_{i+l+1}, m^l_{\mathcal{A}}(a_{i+l}, ..., a_{i+1}), a_i, ..., a_1))
\end{aligned}
$$

If we write the right hand side of the above formula for short as

$$\sum m_{\mathcal{B}}(\mathcal{G}, ..., \mathcal{G}, T, \mathcal{F}, ..., \mathcal{F}) - \sum T(id, ..., id, m_{\mathcal{A}}, id, ..., id),$$

then for $T_0 \in Hom_{\mathcal{Q}}(\mathcal{F}_0, \mathcal{F}_1)$ and $T_1 \in Hom_{\mathcal{Q}}(\mathcal{F}_1, \mathcal{F}_2)$, we have

$$m^2_{\mathcal{Q}}(T_1, T_0) = \sum m_{\mathcal{B}}(\mathcal{F}_2, ..., \mathcal{F}_2, T_1, \mathcal{F}_1, ..., \mathcal{F}_1, T_0, \mathcal{F}_0, ..., \mathcal{F}_0),$$

and similar formulas apply to higher differentials $m^d_{\mathcal{Q}}$ for $d > 2$. Note that there is no $m_{\mathcal{A}}$ involved in $m^d_{\mathcal{Q}}$ for $d \ge 2$.

Those T for which $m^1_{\mathcal{Q}}(T) = 0$ are the *module homomorphisms*, and $H(T)$ in $H(\mathcal{Q})$ descends to a natural transformation between $H(\mathcal{F})$ and $H(\mathcal{G})$ under the map

$$H(nu\text{-}fun(\mathcal{A}, \mathcal{B})) \to Nu\text{-}fun(H(\mathcal{A}), H(\mathcal{B})).$$

Assume $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$ are two A_∞-functors such that $\mathcal{F}(X) = \mathcal{G}(X)$ for every $X \in \mathrm{Ob}(\mathcal{A})$. Then $\mathcal{F}$ and $\mathcal{G}$ is called *homotopic* if there is $T \in hom^{-1}_{\mathcal{Q}}(\mathcal{F}, \mathcal{G})$ such $m^1_{\mathcal{Q}}(T)^d = \mathcal{G}^d - \mathcal{F}^d$. We have $H(\mathcal{F}) = H(\mathcal{G})$ if $\mathcal{F}$ and $\mathcal{G}$ are homotopic.

Let $\mathcal{A}$, $\mathcal{B}$ be c-unital A_∞-categories, a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is called c-unital if $H(\mathcal{F})$ is unital. Then the full subcategory $fun(\mathcal{A}, \mathcal{B}) \subset \mathcal{Q}$ consisting of c-unital functors is a c-unital A_∞-category.

A c-unital functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if $H(\mathcal{F}) : H(\mathcal{A}) \to H(\mathcal{B})$ is an equivalence of categories.

A_∞-modules and Yoneda embedding

In this subsection, we will assume all A_∞-categories to be c-unital.

Define the A_∞-category of left $\mathcal{A}$-modules as

$$l\text{-}mod(\mathcal{A}) = fun(\mathcal{A}, Ch).$$

Explicitly, any $\mathcal{M} \in l\text{-}mod(\mathcal{A})$ assigns a cochain complex $\mathcal{M}(X)$ to each object X and we have

$$m^d_{\mathcal{M}} : Hom_{\mathcal{A}}(X_{d-1}, X_d) \cdots \otimes Hom_{\mathcal{A}}(X_0, X_1) \otimes \mathcal{M}(X_0) \to \mathcal{M}(X_d)[2-d]$$

with the property that

$$\sum m_{\mathcal{M}}(id, .., id, m_{\mathcal{M}}) + \sum m_{\mathcal{M}}(id, ..., id, m_{\mathcal{A}}, id, ..., id) = 0,$$

where there is at least one id after $m_{\mathcal{A}}$ in the second term.

An important example of a left $\mathcal{A}$-module is $\tilde{\mathcal{Y}}_{X_0}$ for $X_0 \in Ob\mathcal{A}$ defined as $\tilde{\mathcal{Y}}_{X_0}(X) = Hom_{\mathcal{A}}(X_0, X)$ and $\tilde{\mathcal{Y}}^d_{X_0}$ coincides with $m^d_{\mathcal{A}}$.

The category of right $\mathcal{A}$-modules $mod(\mathcal{A})$ (following usual convention, we don't denote it by $r\text{-}mod(\mathcal{A})$) can be defined similarly as $fun(\mathcal{A}^{opp}, Ch)$. An important example is $\mathcal{Y}_{X_0}$ defined as $\mathcal{Y}_{X_0}(X) = Hom_{\mathcal{A}}(X, X_0)$ and this gives the *Yoneda embedding*

$$\begin{aligned} \mathcal{Y} : &\mathcal{A} \to mod(\mathcal{A}) \\ &X \mapsto \mathcal{Y}_X. \end{aligned}$$

For $c_i \in Hom_{\mathcal{A}}(Y_{i-1}, Y_i)$, $1 \leq i \leq d$,

$$\mathcal{Y}(c_d, ..., c_1)^k : \mathcal{Y}_{Y_0}(X_k) \otimes Hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \cdots \otimes Hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{Y}_{Y_d}(X_0)$$

is $m^{k+d+1}_{\mathcal{A}}(c_d, ..., c_1, b, a_k, ..., a_1)$ for $b \in \mathcal{Y}_{Y_0}(X_k)$ and $a_i \in Hom_{\mathcal{A}}(X_{i-1}, X_i)$.

Note that $mod(\mathcal{A})$ is a dg-category, and the Yoneda embedding $\mathcal{Y}$ is cohomologically full and faithful. This gives a construction showing that every A_∞-category is quasi-equivalent to a (strictly unital) dg-category, i.e. its image under $\mathcal{Y}$.

For $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, we can define the associated pull-back functor

$$\begin{aligned} \mathcal{F}^* : mod(\mathcal{B})(\text{resp. } l\text{-}mod(\mathcal{B})) &\to mod(\mathcal{A})(\text{resp. } l\text{-}mod(\mathcal{A})) \\ \mathcal{M} &\mapsto \mathcal{M} \circ \mathcal{F}. \end{aligned}$$

A_∞-triangulation

Recall that a triangulated envelope of an A_∞-category $\mathcal{A}$ is a pair $(\mathcal{B}, \mathcal{F})$ of a triangulated A_∞-category and a quasi-embedding $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B}$ is generated by the image of objects in $\mathcal{A}$. We refer the reader to Section 3, Chapter 1 in [26] for the definition of triangulated A_∞-categories. Any two triangulated envelopes of $\mathcal{A}$ are quasi-equivalent.

There are basically two ways of constructing A_∞-triangulated envelope. One is to take the usual triangulated closure of the image of $\mathcal{A}$ under the Yoneda embedding in $mod(\mathcal{A})$, since $mod(\mathcal{A})$ is triangulated. The other is by taking twisted complexes of $\mathcal{A}$ which we denote by $Tw(\mathcal{A})$. The formulation in the definition is a little bit long and messy, which we don't really need in this dissertation, so we refer the reader to consult Seidel [26] Section 3 for a detailed description.

1.3 Infinitesimal Fukaya Categories

In this section, we review the definition of the infinitesimal Fukaya category on a Liouville manifold, originated from [22]. This section is by no means a complete or rigorous exposition of Fukaya categories. One could consult [2] for a comprehensive introduction, and Seidel's book [26] for a complete and rigorous treatment.

Our goal here is to give a rough idea of how the Fukaya category (in the exact setting) is defined, and what kind of extra structures one should put on the ambient symplectic manifold and on the Lagrangian submanifolds so to give a coherent definition of the A_∞-structure. We also include several specific facts about $Fuk(T^*X)$, which will supplement Chapter 2.

Assumptions on the ambient symplectic manifold

Let $(M, \omega = d\theta)$ be a $2n$-dimensional Liouville manifold. By definition, M is obtained by gluing a compact symplectic manifold with contact boundary $(M_0, \omega_0 = d\theta_0)$ with an infinite cone $(\partial M_0 \times [1, \infty), d(r\theta_0|_{\partial M_0}))$ along ∂M_0, where r is the coordinate on $[1, \infty)$. We require that the the *Liouville vector field* Z, defined by the property $\iota_Z \omega = \theta$, is pointing outward along ∂M_0, and the gluing is by identifying Z with $r\partial_r$.

Let J be a ω-compatible almost complex structure on (M, ω), whose restriction to the cone $\partial M_0 \times [S, \infty)$ for $S >> 0$, satisfies that $J\partial_r = R$, where R is the Reeb vector field of $r\theta_0|_{\partial M_0 \times \{r\}}$, and J preserves $\ker(r\theta_0|_{\partial M_0 \times \{r\}})$, on which it is induced from $J|_{\partial M_0 \times \{S\}}$. We will call such a J a *conical* almost complex structure. It is a basic fact that the space of all such almost complex structures is contractible. The compatible metric g will be conical near infinity, i.e. $g = r^{-1}dr^2 + S^{-1}rds^2$, where $ds^2 = \omega(\cdot, J\cdot)|_{\partial M_0 \times \{S\}}$. Let $\mathcal{H}$ be the set of Hamiltonian functions whose restriction to $\partial M_0 \times [S, \infty)$ is r for $S >> 0$. Note that the Hamiltonian vector field X_H of $H \in \mathcal{H}$ near infinity is $-rR$.

One can compactify M using the cone structure, i.e. $\overline{M} = M_0 \cup \{[t_0 x : t_1] | x \in \partial M_0, t_0, t_1 \in \mathbb{R}^+, t_0^2 + t_1^2 \neq 0\}$, here $[t_0 x : t_1]$ denotes the equivalence class of the relation $(t_0 x, t_1) \sim (\lambda t_0 x, \lambda t_1)$ for $\lambda > 0$. It is easy to see that $\overline{M} = M \cup M^\infty$, where we think of an element (x, r) in the cone as $[rx : 1]$ and the points in M^∞ are of the form $[x : 0]$.

Floer theory with $\mathbb{Z}/2\mathbb{Z}$-coefficients and gradings

To obtain well defined Floer theory for noncompact Lagrangian submanifolds, we should be more careful about their behavior near infinity. First we restrict ourselves in some fixed analytic-geometric setting $\mathcal{C}$, and require that the Lagrangians L we are considering satisfy $\overline{L}$ is a $\mathcal{C}$-set in $\overline{M}$ (see 1.1). Second, we need to ensure compactness of holomorphic discs with Lagrangian boundary conditions. A sufficient condition for this is the *tameness* condition following [29]. We will discuss this in more detail in the next section.

Recall the Floer theory defines for each pair of Lagrangians L_1, L_2 in M a $\mathbb{Z}/2\mathbb{Z}$-graded cochain complex

$$CF^*(L_0, L_1) := (\bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}/2\mathbb{Z}\langle p \rangle, \partial_{CF}) \tag{1.2}$$

$$\partial_{CF}(p) = \sum_{q \in L_1 \cap L_2} \sharp \mathcal{M}(p, q; L_0, L_1)^{0\text{-d}} \cdot q, \tag{1.3}$$

where $\mathcal{M}(p, q; L_0, L_1)^{k\text{-d}}$ is the quotient (by $\mathbb{R}$-symmetry) of the $(k+1)$-dimensional locus of the moduli space $\hat{\mathcal{M}}(p, q; L_0, L_1)$ of holomorphic strips, starting from q, ending at p and bounding L_0, L_1, i.e. a map

$$u : \mathbb{R} \times [0, 1] \to M, \text{ such that}$$

$$\lim_{s \to -\infty} u(s, t) = q, \lim_{s \to +\infty} u(s, t) = p \tag{1.4}$$

$$u(\mathbb{R} \times \{0\}) \subset L_0, u(\mathbb{R} \times \{1\}) \subset L_1 \tag{1.5}$$

$$(du)^{0,1} = 0 (\Leftrightarrow \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0). \tag{1.6}$$

There are always several technical issues to be clarified in the above definition.

(a) *Transverse intersections.* Implicit in (1.2) is the step of Hamiltonian perturbation to make L_0 and L_1 transverse. Let L_i^∞ denote $\overline{L_i} \cap M^\infty$. If $L_0^\infty \cap L_1^\infty = \emptyset$, then one chooses a generic Hamiltonian function $\widetilde{H}$, whose Hamiltonian vector field vanishes on L_1 outside a compact region, and replaces L_1 by $\phi_{\widetilde{H}}^t(L_1)$ for small $t > 0$. If $L_0^\infty \cap L_1^\infty \neq \emptyset$, then one replaces L_1 by $\phi_H^t(L_1)$ for a generic $H \in \mathcal{H}$. It can be shown that $\phi_H^t(L_1^\infty)$ will be apart from L_0^∞ for sufficiently small $t > 0$. The invariance of Floer theory under Hamiltonian perturbations ensures that the complex $CF^*(L_0, L_1)$ is well defined up to quasi-isomorphisms.

(b) *Regularity of Moduli space of strips.* One views the $\overline{\partial}$-operator on u, i.e. $(du)^{0,1}$, as a section of a natural Banach vector bundle over a suitable space of maps u satisfying (1.4) and (1.5). Then $\hat{\mathcal{M}}(p, q; L_0, L_1)$ becomes the intersection of $\overline{\partial}$ with the zero section. We need the intersection to be transverse, and this is equivalent to the linearized operator D_u (a Fredholm operator) of $\overline{\partial}$ at any $u \in \overline{\partial}^{-1}(0)$ being surjective. In many good settings (including the cases in $Fuk(T^*X)$), this is true for a generic choice of J, which we will refer as a *regular* (compatible) almost complex structure. Then by Gromov's compactness theorem, $\mathcal{M}(p, q; L_0, L_1)^{0\text{-d}}$ is a compact manifold, so $\sharp \mathcal{M}(p, q; L_0, L_1)^{0\text{-d}}$ is finite. Different choices of regular J's give cobordant moduli spaces, therefore the number doesn't depend on such choices (note that we are working over $\mathbb{Z}/2\mathbb{Z}$, so we don't need any orientation on $\mathcal{M}(p, q; L_0, L_1)$ to conclude this). More generally, one would need to introduce time-dependent almost complex structures and Hamiltonian perturbations to achieve transversality.

(c) $\partial_{CF}^2 = 0$. This is ensured when no sphere or disc bubbling occurs, and it holds for a pair of *exact* Lagrangians L_0, L_1, i.e. $\theta|_{L_j}$ is an exact 1-form for $j = 0, 1$. To verify this, one studies the boundary of the 1-dimensional moduli space $\mathcal{M}(p, q; L_0, L_1)^{1\text{-d}}$ of holomorphic

strips starting at q and ending at p, and realizes that they are broken trajectories corresponding exactly to the terms involving q in $\partial^2_{CF}(p)$. Since the number of boundary points is even, $\partial^2_{CF} = 0$.

(d) *Gradings.* For any holomorphic strip u connecting q to p, the Fredholm index of the linearized Cauchy–Riemann operator D_u "in principle" gives the relative grading between p and q. The index can be calculated by the *Maslov index* of u defined as follows. A strip $\mathbb{R} \times [0,1]$ is conformally identified with the closed unit disc D, with two punctures on the boundary. Then one can trivialize the symplectic vector bundle u^*TM over the closed unit disc, and think of T_pL_j, T_qL_j for $j = 0, 1$ as elements in the Lagrangian Grassmannian $LGr(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$.

By a standard fact from linear symplectic geometry, there is a unique set of numbers $\{\alpha_k \in (-\frac{1}{2}, 0)\}_{k=1,...,n}$ such that relative to an orthonormal basis $\{v_1, ..., v_n\}$ of T_pL_0, T_pL_1 is spanned by $e^{2\pi\sqrt{-1}\alpha_k}v_k$ for $k = 1, ..., n$. One could consult Lemma 3.3 in [1] for a proof. Since we will use it in Proposition 2.5.2, we discuss this in a little more detail. First, this property is invariant under $U(n)$-transformation, so we can assume $T_pL_0 = \mathbb{R}^n \subset \mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^n$. There is a standard way to produce a unitary matrix U such that $T_pL_1 = U \cdot T_pL_0$, namely choose a symmetric matrix A in $GL_n(\mathbb{R})$ for which $T_pL_1 = (A + \sqrt{-1}I) \cdot T_pL_0$, then let $U = (A + \sqrt{-1}I)(A^2 + I^2)^{-\frac{1}{2}}$. Also for any $B + \sqrt{-1}C \in U(n)$ satisfying $T_pL_1 = U \cdot T_pL_0$, we have $B + \sqrt{-1}C = (A + \sqrt{-1}I)(A^2 + I^2)^{-\frac{1}{2}}O$ for some $O \in O(n)$, and $BC^{-1} = A$. Now let $\{v_1, ..., v_n\}$ be an orthonormal collection of eigenvectors of A, hence of U as well, and $e^{2\pi\sqrt{-1}\alpha_1}, ..., e^{2\pi\sqrt{-1}\alpha_k}$, $\alpha_j \in (-\frac{1}{2}, 0)$ be their corresponding eigenvalues of U. Then $\{\alpha_j\}_{j=1,...,n}$ is the desired collection of numbers.

Then $\lambda_p(t) := \mathrm{Span}\{e^{2\pi\sqrt{-1}\alpha_j t}v_j\} \in LGr(\mathbb{R}^{2n}, \omega_0), t \in [0,1]$ is the so called *canonical short path* from T_pL_0 to T_pL_1. Let λ_q be the canonical short path from T_qL_0 to T_qL_1, and $\ell_j, j = 0, 1$ denote the path of tangent spaces to L_j from q to p in $u^*TM|_{\partial D}$. Then the *Maslov index* of u, denoted as $\mu(u)$, is defined to be the Maslov number of the loop by concatenating the paths $\ell_0, \lambda_p, -\ell_1, -\lambda_q$.

In general, $\mu(u)$ depends on the homotopy class of u, so wouldn't give well defined relative degree between p and q. But if L_0, L_1 are both oriented, we have a well defined grading, namely, $\deg(p) = 0$ if λ_p takes the orientation of L_0 into the orientation of L_1, otherwise, $\deg p = 1$. In the next section, we will see that under certain assumptions, we will not only get $\mathbb{Z}/2\mathbb{Z}$-gradings on the Floer complex, but $\mathbb{Z}$-gradings.

(e) *Product structure.* Consider three Lagrangians L_0, L_1, L_2, then one can define a linear map

$$m : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$
$$m(a_1, a_0) = \sum_{a_2 \in L_0 \cap L_2} \sharp\mathcal{M}(a_0, a_1, a_2; L_0, L_1, L_2)^{0\text{-d}} \cdot a_2.$$

$\mathcal{M}(a_0, a_1, a_2; L_0, L_1, L_2)^{0\text{-d}}$ is the 0-dimensional locus of the moduli space of equivalence class of holomorphic maps

$$u : (D, \{0, 1, 2\}) \to (M, \{a_0, a_1, a_2\}), u(\overline{i(i+1)}) \subset L_i, i \in \mathbb{Z}/(3\mathbb{Z})$$

where $0, 1, 2$ are three (counterclockwise) marked points on ∂D, and $\overline{i(i+1)}$ denotes the arc in ∂D connecting i and $i+1$. The equivalence relation is composition with conformal maps of the domain. Since the conformal structure of a disc with three marked points on the boundary is unique (and there is no nontrivial automorphism), we can just fix a conformal structure once for all.

As before one needs to separate L_0, L_1, L_2 near infinity if necessary, and the separation process obeys a principle called *propagating forward in time*. Namely one replaces L_i by $\phi^{t_i}_{H_i}(L_i)$, for some $H_i \in \mathcal{H}, i = 0, 1, 2$, and the choices of $(t_2, t_1, t_0) \in \mathbb{R}^3_+$ should be in a fringed set (see Definition 1.1.6). The regularity issue about $\mathcal{M}(a_0, a_1, a_2; L_0, L_1, L_2)$ is similar to that of (b).

Similarly to (c), by looking at the boundary of $\mathcal{M}(a_0, a_1, a_2; L_0, L_1, L_2)^{1\text{-d}}$, one concludes the following equation

$$m(\partial_{CF}\cdot, \cdot) + m(\cdot, \partial_{CF}\cdot) + \partial_{CF} m(\cdot, \cdot) = 0.$$

This means that m induces a multiplication on the cohomological level HF^*. We will see later that m is not strictly associative, but associative up to homotopy.

(infinitesimal) Fukaya category of M

The preliminary version of the Fukaya category (with $\mathbb{Z}/2\mathbb{Z}$-grading, and over $\mathbb{Z}/2\mathbb{Z}$-coefficients), is an upgrade of the Floer theory, which uncovers much richer structure, the A_∞-structure, of Lagrangian intersection theory. One not only studies ∂_{CF} and m, but also studies for each sequence of $n+1$ Lagrangians the *higher compositions* μ^n

$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_d)[2-d]$$
$$\mu^d(a_{d-1}, \cdots, a_1, a_0) = \sum_{a_d \in L_0 \cap L_d} \sharp \mathcal{M}(a_0, a_1, \cdots, a_d; L_0, L_1, \cdots, L_d)^{0\text{-d}} \cdot a_d,$$

where the moduli space $\mathcal{M}(a_0, a_1, \cdots, a_d; L_0, L_1, \cdots, L_d)^{0\text{-d}}$ is defined similarly as before. Assuming regularity of the moduli spaces and no bubblings (ensured by Lagrangians being exact), the boundary of $\mathcal{M}(a_0, a_1, \cdots, a_d; L_0, L_1, \cdots, L_d)^{1\text{-d}}$ gives us the identity 1.1.

Now let's discuss the (final) version of Fukaya category with $\mathbb{Z}$-gradings, and $\mathbb{C}$-coefficients. We first collect several basic notions about T^*X which we will use in later discussions.

Some basic notions about T^*X

(a) *Almost complex structures.* Given any Riemannian metric on X, there is a ω-compatible almost complex structure, called *Sasaki almost complex structure*, J_{Sas} on T^*X defined as follows. For any point $(x, \xi) \in T^*X$, there is a canonical splitting

$$T_{(x,\xi)} T^*X = T_b \oplus T_f,$$

using the dual Levi-Civita connection on T^*X, where T_f denotes the fiber direction and T_b denotes the horizontal base direction. The metric also gives an identification $j : T_b \to T_f$ and it induces a unique almost complex structure, J_{Sas}, by requiring $J_{Sas}(v) = -j(v)$ for $v \in T_b$.

Since T^*X is a Liouville manifold, one can use the construction in Section 1.3 to get a conical almost complex structure, by requiring $J|_{|\xi|=r} = J_{Sas}|_{|\xi|=r}$ for r sufficiently large, and $J = J_{Sas}$ near the zero section. We will denote any of these almost complex structures by J_{con}.

(b) *Standard Lagrangians.* Given a smooth submanifold $Y \subset X$ and a defining function f for ∂Y which is positive on Y, we define the *standard Lagrangian*

$$L_{Y,f} = T^*_Y X + \Gamma_{d\log f} \subset T^*X|_Y. \tag{1.7}$$

It is easy to check that $L_{Y,f}$ is determined by $f|_Y$.

In Chapter 2, we often restrict ourselves to standard Lagrangians defined by an open submanifold V and a semi-defining function of V (see Remark 1.1.3).

(c) *Variable dilations.* Consider the class of Lagrangians of the form $L = \Gamma_{df}$, where f is a function on an open submanifold U with smooth boundary, and ∂U decomposes into two components $(\partial U)_{in}$ and $(\partial U)_{out}$ such that $\lim_{x\to\partial U_{in}} f(x) = -\infty$ and $\lim_{x\to\partial U_{out}} f(x) = +\infty$.

The *variable dilation* is defined by the following Hamiltonian flow. Choose $0 < A < B < 1$ and a bump function $b_{A,B} : \mathbb{R} \to \mathbb{R}$, such that $b_{A,B}(s) = s$ on $[\log B, -\log B]$ and $|b_{A,B}(s)| = -\log\sqrt{AB}$ outside $[\log A, -\log A]$. We assume that $b_{A,B}$ is odd and nondecreasing. Take a function $D^f_{A,B}$ which extends $b_{A,B} \circ \pi^* f$ to the whole T^*X. The Hamiltonian flow $\varphi^t_{D^f_{A,B}}$ fixes $L|_{X_{|f|>-\log A}}$, dilates $L|_{X_{|f|<-\log B}}$ by the factor $1-t$, and sends L to a new graph.

Compactness of moduli space of holomorphic discs: tame condition and perturbations

As we mentioned in the last section, we need certain tameness condition to ensure the compactness of the moduli space of holomorphic discs bounding a sequence of Lagrangians. The tameness condition adopted here is from Definition 4.1.1 and 4.7.1 in [29]. (M, J) is certainly a tame almost complex manifold in that sense. For a smooth submanifold N, let $d_N(\cdot,\cdot)$ denote the distance function of the metric on N induced from M. The tameness requirement on a Lagrangian submanifold L is the existence of two positive numbers δ_L, C_L, such that within any δ_L-ball in M centered at a point $x \in L$, we have $d_L(x, y) \leq C_L d_M(x, y), y \in L$, and the portion of L in that ball is contractible.

The main consequence of these is the monotonicity property on holomorphic discs from Proposition 4.7.2 (iii) in [29].

Proposition 1.3.1. *There exist two positive constants R_L, a_L, such that for all $\mathfrak{r} < R_L, x \in M$, and any compact J-holomorphic curve $u : (C, \partial C) \to (B_\mathfrak{r}(x), \partial B_\mathfrak{r}(x) \cup L)$ with $x \in u(C)$, we have Area$(u) \geq a_L \mathfrak{r}^2$.*

Remark 1.3.2. As indicated in [22], the argument of this proposition is entirely local, one could replace the pair (M, L) by an open submanifold $U \subset M$ together with a properly embedded Lagrangian submanifold W in U satisfying the tame condition. In particular, if $M = T^*X$, and W is the graph of differential of a function f over an open set which is C^1-*close to the zero section*, i.e. the norm of the partial derivatives of f has uniform bound, then one can dilate W towards the zero section, and get a uniform bound for the family $(\epsilon \cdot U, \epsilon \cdot W)$. More precisely, one could find $R_{\epsilon \cdot W} = \epsilon R_W$ and $a_{\epsilon \cdot W} = a_W$.

With the monotonicity property, one can show the compactness of moduli of discs bounding a sequence of exact Lagrangians $L_1, ..., L_k$ using standard argument. Moreover, assume $M = T^*X$, and consider the class of Lagrangians in Section 1.3 (c), then we have better control of where holomorphic discs can go bounding a sequence of such Lagrangians, see the proof of Lemma 2.4.3 and Section 6.5 in [22] for more details.

Gradings on Lagrangians and $\mathbb{Z}$-grading on CF^*

Let $\mathcal{L}Gr(TM) = \bigcup_{x \in M} LGr(T_xM, \omega_x)$ be the Lagrangian Grassmannian bundle over M. To obtain gradings on Lagrangian vector spaces in TM, we need a *universal* Lagrangian Grassmannian bundle $\widetilde{\mathcal{L}Gr(TM)}$, and this amounts to the condition that $2c_1(TM) = 0$. Choose a trivialization α of the bicanonical bundle $\kappa^{\otimes 2}$, and a grading to $\gamma \in LGr(T_xM, \omega_x)$ is a lifting of the *phase map* $\phi(\gamma) = \frac{\alpha(\Lambda^n \gamma)}{|\alpha(\Lambda^n \gamma)|} \in S^1$ to $\mathbb{R}$.

The condition $2c_1(TM) = 0$ holds if $M = T^*X$ for an n-dimensional compact manifold X. Because the pull back of $\Lambda^n TT^*X$ to the zero section X is just $\mathfrak{or}_X \otimes \mathbb{C}$, where $\mathfrak{or}_X$ is the orientation sheaf on X. Since $\mathfrak{or}_X^{\otimes 2}$ is always trivial, and X is a deformation retract of T^*X, we get $c_1(TT^*X)$ is 2-torsion. In fact, given a Riemannian metric on X, $\mathfrak{or}_X^{\otimes 2}$ is canonically trivialized, and the same for $\kappa^{\otimes 2}$.

For a Lagrangian submanifold L in M, we define a *grading* of L to be a continuous lifting $L \to \mathbb{R}$ to the phase map $\phi_L : L \to S^1$. The obstruction to this is the *Maslov class* $\mu_L = \phi_L^* \beta \in H^1(L, \mathbb{Z})$, where β is the class representing the $1 \in H^1(S^1, \mathbb{Z})$.

Proposition 1.3.3. *Standard Lagrangians and the local Morse brane $L_{x,F}$ in T^*X both admit canonical gradings.*

The reason that all these Lagrangians admit canonical grading is that they are all constructed by (properly embedded) partial graphs over smooth submanifolds. Suppose there is a loop $\Omega \subset L$ such that $(\phi_L)|_\Omega : \Omega \to S^1$ is homotopically nontrivial. Since Ω is contained in a compact subset of L, one can dilate L so that when $\epsilon \to 0$, $T(\epsilon \cdot L)|_\Omega$ is uniformly close to the tangent planes to the zero section if $L = L_{x,F}$ or to $T^*_Y X$ if $L = L_{Y,f}$. It is easy to check that $T^*_Y X$ has constant phase 1 (resp. -1) if Y has even (resp. odd) codimension, so admit canonical grading 0 (resp. 1). Then we get a contradiction, because the homotopy type of the map $(\phi_{\epsilon \cdot L})|_{\epsilon \cdot \Omega} : \epsilon \cdot \Omega \to S^1$ is unchanged under dilation, and L has a canonical grading.

Given two graded Lagrangians $L_i, \theta_i : L_i \to \mathbb{R}$, $i = 0, 1$, then for any $p \in L_0 \cap L_1$ (assuming transverse intersection), we can define an *absolute* $\mathbb{Z}$*-grading* of p:

$$\deg p = \theta_1 - \theta_0 - \sum_{i=1}^{n} \alpha_i, \tag{1.8}$$

where $\alpha_i, i = 1, ..., n$ are constants defining the canonical short path from L_0 to L_1 in Section 1.3 (d).

It is easy to check that $\mathrm{ind}(u) = \deg q - \deg p$ for any holomorphic strip u connecting q to p for $q, p \in L_0 \cap L_1$, and the absolute $\mathbb{Z}$-grading gives the $\mathbb{Z}$-grading of $CF^*(L_0, L_1)$ for two graded Lagrangians. For more details, see Section 4 in [1].

Pin-structures.

Recall that $Pin^+(n)$ is a double cover of $O(n)$ with center $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A *Pin-structure* on a manifold M of dimension n, is a lifting of the classifying map $M \to BO(n)$ of TM to a map $M \to BPin^+(n)$. The obstruction to the existence of a Pin-structure is the second Stiefel-Whitney class $w_2 \in H^2(M, \mathbb{Z}/2\mathbb{Z})$. The choices of Pin-structures form a torsor over $H^1(M, \mathbb{Z}/2\mathbb{Z})$.

For any class $[w] \in H^2(M, \mathbb{Z}/2\mathbb{Z})$, one could define the notion of a $[w]$-twisted Pin-structure on M. Fix a Čech representative w of $[w]$, and a Čech cocycle $\tau \in \check{\mathrm{C}}^1(X, O(n))$ representing the principal $O(n)$-bundle associated to TM. Then choose a Čech cochain $\tilde{w} \in \check{\mathrm{C}}^1(X, Pin^+(n))$ which is a lifting of τ under the exact sequence

$$0 \to \check{\mathrm{C}}^1(X, \mathbb{Z}/2\mathbb{Z}) \to \check{\mathrm{C}}^1(X, Pin^+(n)) \to \check{\mathrm{C}}^1(X, O(n)) \to 0.$$

We say $\tilde{w}$ defines a $[w]$-twisted Pin-structure if the Čech-coboundary of $\tilde{w}$, which obviously lies in the subset $\check{\mathrm{C}}^2(X, \mathbb{Z}/2\mathbb{Z})$, is equal to w. It is clear that the definition doesn't essentially depend on the choice of cocycle representatives, and the set of $[w]$-twisted Pin-structures, if nonempty, forms a torsor over $H^1(M, \mathbb{Z}/2\mathbb{Z})$.

Fix a background class $[w] \in H^2(M, \mathbb{Z}/2\mathbb{Z})$, for any submanifold $L \subset M$, define a *relative Pin-structure* on L to be a $[w]|_L$-twisted Pin-structure. Here we fix a Čech-representative of $[w]$, and use it for all L. Note that the existence of a relative Pin-structure only depends on the homotopy class of the inclusion $L \hookrightarrow M$.

Now let $M = T^*X$ and fix $\pi^* w_2(X)$ as the background class in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ and a relative Pin-structure on the zero section. For any smooth submanifold $Y \subset X$, the metric on X gives a canonical way (up to homotopy) to identify $T^*_Y X$ near the zero section with a tubular neighborhood of Y in X, hence there is a canonical relative Pin-structure on $T^*_Y X$ by pulling back the fixed relative Pin-structure on X. Since the inclusion $L_{Y,f} \hookrightarrow M$ in (1.7) is canonically homotopic to the inclusion $T^*_Y X \hookrightarrow M$ by dilation, and similarly for $L_{x,F} \hookrightarrow M$ with $T^*_U X \hookrightarrow M$, we have the following

Proposition 1.3.4. *The Lagrangians $L_{Y,f}$ and $L_{x,F}$ have canonical Pin-structures.*

Final definition of $Fuk(M)$

Fix a background class in $H^2(M, \mathbb{Z}/2\mathbb{Z})$.

Definition 1.3.5. A *brane structure* b on a Lagrangian submanifold $L \subset M$ is a pair $(\tilde{\alpha}, P)$, where $\tilde{\alpha}$ is a grading on L and P is a relative Pin-structure on L.

Recall that we need tame Lagrangians to ensure compactness of moduli of discs, but there are many Lagrangians, e.g. many standard Lagrangians in T^*X, which are not tame, but admit appropriate perturbations by tame Lagrangians. Therefore the following is introduced in [22].

Definition 1.3.6. A *tame perturbation* of L is a smooth family of tame Lagrangians $L_t, t \in \mathbb{R}$, with $L_0 = L$ such that
(1) Restricted to the cone $\partial M_0 \times [1, \infty)$, the map $t \times r : L_t|_{r>S} \to \mathbb{R} \times (S, \infty)$ is a submersion for $S >> 0$;
(2) Fix a defining function $m_{\overline{L}}$ for $\overline{L} \subset \overline{M}$, we require that for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $L_t \subset N_\epsilon(L) := \{m_{\overline{L}} < \epsilon\}$ for $|t| < t_\epsilon$.

Note it is enough to define the family over an open interval of 0 in $\mathbb{R}$.

Now we define $Fuk(M)$. An object in $Fuk(M)$ is a triple $(L, b, \mathcal{E})$ together with a tame perturbation $\{L_t\}_{t\in\mathbb{R}}$ of L, where (L, b) is an exact Lagrangian brane, $\mathcal{E}$ is a vector bundle with flat connection on L. It is clear that any element in the perturbation family L_t canonically inherits a brane structure, and a vector bundle with flat connection from L. In the following, we still use L to denote an object.

It is proved in Lemma 5.4.5 of [22] that every standard Lagrangian admits a tame perturbation. So for each pair (U, m) of an open submanifold $U \subset X$ and a semi-defining function m of U, there is a *standard object* in $Fuk(T^*X)$, which is the standard Lagrangian $L_{U,m}$ equipped with the canonical brane structures, a trivial rank 1 local system and the perturbation in Lemma 5.4.5 of [22].

The morphism space between L_0 and L_1 is the $\mathbb{Z}$-graded Floer complex enriched by the vector bundles

$$Hom_{Fuk(M)}(L_0, L_1) =: \bigoplus_{p\in L_0\cap L_1} Hom(\mathcal{E}_0|_p, \mathcal{E}_1|_p) \otimes_{\mathbb{C}} \mathbb{C}\langle p\rangle[-\deg(p)].$$

Implicit in the formula, is to first do Hamiltonian perturbations to L_0 and L_1 (as objects in $Fuk(M)$) as in Section 1.3 (a), and then replace the resulting Lagrangians by their sufficiently small tame perturbations. In Section2.4, we choose certain conical perturbations to $L_{x,F}$ and L_V, which combines the Hamiltonian and tame perturbations together.

The relative *Pin*-structures on the Lagrangian branes enable us to define orientations on the moduli space of discs, and gives the (higher) compositions over $\mathbb{C}$.

$$\begin{aligned}\mu^d :& Hom_{Fuk(M)}(L_{d-1}, L_d) \otimes \cdots Hom_{Fuk(M)}(L_1, L_2) \otimes Hom_{Fuk(M)}(L_0, L_1) \\ & \to Hom_{Fuk(M)}(L_0, L_d)[2-d] \\ & \mu^d(\phi_{d-1} \otimes a_{d-1}, \cdots, \phi_1 \otimes a_1, \phi_0 \otimes a_0) = \sum_{a_d \in L_0 \cap L_d} \sum_{u \in \mathcal{M}} \mathrm{sgn}(u) \cdot \phi_u \otimes a_d,\end{aligned}$$

where $\mathcal{M} = \mathcal{M}(a_0, a_1, \cdots, a_d; L_0, L_1, \cdots, L_d)^{\text{0-d}}$, $\phi_i \in Hom(\mathcal{E}_i|_{a_i}, \mathcal{E}_{i+1}|_{a_i})$ for $i = 0, ..., d-1$, and $\phi_u \in Hom(\mathcal{E}_0|_{a_d}, \mathcal{E}_d|_{a_d})$ associated to a holomorphic disc u is the composition of successive parallel transport along the edges of u and ϕ_i on the corresponding vertices.

Chapter 2

Holomorphic Lagrangian branes correspond to perverse sheaves

2.1 Introduction

For a real analytic manifold X, one could associate two invariants which encode the local/global analytic and topological structure of X. One is the constructible derived category of sheaves of $\mathbb{C}$-vector spaces on X, denoted as $D^b_c(X)$, and the other is the Fukaya category of the cotangent bundle of X, denoted as $Fuk(T^*X)$. Roughly speaking, $D^b_c(X)$ is generated by locally constant sheaves supported on submanifolds of X, which we will call (co)standard sheaves. The morphism spaces between these sheaves are naturally identified with relative singular cohomology of certain subsets of X taking values in local systems. On the other hand, $Fuk(T^*X)$ is a realm of studying exact Lagrangian submanifolds of T^*X and the intersection theory of them.

