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Support in Tensor Triangular Geometry

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SUPPORT IN TENSOR TRIANGULAR GEOMETRY

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by

Changhan Zou

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Abstract

Support in tensor triangular geometry

by

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For a rigidly-compactly generated tensor triangulated category, we construct a support theory valued in the homogeneous Zariski spectrum of the graded endomorphism ring of the unit object, extending earlier work of Benson–Iyengar–Krause by removing the Noetherian hypothesis on the endomorphism ring. This allows us to define a non-Noetherian stratification theory. As an example, we show that the stable module category of a finite group is stratified by the canonical action of the Tate cohomology ring, which is typically non-noetherian.

We also compare this support theory with the extension of the Balmer–Favi support by Sanders, which takes values in the Balmer spectrum of the category. When the comparison map from the Balmer spectrum to the homogeneous Zariski spectrum of the endomorphism ring is a homeomorphism, the two support theories agree, as do their corresponding stratification theories.

As an application of this comparison, we show that the dualizable localizing ideals of a rigidly-compactly generated tt-∞-category are classified by the convex subsets of the Balmer spectrum of the category, assuming that the category is locally cohomologically stratified. This generalizes a recent classification theorem of Efimov.

To my parents

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Chapter 1

Introduction*

Tensor triangular geometry provides a unifying framework for understanding the structure of tensor triangulated categories in algebra and topology. Analogous to the classical Hilbert's Nullstellensatz in algebraic geometry, which sets up a correspondence between closed subsets of an affine variety and radical ideals of its coordinate ring, the fundamental theorem of tensor triangular geometry [Bal05] states that radical thick ideals of an essentially small tensor triangulated category $\mathcal{K}$ correspond bijectively to certain subsets of the Balmer spectrum $\operatorname{Spc}(\mathcal{K})$: There is an inclusion-preserving bijection

$$\{\text{radical thick ideals of } \mathcal{K}\} \xrightarrow{\sim} \{\text{Thomason subsets of } \operatorname{Spc}(\mathcal{K})\}.$$

This abstract classification theorem recovers and unifies classical results in stable homotopy theory [HS98], algebraic geometry [Tho97], and modular representation theory [BCR97].

Key to Balmer's theorem is the notion of *support*. Each object $a \in \mathcal{K}$ has a support which is a closed subset in the spectrum $\operatorname{Spc}(\mathcal{K})$ defined as

$$\operatorname{supp}(a) := \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \,\middle|\, a \notin \mathcal{P} \right\},$$

*This dissertation is based on the author's previous papers [Zou25a, Zou25b].

the set of prime ideals not containing a. The bijection in Balmer's theorem sends a radical thick ideal $\mathcal{I}$ to its support $\mathrm{supp}(\mathcal{I}) \coloneqq \bigcup_{a\in\mathcal{I}} \mathrm{supp}(a)$. Thus, two objects generate the same radical thick ideal if and only if they have the same support. Therefore, by computing the supports, we obtain a complete classification of objects in $\mathcal{K}$ up to tensor triangular equivalence.

Moreover, Balmer established a universal property for the space $\mathrm{Spc}(\mathcal{K})$ together with the support supp, which shows that if there is any space X with an assignment σ of a closed subset of X to each object in $\mathcal{K}$ that classifies the radical thick ideals of $\mathcal{K}$ in the way above, then X must be homeomorphic to $\mathrm{Spc}(\mathcal{K})$ and σ coincides with supp under this homemorphism. This indicates that one should consider only the Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ and support supp when one wants to classify objects of $\mathcal{K}$ up to tensor triangular equivalence.

For example, consider $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(R)$, the perfect derived category of a commutative ring R. The cohomological support of a complex $Z \in \mathrm{D}^{\mathrm{perf}}(R)$ is defined as

$$\left\{ \mathfrak{p} \in \mathrm{Spec}(R) \,\middle|\, H^*(Z)_{\mathfrak{p}} \neq 0 \right\}.$$

Hopkins [Hop87] and Neeman [Nee92] showed that the cohomological support induces an inclusion-preserving bijection

$$\{\text{radical thick ideals of } \mathrm{D}^{\mathrm{perf}}(R)\} \xrightarrow{\sim} \{\text{Thomason subsets of } \mathrm{Spec}(R)\}.$$

In other words, the Zariski spectrum $\mathrm{Spec}(R)$ is recovered by the Balmer spectrum $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R))$ and the cohomological support is an instance of the Balmer support supp.

Balmer's work provides a satisfactory solution to the problem of classifying objects in an essentially small tensor triangulated category $\mathcal{K}$ with respect to its tensor triangular structure. However, one often encounters $\mathcal{K}$ as the subcategory $\mathcal{T}^c$ of compact objects of a larger tensor triangulated category $\mathcal{T}$. For example, the category $\mathrm{D}^{\mathrm{perf}}(R)$ sits inside the unbounded derived category $\mathrm{D}(R)$ as the full subcategory of compact objects. A natural question, then, is how to classify the objects of $\mathcal{T}$ with respect to its tensor triangular structure.

Indeed, Neeman answered this question for $\mathrm{D}(R)$ when R is Noetherian. For any complex $Z \in \mathrm{D}(R)$, Foxby's small support of Z is defined as

$$\left\{ \mathfrak{p} \in \mathrm{Spec}(R) \,\middle|\, Z \otimes \kappa(\mathfrak{p}) \neq 0 \right\}$$

where $\otimes$ is the usual derived tensor product and $\kappa(\mathfrak{p})$ denotes the residue field regarded as a stalk complex. In [Nee92], it is proved that Foxby's small support induces an inclusion-preserving bijection

$$\{\text{localizing ideals of } \mathrm{D}(R)\} \xrightarrow{\sim} \{\text{subsets of } \mathrm{Spec}(R)\}.$$

Unfortunately, Neeman [Nee00] also gave an example of a non-Noetherian ring such that the classification of localizing ideals fails. Therefore, for a general $\mathcal{T}$, we should not expect an unconditional classification of the localizing ideals via the subsets of the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$. Nevertheless, there are several support theories on $\mathcal{T}$ developed under certain Noetherian type assumption, and they have been used to successfully establish classification of localizing ideals of tensor triangulated categories other than $\mathrm{D}(R)$.

For example, inspired by the work of Neeman [Nee92] and Hovey–Palmieri–Strickland [HPS97], Benson, Iyengar, and Krause [BIK08, BIK11a, BIK11b] developed a theory of support for objects in a compactly generated triangulated category $\mathcal{T}$ equipped with an action of a graded-commutative Noetherian graded ring R. The BIK support $\mathrm{Supp}_{\mathrm{BIK}}$ assigns to every object $t \in \mathcal{T}$ a subset of the homogeneous Zariski spectrum $\mathrm{Spec}^{\mathrm{h}}(R)$. It, therefore, induces a map

$$\{\text{localizing ideals of } \mathcal{T}\} \xrightarrow{\mathrm{Supp}_{\mathrm{BIK}}} \{\text{subsets of } \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T})\}$$

where $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T}) \coloneqq \bigcup_{t\in\mathcal{T}} \mathrm{Supp}_{\mathrm{BIK}}(t) \subseteq \mathrm{Spec}^{\mathrm{h}}(R)$ is called the space of supports. When this map is a bijection, we say that $\mathcal{T}$ is *BIK-stratified by R*. With this machinery, BIK managed to classify the localizing ideals of the stable module category $\mathrm{StMod}(kG)$ of a finite group G over a field k, equipped with an action by the group cohomology ring $H^*(G,k)$. In short, $\mathrm{StMod}(kG)$ is BIK-stratified by $H^*(G,k)$; see [BIK11a].

In [BF11], Balmer–Favi defined a notion of support for a rigidly-compactly generated tensor triangulated category $\mathcal{T}$. Instead of relying on an auxiliary Noetherian ring action on $\mathcal{T}$, their support takes values in the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ which is required to satisify a topological assumption called weakly Noetherian. Later on, Barthel–Heard–Sanders [BHS23b] established a theory of stratification using the Balmer–Favi support. More precisely, $\mathcal{T}$ is said to be BHS-stratified (or simply, stratified) if the Balmer–Favi support induces an inclusion-preserving bijection

$$\{\text{localizing ideals of } \mathcal{T}\} \xrightarrow{\mathrm{Supp}_{\mathrm{BF}}} \{\text{subsets of } \mathrm{Spc}(\mathcal{T}^c)\}.$$

In [BHS23b], many categories in algebra and topology were shown to be stratified.

Our goal in this dissertation is to study the BIK support and the BF support beyond the Noetherian situation. In [San17], Sanders gave a generalization of the BF support which removes the weakly Noetherian assumption on $\mathrm{Spc}(\mathcal{T}^c)$. We call the support defined by Sanders the BFS support and denote it by Supp. Inspired by this work, we propose a generalization of the BIK support which does not require the ring R acting on the category to be Noetherian. For any rigidly-compactly generated tensor triangulated category $\mathcal{T}$, the endomorphism ring $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ of the unit object canonically acts on $\mathcal{T}$. Therefore, we obtain a canonical BIK support for $\mathcal{T}$ without any additional choice of ring action or any Noetherian hypothesis. We then study how the canonical BIK support and the BFS support relate to each other. By the work of Balmer [Bal10], there is a natural comparison map

$$\rho\colon \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$$

between the spaces in which the BFS and BIK support theories take values. We prove the following:

Theorem A (Theorem 4.2.8)**.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category such that ρ is a homeomorphism. Then $\rho(\mathrm{Supp}(t)) = \mathrm{Supp}_{\mathrm{BIK}}(t)$ for any $t \in \mathcal{T}$.*

We also discuss the two associated stratifcation theories. A surprising result (Theorem 4.1.14) is that if $\mathcal{T}$ is stratified by the BFS support then $\mathrm{Spc}(\mathcal{T}^c)$ is necessarily weakly Noetherian. Therefore, the BFS support cannot provide stratified examples beyond the weakly Noetherian context as discussed in [BHS23b]. However, this support still gives us interesting information for categories whose Balmer spectra are not weakly Noetherian; see Section 3.3. On the other hand, stratifcation by the canonical BIK support does provide us with a non-Noetherian example. In BIK's stratification of $\mathrm{StMod}(kG)$, the action of $H^*(G,k)$ is not the canonical action by the graded endomorphism ring $\mathrm{End}^*_{\mathrm{StMod}(kG)}(k)$ of the unit object. The latter is the Tate cohomology ring $\hat{H}^*(G,k)$, which is typically non-Noetherian. Applying our generalized BIK machinery, we show that $\mathrm{StMod}(kG)$ is indeed BIK-stratified by the Tate cohomology ring $\hat{H}^*(G,k)$ (Theorem 5.1.2).

As an application of the comparison theorem, we explore the notion of convexity in tensor triangular geometry. We prove that for any rigidly-compactly generated tt-∞-category $\mathcal{C}$ that is nicely stratified, the BFS support induces a bijection between the dualizable localizing ideals of $\mathcal{C}$ and the convex subsets of $\mathrm{Spc}(\mathcal{C}^c)$:

Theorem B (Theorem 5.2.17)**.** *Let $\mathcal{C}$ be a rigidly-compactly generated tt-∞-category with $\mathrm{Spc}(\mathcal{C}^c)$ Noetherian. If $\mathcal{C}$ is locally cohomologically stratified then the BFS support induces a bijection*

$$\{\textit{dualizable localizing ideals of } \mathcal{C}\} \xrightarrow{\sim} \{\textit{convex subsets of } \mathrm{Spc}(\mathcal{C}^c)\}.$$

Moreover, in this case, the dualizable localizing ideals of $\mathcal{C}$ are themselves compactly generated.

See Definition 5.2.12 for the definition of locally cohomologically stratified. Theorem B is motivated by Efimov's recent result [Efi24, Theorem 3.10] which treats the special case $\mathcal{C} = \mathrm{D}(R)$ for a commutative Noetherian ring R. However, not all arguments in his proof extend to the tensor triangular setting. A key ingredient is [Efi24, Lemma 3.12], which shows that, under certain assumptions, a complex in $\mathrm{D}(R)$ is perfect. In our setting, the analogous conclusion is that an object $t \in \mathcal{C}$ is compact under similar hypotheses. As our proof of Theorem B demonstrates, the crucial point is that the BFS support of t is specialization closed. Using Theorem A, we establish this property by invoking results on the BIK support from [BIK08].

Outline of the dissertation

Chapter II starts with a review of spectral space. We then define a new notion of small support for modules, based on the notion of weakly associated prime. This will be the key to the proof of Theorem A. We finish this chapter with a review of tensor triangular geometry and related terminology.

Chatper III deals with support theories. We first develop our general BIK support. After a quick review of the BFS support, we study the detection property in terms of this support, showing that the detection property can be checked locally in an algebraic sense.

In Chapter IV, we consider the stratification theory based on the BFS support. In particular, we show that stratification by the BFS support forces the Balmer spectrum to be weakly Noetherian. We then prove Theorem A.

Chapter V is devoted to applications. We first show that the stable module category canonically BIK-stratified by its Tate cohomology ring, and then prove Theorem B.

Chapter 2

Preliminaries

2.1 Spectral spaces

We start by recalling some basic concepts concerning spectral spaces, which will be used throughout this thesis.

2.1.1 Definition. Let X be a spectral space in the sense of [DST19]. The *specialization order* on X is defined by $x \leq y$ if and only if y is a specialization of x, that is, $y \in \overline{\{x\}}$.

2.1.2 Definition. Let S be a subset of a spectral space X. We say that S is *specialization closed* if for all $x \in S$ and every specialization $x \leq y$ in X, we have $y \in S$. We say that S is *generalization closed* if for all $y \in S$ and every specialization $x \leq y$ in X, we have $x \in S$.

2.1.3 Remark. Note that the specialization closed subsets are exactly the unions of closed subsets. By definition, the generalization closed subsets are precisely the complements of the specialization closed subsets.

2.1.4 Definition. Let X be a spectral space. A subset S of X is said to be *convex* if it is convex with respect to the specialization order on X, that is

$$x \leq y \leq z \text{ and } x, z \in S \implies y \in S.$$

2.1.5 Remark. Specialization closed subsets are clearly convex. More generally, we have:

2.1.6 Lemma. *A subset S of a spectral space X is convex if and only if there exist specialization closed subsets $S_2 \subseteq S_1$ of X such that $S = S_1 \setminus S_2$.*

Proof. For the "if" part, suppose $x \leq y \leq z$ in X such that $x, z \in S = S_1 \setminus S_2$. Since S_1 is specialization closed and contains x, it also contains y. Since the complement of S_2 is generalization closed (recall Remark 2.1.3) and contains z, it also contains y. Hence, $y \in S_1 \setminus S_2 = S$. For the "only if" part, we can take S_1 to be the specialization closure of S, that is, $\bigcup_{x \in S} \overline{\{x\}}$, and S_2 to be $S_1 \setminus S$. To see that S_2 is specialization closed, consider a specialization $y \leq z$ with $y \in S_2$. Since $y \in S_1 = \bigcup_{x \in S} \overline{\{x\}}$, we have $z \in S_1$ and there exists some $x \in S$ with $x \leq y$. Hence, if $z \notin S_2$, we must have $z \in S$, but then the convexity of S forces $y \in S$, which is absurd. Thus, $z \in S_2$. This establishes that S_2 is specialization closed. $\square$

2.1.7 Remark. The first part of the proof demonstrates that the set difference $S_1 \setminus S_2$ is convex for any two specialization closed subsets, regardless of whether $S_2 \subseteq S_1$.

2.1.8 Definition. Let X be a spectral space. A subset of X is said to be *Thomason* if it is a union of closed subsets, each of which has quasi-compact complement. The Thomason subsets form the open subsets of a dual spectral topology on X called the *Hochster dual topology**. We write X^* for X equipped with the Hochster

*The Hochster dual topology is called the *inverse topology* in [DST19].

dual topology.

2.1.9 Remark. Every Thomason subset is specialization closed, but not every specialization closed subset is Thomason. However, if X is a Noetherian topological space, then the Thomason subsets coincide with the specialization closed subsets.

2.1.10 Definition. Let X be a spectral space. A subset W of X is said to be *weakly visible* if there exist Thomason subsets U and V such that $W = U \cap V^c$. In particular, we say a point $x \in X$ is weakly visible if the singleton $\{x\}$ is weakly visible. The spectral space X is said to be *weakly Noetherian* if every point of X is weakly visible.

2.1.11 Remark. Every weakly visible subset is convex; the converse holds if X is Noetherian. This follows from Remark 2.1.7 and Remark 2.1.9.

2.1.12 Example. Every Noetherian spectral space and every profinite space is weakly Noetherian; see [BHS23b, Remarks 2.2 and 2.4].