There is a canonical equivalence between these two invariants, established by Nadler and Zaslow in [22] and [21], given by the so called microlocal functor

$$H^0(\mu_X) : D^b_c(X) \to DFuk(T^*X).$$

Here we write $H^0(\mu_X)$ because this functor is induced from taking the 0-th cohomology of the A_∞-functor μ_X on the A_∞-version of these two categories. This equivalence can be seen as an elaboration of the microlocal studies of constructible sheaves, initiated by Kashiwara-Schapira, which assigns to any sheaf $\mathcal{F}$ on X a conical Lagrangian $SS(\mathcal{F})$ in T^*X, called the singular support of $\mathcal{F}$. Indeed, if one uses the $\mathbb{R}_+$-action on the cotangent fibers to dilate the Lagrangian (brane) $\mu_X(\mathcal{F})$, then at the limit one gets

$$SS(\mathcal{F}) \subset \lim_{t\to 0^+} t \cdot \mu_X(\mathcal{F}).$$

In the construction of μ_X, each standard or costandard sheaf is explicitly sent to a Lagrangian graph in T^*X which lives over the submanifold and asymptotically approaches the singular

support of the sheaf near infinity, so that the Floer cohomologies for these branes match with the morphisms on the sheaf side.

In the complex setting, when X is a complex manifold, one could also study $\mathcal{D}$-modules on X. The Riemann–Hilbert correspondence equates the derived category of regular holonomic $\mathcal{D}$-modules $D^b_{rh}(\mathcal{D}_X\text{-mod})$ with $D^b_c(X)$. There are also physical interpretations of the relation of branes (including coisotropic branes) with $\mathcal{D}$-modules, see [14], [15]. These relations together with μ_X connect different approaches to quantizing conical Lagrangians in T^*X.

We are going to investigate the special role of holomorphic Lagrangian branes in $Fuk(T^*X)$ in the complex setting, via the Nadler–Zaslow correspondence. For the notion of holomorphic, we have used the complex structure on T^*X induced from that on X. Recall there is an abelian category sitting inside $D^b_c(X)$, the category of perverse sheaves, which is the image of the standard abelian category (single $\mathcal{D}$-modules) in $D^b_{rh}(\mathcal{D}_X\text{-mod})$ under the Riemann–Hilbert correspondence. Our main result is the following:

Theorem 2.1.1. *Let X be a compact complex manifold and $H^0(\mu_X)^{-1}$ denote the inverse functor of $H^0(\mu_X)$. Then for any holomorphic Lagrangian brane L in T^*X, $H^0(\mu_X)^{-1}(L)$ is a perverse sheaf in $D^b_c(X)$ up to a shift. Equivalently, L gives rise to a single holonomic $\mathcal{D}$-module on X.*

In the following, we will give the motivation and the proof of our result from two aspects: the symplectic geometry and the microlocal geometry. In the symplectic geometry part, we will summarize the Floer cohomology calculations we have for certain classes of Lagrangian branes. Then in the microlocal geometric side, we will introduce the microlocal approach to perverse sheaves and explain why the Floer calculations imply our main theorem. All of the functors below are derived and we will always omit the derived notation R or L unless otherwise specified.

Floer complex calculations

We calculate the Floer complex for two pairs of Lagrangian branes in the cotangent bundle T^*X of a complex manifold X. It involves three kinds of Lagrangians which we briefly describe. Firstly, we have a (exact) holomorphic Lagrangian brane L with grading $-\dim_{\mathbb{C}} X$ (see Lemma 2.5.1). One could dilate L using the $\mathbb{R}_+$-action on the cotangent fibers and take limit to get a conical Lagrangian

$$\text{Conic}(L) := \lim_{t\to 0+} t \cdot L. \tag{2.1}$$

Then for each smooth point $(x,\xi) \in \text{Conic}(L)$, we define a Lagrangian brane, which we will call a *local Morse brane*, depending on the following data. We choose a generic holomorphic function F near x which vanishes at x and has $d\Re(F)_x = \xi$. By the word "generic", we mean the graph $\Gamma_{d\Re(F)}$ should intersect $\text{Conic}(L)$ at (x,ξ) in a transverse way. Then the local Morse brane, denoted as $L_{x,F}$, is defined by extending $\Gamma_{d\Re(F)}$ in an appropriate way, so that $L_{x,F}$ lives over a small neighborhood of x, and $L_{x,F}$ has certain behavior near infinity. Note

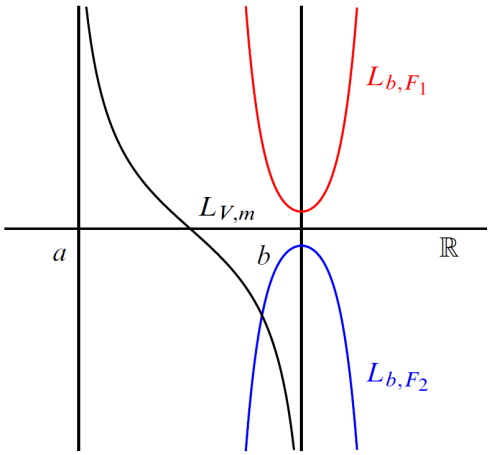


Figure 2.1: A picture illustrating a standard brane, two local Morse branes and their Floer cohomology for $X = \mathbb{R}$.

that the construction of $L_{x,F}$ is completely local; it only knows the local geometry (actually the microlocal geometry) around x. The last kind of Lagrangian we consider is the brane corresponding to a standard sheaf associated to an open set V under the microlocal functor μ_X. The construction is very easy. Take a function m on X with $m = 0$ on ∂V and $m > 0$ on V, then the Lagrangian is the graph $\Gamma_{d\log m}$, which lives over V. We will call such a brane a *standard brane* and denote it by $L_{V,m}$. We have been mixing up the terminology Lagrangian and Lagrangian brane freely, since the Lagrangians $L_{x,F}$ and $L_{V,m}$ will be equipped with canonical brane structures. Our Floer complex calculations show the following:

Theorem 2.1.2. *Under certain assumptions on the boundary of V, we have*

$$HF(L_{x,F}, L_{V,m}) \simeq (\Omega(B_\epsilon(x) \cap V, B_\epsilon(x) \cap V \cap \{\Re(F) < 0\}), d), \tag{2.2}$$

$$HF^\bullet(L_{x,F}, L) = 0 \text{ for } \bullet \neq 0, \tag{2.3}$$

where $B_\epsilon(x)$ is a small ball around x and the first identification is a canonical quasi-isomorphism.

An illustrating picture [1] for the branes $L_{U,m}$ and $L_{x,F}$ in the case of $X = \mathbb{R}$ is presented in Figure 2.1, where $V = (a, b)$, $F_1 = x - b$ and $F_2 = b - x$. The standard brane $L_{V,m}$ corresponds to the sheaf $i_*\mathbb{C}_V$, and one can check that (2.2) holds and compare it with (2.5).

Microlocal Geometry

There are roughly two characterizations of perverse sheaves. One is characterized by the vanishing degrees of the cohomological (co)stalks of sheaves. The other is the microlocal (or Morse theoretic) approach using vanishing property of the microlocal stalks (or

[1] Of course $X = \mathbb{R}$ is not a complex manifold, but it will become clear that the construction of local Morse brane generalizes to the real setting; also see Section 2.2

local Morse groups) of a sheaf. These are due to Beilinson–Bernstein–Deligne[3], Goresky–MacPherson[7], and Kashiwara–Schapira[16]. In this chapter, we will mainly adopt the latter one. We also include a path from (co)stalk characterization to the microlocal characterization in Section 2.2.

The microlocal stalk of a sheaf is a measurement of the change of sections of the sheaf when propagating along the direction determined by a given covector in T^*X. More precisely, let $\mathcal{F}$ be a sheaf whose cohomology sheaf is constructible with respect to some stratification $\mathcal{S}$. There is the standard conical Lagrangian $\Lambda_{\mathcal{S}}$ in T^*X associated to $\mathcal{S}$, which is the union of all the conormals to the strata. Now pick a smooth point (x, ξ) in $\Lambda_{\mathcal{S}}$, and choose a sufficiently generic holomorphic function F near x with $F(x) = 0$ and $dF_x = \xi$ (this is exactly the same condition we put on F when we construct $L_{x,F}$ in 2.1). The microlocal stalk (or local Morse group) $M_{x,F}(\mathcal{F})$ of $\mathcal{F}$ is defined to be

$$M_{x,F}(\mathcal{F}) = \Gamma(B_\epsilon(x), B_\epsilon(x) \cap \{\Re(F) < 0\}, \mathcal{F}) \tag{2.4}$$

for sufficiently small ball $B_\epsilon(x)$. In particular if $\mathcal{F}$ is $i_*\mathbb{C}_V$, the standard sheaf associated an open embedding $i : V \hookrightarrow X$, then one gets

$$M_{x,F}(i_*\mathbb{C}_V) = \Gamma(B_\epsilon(x) \cap V, B_\epsilon(x) \cap V \cap \{\Re(F) < 0\}, \mathbb{C}). \tag{2.5}$$

Recall that $H^0(\mu_X)$ sends $i_*\mathbb{C}_V$ to $L_{V,m}$, and standard sheaves associated to open sets generate the category $D^b_c(X)$. So comparing (2.5) with (2.2), one almost sees that the functor $HF(L_{x,F}, -)$ on $DFuk(T^*X)$ is equivalent to the functor $M_{x,F}(-)$ on $D^b_c(X)$ under the Nadler–Zaslow correspondence. This is confirmed by studying composition maps on the A_∞-level.

With the same assumptions as above plus the further assumption that $\mathcal{S}$ is a complex stratification, the microlocal characterization of a perverse sheaf is very simple. It says that $\mathcal{F}$ is a perverse sheaf if and only if the cohomology of the microlocal stalk $M_{x,F}(\mathcal{F})$ is concentrated in degree 0 for all choices of (x, ξ). For a holomorphic Lagrangian brane L, it is not hard to prove that $H^0(\mu_X)^{-1}(L)$ is a sheaf whose cohomology sheaf is constructible with respect to a complex stratification. Now it is easy to see that (2.3) directly implies our main theorem (Theorem 2.1.1).

Organization

The preliminaries are included in Chapter 1, which we will frequently refer to. Section 2.2 starts from basic definitions and properties of constructible sheaves and perverse sheaves, then heads towards the microlocal characterization of a perverse sheaf. Section 2.3 gives an overview of Nadler–Zaslow correspondence, with detailed discussion on several aspects, including Morse trees and the use of Homological Perturbation Lemma, since similar techniques will be applied in the later sections. Section 2.4 devotes to the construction of the local Morse brane $L_{x,F}$ and the proof that it corresponds to the local Morse group functor $M_{x,F}$ on the sheaf side. In section 2.5, we show the proof of (2.3) and conclude with our main theorem, some consequences and generalizations.

2.2 Perverse sheaves and the local Morse group functor

Constructible sheaves

Let X be an analytic manifold. Throughout the chapter, we will always work in a fixed analytic-geometric setting, and all the stratifications we consider are assumed to be Whitney stratifications (see Section 1.1). A sheaf of $\mathbb{C}$-vector spaces is *constructible* if there exists a stratification $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ such that $i_\alpha^*\mathcal{F}$ is a locally constant sheaf, where i_α is the inclusion $S_\alpha \hookrightarrow X$. Let $D_c^b(X)$ denote the bounded derived category of complexes of sheaves whose cohomology sheaves are all constructible. In the following, we simply call such a complex a sheaf. $D_c^b(X)$ has a natural differential graded enrichment, denoted as $Sh(X)$. The morphism space between two sheaves $\mathcal{F}, \mathcal{G}$ is the complex $RHom(\mathcal{F}, \mathcal{G})$, where $RHom(\mathcal{F}, \cdot)$ is the right derived functor of the usual $Hom(\mathcal{F}, \cdot)$ functor (by taking global section of the sheaf $\mathcal{H}om(\mathcal{F}, \cdot)$). Similarly, we denote by $Sh_{\mathcal{S}}(X)$ the subcategory of $Sh(X)$ consisting of sheaves constructible with respect a fixed stratification $\mathcal{S}$.

There are the standard four functors between $Sh(X)$ and $Sh(Y)$ associated to a map $f : X \to Y$, namely $f_*, f_!, f^*$ and $f^!$. Here and after, all functors are derived, though we omit the derived notation. f_*, f^* are right and left adjoint functors, similar for the pair $f^!$ and $f_!$. More explicitly, we have for $\mathcal{F} \in Sh(X), \mathcal{G} \in Sh(Y)$,

$$\mathcal{H}om(\mathcal{G}, f_*\mathcal{F}) \simeq f_*\mathcal{H}om(f^*\mathcal{G}, \mathcal{F}), f_*\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}) \simeq \mathcal{H}om(f_!\mathcal{F}, \mathcal{G})$$

$$Hom(\mathcal{G}, f_*\mathcal{F}) \simeq Hom(f^*\mathcal{G}, \mathcal{F}), Hom(\mathcal{F}, f^!\mathcal{G}) \simeq Hom(f_!\mathcal{F}, \mathcal{G}).$$

And there is the Verdier duality $\mathbb{D} : Sh(X) \to Sh(X)^{op}$, which gives the relation $\mathbb{D}f_* = f_!\mathbb{D}, \mathbb{D}f^* = f^!\mathbb{D}$. Let $i : U \hookrightarrow X$ be an open inclusion and $j : Y = X - U \hookrightarrow X$ be the closed inclusion of the complement of U, then $i^* = i^!$ and $j_* = j_!$. There are two standard exact triangles, taking global sections of which gives the long exact sequences for the relative hypercohomology of $\mathcal{F}$ for the pair (X, Y) and (X, U) respectively,

$$j_!j^!\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \xrightarrow{[1]}, \; i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \xrightarrow{[1]} . \tag{2.6}$$

The *stalk* of $\mathcal{F}$ at $x \in X$ will mean the complex $i_x^*\mathcal{F}$, where $i_x : \{x\} \hookrightarrow X$ is the inclusion. The i-th cohomology sheaf of a complex $\mathcal{F}$ will be denoted as $\mathcal{H}^i(\mathcal{F})$. Note that the stalk of $\mathcal{H}^i(\mathcal{F})$ at x is isomorphic to $H^i(i_x^*\mathcal{F})$. Also let $\mathrm{supp}(\mathcal{F}) := \overline{\{x \in X : \mathcal{H}^j(\mathcal{F})_x \neq 0 \text{ for some } j\}}$.

According to [22], the *standard* objects, i.e. sheaves of the form $i_*\mathbb{C}_U$, where $i : U \hookrightarrow X$ is an open inclusion, generate $Sh(X)$. The argument goes like the following. It suffices to prove the statement for the subcategory $Sh_{\mathcal{S}}(X)$ for any stratification $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$. Without loss of generality, we can assume each stratum of $\mathcal{S}$ is connected and is a cell. Let $\mathcal{S}_{\leq k}$, $0 \leq k \leq n = \dim X$, denote the union of all strata in $\mathcal{S}$ of dimension less than or equal to k. Let $\mathcal{S}_{>k} = X - \mathcal{S}_{\leq k}$ and $\mathcal{S}_k = \mathcal{S}_{\leq k} - \mathcal{S}_{\leq k-1}$. Denote by $i_k, i_{>k}, j_{\leq k}$ the inclusion of $S_\bullet$

with corresponding subscripts. The standard exact triangle on the left of (2.6) for a sheaf $\mathcal{G}$ supported on $\mathcal{S}_{\leq k}$ gives,

$$j_{\leq k-1!}j^!_{\leq k-1}\mathcal{G} \to \mathcal{G} \to i_{k*}i_k^*\mathcal{G} = i_{>k-1*}i^*_{>k-1}\mathcal{G} \overset{[1]}{\to}. \tag{2.7}$$

We start from $\mathcal{G}_n = \mathcal{F} \in Sh_{\mathcal{S}}(X)$, then use (2.7) inductively for $\mathcal{G}_{k-1} = j_{\leq k-1!}j^!_{\leq k-1}\mathcal{G}_k = j_{\leq k-1!}j^!_{\leq k-1}\mathcal{F}$ from $k = n$ through $k = 1$, and get that $\mathcal{F}$ can be obtained by taking iterated mapping cones of shifts of $i_{\alpha*}\mathbb{C}_{S_\alpha}$, $S_\alpha \in \mathcal{S}$. Let $\mathcal{U}_{\mathcal{S}} = \{X, O_\alpha = X - \overline{S_\alpha}, O'_\alpha = X - \partial S_\alpha : \alpha \in \Lambda\}$. Now the claim is $i_{\alpha*}\mathbb{C}_{S_\alpha}$ can be generated by $i_{U*}\mathbb{C}_U, U \in \mathcal{U}_{\mathcal{S}}$. This follows from a similar argument. Putting $\mathcal{F} = \mathbb{C}_X$, $i = i_{O_\alpha}$ or $i = i_{O'_\alpha}$ on the left of (2.6), we get the generation statement for $j_{\overline{S}_\alpha!}j^!_{\overline{S}_\alpha}\mathbb{C}_X, j_{\partial S_\alpha!}j^!_{\partial S_\alpha}\mathbb{C}_X$. Then letting $\mathcal{G} = j_{\overline{S}_\alpha!}j^!_{\overline{S}_\alpha}\mathbb{C}_X$ in (2.7) for $k = \dim S_\alpha$, and identifying $j_{\leq k-1!}j^!_{\leq k-1}\mathcal{G}$ with $j_{\partial S_\alpha!}j^!_{\partial S_\alpha}\mathcal{G}$, $i_{k*}i_k^*\mathcal{G}$ with $i_{\alpha*}\mathbb{C}_{S_\alpha}$, we get the generation statement for $i_{\alpha*}\mathbb{C}_{S_\alpha}$.

Since $Sh(X)$ is a dg category, it suffices to study the morphisms between any two standard objects associated to open sets, and the composition maps for a triple of standard objects.

Proposition 2.2.1 (Lemma 4.4.1 [22]). *Let $i_0 : U_0 \hookrightarrow X$ and $i_1 : U_1 \hookrightarrow X$ be the inclusion of two open submanifolds of X. Then there is a natural quasi-isomorphism*

$$Hom(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \simeq (\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d).$$

Furthermore, for a triple of open inclusions $i_k : U_k \hookrightarrow X$, $k = 0, 1, 2$, the composition map

$$Hom(i_{1*}\mathbb{C}_{U_1}, i_{2*}\mathbb{C}_{U_2}) \otimes Hom(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \to Hom(i_{0*}\mathbb{C}_{U_0}, i_{2*}\mathbb{C}_{U_2})$$

is naturally identified with the wedge product on (relative) deRham complexes

$$(\Omega(\overline{U}_1 \cap U_2, \partial U_1 \cap U_2), d) \otimes (\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d) \to (\Omega(\overline{U}_0 \cap U_2, \partial U_0 \cap U_2), d)$$

[22] introduced how to perturb U_0 and U_1 to have transverse boundary intersection, and to use the perturbed open sets to calculate $Hom(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1})$. Let m_i be a semi-defining function of U_i for $i = 0, 1$ (see Remark 1.1.3). There exists a fringed set $R \subset \mathbb{R}^2_+$ (see Definition 1.1.6) such that $m_1 \times m_0 : X \to \mathbb{R}^2$ has no critical value in R (by Corollary 1.1.7). In particular, for $(t_1, t_0) \in R$, $X_{m_0=t_0}$ and $X_{m_1=t_1}$ intersect transversely. Then there is a compatible collection of identifications

$$(\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d) \simeq (\Omega(X_{m_0 \geq t_0} \cap X_{m_1 > t_1}, X_{m_0=t_0} \cap X_{m_1>t_1}), d).$$

Perverse sheaves

Let X be a complex analytic manifold of dimension n. In this section, we review some basic definitions and properties of the perverse t-structure $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ (with respect to the "middle perversity"). The exposition is following Section 8.1 of [12].

Definition 2.2.2. Define the full subcategories ${}^pD_c^{\leq 0}(X)$ and ${}^pD_c^{\geq 0}(X)$ in $D_c^b(X)$ as follows. A sheaf $\mathcal{F} \in {}^pD_c^{\leq 0}(X)$ if

$$\dim\{\mathrm{Supp}(\mathcal{H}^j(\mathcal{F}))\} \leq -j, \text{ for all } j \in \mathbb{Z}$$

and $\mathcal{F} \in {}^pD_c^{\geq 0}(X)$ if

$$\dim\{\mathrm{Supp}(\mathcal{H}^j(\mathbb{D}\mathcal{F}))\} \leq -j, \text{ for all } j \in \mathbb{Z}.$$

An object of its heart $Perv(X) = {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$ is called a *perverse sheaf*. Let ${}^p\tau^{\leq k} : D_c^b(X) \to {}^pD^{\leq k}(X) := {}^pD^{\leq 0}(X)[-k]$, ${}^p\tau^{\geq k} : D_c^b(X) \to {}^pD^{\geq k}(X) := {}^pD^{\geq 0}(X)[-k]$ be the corresponding truncation functors. Let ${}^pH^k = {}^p\tau^{\geq k}{}^p\tau^{\leq k}[k] : D_c^b(X) \to Perv(X)$ be the k-th perverse cohomology functor.

Here are several properties of perverse t-structures.

Proposition 2.2.3. *Let $\mathcal{F} \in D_c^b(X)$ and $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ be a complex stratification of X consisting of connected strata with respect to which $H^j(\mathcal{F})$ are constructible. Then*
(i) $\mathcal{F} \in D_c^{\leq 0}(X)$ if and only if $H^j(i_{S_\alpha}^*\mathcal{F}) = 0$ for all $j > -\dim S_\alpha$;
(ii) $\mathcal{F} \in D_c^{\geq}0(X)$ if and only if $H^j(i_{S_\alpha}^!\mathcal{F}) = 0$ for all $j < -\dim S_\alpha$.

Lemma 2.2.4. *(1) A sheaf $\mathcal{F} \in D_c^b(X)$ is isomorphic to zero if and only if ${}^pH^k(\mathcal{F}) = 0$ for all $k \in \mathbb{Z}$.*
(2) A morphism $f : \mathcal{F} \to \mathcal{G}$ in $D_c^b(X)$ is an isomorphism if and only if the induced map ${}^pH^k(f) : {}^pH^k(\mathcal{F}) \to {}^pH^k(\mathcal{G})$ is an isomorphism for all $k \in \mathbb{Z}$.

Proof. (1) Since $\mathcal{F} \in D_c^b(X)$, we have $\mathcal{F} \in {}^pD^{\geq a}(X) \cap {}^pD^{\leq b}(X)$ for some $a, b \in \mathbb{Z}$. By the distinguished triangle,

$${}^p\tau^{\leq b-1}\mathcal{F} \to {}^p\tau^{\leq b}\mathcal{F} \simeq \mathcal{F} \to {}^p\tau^{\leq b}{}^p\tau^{\geq b}\mathcal{F} \simeq 0 \overset{[1]}{\to},$$

we get $\mathcal{F} \in {}^pD^{\leq b-1}(X)$. Inductively, we conclude that $\mathcal{F} \simeq 0$.

(2) is an easy consequence of (1). □

Proposition 2.2.5. *$\mathcal{F} \in D_c^b(X)$ is perverse $\Leftrightarrow$ ${}^pH^*(\mathcal{F})$ is concentrated in degree 0.*

Proof. " $\Rightarrow$ " is clear.
" $\Leftarrow$ ": Consider the exact triangle

$${}^p\tau^{\leq -1}\mathcal{F} \to \mathcal{F} \to {}^p\tau^{\geq 0}\mathcal{F} \overset{[1]}{\to}.$$

It gives rise to a long exact sequence in $Perv(X)$

$$\to {}^pH^k({}^p\tau^{\leq -1}\mathcal{F}) \to {}^pH^k(\mathcal{F}) \to {}^pH^k({}^p\tau^{\geq 0}\mathcal{F}) \to {}^pH^{k+1}({}^p\tau^{\leq -1}\mathcal{F}) \to .$$

Since ${}^pH^*(\mathcal{F})$ is concentrated in degree 0, we have ${}^pH^k({}^p\tau^{\leq -1}\mathcal{F}) = 0$ for all $k \in \mathbb{Z}$. From Lemma 2.2.4, we get $\mathcal{F}$ is isomorphic to ${}^p\tau^{\geq 0}\mathcal{F}$.

Similarly, if we consider the exact triangle

$$ {}^{p}_{\mathcal{T}}{}^{\leq 0}{}^{p}_{\mathcal{T}}{}^{\geq 0}\mathcal{F} \to {}^{p}_{\mathcal{T}}{}^{\geq 0}\mathcal{F} \to {}^{p}_{\mathcal{T}}{}^{\geq 1}\mathcal{F} \overset{+1}{\to} $$

and get a long exact sequence in $Perv(X)$, then we can conclude that ${}^{p}_{\mathcal{T}}{}^{\geq 0}\mathcal{F}$ is isomorphic to ${}^{p}_{\mathcal{T}}{}^{\leq 0}{}^{p}_{\mathcal{T}}{}^{\geq 0}\mathcal{F} = {}^{p}H^0(\mathcal{F})$. Therefore, $\mathcal{F} \simeq {}^{p}H^0(\mathcal{F})$ in $D^b_c(X)$. □

Fix a complex stratification $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ of X with each stratum connected. The perverse t-structure on $D^b_c(X)$ induces the perverse t-structure on $D^b_{\mathcal{S}}(X)$.

Let $\Lambda_{\mathcal{S}} := \bigcup_{\alpha\in\Lambda} T^*_{S_\alpha}X \subset T^*X$ be the standard conical Lagrangian associated to $\mathcal{S}$. For each $S_\alpha \in \mathcal{S}$, let $D^*_{S_\alpha}X = T^*_{S_\alpha}X \cap (\bigcup_{\alpha\neq\beta\in\Lambda} \overline{T^*_{S_\beta}X})$. Then the smooth locus in $\Lambda_{\mathcal{S}}$ is the union $\bigcup_{\alpha\in\Lambda}(T^*_{S_\alpha}X - D^*_{S_\alpha}X)$.

Local Morse group functor $M_{x,F}$ on $D^b_{\mathcal{S}}(X)$

$Perv(X)$ is an abelian subcategory in $D^b_c(X)$. An exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ in $Perv(X)$, though corresponds to an exact triangle in $D^b_c(X)$, *does not* give an exact sequence on the stalks. The correct "stalk" to take in $Perv(X)$, in the sense that it gives an exact sequence, is the microlocal stalk. We now introduce the microlocal stalk under its another name, local Morse group functor.

Let $(x,\xi) \in \Lambda_{\mathcal{S}}$ be a smooth point. Fix a local holomorphic coordinate $\mathbf{z}$ around x with origin at x, and let $\mathrm{r}(\mathbf{z}) = \|\mathbf{z}\|^2$ be the standard distance squared function. Let F be a germ of holomorphic function on X, i.e. defined on some small open ball $B_{2\epsilon}(x) = \{\mathbf{z} : \mathrm{r}(\mathbf{z}) < (2\epsilon)^2\} \subset X$, such that $F(x) = 0$, $d\Re(F)_x = \xi$ and the graph $\Gamma_{d\Re(F)}$ is transverse to $\Lambda_{\mathcal{S}}$ at (x,ξ). We also assume ϵ small enough so that x is the only $\Lambda_{\mathcal{S}}$ -critical point of $\Re(F)$. In the following, we will call such a triple (x,ξ,F) a *test triple*.

Let $\phi_{x,F} : D_{\mathcal{S}}(B_\epsilon(x)) \to D^b_c(F^{-1}(0)\cap B_\epsilon(x))$ be the vanishing cycle functor associated to F (see §8.6 in [16] for the definition of nearby and vanishing cycle functors). Note that for any $\mathcal{F} \in D_{\mathcal{S}}(X)$, $\phi_{x,F}(\mathcal{F})$ is supported on x.

Definition 2.2.6. Given (x,ξ,F), define the *local Morse group functor* $M_{x,F} := j^!_x\phi_{x,F}[-1]l^* = j^*_x\phi_{x,F}[-1]l^* : D^b_{\mathcal{S}}(X) \to D^b(\mathbb{C})$, where $l : B_\epsilon(x) \hookrightarrow X$ and $j_x : \{x\} \hookrightarrow F^{-1}(0)\cap B_\epsilon(x)$ are the inclusions.

It's a standard fact that on $D^b_{\mathcal{S}}(X)$,

$$M_{x,F}(\mathcal{F}) \simeq \Gamma(B_\epsilon(x), B_\epsilon(x)\cap F^{-1}(t), \mathcal{F}) \simeq \Gamma(B_\epsilon(x), B_\epsilon(x)\cap\{\Re(F) < \mu\}, \mathcal{F}), \tag{2.8}$$

where t is any complex number with $0 < |t| << \epsilon$, and $\mu \leq 0$ with $|\mu| << \epsilon$.

More generally, let X be a real analytic manifold with a Riemannian metric, and let $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Lambda}$ be a (real) stratification. Assume a function $g : X \to \mathbb{R}$ satisfies similar

conditions of $\Re(F)$ at a given point $x \in S_\alpha$, with Morse index of $g|_{S_\alpha}$ equal to λ. Then given a sheaf $\mathcal{F}$ in $D^b_{\mathcal{S}}(X)$, the hypercohomology groups

$$\mathbb{H}^i(B_\epsilon(x), \{g < 0\} \cap B_\epsilon(x), \mathcal{F}[\lambda]), i \in \mathbb{Z} \tag{2.9}$$

are independent of the choices of g and x for (x, dg_x) staying in a fixed connected component of $T^*_{S_\alpha}X - D^*_{S_\alpha}X$. For more details, see [19] Theorems 2.29, 2.31 and the references therein. In the complex setting, $T^*_{S_\alpha}X - D^*_{S_\alpha}X$ is always connected, so $M_{x,F}(\mathcal{F})$ are quasi-isomorphic for different choices of (x, ξ) in it (but not in a canonical way since there may be monodromies).

Then the singular support $SS(\mathcal{F})$ of $\mathcal{F}$ can be described as the closure of the set of covectors in $\Lambda_{\mathcal{S}}$ with the relative hypercohomology groups in (2.9) not all equal to 0. For the definition of $SS(F)$, see section 5.1 in [16]; the fact that one can use vanishing cycles to detect singular support is stated in Proposition 8.6.4 of [16].

Lemma 2.2.7. *$M_{x,F} : D^b_{\mathcal{S}}(X) \to D^b(\mathbb{C})$ is t-exact. It commutes with the Verdier duality.*

Proof. It's standard that $\phi_{x,F}[-1]$ and l^* are perverse t-exact. Since $\phi_{x,F}[-1]l^*(\mathcal{F})$ is supported on x, we have

$$M_{x,F}({}^pH^k(\mathcal{F})) = j^!_{\{x\}}\phi_{x,F}[-1]l^*({}^pH^k(\mathcal{F})) \simeq H^k(j^!_{\{x\}}\phi_{x,F}[-1]l^*(\mathcal{F})) = H^k(M_{x,F}(\mathcal{F})).$$

$\phi_{x,F}[-1]$ and l^* commute with $\mathbb{D}$, so it's easy to see that

$$M_{x,F}\mathbb{D} = j^!_{\{x\}}\phi_{x,F}[-1]l^*\mathbb{D} \simeq \mathbb{D}j^*_{\{x\}}\phi_{x,F}[-1]l^* = \mathbb{D}M_{x,F}.$$

□

Lemma 2.2.8. *For each stratum $S_\alpha \in \mathcal{S}$, choose one test triple $(x_\alpha, \xi_\alpha, F_\alpha)$ with $(x_\alpha, \xi_\alpha) \in \Lambda_{S_\alpha}$. If $M_{x_\alpha,F_\alpha}(\mathcal{F}) \simeq 0$ for all $S_\alpha \in \mathcal{S}$, then $\mathcal{F} \simeq 0$.*

Proof. From the previous discussion, $M_{x_\alpha,F_\alpha}(\mathcal{F}) \simeq 0$ for one choice of $(x_\alpha, \xi_\alpha, F_\alpha)$ is equivalent to $M_{x_\alpha,F_\alpha}(\mathcal{F}) \simeq 0$ for all possible choices of $(x_\alpha, \xi_\alpha, F_\alpha)$.

Again, let $\mathcal{S}_{\leq k}$, $0 \leq k \leq n = \dim_{\mathbb{C}} X$, denote the union of all strata in $\mathcal{S}$ of dimension less than or equal to k. Let $\mathcal{S}_{>k} = X - \mathcal{S}_{\leq k}$ and $\mathcal{S}_k = \mathcal{S}_{\leq k} - \mathcal{S}_{\leq k-1}$. Denote by $i_k, i_{>k}, j_{\leq k}$ the inclusion of $S_\bullet$ with corresponding subscripts.

For any test triple $(x, 0, F)$ with $x \in \mathcal{S}_n$, x is a Morse singularity of F with index 0. By basic Morse theory, $M_{x,F}(\mathcal{F}) \simeq 0$ implies $i^*_n\mathcal{F} \simeq 0$. By the adjunction exact triangle (2.6), $\mathcal{F} \simeq j_{\leq n-1!}j^!_{\leq n-1}\mathcal{F}$.

In the following we will only look at $B_\epsilon(x)$, and omit functors related to the open inclusion $l : B_\epsilon(x) \hookrightarrow X$. Note for $x \in \mathcal{S}_{n-1}$, by base change formula, $\phi_{x,F}(j_{\leq n-1!}j^!_{\leq n-1}\mathcal{F}) \simeq \hat{j}_{\leq n-1!}\phi_{x,F_{n-1}}(j^!_{\leq n-1}\mathcal{F})$, where F_{n-1} is the restriction of F to $\mathcal{S}_{n-1}$, and $\hat{j}_{\leq n-1}$ is the inclusion of $F^{-1}(0) \cap \mathcal{S}_{\leq n-1}$ into $F^{-1}(0)$. Therefore $M_{x,F}(j_{\leq n-1!}j^!_{\leq n-1}\mathcal{F}) \simeq M_{x,F_{n-1}}(j^!_{\leq n-1}\mathcal{F})$. Since x is a Morse singularity of F_{n-1} with index 0 on $\mathcal{S}_{n-1}$, by previous argument, $j^!_{n-1}\mathcal{F} \simeq 0$. By Verdier duality, $j^*_{n-1}\mathcal{F} \simeq 0$ as well. Applying the adjunction exact triangle again to the open set $\mathcal{S}_{\geq n-1}$, we get $\mathcal{F} \simeq j_{\leq n-2!}j^!_{\leq n-2}\mathcal{F}$, and by induction, we get $\mathcal{F} \simeq 0$. □

Combining the two lemmas, we immediately get the following (a similar statement can be found in Theorem 10.3.12 of [16]).

Proposition 2.2.9 (Microlocal characterization of perverse sheaves)**.** *For each stratum $S_\alpha \in \mathcal{S}$, choose a test triple $(x_\alpha, \xi_\alpha, F_\alpha)$ with $(x_\alpha, \xi_\alpha) \in \Lambda_{S_\alpha}$. Then $\mathcal{F} \in D^b_{\mathcal{S}}(X)$ is perverse if and only $M_{x_\alpha, F_\alpha}(\mathcal{F})$ has cohomology group concentrated in degree 0 for all $S_\alpha \in \mathcal{S}$.*

$M_{x,F}$ as a functor on the dg-category $Sh_{\mathcal{S}}(X)$

$M_{x,F}$ can be naturally viewed as a dg-functor from $Sh_{\mathcal{S}}(X)$ to Ch, where Ch denotes the dg-category of cochain complexes of vector spaces. And we have a natural identification

$$M_{x,F} \simeq \Gamma(B_\epsilon(x), B_\epsilon(x) \cap \{\Re(F) < \mu\}, -)$$

for sufficiently small $\epsilon > 0$ and $-\epsilon << \mu \leq 0$.

To make future calculations easier, we refine $\mathcal{S}$ into a new (real) stratification $\widetilde{\mathcal{S}}$ with each stratum a cell, and view $Sh_{\mathcal{S}}(X)$ as a subcategory of $Sh_{\widetilde{\mathcal{S}}}(X)$. $M_{x,F}$ is obviously extended to $Sh_{\widetilde{\mathcal{S}}}(X)$, as long as x is not lying in any newly added stratum, and the microlocal characterization for perverse sheaves (Proposition 2.2.9) still applies for $Sh_{\mathcal{S}}(X)$. In the following, to simplify notation, we still denote $\widetilde{\mathcal{S}}$ by $\mathcal{S}$.

We have seen in Section 2.2 that $Sh_{\mathcal{S}}(X)$ is generated by $i_*\mathbb{C}_U$ for $U \in \mathcal{U}_{\mathcal{S}} = \{X, O_\alpha = X - \overline{S_\alpha}, O'_\alpha = X - \partial S_\alpha : S_\alpha \in \mathcal{S}\}$. So to understand $M_{x,F}$, it suffices to understand its interaction with these standard generators. It is easy to see that $M_{x,F}$ is only nontrivial on the finite subcollection of $i_*\mathbb{C}_U$, where

$$U \in \mathcal{U}_{\mathcal{S},x} := \{V \in \mathcal{U}_{\mathcal{S}} : x \in \overline{V}\}. \tag{2.10}$$

Similar to Proposition 2.2.1, we have the following lemma.

Lemma 2.2.10. *For each $i_V : V \hookrightarrow X$ open, consider the dg functor $\Gamma(V, -) : Sh(X) \to Ch$. For any two open embeddings $i_0 : U_0 \hookrightarrow X, i_1 : U_1 \hookrightarrow X$, the composition map*

$$Hom_{Sh(X)}(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \otimes \Gamma(V, i_{0*}\mathbb{C}_{U_0}) \to \Gamma(V, i_{1*}\mathbb{C}_{U_1}) \tag{2.11}$$

is canonically identified with the wedge product on the deRham complexes

$$(\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d) \otimes (\Omega(U_0 \cap V), d) \to (\Omega(U_1 \cap V), d).$$

Corollary 2.2.11. *(1) The functor $M_{x,F}(-)$ on $Sh_{\mathcal{S}}(X)$ fits into the exact triangle $M_{x,F}(-) \longrightarrow \Gamma(B_\epsilon(x), -) \longrightarrow \Gamma(B_\epsilon(x) \cap \{\Re(F) < 0\}, -) \xrightarrow{[1]}$.*
(2) Given U_0, U_1 open in X, the composition map

$$Hom_{Sh(X)}(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \otimes M_{x,F}(i_{0*}\mathbb{C}_{U_0}) \to M_{x,F}(i_{1*}\mathbb{C}_{U_1})$$

is canonically given by the wedge product on the deRham complexes:

$$\begin{aligned}&(\Omega(\overline{U_0} \cap U_1, \partial U_0 \cap U_1), d) \otimes (\Omega(U_0 \cap B_\epsilon(x), U_0 \cap B_\epsilon(x) \cap \{\Re(F) < 0\}), d)\\ &\to (\Omega(U_1 \cap B_\epsilon(x), U_1 \cap B_\epsilon(x) \cap \{\Re(F) < 0\}), d).\end{aligned}$$

2.3 The Nadler–Zaslow Correspondence

Two categories: $Open(X)$ and $Mor(X)$

Let $Open(X)$ be the dg category whose objects are open subsets in X with a semi-defining function (see Remark 1.1.3). For two objects $\mathfrak{U}_0 = (U_0, m_0), \mathfrak{U}_1 = (U_1, m_1)$, define

$$Hom_{Open(X)}(\mathfrak{U}_0, \mathfrak{U}_1) := Hom(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \simeq (\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d).$$

The composition for a triple is the wedge product on deRham complexes as in Proposition 2.2.1. From previous discussions, $Sh(X)$ is a triangulated envelope of $Open(X)$.