2.1.13 Remark. Let X be a spectral space. A Thomason subset of X is a union of Thomason closed subsets of X. This follows from [San13, Lemma 3.3] which states that the Thomason closed subsets are precisely the closed subsets with quasi-compact complements. As a consequence, the closure $\overline{\{y\}}$ of a point $y \in X$ is the intersection of all Thomason closed subsets containing y, since the quasi-compact open subsets form a basis for the spectral topology. Therefore, for any point $x \in X$, the generalization closure

$$\operatorname{gen}(x) \coloneqq \left\{ y \in X \,\middle|\, y \leq x \right\}$$

is the complement of the largest Thomason subset not containing x. Given the discussion above, we see that a point $x \in X$ is weakly visible if and only if

$$\{x\} = Z \cap \operatorname{gen}(x)$$

for some Thomason closed subset Z.

2.1.14 Remark. The notion of a weakly visible subset leads to the following definition introduced by Sanders [San17]:

2.1.15 Definition. Let X be a spectral space. The weakly visible subsets of X form a basis of open subsets for a topology on X called the *localizing topology*. We write X_{loc} for X equipped with the localizing topology, and for any subset S of X we write $\overline{S}^{\text{loc}}$ for the closure of S in X_{loc}.

2.1.16 Remark. Note that a spectral space X is weakly Noetherian if and only if its localizing topology is discrete. We will call a subset of X *localizing closed* if it is closed with respect to the localizing topology. Thus, X is weakly Noetherian if and only if every subset of X is localizing closed.

2.1.17 Remark. Recall that the *constructible topology* on a spectral space X is the topology generated by the sets $U \cap V^c$ with U and V Thomason *closed* subsets of X. We write X_{con} for X equipped with the constructible topology. From the definitions we see that the localizing topology is finer than the constructible topology. Note that the constructible topology is discrete if and only if the spectral space is finite; see [DST19, Example 1.3.12 and Theorem 1.3.14]. Therefore, any infinite weakly Noetherian spectral space provides an example whose localizing topology is strictly finer than the constructible topology. An explicit example is given as follows.

2.1.18 Example. Let S^+ denote the one-point compactification of a discrete infinite space S. We denote the point at infinity by ∞. Observe that the space S^+ is a profinite space. We now define a partial order on S^+ by

$$s \leq t \iff s = \infty \text{ or } s = t \in S.$$

This is a spectral order and hence yields a Priestly space whose associated spectral space is denoted by S_∞; see [DST19, 1.6.13] for details. Note that the constructible topology on S_∞ coincides with the original topology on S^+. However, by [DST19, 1.6.15(iv) and (v)] the localizing topology on S_∞ is discrete and is therefore strictly finer than the constructible topology on S_∞.

2.1.19 Remark. On the other hand, there are examples where the localizing and constructible topologies coincide:

2.1.20 Example. Consider the space $\mathbb{N} \cup \{\infty\}$ of extended natural numbers whose nonempty open subsets are of the form $[n, \infty]$ for $n \in \mathbb{N}$. This is a spectral space and we denote its Hochster dual by X. The space X coincides with the Balmer spectrum $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ of the p-local stable homotopy category $\mathrm{SH}_{(p)}$; see [HS98] and [Bal10, Corollary 9.5]. Since every Thomason subset of X is closed, the localizing topology coincides with the constructible topology.

2.1.21 Remark. In general, the localizing topology is not a spectral topology. In fact, the localizing topology is quasi-compact if and only if it coincides with the constructible topology. Indeed, this follows from the fact that a continuous surjection from a quasi-compact space to a Hausdorff space is a topological quotient, in light of the continuous bijection $X_{\mathrm{loc}} \to X_{\mathrm{con}}$.

2.2 Support for modules

We now introduce a notion of small support for graded modules over graded-commutative graded rings. This will be used to develop a (non-Noetherian) stratification theory in the sense of [BIK11a, BIK11b] The key point is that we do not assume that the rings are Noetherian. Our definition relies on weakly associated primes, which behave better than associated primes in the absence of the

Noetherian assumption.

2.2.1 Notation. For this section, R will denote a $\mathbb{Z}$-graded graded-commutative ring. Ideals and modules will always be graded. The abelian category of R-modules and degree-zero homomorphisms will be denoted $\operatorname{Mod}(R)$. We write $\operatorname{Spec}^{\mathrm{h}}(R)$ for the homogeneous Zariski spectrum of R. It is a spectral space. For any subset S of $\operatorname{Spec}^{\mathrm{h}}(R)$, we write $\operatorname{cl}(S)$ for the specialization closure of S. Given any ideal $\mathfrak{a}$ of R and any prime ideal $\mathfrak{p}$ in $\operatorname{Spec}^{\mathrm{h}}(R)$, we write $\mathcal{V}(\mathfrak{a})$ for the set of prime ideals containing $\mathfrak{a}$ and $\operatorname{gen}(\mathfrak{p})$ for the generalization closure of $\mathfrak{p}$. The complement of $\operatorname{gen}(\mathfrak{p})$ is denoted by $\mathcal{Z}(\mathfrak{p})$. We refer the reader to [BH98, Section 1.5] and [DS13, Section 2] for more on graded commutative algebra.

2.2.2 Definition. The *big support* of an R-module M is defined as

$$\operatorname{Supp}_R M := \left\{ \mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R) \,\middle|\, M_{\mathfrak{p}} \neq 0 \right\}$$

where $M_{\mathfrak{p}}$ is the graded localization of M at $\mathfrak{p}$.

2.2.3 Definition. Let M be an R-module. A prime $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R)$ is said to be *associated* to M if there exists a homogeneous element m in M such that $\mathfrak{p} = \operatorname{Ann}(m)$, the annihilator of m. We denote the set of associated primes of M by $\operatorname{Ass}(M)$. More generally, a prime $\mathfrak{p}$ is said to be *weakly associated* to M if there exists a homogeneous element m in M such that $\mathfrak{p}$ is minimal among the primes containing $\operatorname{Ann}(m)$. The set of weakly associated primes of M is denoted by $\operatorname{WeakAss}(M)$.

2.2.4 Lemma. *Let M be an R-module and $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R)$. The following hold:*

(a) $\operatorname{Ass}(M) \subseteq \operatorname{WeakAss}(M) \subseteq \operatorname{Supp}_R M$. *If R is Noetherian then we have* $\operatorname{Ass}(M) = \operatorname{WeakAss}(M)$.

(b) $M = 0$ *if and only if* $\operatorname{WeakAss}(M) = \varnothing$*.*

(c) $\operatorname{WeakAss}(M_{\mathfrak{p}}) = \operatorname{WeakAss}(M) \cap \operatorname{gen}(\mathfrak{p})$*.*

(d) $\operatorname{cl}(\operatorname{WeakAss}(M)) = \operatorname{Supp}_R M$*.*

Proof. For parts (a), (b), and (c), the proofs in [Sta23, Lemma 0589, Lemma 058A, Lemma 0588, Lemma 05C9] carry over to the graded setting. Part (d) is a direct consequence of (b) and (c) since $\operatorname{Supp}_R M$ is always specialization closed. See also [Bou98, IV, Exercise 17, page 289]. □

2.2.5 Remark. When R is not Noetherian, there can exist a nonzero R-module M such that $\operatorname{Ass}(M) = \varnothing$; see [Sta23, Remark 05BX], for example. We now define the (small) support for modules:

2.2.6 Definition. The *support* of an R-module M is the set

$$\operatorname{supp}_R M \coloneqq \operatorname{WeakAss}(M).$$

2.2.7 Remark. From Lemma 2.2.4(b) we see that the support detects vanishing of a module:

$$\operatorname{supp}_R M = \varnothing \iff M = 0. \tag{2.2.8}$$

2.2.9 Remark. Definition 2.2.6 is different from the small support defined in [BIK08] when R is Noetherian. Assume that R is Noetherian. We have $\operatorname{supp}_R M = \operatorname{Ass}(M)$ by Lemma 2.2.4(a). In [BIK08, Section 2] the authors defined the small support of a module M by choosing a minimal injective resolution I^* of M and then invoking the structure theorem for injective modules over Noetherian rings. More precisely, their small support of M is defined as

$$\left\{ \mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R) \;\middle|\; \begin{array}{c} \exists i \geq 0\colon I^i \text{ has a direct summand isomorphic to} \\ \text{a shifted copy of the injective hull } E(R/\mathfrak{p}) \end{array} \right\}.$$

By [BH98, Theorem 3.6.3] this is equal to $\bigcup_{i\geq 0} \operatorname{Ass}(I^i)$, which contains $\operatorname{Ass}(M)$ since M is a submodule of I^0. Nevertheless, our notion of support still satisfies some of the properties derived in [BIK08, Section 2]:

2.2.10 Lemma. *Let M be an R-module and $\mathcal{V} \subseteq \operatorname{Spec}^{\mathrm{h}}(R)$ a specialization closed subset. The following hold:*

(a) $\operatorname{supp}_R M \subseteq \operatorname{cl}(\operatorname{supp}_R M) = \operatorname{Supp}_R M$.

(b) $\operatorname{supp}_R M_{\mathfrak{p}} = \operatorname{supp}_R M \cap \operatorname{gen}(\mathfrak{p})$.

(c) $\operatorname{supp}_R M \subseteq \mathcal{V} \iff M_{\mathfrak{p}} = 0$ *for all* $\mathfrak{p} \notin \mathcal{V}$.

Proof. The equality in (a) is just Lemma 2.2.4(d) and the inclusion is clear. Part (b) is Lemma 2.2.4(c). Part (c) follows immediately from (a). □

2.2.11 Definition. Let $\mathcal{V}$ be a specialization closed subset of $\operatorname{Spec}^{\mathrm{h}}(R)$. Define the full subcategory

$$\operatorname{Mod}(R)_{\mathcal{V}} := \left\{ M \in \operatorname{Mod}(R) \,\middle|\, \operatorname{supp}_R M \subseteq \mathcal{V} \right\}.$$

2.2.12 Lemma. *If $\mathcal{V} \subseteq \operatorname{Spec}^{\mathrm{h}}(R)$ is specialization closed then $\operatorname{Mod}(R)_{\mathcal{V}}$ is a localizing Serre subcategory of $\operatorname{Mod}(R)$. That is, $\operatorname{Mod}(R)_{\mathcal{V}}$ is closed under coproducts, and for any exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, M is in $\operatorname{Mod}(R)_{\mathcal{V}}$ if and only if M' and M'' are in $\operatorname{Mod}(R)_{\mathcal{V}}$.*

Proof. This follows from Lemma 2.2.10(c) and the fact that the localization functor at any prime $\mathfrak{p}$ is exact and preserves coproducts. □

2.2.13 Definition. Let M be an R-module and $\mathfrak{a}$ an ideal of R. The module M is said to be $\mathfrak{a}$-*torsion* if every element of M is annihilated by a power of $\mathfrak{a}$.

2.2.14 Lemma. *Let M be an R-module and $\mathfrak{a}$ an ideal of R. We have:*

(a) If M is $\mathfrak{a}$-torsion then $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$.

(b) $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$ *if and only if M is $\mathfrak{b}$-torsion for any finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$.*

Proof. For part (a), if M is $\mathfrak{a}$-torsion then we have $M_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \notin \mathcal{V}(\mathfrak{a})$ and thus $\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$. For part (b), recall that by definition $\operatorname{supp}_R M = \operatorname{WeakAss}(M)$. Hence if $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$ then

$$\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{WeakAss}(M)} \mathfrak{p}.$$

Note that for any homogeneous element m of M, we have

$$\sqrt{\operatorname{Ann}(m)} = \bigcap_{\mathfrak{p} \text{ minimal over } \operatorname{Ann}(m)} \mathfrak{p}.$$

By definition, if $\mathfrak{p}$ is minimal over $\operatorname{Ann}(m)$ then $\mathfrak{p} \in \operatorname{WeakAss}(M)$. Therefore

$$\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{WeakAss}(M)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \text{ minimal over } \operatorname{Ann}(m)} \mathfrak{p} = \sqrt{\operatorname{Ann}(m)}.$$

It follows that M is $\mathfrak{b}$-torsion for any finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$. The other direction follows from part (a). $\square$

2.2.15 Remark. We see from the lemma above that for a finitely generated ideal $\mathfrak{a}$, an R-module M is $\mathfrak{a}$-torsion if and only if $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$. However, this does not always hold when $\mathfrak{a}$ is not finitely generated. Indeed, in [Roh19] the author studied two torsion functors for an ideal $\mathfrak{a}$ of a commutative ring R (in the ungraded setting): the *small $\mathfrak{a}$-torsion functor* $\varGamma_{\mathfrak{a}}$ and the *large $\mathfrak{a}$-torsion functor* $\overline{\varGamma}_{\mathfrak{a}}$, which are defined as

$$\varGamma_{\mathfrak{a}} M \coloneqq \left\{ m \in M \,\middle|\, \mathfrak{a}^n \subseteq \operatorname{Ann}(m) \text{ for some } n \in \mathbb{N} \right\}$$

and

$$\overline{\Gamma}_{\mathfrak{a}} M := \left\{ m \in M \,\middle|\, \mathfrak{a} \subseteq \sqrt{\mathrm{Ann}(m)} \right\}.$$

Hence $M = \Gamma_{\mathfrak{a}} M$ if and only if M is $\mathfrak{a}$-torsion. On the other hand, $M = \overline{\Gamma}_{\mathfrak{a}} M$ if and only if $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{a})$; this follows from [Roh19, (3.3)(B)] and Lemma 2.2.10(a). It is clear that $\Gamma_{\mathfrak{a}}$ is a subfunctor of $\overline{\Gamma}_{\mathfrak{a}}$ and that $\Gamma_{\mathfrak{a}} = \overline{\Gamma}_{\mathfrak{a}}$ if $\mathfrak{a}$ is finitely generated. However, these two functors do not coincide in general; see [Roh19, Section 4].

2.3 Tensor triangular geometry

In this section, we recall the basics of tensor triangular geometry. Our main reference is [Bal05].

2.3.1 Definition. A *tensor triangulated category* (or *tt-category*) is a triangulated category equipped with a compatible closed symmetric monoidal structure. Here, compatible means that the tensor product functor is exact in each variable. A *tensor triangulated functor* (or *tt-functor*) is an exact functor which is a strong monoidal functor.

2.3.2 Notation. We fix a tensor triangulated category $\mathcal{K}$ and assume that $\mathcal{K}$ is essentially small for the rest of this section.

2.3.3 Definition. A *thick ideal* $\mathcal{J}$ of $\mathcal{K}$ is a thick subcategory of $\mathcal{K}$ that is also a *tensor ideal* meaning that $a \otimes b \in \mathcal{J}$ whenever $a \in \mathcal{J}$ and $b \in \mathcal{K}$.

2.3.4 Definition. Let $\mathcal{J}$ be a thick ideal of $\mathcal{K}$. The *radical* of $\mathcal{J}$ is defined to be

$$\sqrt{\mathcal{J}} := \left\{ a \in \mathcal{K} \,\middle|\, \exists n \geq 1 \text{ such that } a^{\otimes n} \in \mathcal{J} \right\}.$$

A thick ideal $\mathcal{J}$ is called *radical* if $\sqrt{\mathcal{J}} = \mathcal{J}$.

2.3.5 Definition. A *prime ideal* $\mathcal{P}$ of $\mathcal{K}$ is a proper thick ideal of $\mathcal{K}$ satisfying the prime condition: $a \otimes b \in \mathcal{P} \implies a \in \mathcal{P}$ or $b \in \mathcal{P}$.

2.3.6 Remark. As in commutative algebra, the radical of a thick ideal $\mathcal{I}$ is the intersection of all the prime ideals containing it:

$$\sqrt{\mathcal{I}} = \bigcap_{\mathcal{P} \supseteq \mathcal{I}} \mathcal{P}.$$

2.3.7 Definition. The *Balmer spectrum* of $\mathcal{K}$, denoted by $\mathrm{Spc}(\mathcal{K})$, is a topological space whose points are the prime ideals of $\mathcal{K}$. For any object $a \in \mathcal{K}$, the *Balmer support* of a is a closed subset

$$\mathrm{supp}(a) := \left\{ \mathcal{P} \subset \mathcal{K} \,\middle|\, a \notin \mathcal{P} \right\} \subseteq \mathrm{Spc}(\mathcal{K}).$$

The supports of objects form a basis of closed sets for the topology on $\mathrm{Spc}(\mathcal{K})$.

2.3.8 Remark. As in commutative algebra, a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a continuous map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ by sending a Balmer prime $\mathcal{P}$ to its preimage under F. For every $a \in \mathcal{K}$, we have $\mathrm{supp}(F(a)) = \mathrm{Spc}(F)^{-1}(\mathrm{supp}(a))$.