Define another A_∞-category, denoted as $Mor(X)$ with the same objects as $Open(X)$. The morphism between two objects $\mathfrak{U}_i = (U_i, m_i), i = 0, 1$ is defined by the Morse complex calculation of $Hom(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1})$, using perturbation to smooth transverse boundaries similar to the process at the end of Section 2.2. Let $f_i = \log m_i$ for $i = 0, 1$. Pick a stratification $\mathcal{T}$ compatible with ∂U_0. There is $\bar{t}_1 > 0$ such that m_1 has no $\Lambda_{\mathcal{T}}$-critical value in $(0, \bar{t}_1)$. Fix $t_1 \in (0, \bar{t}_1)$. Let W be a small neighborhood of $\partial U_0 \cap X_{m_1=t_1}$ on which df_0 and df_1 are linearly independent. Since $X_{m_1=t_1} \cap U_0 - W$ is compact, one could dilate df_0 by $\epsilon > 0$ so that $|df_1| > 2\epsilon|df_0|$ on $X_{m_1=t_1} \cap U_0 - W$. There is $\bar{t}_0 > 0$ such that for any $t \in (0, \bar{t}_0)$, $X_{m_0=t}$ intersects $X_{m_1=t_1}$ transversally. Choose $t_0 \in (0, \bar{t}_0)$ such that $|\epsilon \cdot df_0| > 2|df_1|$ on $X_{m_0=t_0} \cap X_{m_1 \geq t_1} - W$. Such a t_0 always exists, since df_1 is bounded on $X_{m_1 \geq t_1}$. There is also a convex space of choices of Riemannian metric g in a neighborhood of $X_{m_0 \geq t_0} \cap X_{m_1 \geq t_1}$, with respect to which the gradient vector field $\nabla(f_1 - \epsilon f_0)$ is pointing outward along $X_{m_0=t_0} \cap X_{m_1 \geq t_1}$, and inward along $X_{m_0 \geq t_0} \cap X_{m_1=t_1}$. After small perturbations, one can perturb the function $f_1 - \epsilon f_0$ to be Morse, and the pair $(f_1 - \epsilon f_0, g)$ to be Morse-Smale.

Let M be an n-dimensional *manifold with corners*. By definition, for every point $y \in \partial M$, there is a local chart $\phi_y : U_y \to \mathbb{R}^n$ identifying an open neighborhood U_y of y with an open subset of a quadrant $\{x_{i_1} \geq 0, ..., x_{i_k} \geq 0\}$. We will say a function f on M is *directed*, if (1) $\phi_{y*}f$ can be extended to be a smooth function on an open neighborhood of $\phi_y(U_y)$, and with respect to some Riemannian metric g, the gradient vector field of the resulting function is pointing either strictly outward or strictly inward along every face of $\phi_y(U_y)$, (2) f is a Morse function on M and the pair (f, g) is Morse-Smale. We will also call (f, g) a *directed pair*. From the above discussion, $f_1 - \epsilon f_0$ is directed on the manifold with corners $X_{m_0 \geq t_0} \cap X_{m_1 \geq t_1}$.

Now define

$$Hom_{Mor(X)}(\mathfrak{U}_0, \mathfrak{U}_1) := Mor^*(X_{m_0 > t_0} \cap X_{m_1 > t_1}, f_1 - \epsilon f_0),$$

where $Mor^*(X_{m_0 > t_0} \cap X_{m_1 > t_1}, f_1 - \epsilon f_0)$ is the usual Morse complex associated to the function $f_1 - \epsilon f_0$ (after small perturbations when necessary). It is clear from the above description that the definition essentially doesn't depend on the choices of t_0, t_1, ϵ and g. There are compatible quasi-isomorphisms between the complexes with different choices.

The (higher) compositions are defined by counting Morse trees as follows.

Definition 2.3.1. A *based metric Ribbon tree* T is an embedded tree into the unit disc consisting of the following data.
1. Vertices: there are $n+1$ points on the boundary of the unit disc in $\mathbb{R}^2$ labeled counterclockwise by $v_0, ..., v_n$, where v_0 is referred as the *root vertex*, and others are referred as *leaf vertices*. There is a finite set of points in the interior of the disc, which are referred as *interior vertices*.
2. Edges: there are straight line segments referred as *edges* connecting the vertices. An edge e connecting to the root or a leaf is called an *exterior* edge, otherwise it is called an *interior* edge. We will use e_i to denote the unique exterior edge attaching to v_i, and e_{in} to denote an interior edge. The resulting graph of vertices and edges should be a connected embedded stable tree in the usual sense, i.e. the edges do not intersect each other in the interior, there are no cycles in the graph, and each interior vertex has at least 3 edges.
3. Metric and orientation: the tree is oriented from the leaves to the root, in the direction of the shortest path (measured by the number of passing edges). Each interior edge e_{in} is given a *length* $\lambda(e_{in}) > 0$. One could parametrize the edges as follows, but the parametrization is not part of the data. Each e_{in} is parametrized by the bounded interval $[0, \lambda(e_{in})]$ respecting the orientation. Every $e_i - \{v_i\}, i \neq 0$ is parametrized by $(-\infty, 0]$ and $e_0 - \{v_0\}$ is parametrized by $[0, \infty)$.
4. Equivalence relation: two based metric Ribbon trees are considered the same if there is an isotopy of the closed unit disc which identifies the above data.

Let $\mathfrak{U}_i = (U_i, f_i), i \in \mathbb{Z}/(k+1)\mathbb{Z}$ be a sequence of objects in $Mor(X)$ (when we compare the magnitude of two indices, we think of them as natural numbers ranging from 0 to k). We can apply the perturbation process as before to produce a *directed sequence* $\tilde{\mathfrak{U}}_i = (\tilde{U}_i, \tilde{f}_i)$, where $\partial\tilde{U}_i$'s are all smooth and transversely intersect with each other, and $\tilde{f}_j - \tilde{f}_i$ is directed on $\overline{\tilde{U}}_i \cap \overline{\tilde{U}}_j$ for $j > i$ (the boundary on which $\nabla(\tilde{f}_j - \tilde{f}_i)$ is pointing outward is understood to be $\partial\tilde{U}_i \cap \overline{\tilde{U}}_j$). A *Morse tree* is a continuous map $\phi : T \to X$ such that
1. $\phi(v_i) \in Cr(\tilde{U}_{i-1} \cap \tilde{U}_i, \tilde{f}_i - \tilde{f}_{i-1})$, for $i \in \mathbb{Z}/(k+1)\mathbb{Z}$;
2. The tree divides the disc into several connected components, and we label these components counterclockwise starting from 0 on the left-hand-side of e_0 (with respect to the given orientation). Let $\ell(e)$ and $r(e)$ denote the label on the left and right-hand-side of an edge e respectively. Then we require that $\phi|_e$ is C^1 and under some parametrization of the edges, we have

$$\frac{d\phi(t)}{dt}|_{e_{in}} = \nabla(\tilde{f}_{\ell(e_{in})} - \tilde{f}_{r(e_{in})}), \text{ for } t \in (0, \lambda(e_{in})),$$
$$\frac{d\phi(t)}{dt}|_{e_i} = \nabla(\tilde{f}_{\ell(e_i)} - \tilde{f}_{r(e_i)}), \text{ for } t \in (-\infty, 0) \text{ if } i \neq 0 \text{ and } t \in (0, \infty) \text{ if } i = 0.$$

For $a_i \in Cr(\tilde{U}_i \cap \tilde{U}_{i+1}, \tilde{f}_{i+1} - \tilde{f}_i), i \in \mathbb{Z}/(k+1)\mathbb{Z}$, let $\mathcal{M}(T; \tilde{f}_0, ..., \tilde{f}_{k-1}; a_0, ..., a_k)$ denote the moduli space of Morse trees with $\phi(v_i) = a_{i-1}$. After a small perturbation of the functions, this moduli space is regular, and the signed count of the 0-dimensional part

$\mathcal{M}(T; \tilde{f}_0, ..., \tilde{f}_k; a_0, ..., a_k)^{\text{0-d}}$ gives the higher compositions

$$m^k_{Mor(X)}(a_{k-1}, a_{k-2}, ..., a_0) = \sum_{b_k \in Cr(\tilde{U}_0 \cap \tilde{U}_k, \tilde{f}_k - \tilde{f}_0)} \sharp \mathcal{M}(T; \tilde{f}_0, ..., \tilde{f}_k; a_0, ..., a_{k-1}, b_k)^{\text{0-d}} \cdot b_k.$$

$Open(X) \simeq Mor(X)$ via Homological Perturbation Lemma

Let's recall the Homological Perturbation Lemma summarized in [26]. Assume we are given an A_∞-category $\mathcal{A}$ and a collection of chain maps F, G on $Hom_{\mathcal{A}}(X_1, X_2)$ for each pair of objects X_1, X_2 such that

$$G \circ F = \text{Id}, \ F \circ G - \text{Id} = m^1_{\mathcal{A}} \circ H + H \circ m^1_{\mathcal{A}},$$

where H is a map on $Hom_{\mathcal{A}}(X_1, X_2)$ of degree -1.

Theorem 2.3.2. *There exists an A_∞-category $\mathcal{B}$ with the same objects, morphism spaces and m^1 as $\mathcal{A}$. This comes with A_∞-morphisms $\mathcal{F} : \mathcal{B} \to \mathcal{A}$, $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ which are identity on objects and $\mathcal{F}^1 = F, \mathcal{G}^1 = G$. There is also a homotopy $\mathcal{H}$ between $\mathcal{F} \circ \mathcal{G}$ and $Id_{\mathcal{A}}$ such that $\mathcal{H}^1 = H$.*

Remark 2.3.3. In this chapter, all A_∞-categories and A_∞-functors are assumed to be c-unital. The Homological Perturbation Lemma generalizes to left A_∞-modules, namely, in addition to the above data, let $\mathcal{M} : \mathcal{A} \to Ch$ be a left A_∞-module over $\mathcal{A}$ and assume for each object X, there are chain maps $\tilde{F}, \tilde{G}$ and a homotopy $\tilde{H}$ on $\mathcal{M}(X)$ satisfying

$$\tilde{G} \circ \tilde{F} = \text{Id}, \ \tilde{F} \circ \tilde{G} - \text{Id} = d \circ \tilde{H} + \tilde{H} \circ d,$$

then one can construct a left A_∞-module $\mathcal{N}$ over $\mathcal{B}$ such that

$$\mathcal{N}^1 = \tilde{G} \circ \mathcal{M}^1 \circ F,$$

and there are module homomorphisms

$$t : \mathcal{N} \to \mathcal{F}^* \mathcal{M}, \ s : \mathcal{M} \to \mathcal{G}^* \mathcal{N}.$$

Then we have the following composition

$$T = (\mathcal{R}_{\mathcal{G}}(t)) \circ s : \mathcal{M} \to \mathcal{N} \circ \mathcal{G} \to \mathcal{M} \circ \mathcal{F} \circ \mathcal{G},$$

where $\mathcal{R}_{\mathcal{G}}$ is taking composition with $\mathcal{G}$ on the right. Using the homotopy between $\mathcal{F} \circ \mathcal{G}$ and $Id_{\mathcal{A}}$, we get a composition of morphisms between the induced cohomological functors

$$H(T) : H(\mathcal{M}) \xrightarrow{H(s)} H(\mathcal{N} \circ \mathcal{G}) \xrightarrow{H(\mathcal{R}_{\mathcal{G}}(t))} H(\mathcal{M}) = H(\mathcal{M} \circ \mathcal{F} \circ \mathcal{G}).$$

Then it is easy to check that $H(T) = id$ and $H(s) \circ H(\mathcal{R}_{\mathcal{G}}(t)) = id$, so

$$H(s) : H(\mathcal{M}) \xrightarrow{\sim} H(\mathcal{G}^* \mathcal{N}).$$

Here we will use the version where G is an idempotent, $Hom_{\mathcal{B}}(X_1, X_2)$ is the image of G and $F : Hom_{\mathcal{B}}(X_1, X_2) \to Hom_{\mathcal{A}}(X_1, X_2)$ is the inclusion; see [17]. The same for $\tilde{G}$ and $\tilde{F}$.

Let (X, g) be a Riemannian manifold with corners. Let (f, g) be a directed pair with φ_t the gradient flow of f. Denote by $H_0 \subset \partial X$ the hypersurface where ∇f is pointing outward, and $H_1 \subset \partial X$ the hypersurface where ∇f is pointing inward. Let $D'(X - H_0, H_1), D'(X - H_1, H_0)$, called *relative currents*, be the dual of $\Omega(X - H_1, H_0)$ and $\Omega(X - H_0, H_1)$ respectively.

In the following, we briefly recall the idempotent functor on $\Omega(X, H_0)$ constructed in [11] and [17] and used by [22] in the manifold-with-corners setting. Consider the functor

$$\begin{aligned} P : \Omega(X - H_1, H_0) &\to D'(X - H_1, H_0) \\ \alpha &\mapsto \sum_{x \in Cr(f)} (\int_{\mathscr{U}_x} \alpha)[\mathscr{S}_x], \end{aligned} \tag{2.12}$$

where $Cr(f)$ is the set of critical points of f, $\mathscr{U}_x$ is the unstable manifold associated to x and $\mathscr{S}_x$ is the stable manifold associated to x.

There is a homotopy functor T between P and the inclusion $I : \Omega(X - H_1, H_0) \hookrightarrow D'(X - H_1, H_0)$ given by the current $\{(\varphi_t(y), y) : t \in \mathbb{R}_{\geq 0}\} \subset X \times X$ in $D'(X \times X)$. To construct a real idempotent functor on $\Omega(X - H_1, H_0)$, one composes it with a smoothing functor. For readers interested in further details, see [17].

A consequence of the functor P is the Morse theory for manifolds with corners:

$$\Omega(X - H_1, H_0) \simeq Mor^*(X, f). \tag{2.13}$$

Following the notations in Section 2.3, this implies the following canonical quasi-isomorphisms

$$\begin{aligned} &Hom_{Mor(X)}(\mathfrak{U}_0, \mathfrak{U}_1) \simeq Mor^*(X_{m_0 > t_0} \cap X_{m_1 > t_1}, f_1 - \epsilon f_0) \\ \simeq &(\Omega(X_{m_0 \geq t_0} \cap X_{m_1 > t_1}, X_{m_0 = t_0} \cap X_{m_1 > t_1}), d) \simeq (\Omega(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1), d) \\ = &Hom_{Open(X)}(\mathfrak{U}_0, \mathfrak{U}_1). \end{aligned}$$

Applying Homological Perturbation Lemma to the dg category $Open(X)$ through the functors P, I and T for each pair of objects, one can show that $Mor(X)$ is exactly the A_∞-category $\mathcal{B}$ in Theorem 2.3.3 constructed out of these data. Using the set-up for defining higher morphisms in $Mor(X)$ of Section 2.3, there is a nice description of the moduli space $\mathcal{M}(T; \tilde{f}_0, ..., \tilde{f}_{k-1}; a_0, ..., a_k)$ in terms of intersections of the stable manifold of a_i for $i \neq k$ and the unstable manifold of a_k, which also involves the functors P and I. After a smoothing functor, one replaces intersection of currents by wedge product on differential forms, then compare this with the formalism of Homological Perturbation Lemma to get the assertion. For more details about the argument, see [17]. We will use the same idea in Section 2.4 for left A_∞-modules.

The microlocalization $\mu_X : Sh(X) \xrightarrow{\sim} TwFuk(T^*X)$

For any $\mathfrak{U} = (U, m) \in Mor(X)$, we can associate the standard brane $L_{U,m}$ in $Fuk(T^*X)$ (see Section 1.3 (b)), and in this way $Mor(X)$ is naturally identified with the A_∞-subcategory of $Fuk(T^*X)$ generated by these standard branes.

Roughly speaking, one does a series of appropriate perturbations and dilations to the branes L_{U_i,m_i}, $i = 1, ..., k$, so that (1) after further variable dilations (see Section 1.3 (c)), one can use the monotonicity properties (Proposition 1.3.1, Remark 1.3.2) to get that all holomorphic discs bounding the (dilating family of) the new branes $\epsilon \cdot L_i, i = 1, ..., k$ have boundary lying in the partial graphs $\epsilon \cdot L_i|_{\tilde{U}_i} = \epsilon \cdot \Gamma_{d\tilde{f}_i}, i = 1, ..., k$, where $\tilde{U}_i$ is a small perturbation of U_i and $\tilde{f}_i : \tilde{U}_i \to \mathbb{R}$ is some function; (2) the sequence $(\tilde{U}_i, \tilde{f}_i), i = 1, ..., k$ is directed, and hence one could use $(\tilde{U}_i, \tilde{f}_i)$ as representatives in the calculation of (higher) morphisms involving $\mathfrak{U}_i = (U_i, m_i), i = 1, ..., k$ in $Mor(X)$. Since we will use the same technique in Section 2.4 and 2.4, we refer the details there.

Recall Fukaya-Oh's theorem.

Theorem 2.3.4 ([6]). *For a compact Riemannian manifold (X, g), and a generic sequence of functions $f_1, ..., f_k$ on X, there is an orientation preserving diffeomorphism between the moduli space of holomorphic discs (with respect to the Sasaki almost complex structure) bounding the sequence of graphs $\epsilon \cdot \Gamma_{df_i}, i = 1, ..., k$ and the moduli space of Morse trees for the sequence $(X, \epsilon f_i), i = 1, ..., k$, for all $\epsilon > 0$ sufficiently small.*

Since the proof of the theorem is local and essentially relies on the C^1-closeness of the graphs to the zero section, one could adapt it to the directed sequence $(\tilde{U}_i, \tilde{f}_i), i = 1, ..., k$ and conclude that the moduli space of discs bounding $\epsilon \cdot L_i, i = 1, ..., k$ is diffeomorphic (as oriented manifolds) to the moduli space of Morse trees for the sequence $(\tilde{U}_i, \tilde{f}_i), i = 1, ..., k$. Therefore we get the quasi-embedding $i : Mor(X) \hookrightarrow Fuk(T^*X)$.

Next one composes i with the quasi-equivalence $\mathcal{P} : Open(X) \to Mor(X)$ from Section 2.3, and get a quasi-embedding $i \circ \mathcal{P} : Open(X) \to Fuk(T^*X)$. Then taking twisted complexes on both sides, we get the microlocal functor $\mu_X : Sh(X) \to TwFuk(T^*X)$. To simplify notation, we will denote $TwFuk(T^*X)$ by $F(T^*X)$. The main idea in [21] of proving that μ_X is a quasi-equivalence is to resolve the conormal to the diagonal in $T^*(X \times X)$ using product of standard branes in T^*X. Since we will only use the statement, we refer interested readers to [21] for details.

For a fixed stratification $\mathcal{S}$, let $Fuk_{\mathcal{S}}(T^*X)$ be the full subcategory of $Fuk(T^*X)$ consisting of branes L with $L^\infty \subset T^\infty_{\mathcal{S}} X$, and let $F_{\mathcal{S}}(T^*X)$ denote its twisted complexes. Then we also have

$$\mu_X|_{Sh_{\mathcal{S}}(X)} : Sh_{\mathcal{S}}(X) \xrightarrow{\sim} F_{\mathcal{S}}(T^*X). \tag{2.14}$$

2.4 Quasi-representing $M_{x,F}$ on $Fuk_{\mathcal{S}}(T^*X)$ by the local Morse brane $L_{x,F}$

Continuing the convention from Section 2.2, for a complex stratification $\mathcal{S}$, we refine it to have each stratum a cell, and denote the resulting stratification by $\mathcal{S}$ as well. The test triples (x, ξ, F) we are considering for $\Lambda_{\mathcal{S}}$ are always away from the newly added strata.

Given a test triple (x, ξ, F), we will construct a Lagrangian brane $L_{x,F}$ supported on a neighborhood of x, such that the functor $Hom_{F(T^*X)}(L_{x,F}, -) : F_{\mathcal{S}}(T^*X) \to Ch$ under pull-back by μ_X is quasi-isomorphic to the local Morse group functor $M_{x,F}$.

Construction of $L_{x,F}$

Consider the function $\mathrm{r} \times \Re(F) : B_{2\epsilon}(x) \to \mathbb{R}^2$, where $\mathrm{r}(\mathbf{z}) = \|\mathbf{z}\|^2$ as before. Let R be an open subset of $\Lambda_{\mathcal{S}}$-regular values of $\mathrm{r} \times \Re(F)$ in $\mathbb{R}^2$, such that it contains $(0, \delta) \times \{0\}$ for some $\delta > 0$, and if $(a, b), (a, c) \in R$ for $b < c$, then $\{a\} \times [b, c] \subset R$ (here we have used Lemma 1.1.5). There exists a $0 < \tilde{r}_2 < \delta$ for which the function r has no $\Lambda_{\mathcal{S}}$-critical value in $(0, \tilde{r}_2)$. Fixing such a $\tilde{r}_2$, choose $0 < \tilde{r}_1 < \tilde{r}_2$ and $\eta > 0$ small enough so that R contains $(\tilde{r}_1, \tilde{r}_2) \times (-2\eta, 2\eta)$, and $\Re(F)$ has no $\Lambda_{\mathcal{S}}$-critical value in $(-2\eta, 0)$ or $(0, 2\eta)$. Also choose $\tilde{r}_1 < r_1 < r_2 < \tilde{r}_2$.

Let

$$\mu = -\frac{1}{2}\eta, \delta_1 = \frac{1}{2}(r_2 - r_1), \delta_2 = \frac{1}{4}\eta,$$

and

$$u(\mathbf{z}) = \mathrm{r}(\mathbf{z}) - (r_2 - \delta_1), v(\mathbf{z}) = \Re(F)(\mathbf{z}) - (\mu - \delta_2).$$

Near $\{u = v = 0\}$, we smooth the corners in

$$W_1 := \{u = 0, v \leq 0\} \cup \{u \leq 0, v = 0\}$$

as follows. Let $\bar{\epsilon}_1 = \frac{1}{2}\min(\delta_1, \delta_2)$. We remove the portion $\{u^2 + v^2 \leq \bar{\epsilon}_1^2\}$ from W_1 and glue in 3/4 of the cylinder $\{u^2 + v^2 = \bar{\epsilon}_1^2\}$, i.e. the part where u, v are not both negative. Then we smooth the connecting region so that its (outward) unit conormal vector is always a linear combination of $d\mathrm{r}$ and $d\Re(F)$, in which at least one of the coefficients is positive. This can be achieved by looking at the local picture in the leftmost corner of Figure 2.2, where we complete u, v to be the coordinates of a local chart. We will denote the resulting hypersurface by $\widetilde{W}_1$.

Now we choose a defining function $m_{\widetilde{W}_1}$ for $\widetilde{W}_1$ such that

(i) in an open neighborhood $\mathcal{U}_1$ of $\widetilde{W}_1$, $m_{\widetilde{W}_1}$ is a function of u, v, and $dm_{\widetilde{W}_1} \neq 0$,

(ii) $m_{\widetilde{W}_1} = u$ on $\{v \leq -\frac{4}{3}\bar{\epsilon}_1, |u| \leq \frac{1}{2}\bar{\epsilon}_1\}$, $m_{\widetilde{W}_1} = v$ on $\{u \leq -\frac{4}{3}\bar{\epsilon}_1, |v| \leq \frac{1}{2}\bar{\epsilon}_1\}$.

Then there exists a $0 < \epsilon_1 < \frac{1}{2}\bar{\epsilon}_1$, so that the set $\{0 \leq m_{\widetilde{W}} \leq \epsilon_1\}$ is contained in $\mathcal{U}_1$, and $dm_{\widetilde{W}_1}$ is a linear combination of $d\mathrm{r}$ and $d\Re(F)$ over that set, in which at least one of the coefficients is positive.

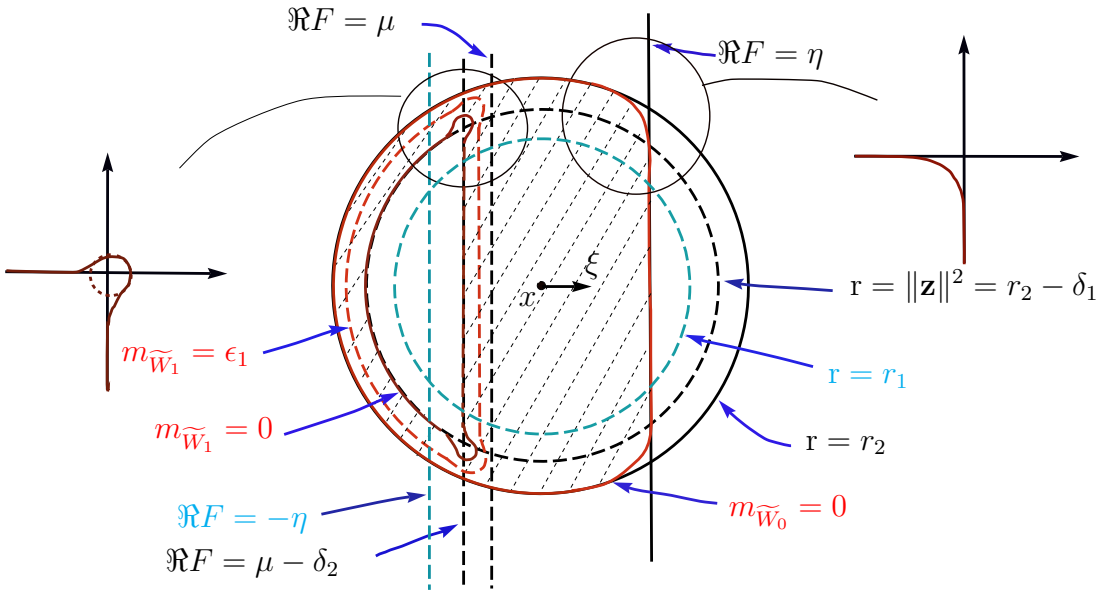


Figure 2.2: Construction of U: U is the shaded area, and the leftmost and rightmost pictures are illustrating the local smoothing process for the corners.

Similarly, we can smooth the corners in $\{\mathrm{r} = r_2, \Re(F) \leq \eta\} \cup \{\mathrm{r} \leq r_2, \Re(F) = \eta\}$, but in the way illustrated on the right hand side of Figure 2.2. We will denote the resulting hypersurface by $\widetilde{W}_0$. We choose a defining function $m_{\widetilde{W}_0}$ and $\epsilon_0 > 0$ in the same fashion as for $m_{\widetilde{W}_1}$ and ϵ_1.

Let $b : (-\infty, \eta) \to \mathbb{R}$ be a nondecreasing C^1-function such that $b(x) = x$ for $x \in (\frac{-\eta}{4}, \frac{\eta}{2})$, $\lim_{x \to \eta^-} b(x) = +\infty$, and the derivative $b' = 0$ exactly on $(-\infty, -\eta]$. Let $c : (0, r_2) \to \mathbb{R}_{\geq 0}$ be a nondecreasing C^1-function such that $\lim_{x \to r_2^-} c(x) = +\infty$, and $c' = 0$ exactly on $(0, r_1]$. Let $d : (-\infty, 0) \to \mathbb{R}_{\geq 0}$ be a nondecreasing C^1-function with $\lim_{x \to 0^-} d(x) = +\infty$, and $d' = 0$ exactly on $(-\infty, -\epsilon_0]$. Let $e : (0, +\infty) \to \mathbb{R}_{\leq 0}$ be a nondecreasing C^1-function with $\lim_{x \to 0^+} e(x) = -\infty$ and $e' = 0$ exactly on $[\epsilon_1, +\infty)$.

Now define

$$f = b \circ \Re(F) + c \circ \mathrm{r} + d \circ m_{\widetilde{W}_0} + e \circ m_{\widetilde{W}_1}$$

on

$$U = \text{ the domain bounded by } \widetilde{W}_0 \text{ and } \widetilde{W}_1.$$

The construction of U is interpreted in Figure 2.2, where U is the shaded area.

Lemma 2.4.1. *Γ_{df} is a closed, properly embedded Lagrangian submanifold in T^*X satisfying $\overline{\Gamma_{df}} \pitchfork \overline{\Lambda_{\mathcal{S}}} = \{(x, \xi)\}$ in $\overline{T^*X}$.*

Proof. Only the part $\Gamma_{df} \cap \Lambda_{\mathcal{S}} = \{(x, \xi)\}$ needs to be proved. The hypersurfaces $\mathrm{r} = r_1$ and $\Re(F) = -\eta$ divide U into three regions

$$U_1 = \{\mathrm{r} \leq r_1\} \cap U,\ U_2 = \{\mathrm{r} > r_1\} \cap \{\Re(F) > -\eta\} \cap U, \text{ and } U_3 = \{\Re(F) \leq -\eta\} \cap U.$$

On U_1 (resp. U_3), df is a positive multiple of $d\Re(F)$ (resp. $d\mathrm{r}$), so $df \notin \Lambda_{\mathcal{S}}$ over $U_1 \cup U_3$ except at x. On U_2, df is a linear combination of $d\Re(F)$ and $d\mathrm{r}$, in which at least one of the coefficients is positive. Since $\mathrm{r} \times \Re(F)$ is $\Lambda_{\mathcal{S}}$-regular on U_2, we get $df \notin \Lambda_{\mathcal{S}}$. □

By Proposition 1.3.3 and 1.3.4, Γ_{df} can be equipped with a canonical brane structure b. Let $L_{x,F}$ denote $(\Gamma_{df}, \mathcal{E}, b, \Phi)$ as an object in $Fuk(T^*X)$, where $\mathcal{E}$ is a trivial local system of rank 1 on Γ_{df} and the perturbation $\Phi = \{L^s_{x,F}\}$ is defined as follows.

For sufficiently small $s > 0$, let

$$U_s = \{x \in U : |f(x)| < -\log s\}, U_0 = U \tag{2.15}$$

and $L^s_{x,F}$ be a Lagrangian over $\overline{U}$ satisfying:
(1) $L^s_{x,F}|_{\overline{U}_s} = \Gamma_{df}|_{\overline{U}_s}$;
(2) $L^s_{x,F}|_{\partial U} = T^*_{\partial U}X|_{|\xi| \geq \beta_s}$, where $\beta_s \to \infty$ as $s \to 0$;
(3) $L^s_{x,F}|_U = \Gamma_{df_s}|_U$ for a function f_s on U;
(4) $f_s|_{U_s} = f|_{U_s}$, and $df_s|_z = \lambda(z)df|_z$ for some $0 < \lambda(z) \leq 1$ for $z \in U - U_s$;
(5) let $K^s_{\bar{\eta}} = \min\{|\xi| : \xi \in L^s_{x,F}|_{\partial U_{\bar{\eta}}}\}$, we require $K^s_{\bar{\eta}_1} > K^s_{\bar{\eta}_2}$ for $0 < \bar{\eta}_1 < \bar{\eta}_2 < s$. The notation $L|_W$ for a Lagrangian L and a subset $W \subset X$ means the set $L \cap \pi^{-1}(W)$, where $\pi : T^*X \to X$ is the standard projection.

Computation of $Hom_{Fuk(T^*X)}(L_{x,F}, L_V)$ for $V \in \mathcal{U}_{\mathcal{S},x}$

Let $V \in \mathcal{U}_{\mathcal{S},x}$ (recall the notation is defined in (2.10)), then ∂V is stratified by a subset $\mathcal{S}_{\partial V}$ of $\mathcal{S}$. Fix a semi-defining function m for V (see Section 1.1). We have the standard Lagrangian brane $L_{V,m}$ associated to V as defined in Section 1.3 (b), for which we will simply denote by L_V. Let

$$V_t = X_{m>t} \text{ for } t \geq 0.$$

Let $(\partial U)_{out}$, $(\partial U)_{in}$ denote $\widetilde{W}_0$ and $\widetilde{W}_1$ respectively. Let A denote the annulus enclosed by $(\partial U)_{in}$, $\mathrm{r} = r_2$ and $\Re(F) = \mu$, including only the boundary component $(\partial U)_{in}$.

Lemma 2.4.2. *For $t > 0$ sufficiently small, there is a compatible collection of quasi-isomorphisms of complexes*

$$\begin{aligned}
&(\Omega((\overline{U} - (\partial U)_{out}) \cap V_t, (\partial U)_{in} \cap V_t), d) \\
\simeq &(\Omega((\overline{U} - (\partial U)_{out}) \cap V_t, A \cap V_t), d) && (2.16)\\
\simeq &(\Omega(B_{r_2}(x) \cap V_t, B_{r_2}(x) \cap \{\Re(F) < \mu\} \cap V_t), d) && (2.17)\\
\simeq &(\Omega(B_{r_2}(x) \cap V, B_{r_2}(x) \cap \{\Re(F) < \mu\} \cap V), d) && (2.18)
\end{aligned}$$

Proof. For (2.16), we only need to prove that $A \cap V_t$ deformation retracts onto $(\partial U)_{in} \cap V_t$. First, we can construct a smooth vector field on an open neighborhood of $(\overline{U} - (\partial U)_{out}) \cap \{\mu - \delta_2 < \Re(F) < \mu\}$ integrating along which gives a deformation retraction from A onto $U \cap \{\Re(F) \leq \mu - \delta_2\} \cap \{m_{\widetilde{W}_1} \geq 0\} \cup (\partial U)_{in}$. For example, we can choose the vector field $\mathbf{v}$ such that

$$\mathbf{v}(\Re(F)) = -1; \mathbf{v}(\mathrm{r}) = 0 \text{ near } \partial B_{r_2}(x); \mathbf{v}(m) = 0 \text{ near } \partial V_t; \mathbf{v}(m_{\widetilde{W}_1}) \neq 0 \text{ on } \widetilde{W}_1.$$

Similarly, we can construct a deformation retraction from $U \cap \{\Re(F) \leq \mu - \delta_2\} \cap \{m_{\widetilde{W}_1} \geq 0\} \cup (\partial U)_{in}$ onto $(\partial U)_{in} \cap V_t$.

The identification in (2.17) is by excision on the triple $(V_t \cap \{m_{\widetilde{W}_1} < 0\}) \subset V_t \cap B_{r_2}(x) \cap \{\Re(F) < \mu\} \subset V_t \cap B_{r_2}(x)$, and a deformation retraction from $V_t \cap B_{r_2}(x)$ onto $V_t \cap \{m_{\widetilde{W}_0} < 0\}$. One can construct a similar vector field for this and we omit the details. The quasi-isomorphism (2.18) can be also obtained in a similar way. □

Let $L_V^t, t > 0$ small, be a family of perturbations of L_V supported over $\overline{V_t}$ satisfying:
(1) $L_V^t|_{\overline{V}_{2t}} = L_V|_{\overline{V}_{2t}}$,
(2) $L_V^t|_{\partial V_t} = T^*_{\partial V_t} X|_{|\xi| \geq \lambda_t}$, for some $\lambda_t > 0$;
(3) $L_V^t|_{V_t} = \Gamma_{dh_{V_t}}$, where h_{V_t} is a function on V_t such that $h_{V_t}|_{V_{2t}} = \log m|_{V_{2t}}$, dh_{V_t} and $d\log m$ are colinear on V_t and $1 \leq \frac{dh_{V_t}}{d\log m} \leq 1.2$.

Again by by Proposition 1.3.3 and 1.3.4, L_V carries a canonical brane structure. Let L_V also denote the object in $Fuk(T^*X)$ consisting of the canonical brane structure, a trivial local system of rank 1 on it and the above perturbation $\{L_V^t\}_{t \geq 0}$. The proof of the following lemma is essentially the same as in Section 6 of [22]. The only difference is that we use the above conical perturbations and avoid geodesic flows.

Lemma 2.4.3. *There is a fringed set $R \subset \mathbb{R}^2_+$, such that for $(t, s) \in R$, there is a compatible collection of quasi-isomorphisms*

$$Hom_{Fuk(T^*X)}(L^s_{x,F}, L^t_V) \simeq \Omega(V \cap B_{r_2}(x), V \cap B_{r_2}(x) \cap \{\Re(F) < \mu\}).$$

Proof. Step 1. *Perturbations and dilations.*
This step is essentially the same as the perturbation process in $Mor(X)$ stated at the beginning of Section 2.3. Nevertheless, we repeat it to set up notations. There is an (nonempty) open interval $(0, \eta_0)$ such that for all $t \in (0, \eta_0)$, ∂V_t and ∂U intersect transversally. Fixing any $t \in (0, \eta_0)$, there is an open neighborhood W_t of $\partial V_t \cap \partial U$ such that the covectors $d\log m|_z$ and $L^s_{x,F}|_z$ are linearly independent for all $s > 0$ and $z \in W_t \cap \overline{V_t} \cap \overline{U}$. Choose $t < \bar{t} < \eta_0$ such that $X_{t \leq m \leq \bar{t}} \cap \partial U \subset W_t$. Since $X_{t \leq m \leq \bar{t}} - W_t$ is compact, we can find $\epsilon_t > 0$ such that $|dh_{V_t}| > \epsilon_t |df|$ on $(X_{t \leq m_V \leq \bar{t}} - W_t) \cap \overline{U}$. There is a small $\eta_1 > 0$ such that $(\overline{U} - U_{\eta_1}) \cap X_{t \leq m_V \leq \bar{t}} \subset W_t$ and on $(\overline{U} - U_{\eta_1}) \cap \overline{V}_t - W_t$ we have $|dh_{V_t}|$ bounded above by some M_t, so we can find $\eta_1 > \bar{s} > s > 0$ small enough so that $\{\epsilon_t |\xi| : \xi \in L^s_{x,F}|_{\overline{U} - U_{\bar{s}}}\}$ is bounded below by $2M_t$. In summary, we first choose $t, \bar{t}$, then ϵ_t and lastly $s, \bar{s}$, and it's clear that the collection of such (t, s) forms a fringed set in $\mathbb{R}^2_+$. It is also clear that we can choose $\bar{s}$ small so that $(t, \bar{t}) \times (s, \bar{s}) \subset R$.