2.3.9 Definition. A *support datum* on $\mathcal{K}$ is a pair (X, σ) where X is a topological space and σ is an assignment which associates to any object $a \in \mathcal{K}$ a closed subset $\sigma(a)$ of X satisfying the following axioms:

(a) $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$;

(b) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$;

(c) $\sigma(\Sigma a) = \sigma(a)$;

(d) $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$ for any exact triangle $a \to b \to c \to \Sigma a$;

(e) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

A *morphism* $(X, \sigma) \to (Y, \tau)$ of support data on $\mathcal{K}$ is given by a continuous map $f\colon X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all objects $a \in \mathcal{K}$.

2.3.10 Theorem. *The Balmer support datum* $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is the final support datum on* $\mathcal{K}$*. In other words, it satisfies the axioms listed above, and for any support datum* (X, σ) *on* $\mathcal{K}$ *there exists a unique continuous map* $\varphi\colon X \to \mathrm{Spc}(\mathcal{K})$ *such that* $\sigma(a) = \varphi^{-1}(\mathrm{supp}(a))$ *for all objects* $a \in \mathcal{K}$*.*

Proof. See [Bal05, Theorem 3.2]. □

2.3.11 Remark. The Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ is a spectral space. The Thomason closed subsets of $\mathrm{Spc}(\mathcal{K})$ are precisely the supports $\mathrm{supp}(a)$ of objects $a \in \mathcal{K}$. By Remark 2.1.13, the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ are sets of the form

$$\mathrm{supp}(\mathcal{E}) \coloneqq \bigcup_{a \in \mathcal{E}} \mathrm{supp}(a)$$

for a collection $\mathcal{E} \subseteq \mathcal{K}$ of objects.

2.3.12 Theorem. *The Balmer support induces an inclusion-preserving bijection*

$$\{\textit{radical thick ideals of } \mathcal{K}\} \xrightarrow{\sim} \{\textit{Thomason subsets of } \mathrm{Spc}(\mathcal{K})\}$$

by sending a radical thick ideal $\mathcal{I}$ *to* $\mathrm{supp}(\mathcal{I})$*. The converse sends a Thomason subset* Y *to the radical thick ideal* $\mathcal{K}_Y \coloneqq \left\{ a \in \mathcal{K} \,\middle|\, \mathrm{supp}(a) \subseteq Y \right\}$*.*

Proof. See [Bal05, Theorem 4.10]. □

2.3.13 Remark. If $\mathcal{K}$ is rigid, i.e., each object of $\mathcal{K}$ is dualizable, then every thick ideal of $\mathcal{K}$ is automatically radical. See [Bal05, Remark 4.3].

2.3.14 Definition. If a support datum (X, σ) classifies the radical thick ideals of $\mathcal{K}$ as in Theorem 2.3.12, then we call it a *classifying support datum*.

2.3.15 Theorem. *Any classifying support datum (X, σ) on $\mathcal{K}$ with X spectral is isomorphic to the Balmer support datum* $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$*. In other words, if* (X, σ) *is a classifying support datum, then the map $\varphi\colon X \to \mathrm{Spc}(\mathcal{K})$ in Theorem 2.3.10 is a homeomorphism and hence σ coincides with the Balmer support under this homeomorphism.*

Proof. The was originally proved in [Bal05, Theorem 5.2] under the hypothesis that X is a Noetherian space. For the general case, see [BKS07, Corollary 5.2]. $\square$

2.3.16 Example. Let X be a quasi-compact, quasi-separated scheme. Consider the derived category $\mathrm{D}^{\mathrm{perf}}(X)$ of perfect complexes over X. This is a tensor triangulated category whose monoidal structure is given by the usual derived tensor product $\otimes^L_{\mathcal{O}_X}$. For any perfect complex $a \in \mathrm{D}^{\mathrm{perf}}(X)$, we write

$$\mathrm{supp}^h(a) \coloneqq \left\{ x \in X \,\middle|\, a_x \neq 0 \text{ in } \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x}) \right\}.$$

By [Tho97, Theorem 3.15], the pair (X, supp^h) is a classifying support datum on $\mathrm{D}^{\mathrm{perf}}(X)$. Consequently, we obtain a homeomorphism $X \cong \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ by Theorem 2.3.15.

2.3.17 Example. Let G be a finite group (or more genearlly, a finite group scheme) over a field k of characteristic p. Consider the stable module category $\mathrm{stmod}(kG)$ of finitely generated kG-modules. This is a tensor triangulated category whose monoidal structure is given by tensoring over k. We denote by $H^*(G, k)$ the graded-commutative cohomology ring. For any finitely generated kG-module M, we write $\sigma(M) \subseteq \mathrm{Proj}(H^*(G, k))$ for its Π-support as defined in [FP07, Definition 4.1]. By [FP07, Proposition 4.2 and Theorem 5.3], the pair $(\mathrm{Proj}(H^*(G, k)), \sigma)$ is a classifying support datum. Therefore, by Theorem 2.3.15, we have a homemorphism $\mathrm{Proj}(H^*(G, k)) \cong \mathrm{Spc}(\mathrm{stmod}(kG))$.

2.3.18 Remark. In fact, there is a sheaf $\mathcal{O}_{\mathcal{K}}$ of commutative rings on $\mathrm{Spc}(\mathcal{K})$ making it a locally ringed space. The homeomorphisms in the two examples above can be upgraded to isomorphisms of schemes. See [Bal05, Theorem 6.3].

2.3.19 Example. Let SH^c denote the stable homotopy category of finite spectra. By [HS98] and [Bal10, Corollary 9.5], the Balmer spectrum $\mathrm{Spc}(\mathrm{SH}^c)$ can be depicted as the space

$$
\begin{tikzcd}
\mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{p,\infty} & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
\mathcal{P}_{2,n+1} \arrow[u] & \mathcal{P}_{3,n+1} \arrow[u] & \cdots & \mathcal{P}_{p,n+1} \arrow[u] & \cdots \\
\mathcal{P}_{2,n} \arrow[u] & \mathcal{P}_{3,n} \arrow[u] & \cdots & \mathcal{P}_{p,n} \arrow[u] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
\mathcal{P}_{2,1} \arrow[u] & \mathcal{P}_{3,1} \arrow[u] & \cdots & \mathcal{P}_{p,1} \arrow[u] & \cdots \\
& & \mathcal{P}_{p,0} \arrow[ull] \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

where each arrow $a \to b$ indicates that b is a specialization of a. The prime $\mathcal{P}_{p,n}$ is defined to be the kernel in SH^c of the p-local n-th Morava K-theory. In other words, $\mathcal{P}_{p,n} = \left\{ x \in \mathrm{SH}^c \,\middle|\, x \otimes K(p,n) = 0 \right\}$ where $K(p,n)$ denotes the p-local n-th Morava K-theory spectrum. In particular, $\mathcal{P}_{p,0} = \mathrm{SH}^c_{\mathrm{tor}}$ is the subcategory of finite torsion spectra, which is independent of p.

2.4 Big tt-categories

In this section, we recall the notion of a big tt-category. Our main references are [BF11, BHS23b].

2.4.1 Definition. A *big tt-category* is a tt-category which admits coproducts.

2.4.2 Definition. A morphism of big tt-categories is a tt-functor $F\colon \mathcal{T} \to \mathcal{S}$ preserving coproducts. Such functor is called a *geometric functor*.

2.4.3 Definition. Let $\mathcal{T}$ be a big tt-category. A *localizing subcategory* of $\mathcal{T}$ is a thick subcategory closed under coproducts. A *localizing ideal* of $\mathcal{T}$ is a localizing subcategory which is also a tensor ideal. We write $\mathrm{Loc}\langle\mathcal{E}\rangle$ and $\mathrm{Loc}_{\otimes}\langle\mathcal{E}\rangle$ for the localizing subcategory and localizing ideal generated by a collection $\mathcal{E} \subseteq \mathcal{T}$ of objects.

2.4.4 Definition. A localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is *strictly localizing* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ admits a right adjoint. In other words, $\mathcal{L}$ is the kernel of a Bousfield localization on $\mathcal{T}$; see [Nee01, Proposition 9.1.8], for example.

2.4.5 Notation. Let $\mathcal{T}$ be a big tt-category. We denote by $\mathcal{T}^c$ the thick subcategory of compact objects, i.e., objects x such that $\mathrm{Hom}_{\mathcal{T}}(x,-)\colon \mathcal{T} \to \mathrm{Ab}$ preserves coproducts. We write $\mathcal{T}^d$ for the subcategory of dualizable objects, that is, objects x with the property that for all $y \in \mathcal{T}$, the canonical map $\hom(x,\mathbb{1})\otimes y \to \hom(x,y)$ is an isomorphism where hom denotes the interal hom functor. The subcategory $\mathcal{T}^d$ is again a tt-category.

2.4.6 Remark. A geometric functor $F\colon \mathcal{T} \to \mathcal{S}$ restricts to a tt-functor $\mathcal{T}^d \to \mathcal{S}^d$ between dualizable objects. However, it does not necessarily restrict to a functor $\mathcal{T}^c \to \mathcal{S}^c$ between compact objects.

2.4.7 Definition. Let $\mathcal{T}$ be a big tt-category. We say that $\mathcal{T}$ is *compactly generated* if there exists a set $\mathcal{E} \subseteq \mathcal{T}^c$ of compact objects such that $\mathrm{Loc}\langle\mathcal{E}\rangle = \mathcal{T}$. We call $\mathcal{T}$ *rigidly-compactly generated* is it is compactly generated and $\mathcal{T}^d = \mathcal{T}^c$.

2.4.8 Recollection. A geometric functor $F\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tt-categories enjoys nice properties. For example, it restricts to a tt-functor

$\mathcal{T}^c = \mathcal{T}^d \to \mathcal{S}^d = \mathcal{S}^c$ between rigid-compact objects, thereby inducing a continuous map $\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$ between Balmer spectra. Moreover, F admits a right adjoint U such that the projection formula

$$U(F(x) \otimes y) \simeq x \otimes U(y) \tag{2.4.9}$$

holds for all $x \in \mathcal{T}$ and $y \in \mathcal{S}$.

2.4.10 Notation. Let $\mathcal{T}$ be a rigidly-compactly generated tt-category and Y a Thomason subset of $\operatorname{Spc}(\mathcal{T}^c)$. By Theorem 2.3.12, Y give rises to a thick ideal

$$\mathcal{T}^c_Y = \left\{ a \in \mathcal{T}^c \,\middle|\, \operatorname{supp}(a) \subseteq Y \right\}$$

of $\mathcal{T}^c$. We write $\mathcal{T}_Y \coloneqq \operatorname{Loc}_\otimes\langle \mathcal{T}^c_Y \rangle$ for the localizing ideal of $\mathcal{T}$ generated by $\mathcal{T}^c_Y$.

2.4.11 Recollection. Let $\mathcal{T}$ and Y be as above. By the results of [Nee01] (see [BCHS25, Theorem 2.13] for more detail), $\mathcal{T}_Y$ is strictly localizing. The associated Bousfield localization

$$\mathcal{T}_Y \overset{i}{\hookrightarrow} \mathcal{T} \overset{q}{\twoheadrightarrow} \mathcal{T}/\mathcal{T}_Y$$

is called a finite localization. Denoting $e_Y \coloneqq ii^R(\mathbb{1})$ and $f_Y \coloneqq q^R q(\mathbb{1})$, we obtain a localization triangle

$$e_Y \to \mathbb{1} \to f_Y \to \Sigma e_Y \tag{2.4.12}$$

in $\mathcal{T}$ such that

$$\mathcal{T}_Y = e_Y \otimes \mathcal{T} = \operatorname{Ker}(f_Y \otimes -) \quad \text{and} \quad \mathcal{T}/\mathcal{T}_Y \cong f_Y \otimes \mathcal{T}. \tag{2.4.13}$$

Moreover, $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{T}_Y$ is a geometric functor whose induced map on spectra $\varphi\colon \operatorname{Spc}((\mathcal{T}/\mathcal{T}_Y)^c) \hookrightarrow \operatorname{Spc}(\mathcal{T}^c)$ is a topological embedding with image being the complement of Y in $\operatorname{Spc}(\mathcal{T}^c)$.

2.4.14 Example. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$. We write $Y_\mathcal{P} \coloneqq \operatorname{gen}(\mathcal{P})^c$ for the complement of the set of generalizations of $\mathcal{P}$. Note that $Y_{\mathcal{P}} = \mathrm{supp}(\mathcal{P})$ and hence

$$\mathrm{Ker}(f_{Y_{\mathcal{P}}} \otimes -) = \mathcal{T}_{Y_{\mathcal{P}}} = \mathrm{Loc}_{\otimes}\langle \mathcal{T}^c_{Y_{\mathcal{P}}} \rangle = \mathrm{Loc}_{\otimes}\langle \mathcal{P} \rangle \tag{2.4.15}$$

by (2.4.13) and (2.3.12). Therefore, the finite localization

$$\mathcal{T} \to \mathcal{T}/\mathcal{T}_{Y_{\mathcal{P}}} =: \mathcal{T}_{\mathcal{P}}$$

is obtained by killing $\mathrm{Loc}_{\otimes}\langle \mathcal{P} \rangle$. The induced spectral map $\varphi_{\mathcal{P}} \colon \mathrm{Spc}(\mathcal{T}^c_{\mathcal{P}}) \hookrightarrow \mathrm{Spc}(\mathcal{T}^c)$ identifies $\mathrm{Spc}(\mathcal{T}^c_{\mathcal{P}})$ with its image $\mathrm{gen}(\mathcal{P})$. Hence, the category $\mathcal{T}_{\mathcal{P}}$ is *local* in the sense that the spetrum $\mathrm{Spc}(\mathcal{T}^c_{\mathcal{P}})$ admits a unique closed point. We call $\mathcal{T}_{\mathcal{P}}$ the *localization of* $\mathcal{T}$ *at* $\mathcal{P}$ and write $t_{\mathcal{P}}$ for the image of an object $t \in \mathcal{T}$ in $\mathcal{T}_{\mathcal{P}}$.

2.4.16 Example. Let A be a commutative ring. The derived category $\mathrm{D}(A)$ is a rigidly-compactly generated tt-category. Its rigid-compact objects are the perfect complexes over A. By Example 2.3.16, we have $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(A)) \cong \mathrm{Spec}(A)$. For a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, let $\mathcal{P}$ be the corresponding Balmer prime. The local category $\mathrm{D}(A)_{\mathcal{P}}$ is equivalent to the derived category $\mathrm{D}(A_{\mathfrak{p}})$ of the local ring $A_{\mathfrak{p}}$. See [BHS23b, Example 1.36].

2.5 Higher tt-categories

We now introduce the higher-categorical counterparts of the notions in the previous section. We refer the reader to [Lur09, Lur17] for the basics of ∞-categories.

2.5.1 Notation. For any ∞-category $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}$ for the mapping space and $\mathrm{Ho}(\mathcal{C})$ for its homotopy category. If $\mathcal{C}$ admits a symmetric monoidal structure, we write $\mathrm{CAlg}(\mathcal{C})$ for the ∞-category of commutative algebras in $\mathcal{C}$. We denote by $\mathcal{S}p$ the ∞-category of spectra.

2.5.2 Notation. Let $\mathsf{Pr}^L_{\mathrm{st}}$ denote the ∞-category of presentable stable ∞-categories and colimit-preserving functors.

2.5.3 Definition. A *big tt-∞-category* is an object $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$. In other words, a big tt-∞-category is a symmetric monoidal presentable stable ∞-category whose tensor commutes with colimits in each variable. A morphism of big tt-∞-categories is a morphism in $\mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$, that is, a colimit-preserving symmetric monoidal functor.

2.5.4 Remark. Any big tt-∞-category $\mathcal{C}$ is automatically closed symmetric monoidal. This follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9], since the tensor on $\mathcal{C}$ is assumed to preserve colimits. We denote by $\hom_{\mathcal{C}}$ the interal hom.

2.5.5 Definition. Let $\mathcal{C}$ be a big tt-∞-category. A *localizing subcategory* of $\mathcal{C}$ is a stable full subcategory closed under colimits (or equivalently, coproducts). A *localizing ideal* of $\mathcal{C}$ is a localizing subcategory of $\mathcal{C}$ that is closed under tensor product with objects in $\mathcal{C}$.

2.5.6 Definition. Let $\mathcal{C}$ be a big tt-∞-category. An object $x \in \mathcal{C}$ is *compact* if the functor $\hom_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathcal{C}$ preserves filtered colimits (or equivalently, coproducts). The subcategory $\mathcal{C}^c$ of compact objects is a stable subcategory closed under retracts. We say that $\mathcal{C}$ is a *compactly generated tt-∞-category* if it is compactly generated as an ∞-category, that is, $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^c)$.

2.5.7 Definition. Let $\mathcal{C}$ be a big tt-∞-category. An object $x \in \mathcal{C}$ is *dualizable* if the canonical map $\hom_{\mathcal{C}}(x, \mathbb{1}) \otimes y \to \hom_{\mathcal{C}}(x, y)$ is an equivalence for all $y \in \mathcal{C}$.