There is a Riemannian metric g on X such that after a small perturbation, $(h_{V_t}-\epsilon_t f_s, g)$ is a directed pair on the manifold with corners $\overline{V_{\bar{t}}}\cap\overline{U_{\bar{s}}}$, and choices of such metric form an open convex subset. And this also holds for any $(\tilde{s},\tilde{t})\in(t,\bar{t})\times(s,\bar{s})$.

Step 2. *Energy bound.*
Choose $t<t_1<t_2<t_3<\bar{t}$ and $s<s_1<s_2<s_3<\bar{s}$.

Let $G_1=L_V^t|_{X_{t_2<m<t_3}}$ and $G_2=\epsilon_t\cdot L_{x,F}^s|_{U_{s_2}-\overline{U}_{s_3}}$. Choose some very small $\delta_{i,v}>0$ and define the tube-like open set

$$T_i=\bigcup_{(x,\xi)\in G_i} B^v_{\delta_{i,v}}(x,\xi),$$

where $B^v_{\delta_{i,v}}(x,\xi)$ means the vertical ball in the cotangent fiber T^*_xX of radius $\delta_{i,v}$ centered at (x,ξ). With small enough $\delta_{i,v}$, we have $T_1\cap T_2=\emptyset$.

Let $L_V^{t,\ell}=\varphi^{\ell}_{D^{\log m}_{t_3,\bar{t}}}(L_V^t)$ and $L_{x,F}^{s,\ell}=\varphi^{\epsilon_t\ell}_{D^f_{s_3,\bar{s}}}(\epsilon_t\cdot L_{x,F}^s)$ for $0<\ell<1$, where $\varphi^{\ell}_{D^{\log m}_{t_3,\bar{t}}}$ and $\varphi^{\epsilon_t\ell}_{D^f_{s_3,\bar{s}}}$ are the variable dilations defined in Section 1.3 (c). Note that the variable dilations fix G_1,G_2 and $\overline{L_V^{t,\ell}}\cap\overline{L_{x,F}^{s,\ell}}=(1-\ell)\cdot(\overline{L_V^t}\cap\epsilon_t\cdot\overline{L_{x,F}^s})$.

By Proposition 1.3.1 and Remark 1.3.2, for ℓ sufficiently close to 1, all the discs bounding $L_V^{t,\ell}$ and $L_{x,F}^{s,\ell}$ have boundaries lying in $L_V^{t,\ell}|_{X_{m>t_2}}\cup L_{x,F}^{s,\ell}|_{U_{s_2}}$. Fixing such an ℓ, the same holds for the family of uniform dilations $\epsilon\cdot L_V^{t,\ell}$ and $\epsilon\cdot L_{x,F}^{s,\ell}$, $0<\epsilon\leq 1$.

It is easy to see that $L_{x,F}^{s,\ell}|_{U_{s_1}}$ and $L_V^{t,\ell}|_{V_{t_1}}$ are the graph of differentials of a directed sequence $(U_{s_1},f_1),(V_{t_1},f_2)$, and by Fukaya-Oh's theorem (Theorem 2.3.4) and Morse theory for manifolds with corners (2.13), we have for $\epsilon>0$ sufficiently small,

$$\begin{aligned}&Hom_{Fuk(T^*X)}(L_{x,F}^s,L_V^t)\simeq Hom_{Fuk(T^*X)}(\epsilon\cdot L_{x,F}^{s,\ell},\epsilon\cdot L_V^{t,\ell})\\ \simeq&Mor^*(\overline{U}_{s_1}\cap\overline{V}_{t_1},\epsilon(f_2-f_1))\simeq(\Omega((\overline{U}-(\partial U)_{out})\cap V_{t_1},(\partial U)_{in}\cap V_{t_1}),d)\\ \simeq&(\Omega(B_{r_2}(x)\cap V,B_{r_2}(x)\cap\{\Re(F)<\mu\}\cap V),d).\end{aligned}$$

The last identity is from Lemma 2.4.2. □

$H(M_{x,F})\cong H(\mu_X^*Hom_{F(T^*X)}(L_{x,F},-))$

Given a sequence of open submanifolds with semi-defining functions $(V_i,m_i), i=1,...,k$, there is a fringed set $R\subset\mathbb{R}_+^{k+1}$ such that for all $(t_k,...,t_1,t_0)\in R$, there exist $\epsilon_k,...,\epsilon_1,\epsilon_0>0$ and $(t_k,...,t_1,t_0)<(\bar{t}_k,...,\bar{t}_1,\bar{t}_0)\in R$, satisfying
(1) $\partial U(\text{resp.}\partial U_{t_0}),\partial V_{t_1},...,\partial V_{t_k}$ intersect transversally, i.e. the unit conormal vectors to them are linearly independent at any intersection point;
(2) Let $\Gamma_i^{t_i}=\epsilon_i\cdot L_{V_i}^{t_i}|_{V_{i,\bar{t}_i}}$ for $i=1,...,k$, and $\Gamma_0^{t_0}=\epsilon_0\cdot L_{x,F}^{t_0}|_{U_{\bar{t}_0}}$. Then $\epsilon_i\cdot L_{V_i}^{t_i}\cap\epsilon_j\cdot L_{V_j}^{t_j}=\Gamma_i^{t_i}\cap\Gamma_j^{t_j}$ for $0<i<j$ and $\epsilon_0\cdot L_{x,F}^{t_0}\cap\epsilon_i\cdot L_{V_i}^{t_i}=\Gamma_0^{t_0}\cap\Gamma_i^{t_i}$, for $i>0$;
(3) For all $(p_i)_{i=1}^k\in R$ with $(t_i)_{i=0}^k<(p_i)_{i=0}^k<(\bar{t}_i)_{i=0}^k$, $(U_{\bar{t}_0},\epsilon_0 f_{t_0}),(V_1,\epsilon_1 h_{V_{1,t_1}}),...,(V_k,\epsilon_k h_{V_{k,t_k}})$ is a directed sequence.
The notation $(t_i)_{i=0}^k<(s_i)_{i=0}^k$ means $t_i<s_i$ for all $0\leq i\leq k$. We start by choosing

appropriate t_k and ϵ_k and then do induction. First, let

$$\Lambda_{\neq k} = \bigcup_{i<k} \Lambda_i.$$

There is $\eta_k > 0$ such that on $(0, \eta_k)$, m_k has no $\Lambda_{\neq k}$-critical value. Pick any small $t_k \in (0, \eta_k)$, and form $L_{V_k}^{t_k}$. Let $\epsilon_k = 1$.

Suppose we have chosen $t_k, ..., t_{i+1}$ and $\epsilon_k, ..., \epsilon_{i+1}$ for $i > 0$, let Λ_{j,t_j} be the associated conical Lagrangian of the stratification compatible with $\{X_{m_j=t_j}\}$, for $j = i+1, ..., k$. Let

$$\Lambda_{\neq i} = (\bigcup_{j<i} \Lambda_j) \bigcup (\bigcup_{j>i} \Lambda_{j,t_j}).$$

There is $\eta_i > 0$ so that m_i has no $\Lambda_{\neq i}$-critical value in $(0, \eta_i)$. On $V_i \cap X_{m_j \geq t_j}$ for each $j > i$, there is an open neighborhood W_{ij} of $X_{m_i=0} \cap X_{m_j=t_j}$ on which $d \log m_i$ and $d \log m_j$ are everywhere linearly independent. Choose $t_j < \eta'_{ji} < \eta_j$ such that $X_{t_j \leq m_j \leq \eta'_{ji}} \cap \partial V_i \subset W_{ij}$ for $j > i$. On $X_{t_j \leq m_j \leq \eta'_{ji}} \cap V_i - W_{ij}$, we have the covectors in $\epsilon_j \cdot L_{V_j}^{t_j}$ bounded from below by some $N_{ij} > 0$. Choose $\epsilon_i > 0$ such that on this region, the covectors in $\epsilon_i \cdot L_{V_i}$ are bounded above by $\frac{1}{2} N_{ij}$ for all $j > i$. Next, choose $0 < \eta'_{ij} < \eta_i$ so that $X_{m_i \leq \eta'_{ij}} \cap X_{m_j \leq \eta'_{ji}} \subset W_{ij}$ for all $j > i$. Then covectors in $\epsilon_j \cdot L_{V_j}^{t_j}$ over $X_{m_i \leq \eta'_{ij}} \cap X_{m_j \geq t_j} - W_{ij}$ are bounded above by some M_{ij}. Choose $0 < t_i < \eta'_{ij}$ for all $j > i$ so that the covectors on the graph $\epsilon_i \cdot d \log m_i$ over $X_{m_i = t_i} \cap X_{m_j \geq t_j}$ are bounded below by $2M_{ij}$. Now we have t_i, ϵ_i and $L_{V_i}^{t_i}$.

Finally, having chosen $t_k, ..., t_1$, and $\epsilon_k, ..., \epsilon_1$, to choose ϵ_0, we do the same thing as before. However, to choose t_0, we don't shrink U. Instead, we find t_0 small enough so that on $U - U_{t_0} \cap X_{m_j \geq t_j}$, $\epsilon_0 \cdot L_{x,F}^{t_0}$ is bounded below by $2M_{0j}$ for all $j > 0$. Clearly, the choices of $(t_k, ..., t_0)$ form a fringed set $R \subset \mathbb{R}_+^{k+1}$.

The choices of $\bar{t}_k, ..., \bar{t}_0$ can be made as follows. First, $\bar{t}_k$ can be anything satisfying $t_k < \bar{t}_k < \eta'_{ki}$ for all $i < k$. Once we have chosen $\bar{t}_k, ..., \bar{t}_{i+1}$ for $i > 0$, $\bar{t}_i$ should satisfy $t_i < \bar{t}_i < \eta'_{ij}$ for all $j \neq i$ and on $X_{t_i \leq m_i \leq \bar{t}_i}$, the covectors $d \log m_i$ are bounded below by $1.5 M_{ij}$ for all $j > i$. Similar choice can be made for $\bar{t}_0$. Also we can make $(\bar{t}_k, ..., \bar{t}_0)$ belong to R.

Again, we do variable dilations to $\epsilon_0 \cdot L_{x,F}^{t_0}, \epsilon_i \cdot L_{V_i}^{t_i}$ and run the energy bound argument on holomorphic discs. Choose $(t_i)_{i=0}^k < (p_i)_{i=0}^k < (q_i)_{i=0}^k < (s_i)_{i=0}^k < (\bar{t}_i)_{i=0}^k$ in R. Let

$$\tilde{L}_{x,F}^{t_0,\ell} = \varphi^{\epsilon_0 \ell}_{D^f_{s_0, \bar{t}_0}}(\epsilon_0 \cdot L_{x,F}^{t_0}),\ \tilde{L}_{V_i}^{t_i,\ell} = \varphi^{\epsilon_i \ell}_{D^{\log m_i}_{s_i, \bar{t}_i}}(\epsilon_i \cdot L_{V_i}^{t_i}) \text{ for } i > 1,$$

and

$$\tilde{L}_{x,F}^{t_0,\ell}|_{U_{p_0}} = \Gamma_{d\tilde{f}_{t_0,\ell}},\ \tilde{L}_{V_i}^{t_i,\ell}|_{V_{i,p_i}} = \Gamma_{d\tilde{h}_{i,t_i}}$$

for some function $\tilde{f}_{t_0,\ell}$ on U_{p_0}, and $\tilde{h}_{i,t_i,\ell}$ on V_{i,p_i} for $1 \leq i \leq k$.

For ℓ sufficiently close to 1, all holomorphic discs bounding these Lagrangians have boundaries lying in $\tilde{L}_{x,F}^{t_0,\ell}|_{U_{q_0}} \cup \bigcup_{i=1}^k \tilde{L}_{V_i}^{t_i,\ell}|_{V_{i,q_i}}$. Since the sequence $(U_{p_0}, \tilde{f}_{t_0,\ell}), (V_{1,p_1}, \tilde{h}_{1,t_1,\ell}), ..., (V_{k,p_k}, \tilde{h}_{k,t_k,\ell})$ is directed, using Fukaya-Oh's theorem, we get the following.

Lemma 2.4.4. *For ℓ sufficiently close to 1,*

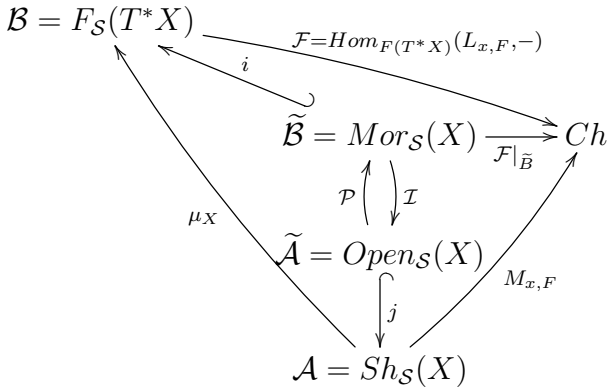
$$\begin{aligned} m^k_{Fuk(T^*X)} :& Hom_{Fuk(T^*X)}(\tilde{L}^{t_{k-1},\ell}_{V_{k-1}}, \tilde{L}^{t_k,\ell}_{V_k}) \otimes \cdots \otimes Hom_{Fuk(T^*X)}(\tilde{L}^{t_0,\ell}_{x,F}, \tilde{L}^{t_1,\ell}_{V_1}) \\ &\to Hom_{Fuk(T^*X)}(\tilde{L}^{t_0,\ell}_{x,F}, \tilde{L}^{t_k,\ell}_{V_k})[2-k] \end{aligned}$$

is given by counting Morse trees:

$$m^k_{Fuk(T^*X)}(a_{k-1}, \cdots, a_0) = \sum_T \sum_{a_k \in S} \#\mathcal{M}(T; \epsilon\tilde{f}_{t_0,\ell}, \epsilon\tilde{h}_{1,t_1,\ell}, \cdots, \epsilon\tilde{h}_{k,t_k,\ell}; \pi(a_0), \cdots, \pi(a_k))^{0\text{-}d} \cdot a_k,$$

where $S = \epsilon \cdot (\Gamma_{d\tilde{f}_{t_0,\ell}} \cap \Gamma_{d\tilde{h}_{k,t_k,\ell}})$ for all ϵ sufficiently close to 0, where $\pi : T^*X \to X$ is the standard projection.

Consider the following diagram:



Here $i : \widetilde{\mathcal{B}} \hookrightarrow \mathcal{B}$ and $j : \widetilde{\mathcal{A}} \hookrightarrow \mathcal{A}$ are embeddings into triangulated envelopes; the functors $\mathcal{I}, \mathcal{P}$ are from applying the homological perturbation formalism to the functor P in (2.12); $\mu_X : \mathcal{A} \to \mathcal{B}$ is the microlocal functor in Section 2.3.

In Remark 2.3.3, putting G and $\tilde{G}$ to be the idempotent P in (2.12) on corresponding complexes, $\mathcal{M}$ to be $M_{x,F}|_{\widetilde{\mathcal{A}}}$ and $\mathcal{F}$ to be $\mathcal{I}$, gives us the $\mathcal{N}$ exactly the same as $\mathcal{F}|_{\widetilde{\mathcal{B}}}$. This follows directly from Lemma 2.4.4, and we have

$$H(j^*M_{x,F}) \cong H(\mathcal{P}^*\mathcal{F}|_{\mathcal{B}}) \cong H(j^*\mu_X^*\mathcal{F}).$$

Since the functors $\mu_X^*\mathcal{F}$ and $M_{x,F}$ both respect forming cones, we have

$$H(\mu_X^*\mathcal{F}) \cong H(M_{x,F}). \tag{2.19}$$

2.5 Computation of $Hom_{Fuk(T^*X)}(L_{x,F}, -)$ on holomorphic branes in $Fuk_{\mathcal{S}}(T^*X)$

Holomorphic Lagrangian Branes.

Let X be a compact complex manifold of dimension n. Let $T^*X_{\mathbb{C}}$ denote the holomorphic cotangent bundle of X equipped with the standard holomorphic symplectic form $\omega_{\mathbb{C}}$. Like

in the real case, there is a complex projectivization of $T^*X_{\mathbb{C}}$, namely

$$\overline{T^*X_{\mathbb{C}}} = (T^*X_{\mathbb{C}} \times \mathbb{C} - T^*_X X_{\mathbb{C}} \times \{0\})/\mathbb{C}^*.$$

For a holomorphic (complex analytic) Lagrangian L in $T^*X_{\mathbb{C}}$ which is by assumption a $\mathcal{C}$-set in $\overline{T^*X}$, using Theorem 4.4 in [23], one sees that $\overline{L}$ is complex analytic in $\overline{T^*X}_{\mathbb{C}}$. Note if X is a proper algebraic variety, then L is algebraic in $T^*X_{\mathbb{C}}$.

There is the standard identification (of real vector bundles) $\phi : T^*X_{\mathbb{C}} \to T^*X$ as follows. In local coordinates (q_{z_j}, p_{z_j}) on $T^*X_{\mathbb{C}}$ and $(q_{x_j}, q_{y_j}, p_{x_j}, p_{y_j})$ on T^*X, where $z_j = x_j + \sqrt{-1}y_j$ on X, we have $q_{x_j} = \Re q_{z_j}, q_{y_j} = \Im q_{z_j}, p_{x_j} = \Re p_{z_j}, p_{y_j} = -\Im p_{z_j}$. It's easy to check that $\phi^*\omega = \Re\omega_{\mathbb{C}}$, so ϕ sends every holomorphic Lagrangian to a Lagrangian. In the following, by a holomorphic Lagrangian in T^*X, we mean an *exact* Lagrangian which is the image $\phi(L)$ of a holomorphic Lagrangian L in $T^*X_{\mathbb{C}}$ under the identification ϕ. We will write L instead of $\phi(L)$ when there is no cause of confusion.

Equip T^*X with the Sasaki almost complex structure J_{Sas} and let η be the canonical trivialization of the bicanonical bundle κ (See Section 1.3).

First we have the following lemma on the flat case $X = \mathbb{C}^n$ (we don't need X to be compact here), where η is the volume form $\Omega = \bigwedge_{i=1}^n (dq_{x^i} + \sqrt{-1}dp_{x^i}) \wedge (dq_{y^i} + \sqrt{-1}dp_{y^i})$ up to a positive scalar.

Proposition 2.5.1. *Every holomorphic Lagrangian brane in T^*X ($X = \mathbb{C}^n$) has an integer grading with respect to J_{Sas}.*

Proof. Let L be a holomorphic Lagrangian in $T^*X_{\mathbb{C}}$. For any $(x, \xi) \in L$, let v_1,..., v_k, w_1,..., w_{n-k} be a basis of $T_{(x,\xi)}L$. After a change of coordinate and basis, we can assume that $v_i = \partial_{q_{z^i}} + \sum_{\mu=1}^{n} v_i^\mu \partial_{p_{z^\mu}}$ and $w_j = \sum_{\mu=1}^{n} w_j^\mu \partial_{p_{z^\mu}}$ for $i = 1, ..., k$, and $j = 1, ..., n-k$.

Then the condition of L being a Lagrangian implies that $w_1, ..., w_{n-k}$ generate $\langle \partial_{p_{z^i}} \rangle_{i=k+1,...,n}$ and after another change of basis, we could get $v_i = \partial_{q_{z^i}} + \sum_{\mu=1}^{k} v_i^\mu \partial_{p_{z^\mu}}$ with $(v_i^\mu)_{i,\mu \in \{1,...,k\}}$ a symmetric $k \times k$ matrix.

Let $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $J_m = \begin{pmatrix} J_1 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_1 \end{pmatrix}$ of size $2m \times 2m$. Then $T_{\phi(x,\xi)}\phi(L)$ has the form $\left(\begin{array}{cc|cc} I_k & \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K \end{array}\right)$ (by this, we mean $T_{\phi(x,\xi)}\phi(L)$ is spanned by the row vectors of the matrix under the basis $\partial_{q_{x^1}}, \partial_{q_{y^1}}, ..., \partial_{q_{x^n}}, \partial_{q_{y^n}}, \partial_{p_{x^1}}, \partial_{p_{y^1}}, ..., \partial_{p_{x^n}}, \partial_{p_{y^n}}$) where I_k is the identity matrix of size $2k \times 2k$, A is a symmetric matrix satisfying $AJ_k =$

$-J_kA$, and $K = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$ of size $2(n-k) \times 2(n-k)$. In particular, $KJ_{n-k} = -J_{n-k}K$.

Let $\Omega = \bigwedge_{i=1}^n (dq_{x^i} + \sqrt{-1}dp_{x^i}) \wedge (dq_{y^i} + \sqrt{-1}dp_{y^i})$ be a holomorphic volume form on T^*X with respect to J_{Sas}. Then for any basis $u_1, ..., u_{2n}$ of $T_{\phi(x,\xi)}\phi(L)$, $(\Omega(u_1 \wedge \cdots \wedge u_{2n}))^2 = C \cdot (\det(I_k + \sqrt{-1}A)\det(\sqrt{-1}K))^2$ where $C > 0$. Since $AJ_k = -J_kA$, for any eigenvector v of A with eigenvalue λ, we have $A(J_kv) = -\lambda(J_kv)$. In particular, if $1 + \lambda\sqrt{-1}$ is an eigenvalue of $I_k + \sqrt{-1}A$ then $1 - \lambda\sqrt{-1}$ is an eigenvalue of it as well, and they are of the same multiplicity. So $(\Omega(u_1 \wedge \cdots \wedge u_{2n}))^2$ is always a positive number, which implies that L has integer grading. □

In the general case of X, for any small disc $D = \{\sum_i |z_i|^2 < \epsilon\} \subset X$, let J_D be the Sasaki almost complex structure induced from a metric on X which is flat on D. Given a graded holomorphic Lagrangian L, deform J_{con} (relative to infinity) to agree with J_D on a relatively compact subset of $T^*X|_D$. Proposition 2.5.1 says that it gives a new grading on L which has integer value on that subset. Since the space of compatible almost complex structures which agree with J_{con} near infinity is contractible and X is connected, the integer on each connected component of L is independent of D, and this constant has the same amount of information as the original grading of L. Because of this, we will by some abuse of language say that every holomorphic Lagrangian has integer grading.

Proposition 2.5.2. *Let L_0, L_1 be two holomorphic Lagrangians in T^*X with integer gradings θ_0, θ_1 respectively. Assume that L_0 and L_1 intersect transversally. Then $HF^\bullet(L_0, L_1)$ is concentrated in degree $\theta_1 - \theta_0 + n$.*

Proof. Let $p \in L_0 \cap L_1$. By the proof of Proposition 2.5.1 and transversality, under one coordinate system T_pL_0 has the form $\left(\begin{array}{cc|cc} I_k & \mathbf{0} & A_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K_0 \end{array}\right)$ and T_pL_1 has the form $\left(\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & K_1 & \mathbf{0} \\ \mathbf{0} & I_l & \mathbf{0} & A_1 \end{array}\right)$ where $k + l \geq n$, $A_i, i = 0, 1$ is of the same type as A and $K_i, i = 0, 1$ is of the same type as K in the proof of Proposition 2.5.1.

We find the degree of p using (1.8). First, let

$$M_0 = \begin{pmatrix} I + \sqrt{-1}A_0 & \mathbf{0} \\ \mathbf{0} & \sqrt{-1}K_0 \end{pmatrix}, \; M_1 = \begin{pmatrix} \sqrt{-1}K_1 & \mathbf{0} \\ \mathbf{0} & I + \sqrt{-1}A_1 \end{pmatrix},$$

and

$$U_i = M_i(\Re M_i^2 + \Im M_i^2)^{-\frac{1}{2}}, i = 0, 1, \; \widetilde{U} = U_0U_1U_0^{-1},$$
$$C = \Re\widetilde{U}\Im\widetilde{U}^{-1}, U = (C + \sqrt{-1}I)(C^2 + I)^{-\frac{1}{2}}.$$

It is easy to see that $U_i, U \in U(2n)$, under an orthonormal basis of T_pL_0, we have

$$T_pL_1 = U \cdot T_pL_0,$$

and the eigenvalues together with eigenvectors of U will give the canonical short path from T_pL_0 to T_pL_1.

Let $S' = \{B \in M_{2n\times 2n}(\mathbb{C}) : BJ_n = -J_n\overline{B}\}$. Then for a matrix in S', its eigenvalues are of the form $\lambda_i, -\overline{\lambda}_i, i = 1, ..., n$. It's straightforward to check that $U \in S'$ (since $U_i \in S'$), so the eigenvalues of U are $e^{2\pi\sqrt{-1}\alpha_i}, e^{2\pi\sqrt{-1}(-\frac{1}{2}-\alpha_i)}, i = 1, ..., n$, for some $\alpha_i \in (-\frac{1}{2}, 0)$. Therefore

$$\deg(p) = \theta_1 - \theta_0 - 2\sum_{i=1}^{n}(\alpha_i - \frac{1}{2} - \alpha_i)) = \theta_1 - \theta_0 + n.$$

□

Let $Lag(T^*X)$ be the set of all Lagrangian submanifolds in T^*X. Let $Lag_{\mathcal{S}}(T^*X) = \{L \in Lag(T^*X) : L^\infty \subset \Lambda^\infty_{\mathcal{S}}\}$.

Transversality of $L_{x,F}$ with $t \cdot L$ for $L \in Lag_{\mathcal{S}}(T^*X)$ and $t > 0$ sufficiently small

For any $L \in Lag(T^*X)$, consider $\mathcal{L}_{t>0} = \{((x, \xi), t) : (x, \xi) \in t \cdot L, t > 0\} \subset T^*X \times \mathbb{R}$ and denote each fiber over t as $\mathcal{L}_t$. Define $\text{Conic}(L) = \overline{\mathcal{L}_{t>0}} - \mathcal{L}_{t>0} \subset T^*X \times \{0\}$, and we also view it inside T^*X. Similarly, if X is a proper algebraic variety and L is a holomorphic Lagrangian (hence algebraic) in $T^*X_{\mathbb{C}}$, consider $\mathcal{L}_{w\in\mathbb{C}^*} = \{((x, \xi), w) : (x, \xi) \in w \cdot L, w \in \mathbb{C}^*\} \subset T^*X_{\mathbb{C}} \times \mathbb{C}^*$. Define $\text{Conic}(L^{alg})$ to be the fiber at 0 of the algebraic closure of $\mathcal{L}_{w\in\mathbb{C}^*}$ in $T^*X_{\mathbb{C}} \times \mathbb{P}^1$.

Let

$$Cone(L^\infty) = \text{Cl}\{(x, \xi) \in T^*X : \lim_{s\in\mathbb{R}_+, s\to\infty}(x, s\xi) \in L^\infty \text{ in } \overline{T^*X}\} \subset T^*X.$$

For a holomorphic Lagrangian L, $L^\infty_{\mathbb{C}}$ will denote $\overline{L} \cap T^\infty X_{\mathbb{C}} \subset \overline{T^*X}_{\mathbb{C}}$, and let

$$Cone_{\mathbb{C}^*}(L^\infty_{\mathbb{C}}) = \text{Cl}\{(x, \xi) \in T^*X_{\mathbb{C}} : \lim_{\lambda\in\mathbb{C}^*, \lambda\to\infty}(x, \lambda\xi) \in L^\infty_{\mathbb{C}} \text{ in } \overline{T^*X}_{\mathbb{C}}\} \subset T^*X_{\mathbb{C}},$$

where Cl means taking closure. In particular, $Cone(L^\infty) \subset Cone_{\mathbb{C}^*}(L^\infty_{\mathbb{C}})$, so $L \in Lag_{\mathcal{S}}(T^*X)$ for a complex stratification $\mathcal{S}$.

In the following, $\mathcal{S}$ is a refinement of a complex stratification, with each stratum a cell, and $L \in Lag_{\mathcal{S}}(T^*X)$.

Lemma 2.5.3. *Conic(L) is a closed (possibly singular) conical Lagrangian in T^*X.*

Proof. $\mathrm{Conic}(L) = Cone(L^\infty) \cup \overline{\pi(L)}$. In fact, $\mathrm{Conic}(L) \cap T^*_X X = \overline{\pi(L)}$ and $(x,\xi) \in \mathrm{Conic}(L) - T^*_X X \Leftrightarrow \exists (x_n,\xi_n) \in L$ and $t_n \to 0^+$ such that $\lim\limits_{n\to\infty}(x_n, t_n\xi_n) = (x,\xi) \Leftrightarrow \lim\limits_{t\to\infty}(x,t\xi) = \lim\limits_{n\to\infty}(x_n,\xi_n) \in L^\infty$. □

Since $\mathcal{L}_{t>0} = \{((x,\xi),t) : (x,\xi) \in t\cdot L, t>0\}$ is a $\mathcal{C}$-set in $T^*X \times \mathbb{R}$, $\mathrm{Conic}(L)$ is a $\mathcal{C}$-set. Note that $\mathrm{Conic}(L) \subset \Lambda_{\mathcal{S}}$, so we can take a stratification of $\Lambda_{\mathcal{S}}$ which is compatible with $\mathrm{Conic}(L)$. Then choose a stratification $\mathcal{T}$ of $\mathcal{L}_{t\geq 0} := \overline{\mathcal{L}_{t>0}}$ compatible with the above stratification restricted to $\mathrm{Conic}(L)$. It is clear that for any covector (x,ξ) in an open stratum in $\Lambda_{\mathcal{S}}$ away from $\mathrm{Conic}(L)$, $L_{x,F} \cap t\cdot L = \emptyset$ for $t>0$ sufficiently small, for any test triple (x,ξ,F). So we only need to look at (x,ξ) in an open stratum of $\mathrm{Conic}(L)$.

Let T_α be an open stratum of $\mathrm{Conic}(L)$. For any $((x,\xi),0) \in T_\alpha$, there is some open neighborhood of it that only intersects open strata in $\mathcal{L}_{t>0}$, and let (x,ξ,F) be a test triple for $\Lambda_{\mathcal{S}}$. Denote $\mathcal{L}_{x,F} = L_{x,F} \times \mathbb{R} \subset T^*X \times \mathbb{R}$.

Lemma 2.5.4. *In a neighborhood of $((x,\xi),0)$, $\mathcal{L}_{x,F}$ intersects $\mathcal{L}_{t>0}$ transversally.*

Proof. For a small (open) ball $B_r(x)$ with center x, $\pi^{-1}(B_r(x)) \subset T^*X$ is diffeomorphic to $D^n \times \mathbb{R}^n$, where D^n is the (open) unit disc in $\mathbb{R}^n$. So we have two $\mathcal{C}$-maps by taking tangent spaces:

$$f_1 : \mathcal{L}_{t>0} \to \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1}), f_2 : \mathcal{L}_{x,F} \to \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1})$$

and by restriction, these give the following map

$$f = (f_1, f_2) : \mathcal{L}_{t>0} \cap \mathcal{L}_{x,F} \to \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1}) \times \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1}).$$

Let $N = \{(A,B) \in \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1}) \times \mathrm{Gr}_{n+1}(\mathbb{R}^{2n+1}) : A + B \neq \mathbb{R}^{2n+1}\}$. It is clear that N is a closed $\mathcal{C}$-set.

Suppose there is a sequence of points $((x_n,\xi_n),t_n) \in T_\beta, t_n > 0$ approaching $((x,\xi),0)$, where T_β is an open stratum in $\mathcal{L}_{t>0}$, on which $\mathcal{L}_{x,F}$ and $\mathcal{L}_{t>0}$ intersect nontransversally, then $f((x_n,\xi_n),t_n) \in N$. Since N is compact, there exists a subsequence $((x_{n_k},\xi_{n_k}),t_{n_k})$ such that $f((x_{n_k},\xi_{n_k}),t_{n_k})$ converges to a point in N and $\lim\limits_{k\to\infty} T_{((x_{n_k},\xi_{n_k}),t_{n_k})}\mathcal{L}_{t>0}$ exists.

By the Whitney property, $T_{(x,\xi)}\mathrm{Conic}(L) \subset \lim\limits_{k\to\infty} T_{(x_{n_k},\xi_{n_k},t_{n_k})}\mathcal{L}_{t>0}$. This implies that L is not transverse to $L_{x,F}$ at (x,ξ), which is a contradiction. □

Lemma 2.5.5. *Let $((x,\xi),0) \in T_\alpha$ as in Lemma 2.5.4. Then $L_{x,F}$ intersects $t\cdot L$ transversally for all sufficiently small $t>0$, and the intersections are within the holomorphic portion of $L_{x,F}$.*

Proof. First, by curve selection lemma, given any neighborhood W of $((x,\xi),0)$, we have $\mathcal{L}_{x,F} \cap \mathcal{L}_{t_0>t>0} \subset W$ for t_0 sufficiently small.

We only need to prove that there is a neighborhood of $((x,\xi),0)$ contained in W, such that for any $((x_t,\xi_t),t) \in \mathcal{L}_t \cap \mathcal{L}_{x,F}$, $0<t<t_0$, we have $\pi_* T_{(x_t,\xi_t,t)}(\mathcal{L}_{t>0} \cap \mathcal{L}_{x,F}) \neq \{0\}$

where $\pi : \mathcal{L}_{t>0} \cap \mathcal{L}_{x,F} \to \mathbb{R}$ is the projection to t. In fact, since $\mathcal{L}_{t>0} \pitchfork \mathcal{L}_{x,F}$ in W, $\pi_* T_{(x_t,\xi_t,t)}(\mathcal{L}_{t>0} \cap \mathcal{L}_{x,F}) \neq \{0\}$ implies the transversality of L_t and $L_{x,F}$.

The assertion is true because the function t on $\mathcal{L}_{t>0} \cap \mathcal{L}_{x,F}$ has no critical value in $(0, \eta)$ for some $\eta > 0$ sufficiently small. □

Now we are ready to prove the main theorem.

Theorem 2.5.6. *If L is a holomorphic Lagrangian brane of constant grading $-n$ and $\mathcal{F} \in Sh(X)$ quasi-represents L, i.e. $\mu_X(\mathcal{F}) \simeq L$, then $\mathcal{F}$ is a perverse sheaf.*

Proof. From (2.14), $\mathcal{F} \in Sh_{\mathcal{S}}(X)$ for a complex analytic stratification $\mathcal{S}$. Let $\tilde{\mathcal{S}}$ be a refinement of $\mathcal{S}$ with each stratum a cell. By Proposition 2.5.2, for generic choices of test triple (x, ξ, F) of $\Lambda_{\tilde{\mathcal{S}}}$, we have the cohomology of $M_{x,F}(\mathcal{F}) \simeq Hom_{Fuk(T^*X)}(L_{x,F}, L)$ concentrated in degree 0, so $\mathcal{F}$ is perverse. □

Remark 2.5.7. One could easily deduce from the above discussions that if $\mathcal{F} \in Perv(X)$ quasi-represents a holomorphic brane L, then $Conic(L) = SS(\mathcal{F})$, and in particular we have $Cone(L^\infty) = \phi(Cone_{\mathbb{C}^*}(L^\infty_{\mathbb{C}}))$.

Recall the Morse-theoretic definition of the characteristic cycle $CC(\mathcal{F})$ for $\mathcal{F} \in Sh_{\mathcal{S}}(X)$ (see chapter IX of [16] or section 2 of [25]; in general X only needs to be a real oriented analytic manifold). Consider $\bigcup_{S_\alpha \in \mathcal{S}} T^*_{S_\alpha} X - D^*_{S_\alpha} X = \bigcup_{i \in I} \Lambda_i$, where $\Lambda_i, i \in I$ are disjoint connected components.

Definition 2.5.8. The *characteristic cycle* $CC(\mathcal{F})$ of a sheaf $\mathcal{F} \in Sh_{\mathcal{S}}(X)$ is the Lagrangian cycle with values in the orientation sheaf of X defined as follows:
(1) The orientation on $\Lambda_i, i \in I$ are induced from the canonical orientation of $T^*_{S_\alpha} X$;
(2) The multiplicity of Λ_i is equal to $\chi(M_{x,F}(\mathcal{F}))$, where $(x, dF_x) \in \Lambda_i$ and (x, dF_x, F) is a test triple for $\Lambda_{\mathcal{S}}$.

Corollary 2.5.9. *If X is a proper algebraic variety, $\mathcal{F}$ and L are as in Theorem 2.5.6, and L is equipped with a vector bundle of rank d with flat connection, then $CC(\mathcal{F}) = d \cdot Conic(L^{alg})^{sm}$.*

Proof. If X is a proper algebraic variety, then $Conic(L^{alg})$ is an algebraic cycle whose multiplicity at a smooth point (x, ξ) is equal to the intersection number of $L_{x,F}$ with $\mathcal{L}_t$ for $t > 0$ sufficiently small, and this is equal to the Euler characteristic of the local Morse group at (x, ξ) quotient by the rank of the vector bundle. □

A generalization

Holomorphic branes are very restrictive. First, they have strong conditions on $CC(\mathcal{F})$ for the sheaf $\mathcal{F}$ it represents or equivalently $Conic(L^{alg})$ if X is proper algebraic. For example, on $T^*\mathbb{P}^1$, we cannot have a connected L with $Conic(L^{alg})$ equal to the sum of the zero section

and one cotangent fiber, each of which has multiplicity 1. Second, fixing $Conic(L^{alg})$, they cannot produce all the perverse sheaves with this characteristic cycle. For example, on $T^*\mathbb{P}^1$, let $Conic(L^{alg}) = T^*_X X + T^*_{z=0} X + T^*_{z=\infty} X$, then the only candidates for connected L are meromorphic sections of the holomorphic bundle $T^*\mathbb{P}^1 \to \mathbb{P}^1$ which have simple poles at 0 and ∞. One can show that up to a positive multiple, only $\Gamma_{\frac{dz}{z}}$ and $\Gamma_{-\frac{dz}{z}}$ are exact Lagrangians. So there are only two kinds of perverse sheaves coming in this way: one is $i_*\mathcal{L}_U[1]$ on $\mathbb{P}^1 - \{\infty\}$ and $i_!\mathcal{L}_U[1]$ on $\mathbb{P}^1 - \{0\}$, where $\mathcal{L}_U$ is a rank 1 local system on $U = \mathbb{P}^1 - (\{0\} \cup \{\infty\})$, and the other is its Verdier dual.