2.5.8 Notation. For $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ we write $\mathrm{Mod}_{\mathcal{C}} \coloneqq \mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^L_{\mathrm{st}})$ for the ∞-category of $\mathcal{C}$-modules. The relative Lurie tensor product $\otimes_{\mathcal{C}}$ makes $\mathrm{Mod}_{\mathcal{C}}$ into a closed symmetric monoidal ∞-category with unit $\mathcal{C}$ and internal hom $\mathrm{Fun}^L_{\mathcal{C}}$.

2.5.9 Definition. We say that a $\mathcal{C}$-module $\mathcal{M}$ is *dualizable over* $\mathcal{C}$ if $\mathcal{M}$ is a dualizable object in $\mathrm{Mod}_{\mathcal{C}}$, that is, the canonical map $\mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \otimes \mathcal{N} \to \mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is an equivalence for every $\mathcal{C}$-module $\mathcal{N}$. In particular, we say that $\mathcal{M} \in \mathsf{Pr}^L_{\mathrm{st}} \simeq \mathrm{Mod}_{\mathcal{S}p}$ is *dualizable* when it is dualizable over the ∞-category of spectra.

2.5.10 Example. Every compactly generated $\mathcal{M} \in \mathsf{Pr}^L_{\mathrm{st}}$ is dualizable. Conversely, every dualizable $\mathcal{M}$ is a retract of a compactly generated category. See [Lur18, D.7.3.1].

2.5.11 Definition. A big tt-∞-category $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ is *locally rigid* if $\mathcal{C}$ is dualizable and the multiplication $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C} \otimes \mathcal{C}}$. We say that $\mathcal{C}$ is *rigid* if it is locally rigid and its tensor unit $\mathbb{1}$ is compact.

2.5.12 Remark. Suppose that $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ is compactly generated. Then $\mathcal{C}$ is locally rigid if and only if $\mathcal{C}^c \subseteq \mathcal{C}^d$. This follows from [Ram24, Example 4.6]. Hence, $\mathcal{C}$ is rigid if and only if $\mathcal{C}^c \subseteq \mathcal{C}^d$ and $\mathbb{1} \in \mathcal{C}^c$. This is the case exactly when $\mathcal{C}^c = \mathcal{C}^d$, by [Ram24, Lemma 4.50].

2.5.13 Definition. We say that a big tt-∞-category $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ is *rigidly-compactly generated* if it is both rigid and compactly generated.

2.5.14 Remark. The terminology above is compatible with the terminology commonly used in the literature on tensor triangular geometry. Recall that a stable ∞-category $\mathcal{C}$ is presentable if and only if the triangulated category $\mathrm{Ho}(\mathcal{C})$ is well generated, and that $\mathcal{C}$ is compactly generated as an ∞-category if and only if $\mathrm{Ho}(\mathcal{C})$ is compactly generated as a triangulated category; see [Efi24, Section 1.3] and [DM24, Lemma A.8 and Remark A.9]. Moreover, it follows from Remark 2.5.12 that a big tt-∞-category $\mathcal{C}$ is a rigidly-compactly generated tt-∞-category if and only if $\mathrm{Ho}(\mathcal{C})$ is a rigidly-compactly generated tt-category*. Furthermore, a func-

*In the terminology of [HPS97], this is equivalent to $\mathrm{Ho}(\mathcal{C})$ being a unital algebraic stable tor $\mathcal{C} \to \mathcal{D}$ is a morphism of big tt-∞-categories if and only if the induced functor $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is a geometric functor; a stable full subcategory $\mathcal{L}$ of $\mathcal{C}$ is a localizing ideal if and only if $\mathrm{Ho}(\mathcal{L})$ is a localizing ideal of $\mathrm{Ho}(\mathcal{C})$.

2.5.15 Lemma. *Let $\mathcal{C} \to \mathcal{D}$ be a morphism of big tt-∞-categories. Suppose that $\mathcal{D}$ is dualizable and $\mathcal{C}$ is locally rigid. The relative tensor product functor $- \otimes_{\mathcal{C}} \mathcal{D} \colon \mathrm{Mod}_{\mathcal{C}} \to \mathrm{Mod}_{\mathcal{D}}$ preserves fully faithful morphisms.*

Proof. By [Ram24, Proposition 4.17], $\mathcal{D}$ is dualizable over $\mathcal{C}$, so we have an equivalence $- \otimes_{\mathcal{C}} \mathcal{D} \simeq \mathrm{Fun}^L_{\mathcal{C}}(\mathrm{Fun}^L_{\mathcal{C}}(\mathcal{D}, \mathcal{C}), -)$. The latter preserves fully faithful morphisms, which completes the proof. $\square$

2.5.16 Lemma. *Consider $\mathcal{C} \in \mathrm{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$ and $\mathcal{C}$-modules $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ are dualizable, then so is $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$.*

Proof. Recall from [Lur17, Section 4.4] that the relative tensor product is computed as the colimit of the geometric realization of a two-sided Bar construction in which the tensors are over $\mathcal{S}p$. Therefore, $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ is a colimit of dualizable categories and hence is itself dualizable by [Efi24, Proposition 1.65]. $\square$

homotopy category. More generally, $\mathcal{C}$ is locally rigid and compactly generated if and only if $\mathrm{Ho}(\mathcal{C})$ is an algebraic stable homotopy category.

Chapter 3

Support theories

3.1 BIK support

Benson, Iyengar, and Krause [BIK08] developed a theory of support for any compactly generated triangulated category equipped with a central action by a Noetherian graded-commutative graded ring. In this section we show how the Noetherian hypothesis can be removed.

3.1.1 Terminology. For the rest of the section, we fix a *compactly generated R-linear triangulated category* $\mathcal{T}$. That is, a compactly generated triangulated category $\mathcal{T}$ equipped with a homomorphism of graded rings $R \to Z(\mathcal{T})$ where $Z(\mathcal{T})$ is the graded center of $\mathcal{T}$. The full subcategory of compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$. Given any two objects x and t in $\mathcal{T}$, the graded abelian group

$$\operatorname{Hom}^*_{\mathcal{T}}(x,t) \coloneqq \coprod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(x, \Sigma^n t)$$

has a graded $Z(\mathcal{T})$-module structure and hence is a graded module over R; see [BIK08, Section 4] for further details.

3.1.2 Lemma. *Let $\mathcal{V} \subseteq \operatorname{Spec}^{\mathrm{h}}(R)$ be specialization closed. The subcategory*

$$\mathcal{T}_{\mathcal{V}} \coloneqq \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t) \subseteq \mathcal{V} \textit{ for each } x \in \mathcal{T}^c \right\}$$

is strictly localizing.

Proof. Note that $\mathcal{T}_{\mathcal{V}}$ is a localizing subcategory of $\mathcal{T}$ by Lemma 2.2.12. The proof of [BIK08, Proposition 4.5] then carries over verbatim, due to Lemma 2.2.10(c). □

3.1.3 Notation. Let $t \in \mathcal{T}$ and $\mathcal{V} \subseteq \operatorname{Spec}^{\mathrm{h}}(R)$ be specialization closed. Since $\mathcal{T}_{\mathcal{V}}$ is strictly lcoalizing by the lemma above, we have a corresponding localization triangle

$$\varGamma_{\mathcal{V}} t \to t \to L_{\mathcal{V}} t$$

where $L_{\mathcal{V}}$ and $\varGamma_{\mathcal{V}}$ are the Bousfield localization and colocalization functor, respectively. We think of $\varGamma_{\mathcal{V}} t$ as the part of t supported on $\mathcal{V}$ and $L_{\mathcal{V}} t$ as the part of t supported away from $\mathcal{V}$.

3.1.4 Remark. Recall from Remark 2.2.9 that our supp_R differs from the small support defined in [BIK08] even when R is Noetherian. In any case, they have the same specialization closure, namely Supp_R; see Lemma 2.2.10(a) and [BIK08, Lemma 2.2(1)]. Thus,

$$\mathcal{T}_{\mathcal{V}} = \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t) \subseteq \mathcal{V} \text{ for each } x \in \mathcal{T}^c \right\}$$

is the same as the category defined in [BIK08, Lemma 4.3]. Therefore, the localization functor $L_{\mathcal{V}}$ is equal to the one in [BIK08, Definition 4.6]. In particular, these localization functors satisfy the following composition rules:

3.1.5 Lemma. *Let $\mathcal{V}$ and $\mathcal{W}$ be specialization closed subsets of $\operatorname{Spec}^{\mathrm{h}}(R)$. The following hold:*

(a) $\Gamma_{\mathcal{V}}\Gamma_{\mathcal{W}} \cong \Gamma_{\mathcal{V}\cap\mathcal{W}} \cong \Gamma_{\mathcal{W}}\Gamma_{\mathcal{V}}$.

(b) $L_{\mathcal{V}}L_{\mathcal{W}} \cong L_{\mathcal{V}\cup\mathcal{W}} \cong L_{\mathcal{W}}L_{\mathcal{V}}$.

(c) $\Gamma_{\mathcal{V}}L_{\mathcal{W}} \cong L_{\mathcal{W}}\Gamma_{\mathcal{V}}$.

Proof. See [BIK08, Proposition 6.1]. □

3.1.6 Remark. In Remark 3.1.4 we have noted that the construction of the localization functors in [BIK08, Definition 4.6] depends only on the notion of big support of modules, which does not require the ring R to be Noetherian. However, to show that such a localization functor is a finite localization we want to have a more concrete description of the category $\mathcal{T}_{\mathcal{V}}$. For example, the objects in $\mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ for a finitely generated ideal $\mathfrak{a}$ should be those $t \in \mathcal{T}$ with the property that $\mathrm{Hom}^*_{\mathcal{T}}(x,t)$ is $\mathfrak{a}$-torsion for every $x \in \mathcal{T}^c$. This was established in [BIK08, Lemma 2.4(2)] under the Noetherian hypothesis on the ring R. Thanks to Lemma 2.2.14, this remains true for general commutative rings.

3.1.7 Remark. Our next goal is to show that for any finitely generated ideal $\mathfrak{a}$ of R and any prime ideal $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(R)$ the localization functors corresponding to $\mathcal{V}(\mathfrak{a})$ and $\mathcal{Z}(\mathfrak{p})$ are finite localizations, that is, the localizing subcategories $\mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ and $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ are generated by compact objects in $\mathcal{T}$. Let us first recall the notion of Koszul objects.

3.1.8 Definition. Let $r \in R$ be a homogeneous element of degree d and let t be an object of $\mathcal{T}$. We denote by $t/\!\!/r$ any object that fits into an exact triangle

$$t \xrightarrow{r} \Sigma^d t \to t/\!\!/r.$$

This is called a *Koszul object of* r *on* t. Given a finite sequence $\underline{r} = (r_1, \ldots, r_n)$ of homogeneous elements, a *Koszul object of* $\underline{r}$ *on* t is defined iteratively and denoted by $t/\!\!/\underline{r}$. For a finitely generated ideal $\mathfrak{a}$ of R, we write $t/\!\!/\mathfrak{a}$ for any Koszul object of any finite sequence of homogeneous generators for $\mathfrak{a}$; see [BIK08, Definition 5.10] for further discussion. A Koszul object $t/\!\!/\mathfrak{a}$ depends on the choice of generating sequence for the ideal $\mathfrak{a}$. Nevertheless, the thick subcategory generated by $t/\!\!/\mathfrak{a}$ depends only on the radical of $\mathfrak{a}$ by [BIK11a, Lemma 3.4(2)]. Note also that $t/\!\!/\mathfrak{a}$ is compact if t is compact.

3.1.9 Lemma. *For every object $t \in \mathcal{T}$ and every finitely generated ideal $\mathfrak{a}$ of R we have $t/\!\!/\mathfrak{a} \in \mathcal{T}_{\mathcal{V}(\mathfrak{a})}$.*

Proof. By [BIK08, Lemma 5.11(1)] the R-module $\operatorname{Hom}^*_{\mathcal{T}}(x, t/\!\!/\mathfrak{a})$ is $\mathfrak{a}$-torsion for every $x \in \mathcal{T}^c$. Lemma 2.2.14 then yields the desired result. $\square$

3.1.10 Proposition. *For any finitely generated ideal $\mathfrak{a}$ of R and any $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R)$, the categories $\mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ and $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ are compactly generated subcategories of $\mathcal{T}$:*

(a) $\mathcal{T}_{\mathcal{V}(\mathfrak{a})} = \operatorname{Loc}\langle x/\!\!/\mathfrak{a} \mid x \in \mathcal{T}^c\rangle$.

(b) $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})} = \operatorname{Loc}\langle x/\!\!/r \mid x \in \mathcal{T}^c$ *and* $r \in R \setminus \mathfrak{p}$ *homogeneous*$\rangle$.

Proof. By Lemma 3.1.9 we have $\mathcal{S} := \operatorname{Loc}\langle x/\!\!/\mathfrak{a} \mid x \in \mathcal{T}^c\rangle \subseteq \mathcal{T}_{\mathcal{V}(\mathfrak{a})}$. Since $\mathcal{S}$ is a compactly generated subcategory of $\mathcal{T}_{\mathcal{V}(\mathfrak{a})}$, it is strictly localizing [Nee01, Proposition 9.1.19]. Denoting the corresponding colocalization functor by $\varGamma$, we have for any $t \in \mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ an exact triangle

$$\varGamma t \to t \to s$$

for some $s \in \mathcal{S}^{\perp}$. It remains to prove $s = 0$. Let $r_1, \ldots, r_n$ be a sequence of homogeneous generators for $\mathfrak{a}$. Note that for any $x \in \mathcal{T}^c$ we have

$$\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/(r_1, \ldots, r_n), s) = 0.$$

We claim

$$\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/(r_1,\dots,r_{n-1}),s)=0.$$

Let $m\in\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/(r_1,\dots,r_{n-1}),s)$. Since $s\in\mathcal{T}_{\mathcal{V}(\mathfrak{a})}=\bigcap_{i=1}^n\mathcal{T}_{\mathcal{V}(r_i)}\subseteq\mathcal{T}_{\mathcal{V}(r_n)}$, there exists a positive integer k with $r_n^k m=0$ by Lemma 2.2.14. Let d be the degree of r_n. Applying $\operatorname{Hom}^*_{\mathcal{T}}(-,s)$ to the exact triangle

$$x/\!\!/(r_1,\dots,r_{n-1})\xrightarrow{r_n}\Sigma^d x/\!\!/(r_1,\dots,r_{n-1})\to x/\!\!/(r_1,\dots,r_n)$$

yields an exact sequence

$$\begin{aligned}0=\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a},s)\to\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/(r_1,\dots,r_{n-1}),s)[-d]&\\ \xrightarrow{r_n}\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/(r_1,\dots,r_{n-1}),s)\to\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a},s)[-1]&=0.\end{aligned}$$

Thus, multiplying by r_n is an isomorphism, so $m=0$. The claim follows. An induction yields $\operatorname{Hom}^*_{\mathcal{T}}(x,s)=0$. This is true for all $x\in\mathcal{T}^c$. Therefore $s=0$, which establishes (a).

For part (b), set $\mathcal{S}:=\operatorname{Loc}\langle x/\!\!/r\mid x\in\mathcal{T}^c \text{ and } r\in R\setminus\mathfrak{p} \text{ is homogeneous}\rangle\subseteq\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. Similarly, for any $t\in\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ we have an exact triangle

$$\varGamma t\to t\to s$$

with $s\in\mathcal{S}^{\perp}$, where $\varGamma$ is the corresponding colocalization functor. It remains to show that $s=0$. Since $s\in\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$, we have $\operatorname{supp}_R\operatorname{Hom}^*_{\mathcal{T}}(x,s)\subseteq\mathcal{Z}(\mathfrak{p})$ and hence $\operatorname{Hom}^*_{\mathcal{T}}(x,s)_{\mathfrak{p}}=0$ by Lemma 2.2.10(b) and (2.2.8), for every $x\in\mathcal{T}^c$. It follows that for any homogeneous element $m\in\operatorname{Hom}^*_{\mathcal{T}}(x,s)$ there exists some homogeneous element $r\notin\mathfrak{p}$ with $rm=0$. Let d be the degree of r. The exact sequence

$$0=\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/r,s)[-1]\to\operatorname{Hom}^*_{\mathcal{T}}(x,s)[-d]\xrightarrow{r}\operatorname{Hom}^*_{\mathcal{T}}(x,s)\to\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/r,s)=0$$

implies that $m = 0$. Therefore $s = 0$, which completes the proof. $\square$

3.1.11 Remark. We are now ready to define the BIK support for objects in $\mathcal{T}$.

3.1.12 Definition. The *BIK support* of an object t in $\mathcal{T}$ is defined as

$$\mathrm{Supp}_{\mathrm{BIK}}(t) \coloneqq \left\{ \mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(R) \;\middle|\; \begin{array}{c} \mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} t \neq 0 \text{ for any finitely} \\ \text{generated ideal } \mathfrak{a} \text{ contained in } \mathfrak{p} \end{array} \right\}.$$

3.1.13 Remark. If $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(R)$ is finitely generated then Lemma 3.1.5(a) implies that $\mathrm{Supp}_{\mathrm{BIK}}(t) = \left\{ \mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(R) \,\middle|\, \mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} t \neq 0 \right\}$. Therefore, the support $\mathrm{Supp}_{\mathrm{BIK}}$ coincides with that in [BIK08] when R is Noetherian, keeping in mind Remark 3.1.4.

3.1.14 Remark. We give the following more flexible description of the BIK support.

3.1.15 Proposition. *We have*

$$\mathrm{Supp}_{\mathrm{BIK}}(t) = \left\{ \mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(R) \;\middle|\; \begin{array}{c} \mathit{\Gamma}_{\mathcal{V}} L_{\mathcal{Z}} t \neq 0 \textit{ for any Thomason} \\ \textit{subsets } \mathcal{V}, \mathcal{Z} \textit{ such that } \mathfrak{p} \in \mathcal{V} \cap \mathcal{Z}^{c} \end{array} \right\}$$

for any $t \in \mathcal{T}$.

Proof. Recall from Remark 2.1.13 that a Thomason subset of a spectral space is a union of Thomason closed subsets and that $\mathcal{Z}(\mathfrak{p}) = \mathrm{gen}(\mathfrak{p})^{c}$ is the largest Thomason subset not containing $\mathfrak{p}$. Moreover, the Thomason closed subsets of $\mathrm{Spec}^{\mathrm{h}}(R)$ are exactly subsets of the form $\mathcal{V}(\mathfrak{a})$ for some finitely generated ideal $\mathfrak{a}$. Therefore the result follows from Lemma 3.1.5. $\square$

3.1.16 Notation. For $S \subseteq \mathrm{Spec}^{\mathrm{h}}(R)$ we write $\min S$ for the set of prime ideals in S that are minimal (with respect to inclusion) among the prime ideals in S.

3.1.17 Theorem. *For any $t \in \mathcal{T}$ we have*

$$\bigcup_{x \in \mathcal{T}^c} \min \operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t) \subseteq \operatorname{Supp}_{\mathrm{BIK}}(t) \subseteq \bigcup_{x \in \mathcal{G}} \operatorname{Supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t)$$

where $\mathcal{G}$ is any set of compact generators for $\mathcal{T}$.

Proof. Suppose $\mathfrak{p} \in \min \operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t)$ for some $x \in \mathcal{T}^c$. By Lemma 2.2.10(a) we have $\operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}} \neq 0$ and hence $\operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}} = \{\mathfrak{p}\}$ by the minimality of $\mathfrak{p}$ and Lemma 2.2.10(b). It then follows from Lemma 2.2.14 that $\operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}}$ is a-torsion for any homogeneous element $a \in \mathfrak{p}$. Now let $r \in \mathfrak{p}$ be a homogeneous element of degree d. Applying $\operatorname{Hom}^*_{\mathcal{T}}(-,t)_{\mathfrak{p}}$ to the exact triangle

$$x \xrightarrow{r} \Sigma^d x \to x/\!\!/r$$

yields a long exact sequence

$$\cdots \to \operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}}[-1] \to \operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/r,t)_{\mathfrak{p}} \to \operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}}[-d] \xrightarrow{r} \operatorname{Hom}^*_{\mathcal{T}}(x,t)_{\mathfrak{p}} \to \cdots .$$

Hence $\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/r,t)_{\mathfrak{p}}$ is also a-torsion for any homogeneous element $a \in \mathfrak{p}$ and $\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/r,t)_{\mathfrak{p}} \neq 0$. An induction shows that $\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a},t)_{\mathfrak{p}} \neq 0$ for any finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{p}$. On the other hand, since $\mathit{\Gamma}_{\mathcal{Z}(\mathfrak{p})} t \in \mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ it follows that

$$\operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, \mathit{\Gamma}_{\mathcal{Z}(\mathfrak{p})} t) \subseteq \mathcal{Z}(\mathfrak{p})$$

which implies $\operatorname{supp}_R \operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, \mathit{\Gamma}_{\mathcal{Z}(\mathfrak{p})} t)_{\mathfrak{p}} = \varnothing$ by Lemma 2.2.10(b). In view of (2.2.8), we then have $\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, \mathit{\Gamma}_{\mathcal{Z}(\mathfrak{p})} t)_{\mathfrak{p}} = 0$ and thus

$$\operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, L_{\mathcal{Z}(\mathfrak{p})} t)_{\mathfrak{p}} \cong \operatorname{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, t)_{\mathfrak{p}} \neq 0.$$

Since $x/\!\!/\mathfrak{a} \in \mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ (Lemma 3.1.9), we have

$$\mathrm{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, \varGamma_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} t)_{\mathfrak{p}} \cong \mathrm{Hom}^*_{\mathcal{T}}(x/\!\!/\mathfrak{a}, L_{\mathcal{Z}(\mathfrak{p})} t)_{\mathfrak{p}} \neq 0$$

and therefore $\varGamma_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} t \neq 0$. This is true for every finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in \mathrm{Supp}_{\mathrm{BIK}}(t)$, which establishes the first inclusion. For the second, note that

$$\mathfrak{p} \notin \bigcup_{x \in \mathcal{G}} \mathrm{Supp}_R \mathrm{Hom}^*_{\mathcal{T}}(x, t) \iff t \in \mathcal{T}_{\mathcal{Z}(\mathfrak{p})} \iff L_{\mathcal{Z}(\mathfrak{p})} t = 0.$$

This implies $\mathfrak{p} \notin \mathrm{Supp}_{\mathrm{BIK}}(t)$. □

3.1.18 Remark. The theorem above generalizes [BIK08, Corollary 5.3]. In [BIK08, Theorem 5.2] it was shown that the first inclusion is an equality under the assumption that R is Noetherian. It is not clear if the equality still holds without the Noetherian assumption.

3.1.19 Definition. We say that the BIK support satisfies the *detection property* if $\mathrm{Supp}_{\mathrm{BIK}}(t) = \varnothing$ implies $t = 0$ for every $t \in \mathcal{T}$.

3.1.20 Corollary. *If the descending chain condition holds for the prime ideals of R then the BIK support satisfies the detection property.*

Proof. The hypothesis implies that if a subset S of $\mathrm{Spec}^{\mathrm{h}}(R)$ is nonempty then $\min S$ is nonempty. The result then follows from Theorem 3.1.17 and (2.2.8). □

3.1.21 Remark. If R is Noetherian then the descending chain condition on the prime ideals of R holds (see [DST19, Corollary 12.4.5(1)], for example) and thus the BIK support has the detection property. However, we do not know if the detection property is always satisfied when R is not Noetherian.

3.1.22 Remark. In the following we record some basic properties of the BIK support, which are inspired by [San17, Theorems 4.2 and 4.7].

3.1.23 Proposition. *The following hold:*

(a) $\operatorname{Supp}_{\mathrm{BIK}}(0) = \varnothing$.

(b) $\operatorname{Supp}_{\mathrm{BIK}}(t) = \operatorname{Supp}_{\mathrm{BIK}}(\Sigma t)$ *for every* $t \in \mathcal{T}$.

(c) $\operatorname{Supp}_{\mathrm{BIK}}(c) \subseteq \operatorname{Supp}_{\mathrm{BIK}}(a) \cup \operatorname{Supp}_{\mathrm{BIK}}(b)$ *if* $a \to b \to c \to \Sigma a$ *is an exact triangle in* $\mathcal{T}$.

(d) $\operatorname{Supp}_{\mathrm{BIK}}(t_1 \oplus t_2) = \operatorname{Supp}_{\mathrm{BIK}}(t_1) \cup \operatorname{Supp}_{\mathrm{BIK}}(t_2)$ *for any* $t_1, t_2 \in \mathcal{T}$.

(e) $\operatorname{Supp}_{\mathrm{BIK}}(\varGamma_{\mathcal{V}} t) = \mathcal{V} \cap \operatorname{Supp}_{\mathrm{BIK}}(t)$ *and* $\operatorname{Supp}_{\mathrm{BIK}}(L_{\mathcal{V}} t) = \mathcal{V}^c \cap \operatorname{Supp}_{\mathrm{BIK}}(t)$ *for any* $t \in \mathcal{T}$ *and any specialization closed subset* $\mathcal{V}$ *of* $\operatorname{Spec}^{\mathrm{h}}(R)$.

Proof. Part (a) is clear. Parts (b) and (d) follow from the fact that the localization and colocalization functors preserve finite direct sums and are exact.

For part (c), if $\mathfrak{p} \notin \operatorname{Supp}_{\mathrm{BIK}}(a) \cup \operatorname{Supp}_{\mathrm{BIK}}(b)$, then there exist finitely generated ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{p}$ such that $\varGamma_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} a = 0$ and $\varGamma_{\mathcal{V}(\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} b = 0$. Thus by Lemma 3.1.5 we have

$$\varGamma_{\mathcal{V}(\mathfrak{a}+\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} a = \varGamma_{\mathcal{V}(\mathfrak{a})} \varGamma_{\mathcal{V}(\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} a = 0 = \varGamma_{\mathcal{V}(\mathfrak{a})} \varGamma_{\mathcal{V}(\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} b = \varGamma_{\mathcal{V}(\mathfrak{a}+\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} b.$$

It then follows from the exactness of localization functors that $\varGamma_{\mathcal{V}(\mathfrak{a}+\mathfrak{b})} L_{\mathcal{Z}(\mathfrak{p})} c = 0$. Therefore $\mathfrak{p} \notin \operatorname{Supp}_{\mathrm{BIK}}(c)$, which establishes (c).

For (e), note that $\varGamma_{\operatorname{Spec}^{\mathrm{h}}(R)} L_{\mathcal{V}}(\varGamma_{\mathcal{V}} t) = 0 = \varGamma_{\mathcal{V}} L_{\varnothing}(L_{\mathcal{V}} t)$. Thus $\operatorname{Supp}_{\mathrm{BIK}}(\varGamma_{\mathcal{V}} t) \subseteq \mathcal{V}$ and $\operatorname{Supp}_{\mathrm{BIK}}(L_{\mathcal{V}} t) \subseteq \mathcal{V}^c$. Applying (c) to the exact triangle $\varGamma_{\mathcal{V}} t \to t \to L_{\mathcal{V}} t$ we obtain

$$\operatorname{Supp}_{\mathrm{BIK}}(\varGamma_{\mathcal{V}} t) \subseteq \operatorname{Supp}_{\mathrm{BIK}}(t) \cup \operatorname{Supp}_{\mathrm{BIK}}(L_{\mathcal{V}} t) \subseteq \operatorname{Supp}_{\mathrm{BIK}}(t) \cup \mathcal{V}^c.$$

Intersecting with $\mathcal{V}$ leads to $\operatorname{Supp}_{\mathrm{BIK}}(\varGamma_{\mathcal{V}} t) = \operatorname{Supp}_{\mathrm{BIK}}(t) \cap \mathcal{V}$. A similar argument shows that $\operatorname{Supp}_{\mathrm{BIK}}(L_{\mathcal{V}} t) = \operatorname{Supp}_{\mathrm{BIK}}(t) \cap \mathcal{V}^c$, which completes the proof. $\square$

3.1.24 Notation. For any class $\mathcal{E}$ of objects in $\mathcal{T}$ we write $\mathrm{Loc}\langle\mathcal{E}\rangle$ for the localizing subcategory generated by $\mathcal{E}$ and define $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E}) \coloneqq \bigcup_{t\in\mathcal{E}} \mathrm{Supp}_{\mathrm{BIK}}(t)$.

3.1.25 Proposition. *The following hold:*

(a) $\mathrm{Supp}_{\mathrm{BIK}}(t)$ *is localizing closed (recall Remark 2.1.16) for any* $t \in \mathcal{T}$.

(b) $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{L})$ *is localizing closed for any localizing subcategory* $\mathcal{L}$ *of* $\mathcal{T}$.

(c) If the BIK support satisfies the detection property then for any class of objects $\mathcal{E}$ *of* $\mathcal{T}$ *we have*

$$\mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle\mathcal{E}\rangle) = \overline{\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})}^{\mathrm{loc}}$$

where $\overline{\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})}^{\mathrm{loc}}$ *denotes the closure of* $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})$ *in* $\mathrm{Spec}^{\mathrm{h}}(R)$ *with respect to the localizing topology.*

(d) If the BIK support satisfies the detection property then for any set of objects $\{t_i\}_{i\in I}$ *of* $\mathcal{T}$ *we have*

$$\mathrm{Supp}_{\mathrm{BIK}}(\coprod_{i\in I} t_i) = \overline{\bigcup_{i\in I} \mathrm{Supp}_{\mathrm{BIK}}(t_i)}^{\mathrm{loc}}. \tag{3.1.26}$$

Proof. (a): If $\mathfrak{p} \notin \mathrm{Supp}_{\mathrm{BIK}}(t)$ then by Proposition 3.1.15 there exist Thomason subsets $\mathcal{V}$ and $\mathcal{Z}$ with $\mathfrak{p} \in \mathcal{V} \cap \mathcal{Z}^c$ and $\varGamma_{\mathcal{V}} L_{\mathcal{Z}} t = 0$. Hence $\mathcal{V} \cap \mathcal{Z}^c$ is an open neighborhood of $\mathfrak{p}$ (with respect to the localizing topology) for which $\mathcal{V} \cap \mathcal{Z}^c \cap \mathrm{Supp}_{\mathrm{BIK}}(t) = \varnothing$. Therefore, $\mathrm{Supp}_{\mathrm{BIK}}(t)$ is localizing closed.

(b): We write S for $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{L})$. For every $\mathfrak{p} \in S$ we choose an object $t(\mathfrak{p}) \in \mathcal{L}$ such that $\mathfrak{p} \in \mathrm{Supp}_{\mathrm{BIK}}(t(\mathfrak{p}))$. Now we have

$$S = \bigcup_{\mathfrak{p}\in S} \mathrm{Supp}_{\mathrm{BIK}}(t(\mathfrak{p})) = \mathrm{Supp}_{\mathrm{BIK}}(\coprod_{\mathfrak{p}\in S} t(\mathfrak{p})),$$

where the second equality follows from Proposition 3.1.23(d) and that $\coprod_{\mathfrak{p}\in S} t(\mathfrak{p})$ belongs to $\mathcal{L}$. Therefore, $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{L})$ is localizing closed by (a).

(c): Note that (b) implies

$$\mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle\mathcal{E}\rangle) \supseteq \overline{\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})}^{\mathrm{loc}}.$$

If $\mathfrak{p} \notin \overline{\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})}^{\mathrm{loc}}$ then there exist Thomason subsets $\mathcal{V}$ and $\mathcal{Z}$ with $\mathfrak{p} \in \mathcal{V} \cap \mathcal{Z}^c$ and $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E}) \cap \mathcal{V} \cap \mathcal{Z}^c = \varnothing$. By Proposition 3.1.23(e) we have

$$\mathrm{Supp}_{\mathrm{BIK}}(\mathit{\Gamma}_{\mathcal{V}} L_{\mathcal{Z}}(\mathcal{E})) = \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E}) \cap \mathcal{V} \cap \mathcal{Z}^c = \varnothing.$$

The detection property then implies $\mathit{\Gamma}_{\mathcal{V}} L_{\mathcal{Z}}(\mathcal{E}) = 0$ and hence $\mathit{\Gamma}_{\mathcal{V}} L_{\mathcal{Z}}(\mathrm{Loc}\langle\mathcal{E}\rangle) = 0$, which implies $\mathfrak{p} \notin \mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle\mathcal{E}\rangle)$. We thus obtain

$$\mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle\mathcal{E}\rangle) = \overline{\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{E})}^{\mathrm{loc}}.$$

(d): Let $\{t_i\}_{i\in I}$ be a set of objects in $\mathcal{T}$. Observe that

$$\begin{aligned} \overline{\bigcup_{i\in I} \mathrm{Supp}_{\mathrm{BIK}}(t_i)}^{\mathrm{loc}} &= \mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle t_i \mid i \in I\rangle) = \mathrm{Supp}_{\mathrm{BIK}}(\mathrm{Loc}\langle \coprod_{i\in I} t_i\rangle) \\ &= \overline{\mathrm{Supp}_{\mathrm{BIK}}(\coprod_{i\in I} t_i)}^{\mathrm{loc}} = \mathrm{Supp}_{\mathrm{BIK}}(\coprod_{i\in I} t_i). \end{aligned}$$ □

3.2 BFS support

In [BF11], Balmer–Favi defined a notion of support for objects in any rigidly-compactly generated tt-category $\mathcal{T}$. This support takes values in the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ but requires the spectrum to be (weakly) Noetherian. In [San17], Sanders proposed a generalization of the Balmer–Favi support theory which does not require any topological assumption. In this section, we summarize some basic properties of support theory by Sanders and then study how the support behaves under geometric functors.