For this reason, we consider a broader class of branes which may produce more perverse sheaves, namely, the branes which are holomorphic near infinity, and are multi-graphs near the zero section.

Proposition 2.5.10. *Let L be a connected Lagrangian brane in T^*X. Assume that $L^\infty \neq \emptyset$, there is $r > 0$ such that $L \cap T^*X|_{|\xi|>r}$ is complex analytic on which it has grading $-n$, and $L \cap T^*X|_{|\xi|\leq r+\epsilon}$ is a multi-graph if $\overline{\pi(L)} = T^*_X X$, i.e. $\pi|_{L\cap T^*X|_{|\xi|\leq r}}$ is a submersion. Then L quasi-represents a perverse sheaf $\mathcal{F}$.*

Proof. We only need to check the local Morse group on the zero section.

If $\overline{\pi(L)} \neq T^*_X X$, then $M_{x,F}(\mathcal{F}) \simeq 0$ for $(x, dF_x = 0)$ a generic point on the zero section.

If $\overline{\pi(L)} = T^*_X X$, take a generic point $(x, 0)$ on the zero section and construct a local Morse brane $L_{x,F}$. Over a small ball $B_r(x)$ of x in X, $\pi^{-1}(B_r(x)) \cap L$ is a finite covering plus some holomorphic portion of L. Consider $HF(L_{x,F}, t \cdot L)$ for $t > 0$ sufficiently small. Since each sheet in the covering connects to a holomorphic part of grading $-n$ by a path along which there is no critical change of the grading, $HF^\bullet(L_{x,F}, t \cdot L)$ is concentrated in degree 0. □

Although one could not represent every perverse sheaf by a holomorphic Lagrangian brane, it is speculative that locally every indecomposable perverse sheaf can be represented by a holomorphic brane.

Chapter 3

Symplectomorphism group of $T^*(G_{\mathbb{C}}/B)$ and the braid group

3.1 Introduction

For a semisimple Lie group $G_{\mathbb{C}}$ over $\mathbb{C}$, the cotangent bundle of the flag variety $T^*\mathcal{B}$ and its relation to the braid group have led to numerous active research directions in geometric representation theory, algebraic geometry and symplectic topology. The main driving force for these is due to the fruitful structures underlying the adjoint quotient maps and the Springer resolutions.

This chapter is an attempt to study the homotopy type of the symplectomorphism group of $T^*\mathcal{B}$ and its relation to the braid group, from a purely geometric point of view. We especially focus on the case of $G_{\mathbb{C}} = SL_3(\mathbb{C})$.

Motivation and set-up

The motivation is from the (strong) categorical braid group action on $D(\mathcal{B})$, the derived category of constructible sheaves on $\mathcal{B}$, by Deligne [5] and Rouquier[24]. This action gives rise to $G_{\mathbb{C}}$-equivariant automorphisms of $D(\mathcal{B})$. One can translate the result to symplectic geometry via the Nadler-Zaslow correspondence[22]. Recall that the Nadler-Zaslow correspondence gives a categorical equivalence between $D(X)$ and $DFuk(T^*X)$, the derived Fukaya category of T^*X, for any compact analytic manifold X (see Section 3.2 for more details). Since symplectomorphisms of $T^*\mathcal{B}$ with reasonable behavior near infinity induce automorphisms of $DFuk(T^*\mathcal{B})$, it is natural to form the following conjecture.

Conjecture 3.1.1. *The "$G_{\mathbb{C}}$-equivariant" symplectomorphism group of $T^*\mathcal{B}$ is homotopy equivalent to the braid group.*

To rigorously state the conjecture, one has to give a definition of "$G_{\mathbb{C}}$-equivariancy" on symplectomorphisms. The global $G_{\mathbb{C}}$-equivariancy condition on a symplectomorphism would

force it to be the identity. The reason is the following. The Springer resolution (see (3.2) for the definition)

$$\mu_{\mathbb{C}} : T^*\mathcal{B} \to \mathcal{N},$$

gives a $G_{\mathbb{C}}$-equivariant isomorphism from the dense $G_{\mathbb{C}}$-orbit in $T^*\mathcal{B}$ to $\mathcal{N}_{reg}$, the orbit of regular nilpotent elements in $\mathcal{N}$. Suppose φ is a $G_{\mathbb{C}}$-equivariant symplectomorphism, then the graph of $\varphi|_{\mu_{\mathbb{C}}^{-1}(\mathcal{N}_{reg})}$ is a complex Lagrangian submanifold in $T^*\mathcal{B}^- \times T^*\mathcal{B}$, hence the graph of φ is a closed complex Lagrangian. Therefore, φ preserves the holomorphic symplectic form and then preserves $\mu_{\mathbb{C}}$ (see Lemma 3.2.4), so we can conclude that $\varphi = \mathrm{id}$.

A natural replacement of the global $G_{\mathbb{C}}$-equivariancy condition is to require φ to be *$G_{\mathbb{C}}$-equivariant at infinity*. It can be formulated via the Lagrangian correspondence L_φ, i.e. the graph of φ, in $T^*\mathcal{B}^- \times T^*\mathcal{B} \simeq T^*(\mathcal{B} \times \mathcal{B})$ and its relation to the Steinberg variety $\mathcal{Z}$. Recall that the Steinberg variety $\mathcal{Z}$ is the union of the conormal varieties to the $G_{\mathbb{C}}$-orbits in $\mathcal{B}\times\mathcal{B}$ under the diagonal action. As we have seen in Chapter 1 and 2, using the $\mathbb{R}_+$-action on $T^*(\mathcal{B}\times\mathcal{B})$, one can projectivize the cotangent bundle and get a compact symplectic manifold with a contact boundary. We denote the boundary by $T^\infty(\mathcal{B} \times \mathcal{B})$, and for any Lagrangian L in the cotangent bundle, we use L^∞ to denote for $\overline{L} \cap T^\infty(\mathcal{B} \times \mathcal{B})$. Then we make the following definition (see Section 3.2 for more discussions).

Definition 3.1.2. A symplectomorphism φ of $T^*\mathcal{B}$ is *$G_{\mathbb{C}}$-equivariant at infinity* if $L_\varphi^\infty \subset \mathcal{Z}^\infty$. We denote by $\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B})$ for the group of symplectomorphisms that are $G_{\mathbb{C}}$-equivariant at infinity.

We are content with this definition since the Steinberg variety is one of the key players in geometric representation theory, and this definition builds a natural bridge between geometric representation theory and symplectic geometry.

For example, if $G_{\mathbb{C}} = SL_2(\mathbb{C})$, then the symplectomorphisms that we are considering are the compactly supported ones. For general $G_{\mathbb{C}}$, φ has to preserve the Springer fibers, i.e. fibers of $\mu_{\mathbb{C}}$, at infinity. If we fix a maximal compact subgroup G in $G_{\mathbb{C}}$ (e.g. $SU(n)$ inside $SL_n(\mathbb{C})$ and identify $\mathcal{B}$ with G/T), then we can consider the subgroup $\mathrm{Sympl}_{\mathcal{Z}}^G(T^*\mathcal{B})$ of (genuine) G-equivariant symplectomorphisms. We make the following conjecture.

Conjecture 3.1.3. *There is a sequence of homotopy equivalences*

$$\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B}) \simeq \mathrm{Sympl}_{\mathcal{Z}}^G(T^*\mathcal{B}) \simeq B_{\mathbf{W}}.$$

Our main results provide evidence for this conjecture.

Main Theorem

Theorem 3.1.4. *(1) There is a natural surjective group homomorphism*

$$\beta_G : \mathrm{Sympl}_{\mathcal{Z}}^G(T^*\mathcal{B}) \to B_{\mathbf{W}}, \text{ for } G = SU(n).$$

(2) β_G is a homotopy equivalence for $G = SU(2), SU(3)$.

The construction of β_G is purely geometric as apposed to the alternative categorical construction (see Remark 3.1.5 below). As mentioned before, every $\varphi \in \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$ must preserve each reduced space of the Hamiltonian G-action. So the problem of studying the (weak) homotopy type of $\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$ can be roughly reduced to the study of homotopy classes (and homotopy between homotopies and so on) of the symplectomorphisms on the Hamiltonian reductions over a Weyl chamber W in the dual of the Cartan subalgebra $\mathfrak{t}^* \cong i\mathfrak{t}$, with some further restrictions at infinity.

For $n = 2$, the reduced space over each element $p \in W$ is a point. However, we have to divide them into two cases. If $p \neq 0$, then $\mu^{-1}(p)$ is an orbit of the T-action, so $\varphi|_{\mu^{-1}}(p)$ is a rotation and corresponds to an element in S^1. If $p = 0$, then the restriction of φ on $\mu^{-1}(0) = T^*_{\mathcal{B}}\mathcal{B}$ is a G-equivariant automorphism of G/T. Since $\mathrm{Aut}^G(G/T) \cong \mathbf{W}$, $\varphi|_{\mu^{-1}(0)}$ corresponds to an element in $\mathbf{W} \cong \mathbb{Z}_2$. Note that the circles over the interior of W approach the zero section to a big circle, we see that φ corresponds to a path in S^1 starting from ± 1 and ending at 1, and that φ is a (iterated) Dehn twist. It is then easy to see that $\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$ is homotopy equivalent to $B_2 = \mathbb{Z}$, which is weakly homotopy equivalent to $\mathrm{Sympl}^c(T^*S^2)$, by the result of Seidel[27].

For $n \geq 3$, things are more interesting and we will not have all φ being compactly supported. The picture in the case of $G = SU(3)$ is very illustrating. Let $\mu : T^*\mathcal{B} \to i\mathfrak{su}(3) \cong \mathfrak{su}(3)^*$ be the moment map. Along the ray generated by $p = \mathrm{diag}(1, 0, -1) \in i\mathfrak{t}$, the reduced spaces are all S^2 with three distinguished points corresponding to the singular loci of μ. There are exactly two types Springer fibers contained in $\mu^{-1}(p)$: one is the Springer fiber over a regular nilpotent element (a 3×3-nilpotent matrix having one single Jordan block in its Jordan normal form), which is just a point; the other is the Springer fiber over a subregular nilpotent element (a 3×3-nilpotent matrix having two Jordan blocks), which is the wedge of two 2-spheres. The union of subregular Springer fibers in $\mu^{-1}(p)$ projects to two line segments connecting the three special points in the reduced space M_p. Now we draw a small disc $\mathcal{U}_s$ around these line segments in $\mu^{-1}(s \cdot p)$ for each $s > 0$, which forms a $\mathbb{R}_+$-invariant family. Let φ_s be the induced map on $M_{s \cdot p}$ by φ. As $s \to \infty$, φ_s tends to fix all the points outside of $\mathcal{U}_s$, hence after a small homotopy near $\partial(\mathcal{U}_s)$, $\varphi_s|_{\mathcal{U}_s}$ becomes a symplectomorphism of $\mathcal{U}_s$, which permutes the three marked points and fixes each point on the boundary. Therefore, it gives rise to an element in B_3, the braid group of three strands.

For $G = SU(n)$, we focus on certain region in $\mu^{-1}(p_n)$, where $p_n = \mathrm{diag}(1, -1, 0, ..., 0) \in i\mathfrak{su}(n)$, and use similar argument. To prove surjectivity of β_G, we explicitly construct fiberwise Dehn twists associated to each simple root α (see Remark 3.1.6 below), and we show that their image under β_G generates $B_{\mathbf{W}}$.

Remark 3.1.5. One could compare the map in Theorem 3.1.4 with the composition

$$\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B}) \to \mathrm{Aut}(DFuk(T^*\mathcal{B})) \cong \mathrm{Aut}(D(\mathcal{B}))$$

through the categorical action of $\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$. Conjecture 3.1.3 implies that this construction gives β_G as well. The reason is that the Lagrangian correspondences for the fiberwise

Dehn twists in $T^*(\mathcal{B} \times \mathcal{B})$ represent exactly the integral kernels for the braid group action on $D(\mathcal{B})$.

For part (2) of Theorem 3.1.4, we have seen the proof when $G = SU(2)$. The proof for $G = SU(3)$ consists of two steps. The first step is to construct local symplectic charts for $\mu^{-1}(W)$ and "trivialize" each chart by certain reduced spaces. The main techniques are the Duistermaat-Heckman theorem on the normal form of a moment map near a regular value (see [10]), and Weinstein's Lagrangian tubular neighborhood theorem. The second step is to carefully choose isotopies for each φ over the local charts and to construct relevant fibrations with contractible fibers and/or contractible base, so then we deduce a homotopy equivalence between $\ker \beta_G$ and a certain path space of the loop space of $\mathrm{Sympl}(S^2)$. By some further restrictions and the fact that $\mathrm{Sympl}(S^2)$ is homotopy equivalent to $SO(3)$, we conclude that the space we have finally arrived is contractible. One of the difficulties along the way is to take special care for the singular loci of the moment map.

Remark 3.1.6. This is a remark on some related result by Seidel-Smith and Thomas. Seidel-Smith [28] considered symplectic fibrations that naturally arise in the adjoint quotient maps in Lie theory, and constructed link invariants by the symplectic monodromies. Thomas [31] considered the case where $T^*\mathcal{B}$ is a symplectic fiber, and deduced an inclusion $B_{\mathbf{W}} \hookrightarrow \pi_0(\mathrm{Sympl}(T^*\mathcal{B}))$. It is described in [31] that the braid group actions are exactly the "family Dehn twists" about the family of isotropic spheres over $T^*(G_{\mathbb{C}}/P)$, which are the image of the left map in the standard correspondence

$$T^*(G_{\mathbb{C}}/B) \leftarrow G_{\mathbb{C}}/B \times_{G_{\mathbb{C}}/P} T^*(G_{\mathbb{C}}/P) \to T^*(G_{\mathbb{C}}/P),$$

associated to the $\mathbb{P}^1$-fibration $G_{\mathbb{C}}/B \to G_{\mathbb{C}}/P$, for a minimal parabolic subgroup P. This is essentially the same as the fiberwise Dehn twists that we consider here, though we identify $T^*\mathcal{B}$ as a symplectic fiber bundle over $T^*(G_{\mathbb{C}}/P)$ using the Killing form on $\mathfrak{g}$ (rather than $\mathfrak{g}_{\mathbb{C}}$), and we explicitly make the fiberwise Dehn twists all G-equivariant.

3.2 Preliminaries and Set-ups

Notations: Throughout this chapter, we will use $G_{\mathbb{C}}$ to denote a semisimple Lie group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and G to denote for a maximal compact subgroup in $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. We will mostly focus on type A, e.g. $G_{\mathbb{C}} = SL_n(\mathbb{C})$ and $G = SU(n)$. Fix a Borel subgroup B in $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$ and nilradical $\mathfrak{n}$, and let $\mathcal{B}$ denote for $G_{\mathbb{C}}/B$. Then $T := B \cap G$ is a maximal torus in G, with Lie algebra $\mathfrak{t}$, and we have the canonical identification $\mathcal{B} \cong G/T$. For $G_{\mathbb{C}} = SL_n(\mathbb{C})$, we will mostly take B to be the subgroup of upper triangular matrices, then T consists of diagonal matrices in $SU(n)$.

Set-up for the symplectomorphism group

We consider $T^*\mathcal{B}$ as a *real* symplectic manifold, and would like to study the homotopy type of its symplectomorphism group. Since $T^*\mathcal{B}$ is noncompact, we must put some restrictions on

the behavior of the symplectomorphisms near the infinity of $T^*\mathcal{B}$, so that the resulting group has "nice" structures. A typical restriction is to make the symplectomorphisms compactly supported, which will turn out to be too small (see the discussion below). Instead we pose the condition that the symplectomorphisms are $G_{\mathbb{C}}$-equivariant at infinity, where the $G_{\mathbb{C}}$-action is the standard Hamiltonian action induced from the left action of $G_{\mathbb{C}}$ on $\mathcal{B}$. We will make the restriction more precise after a brief discussion of the motivation.

Motivation for the definition

Let $D(\mathcal{B})$ be the constructible derived category of sheaves on $\mathcal{B}$, and let $DFuk(T^*\mathcal{B})$ be the derived Fukaya category of $T^*\mathcal{B}$. We have seen in Chapter 1 and 2 that there is a categorical equivalence (the Nadler-Zaslow correspondence) between $D(M)$ and $DFuk(T^*M)$, for any real analytic manifold M. Motivated by the results of [5] and [24] on the braid group action on $D(\mathcal{B})$, which are $G_{\mathbb{C}}$-equivariant automorphisms of the category, and the Nadler-Zaslow correspondence between $D(\mathcal{B})$ and $DFuk(T^*\mathcal{B})$, we would like to study the group of "$G_{\mathbb{C}}$-equivariant" symplectomorphisms of $T^*\mathcal{B}$, and to see its relation to the braid group. As discussed in the Introduction, the most natural interpretation of "$G_{\mathbb{C}}$-equivariancy" is to impose that φ is $G_{\mathbb{C}}$-equivariant at infinity.

Definition of $\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$

Let φ be any symplectomorphism of $T^*\mathcal{B}$, then its graph L_φ is a Lagrangian correspondence in $(T^*\mathcal{B})^- \times T^*\mathcal{B} \cong T^*(\mathcal{B}\times\mathcal{B})$. Using the $\mathbb{R}_+$-action on the cotangent fibers of $T^*(\mathcal{B}\times\mathcal{B})$, we can projectivize the space with the boundary divisor $T^\infty(\mathcal{B}\times\mathcal{B})$ being a contact manifold, with contact form θ^∞. We require φ to be well-behaved near the infinity divisor, in the sense that $L^\infty_\varphi := \overline{L_\varphi} \cap T^\infty(\mathcal{B}\times\mathcal{B})$ is θ^∞-isotropic.

As discussed in the Introduction, global $G_{\mathbb{C}}$-equivariancy on a symplectomorphism φ forces φ to preserve each Springer fiber, which implies that φ must be the identity. However, if we only require the $G_{\mathbb{C}}$-equivariancy condition "at infinity", this would give a reasonable constraint on φ by

$$L^\infty_\varphi \subset (\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})^\infty,$$

where $\tilde{\mathcal{N}} = T^*\mathcal{B}$ and the fiber product is taken for the Springer resolution $\mu_{\mathbb{C}}$. Note that $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is just the *Steinberg variety* $\mathcal{Z}$, which is a Lagrangian subvariety by an alternative description as the union of conormal varieties to the diagonal $G_{\mathbb{C}}$-orbits ($\#\mathbf{W}$-many) in $\mathcal{B}\times\mathcal{B}$. Thus, we make the following definition.

Definition 3.2.1. A symplectomorphism φ of $T^*\mathcal{B}$ is *$G_{\mathbb{C}}$-equivariant at infinity* if $L^\infty_\varphi \subset \mathcal{Z}^\infty$. And we denote by $\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B})$ for the group of symplectomorphisms with $G_{\mathbb{C}}$-equivariancy at infinity.

We define the C^1-topology on $\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B})$ as follows.

$$\lim_{n\to\infty} f_n = f \in \mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B}) \Leftrightarrow \tag{3.1}$$

(a) $\lim_{n\to\infty} f_n|_K = f|_K$ in $C^1(K, T^*\mathcal{B})$ for all compact subdomain K;

(b) for any sequence of points $y_n \in L_{f_n}$, if $\lim_{n\to\infty} y_n$ exists in $T^\infty(\mathcal{B}\times\mathcal{B})$, then it lies in $\mathcal{Z}^\infty$.

It is easy to see that $\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B})$ endowed with this topology is a topological group. The main concern about the topology of symplectomorphisms on a non-compact symplectic manifold M is that the induced automorphisms on the Fukaya category $Fuk(M)$ of a continuous family of symplectomorphisms should remain the same, i.e. there should be a well defined map $\pi_0(\mathrm{Sympl}(M)) \to \mathrm{Aut}(Fuk(M))$. In our setting, we view each $\varphi \in \mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B})$ as a Lagrangian correspondence L_φ with $L_\varphi^\infty \subset \mathcal{Z}^\infty$, and it corresponds to a sheaf (or an integral kernel) $\mathcal{F}_\varphi$ in $Sh_{\mathcal{Z}}(\mathcal{B}\times\mathcal{B})$, the full subcategory of sheaves with singular support contained in $\mathcal{Z}$ (cf. [22], [21]). Now if we have a continuous family $\{\varphi_s\}_{0\le s\le 1}$ in the C^1-topology defined by (3.1), then the family of sheaves $\mathcal{F}_{\varphi_s}$ remain the same. This can be argued using the test branes representing the micolocal stalk functors in $Fuk(T^*\mathcal{B})$ and the fact that the isotopy of the branes L_{φ_s} is non-characteristic with respect to any fixed finite set of test branes; for more details see Section 2.4 and [21].

We also consider the subgroup of G-equivariant symplectomorphisms, denoted as $\mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$. As stated in the Introduction, we conjecture that

$$\mathrm{Sympl}_{\mathcal{Z}}(T^*\mathcal{B}) \simeq \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B}) \simeq B_{\mathbf{W}}.$$

Moment maps

Moment maps for the $G_{\mathbb{C}}$-action and G-action on $T^*\mathcal{B}$

For any element $x \in G_{\mathbb{C}}$, let L_x (resp. R_x) denote the action of left (resp. right) multiplication by x on $G_{\mathbb{C}}$. We will use the left action to identify $G_{\mathbb{C}} \times \mathfrak{g}^*_{\mathbb{C}}$ with $T^*G_{\mathbb{C}}$:

$$\begin{array}{ccc} G_{\mathbb{C}} \times \mathfrak{g}^*_{\mathbb{C}} & \to & T^*G_{\mathbb{C}} \\ (x,\xi) & \mapsto & (x, L^*_{x^{-1}}\xi) \end{array}.$$

Using the Killing form to identify $\mathfrak{g}^*_{\mathbb{C}}$ with $\mathfrak{g}_{\mathbb{C}}$, the moment maps for the left and right $G_{\mathbb{C}}$-action (with respect to the holomorphic symplectic form) under the above identification are given by

$$\mu_L: \begin{array}{ccc} G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} & \to & \mathfrak{g}_{\mathbb{C}} \\ (x,\xi) & \mapsto & \mathrm{Ad}_x\xi \end{array}, \text{ and } \mu_R: \begin{array}{ccc} G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} & \to & \mathfrak{g}_{\mathbb{C}} \\ (x,\xi) & \mapsto & \xi \end{array}, \text{ respectively.}$$

For the right Hamiltonian B-action on $T^*G_{\mathbb{C}}$ induced from the right $G_{\mathbb{C}}$-action, the moment map is given by

$$\mu_{R,B}: \begin{array}{ccc} G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} & \to & \mathfrak{b}^* \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{n} \\ (x,\xi) & \mapsto & \overline{\xi} \end{array},$$

where $\overline{\xi}$ means the image of ξ under the quotient map $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{n}$. Then we have $T^*\mathcal{B} = \mu_{R,B}^{-1}(0)/B = G_{\mathbb{C}} \times_B \mathfrak{n}$, where B acts on the right on $G_{\mathbb{C}}$ and acts adjointly on $\mathfrak{n}$ in the last twisted product. In the following, we will also use $(x, \xi), \xi \in \mathfrak{n}$, to denote a point in $T^*\mathcal{B}$, though it should be understood as a representative in the equivalence class under the relation $(x, \xi) \sim (xb, \mathrm{Ad}_{b^{-1}}\xi)$. Now the moment map for the left $G_{\mathbb{C}}$-action on $T^*\mathcal{B}$ is given by

$$\begin{array}{rccc} \mu_{\mathbb{C}}: & T^*\mathcal{B} & \to & \mathfrak{g}_{\mathbb{C}} \\ & (x,\xi) & \mapsto & \mathrm{Ad}_x\xi \end{array}. \tag{3.2}$$

Since the image of $\mu_{\mathbb{C}}$ is the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, we will sometimes write the codomain of $\mu_{\mathbb{C}}$ as $\mathcal{N}$, and then it becomes the *Springer resolution*. The fiber of the Springer resolution over $u \in \mathcal{N}$ is called a *Springer fiber*, and is denoted by $\mathcal{B}_u$. Here we recall some basic facts about the Springer resolutions.

The nilpotent cone $\mathcal{N}$ is stratified by $G_{\mathbb{C}}$-orbits, and they form a partially ordered set. The greatest one in the poset is the open dense orbit consisting of regular nilpotent elements, and is denoted by $\mathcal{N}_{reg}$. $\mathcal{N}_{reg}$ covers a unique orbit called the *subregular* orbit and is denoted by $\mathcal{N}_{sub}$. The least element in the poset is the zero orbit and it is covered by a unique orbit called the *minimal orbit*, denoted by $\mathcal{N}_{min}$. For $G_{\mathbb{C}} = SL_n(\mathbb{C})$, $\mathcal{N}$ is the set of all nilpotent matrices, the orbits are determined by the Jordan normal form, and are classified by partitions of n. We will use the notation $(n_1^{k_1}, n_2^{k_2}, ..., n_\ell^{k_\ell})$ to denote the partition of n by k_i copies of n_i, for $i = 1, ..., \ell$ and $n_1 > n_2 > \cdots > n_\ell \geq 1$.

The Springer fibers $\mathcal{B}_u$ have irreducible components indexed by Young tableaux, and over the above mentioned orbits, the geometry is well-known: if $u \in \mathcal{N}_{reg}$, then $\mathcal{B}_u$ is a point; for $u \in \mathcal{N}_{sub}$, $\mathcal{B}_u$ is the Dynkin curve determined by the root system; for $u = 0$, $\mathcal{B}_u = \mathcal{B}$; for $u \in \mathcal{N}_{min}$, if $G_{\mathbb{C}} = SL_n(\mathbb{C})$, then each component is a fiber bundle over the Grassmannian of k-planes in $\ker u$ with fiber a product of flag varieties determined by the k-plane, $0 \leq k < n-1$. Except for some specific types, the geometry and topology of Springer fibers (mostly about their singularities) are largely unknown. The celebrated Springer correspondence gives a correspondence between the irreducible representations of the Weyl group and the Weyl group action on the top homology of the Springer fibers.

Similar formulas for the above moment maps apply to the left and right G-action on T^*G and $T^*\mathcal{B} \cong T^*(G/T)$, with respect to the real symplectic forms. In particular, we have the identification $T^*\mathcal{B} \cong G \times_T \mathfrak{t}^\perp$, and we will use $(x, \xi), \xi \in \mathfrak{t}^\perp$ to denote a point in $T^*\mathcal{B}$, and the moment map for the left G-action, after identifying $\mathfrak{g}^*$ with $i\mathfrak{g}$, is given by

$$\begin{array}{rccc} \mu: & T^*\mathcal{B} & \to & i\mathfrak{g} \\ & (x,\xi) & \mapsto & \mathrm{Ad}_x\xi \end{array}. \tag{3.3}$$

Lemma 3.2.2. *For $G = SU(n)$, the singular values of μ in a (open) Weyl chamber of $i\mathfrak{t}$ are exactly those p such that p has a proper subset of eigenvalues that sum up to zero.*

Proof. Let $\overset{\circ}{W}$ be a (open) Weyl chamber in $i\mathfrak{t}$. Then $\mu^{-1}(\overset{\circ}{W})$ is a symplectic manifold with a Hamiltonian T-action, and the restriction of μ is just the moment map for the T-action.

The proof is contained in the proof of Lemma 3.4.5 and we omit the details here. Since there is a nontrivial center in G, to make the T-action quasi-free (i.e. the stabilizer of any point is a connected subgroup of T), we quotient out the center in G and consider the action by the adjoint group. Then $(x,\xi) \in \mu^{-1}(\overset{\circ}{W})$ is a singular point of μ if and only if (x,ξ) has a nontrivial stabilizer by the T-action. This is exactly when ξ has a nontrivial stabilizer in T.

If p has a proper subset of eigenvalues that sum up to zero, then $p = \begin{bmatrix} p_1 & \\ & p_2 \end{bmatrix}$ up to conjugation, for some $p_1 \in i\mathfrak{su}(k)$ and $p_2 \in i\mathfrak{su}(n-k)$, then we can find $\xi \in \mu^{-1}(p)$ of the form $\begin{bmatrix} \xi_1 & \\ & \xi_2 \end{bmatrix}$ (up to conjugation) with $\xi_i \in \mu^{-1}(p_i)$, $i = 1, 2$. Then ξ has nontrivial stabilizers containing $\mathrm{diag}(e^{i\theta}, ..., e^{i\theta}, e^{i\rho}, ..., e^{i\rho})$ with $k\theta + (n-k)\rho \in 2\mathbb{Z}\pi$. Conversely, assume ξ is fixed by an element of the form $\mathrm{diag}(e^{i\theta}, ..., e^{i\theta}, e^{i\theta_1}, ..., e^{i\theta_{n-k}})$ (up to conjugation), where the first $0 < k < n$ entries are all $e^{i\theta}$, and $\theta_j - \theta \notin 2\mathbb{Z}\pi$ for $j = 1, ..., n-k$. Then we have $\xi_{j\ell} = 0$ for $j \in \{1, ..., k\}, \ell \in \{k+1, ..., n\}$, thus ξ is of the form $\begin{bmatrix} \xi_1 & \\ & \xi_2 \end{bmatrix}$ (up to conjugation), where $\xi_i \in \mu^{-1}(p_i), i = 1, 2$ for some $p_1 \in i\mathfrak{su}(k)$ and $p_2 \in i\mathfrak{su}(n-k)$. □

For any $p \in i\mathfrak{g}$, we will use G_p to denote the stabilizer of p in the coadjoint action by G, and M_p to denote for the (possibly singular) reduced space $\mu^{-1}(p)/G_p$.

Proposition 3.2.3. *For any $p \in i\mathfrak{t}$, M_p is naturally identified with the reduced space at zero of the T-action on the coadjoint orbit $\mathcal{O}(p)$.*

Proof. Note that $\mathcal{O}(p)$ is the reduced space at p of the left G-action on T^*G. Since the left and right G-actions on T^*G commute, taking 2-step Hamiltonian reductions in both orders are the same. □

Lemma 3.2.4. *Any G-equivariant symplectomorphism φ of $T^*\mathcal{B}$ must preserve μ, i.e. $\mu \circ \varphi = \mu$.*

Proof. Since φ is G-equivariant, $\mu \circ \varphi$ is also a moment map for the G-action on $T^*\mathcal{B}$. Note the dual of the moment map $\mathfrak{g} \to C^\infty(T^*\mathcal{B})$ is unique up to a functional $\sigma \in \mathfrak{g}^*$ such that σ vanishes on $[\mathfrak{g}, \mathfrak{g}]$ (see 5.2 in [20]). By semisimplicity of $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so $\sigma = 0$. Therefore, $\mu \circ \varphi = \mu$.

□

3.3 Construction of the surjective homomorphism $\beta_G : \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B}) \to B_{\mathbf{W}}, G = SU(n)$

Since the moment map $\mu : T^*\mathcal{B} \to i\mathfrak{g}$ factors through $\mu_{\mathbb{C}} : T^*\mathcal{B} \to \mathcal{N}$, every Spinger fiber is contained in $\mu^{-1}(p)$ for some p. For $G = SU(n)$, μ is the composition of $\mu_{\mathbb{C}}$ with the map $\mathcal{N} \to i\mathfrak{su}(n), u \mapsto u + u^*$. Also the two descriptions of $T^*\mathcal{B}$ by $G_{\mathbb{C}} \times_B \mathfrak{n}$ and $G \times_T i\mathfrak{t}$ are

identified by $(x,u)\mapsto(x,u+u^*)$, where we only choose $x\in G$. In the following, we will call a Springer fiber *nontrivial* if it is not a point, and we will denote its type by the type of the nilpotent orbit it corresponds to.

Lemma 3.3.1. *If $p\in i\mathfrak{su}(n)$ has $n-1$ positive eigenvalues or $n-1$ negative eigenvalues, then $\mu^{-1}(p)$ does not contain any nontrivial Springer fibers.*

Proof. For an element $u=[a_{ij}]\in\mathfrak{n}$, u is nonregular exactly when $a_{i,i+1}=0$ for some i. Then by conjugation of some permutation matrix, $u+u^*$ has the 2×2 submatrix on the upperleft corner to be zero. If $u+u^*$ has $n-1$ positive eigenvalues or $n-1$ negative eigenvalues, then the top 2×2 submatrix must have one positive eigenvalue and one negative eigenvalue. So the lemma follows. □

A study of certain loci in $\mu^{-1}(\mathrm{diag}(1,-1,0,...,0))$

Let $p_n=\mathrm{diag}(1,-1,0,...,0)\in i\mathfrak{su}(n)$. Given $(x,\xi)\in\mu^{-1}(p_n)$, the submatrix ξ_{n-1} of ξ, consisting of the first $n-1$ rows and $n-1$ columns, lies in $\mathcal{O}(\epsilon p_{n-1})$ for some $\epsilon\geq 0$, by the Gelfand-Cetlin pattern or basic facts about Hermitian matrices. Therefore, ξ can be conjugated to the matrix z_n in (3.4) below, by a matrix $y_{n-1}\in SU(n-1)$ under the obvious embedding $SU(n-1)\hookrightarrow SU(n)$ (taking y_{n-1} to $\begin{bmatrix} y_{n-1} & \\ & 1\end{bmatrix}$). Now we calculate the characteristic polynomial of z_n and see the possible values for $a_1,...,a_{n-1}$ in (3.4).

$$\det(z_n-\lambda I)=\det\begin{bmatrix}\epsilon-\lambda & & & & & a_1\\ & -\epsilon-\lambda & & & & a_2\\ & & -\lambda & & & a_3\\ & & & \ddots & & \vdots\\ & & & & -\lambda & a_{n-1}\\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_{n-1} & -\lambda\end{bmatrix} \tag{3.4}$$

$$=\det\begin{bmatrix}\epsilon-\lambda & & & & & a_1 & \\ & -\epsilon-\lambda & & & & a_2 & \\ & & -\lambda & & & a_3 & \\ & & & \ddots & & \vdots & \\ & & & & -\lambda & a_{n-1} & \\ 0 & 0 & 0 & \cdots & 0 & -\lambda+\frac{|a_1|^2}{\lambda-\epsilon}+\frac{|a_2|^2}{\lambda+\epsilon}+\frac{1}{\lambda}\sum_{i=3}^{n-1}|a_i|^2\end{bmatrix}$$

$$=(-1)^n(\lambda-\epsilon)(\lambda+\epsilon)\lambda^{n-3}(\lambda-\frac{|a_1|^2}{\lambda-\epsilon}-\frac{|a_2|^2}{\lambda+\epsilon}-\frac{1}{\lambda}\sum_{i=3}^{n-1}|a_i|^2).$$

There are three cases.
(1) If $\epsilon \neq 0, 1$, then we must have

$$1 - \frac{|a_1|^2}{1-\epsilon} - \frac{|a_2|^2}{1+\epsilon} - \sum_{i=3}^{n-1} |a_i|^2 = 0, \ -1 - \frac{|a_1|^2}{-1-\epsilon} - \frac{|a_2|^2}{-1+\epsilon} + \sum_{i=3}^{n-1} |a_i|^2 = 0$$
$$\frac{1}{\epsilon}|a_1|^2 - \frac{1}{\epsilon}|a_2|^2 = 0, \ \sum_{i=3}^{n-1} |a_i|^2 = 0.$$

These are equivalent to

$$|a_1|^2 = |a_2|^2 = \frac{1}{2}(1-\epsilon^2), a_3 = \cdots = a_{n-1} = 0.$$

Since we only care about ξ up to the adjoint T-action, we can quotient out the adjoint actions by $\{\mathrm{diag}(e^{i\alpha}, e^{i\alpha}, \cdots, e^{i\alpha}, e^{-i(n-1)\alpha}), \alpha \in [0, 2\pi)\}$ on z_n, which commute with the image of $SU(n-1)$ in $SU(n)$, and assume that $a_1 = \sqrt{\frac{1}{2}(1-\epsilon^2)}$ and $a_2 = \sqrt{\frac{1}{2}(1-\epsilon^2)}e^{i\theta}$.
(2) If $\epsilon = 0$, then we have

$$\sum_{i=1}^{n-1} |a_i|^2 = 1.$$

(3) If $\epsilon = 1$, then

$$a_1 = \cdots = a_{n-1} = 0.$$

In summary, if $\epsilon \neq 0, 1$, then

$$z_n = \begin{bmatrix} \epsilon & & & & & \sqrt{\frac{1}{2}(1-\epsilon^2)} \\ & -\epsilon & & & & \sqrt{\frac{1}{2}(1-\epsilon^2)}e^{i\theta} \\ & & 0 & & & 0 \\ & & & \ddots & & \vdots \\ & & & & 0 & 0 \\ \sqrt{\frac{1}{2}(1-\epsilon^2)} & \sqrt{\frac{1}{2}(1-\epsilon^2)}e^{-i\theta} & 0 & \cdots & 0 & 0 \end{bmatrix}; \tag{3.5}$$

if $\epsilon = 0$, then $\xi_{n-1} = \mathbf{0}$ and the last column of ξ has length square equal to 1; if $\epsilon = 1$, then $\xi_{n-1} = p_{n-1}$ and the last column and row of ξ are zero. In addition, if $\epsilon \neq 0, 1$, by a direct calculation, we see that if $\xi = y_{n-1} z_n y_{n-1}^*$ and $y_{n-1} = [b_{jk}]_{1 \leq j,k \leq n-1}$, then $|b_{j1}| = |b_{j2}|$ for all j.

Lemma 3.3.2. *The projection of the union of subregular Springer fibers in the reduced space of $p_n = \mathrm{diag}(1, -1, 0, ..., 0)$ is of real codimension 1.*

Proof. We prove this by induction. First, for $G = SU(3)$, the reduced space is a $\mathbb{P}^1$ with three distinguished points Q_1, Q_2, Q_3 whose preimage in $\mu^{-1}(p_3)$ are of the form $(x_1, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$, $(x_2, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix})$ and $(x_3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix})$ respectively. This can be easily seen from Lemma 3.2.2. The subregular Springer fibers in $\mu^{-1}(p_3)$ consists of points of the form $(x, \begin{bmatrix} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{bmatrix})$ and $(y, \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & a \\ \bar{b} & \bar{a} & 0 \end{bmatrix})$, where $|a|^2 + |b|^2 = 1$. So in the reduced space, they project to two line segments connecting Q_1 with Q_2 and Q_2 with Q_3.