3.2.1 Hypothesis. Throughout this section, we fix a rigidly-compactly generated tt-category $\mathcal{T}$.

3.2.2 Notation. Let $W = U \cap V^c$ be a weakly visible subset of $\mathrm{Spc}(\mathcal{T}^c)$ where U and V are Thomason subsets. Define $g_W \coloneqq e_U \otimes f_V$ (recall Recollection 2.4.11). By [BF11, Remark 7.6] the idempotent object g_W does not depend on the choice of U and V up to isomorphism. Moreover, it follows from [BHS23b, Lemma 1.27] that

$$g_W = 0 \iff W = \varnothing. \tag{3.2.3}$$

3.2.4 Remark. Let $F\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor between rigidly-compactly generated tt-categories and let $\varphi\colon \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$ be the induced spectral map. By [BS17, Proposition 5.11] we have

$$F(g_W) \simeq g_{\varphi^{-1}(W)} \tag{3.2.5}$$

for any weakly visible subset W of $\mathrm{Spc}(\mathcal{T}^c)$.

3.2.6 Remark. A point $\mathcal{P}$ in the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ is not always weakly visible. However, since the Thomason closed subsets of $\mathrm{Spc}(\mathcal{T}^c)$ form a basis of closed subsets it follows that

$$\overline{\{\mathcal{P}\}} = \bigcap_{\substack{a \in \mathcal{T}^c \\ a \notin \mathcal{P}}} \mathrm{supp}(a) \quad \text{and hence} \quad \{\mathcal{P}\} = \bigcap_{\substack{a \in \mathcal{T}^c \\ a \notin \mathcal{P}}} \mathrm{supp}(a) \cap \mathrm{gen}(\mathcal{P}).$$

In other words, $\{\mathcal{P}\}$ is always an intersection of weakly visible subsets. This leads to the following:

3.2.7 Definition. The *BFS support* of an object $t \in \mathcal{T}$ is

$$\operatorname{Supp}(t) \coloneqq \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) \;\middle|\; \begin{array}{l} g_W \otimes t \neq 0 \text{ for every weakly} \\ \text{visible subset } W \text{ containing } \mathcal{P} \end{array} \right\}.$$

3.2.8 Remark. By [BHS23b, Lemma 1.27] we have

$$g_{W_1 \cap W_2} = g_{W_1} \otimes g_{W_2} \tag{3.2.9}$$

for any weakly visible subsets W_1 and W_2. By Remark 2.1.13 every Thomason subset of $\operatorname{Spc}(\mathcal{T}^c)$ is a union of Thomason closed subsets and $Y_\mathcal{P} = \operatorname{gen}(\mathcal{P})^c$ is the largest Thomason subset not containing $\mathcal{P}$. Therefore, a Balmer prime $\mathcal{P}$ is in $\operatorname{Supp}(t)$ if and only if $e_{\operatorname{supp} a} \otimes f_{Y_\mathcal{P}} \otimes t \neq 0$ for every Thomason closed subset $\operatorname{supp} a$ that contains $\mathcal{P}$ (cf. Definition 3.1.12).

3.2.10 Remark. If $\operatorname{Spc}(\mathcal{T}^c)$ is weakly Noetherian, that is, every point of $\operatorname{Spc}(\mathcal{T}^c)$ is weakly visible, then the BFS support coincides with the Balmer–Favi support in [BHS23b, Definition 2.11]. Indeed, if $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$ is weakly visible then (3.2.9) implies that $\mathcal{P} \in \operatorname{Supp}(t)$ if and only if $g_\mathcal{P} \otimes t \neq 0$.

3.2.11 Proposition. *The BFS support satisfies the following basic properties:*

(a) $\operatorname{Supp}(0) = \varnothing$ *and* $\operatorname{Supp}(\mathbb{1}) = \operatorname{Spc}(\mathcal{T}^c)$.

(b) $\operatorname{Supp}(\Sigma t) = \operatorname{Supp}(t)$ *for every* $t \in \mathcal{T}$.

(c) $\operatorname{Supp}(c) \subseteq \operatorname{Supp}(a) \cup \operatorname{Supp}(b)$ *for any exact triangle* $a \to b \to c \to \Sigma a$ *in* $\mathcal{T}$.

(d) $\operatorname{Supp}(t_1 \oplus t_2) = \operatorname{Supp}(t_1) \cup \operatorname{Supp}(t_2)$ *for any* $t_1, t_2 \in \mathcal{T}$.

(e) $\operatorname{Supp}(t_1 \otimes t_2) \subseteq \operatorname{Supp}(t_1) \cap \operatorname{Supp}(t_2)$ *for any* $t_1, t_2 \in \mathcal{T}$.

(f) $\operatorname{Supp}(t \otimes e_Y) = \operatorname{Supp}(t) \cap Y$ *and* $\operatorname{Supp}(t \otimes f_Y) = \operatorname{Supp}(t) \cap Y^c$ *for every object* $t \in \mathcal{T}$ *and every Thomason subset* $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$.

Proof. Parts (a), (b), (d), and (e) are immediate from the definitions. The proofs for parts (c) and (f) can be found in [San17, Theorem 4.2]. $\square$

3.2.12 Remark. The half-tensor formula [BHS23b, Lemma 2.18] still holds without the weakly Noetherian assumption:

3.2.13 Lemma. *For any compact object $x \in \mathcal{T}^c$ and arbitrary object $t \in \mathcal{T}$ we have*

$$\mathrm{Supp}(x \otimes t) = \mathrm{supp}(x) \cap \mathrm{Supp}(t).$$

In particular, for any compact object $x \in \mathcal{T}^c$, the BFS support coincides with the Balmer support: $\mathrm{Supp}(x) = \mathrm{supp}(x)$.

Proof. Observe that

$$\begin{aligned} \mathcal{P} \notin \mathrm{Supp}(x \otimes t) &\iff x \otimes t \otimes g_W = 0 \text{ for some weakly visible } W \ni \mathcal{P} \\ &\iff e_{\mathrm{supp}(x)} \otimes t \otimes g_W = 0 \quad (\mathrm{Loc}_{\otimes}\langle x \rangle = e_{\mathrm{supp}(x)} \otimes \mathcal{T}) \\ &\iff \mathcal{P} \notin \mathrm{Supp}(e_{\mathrm{supp}(x)} \otimes t) = \mathrm{supp}(x) \cap \mathrm{Supp}(t) \end{aligned}$$

where the last equality is due to Proposition 3.2.11(f). $\square$

3.2.14 Remark. By Remark 2.1.16, $\mathrm{Spc}(\mathcal{T}^c)$ is weakly Noetherian if and only if its localizing topology is discrete. Thus, the localizing topology becomes relevant when $\mathrm{Spc}(\mathcal{T}^c)$ is not weakly Noetherian, as the following shows.

3.2.15 Proposition. *For any object $t \in \mathcal{T}$ and any localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ we have*

(a) $\mathrm{Supp}(t)$ *is localizing closed.*

(b) $\mathrm{Supp}(\mathcal{L}) \coloneqq \bigcup_{s \in \mathcal{L}} \mathrm{Supp}(s)$ *is localizing closed.*

Proof. See [San17, Theorems 4.2]. □

3.2.16 Remark. We now study how the BFS support behaves under geometric functors.

3.2.17 Proposition. *Let $F\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor between rigidly-compactly generated tt-categories, U its right adjoint, and $\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$ the induced spectral map. We have*

$$\operatorname{Supp}(U(\mathbb{1})) = \overline{\operatorname{im}\varphi}^{\mathrm{loc}}$$

where $\overline{\operatorname{im}\varphi}^{\mathrm{loc}}$ denotes the closure of $\operatorname{im}\varphi$ in $\operatorname{Spc}(\mathcal{T}^c)$ with respect to the localizing topology.

Proof. If $\mathcal{Q} \notin \operatorname{Supp}(U(\mathbb{1}))$ then there exists some weakly visible subset $W \ni \mathcal{Q}$ such that $g_W \otimes U(\mathbb{1}) = 0$ and thus $U(F(g_W)) = 0$ by the projection formula (2.4.9). It follows that

$$0 = \operatorname{Hom}(g_W, UF(g_W)) \simeq \operatorname{Hom}(F(g_W), F(g_W))$$

and hence $F(g_W) = 0$. We then have $\varphi^{-1}(W) = \varnothing$ in view of (3.2.3) and (3.2.5). Therefore $\mathcal{Q} \notin \operatorname{im}\varphi$. We have established $\operatorname{im}\varphi \subseteq \operatorname{Supp}(U(\mathbb{1}))$. This implies, by Proposition 3.2.15, that $\overline{\operatorname{im}\varphi}^{\mathrm{loc}} \subseteq \operatorname{Supp}(U(\mathbb{1}))$.

For the other inclusion, if $\mathcal{Q} \notin \overline{\operatorname{im}\varphi}^{\mathrm{loc}}$ then there exists a weakly visible subset $W \ni \mathcal{Q}$ such that $W \cap \operatorname{im}\varphi = \varnothing$. This means $\varphi^{-1}(W) = \varnothing$, which implies $F(g_W) = 0$. By the projection formula we obtain $g_W \otimes U(\mathbb{1}) = 0$ and hence $\mathcal{Q} \notin \operatorname{Supp}(U(\mathbb{1}))$. □

3.2.18 Remark. In [BCHS25, Corollary 13.15] it was shown that if $\operatorname{Spc}(\mathcal{T}^c)$ is weakly Noetherian then $\operatorname{Supp}(U(\mathbb{1})) = \operatorname{im}\varphi$, which is a special case of the proposition above.

3.2.19 Proposition. *Let F, U, and φ be as in Proposition 3.2.17. The following hold:*

(a) $\operatorname{Supp}(F(t)) \subseteq \varphi^{-1}(\operatorname{Supp}(t))$ *for any* $t \in \mathcal{T}$.

(b) $\varphi(\operatorname{Supp}(s)) \subseteq \operatorname{Supp}(U(s))$ *for any* $s \in \mathcal{S}$ *if* U *is conservative.*

Proof. For part (a), let $\mathcal{P} \in \operatorname{Supp}(F(t))$. If $\mathcal{Q} := \varphi(\mathcal{P}) \notin \operatorname{Supp}(t)$ then there exists a weakly visible subset $W \ni \mathcal{Q}$ with $g_W \otimes t = 0$. Hence

$$0 = F(g_W \otimes t) \simeq F(g_W) \otimes F(t) \simeq g_{\varphi^{-1}(W)} \otimes F(t)$$

which contradicts $\mathcal{P} \in \operatorname{Supp}(F(t))$. This establishes (a).

To prove part (b), suppose that $\mathcal{P} \in \operatorname{Supp}(s)$ but $\mathcal{Q} := \varphi(\mathcal{P}) \notin \operatorname{Supp}(U(s))$. By definition there exists a weakly visible subset $W \ni \mathcal{Q}$ such that $g_W \otimes U(s) = 0$. We then have $U(F(g_W) \otimes s) = 0$ by the projection formula and thus $g_{\varphi^{-1}(W)} \otimes s = 0$ by the conservativity of U. Proposition 3.2.11(f) then implies $\varphi^{-1}(W) \cap \operatorname{Supp}(s) = \varnothing$, which contradicts $\mathcal{P} \in \operatorname{Supp}(s)$. $\square$

3.2.20 Corollary. *Let F, U, and φ be as in Proposition 3.2.17. If F is a finite localization then we have*

(a) $\varphi(\operatorname{Supp}(s)) = \operatorname{Supp}(U(s))$ *for any* $s \in \mathcal{S}$.

(b) $\operatorname{Supp}(F(t)) = \varphi^{-1}(\operatorname{Supp}(t))$ *for any* $t \in \mathcal{T}$.

Proof. For part (a), it suffices to show that $\operatorname{Supp}(U(s)) \subseteq \varphi(\operatorname{Supp}(s))$ by Proposition 3.2.19(b). To this end, suppose $\mathcal{Q} = \varphi(\mathcal{P}) \in \operatorname{Supp}(U(s))$ but $\mathcal{P} \notin \operatorname{Supp}(s)$. By definition there exists a weakly visible subset $W \subseteq \operatorname{Spc}(\mathcal{S}^c)$ such that $\mathcal{P} \in W$ and $g_W \otimes s = 0$. The map $\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \hookrightarrow \operatorname{Spc}(\mathcal{T}^c)$ exhibits $\operatorname{Spc}(\mathcal{S}^c)$ as a spectral subspace of $\operatorname{Spc}(\mathcal{T}^c)$ since F is a finite localization. We thus have $W = \varphi^{-1}(Z)$ for some weakly visible subset $Z \subseteq \mathrm{Spc}(\mathcal{T}^c)$ by [DST19, Theorem 2.1.3]. It follows from the projection formula and (3.2.5) that

$$g_Z \otimes U(s) \simeq U(F(g_Z) \otimes s) \simeq U(g_W \otimes s) = 0.$$

This contradicts $\mathcal{Q} \in \mathrm{Supp}(U(s))$.

For part (b), suppose that F is the finite localization associated to a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ and let V be the complement of Y. We then have an induced embedding $\varphi\colon \mathrm{Spc}(\mathcal{T}(V)^c) \hookrightarrow \mathrm{Spc}(\mathcal{T}^c)$ with image V. For any $t \in \mathcal{T}$ we have $\mathrm{Supp}(F(t)) = \varphi^{-1}(\mathrm{Supp}(UF(t)))$ by part (a). Observe that

$$\begin{aligned}\varphi^{-1}(\mathrm{Supp}(UF(t))) &= \varphi^{-1}(\mathrm{Supp}(U(\mathbb{1}) \otimes t)) = \varphi^{-1}(\mathrm{Supp}(f_Y \otimes t)) \\ &= \varphi^{-1}(Y^c \cap \mathrm{Supp}(t)) = \varphi^{-1}(\mathrm{Supp}(t))\end{aligned}$$

where the first equality uses the projection formula, the second equality uses the fact that $U(\mathbb{1}) = UF(\mathbb{1}) = f_Y$, and the third equality is due to Lemma 3.2.13. This completes the proof. □

3.3 The detection property

In this section, we use algebraic localizations of a rigidly-compactly generated tt-category $\mathcal{T}$ to study the detection property for the BFS support. Our main result is that the detection property can be checked at the algebraic localizations of $\mathcal{T}$ at the closed points of the Zariski spectrum of the graded endomorphism ring of the unit object.

3.3.1 Notation. Throughout this section, $\mathcal{T}$ denotes a rigidly-compactly generated tt-category. Let $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ denote the graded endomorphism ring of the unit object in $\mathcal{T}$. Denote by $\mathrm{End}^{\mathrm{hom}}_{\mathcal{T}}(\mathbb{1})$ the set of homogeneous elements in $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$. By

[Bal10, Theorem 5.3], there is a natural continuous map

$$\begin{aligned}\rho\colon \operatorname{Spc}(\mathcal{T}^c) &\to \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))\\ \mathcal{P} &\mapsto \langle f \in \operatorname{End}^{\hom}_{\mathcal{T}}(\mathbb{1}) \mid \operatorname{cone}(f) \notin \mathcal{P}\rangle\end{aligned}$$

such that $\rho^{-1}(\mathcal{V}(s)) = \operatorname{supp}(\operatorname{cone}(s))$ for any $s \in \operatorname{End}^{\hom}_{\mathcal{T}}(\mathbb{1})$.

3.3.2 Definition (Graded algebraic localization). Let $S \subseteq \operatorname{End}^{\hom}_{\mathcal{T}}(\mathbb{1})$ be a multiplicative subset of homogeneous elements. The *algebraic localization of* $\mathcal{T}$ *with respect to* S is the finite localization of $\mathcal{T}$ associated to the Thomason subset

$$\rho^{-1}\Big(\bigcup_{s\in S} \mathcal{V}(s)\Big) = \bigcup_{s\in S} \operatorname{supp}(\operatorname{cone}(s)).$$

We denote this localization by $\mathcal{T} \to S^{-1}\mathcal{T}$. The corresponding localization functor and colocalization functor are denoted by L_S and $\mathit{\Gamma}_S$, respectively.

In particular, for a prime ideal $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$ we define the multiplicative subset $S_{\mathfrak{p}} := \left\{ s \in \operatorname{End}^{\hom}_{\mathcal{T}}(\mathbb{1}) \,\middle|\, s \notin \mathfrak{p} \right\}$ and call $\mathcal{T}_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}\mathcal{T}$ the *algebraic localization of* $\mathcal{T}$ *at* $\mathfrak{p}$.

3.3.3 Notation. For any $t \in \mathcal{T}$ we write $\pi_*(t)$ for

$$\operatorname{Hom}^{-*}_{\mathcal{T}}(\mathbb{1}, t) = \coprod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^{-n} t).$$

3.3.4 Remark. The following proposition generalizes both [HPS97, Theorem 3.3.7] and [Bal10, Corollary 3.10]. The ungraded version of it can be found in [Del10, Theorem 2.33(h)].

3.3.5 Proposition. *Let* $S \subseteq \operatorname{End}^{\hom}_{\mathcal{T}}(\mathbb{1})$ *be a multiplicative subset of homogeneous elements. There is a natural isomorphism*

$$\operatorname{Hom}^*_{\mathcal{T}}(x, L_S t) \cong S^{-1} \operatorname{Hom}^*_{\mathcal{T}}(x, t)$$

for $x \in \mathcal{T}^c$ and $t \in \mathcal{T}$. In particular, we have

$$\pi_*(L_S t) \cong S^{-1}\pi_*(t)$$

for any $t \in \mathcal{T}$.

Proof. If x is compact then $S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(x,-)$ is a coproduct-preserving homological functor on $\mathcal{T}$ which vanishes on $t \otimes \operatorname{cone}(s)$ for all $t \in \mathcal{T}$ and $s \in S$ and hence on $\operatorname{Loc}\langle \mathcal{T} \otimes \operatorname{cone}(s) \mid s \in S\rangle = \operatorname{Loc}_{\otimes}\langle \operatorname{cone}(s) \mid s \in S\rangle = \varGamma_S \mathcal{T}$. It follows from the localization triangle

$$\varGamma_S t \to t \to L_S t$$

that there is a natural isomorphism

$$S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(x,t) \cong S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(x,L_S t)$$

for any $x \in \mathcal{T}^c$ and $t \in \mathcal{T}$. Let d be the degree of s. Applying $\operatorname{Hom}^*_{\mathcal{T}}(-,t)$ to the exact triangle

$$x \xrightarrow{s} \Sigma^d x \to \operatorname{cone}(s) \otimes x$$

yields an exact sequence

$$\begin{aligned} 0 = \operatorname{Hom}^*_{\mathcal{T}}(\operatorname{cone}(s) \otimes x, L_S t) &\to \operatorname{Hom}^*_{\mathcal{T}}(x, L_S t)[-d] \\ &\xrightarrow{s} \operatorname{Hom}^*_{\mathcal{T}}(x, L_S t) \to \operatorname{Hom}^*_{\mathcal{T}}(\operatorname{cone}(s) \otimes x, L_S t)[1] = 0. \end{aligned}$$

Thus multiplying by s is an isomorphism on $\operatorname{Hom}^*_{\mathcal{T}}(x, L_S t)$. Therefore we have

$$S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(x, L_S t) \cong \operatorname{Hom}^*_{\mathcal{T}}(x, L_S t)$$

which completes the proof. $\square$

3.3.6 Example. Recall from [Bal10, Corollary 9.5] that the Balmer spectrum

$\operatorname{Spc}(\mathrm{SH}^c)$ of the stable homotopy category together with its comparison map can be depicted as follows:

$$
\begin{tikzcd}
 & \mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{p,\infty} & \cdots \\
\operatorname{Spc}(\mathrm{SH}^c) = \arrow[dddddd, "\rho_{\mathrm{SH}^c}"'] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
 & \mathcal{P}_{2,n+1} \arrow[u] & \mathcal{P}_{3,n+1} \arrow[u] & \cdots & \mathcal{P}_{p,n+1} \arrow[u] & \cdots \\
 & \mathcal{P}_{2,n} \arrow[u] & \mathcal{P}_{3,n} \arrow[u] & \cdots & \mathcal{P}_{p,n} \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
 & \mathcal{P}_{2,1} \arrow[u] & \mathcal{P}_{3,1} \arrow[u] & \cdots & \mathcal{P}_{p,1} \arrow[u] & \cdots \\
 & & & \mathcal{P}_{p,0} \arrow[ull] \arrow[ul] \arrow[ur] & & \\
\operatorname{Spec}(\mathbb{Z}) = & 2\mathbb{Z} & 3\mathbb{Z} & \cdots & p\mathbb{Z} & \cdots \\
 & & & (0) \arrow[ull] \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

See Example 2.3.19 for the explanation of the notion. Recall that for any prime number p, the p-local stable homotopy category $\mathrm{SH}_{(p)}$ is defined as the Bousfield localization of the stable homotopy category SH with respect to the mod-p Moore spectrum. By Proposition 3.3.5, $\mathrm{SH}_{(p)}$ can be realized as the algebraic localization at $p\mathbb{Z} \in \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathrm{SH}}(\mathbb{1})) \cong \operatorname{Spec}(\mathbb{Z})$ of SH. Moreover, since we have

$$\mathcal{P}_{p,\infty} = \operatorname{thick}_{\otimes}\langle \operatorname{cone}(s) \mid s \notin p\mathbb{Z} \rangle$$

by [Bal10, Corollary 9.5(c)], the algebraic localization at $p\mathbb{Z}$ coincides with the localization at $\mathcal{P}_{p,\infty}$ in the sense of Example 2.4.14. Therefore, the Balmer spectrum

$\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ can be identified with

$$\mathrm{gen}(\mathcal{P}_{p,\infty}) = \{\mathcal{P}_{p,n} \mid 0 \leq n \leq \infty\} \subset \mathrm{Spc}(\mathrm{SH}^c).$$

For any spectrum $t \in \mathrm{SH}$, we write $t_{(p)} \coloneqq t_{p\mathbb{Z}} = t_{\mathcal{P}_{p,\infty}}$ for the corresponding p-local spectrum.

3.3.7 Remark. The space $\mathrm{Spc}(\mathrm{SH}^c)$ is not weakly Noetherian. Indeed, by [Bal10, Corollary 9.5] any Thomason subset of $\mathrm{Spc}(\mathrm{SH}^c)$ is the union of subsets of the form $\overline{\{\mathcal{P}_{p,n_p}\}}$ where p is a prime number and $0 \leq n_p < \infty$. Thus the closed point $\mathcal{P}_{p,\infty}$ is not weakly visible for any prime number p. In particular, any Thomason subset of $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ is of the form $\overline{\{\mathcal{P}_{p,n}\}}$ for $0 \leq n < \infty$ and therefore $\mathcal{P}_{p,\infty}$ is the only point in $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ that is not weakly visible.

3.3.8 Remark. In [BHS23b, Remark 11.11] the authors considered the Balmer–Favi support for $\mathrm{SH}_{(p)}$ excluding $\mathcal{P}_{p,\infty}$ since it is not weakly visible. More precisely, for any $t_{(p)} \in \mathrm{SH}_{(p)}$ they defined

$$\mathrm{Supp}_{<\infty}(t_{(p)}) \coloneqq \left\{ \mathcal{P}_{p,n} \,\middle|\, 0 \leq n < \infty \text{ and } g_{\mathcal{P}_{p,n}} \otimes t_{(p)} \neq 0 \right\}.$$

They then extended this support to the point $\mathcal{P}_{p,\infty}$ by declaring that $\mathcal{P}_{p,\infty}$ is in the support of a p-local spectrum $t_{(p)}$ if and only if $\mathrm{H}\mathbb{F}_p \otimes t_{(p)} \neq 0$. We denote their extended support function by $\mathrm{Supp}_{\leq\infty}$.

Let $I \in \mathrm{SH}$ be the Brown-Comenetz dual of the sphere spectrum. Note that the p-local Brown-Comenetz dual of the sphere spectrum $I_{(p)} \in \mathrm{SH}_{(p)}$ is isomorphic to the Brown-Comenetz dual of the p-local sphere spectrum as defined in [HP99, Section 7]. In [BHS23b, Remark 11.11] it was explained that $\mathrm{Supp}_{<\infty}(I_{(p)}) = \varnothing$ and $\mathrm{H}\mathbb{F}_p \otimes I_{(p)} = 0$. Hence $\mathrm{Supp}_{\leq\infty}(I_{(p)}) = \varnothing$. This motivates the following:

3.3.9 Definition (The detection property). We say that $\mathcal{T}$ has the *detection prop-*

erty if $\mathrm{Supp}(t) = \varnothing$ implies $t = 0$ for every $t \in \mathcal{T}$.

3.3.10 Remark. If the space $\mathrm{Spc}(\mathcal{T}^c)$ is Noetherian then $\mathcal{T}$ has the detection property by [BHS23b, Theorem 3.22 and Remark 3.9]. For example, the derived category $\mathrm{D}(R)$ of a commutative Noetherian ring has the detection property. However, we do not know in what generality the detection property holds; see, for example, the discussion in [San17, Section 8.1] and [BCHS25, Remark 6.6].

3.3.11 Example. In Remark 3.3.8 we see that the function $\mathrm{Supp}_{\leq\infty}$ does not detect vanishing of objects in $\mathrm{SH}_{(p)}$. In contrast, the BFS support Supp does:

3.3.12 Proposition. *The category* $\mathrm{SH}_{(p)}$ *satisfies the detection property.*

Proof. Suppose $\mathrm{Supp}(t_{(p)}) = \varnothing$ for some $t_{(p)} \in \mathrm{SH}_{(p)}$. Since $\mathcal{P}_{p,\infty} \notin \mathrm{Supp}(t_{(p)})$, we have $e_{\overline{\{\mathcal{P}_{p,n}\}}} \otimes t_{(p)} = 0$ for some $0 \leq n < \infty$ and thus $t_{(p)} \simeq f_{\overline{\{\mathcal{P}_{p,n}\}}} \otimes t_{(p)}$. If $n = 0$ then $t_{(p)} = 0$ as desired. Now suppose $n > 0$. By $\mathcal{P}_{p,n-1} \notin \mathrm{Supp}(t_{(p)})$ we have

$$0 = e_{\overline{\{\mathcal{P}_{p,n-1}\}}} \otimes f_{\overline{\{\mathcal{P}_{p,n}\}}} \otimes t_{(p)} \simeq e_{\overline{\{\mathcal{P}_{p,n-1}\}}} \otimes t_{(p)}.$$

An induction shows that $0 = e_{\overline{\{\mathcal{P}_{p,0}\}}} \otimes t_{(p)} \simeq t_{(p)}$, which completes the proof. $\square$

3.3.13 Example. By Remark 3.3.8 and Proposition 3.3.12, the p-local Brown-Comenetz dual of the sphere spectrum has support $\mathrm{Supp}(I_{(p)}) = \{\mathcal{P}_{p,\infty}\}$.

3.3.14 Remark. The following corollary says that the BFS support of an object can be computed at its localizations at all closed points in the Balmer spectrum. Keep in mind the notation introduced in Example 2.4.14.

3.3.15 Corollary. *For every object* $t \in \mathcal{T}$ *we have*

$$\mathrm{Supp}(t) = \bigcup_{\substack{\mathcal{M} \in \mathrm{Spc}(\mathcal{T}^c) \\ \mathcal{M} \textit{ closed}}} \varphi_{\mathcal{M}}(\mathrm{Supp}(t_{\mathcal{M}}))$$

where $\varphi_{\mathcal{M}} \coloneqq \mathrm{Spc}(\mathcal{T}^c_{\mathcal{M}}) \hookrightarrow \mathrm{Spc}(\mathcal{T}^c)$ *is the map induced by the localization at* $\mathcal{M}$.

Proof. By Corollary 3.2.20(b), for any $t \in \mathcal{T}$ we have $\mathrm{Supp}(t_{\mathcal{M}}) = \varphi_{\mathcal{M}}^{-1}(\mathrm{Supp}(t))$ and therefore $\varphi_{\mathcal{M}}(\mathrm{Supp}(t_{\mathcal{M}})) = \mathrm{Supp}(t) \cap \mathrm{im}\,\varphi_{\mathcal{M}} = \mathrm{Supp}(t) \cap \mathrm{gen}(\mathcal{M})$. The result then follows from [Bal05, Proposition 2.11]. $\square$

3.3.16 Example. From Example 3.3.6, Example 3.3.13, and Corollary 3.3.15 we obtain $\mathrm{Supp}(I) = \left\{ \mathcal{P}_{p,\infty} \,\middle|\, p \text{ prime} \right\}$. In other words, the Brown-Comenetz dual of the sphere spectrum I is supported on the line at infinity in the picture of $\mathrm{Spc}(\mathrm{SH}^c)$.

3.3.17 Theorem. *The category $\mathcal{T}$ has the detection property if and only if for every closed point $\mathfrak{m} \in \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ the algebraic localization $\mathcal{T}_{\mathfrak{m}}$ has the detection property .*

Proof. If $\mathcal{T}$ satisfies the detection property then so does the algebraic localization $\mathcal{T}_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ by Corollary 3.2.20(a). Conversely, suppose that $\mathcal{T}_{\mathfrak{m}}$ satisfies the detection property for every closed point $\mathfrak{m} \in \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$. If $t \in \mathcal{T}$ with $\mathrm{Supp}(t) = \varnothing$, then by Proposition 3.2.19(a) we have $\mathrm{Supp}(t_{\mathfrak{m}}) = \varnothing$ and thus $t_{\mathfrak{m}} = 0$. It follows from Proposition 3.3.5 that $\mathrm{Hom}(x,t)_{\mathfrak{m}} \cong \mathrm{Hom}(x, t_{\mathfrak{m}}) = 0$ for every $x \in \mathcal{T}^c$. Since $\mathfrak{m}$ ranges over all the closed points, we have $\mathrm{Hom}(x,t) = 0$ for every $x \in \mathcal{T}^c$ and therefore $t = 0$, which completes the proof. $\square$

3.3.18 Example. The stable homotopy category $\mathrm{SH}_{(p)}$ of p-local spectra satisfies the detection property (Proposition 3.3.12). The theorem above thus implies that the stable homotopy category SH of all spectra satisfies the detection property.

3.3.19 Corollary. *Any spectrum whose support is contained in $\left\{ \mathcal{P}_{p,\infty} \,\middle|\, p \text{ prime} \right\}$ is dissonant.*

Proof. According to [Rav84, Definition 4.1], a spectrum $t \in \mathrm{SH}$ is called dissonant if $t \otimes K(p,n) = 0$ for every prime number p and every n with $0 \leq n < \infty$. Suppose that $t \in \mathrm{SH}$ is a spectrum with $\mathrm{Supp}(t) \subseteq \left\{ \mathcal{P}_{p,\infty} \,\middle|\, p \text{ prime} \right\}$. It follows that for any prime number p and any n with $0 \leq n < \infty$ we have

$$\varnothing = \mathrm{Supp}(t) \cap \mathrm{gen}(\mathcal{P}_{p,n}) = \mathrm{Supp}(t \otimes f_{Y_{\mathcal{P}_{p,n}}})$$

where the second equality is due to Proposition 3.2.11(f). By Example 3.3.18, we have $t \otimes f_{Y_{\mathcal{P}_{p,n}}} = 0$, which is equivalent to $t \in \mathrm{Loc}_{\otimes}\langle \mathcal{P}_{p,n} \rangle$ by (2.4.15). It then follows from the definition of $\mathcal{P}_{p,n}$ that $t \otimes K(p,n) = 0$. This is true for every prime number p and every n with $0 \leq n < \infty$. Therefore t is dissonant. $\square$

Chapter 4

Stratification

4.1 Stratification

In this section, we use the BFS support theory in Section 3.2 to define a notion of stratification as in [BHS23b]. Our main result (Theorem 4.1.14) justifies the weakly Noetherian hypothesis in [BHS23b].

4.1.1 Hypothesis. Throughout this section, $\mathcal{T}$ denotes a rigidly-compactly generated tt-category.

4.1.2 Definition. We say that $\mathcal{T}$ is *stratified* if the map

$$\Big\{\text{localizing ideals of } \mathcal{T}\Big\} \to \Big\{\text{localizing closed subsets of } \operatorname{Spc}(\mathcal{T}^c)\Big\}$$
$$\mathcal{L} \mapsto \operatorname{Supp}(\mathcal{L})$$

is a bijection.

4.1.3 Remark. The map above is well-defined by Proposition 3.2.15(b). If $\operatorname{Spc}(\mathcal{T}^c)$ is weakly Noetherian then Definition 4.1.2 recovers [BHS23b, Definition 4.4]. In fact, we will see that if $\mathcal{T}$ is stratified in the sense of Definition 4.1.2 then $\operatorname{Spc}(\mathcal{T}^c)$ is necessarily weakly Noetherian. Our proof is based on the comparison between the BFS support and the *homological support*, which was studied in detail under the weakly Noetherian assumption in [BHS23a].