For $G = SU(n), n \geq 4$, given $(x, \xi) \in \mu^{-1}(p_n)$, as we have seen before $\xi = y_{n-1} z_n y_{n-1}^*$ for some $y_{n-1} \in SU(n-1)$.

Using induction we see that the projection of the regular Springer fibers is an open dense subset in the reduced space M_{p_n}. In fact, starting from any point (x', ξ') in $\mu^{-1}(p_{n-1}), \xi' \neq \mathbf{0}$, the set of G_{p_n}-orbits of elements $(x, \xi) \in \mu^{-1}(p_n)$ with $\xi_{n-1} = \epsilon \xi', \epsilon \in (0, 1]$ is a disc (with $\epsilon = 1$ being the center). If (x', ξ') is regular, then the set of G_{p_n}-orbits of regular Springer fibers above it form an annulus with one radial line being removed. This is because by regularity of ξ', the first two entries in the last row of y_{n-1} has the same nonzero modulus. So (x, ξ) is subregular if and only if $\epsilon = 1$ or $0 < \epsilon < 1$ and $\bar{b}_{n-1,1} + e^{i\theta} \bar{b}_{n-1,2} = 0$; (x, ξ) is of type $(2, 1^{n-2})$ (i.e. the partition type of its corresponding nilpotent element) if $\epsilon = 0$, and such locus has dimension $n - 2$ in M_{p_n}; otherwise, (x, ξ) is regular.

The above description also shows that the set of subregular elements in M_{p_n} is of codimension 1. This is because for any ξ corresponding to a subregular element (this can be seen using Young tableaux), after deleting the i-th row and i-th column for some i, we get a regular element $\xi_{i,i}$ of one size smaller. If we apply the above argument to $\xi_{i,i}$ for $i = 1, ..., n$, we still get a disc above it, in which the subregular locus forms two radial lines for $i \neq 1, n$ and one radial line otherwise.

Note that for $\epsilon = 1$, the corresponding elements are fixed points of an S^1-action, while for $0 < \epsilon < 1$, the isotropy groups of the elements are all trivial. □

Following the proof of Lemma 3.3.2, we restrict ourselves to a small ball B_ϵ in the regular locus of $M_{p_{n-1}}$, and look at its n different (open) disc bundles in M_{p_n}. We denote the total space of the bundle by $\mathcal{R}_i$ if it corresponds to deleting the i-th row and column of ξ. Let S be the union of projections of subregular Springer fibers which pass through the zero section in $\mathcal{R}_1$, which in particular also pass through the zero section of $\mathcal{R}_i$ for all i. One can cover a tubular neighborhood $\mathcal{U}$ of S by $2n - 1$ charts, n of which come from $\mathcal{R}_i$ for all i, and the other $n - 1$ charts can be constructed from the normal bundle to S with the S^1-fixed loci deleted.

Corollary 3.3.3. *Every subregular Springer fiber in $\mu^{-1}(p_n)$ has n fixed points by some S^1-action.*

Proof. First, for any point in a subregular Springer fiber in $\mu^{-1}(p_n)$, we can choose S and $\mathcal{U}$ as above such that $\mathcal{U}$ contains the image of the point. Let S' be the intersection of the image of all the subregular Springer fibers with $\mathcal{U}$. By the "disc bundle" description above, the image of the union of subregular Springer fibers that contain n fixed points by an S^1-action intersects S' in a relatively open subset. On the other hand, the singular points of μ in $\mu^{-1}(p_n)$ form a closed subset, and it intersects the union of subregular Springer fibers in the locus of fixed points by some S^1-action, which is G_{p_n}-invariant. Therefore, the projection of these fibers to M_{p_n} intersect S' in a relatively closed subset. Since S' is obviously connected, the Corollary follows.

□

Construction of $\beta_G : \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B}) \to B_{\mathbf{W}}$

Now we restrict ourselves to the image of any particular Springer fiber in S, denoted by σ. Note that $S \simeq \sigma \times B_\epsilon$, we can cut $\mathcal{U}$ by a 2-dimensional surface Σ that contains σ and is transverse to S (one can stratify S according the isotropy groups). Then

$$\mathcal{U} \simeq \Sigma \times B_\epsilon. \tag{3.6}$$

Let $\mathcal{U}_s$ be a family of open sets in $M_{s\cdot p_n}, s > 0$, which is invariant under the $\mathbb{R}_+$-action, and which has $\mathcal{U}_1 = \mathcal{U}$. Given any $\varphi \in \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B})$, we look at $\varphi_s|_{\mathcal{U}_s}$ as $s \to \infty$, where φ_s is the induced symplectomorphism on $M_{s\cdot p_n}$. Since φ has to preserve each Springer fiber at infinity and has to preserve the isotropy group of each point, we see that for s very large, after a small homotopy, the push-forward of $\varphi_s|_{\mathcal{U}_s}$ under (3.6) to $\Sigma \times B_\epsilon$ can be identified as the product map of $\varphi_s|_\Sigma$ and id_{B_ϵ}. In addition, $\varphi_s|_\Sigma$ is a symplectomorphism [1] of a disc that fixes the boundary pointwise and permutes a set of n marked points. Therefore, the homotopy class of φ_s for s sufficiently large corresponds to an element in B_n, the braid group of n-strands, and this gives the desired homomorphism for $G = SU(n)$:

$$\begin{array}{ccc} \beta_G : \mathrm{Sympl}^G_{\mathcal{Z}}(T^*\mathcal{B}) & \longrightarrow & B_{\mathbf{W}} \\ \varphi & \mapsto & [\varphi_s|_\Sigma] \end{array} \tag{3.7}$$

Remark 3.3.4. (a) It is interesting to ask what kinds of Springer fibers occur in $\mu^{-1}(p)$ for any $p \in i\mathfrak{t}$, and what are their moduli. This would lead to some interesting relation between the geometry/topology of the Springer fibers with the wall-crossing phenomena (or blow-up/down patterns) for the change of reduced spaces of μ, equivalently for the moment map of the coadjoint orbits in $i\mathfrak{su}(n)$.

[1] Since there are singularities in Σ, one might want to consider the homotopy class of $\varphi_s|_\Sigma$ in the group of (orientation preserving) homeomorphisms of the disc, which causes no difference.

By more detailed study of the Gelfand-Cetlin pattern for $\xi \in \mu^{-1}(p_n)$, we can show that the types of Springer fibers that occur in $\mu^{-1}(p_n)$ are exactly those of nondegenerate hook-type, i.e. have partition type $(k, 1^{n-k}), k > 1$. Since this is a little bit far from the main theme of this chapter, we leave the statement and details in a later work focusing on the above mentioned question.

(b) It is reasonable to extend (3.7) to Lie groups of other types. We leave this question for a future work.

Fiberwise Dehn twists and the surjectivity of β_G

Fix a Borel subgroup B in $G_{\mathbb{C}}$. Let α be a simple root, and $P_\alpha^{\mathbb{C}}$ be the corresponding minimal parabolic subgroup. Let $P_\alpha = P_\alpha^{\mathbb{C}} \cap G$. Since $T^*(G/T) \cong G \times_T \mathfrak{t}^\perp$ and $T^*(G/P_\alpha) \cong G \times_{P_\alpha} {\mathfrak{p}_\alpha}^\perp$ by the Killing form, we have a natural smooth fibration

$$\begin{array}{ccc} T^*\mathbb{P}^1 & \longrightarrow & T^*(G/T) \\ & & \downarrow \\ & & T^*(G/P_\alpha), \end{array} \tag{3.8}$$

where the vertical arrow is given by the orthogonal projection from $\mathfrak{t}^\perp$ to ${\mathfrak{p}_\alpha}^\perp$.

Lemma 3.3.5. *The fibration (3.8) is a symplectic fibration.*

Proof. By standard result in symplectic fibrations (cf. [9]), if the total space is a symplectic manifold and each fiber is a symplectic submanifold, then there is a symplectic connection on the fibration. If there is no issue about integrating along the horizontal lifting of a vector field on the base, then the fibration is a symplectic fibation.

It is easy to check that each fiber of (3.8) is a symplectic submanifold and is isomorphic to $T^*\mathbb{P}^1$. Also, there is a contraction by the $\mathbb{R}_+$-action on the total space to a compact region, so there is no issue of integrating along horizontal vector fields. □

As before, we will use $(x, \xi), \xi \in \mathfrak{t}^\perp$(resp. $\mathfrak{p}_\alpha^\perp$) to denote a point (up to equivalence relation) in $G \times_T \mathfrak{t}^\perp \cong T^*(G/T)$ (resp. $G \times_{P_\alpha} {\mathfrak{p}_\alpha}^\perp \cong T^*(G/P_\alpha)$). For each simple root α and $\xi \in \mathfrak{t}^\perp$, let ξ_α denote the kernel of ξ under the projection $\mathfrak{t}^\perp \to \mathfrak{p}_\alpha^\perp$. Now we can define a *fiberwise Dehn twist* (the justification of the notion is included in the proof of Lemma 3.3.6).

$$\tau_\alpha(x, \xi) = \begin{cases} (x \exp(h(|\xi_\alpha|)\frac{\xi_\alpha}{|\xi_\alpha|}), \mathrm{Ad}_{\exp(-h(|\xi_\alpha|)\frac{\xi_\alpha}{|\xi_\alpha|})}\xi), \text{ if } \xi_\alpha \neq 0 \\ (x \exp(\frac{\pi}{2}E_\alpha), \mathrm{Ad}_{\exp(-\frac{\pi}{2}E_\alpha)}\xi), \text{ otherwise} \end{cases}, \tag{3.9}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a smooth increasing function satisfying $h(t) + h(-t) = \pi$ and $h(t) = \pi$ for $t >> 0$, and E_α is any unit vector in $\mathfrak{p}_\alpha$. Note that one needs to choose the right Killing form, so that for any unit vector $v \in \mathfrak{p}_\alpha$, $\exp(tv) \in T$ if and only if $t \in \mathbb{Z} \cdot \pi$, otherwise, one has to change the range of h accordingly. For example, if $G = SU(n)$, then E_α is of the form

$e^{i\theta}\epsilon_{ij} - e^{-i\theta}\epsilon_{ji}$ for some i, j with $i - j = 1$, where ϵ_{ij} is the elementary matrix with all entries being zero except that the (i, j)-entry is 1. It is easy to check that τ_α is well-defined, i.e. it doesn't depend on the representative for a point in $G \times_T \mathfrak{t}^\perp$, and it preserves the fibration. The proof of the following Lemma also implies that parallel transport with respect to the canonical symplectic connection preserves τ_α, and in particular, τ_α is smooth.

Lemma 3.3.6. *τ_α is a G-equivariant symplectomorphism of $T^*\mathcal{B}$.*

Proof. The G-action is simply given by $g \cdot (x, \xi) = (gx, \xi)$ for $g \in G$, so it is clear that τ_α is G-equivariant. Away from the locus where $\xi_\alpha = 0$, we can add a parameter t in all the parentheses of $\exp(\cdot)$ in (3.9) to get a one parameter family of diffeomorphism. Then it becomes the integral of some vector field X. We claim that X is the Hamiltonian vector field of the Hamiltonian function $H = \tilde{h}(|\xi_\alpha|)$, where $\tilde{h}$ is an antiderivative of h, so τ_α is the time-1 map of the Hamiltonian flow. To see this, we only need to check for every vertical vector v in $T_{(x,\xi)}T^*\mathcal{B}$, because X is G-equivariant and it preserves μ, and this follows from the computation

$$dH(v) = h(|\xi_\alpha|)\frac{\langle v, \xi_\alpha\rangle}{|\xi_\alpha|},\ \omega(X, v) = \langle h(|\xi_\alpha|)\frac{\xi_\alpha}{|\xi_\alpha|}, v\rangle.$$

□

Lemma 3.3.7. *τ_α is $G_{\mathbb{C}}$-equivariant at infinity.*

Proof. Let (x_n, ξ_n) be a sequence of points approaching $(x_\infty, \xi_\infty) \in T^\infty\mathcal{B}$, i.e. with appropriate choices of representatives, we have $\lim_{n\to\infty} x_n = x_\infty$, $\lim_{n\to\infty} |\xi_n| = \infty$ and $\lim_{n\to\infty} \frac{\xi_n}{|\xi_n|} = \xi_\infty$. Here we have identified $T^\infty\mathcal{B}$ with the unit co-sphere bundle.

There are two cases. The first case is $\lim_{n\to\infty} \frac{|\xi_{n,\alpha}|}{|\xi_n|} \neq 0$, then $\lim_{n\to\infty} \tau_\alpha(x_n, \xi_n) = (x_\infty, \xi_\infty)$. The other case is $\lim_{n\to\infty} \frac{|\xi_{n,\alpha}|}{|\xi_n|} = 0$. Let Φ be the set of roots, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the root space decomposition, and Δ(resp. Δ^-) be the set of positive(resp. negative) roots. By standard result, one can choose a basis for $\mathfrak{g}_{\mathbb{C}}$ as $\{e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{h}_{\mathbb{C}}\}_{\alpha\in\Delta}$, where $(e_\alpha, f_\alpha, h_\alpha)$ forms a $\mathfrak{sl}_2$-triple for all $\alpha \in \Delta$. Then $\mathfrak{t}^\perp$ is generated (over $\mathbb{R}$) by $\{e_\alpha + f_\alpha, i(e_\alpha - f_\alpha)\}_{\alpha\in\Delta}$.

For any $\xi \in i\mathfrak{g}$, let ξ^+ be the portion of ξ in $\mathfrak{n}$ under the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{n} + \mathfrak{n}^-$. Now we need to show that

$$|\mu_{\mathbb{C}}(\tau_\alpha(x, \xi_n)) - \mu_{\mathbb{C}}(x, \xi_n)|/|\xi_n| \to 0 \text{ as } n \to \infty, \tag{3.10}$$

for any norm on $\mathfrak{g}_{\mathbb{C}}$. Given any $\alpha \in S$ (the set of simple roots) and $\beta(\neq \alpha) \in \Delta$, we have

$$(\mathrm{Ad}_{\exp(ae_\alpha - \bar{a}f_\alpha)}(be_\beta + \bar{b}f_\beta))^+ = \mathrm{Ad}_{\exp(ae_\alpha - \bar{a}f_\alpha)}(be_\beta).$$

This holds by the standard formula

$$\mathrm{Ad}_{\exp(X)}Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots,$$

and the fact that $\beta - n\alpha \notin \{0\} \cup \Delta^-$ for any $n \in \mathbb{Z}_{\geq 0}$. Write

$$\xi_n = \xi_{n,\alpha} + \sum_{\beta(\neq\alpha)\in\Delta} (b_\beta e_\beta + \bar{b}_\beta f_\beta)),$$

then

$$\mu_{\mathbb{C}}(\tau_\alpha(x,\xi_n)) - \mu_{\mathbb{C}}(x,\xi_n) = \begin{cases} \mathrm{Ad}_x(\mathrm{Ad}_{h(|\xi_{n,\alpha}|)\frac{\xi_{n,\alpha}}{|\xi_{n,\alpha}|}}\xi^+_{n,\alpha} - \xi^+_{n,\alpha}), \text{ if } \xi_{n,\alpha} \neq 0, \\ 0, \text{ otherwise,} \end{cases}$$

hence (3.10) holds.

□

Corollary 3.3.8. *β_G is surjective for $G = SU(n)$.*

Proof. First we prove for the case of $G = SU(3)$. As we have seen before, the projection of the union of subregular Springer fibers and their image under the Weyl group action in $M_{N\cdot p_3}$ is a triangle with vertices $Q_1 = (x_1, N\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix})$, $Q_2 = (x_2, N\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix})$, and $Q_3 = (x_3, N\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix})$. The edges are $\overline{Q_1Q_2} = \{(x, N\begin{bmatrix}0&a&b\\\bar{a}&0&0\\\bar{b}&0&0\end{bmatrix}) : |a|^2 + |b|^2 = 1\}$, $\overline{Q_2Q_3} = \{(x, N\begin{bmatrix}0&0&a\\0&0&b\\\bar{a}&\bar{b}&0\end{bmatrix}) : |a|^2+|b|^2 = 1\}$, and $\overline{Q_3Q_1} = \{(x, N\begin{bmatrix}0&a&0\\\bar{a}&0&b\\0&\bar{b}&0\end{bmatrix}) : |a|^2+|b|^2 = 1\}$, the first two of which are the projections of the subregular Springer fibers.

Now let us see how τ_{α_1} acts on the edges $\overline{Q_1Q_2}$ and $\overline{Q_2Q_3}$, where α_1 is the simple root whose simple reflection corresponds to the Weyl group element that permutes the second and the third rows and columns. It is easy to see then $\tau_{\alpha_1}(\overrightarrow{Q_2Q_3}) = \overrightarrow{Q_3Q_2}$. For $\overrightarrow{Q_1Q_2}$, we have

$$\tau_{\alpha_1} : N\begin{bmatrix}0&ia&b\\-ia&0&0\\b&0&0\end{bmatrix} \mapsto N\begin{bmatrix}0&ia&b\cos(h(Na))\\-ia&0&-b\sin(h(Na))\\b\cos(h(Na))&-b\sin(h(Na))&0\end{bmatrix},$$

where we only record the ξ-component of the points, and a, b are all nonnegative real numbers in the representatives. Note that the image never intersects the interior of $\overline{Q_2Q_3}$ or $\overline{Q_3Q_1}$, and it intersects $\overline{Q_1Q_2}$ on the interval where $h(Na) = \pi$ and this is exactly when b is sufficiently small. Using the same method, one can test the intersection of $\tau_{\alpha_1}(\overrightarrow{Q_1Q_2})$ with $\overline{Q_iQ_j}, i \neq j$. From these, we can conclude that the picture for large N is the following (cf. Figure 3.1):

One gets a similar picture for τ_{α_2} for the other simple root α_2, thus we complete the proof for $G = SU(3)$.

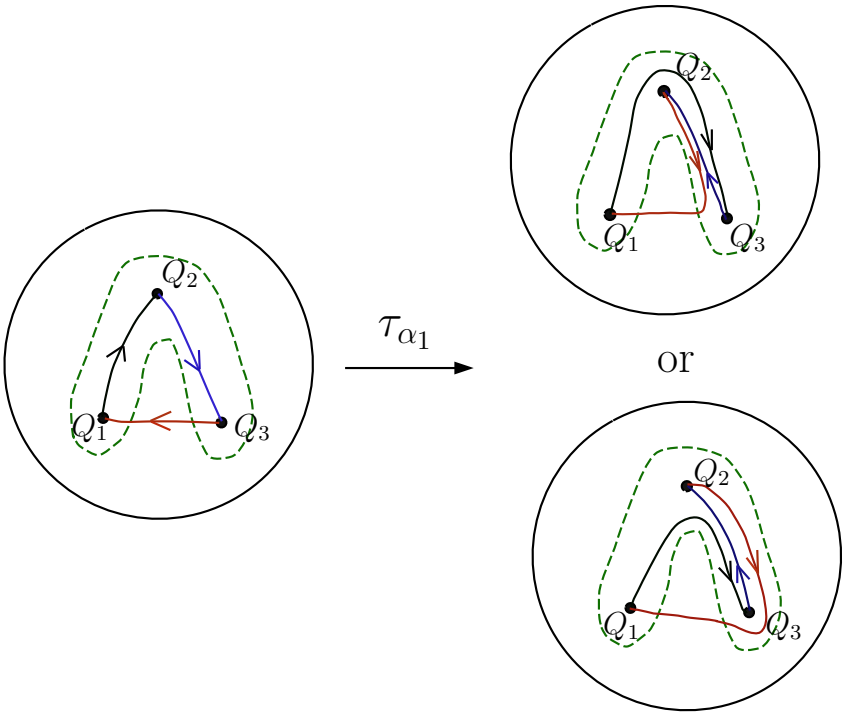


Figure 3.1: The transformation of the triangle $\overline{Q_1Q_2Q_3}$ under τ_{α_1}.

For $G = SU(n)$, one starts with a point (x, ξ) in the subregular locus of M_{p_n} that is fixed by some S^1-action. By the proof in Lemma 3.3.2, ξ has the k-th row and column both be zero for some k, and $\xi_{[k]^c}$–the $(n-1)\times(n-1)$-matrix by deleting the k-th row and column of ξ–corresponds to a regular point in $M_{p_{n-1}}$. Here and after, we use the notation $[i, j]$ for $i < j$ or $[i]$ for $i = j$ to denote the consecutive sequence between i and j, and $\xi_{[i,j]}$ to denote for the submatrix of ξ consisting of the rows and columns indexed by $[i, j]$. We also use $[i, j]^c$ (resp. $[i]^c$) to denote the sequence of the complement to $[i, j]$ (resp. $[i]$) in $[1, n]$ under the numerical ordering. The orbit of (x, ξ) under the right Weyl group action gives the n vertices in the projection of the Springer fiber σ that it belongs to, and we index them by $Q_k = (x_k, \xi_k)$, $k = 1, ..., n$ such that $(\xi_k)_{[k]^c} \in M_{p_{n-1}}$. For any $2 \le j \le n-1$, taking an appropriate section of the projection

$$\begin{array}{rccc} f_i: & M_{p_n} - \{|\xi_{[i-1,i+1]}| = 0\} & \longrightarrow & M_{p_3} \\ & (x, \xi) & \mapsto & (\tilde{x}, \frac{\xi_{[i-1,i+1]}}{|\xi_{[i-1,i+1]}|}) \end{array}$$

near $\overline{Q_{i-1}Q_i} \cup \overline{Q_iQ_{i+1}}$ gives a disc Σ_i transverse to S, and the obvious gluing of $\Sigma_i, i = 2, ..., n-1$ gives a normal surface Σ as in (3.6). The following picture (cf. Figure 3.2) illustrates how Σ_i and Σ_{i+1} are glued, and how the vertices are identified.

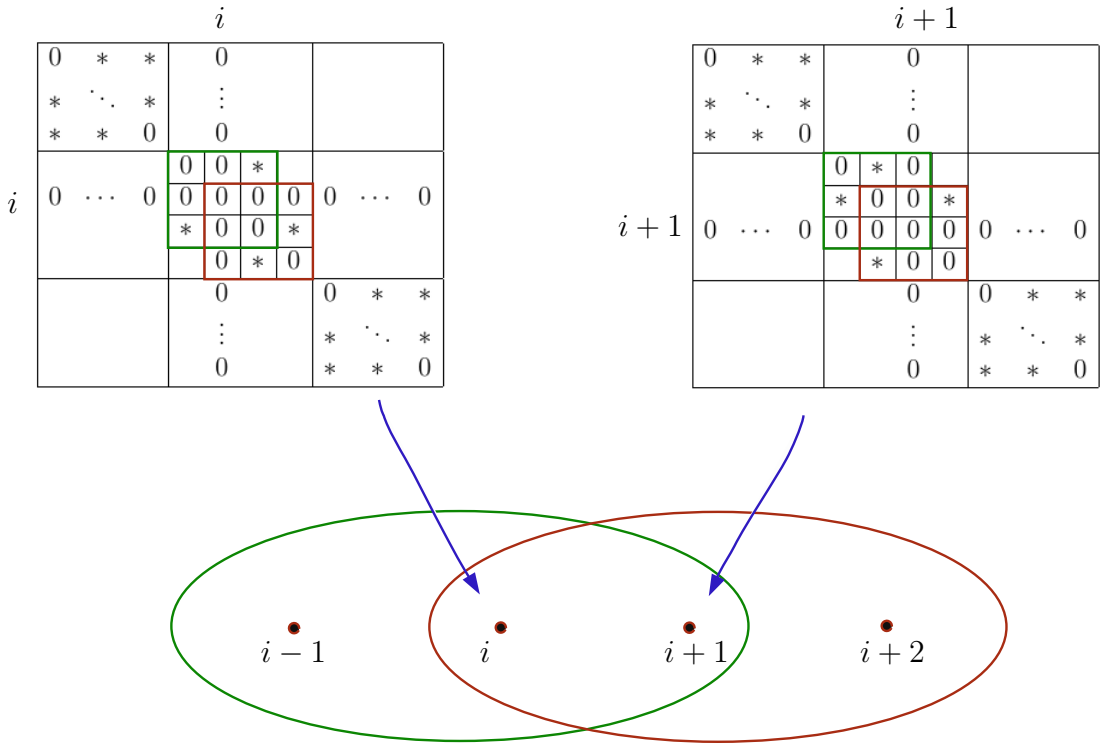


Figure 3.2: Gluing of Σ_i with Σ_{i+1}.

In this way, we reduce the situation to $G = SU(3)$. It is straightforward to check that τ_{α_i}, for the simple root α_i whose simple reflection corresponds to $(i, i+1) \in S_n \cong \mathbf{W}$, reverses $Q_i\vec{Q}_{i+1}$, keeps the homotopy class of $Q_j\vec{Q}_{j+1}$ for $j < i-1$ and $j > i+1$, and there are two possibilities of the image of Q_jQ_{j+1} for $j = i-1, i+1$ as illustrated in Figure 3.1. Therefore, $\{\tau_{\alpha_i}\}_{i=1}^{n-1}$ generates $B_{\mathbf{W}}$.

□

3.4 β_G is a homotopy equivalence for $G = SU(3)$

In this section, we will prove that $\ker\beta_G$ is contractible for $G = SU(3)$. We first review the Duistermaat-Heckman theorem and prove some basic statement for equivariant symplectomorphisms in Section 3.4. Then we divide a Weyl chamber W into three parts: one around the walls, one near the singular values of μ, and the other for the regular subcones. We construct symplectic local charts for their preimages under μ and trivialize the reduced spaces via the Duistermaat-Heckman theorem. We also use the technique of real blowing up to study the "symplectomorphisms" of the reduced space over a singular value. These are done in Section 3.4. Lastly, in Section 3.4, we give the proof that $\ker\beta_G$ is contractible. We deduce a homotopy equivalence between $\ker\beta_G$ and a certain path space of the based loop space of

$\mathrm{Sympl}(S^2)$ by a combination of making isotopies of symplectomorphisms supported on some local charts and constructing certain fibrations with contractible fibers and/or bases. By the fact that $\mathrm{Sympl}(S^2)$ is homotopy equivalent to $SO(3)$, it is easy to show that the path space that we have finally arrived is contractible.

Duistermaat-Heckman theorem and equivariant symplectomorphisms

Let's briefly recall the Duistermaat-Heckman theorem (c.f. [10]) on the local model of the moment map near a regular value for a quasi-free Hamiltonian T-action. Here *quasi-free* means that the stabilizer of any point is a connected subgroup of T.

First, the local model is the following. Let $\pi : P \to M$ be a principal T-bundle over a symplectic manifold (M, ω_0), with a connection form α. Equip $P \times \mathfrak{t}^*$ with the closed 2-form $\omega = \pi^*\omega_0 + d(\tau \cdot \alpha)$, where $\pi^*\omega_0$ also denotes the pull-back form under the projection $P \times \mathfrak{t}^* \to P$, and τ denotes a point in $\mathfrak{t}^*$. Since ω is nondegenerate on $\tau = 0$, there is a neighborhood $U \subset \mathfrak{t}^*$ around 0 such that ω is a symplectic form on $P \times U$. Then the moment map for the T-action on $P \times U$ is given by the projection to the second factor.

Now suppose 0 is a regular value of a moment map $\mu : X \to \mathfrak{t}^*$ for a quasi-free Hamiltonian T-action on a symplectic manifold (X, ω_X). We assume that μ is proper. Then $P = \mu^{-1}(0)$ is a principal T-bundle over the reduced space M_0. Any connection form α on P defines a trivial T-invariant normal bundle F, by $\omega : TX \xrightarrow{\sim} T^*X$. Then there is a T-equivariant diffeomorphism (a fiber bundle map over U) ψ between $\mu^{-1}(U)$ and $P \times U$, for a small neighborhood $U \subset \mathfrak{t}^*$ of 0, such that $\psi|_{P\times\{0\}} = \mathrm{id}$ and $dp_1 \circ d\psi(v) = 0$ for any normal vector in F, where $p_1 : P \times U \to P$ is the projection to the first factor. Now take the above constructed ω on $P \times U$ from α. We have $\psi^*\omega$ and ω_X agree on $P \times \{0\}$. Therefore, by the equivariant version of Moser's argument, the two manifolds are T-equivariantly symplectomorphic in a neighborhood of $P \times \{0\}$, and the symplectomorphism can be chosen to be the identity on $P \times \{0\}$.

Lemma 3.4.1. *Let $\pi_P : P \to B$ be any principal T-bundle with two connection forms A, A' with curvatures $F_A = F_{A'}$, and assume that $H^1(B, \mathbb{R}) = 0$. Then any diffeomorphism f of B which preserves the curvature form F_A can be lifted to a T-equivariant diffeomorphism $\tilde{f}$ of P such that $\tilde{f}^*A = A'$. Such a lifting is unique up to the global T-action on P.*

Proof. Since every principal T-bundle is a fiber-product of principal S^1-bundles, we can assume $T = S^1$. First we show that f can always be lifted to a T-equivariant diffeomorphism $\tilde{f}$ on P. Let $\mathcal{L}$ be the associated line bundle to P. Then it is not hard to see that $\tilde{f}$ corresponds to a nonvanishing section in $\mathcal{L} \otimes f^*\mathcal{L}^{-1}$. Since f preserves $c_1(\mathcal{L})$, $\mathcal{L} \otimes f^*\mathcal{L}^{-1}$ is isomorphic to the trivial bundle.

Take $\tilde{f}$ to be any lifting of f, then $F_A = F_{\tilde{f}^*A}$ and $\tilde{f}^*A - A'$ is the pull-back under π of a closed 1-form in $\Omega^1(B, i\mathfrak{t})$. The gauge transformation by $u \in C^\infty(B, T)$ gives $u^*A' = \pi^*u^{-1}du + u^{-1}A'u = \pi^*u^{-1}du + A'$. Since $H^1(B, \mathbb{R}) = 0$, we can find a global u such that

$\pi^* u^{-1} du = \tilde{f}^* A - A'$, thus $u^* A' = \tilde{f}^* A$ which is equivalent to $A' = (\tilde{f} u^{-1})^* A$. Now replace $\tilde{f}$ by $\tilde{f} \circ u^{-1}$. The last statement on uniqueness is obvious. □

Next, we continue on the local model $P \times U$, and further assume that $H^1(M, \mathbb{R}) = 0$.

Corollary 3.4.2. *Given a smooth family of symplectomorphisms $\{\varphi_\tau\}_{\tau \in U}$ of M which preserve F_α, there exists a T-equivariant symplectomorphism φ of $P \times U$ such that its induced map on the reduced space at τ is φ_τ. Two such φ differ by an element in $C^\infty(U, T)$, or equivalently, the space of such φ is a torsor of $C^\infty(U, T)$.*

Proof. Fix a linear coordinate $\{t_i\}_{i=1}^m$ on $\mathfrak{t}^*$. For any $\tau = (\tau_0, ..., \tau_m) \in U_0$, the derivatives $\frac{d}{dt_i}|_{t_i=0} \varphi_\tau^{-1} \varphi_{\tau + (0,...,t_i,...,0)}$ give m Hamiltonian vector fields $X_i(\tau)$ on M (since we have assumed $H^1(M, \mathbb{R}) = 0$). Then $\pi^*(\iota_{\varphi_{\tau *} X_i(\tau)} \omega), i = 1, ..., m$, give an ($T$-equivariant) exact 1-form $d\beta_\tau \in \Omega^1(P \times \{\tau\}, i\mathfrak{t})$. By direct calculations, a lifting $\{\tilde{\varphi}_\tau\}$ gives a T-equivariant symplectomorphism on $P \times U$ if and only if

$$\tilde{\varphi}_\tau^* \alpha = \alpha - \varphi_\tau^* d\beta_\tau,$$

for all $\tau \in U$. By Lemma 3.4.1, a lifting exists and it is unique up to a composition with an element in $C^\infty(U, T)$. □

Trivialization of the reduced spaces over a Weyl chamber

Now we focus on $G = SU(3)$. Let $w_0 = \mathrm{diag}(1, 1, -2)$, $w_1 = \mathrm{diag}(1, 0, -1)$, and $w_2 = \mathrm{diag}(2, -1, -1)$. Let W be the Weyl chamber in $\mathfrak{t}^* \cong i\mathfrak{t}$ bounded by the rays $\mathbb{R}_{\geq 0} \cdot w_0$ and $\mathbb{R}_{\geq 0} \cdot w_2$. Also, let W_{ij} denote the subcone of W bounded by $\mathbb{R}_{\geq 0} \cdot w_i$ and $\mathbb{R}_{\geq 0} \cdot w_j$ for $(i, j) = (0, 1)$ and $(1, 2)$. For any $p \in i\mathfrak{t}$, we will denote the reduced space by M_p, and we will use φ_p to denote the induced map on M_p by any $\varphi \in \mathrm{Sympl}_{\mathcal{Z}}^G(T^*\mathcal{B})$.

The action by T is not quasi-free, since the center in $SU(3)$ fixes every point. This can be resolved by replacing $G = SU(3)$ by $G_{\mathrm{Ad}} = PSU(3) = SU(3)/\mu_3$, where μ_3 is the center. This is the same as replacing the integral lattice (the root lattice) by the weight lattice in $i\mathfrak{t}$.

Trivialization around the ray $\mathbb{R}_{\geq 0} \cdot w_0$

Fix a $p \in \mathbb{R}_{>0} \cdot w_0$. Then the Lie algebra of G_p is $\mathfrak{g}_p = \{x \in \mathfrak{g} : [x, p] = 0\} \cong \mathfrak{u}(2)$ (we fix such an identification once for all).

Lemma 3.4.3. *For $\epsilon > 0$ small, $\mu^{-1}(p + B_\epsilon(0, i\mathfrak{g}_p))$ is $U(2)$-equivariantly symplectomorphic to a neighborhood of the zero section of $T^*(U(2)/\mu_3)$, where $\mu_3 = \{ \begin{bmatrix} e^{i\frac{2k\pi}{3}} & \\ & e^{i\frac{2k\pi}{3}} \end{bmatrix} : 0 \leq k \leq 2\}$.*

Proof. First, we show that $\mu^{-1}(p + B_\epsilon(0, i\mathfrak{g}_p))$ is a symplectic submanifold with symplectic complement at each point (x, ξ) consisting of the Hamiltonian vector fields $L_\eta, \eta \in i\mathfrak{g}_p^\perp$.

The map $T^*\mathcal{B} \to i\mathfrak{g} \to i\mathfrak{g}/i\mathfrak{g}_p \cong \mathfrak{g}_p^\perp$ is a submersion restricted to $\mu^{-1}(p + B_\epsilon(0, i\mathfrak{g}_p))$, for $\epsilon > 0$ small enough. This is because $d\mu_{(x,\xi)}(L_\eta) = [\eta, \mu(x, \xi)]$ for any $\eta \in \mathfrak{g}$, and $[i\mathfrak{g}_p^\perp, p] = \mathfrak{g}_p^\perp$. Therefore, $\mu^{-1}(p + B_\epsilon(0, i\mathfrak{g}_p))$ is a smooth submanifold and $\{L_\eta(x, \xi) : \eta \in i\mathfrak{g}_p^\perp\}$ is a complement to its tangent space at any point (x, ξ). The tangent space of $\mu^{-1}(p+B_\epsilon(0, i\mathfrak{g}_p))$ at any point (x, ξ) is spanned by $L_\eta, \eta \in \mathfrak{g}_p$ and the vertical vectors $L^*_{x^{-1}}\zeta, \zeta \in \mathrm{Ad}_{x^{-1}} i\mathfrak{g}_p$. Then it is easy to check that ω is nondegenerate on $\mu^{-1}(p+B_\epsilon(0, i\mathfrak{g}_p))$. Also, $i_{L_\eta}\omega = dH_\eta, \eta \in i\mathfrak{g}_p^\perp$ vanishes on $\mu^{-1}(p + B_\epsilon(0, i\mathfrak{g}_p))$, so the space of Hamiltonian vector fields $L_\eta, \eta \in i\mathfrak{g}_p^\perp$ at each point is its symplectic complement.

Next, since $\mu^{-1}(p) \cong U(2)/\mu_3$ is ω-isotropic, by the equivariant version of Weinstein's Lagrangian tubular neighborhood theorem, we get the desired result. □

Let $w_\epsilon = w_0 + \epsilon \cdot \mathrm{diag}(1, -1, 0)$, and $W_{\pm\epsilon} \subset \mathfrak{t}$ be the cone bounded by $\mathbb{R}_{\geq 0} \cdot w_\epsilon$ and $\mathbb{R}_{\geq 0} \cdot w_{-\epsilon}$ for $\epsilon > 0$ small. Identifying $\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon})$ with a cone in $i\mathfrak{u}(2)$ via the map $\mathfrak{g}_p \cong \mathfrak{u}(2)$, we have

Corollary 3.4.4. *$\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon})) \cong U(2)/\mu_3 \times \mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon})$ as Hamiltonian $U(2)$-spaces, where the latter space is equipped with the symplectic form induced from $T^*(U(2)/\mu_3) \cong U(2)/\mu_3 \times i\mathfrak{u}(2)$.*

Proof. The symplectic form on $U(2)/\mu_3 \times i\mathfrak{u}(2)$ is invariant under the translation map $(\cdot, \cdot + v)$ for any $v \in \mathbb{R} \cdot \mathrm{diag}(1, 1)$, and it is getting scaled under the $\mathbb{R}_+$-action on $i\mathfrak{su}(2)$, where we use the standard identification $\mathfrak{u}(2) \cong \mathfrak{su}(2) \times \mathbb{R}$, where $\{\mathbf{0}\} \times \mathbb{R}$ on the right-hand-side corresponds to the centralizer of $\mathfrak{u}(2)$. Note that such change of the symplectic form is compatible with the $\mathbb{R}_+$-action on $\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon}))$, so combining with Lemma 3.4.3, we complete the proof. □

Trivialization along $\overset{\circ}{W}$ and $\overset{\circ}{W}_{ij}$, $(i, j) = (0, 1)$ and $(1, 2)$

Similarly to the proof of Lemma 3.4.3, we have

Lemma 3.4.5. *$\mu^{-1}(\overset{\circ}{W})$ is a symplectic submanifold with symplectic complement consisting of the tangent vectors to the $\exp(B_\epsilon(0, -i\mathfrak{t}^\perp))$-orbits. In particular, the same holds for $\mu^{-1}(\overset{\circ}{W}_{ij})$, for $(i, j) = (0, 1)$ and $(1, 2)$.*

For any $p \in \overset{\circ}{W}_{01}$, $\mu^{-1}(p) \cong U(2)/\mu_3$ as a principal T-bundle over $\mathbb{P}^1 (\cong T\backslash U(2)$, the quotient of $U(2)$ by the left action of T). Let $A \in \Omega^1(U(2)/\mu_3, i\mathfrak{t})$ be the unique right $U(2)$-invariant connection form on $U(2)/\mu_3$ determined by the Killing form, i.e. one takes the Maurer-Cartan form and projects it to $i\mathfrak{t}$. Applying Duistermaat-Heckman theorem (see [10]), we get the following.

Proposition 3.4.6. *$\mu^{-1}(\overset{\circ}{W}_{01})$ is T-equivariantly symplectomorphic to $U(2)/\mu_3 \times \overset{\circ}{W}_{01}$ with symplectic form $c \cdot d(A \cdot \tau)$, where $\tau \in \mathfrak{t}^*$ and c is some positive constant. The symplectomorphism can be chosen to respect the $\mathbb{R}_+$-action.*

Proof. The only thing to be careful is that we have a global identification over $\overset{\circ}{W}_{01}$ rather than a local identification near some point. First, on each reduced space, the cohomology class of $cdA \cdot \tau$ agrees with that of the induced symplectic form, for some fixed $c > 0$. This is because the latter depends linearly on τ and the class vanishes on $\mathbb{R}_{\geq 0} \cdot w_0$.

Fix a T-equivariant isomorphism $\phi : \mu^{-1}(\overset{\circ}{W}_{01}) \to U(2)/\mu_3 \times \overset{\circ}{W}_{01}$. The fact that the reduced spaces are all $\mathbb{P}^1$ ensures that we can apply the equivariant version of Moser's argument on the family of symplectic forms $(1-t)\omega|_{\mu^{-1}(\overset{\circ}{W}_{01})} + t\phi^* d(A \cdot \tau), t \in [0,1]$, and get the statement. □

Real blow-ups and some treatment near the singular loci

3.4.1. *Real blowing up operations and local charts near the singular loci of μ.* The material below on real blow-ups is following [10], section 10. Suppose we have a Hamiltonian S^1-action on $\mathbb{C} \times \mathbb{C}^n$ (equipped with the product of the standard Kahler forms), given by

$$e^{i\theta} \cdot (z_0, z) = (e^{-i\theta} z_0, e^{i\theta} z). \tag{3.11}$$

Then the moment map is $\Phi(z_0, z) = -|z_0|^2 + |z|^2$. The real blowing up is a local surgery to $\mathbb{C} \times \mathbb{C}^n$, so that $\Phi^{-1}(-\infty, 0)$ is unchanged and the new moment map is regular over $(-\infty, \delta)$ for some $\delta > 0$. The construction is as follows.

Let (t, s) be the standard coordinate on $T^*S^1 \cong S^1 \times \mathbb{R}$. Choose $\epsilon, \delta > 0$ very small, remove the set $\{|z_0|^2 < \frac{\epsilon}{2}, -|z_0|^2 + |z|^2 < \delta\}$ in $\mathbb{C} \times \mathbb{C}^n$ and glue with the set $\{s < \epsilon, -s + |z|^2 < \delta\} \subset T^*S^1 \times \mathbb{C}^n$ using the identification $\{\frac{\epsilon}{2} \leq |z_0|^2 < \epsilon, -|z_0|^2 + |z|^2 < \delta\} \cong \{\frac{\epsilon}{2} \leq s < \epsilon, -s + |z|^2 < \delta\}$. We will denote the resulting manifold by $Bl_{\epsilon,\delta}(\mathbb{C} \times \mathbb{C}^n)$. Since the real blowing up can be done within an arbitrarily small ball around the origin for ϵ, δ sufficiently small, we can globalize this procedure to any quasi-free Hamiltonian S^1-action on a symplectic manifold M with the moment map having Morse-Bott singularities of index $(2, 2k)$.

Now let T^w denote the subgroup $\exp(\mathbb{R} \cdot (iw))$ for any $w \in i\mathfrak{t}$. As mentioned before, we have to replace G by G_{Ad} to ensure the action by T to be quasi-free. This is equivalent to replacing the integral lattice in $i\mathfrak{t}$ by the lattice generated by $\frac{1}{3}w_0$ and $\frac{1}{3}w_2$ (i.e. the weight lattice). Let

$$u_1 = \frac{1}{3}\mathrm{diag}(1, -2, 1).$$

For $\nu > 0$ small, let C_ν be the cone bounded by $\mathbb{R}_{\geq 0}(w_1 \pm \nu \cdot u_1)$. Along $\mu^{-1}(\overset{\circ}{C}_\nu)$, T^{w_1} acts freely, so the moment map

$$\mu_{w_1,\nu} : \mu^{-1}(\overset{\circ}{C}_\nu) \xrightarrow{\mu} \overset{\circ}{C}_\nu \to \overset{\circ}{C}_\nu/\langle u_1 \rangle \cong \mathbb{R}_+ \cdot w_1$$

for the T^{w_1}-action is regular. Therefore the reduced space at any $p \in \mathbb{R}_+ \cdot w_1$

$$M^{w_1}_{p,\nu} := T^{w_1} \backslash \mu^{-1}_{w_1,\nu}(p)$$

is a 4-dimensional symplectic manifold with a Hamiltonian T^{u_1}-action. The moment map for the T^{u_1}-action on $M^{w_1}_{p,\nu}$ is denoted by

$$\mu^{w_1}_{p,\nu} : M^{w_1}_{p,\nu} \to (\mathbb{R} \cdot u_1)^* \cong \mathbb{R}.$$

By Lemma 3.3.2, T^{u_1} has exactly three fixed points $Q_j, j = 1, 2, 3$ on $M^{w_1}_{p,\nu}$, which are of the form $(x_1, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$, $(x_2, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix})$ and $(x_3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix})$, respectively. The tangent space $T_{Q_j} M^{w_1}_{p,\nu}$ at Q_j is a 4-dimensional symplectic vector space with a linear symplectic T^{u_1}-action. We can turn it into a Hermitian vector space by equipping it with a compatible inner product so that we have an identify $T_{Q_j} M^{w_1}_{p,\nu} \cong \mathbb{C}^2$ with a (complex-linear) T^{u_1}-action. Here $\mathbb{C}^2$ with coordinates z_0, z_1 is endowed with the standard Kahler form $\omega_{\text{st}} = \frac{i}{2}(dz_0 \wedge d\overline{z}_0 + dz_1 \wedge d\overline{z}_1)$.

Lemma 3.4.7. *Under some unitary change of coordinates on $\mathbb{C}^2$, the action is given by (3.11).*

Proof. By quasi-freeness, the weights of the action must be included in $\{0, 1, -1\}$. Now using the equivariant version of Moser's argument (or the Darboux-Weinstein Theorem), we can identify a neighborhood of 0 in $\mathbb{C}^2$ to a neighborhood of Q_i in $M^{w_1}_{p,\nu}$, in a T^{u_1}-equivariant way. Then we see that the image of the moment map in $(\mathbb{R} \cdot u_1)^* \cong \mathbb{R}$ is symmetric about 0, therefore the weights are 1 and -1, and under some unitary change of coordinates, we have the action be given by (3.11). □

Now we can desingularize the action by T^{u_1} along $\mathbb{R}_{>0} \cdot w_1$, and replace $\mu|_{\mu^{-1}(\overset{\circ}{W})}$ by $\tilde{\mu}$, then $\tilde{\mu}$ is regular over the interior of the cone $W_{01,\delta}$ bounded by $\mathbb{R}_{\geq 0} \cdot w_0$ and $\mathbb{R}_{\geq 0} \cdot (w_1 - \delta \cdot u_1)$, for some $\delta > 0$. Similarly to Proposition 3.4.6, we have $\tilde{\mu}^{-1}(\overset{\circ}{W}_{01,\delta}) \cong (U(2)/\mu_3 \times \overset{\circ}{W}_{01,\delta}, c \cdot d(A \cdot \tau))$.

Remark 3.4.8. Since the blowing down map from $Bl_{\epsilon,\delta}(\mathbb{C} \times \mathbb{C}^n)$ to $\mathbb{C} \times \mathbb{C}^n$ identifies the reduced spaces at 0, this gives a way to identify the reduced spaces over $\mathbb{R}_{>0} \cdot w_1$ with the others over $\overset{\circ}{W}_{01}$.

3.4.2. *The equivariant linear Symplectic group* $Sp(4)^{S^1}$.

Let

$$Sp(4)^{S^1} := \{P \in Sp(4) : P \text{ commutes with the } S^1\text{-action in (3.11)}\},$$

where P is relative to the standard basis $\partial_{x_0}, \partial_{y_0}, \partial_{x_1}, \partial_{y_1}$ and $z_j = x_j + iy_j$ for $j = 0, 1$.

Lemma 3.4.9.

$$Sp(4)^{S^1} = \{\begin{bmatrix} \lambda_1 e^{i\theta_1} & \lambda_2\sigma \circ e^{i\theta_2} \\ \lambda_2\sigma \circ e^{i\theta_3} & \lambda_1 e^{i\theta_4} \end{bmatrix}, \lambda_i \geq 0 \textit{ for } i = 1, 2, \lambda_1^2 - \lambda_2^2 = 1, \theta_1 + \theta_2 = \theta_3 + \theta_4 \textit{ if } \lambda_2 \neq 0\},$$

where σ means taking complex conjugate.

Proof. Let $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, D are all 2×2-matrices. Let R_θ denote the standard rotation matrix on $\mathbb{R}^2$ by angle θ. Then P is S^1-equivariant implies that

$$[\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} R_{-\theta} & 0 \\ 0 & R_\theta \end{bmatrix}] = 0,$$

and this is equivalent to that P is of the form

$$\begin{bmatrix} \lambda_1 e^{i\theta_1} & \lambda_2\sigma \circ e^{i\theta_2} \\ \lambda_3\sigma \circ e^{i\theta_3} & \lambda_4 e^{i\theta_4}, \end{bmatrix}, \lambda_i \geq 0 \text{ for } i = 1, ..., 4 \tag{3.12}$$

relative to the standard basis $\partial_{z_0}, \partial_{z_1}$. Now we need P to be symplectic, i.e. it preserves the Kahler form $\frac{i}{2}(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1)$. By direct calculations, the undetermined quantities in (3.12) should satisfy

$$\lambda_1 = \lambda_4, \lambda_2 = \lambda_3, \lambda_1^2 - \lambda_2^2 = 1, \text{ and } \theta_1 + \theta_2 = \theta_3 + \theta_4 \text{ if } \lambda_2 \neq 0,$$

and we finish the proof. □

Let C be the center of $Sp(4)^{S^1}$, i.e. $\{\begin{bmatrix} e^{-i\theta} & & \\ & & \\ & & e^{i\theta} \end{bmatrix}, \theta \in [0, 2\pi)\}$. By the above Lemma,

$$Sp(4)^{S^1}/C \cong \{\begin{bmatrix} \lambda_1 & \lambda_2\sigma \circ e^{i\theta_2} \\ \lambda_2\sigma \circ e^{i\theta_3} & \lambda_1 e^{i\theta_4} \end{bmatrix} : \lambda_1^2 - \lambda_2^2 = 1, \lambda_i \geq 0 \text{ for } i = 1, 2; \theta_2 = \theta_3 + \theta_4 \text{ if } \lambda_2 \neq 0\}. \tag{3.13}$$

There is an S^1-action by the left multiplication of the subgroup $\{\begin{bmatrix} 1 & & \\ & & \\ & & e^{i\theta} \end{bmatrix}\}$, and the projection

$$\begin{array}{ccc} Sp(4)^{S^1}/C & \longrightarrow & S^1 \\ \begin{bmatrix} \lambda_1 & \lambda_2\sigma \circ e^{i\theta_2} \\ \lambda_2\sigma \circ e^{i\theta_3} & \lambda_1 e^{i\theta_4} \end{bmatrix} & \mapsto & \{\begin{bmatrix} 1 & & \\ & & \\ & & e^{i\theta_4} \end{bmatrix}\} \end{array}$$

is an S^1-equivariant fiber bundle, with each fiber homeomorphic to a disc, so in particular, this map is a homotopy equivalence.

The reduced space at 0 for the Hamiltonian action in (3.11) can be identified with $\mathbb{C}$ (with the standard Kahler form) by taking the slice in $\{|z_0|^2 = |z_1|^2\}$ in which $z_0 \geq 0$ and z_1 is used to be the linear coordinate on $\mathbb{C}$.

Lemma 3.4.10. *Let $z_+(t) = t$ and $z_-(t) = -t$, $t \geq 0$ be the two opposite rays emitting from the origin in the reduced space $\mathbb{C}$ at 0. Let $P \in Sp(4)^{S^1}$ and $\widetilde{P}$ be the induced map on $\mathbb{C}$. Then*
(a) there exists a P for any prescribed values of $\arg(\frac{d}{dt}|_{t=0}\widetilde{P}(z_+(t)))$ and $\arg(\frac{d}{dt}|_{t=0}\widetilde{P}(z_-(t)))$, except for $\arg(\frac{d}{dt}|_{t=0}\widetilde{P}(z_+(t))) = \arg(\frac{d}{dt}|_{t=0}\widetilde{P}(z_-(t)))$.
(b) If $\widetilde{P} \in Sp(2)$, then $P = \begin{bmatrix} 1 & \\ & e^{i\theta_4} \end{bmatrix}$ for some θ_4 modulo the center C. In particular, if P satisfies

$$\frac{d}{dt}|_{t=0}\widetilde{P}(z_+(t)) = 1, \frac{d}{dt}|_{t=0}\widetilde{P}(z_-(t)) = -1, \tag{3.14}$$

then $P \in C$. Therefore, the map

$$\begin{array}{ccc} Sp(4)^{S^1}/C & \longrightarrow & \mathbb{C} \times \mathbb{C} \\ P & \mapsto & (\frac{d}{dt}|_{t=0}\widetilde{P}(z_+(t)), \frac{d}{dt}|_{t=0}\widetilde{P}(z_-(t))) \end{array} \tag{3.15}$$

is an injection.
(c) the map

$$\begin{array}{ccc} Sp(4)^{S^1}/C & \longrightarrow & S^1 \\ P & \mapsto & \exp(i \arg(\frac{d}{dt}|_{t=0}\widetilde{P}(z_+(t)))) \end{array} \tag{3.16}$$

is a homotopy equivalence of spaces.

Proof. (a) One lifting of the tangent vector at 0 of the two rays $z = t$ and $z = -t$ to $\mathbb{C}^2$ is $v_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Take P as in (3.13), then

$$Pv_+ = \begin{bmatrix} \lambda_1 + \lambda_2 e^{-i\theta_2} \\ (\lambda_1 + \lambda_2 e^{-i\theta_2})e^{i\theta_4} \end{bmatrix}, Pv_- = \begin{bmatrix} \lambda_1 - \lambda_2 e^{-i\theta_2} \\ -(\lambda_1 - \lambda_2 e^{-i\theta_2})e^{i\theta_4} \end{bmatrix}.$$

Let $\beta_\pm = \arg(\lambda_1 \pm \lambda_2 e^{-i\theta_2})$, and $w_\pm$ denote $\frac{d}{dt}|_{t=0}\widetilde{P}(z_\pm(t))$. Then

$$\arg(w_+) = \theta_4 + 2\beta_+, \arg(w_-) = \theta_4 + 2\beta_- + \pi.$$

It is not hard to see that $\beta_+ - \beta_-$ ranges in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then we can use θ_4 to adjust $\arg(w_\pm)$ to the prescribed values.
(b) The claim follows by direction calculations. (c) It is obvious by looking at the image of the subgroup $\{\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \theta \in [0, 2\pi)\}$. □

3.4.3. *A deformation retraction of equivariant symplectomorphism group of $M^{w_1}_{p,\nu}$ to a point.* Let $\mathrm{Sympl}^{T^{u_1}}(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$ denote the subgroup of T^{u_1}-equivariant symplectomorphisms $\mathrm{Sympl}^{T^{u_1}}(M^{w_1}_{p,\nu})$ of $M^{w_1}_{p,\nu}$ that *fix each* $Q_j, j = 1, 2, 3$. Let $\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}^3_{j=1})$ denote the group of automorphisms of the reduced space M_p induced from $\mathrm{Sympl}^{T^{u_1}}(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$. For each $Q_j, j = 1, 2, 3$, we fix an identification between a neighborhood of 0 in $\mathbb{C}^2$ with a neighborhood of Q_j in $M^{w_1}_{p,\nu}$ as in the proof of Lemma 3.4.7, and this induces an identification between a neighborhood of 0 in the reduced space $\mathbb{C}$ with a neighborhood of the image of Q_j, which we will denote by Q_j as well, in M_p.

Lemma 3.4.11. $\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}^3_{j=1})$ *is contractible.*

Proof. Step 1. *A fibration* $\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}^3_{j=1}) \to (Sp(4)^{S^1}/C)^3$.
There is an obvious group homomorphism

$$\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}^3_{j=1}) \to (Sp(4)^{S^1}/C)^3, \tag{3.17}$$

by sending each automorphism to the tangent maps (modulo C) at $0 \in \mathbb{C}^2$ any of its lifting near $Q_i, i = 1, 2, 3$ with respect to the fixed trivializations. Let

$$\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}^3_{j=1}) =: \text{kernel of (3.17)},$$

It is easy to see that (3.17) is a principal $\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}^3_{j=1})$-bundle.

Let

$$\mathrm{Sympl}_0(M_p, \{Q_j\}^3_{j=1}) := \{\varphi \in \mathrm{Sympl}(M_p, \{Q_j\}^3_{j=1}) : (d\varphi)_{Q_j} = id, j = 1, 2, 3\},$$

where $\mathrm{Sympl}(M_p, \{Q_j\}^3_{j=1})$ is the true symplectomorphism group of M_p fixing the three special points. The next step shows that $\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}^3_{j=1})$ is homotopy equivalent $\mathrm{Sympl}_0(M_p, \{Q_j\}^3_{j=1})$.

Step 2. $\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}^3_{j=1}) \simeq \mathrm{Sympl}_0(M_p, \{Q_j\}^3_{j=1})$.
Let $\mathrm{Sympl}^{T^{u_1}}_\sharp(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$ be the subgroup in $\mathrm{Sympl}^{T^{u_1}}(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$ consisting of elements $\hat{\phi}$ such that $\hat{\phi}$ restricted to a sufficiently small neighborhood of each Q_j (within the fixed local chart) is the linear transformation $\begin{bmatrix} e^{-i\theta_j} & \\ & e^{i\theta_j} \end{bmatrix}$, for some $\theta_j \in [0, 2\pi)$. Also let $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}^3_{j=1})$ be the image of $\mathrm{Sympl}^{T^{u_1}}_\sharp(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$ in $\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}^3_{j=1})$.

Now we can construct a deformation retraction from the group $\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}^3_{j=1})$ to $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}^3_{j=1})$. Near Q_j, the graph of $\begin{bmatrix} e^{i\theta_j} & \\ & e^{-i\theta_j} \end{bmatrix} \circ \hat{\phi}$ is a Lagrangian in $(\mathbb{C}^2)^- \times \mathbb{C}^2 \cong T^*\Delta_{\mathbb{C}^2}$, which is tangent to the zero section at $((Q_j, Q_j), 0) \in T^*\Delta_{\mathbb{C}^2}$. Equivalently, in

a smaller neighborhood of $(0,0)$, it is the graph of the differential[2] of a function f_j with $Df_j(0)=0$ and $D^2f_j(0)=0$, where (Q_j,Q_j) is regarded as the origin in $\Delta_{\mathbb{C}^2}\cong\mathbb{C}^2$. Let $\mathrm{r}(\mathbf{z})=\|\mathbf{z}\|^2$ and fix a small ball $B_j(\epsilon)=\{\mathrm{r}<\epsilon^2\}\subset\Delta_{\mathbb{C}^2}$, and let $\mathbb{D}_j(\frac{1}{16})\subset B_j(\epsilon)$ be the connected component containing 0 where $|D^2f_j|<\frac{1}{16}$. Here for a function f on a domain, we adopt the following notations

$$|D^2f|=:\sup_x\sum_{m,n}|\frac{\partial^2 f}{\partial x_m\partial x_n}(x)|,\ |Df|=:\sup_x\sum_n|\frac{\partial f}{\partial x_n}(x)|.$$

Now let $\epsilon_0=\sup\{\epsilon\in\mathbb{R}_+ : B_j(\epsilon)\subset\mathbb{D}_j(\frac{1}{16})\}$, then we have $|Df_j|<\frac{1}{16}\epsilon_0$ and $|f_j|<\frac{1}{16}\epsilon_0^2$ on $B_j(\epsilon_0)$, if we make $f_j(0)=0$. Consider a C^∞-function $b_{j,\epsilon_0}(x_1,x_2)$ on the square $[0,\epsilon_0^2)\times(-\frac{1}{16}\epsilon_0^2,\frac{1}{16}\epsilon_0^2)$ satisfying $b_{j,\epsilon_0}(x_1,x_2)=0$ for $|x_1|<\frac{1}{32}\epsilon_0^2$, $b_{j,\epsilon_0}(x_1,x_2)=x_2$ for $|x_1|>\frac{31}{32}\epsilon_0^2$, $b_{j,\epsilon_0}(x_1,0)=0$, and

$$|D^2b_{j,\epsilon_0}|(|D\mathrm{r}|^2+2|D\mathrm{r}|\cdot|Df_j|+|Df_j|^2)+|D_{x_1}b_{j,\epsilon_0}|\cdot|D^2\mathrm{r}|+|D_{x_2}b_{j,\epsilon_0}|\cdot|D^2f_j|<\frac{5}{6}. \quad (3.18)$$

Then the graph of the differential of

$$\begin{bmatrix}e^{-i\theta_j} & \\ & e^{i\theta_j}\end{bmatrix}\circ(s\cdot b_{j,\epsilon_0}\circ(\mathrm{r},f_j)+(1-s)\cdot f_j)|_{B_j(\epsilon_0)},0\le s\le 1$$

glues well with the graph of $\hat\phi$ outside of $B_j(\sqrt{\frac{31}{32}}\epsilon_0)$, and gives a family $\{\hat\phi_s\}_{s\in[0,1]}$ whose induced maps on M_p lie in $\widetilde{\mathrm{Sympl}}_0(M_p,\{Q_j\}_{j=1}^3)$, with $\hat\phi_0=\hat\phi$ and $\hat\phi_1\in\mathrm{Sympl}_\sharp^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)$. Note that for (3.18), if we start with ϵ small enough, then $|D_{x_1}b_{j,\epsilon_0}|<\frac{1}{12}$, $|D_{x_2}b_{j,\epsilon_0}|<2$ and $|D^2b_{j,\epsilon_0}|<5$, for instance, are sufficient for it to hold. We can fix such a small ϵ once for all, and make b_{j,ϵ_0} continuously depend on ϵ_0 (in the C^∞-topology). Thus we have a deformation retraction from $\widetilde{\mathrm{Sympl}}_0(M_p,\{Q_j\}_{j=1}^3)$ to $\widetilde{\mathrm{Sympl}}_\sharp(M_p,\{Q_j\}_{j=1}^3)$.

Step 3. $\widetilde{\mathrm{Sympl}}(M_p,\{Q_j\}_{j=1}^3)$ *is contractible.*
There is a natural fiber bundle

$$\begin{array}{ccc}\mathrm{Sympl}_0(M_p,\{Q_j\}_{j=1}^3) & \longrightarrow & \mathrm{Sympl}(M_p,\{Q_j\}_{j=1}^3)\\ & & \downarrow\\ & & (Sp(2))^3\end{array}$$

by the same construction as in (3.17). By standard results (c.f. [4]), $\mathrm{Sympl}(M_p,\{Q_j\}_{j=1}^3)$ is contractible, therefore

$$B\mathrm{Sympl}_0(M_p,\{Q_j\}_{j=1}^3)\simeq(Sp(2))^3\simeq(S^1)^3.$$

[2]Here we need to fix a special Darboux coordinate system on the neighborhood; see the discussion right before Lemma 3.4.17

In particular, the preimage of the fiber bundle over $(S^1)^3 \cong (U(1))^3 \subset (Sp(2))^3$, for which we will denote by $\mathrm{Sympl}^\dagger(M_p, \{Q_j\}_{j=1}^3)$, is homotopy equivalent to $\mathrm{Sympl}(M_p, \{Q_j\}_{j=1}^3)$ via the inclusion.

On the other hand, there is an inclusion[3] of the fibration involving $\mathrm{Sympl}^\dagger(M_p, \{Q_j\}_{j=1}^3)$ into the fibration (3.17). Since $Sp(4)^{S^1}/C \simeq S^1$ by Lemma 3.4.10 (c), using Step 2 and standard facts about classifying spaces, we deduce that $\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3)$ must be contractible as well.

□

Let

$$\gamma_p : \mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1}, \{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu), T^{u_1}) \to \widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3)$$

be the projection map.

Proposition 3.4.12. *γ_p is a homotopy equivalence, hence*

$$\mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1}, \{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu), T^{u_1})$$

is contractible.

Proof. First, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1}, \{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu), T^{u_1}) & \xrightarrow{\gamma_p} & \widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3) \\ \downarrow & & \downarrow \\ (Sp(4)^{S^1}/C)^3 & \xrightarrow{\quad id \quad} & (Sp(4)^{S^1}/C)^3 \end{array} , \tag{3.19}$$

where the vertical arrows are both the restriction of the tangent maps at $Q_j, j = 1,2,3$ (modulo C), and they give two fiber bundles. The kernel of the left map is the subgroup in $\mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1}, \{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu), T^{u_1})$ consisting of all liftings of elements in $\widetilde{\mathrm{Sympl}}_0(M_p, \{Q_j\}_{j=1}^3)$. The proof of Lemma 3.4.11 shows that this group deformation retracts onto all liftings of elements in $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}_{j=1}^3)$. We will apply the technique of real blow-ups to show that the latter subgroup deformation retracts onto $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}_{j=1}^3)$. Therefore γ_p is a homotopy equivalence. In the following, we will keep using the notations from the proof of Lemma 3.4.11.

For any $\phi \in \widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}_{j=1}^3)$, let $B_j, j = 1,2,3$ be a small ball around Q_j in $M_{p,\nu}^{w_1}$ on which one of the liftings $\hat{\phi}$ is the linear transformation $\begin{bmatrix} e^{-i\theta_j} & \\ & e^{i\theta_j} \end{bmatrix}$ for some θ_j. For $\epsilon, \delta > 0$

[3]To be more rigorous, one should replace $\mathrm{Sympl}^\dagger(M_p, \{Q_j\}_{j=1}^3)$ by $\mathrm{Sympl}^\dagger(M_p, \{Q_j\}_{j=1}^3) \cap \widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3)$ for the inclusion, but the resulting space is homotopy equivalent to $\mathrm{Sympl}(M_p, \{Q_j\}_{j=1}^3)$, by the same technique in Step 2.

small enough, the surgery for the real blow-up to $Bl_{\epsilon,\delta}(M^{w_1}_{p,\nu})$ around each Q_j is taken within a smaller ball $B'_j \subset B_j, j = 1, 2, 3$ satisfying $\overline{B'}_j \subset B_j$, and we denote the resulting moment map for T^{u_1} by

$$\bar{\mu}_{\epsilon,\delta} : Bl_{\epsilon,\delta}(M^{w_1}_{p,\nu}) \to \mathbb{R},$$

where $\bar{\mu}_{\epsilon,\delta}$ is regular over $(-\nu, \delta)$. Then $\hat{\phi}$ induces a symplectomorphism $\hat{\phi}_{\epsilon,\delta}$ on $\bar{\mu}^{-1}_{\epsilon,\delta}(-\nu, \delta)$, whose restriction to the blow-up region near Q_j is the action by $\exp(i\theta_j u_1)$. Conversely, given any $\hat{\phi}_{\epsilon,\delta}$ on $\bar{\mu}^{-1}_{\epsilon,\delta}(-\nu, \delta)$ of this form, we can recover $\hat{\phi}$ on $(\mu^{w_1}_{p,\nu})^{-1}(-\nu, \delta)$.

Now we can describe the space of all liftings of $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}^3_{j=1})$ in $\mathrm{Sympl}^{T^{u_1}}_\sharp(M^{w_1}_{p,\nu}, \{Q_j\}^3_{j=1})$ as an inductive limit of spaces $X_{\epsilon,\delta}$ over $(\epsilon, \delta) \in (\mathbb{R}_+)^2$, where we have a natural inclusion $X_{\epsilon_1,\delta_1} \hookrightarrow X_{\epsilon_2,\delta_2}$, when $\epsilon_1 > \epsilon_2$ and $\delta_1 > \delta_2$. The space $X_{\epsilon,\delta}$ consists of $\hat{\phi}$ whose restriction to a neighborhood of the three blow-up regions for $Bl_{\epsilon,\delta}(M^{w_1}_{p,\nu})$ near each Q_j is given by the action of $\exp(i\theta_j u_1)$ for some $\theta_j \in \mathbb{R}$. By Corollary 3.4.2, after trivializing the reduced spaces of $\bar{\mu}_{\epsilon,\delta}$ over $(-\nu, \delta)$ and the reduced spaces of $\mu^{w_1}_{p,\nu}$ over $(\frac{\delta}{2}, \nu)$, we see that $X_{\epsilon,\delta}$ has a free $C^\infty((-\nu, \nu), T^{u_1})$-action, and $X_{\epsilon,\delta}/C^\infty((-\nu, \nu), T^{u_1})$ corresponds to the space of pairs of paths (ρ_1, ρ_2), where $\rho_1 : (-\nu, \delta) \to \mathrm{Sympl}(S^2)$, $\rho_2 : (\frac{\delta}{2}, \nu) \to \mathrm{Sympl}(S^2)$ satisfy that ρ_1 restricts to the identity on a neighborhood of the blowing up loci and $\rho_1|_{(\frac{\delta}{2},\delta)} = \rho_2|_{(\frac{\delta}{2},\delta)}$ if they are transformed to be relative to a fixed trivialization of reduced spaces for $\mu^{w_1}_{p,\nu}$ (for this one needs to use Remark 3.4.8; see below for more details). Then $(\varinjlim X_{\epsilon,\delta})/C^\infty((-\nu, \nu), T^{u_1})$ deformation retracts onto $\widetilde{\mathrm{Sympl}}_\sharp(M_p, \{Q_j\}^3_{j=1})$, by deforming each pair of paths (ρ_1, ρ_2) to the constant path determined by $\rho_1(0)$.

One needs a little bit more detailed description and control of $X_{\epsilon,\delta}$ to write down a deformation retraction. First, we only need to consider the spaces $X_{\frac{1}{n},\frac{1}{n}}$ for $n \in \mathbb{N}$ sufficiently large, since $\varinjlim X_{\epsilon,\delta} = \lim_{n\to\infty} X_{\frac{1}{n},\frac{1}{n}}$. Now we require any $\varphi \in X_{\frac{1}{n},\frac{1}{n}}$ restricted to the action by $\exp(i\theta_j u_1)$ on $\{|z_0|^2 < \frac{2}{n-2}, -|z_0|^2 + |z_1|^2 < \frac{2}{n-2}\}$ near each Q_j, for some θ_j, and when we trivialize the reduced spaces of $Bl_{\frac{1}{n},\frac{1}{n}}(M^{w_1}_{p,\nu})$ over $(-\nu, \frac{1}{n})$, we require that it agrees with the corresponding trivialization for $Bl_{\frac{1}{n-1},\frac{1}{n-1}}(M^{w_1}_{p,\nu})$ outside the neighborhood $\{|z_0|^2 < \frac{2}{n-1}, -|z_0|^2+|z_1|^2 < \frac{2}{n-1}\}$ (this can be done inductively). In addition, we also require the restriction of the trivialization over $(0, \frac{1}{n})$ to agree with the fixed trivialization over $\overset{\circ}{W}_{12}$ outside the neighborhood $\{|z_0|^2 < \frac{2}{n}, -|z_0|^2+|z_1|^2 < \frac{2}{n}\}$ of each Q_j. For the last requirement, we need to use Remark 3.4.8. After these set-ups, we can simultaneously deformation retract $X_{\frac{1}{n},\frac{1}{n}}$ by deforming their associated paths ρ_1, ρ_2 as above to the constant paths determined by $\rho_1(0)$.

□

Now using the $\mathbb{R}_+$-action, we can trivialize $\mu^{-1}(\overset{\circ}{C}_\nu)$ as follows.

Lemma 3.4.13. *$\mu^{-1}(\overset{\circ}{C}_\nu)$ is T-equivariantly symplectomorphic to $\mu^{-1}(w_1 + B_\nu(0, \langle u_1\rangle)) \times \mathbb{R}_+$ equipped with the symplectic form $d(t\alpha)$, where α is any (it exists!) T^{u_1}-invariant connection*

1-form on $\mu^{-1}(w_1+B_\nu(0,\langle u_1\rangle))$ as a principal T^{w_1}-bundle, whose curvature is the restriction of ω, and t is the coordinate of $\mathbb{R}_+$.

Proof. By the $\mathbb{R}_+$-action, we naturally have an identification of $\mu^{-1}(\overset{\circ}{C}_\nu)$ with $\mu^{-1}(w_1 + B_\nu(0,\langle u_1\rangle)) \times \mathbb{R}_+$. Here $\mu^{-1}(w_1 + B_\nu(0,\langle u_1\rangle))$ in the former space is identical to $\mu^{-1}(w_1 + B_\nu(0,\langle u_1\rangle)) \times \{1\}$ in the latter space.

Let ω also denote its restriction on $\mu^{-1}(w_1 + B_\nu(0,\langle u_1\rangle))$. Then the induced symplectic form on $\mu^{-1}(p + B_\nu(0,\langle u_1\rangle)) \times \mathbb{R}_+$ must be of the form $t\omega + dt \wedge \alpha$, for some 1-form α (doesn't depend on t) on $\mu^{-1}(w_1 + B_\nu(0,\langle u_1\rangle))$, which is obviously T-equivariant and is a connection form for the T^{w_1}-principal bundle. Since $t\omega + dt\wedge\alpha$ is closed, we have the identity $dt \wedge (\omega - d\alpha) = 0$, so $\omega = d\alpha$. Thus the lemma follows. □

A deformation retraction for $\ker\beta_G$ supported near $T^*_{\mathcal{B}}\mathcal{B}$

We start with a general set-up for the statement of Lemma 3.4.14 below. Let (X, ω_X) be a Kahler manifold, and $X^- \times X$ be equipped with the symplectic form $\omega_0 = (-\omega_X) \times \omega_X$. Let N_{Δ_X} be the normal bundle to the diagonal with respect to the Kahler metric $g \times g$ (which is the anti-diagonal in the tangent bundle restricted to Δ_X). Then the product symplectic form gives a natural identification of N_{Δ_X} with $T^*\Delta_X$, thus induces a symplectic form ω_1 on N_{Δ_X}. By the Lagrangian tubular neighborhood theorem, there is a symplectomorphism mapping a tubular neighborhood of the zero section in $T^*\Delta_X$ to a tubular neighborhood of Δ_X in $X^- \times X$, which fixes each point in Δ_X. We state a slightly stronger statement in the following lemma.

Lemma 3.4.14. *There exists a symplectomorphism ψ from a tubular neighborhood $N^\epsilon_{\Delta_X}$ of the zero section in N_{Δ_X} to a tubular neighborhood $U_\epsilon(\Delta_X)$ of Δ_X in $X^- \times X$, such that $\psi|_{\Delta_X} = id$ and $d\psi|_{\Delta_X} = id$.*

Proof. We first identify $N^\epsilon_{\Delta_X}$ with $U_\epsilon(\Delta_X)$ using the exponential map ψ with respect to $g \times g$. Then it suffices to show that $\|\psi^*\omega_0 - \omega_1\|_{((x,x),(tv,-tv))} \sim o(t)$ for any fixed x and v, since by Moser's argument, the vector field generating an isotopy between $\psi^*\omega_0$ and ω_1 will have length at most proportional to $o(t)$ in the direction of v, so the resulting diffeomorphism by integrating this vector field will have differential equal to the identity on the zero section.

For any $v, u, w \in T_xX$, the push-forward of the vertical vector $(u, -u)$ and the horizontal lifting $(w(t), w(t))$ of $(w, w) \in T_{(x,x)}\Delta_X$ at $((x, x), (tv, -tv))$ to $X^- \times X$ under the exponential map is $((d\exp_x)|_{tv}(u), (d\exp_x)|_{(-tv)}(-u))$ and $(J_{w,v}(t), J_{w,-v}(t))$ respectively, where $J_{w,v}(t)$ denotes for the Jacobi vector field for the family of geodesics $\exp_{\exp_x(\tau w)}(t\Gamma(\exp_x(sw))^\tau_0(v))$, where $\Gamma(\exp_x(sw))^\tau_0$ means the parallel transport along the geodesic $\exp_x(sw)$ from time 0 to time τ.

The Kahler property implies that the covariant derivative $D_{\exp_x(tv)}(\omega) = 0$, thus

$$\omega(\Gamma(\exp_x(sv))^t_0(u), \Gamma(\exp_x(sv))^t_0(w)) = \omega(u, w).$$

Now we only need to show that

$$\begin{aligned}&\|(d\exp_x)|_{tv}(u) - \Gamma(\exp_x(sv))_0^t(u)\| \sim o(t),\\ &\|J_{w,v}(t) - \Gamma(\exp_x(sv))_0^t(w)\| \sim o(t),\\ &\omega_1(w_1(t), w_2(t)) = \omega_1(w_1, w_2), \text{ for any two vectors } w_1, w_2 \in T_xX.\end{aligned}$$

These properties hold for any Riemannian manifold X. In fact, one can take the geodesic coordinate at x, and use the fact that the Christoffel symbols vanish at x to deduce that the covariant derivative of the first two of the above vectors in the norm along $\exp_x(tv)$ has norm $o(1)$. One can prove the last equality similarly. □

Let $\ker\beta_G^\sharp$ be the subgroup of $\ker\beta_G$ consisting of φ that restricts to the identity in a neighborhood of $T^*_{\mathcal{B}}\mathcal{B}$.

Lemma 3.4.15. *There is a deformation retraction from* $\ker\beta_G$ *to* $\ker\beta_G^\sharp$.

Proof. Let X denote for $T^*\mathcal{B}$ equipped with a fixed G-invariant metric g. As before, we view L_φ as a Lagrangian correspondence in $X^- \times X$. Let ψ be a fixed symplectomorphism as in Lemma 3.4.14, then ψ is G-equivariant. Note that any $\varphi \in \ker\beta_G$ satisfies $\varphi|_{T^*_{\mathcal{B}}\mathcal{B}} = \mathrm{id}$. If we can show that the tangent spaces of L_φ at $\Delta_X \cap (T^*_{\mathcal{B}}\mathcal{B} \times T^*_{\mathcal{B}}\mathcal{B})$ are transverse to the fiber direction in the normal bundle N_{Δ_X}, then the pull-back of L_φ under ψ is the graph of an exact 1-form in a small neighborhood of $\Delta_X \cap (T^*_{\mathcal{B}}\mathcal{B} \times T^*_{\mathcal{B}}\mathcal{B})$, hence we can run a similar argument as in Lemma 3.4.11 to give a deformation retraction.

To see that the tangent spaces of L_φ at $\Delta_X \cap (T^*_{\mathcal{B}}\mathcal{B} \times T^*_{\mathcal{B}}\mathcal{B})$ are transverse to the fiber direction, we only need to look at the image of $\frac{d}{dt}|_{t=0}(x, \xi_t) \in T_{(x,0)}X$ under $d\varphi$ with $\xi_0 = 0$ and $\frac{d}{dt}|_{t=0}\xi_t \neq 0$. Since φ has to preserve the moment map, the image of the vector cannot be its opposite, so we are done. □

$\ker\beta_G$ is contractible

This section is devoted to the proof that $\ker\beta_G$ is contractible. By Lemma 3.4.15, we only need to prove that $\ker\beta_G^\sharp$ is contractible. In the following, we identity $M_{s\cdot p}$ with M_p for all $s > 0$ using the $\mathbb{R}_+$-action, where p is usually reserved for denoting any fixed element in $\mathbb{R}_+ \cdot w_1$, unless otherwise specified. Let σ denote the projection of the subregular Springer fibers in M_p (cf. Lemma 3.3.2). Fix T_0 to be the union of σ with its image under the right Weyl group action on $T^*\mathcal{B}$ (induced from the right $\mathbf{W}$-action on G/T), using some fixed appropriate parametrization. Under the natural complex structure on $T^*\mathcal{B}$ (induced from that of $\mathcal{B}$), any subregular Springer fiber is a chain of $\mathbb{P}^1$'s with normal crossing singularities, thus for appropriate trivializations near $Q_j, j = 1, 2, 3$, we can make T_0 a smooth curve in M_p. Not only that, we can also fix a parametrization of T_0, denoted by T_0 as well:

$$T_0 : S^1 \cong [0, 1]/(0 \sim 1) \longrightarrow M_p,$$

such that $T_0(\frac{j}{3}) = Q_{j+1}, T_0(t) = t - \frac{j}{3}$ near Q_{j+1} with respect to the fixed identification of the neighborhood of Q_{j+1} to an open neighborhood of 0 in $\mathbb{C}$, for $j = 0, 1, 2$. In the following, we may cut a given parametrized curve into some pieces. If we consider one of the pieces as a single parametrized curve, we will shift the induced parametrization from the original curve *without any rescaling* so that the parametrization starts at 0. When we do concatenation of two parametrized curves, we simply join their parametrization together (after necessary inversions of their parametrizations so that the concatenation is well-defined and has the required orientation), again without any rescaling, and we will specify a starting point if the resulting concatenation is a loop. For each path space considered below, we will specify the (fixed) interval over which every path is defined.

Fix an open tubular neighborhood of σ with a smooth boundary in M_p and denote it by $\mathcal{U}_\sigma$. We assume that $T_0 \pitchfork \partial\mathcal{U}_\sigma = \{P_1, P_3\}$, where P_1 (resp. P_3) is near Q_1 (resp. Q_3); see Figure 3.3. We fix a parametrization of $\partial\mathcal{U}_\sigma$, then $T_0 \cap \mathcal{U}_\sigma$ divides $\mathcal{U}_\sigma$ into two parts, whose boundaries (with the usual orientation) both inherit parametrizations from T_0 and $\partial\mathcal{U}_\sigma$ with starting point at P_1. We denote their boundary paths by Γ_1 and Γ_2. We also view σ as a path on its own.

Let

$$\begin{aligned}\mathcal{S}(\overset{\circ}{C}_\nu) :=&\{\varphi \in \mathrm{Sympl}^T(\mu^{-1}(\overset{\circ}{C}_\nu)) : \varphi \text{ preserves the Springer fibers at infinity} \\ &\text{and } \varphi = id \text{ near the vertex of } C_\nu\}.\end{aligned}$$

Similarly to β_G, we have a group homomorphism

$$\beta_{G,\overset{\circ}{C}_\nu} : \mathcal{S}(\overset{\circ}{C}_\nu) \longrightarrow B_{\mathbf{W}}.$$

For every $\varphi \in \ker \beta_{G,\overset{\circ}{C}_\nu}$, the induced map of φ on $M_{s\cdot p}$ gives a path

$$\rho_\varphi : (0, \infty) \to \widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3),$$

satisfying that[4]

(1) $\rho_\varphi(s) = id$ for s sufficiently small,

(2) for s sufficiently large, $\rho_\varphi(s)|_{M_p - \mathcal{U}_\sigma}$ is close to the identity map,

(3) for s sufficiently large, $\rho_\varphi(s)(T_0 \cap \mathcal{U}_\sigma)$ is contained in $\mathcal{U}_\sigma$ and can be isotoped to $T_0 \cap \mathcal{U}_\sigma$ within $\mathcal{U}_\sigma$, relative to $Q_j, j = 1, 2, 3$ and P_1, P_3.

After an easy deformation retraction of $\ker \beta_{G,\overset{\circ}{C}_\nu}$, we can assume that there is a uniform bound of s for (3.4) and (3.4) to hold. Moreover, we can shrink the neighborhood $\mathcal{U}_\sigma$ in a uniform way along the s-direction, using a vector field X_σ which is tangent to T_0 on the

[4]The conditions are necessary conditions on ρ_φ.

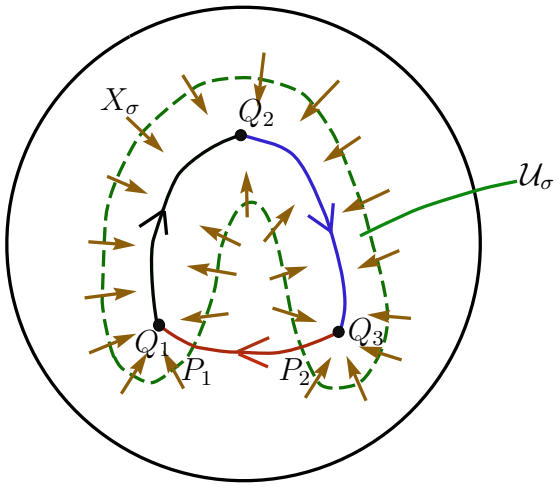


Figure 3.3: A vector field X_σ on $\mathcal{U}_\sigma$ shrinking it towards σ.

portion connecting P_1 (resp. Q_3) and Q_1 (resp. P_3), and whose time t flow $\phi^t_{X_\sigma}$ scales the symplectic form on $\mathcal{U}_\sigma$ by e^{-t} and deformation retracts $\mathcal{U}_\sigma$ onto σ as $t \to \infty$; see Figure 3.3 below. Thus after another easy deformation retraction of $\ker \beta_{\overset{\circ}{G,C_\nu}}$, we can replace the above conditions by

(1') $\rho_\varphi(s) = id$ for s sufficiently small,

(2') for $s \geq 1$, $\rho_\varphi(s)|_{M_p - \phi^{s-1}_{X_\sigma}(\mathcal{U}_\sigma)}$ is close to the identity map and $\rho_\varphi(s)$ restricts to the identity in $\phi^{s-1}_{X_\sigma}(\mathcal{V}_\sigma)$, for a fixed tubular neighborhood of $\partial\mathcal{U}_\sigma$,

(3') for $s \geq 1$, $\rho_\varphi(s)(T_0 \cap \mathcal{U}_\sigma)$ is contained in $\mathcal{U}_\sigma$ and it can be isotoped to $T_0 \cap \mathcal{U}_\sigma$ within $\phi^{s-1}_{X_\sigma}(\mathcal{U}_\sigma)$, relative to $Q_j, j = 1, 2, 3$ and P_1, P_3.

Now we claim that the space of ρ_φ is homotopy equivalent to the fiber product of two path spaces. The first path space is the standard based path space $\mathcal{P}_*(\widetilde{\mathrm{Sympl}}(M_p, \{Q_j\}_{j=1}^3))$, i.e. the domain of each path is $[0, 1]$. Let $\mathcal{P}(T_0, \mathcal{U}_\sigma)$ denote the space of embedded smooth paths connecting P_1, Q_1, Q_2, Q_3, P_3 successively (with domain determined by that of $T_0 \cap \mathcal{U}_\sigma$), which have parametrization $e^{i\theta_j}(t - \frac{j}{3})$ within some small neighborhood of Q_{j+1} with respect to the fixed identification to an open neighborhood of 0 in $\mathbb{C}$, for some θ_j and $j = 1, 2, 3$, and which belong to the homotopy class of $T_0 \cap \mathcal{U}$ as paths in $\mathcal{U}_\sigma$ relative to $\{Q_1, Q_2, Q_3\} \cup \mathcal{V}_\sigma$. Then the second path space is the space of paths

$$\rho : [1, \infty) \to \mathcal{P}(T_0, \mathcal{U}_\sigma),$$

denoted as $\mathcal{P}^{\diamond}(\mathcal{P}(T_0,\mathcal{U}_\sigma))$. There is an obvious map from $\mathcal{P}_*(\widetilde{\mathrm{Sympl}}(M_p,\{Q_j\}_{j=1}^3))$ to $\mathcal{P}(T_0,\mathcal{U}_\sigma)$, defined by evaluating at the end point and sending it to the image of $T_0\cap\mathcal{U}_\sigma$. There is also an obvious map from $\mathcal{P}^{\diamond}(\mathcal{P}(T_0,\mathcal{U}_\sigma))$ to $\mathcal{P}(T_0,\mathcal{U}_\sigma)$ by evaluating at the starting point.

Lemma 3.4.16.

$$\{\textit{the space of } \rho_\varphi\}\simeq \mathcal{P}_*(\widetilde{Sympl}(M_p,\{Q_j\}_{j=1}^3))\times_{\mathcal{P}(T_0,\mathcal{U}_\sigma)}\mathcal{P}^{\diamond}(\mathcal{P}(T_0,\mathcal{U}_\sigma)). \tag{3.20}$$

Proof. The proof of the claim is not hard, though a little bit technical. First, φ restricted to $\mu^{-1}(\overset{\circ}{C}_\nu)$ is determined by a smooth path

$$\rho_{\varphi,\nu}:(0,\infty)\to \mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu),T^{u_1}),$$

up to $C^\infty(\overset{\circ}{C}_\nu,T)$. Let $\mathrm{Sympl}_\dagger^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu),T^{u_1})$ denote for the preimage of $(U(1))^3\subset(Sp(2))^3$ in the left fiber bundle in (3.19). By Proposition 3.4.12, there is a homotopy equivalence

$$\mathrm{Sympl}_\dagger^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu),T^{u_1})\simeq \mathrm{Sympl}^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu),T^{u_1}),$$

therefore, we can restrict ourselves to φ determined by a smooth path

$$\rho_{\varphi,\nu}:(0,\infty)\to \mathrm{Sympl}_\dagger^{T^{u_1}}(M_{p,\nu}^{w_1},\{Q_j\}_{j=1}^3)/C^\infty((-\nu,\nu),T^{u_1}),$$

up to $C^\infty(\overset{\circ}{C}_\nu,T)$.

Using the same techniques as in Step 2 of the proof of Lemma 3.4.11, one can realize the left-hand-side of (3.20) as a fiberation over the right-hand-side after an appropriate deformation retraction, where $\rho_\varphi|_{(0,1]}$ corresponds to the first factor of the fiber product, and $\rho_\varphi|_{[1,\infty)}$ corresponds to the second factor. The fiber is homotopy equivalent to the path space of the product of spaces of compactly supported symplectomorphisms on a 2-disc, which is contractible. Next we will show that the fibration is a fiber bundle, by presenting local sections near any point in the base[5].

Fix a compatible complex structure J on an open neighborhood of $\overline{\mathcal{U}}_\sigma$ (thus a Riemannian metric on it), whose restriction near $Q_j, j=1,2,3$, is the standard complex structure with respect to the fix identification to $\mathbb{C}$. Consider the normal bundle (with respect to the metric) of $T_0\cap\mathcal{U}_\sigma$ with a framing given by $\frac{1}{\|v_x\|^2}Jv_x$, where v_x is the tangent vector to T_0 at x. We identify a tubular neighborhood of $T_0\cap\mathcal{U}_\sigma$, denoted as $\mathcal{V}_{T_0}^\epsilon$ for $\epsilon>0$ small, with the ϵ-disc bundle of the normal bundle of $T_0\cap\mathcal{U}_\sigma$ with respect to the framing (not the length function) via the exponential map. Similarly, we can identify a tubular neighborhood of $\partial\mathcal{U}_\sigma$ (contained in $\mathcal{V}_\sigma$), denoted as $\mathcal{V}^\epsilon$, with the ϵ-disc normal bundle of $\partial\mathcal{U}_\sigma$ in the same way using the exponential map.

Now for any $\tau\in\mathcal{P}(T_0,\mathcal{U}_\sigma)$, the identification of $T_0\cap\mathcal{U}_\sigma$ with τ (via their parametrizations) extends to an identification ι, sending $\mathcal{V}_{T_0}^\epsilon$ to a neighborhood $\mathcal{V}_\tau^\epsilon$ of τ defined similarly as

[5]Actually, it is enough to give a local section near just one point.

for $T_0 \cap \mathcal{U}_\sigma$. One might need to reduce ϵ so that the geodesics starting from τ with the designated lengths are mutually disjoint. Such choices of ϵ can be made continuously with respect to τ. Now one can apply Moser's argument to $t\omega+(1-t)(\iota^{-1})^*\omega$ to give a symplectic identification

$$\iota_\tau : \mathcal{V}_{T_0}^{\epsilon_\tau} \xrightarrow{\sim} \mathcal{V}_\tau, \tag{3.21}$$

where ϵ_τ depends continuously on τ, and $\mathcal{V}_\tau$ denotes for the image of ι_τ. By assumption, $(\iota^{-1})^*\omega$ agrees with ω on τ and on some small neighborhood of the special points P_1, Q_1, Q_2, Q_3, P_3, therefore ι_τ restricts to the original identification between $T_0 \cap \mathcal{U}_\sigma$ and τ, and it is the linear map of the form $\begin{bmatrix} 1 & \\ & e^{i\theta_j} \end{bmatrix}$ near each Q_j, and it is the identity map near P_1, P_3.

Now we can define a local section of the fibration around any point in the base. First pick the loop in $\mathcal{V}^\epsilon$ corresponding to $\frac{1}{2}\epsilon$ times the framing in the normal bundle of $\partial\mathcal{U}_\sigma$, for which we denote by ℓ_ϵ, and pick the paths in $\mathcal{V}_{T_0}^{\epsilon_\tau}$ which correspond to $\pm\frac{1}{2}\epsilon_\tau$ times the framing in the normal bundle of τ, for which we denote by $\varrho_{\epsilon_\tau}^{\pm}$ respectively. After further reducing ϵ appropriately (and continously with respect to τ), the union of $\varrho_{\epsilon_\tau}^{\pm}$ cut ℓ_ϵ into three pieces, denoted by $\ell_\epsilon^j, j = +, 0, -$, in the order from the left-hand-side to the right-hand-side of σ (with respect to the fixed orientation on $\mathcal{U}_\sigma$). We denote the parts bounded by $\ell_\epsilon^\pm$ and a (majority of) portion of $\varrho_{\epsilon_\tau}^{\pm}$ by $\mathcal{U}_{\sigma,\pm}$ respectively, and the rest of $\mathcal{U}_\sigma$ by $\mathcal{U}_{\sigma,0}$. The corresponding three parts in $\mathcal{U}_\sigma$ bounded by the images of $\partial\mathcal{U}_{\sigma,\pm}, \partial\mathcal{U}_{\sigma,0}$ under ι_τ joining with $id|_{\overline{\mathcal{V}}^\epsilon}$ are denoted by $\mathcal{U}_{\sigma,\pm}^\tau$ and $\mathcal{U}_{\sigma,0}^\tau$, respectively. For simplicity, we will use ι_τ to denote for the identification from $\mathcal{U}_{\sigma,0}$ to $\mathcal{U}_{\sigma,0}^\tau$. One can choose smooth loops $\Gamma_\pm^\tau$ enclosing small open neighborhoods of $\mathcal{U}_{\sigma,\pm}^\tau$ disjoint from τ respectively, in a uniform way continuously depending on τ. We denote the open set enclosed by $\Gamma_\pm^\tau$ by $\mathcal{U}^{\Gamma_\pm^\tau}$ respectively. Let $\mathcal{U}_{T_0}^{\Gamma_\pm^\tau}$ be the open sets enclosed by the preimage curves of $\Gamma_\pm^\tau$ under ι_τ, respectively.

Next for any fixed τ, choose symplectomorphisms

$$\psi_{\tau,\pm} : \mathcal{U}^{\Gamma_\pm^\tau} \to \mathcal{U}_{T_0}^{\Gamma_\pm^\tau}, \tag{3.22}$$

which extend to be symplectomorphisms between a small neighborhood of the closure of the corresponding domain and codomain, and which restrict to ι_τ in a tubular neighborhood of $\Gamma_\pm^\tau$. Fix a smooth vector field Y_τ on $\mathcal{U}_\sigma$ supported near $\Gamma_\pm^\tau$, which are nonvanishing and normal to $\Gamma_\pm^\tau$ with respect to the fixed metric.

There exists an open neighborhood of τ in $\mathcal{P}(T_0, \mathcal{U}_\sigma)$, such that for any $\hat{\tau}$ in it, the image of $\iota_\tau^{-1}(\Gamma_\pm^\tau)$ under $\iota_{\hat{\tau}}$ is a curve which is contained in the (enlarged) domain of $\psi_{\tau,\pm}$ and is transverse to the vector field Y_τ (in particular, the curve is contained in the nonvanishing loci of Y_τ). We can shrink the open neighborhood of τ such that under a rescaling of Y_τ by a function[6] on $\mathcal{U}_\sigma$ depending on $\hat{\tau}$, the time-1-flow of the vector field will take $\iota_{\hat{\tau}} \circ \iota_\tau^{-1}(\Gamma_\pm^\tau)$ to $\Gamma_\pm^\tau$ as a *nonparametrized* smooth curve. The rescaling procedure can be made continuously depend on $\hat{\tau}$. The reason for this is that Y_τ defines a local coordinate system near $\Gamma_\pm^\tau$, and for

[6]The function can take zero or negative values.

$\hat{\tau}$ sufficiently close to τ, $\iota_{\hat{\tau}} \circ \iota_{\tau}^{-1}(\Gamma_{\pm}^{\tau})$ looks like a graph of a function on $\Gamma_{\pm}^{\tau}$. In this way, the time-1-flow gives a diffeomorphism from the open set enclosed by $\iota_{\hat{\tau}} \circ \iota_{\tau}^{-1}(\Gamma_{\pm}^{\tau})$ to $\mathcal{U}_{T_0}^{\Gamma_{\pm}^{\tau}}$, which can be made to be $\iota_{\hat{\tau}}^{-1}$ on the boundary, by an easy reparametrization operation. By making further modifications to the diffeomorphism and applying Moser's argument, one can obtain a symplectomorphism $\phi_{\hat{\tau}}$ between the two open regions, which agrees with $\iota_{\hat{\tau}}^{-1}$ in a tubular neighborhood of the boundary. Therefore, $\phi_{\hat{\tau}}^{-1}$ gives an extension of $\iota_{\hat{\tau}}$ (after reducing the original domain) to all of $\mathcal{U}_{\sigma}$. Since this construction is compatible with the flow of X_{σ}, we have a local section near any point in the space on the right-hand-side of (3.20). □

For any $\varphi \in \ker \beta_{G,\overset{\circ}{C}_{\nu}}$, when a point q in any fixed ray in $\overset{\circ}{W}_{01} \cup \overset{\circ}{W}_{12}$ has norm sufficiently large, the induced map of φ on M_q, denoted by φ_q as before, satisfies that the graph of φ_q in $M_q^- \times M_q$ is contained in a tubular neighborhood of the diagonal, which can be identified with a tubular neighborhood of the zero section in $T^*\Delta_{M_q}$, and that it is the graph of the differential of a function (a generating function), denoted by F_{φ_q}. Note that by the trivialization result in Proposition 3.4.6 and Remark 3.4.8, one can have a uniform way of making the above identifications between tubular neighborhoods and to define F_{φ_q} up to a constant.

Now fix a finite covering of open Darboux charts $\{U_{\alpha}\}_{\alpha \in I}$ with Darboux coordinates $x_{\alpha}^{(i)}, i = 1, 2$ of M_q. Then the induced coordinates on Δ_{M_q} extend to natural Darboux coordinate systems near the zero section in $T^*\Delta_{M_q}$, which is identified with a tubular neighborhood of Δ_{M_q} via the Lagrangian neighborhood theorem. More explicitly, on each piece $T^*U_{\alpha} \subset T^*M_q$, we have coordinates $\frac{1}{\sqrt{2}}(x_{\alpha}^{(i)}, x_{\alpha}^{(i)}), i = 1, 2$ on the base and $\frac{1}{\sqrt{2}}(-x_{\alpha}^{(i)}, x_{\alpha}^{(i)}), i = 1, 2$ on the cotangent direction. We fix a tubular neighborhood of the zero section in $T^*\Delta_{M_q}$ once for all, and thus a tubular neighborhood of the zero section in $T^*\Delta_{U_{\alpha}}$ for each α. We will only focus on F_{φ_q} defined as above whose graphs entirely lie in this neighborhood (and will make this implicit in the following), and if the Darboux charts $\{U_{\alpha}\}$ are fine enough, then $|D^2 F_{\varphi_q}| < \frac{5}{6}$ on all $\Delta_{U_{\alpha}}$ is a sufficient condition for the graph of F_{φ_q} to represent the graph of a symplectomorphism of M_q.

Lemma 3.4.17. $\ker \beta_{G,\overset{\circ}{C}_{\nu}}/C^{\infty}(\overset{\circ}{C}_{\nu}, T)$ *is contractible.*

Proof. Fix a parabola-like curve Γ_{ν}^1 contained in $\overset{\circ}{W}_{01} \cap \overset{\circ}{C}_{\frac{1}{2}\nu}$, which is asymptotic to the rays generated by w_1 and $w_1 + \frac{1}{2}\nu \cdot u_1$. Let Γ_{ν}^2 be the parabola-like curve in $\overset{\circ}{W}_{12} \cap \overset{\circ}{C}_{\frac{1}{2}\nu}$ symmetric to Γ_{ν}^1 about $\mathbb{R}_+ \cdot w_1$. Also fix another parabola-like curve $\widetilde{\Gamma}_{\nu}^1$ (resp. $\widetilde{\Gamma}_{\nu}^2$) contained in the open region $\mathcal{V}_{\Gamma_{\nu}^1}$ (resp. $\mathcal{V}_{\Gamma_{\nu}^2}$) bounded by Γ_{ν}^1 (resp. Γ_{ν}^2), which is also asymptotic to the rays generated by w_1 and $w_1 + \frac{1}{2}\nu \cdot u_1$ (resp. $w_1 - \frac{1}{2}\nu \cdot u_1$). Choose a smooth bump function χ_{01} (resp. χ_{12}) with range $[0, 1]$ on $\overset{\circ}{W}_{01}$ (resp. $\overset{\circ}{W}_{12}$) such that $\chi_{01} = 1$ (resp. $\chi_{12} = 1$) on the locus bounded by $\widetilde{\Gamma}_{\nu}^1$, (resp. $\widetilde{\Gamma}_{\nu}^2$) and $\chi_{01} = 0$ (resp. $\chi_{12} = 0$) in a tubular neighborhood of Γ_{ν}^1 (resp. Γ_{ν}^2).

As we view symplectomorphisms of M_q that are sufficiently close to the identity as the graph of the differential of a function F in the fixed tubular neighborhood of $T^*\Delta_{M_q}$ such that $|D^2F_{\varphi_q}| < \frac{1}{2}$ on all Δ_{U_α}, we can define a smooth bump function κ_q with range $[0,1]$ on this locus such that $\kappa_q(F) = 0$ for all F with $|D^2F| > \frac{1}{3}$ on some Δ_{U_α}, and $\kappa_q = 1$ on the locus where $|D^2F| \leq \frac{1}{4}$ on all Δ_{U_α}.

For a fixed $\varphi \in \ker\beta^\sharp_G$, consider the set

$$\mathbb{D}_{\varphi,\nu}(\frac{1}{4}) =: \{q \in \mathcal{V}_{\Gamma^1_\nu} \cup \mathcal{V}_{\Gamma^2_\nu} : |D^2F_{\varphi_q}| < \frac{1}{4} \text{ on all } \Delta_{U_\alpha}\}.$$

Choose another smooth function

$$g : (-\epsilon, 1+\epsilon)^2 \to [0,1],$$

such that $g|_{[\frac{2}{3},1]^2} = 1$ and $g(x_1,x_2) = 0$ if $x_1 \leq \frac{2}{3}$ or $x_2 \leq \frac{2}{3}$, where $\epsilon > 0$ is any small number.

We define a deformation retraction on $\ker \beta_{G,\overset{\circ}{C}_\nu}$, which is supported on $\mathbb{D}_{\varphi,\nu}(\frac{1}{4})$ as follows. We need to specify where F_{φ_q} goes at time $s \in [0,1]$. On $\mathbb{D}_{\varphi,\nu}(\frac{1}{4}) \cap \mathcal{V}_{\Gamma^1_\nu}$ (resp. $\mathbb{D}_{\varphi,\nu}(\frac{1}{4}) \cap \mathcal{V}_{\Gamma^2_\nu}$), F_{φ_q} is sent to $(1 - g(\kappa_q, \chi_{01})) \cdot s)F_{\varphi_q}$ (resp. $(1 - g(\kappa_q, \chi_{12})) \cdot s)F_{\varphi_q}$) at time s. This is clearly supported on the closure of an open set inside $\mathbb{D}_{\varphi,\nu}(\frac{1}{4})$, so trivially extends to all of $\overset{\circ}{C}_\nu$. Let $C_{\frac{1}{6}\nu,\frac{1}{3}\nu}$ be the cone bounded by $w_1 + \frac{1}{6}\nu$ and $w_1 + \frac{1}{3}\nu$, and $C_{-\frac{1}{6}\nu,-\frac{1}{3}\nu}$ be the cone symmetric to $C_{\frac{1}{6}\nu,\frac{1}{3}\nu}$ about $\mathbb{R}_+ \cdot w_1$. Using the $\mathbb{R}_+$-action, one can make a further deformation retraction onto the space of φ such that $\varphi_q = id$ for all $q \in (C_{\frac{1}{6}\nu,\frac{1}{3}\nu} \cup C_{-\frac{1}{6}\nu,-\frac{1}{3}\nu}) \cap \{\|q\| \geq 1\}$, for a fixed norm $\|\cdot\|$ on the Lie algebra.

Let

$$\mathrm{Sympl}^\dagger(M_q, \{Q_j\}_{j=1}^3) =: \mathrm{Sympl}^\dagger(M_p, \{Q_j\}_{j=1}^3),$$

under a fixed identification between M_p and M_q as in Remark 3.4.8. Let $\Omega^\dagger_\lambda(\mathrm{Sympl}(M_q))$ be the subspace of $\Omega(\mathrm{Sympl}^\dagger(M_q, \{Q_j\}_{j=1}^3))$ consisting of loops

$$\ell : [-1,1]/(-1 \sim 1) \longrightarrow \mathrm{Sympl}^\dagger(M_q, \{Q_j\}_{j=1}^3),$$

such that $\ell|_{[1-\lambda,1]\cup[-1,-1+\lambda]} = id$, and $\ell|_{(-\epsilon,\epsilon)}$ is constant, for some $\epsilon > 0$ depending on ℓ. Combining the above deformation retraction with the deformation retraction constructed similarly as in the proof of Lemma 3.20 and Proposition 3.4.12, we can describe the resulting space as the space of paths

$$\varrho : (0,\infty) \longrightarrow \mathcal{P}(\mathrm{Sympl}^\dagger(M_q, \{Q_j\}_{j=1}^3)),$$

such that $\varrho(t)$ is the constant identity path if t is sufficiently small, and $\varrho(t) \in \Omega^\dagger_{\frac{1}{2}t}(\mathrm{Sympl}(M_q))$ for $t \geq 1$. Since $\mathrm{Sympl}^\dagger(M_q, \{Q_j\}_{j=1}^3)$ is contractible, the space of ϱ is contractible as well. □

Proposition 3.4.18. $\ker\beta^\sharp_G$ *is contractible.*

Proof. In the following, we use p to denote for any fixed element in $\mathbb{R}_+ \cdot w_0$.

Step 1. *A deformation retraction of* $\ker \beta_G^\sharp$ *supported on (the G-orbits of)* $\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon}))$

Note that $\mathbb{R}_+ \cdot w_0$ can be viewed as the image of the moment map μ_{w_0} for the $T^{\frac{1}{3}w_0}$-action on $\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon}))$. Since $\varphi|_{\mu^{-1}(p)}$ is the right multiplication by an element in $U(2)/\mu_3$, $\varphi|_{\mu^{-1}(\mathbb{R}_+\cdot w_0)}$ corresponds to a loop in $U(2)/\mu_3$. On the other hand, given any C^∞-map $\Lambda : \mathbb{R}_+ \cdot w_0 \to U(2)/\mu_3$ with $\Lambda(s) = 1$ when s is close to 0, and $\lim_{s\to\infty} \Lambda(s) = 1$, the following construction gives a *unique* $U(2)$-equivariant symplectomorphism ϕ_Λ on $\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon}))$, such that $\phi_\Lambda|_{\mu^{-1}(p)} = R_{\Lambda(p)}, p \in \mathbb{R}_+ \cdot w_0$, where $R_{\Lambda(p)}$ means the right multiplication by $\Lambda(p)$. One starts with the automorphism whose restriction to $\mu_{w_0}^{-1}(p)$ is $R_{\Lambda(p)}$, and modifies it to be a $U(2)$-equivariant symplectomorphism, by applying similar argument as in Corollary 3.4.2 and using the fact that the form $d\beta_\tau|_{\mu^{-1}(p)} = 0$ here.

Now let $\Lambda_\varphi : \mathbb{R}_+ \cdot w_0 \to U(2)/\mu_3$ be the loop defined by φ, then $\phi_{\Lambda_\varphi}^{-1}\varphi$ becomes the identity on $\mu^{-1}(\mathbb{R}_+ \cdot w_0)$. Now by Lemma 3.4.3, and the fact that $T^*(U(2)/\mu_3)$ has a natural Kahler structure, we can run the same argument as in Lemma 3.4.15 to make $\phi_{\Lambda_\varphi}^{-1}\varphi$ isotopic to the identity in a neighborhood of $\mu^{-1}(\mathbb{R}_+ \cdot w_0)$. Then composing it back with ϕ_{Λ_φ}, we get a symplectomorphism of $\mu^{-1}(\mathrm{Ad}_{G_p}(\overset{\circ}{W}_{\pm\epsilon}))$, which agrees with φ near the boundary and is ϕ_{Λ_φ} near $\mu^{-1}(\mathbb{R}_+ \cdot w_0)$. Since φ is very close to the identity map near the infinity, the region of ϕ_{Λ_φ} contains a conical neighborhood of $\mu^{-1}(\mathbb{R}_+ \cdot w_0)$, and we can push it to contain a fixed conical neighborhood of $\mu^{-1}(\mathbb{R}_+ \cdot w_0)$, say $\mu^{-1}(\overset{\circ}{W}_{\pm\frac{\epsilon}{2}})$. Similarly, we can deform φ over $\mu^{-1}(\mathrm{Ad}_{G_{w_2}}(\overset{\circ}{W'}_{\pm\frac{\epsilon}{2}}))$ in the same way, where $W'_{\pm\frac{\epsilon}{2}}$ is the cone bounded by $\mathbb{R}_{\geq 0} \cdot (w_2 \pm \frac{\epsilon}{2} \cdot \mathrm{diag}(0, 1, -1))$.

From now on, we can restrict ourselves to the space of $\varphi \in \ker \beta_G^\sharp$ where φ restricted to $\mu^{-1}(\mathrm{Ad}_{G_{w_0}}(\overset{\circ}{W}_{\pm\frac{\epsilon}{2}}))$ is ϕ_{Λ_φ} and it has similar behavior over $\mu^{-1}(\mathrm{Ad}_{G_{w_2}}(\overset{\circ}{W'}_{\pm\frac{\epsilon}{2}}))$. For simplicity, we still denote this space by $\ker \beta_G^\sharp$. Let $\mathcal{P}_*(\Omega(\mathrm{Sympl}(S^2)))$ be the based path space of the based loop space $\Omega(\mathrm{Sympl}(S^2))$, i.e. the space of maps $\rho : [0, 1] \to \Omega(\mathrm{Sympl}(S^2))$, with $\rho(0)$ being the constant identity loop.

Step 2. *A fibration* $\ker \beta_G^\sharp / C^\infty(\overset{\circ}{W}, T) \to \ker \beta_{G,\overset{\circ}{C}_\nu} / C^\infty(\overset{\circ}{C}_\nu, T)$ *with contractible fibers.*

First, we replace W_{01} (resp. W_{12}) by a proper subcone bounded by $\mathbb{R}_{\geq 0}\cdot w_0$ (resp. $\mathbb{R}_{\geq 0}\cdot w_2$) and a ray in $\overset{\circ}{C}_\nu \cap \overset{\circ}{W}_{01}$ (resp. $\overset{\circ}{C}_\nu \cap \overset{\circ}{W}_{01}$). The natural map

$$\begin{array}{rccc} Res_\nu : \ker \beta_G^\sharp / C^\infty(\overset{\circ}{W}, T) & \to & \ker \beta_{G,\overset{\circ}{C}_\nu} / C^\infty(\overset{\circ}{C}_\nu, T) \\ \varphi & \mapsto & \varphi|_{\mu^{-1}(\overset{\circ}{C}_\nu)} \end{array}$$

is surjective. In fact, once we know its restriction to $\mu^{-1}(\overset{\circ}{C}_\nu)$, any $\varphi \in \ker \beta_G^\sharp$ can be constructed and is determined by a C^∞-map from $\overset{\circ}{W}_{01}$ to $\mathrm{Sympl}(S^2)$ and a map from $\overset{\circ}{W}_{12}$ to $\mathrm{Sympl}(S^2)$, subject to the obvious matching conditions, the constraints near $\mathrm{Ad}_{G_{w_0}}(\overset{\circ}{W}_{\pm\frac{\epsilon}{2}})$

and $\mathrm{Ad}_{G_{w_2}}(\overset{\circ}{W}{}'_{\pm\frac{\epsilon}{2}})$ as in Step 2, and that the maps converge to $\mathrm{id} \in \mathrm{Sympl}(S^2)$ at infinity. Thus, the pair of maps corresponds exactly to a pair of paths in the identity component of the based loop space $\Omega_0(\mathrm{Sympl}(S^2))$, with the initial condition given and the end points lying in $\Omega(SO(3))$ ($SO(3)$ comes from $U(2)/\mu_3$ modulo the center). Since $\mathrm{Sympl}(S^2)$ is homotopy equivalent to $SO(3)$, we see that Res_ν is a fibration with contractible fibers.

In fact, Res_ν is a fiber bundle with a global section based on a strong deformation retraction from $\mathrm{Diff}^+(S^2)$ to $SO(3)$ established in [30]. More precisely, given such a deformation retraction, and a deformation retraction from $\mathrm{Diff}^+(S^2)$ to $\mathrm{Sympl}(S^2)$, one can construct a deformation retraction from $\mathrm{Sympl}(S^2)$ to $SO(3)$. Using the latter deformation retraction, there is an obvious way to construct a global section of Res_ν.

Step 3. $\ker \beta^\sharp_G$ *is contractible.*

From the previous steps, we see that $\ker \beta^\sharp_G / C^\infty(\overset{\circ}{W}, T)$ is contractible. Now we show that the fiber over id for the projection $\ker \beta^\sharp_G \to \ker \beta^\sharp_G / C^\infty(\overset{\circ}{W}, T)$ is again contractible. Any φ lying in the kernel corresponds to a path $\rho_\varphi : [0,1] \to \Omega(T^2)$. However, there are strong constraints on the end points of such paths: φ induces the identity on the reduced spaces near $\mathbb{R}_+ \cdot w_0$ and $\mathbb{R}_+ \cdot w_2$, therefore $\rho_\varphi(0) \in \Omega(T^{\frac{1}{3}w_0})$ and $\rho_\varphi(1) \in \Omega(T^{\frac{1}{3}w_2})$, and these force ρ_φ to lie in $\Omega^2(T^2)$. Since $\Omega^2(T^2)$ is contractible, we can conclude that the fiber is contractible.

Similarly as before, the quotient map $\ker \beta^\sharp_G \to \ker \beta^\sharp_G / C^\infty(\overset{\circ}{W}, T)$ is in fact a fiber bundle with a global section, thus $\ker \beta^\sharp_G$ is contractible. To see this, one first horizontally lift, with respect to the connection form α, the (Hamiltonian) vector field on the reduced space at τ, coming from differentiating φ_τ in the conical direction, and then integrating the resulting vector field along the conical direction gives a T-equivariant diffeomorphism. This procedure is well-defined after the previous steps of deformation retraction of $\ker \beta^\sharp_G$. Then applying the equivariant version of Moser's argument to the diffeomorphism, we get a T-equivariant symplectomorphism which is a lifting of $\{\varphi_\tau\}_{\tau \in \overset{\circ}{W}}$.

□

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