4.1.4 Recollection. Recall that each homological prime $\mathcal{B} \in \operatorname{Spc}^{\mathrm{h}}(\mathcal{T}^c)$ gives rise to a pure-injective object $E_{\mathcal{B}} \in \mathcal{T}$ and the homological support of an object $t \in \mathcal{T}$ is given by

$$\operatorname{Supp}^{\mathrm{h}}(t) \coloneqq \left\{ \mathcal{B} \in \operatorname{Spc}^{\mathrm{h}}(\mathcal{T}^c) \,\middle|\, [t, E_{\mathcal{B}}] \neq 0 \right\}$$

where $[-,-]$ denotes the internal hom. For every homological prime $\mathcal{B} \in \operatorname{Spc}^{\mathrm{h}}(\mathcal{T}^c)$ we have $\operatorname{Supp}^{\mathrm{h}}(E_{\mathcal{B}}) = \{\mathcal{B}\}$. By [Bal20a, Theorem 1.2] the homological support satisfies the tensor product formula

$$\operatorname{Supp}^{\mathrm{h}}(t_1 \otimes t_2) = \operatorname{Supp}^{\mathrm{h}}(t_1) \cap \operatorname{Supp}^{\mathrm{h}}(t_2) \quad \text{for any } t_1, t_2 \in \mathcal{T}.$$

Moreover, there exists a surjective continuous map $\phi\colon \operatorname{Spc}^{\mathrm{h}}(\mathcal{T}^c) \twoheadrightarrow \operatorname{Spc}(\mathcal{T}^c)$; see [Bal20b, Corollary 3.9].

4.1.5 Lemma. *If $W \subseteq \operatorname{Spc}(\mathcal{T}^c)$ is weakly visible then $\operatorname{Supp}^{\mathrm{h}}(g_W) = \phi^{-1}(W)$.*

Proof. Apply [BHS23a, Lemma 3.8] and the tensor-product formula. □

4.1.6 Lemma. *For every $t \in \mathcal{T}$ we have $\phi(\operatorname{Supp}^{\mathrm{h}}(t)) \subseteq \operatorname{Supp}(t)$.*

Proof. If $\mathcal{B} \in \operatorname{Supp}^{\mathrm{h}}(t)$ then for any weakly visible subset $W \ni \mathcal{P} \coloneqq \phi(\mathcal{B})$ we have $\mathcal{B} \in \phi^{-1}(W) = \operatorname{Supp}^{\mathrm{h}}(g_W)$ by Lemma 4.1.5 and hence $\mathcal{B} \in \operatorname{Supp}^{\mathrm{h}}(t \otimes g_W)$ in view of the tensor-product formula. In particular, $t \otimes g_W \neq 0$. This is true for every weakly visible subset W containing $\mathcal{P}$, so $\mathcal{P} \in \operatorname{Supp}(t)$. □

4.1.7 Lemma. *For every $\mathcal{B} \in \operatorname{Spc}^{\mathrm{h}}(\mathcal{T}^c)$ we have $\operatorname{Supp}(E_{\mathcal{B}}) = \{\phi(\mathcal{B})\}$.*

Proof. Let $\mathcal{P} \coloneqq \phi(\mathcal{B})$. First we show that $E_{\mathcal{B}}$ is $\mathcal{P}$-local, i.e., $E_{\mathcal{B}} \simeq E_{\mathcal{B}} \otimes f_{Y_{\mathcal{P}}}$. For any object $x \in \mathcal{P}$ we have $\mathcal{P} \notin \operatorname{supp}(x)$. By Lemma 4.1.6 we have $\mathcal{B} \notin \operatorname{Supp}^{\mathrm{h}}(x)$, that is, $[x, E_{\mathcal{B}}] = 0$. This holds for every $x \in \mathcal{P}$, so $E_{\mathcal{B}} \in \operatorname{Loc}\langle\mathcal{P}\rangle^{\perp} = f_{Y_{\mathcal{P}}} \otimes \mathcal{T}$

is $\mathcal{P}$-local. Thus $\mathrm{Supp}(E_{\mathcal{B}}) \subseteq \mathrm{gen}(\mathcal{P})$. On the other hand, we claim that $E_{\mathcal{B}} \in \mathrm{Loc}_{\otimes}\langle e_{\mathrm{supp}(a)}\rangle$ for any object $a \notin \mathcal{P}$. Indeed, we have

$$\begin{aligned} a \notin \mathcal{P} &\iff \mathcal{P} \in \mathrm{supp}(a) \\ &\iff \mathcal{B} \notin \phi^{-1}(\mathrm{supp}(a)^c) = \mathrm{Supp}^{\mathrm{h}}(f_{\mathrm{supp}(a)}) && \text{by Lemma 4.1.5} \\ &\iff E_{\mathcal{B}} \otimes f_{\mathrm{supp}(a)} = 0 \\ &\iff E_{\mathcal{B}} \in \mathrm{Loc}_{\otimes}\langle e_{\mathrm{supp}(a)}\rangle && \text{by (2.4.13)} \end{aligned}$$

where the second-to-last equivalence is due to [Bal20a, Theorem 1.8] and the fact that $f_{\mathrm{supp}(a)}$ is a (weak) ring object. Therefore we have $E_{\mathcal{B}} \otimes e_{\mathrm{supp}(a)} \simeq E_{\mathcal{B}}$ for any $a \notin \mathcal{P}$. Invoking the half-tensor formula (Lemma 3.2.13) we obtain

$$\mathrm{Supp}(E_{\mathcal{B}}) \subseteq \bigcap_{a \notin \mathcal{P}} \mathrm{supp}(a) = \overline{\{\mathcal{P}\}}.$$

Therefore

$$\mathrm{Supp}(E_{\mathcal{B}}) \subseteq \overline{\{\mathcal{P}\}} \cap \mathrm{gen}(\mathcal{P}) = \{\mathcal{P}\}.$$

It remains to prove $\mathcal{P} \in \mathrm{Supp}(E_{\mathcal{B}})$. From what we have shown it follows that $0 \neq E_{\mathcal{B}} \simeq E_{\mathcal{B}} \otimes f_{Y_{\mathcal{P}}} \simeq E_{\mathcal{B}} \otimes e_{\mathrm{supp}(a)} \otimes f_{Y_{\mathcal{P}}}$ for any $a \notin \mathcal{P}$. By Remark 3.2.8 we conclude that $\mathcal{P} \in \mathrm{Supp}(E_{\mathcal{B}})$. □

4.1.8 Example. By [BC21, Corollary 3.6], the Morava K-theory spectrum $K(p,n)$, for a prime p and $0 \leq n \leq \infty$, is isomorphic to $E_{\mathcal{B}_{p,n}}$, where $\mathcal{B}_{p,n} \in \mathrm{Spc}^{\mathrm{h}}(\mathrm{SH}^c)$ is the homological prime corresponding to the Balmer prime $\mathcal{P}_{p,n} \in \mathrm{Spc}(\mathrm{SH}^c)$. We then have $\mathrm{Supp}(K(p,n)) = \{\mathcal{P}_{p,n}\}$ by Lemma 4.1.7. In particular, the mod-p Eilenberg-Maclane spectrum $\mathrm{H}\mathbb{F}_p = K(p,\infty)$ is supported on the singleton $\{\mathcal{P}_{p,\infty}\}$.

4.1.9 Corollary. *If $\mathcal{T}$ satisfies the detection property then the map in Definition 4.1.2 is surjective.*

Proof. Let $F \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a localizing closed subset. Choosing a homological prime $\mathcal{B}_{\mathcal{P}} \in \phi^{-1}(\{\mathcal{P}\})$ for every $\mathcal{P} \in F$, by Lemma 4.1.7 and [San17, Theorem 4.7(4)] we have

$$\mathrm{Supp}(\mathrm{Loc}_{\otimes}\langle E_{\mathcal{B}_{\mathcal{P}}} \mid \mathcal{P} \in F\rangle) = \overline{\bigcup_{\mathcal{P}\in F} \mathrm{Supp}(E_{\mathcal{B}_{\mathcal{P}}})}^{\mathrm{loc}} = \overline{F}^{\mathrm{loc}} = F. \qquad \square$$

4.1.10 Remark. In [BHS23b, Lemma 3.4] it was proved that if $\mathrm{Spc}(\mathcal{T}^c)$ is weakly Noetherian then the map above is surjective (without assuming the detection property).

4.1.11 Example. The Balmer spectrum $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ is not weakly Noetherian (Remark 3.3.7) but $\mathrm{SH}_{(p)}$ satisfies the detection property (Proposition 3.3.12). Thus every localizing closed subset of $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ can be realized as the support of some localizing ideal of $\mathrm{SH}^c_{(p)}$. Moreover, a subset $S \subseteq \mathrm{Spc}(\mathrm{SH}^c_{(p)})$ is localizing closed if and only if either $\mathcal{P}_{p,\infty} \in S$ or $\mathcal{P}_{p,\infty} \notin S$ and S is finite. This follows from that fact that the Thomason subsets of $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$ are of the form $\overline{\{\mathcal{P}_{p,i}\}}$ for $0 \leq i < \infty$; see Remark 3.3.7.

4.1.12 Remark. The following result indicates that the weakly Noetherian hypothesis in [BHS23a, Theorem 4.7] is unnecessary.

4.1.13 Proposition. *If $\mathcal{T}$ is stratified then $\phi(\mathrm{Supp}^{\mathrm{h}}(t)) = \mathrm{Supp}(t)$ for any $t \in \mathcal{T}$.*

Proof. By Lemma 4.1.6 the inclusion $\phi(\mathrm{Supp}^{\mathrm{h}}(t)) \subseteq \mathrm{Supp}(t)$ always holds. To prove the other inclusion, let $\mathcal{P} \in \mathrm{Supp}(t)$ and choose any $\mathcal{B} \in \phi^{-1}(\{\mathcal{P}\})$. In light of Lemma 4.1.7, we have $\mathrm{Supp}(E_{\mathcal{B}}) = \{\mathcal{P}\} \subseteq \mathrm{Supp}(t)$ and thus $E_{\mathcal{B}} \in \mathrm{Loc}_{\otimes}\langle t\rangle$ due to stratification, which implies $[t, E_{\mathcal{B}}] \neq 0$ since $[E_{\mathcal{B}}, E_{\mathcal{B}}] \neq 0$. This completes the proof. $\square$

4.1.14 Theorem. *If $\mathcal{T}$ is stratified then $\mathrm{Spc}(\mathcal{T}^c)$ is weakly Noetherian.*

Proof. Let W be any subset of $\mathrm{Spc}(\mathcal{T}^c)$. It suffices to show that W is localizing closed by Remark 2.1.16. To see this, we choose a $\mathcal{B}_\mathcal{P} \in \phi^{-1}(\{\mathcal{P}\})$ for every $\mathcal{P} \in W$ and consider the object $t_W := \coprod_{\mathcal{P}\in W} E_{\mathcal{B}_\mathcal{P}}$. By [Bal20a, Proposition 4.3(b)] we have $\mathrm{Supp}^{\mathrm{h}}(t_W) = \left\{ \mathcal{B}_\mathcal{P} \,\middle|\, \mathcal{P} \in W \right\}$. It then follows from Proposition 4.1.13 and Proposition 3.2.15(a) that $W = \mathrm{Supp}(t_W)$ is localizing closed. □

4.1.15 Remark. The theorem above shows that the generalization of the Balmer–Favi support to the BFS support does not broaden the scope of the stratification theory developed in [BHS23b].

4.1.16 Example. The Balmer spectrum $\mathrm{Spc}(\mathrm{SH}^c)$ is not weakly Noetherian and therefore the stable homotopy category SH is not stratified.

4.2 BIK vs BFS

In this section, we apply the machinery developed in Section 3.1 to a rigidly-compactly generated tt-category $\mathcal{T}$. This gives rise to a stratification theory in the sense of [BIK11b, BIK11a] but without requiring any additional Noetherian assumptions. We then compare it with the stratifcation theory in the previous section.

4.2.1 Notation. Throughout this section, $\mathcal{T}$ will denote a rigidly-compactly generated tt-category. Recall that the graded endomorphism ring $\mathrm{End}^*_\mathcal{T}(\mathbb{1})$ of the unit object is graded-commutative and it canonically acts on $\mathcal{T}$; see [DS13, Example 2.5(2)] and [BIK11b, Section 7].

4.2.2 Remark. By applying Section 3.1 to the canonical action by $\mathrm{End}^*_\mathcal{T}(\mathbb{1})$, we obtain a BIK support function $\mathrm{Supp}_{\mathrm{BIK}}$ on $\mathcal{T}$ which takes values in $\mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_\mathcal{T}(\mathbb{1}))$. Moreover, the tensor structure gives another way of describing the BIK support:

4.2.3 Proposition. *For any object $t \in \mathcal{T}$ we have*

$$\operatorname{Supp}_{\mathrm{BIK}}(t) = \left\{ \mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})) \;\middle|\; \begin{array}{c} \mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}\mathbb{1} \otimes L_{\mathcal{Z}(\mathfrak{p})}\mathbb{1} \otimes t \neq 0 \textit{ for any finitely} \\ \textit{generated ideal } \mathfrak{a} \textit{ contained in } \mathfrak{p} \end{array} \right\}.$$

Proof. By Proposition 3.1.10 we have

$$\mathcal{T}_{\mathcal{V}(\mathfrak{a})} = \operatorname{Loc}\langle x /\!\!/ \mathfrak{a} \mid x \in \mathcal{T}^c \rangle$$

for any finitely generated ideal $\mathfrak{a}$ and

$$\mathcal{T}_{\mathcal{Z}(\mathfrak{p})} = \operatorname{Loc}\langle x /\!\!/ r \mid x \in \mathcal{T}^c \text{ and } r \in \operatorname{End}^*_{\mathcal{T}}(\mathbb{1}) \setminus \mathfrak{p} \text{ homogeneous} \rangle$$

for any prime ideal $\mathfrak{p}$. Note that for any $x \in \mathcal{T}^c$ we have $x /\!\!/ \mathfrak{a} \simeq x \otimes \mathbb{1} /\!\!/ \mathfrak{a}$. This follows from the definition of Koszul object and the exactness of tensor product. Hence $\operatorname{Loc}\langle x /\!\!/ \mathfrak{a} \mid x \in \mathcal{T}^c \rangle = \operatorname{Loc}_{\otimes}\langle \mathbb{1} /\!\!/ \mathfrak{a} \rangle$. Therefore $\mathcal{T}_{\mathcal{V}(\mathfrak{a})}$ is a compactly generated localizing ideal of $\mathcal{T}$ and thus a smashing localizing ideal of $\mathcal{T}$ by [BF11, Theorem 4.1]. Similarly, $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ is a smashing localizing ideal of $\mathcal{T}$. It then follows from [BF11, Theorem 2.13] that the functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$ is isomorphic to $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}\mathbb{1} \otimes -$ and that the functor $L_{\mathcal{Z}(\mathfrak{p})}$ is isomorphic to $L_{\mathcal{Z}(\mathfrak{p})}\mathbb{1} \otimes -$. This finishes the proof. □

4.2.4 Remark. Next, we relate the BIK support to the BFS support via the comparison map $\rho : \operatorname{Spc}(\mathcal{T}^c) \to \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$.

4.2.5 Lemma. *Let $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$. The BIK localization functor $L_{\mathcal{Z}(\mathfrak{p})}$ is the finite localization functor associated to the Thomason subset $\rho^{-1}(\mathcal{Z}(\mathfrak{p}))$.*

Proof. Observe that

$$\mathcal{T}_{\mathcal{Z}(\mathfrak{p})} = \operatorname{Loc}\langle x /\!\!/ s \mid x \in \mathcal{T}^c, s \in S_{\mathfrak{p}} \rangle = \operatorname{Loc}_{\otimes}\langle \operatorname{cone}(s) \mid s \in S_{\mathfrak{p}} \rangle$$

where the first equality follows from Proposition 3.1.10(b) and the second equality is explained in the proof of Proposition 4.2.3. Therefore, the localization functor $L_{\mathcal{Z}(\mathfrak{p})}$ is identical to the algebraic localization functor $L_{S_\mathfrak{p}}$ which is associated to the Thomason subset $\rho^{-1}(\bigcup_{s\in S_\mathfrak{p}} \mathcal{V}(s)) = \rho^{-1}(\mathcal{Z}(\mathfrak{p}))$. $\square$

4.2.6 Lemma. *Let $\mathfrak{a}$ be a finitely generated homogeneous ideal of $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$. The BIK colocalization functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$ is the finite colocalization functor associated to the Thomason subset $\rho^{-1}(\mathcal{V}(\mathfrak{a}))$.*

Proof. Let $\mathfrak{a} = (x_1, \dots, x_n)$ be a finitely generated homogeneous ideal of $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ with homogeneous generators $\{x_i\}_{1\le i\le n}$. We then have

$$\mathcal{T}_{\mathcal{V}(\mathfrak{a})} = \mathrm{Loc}\langle x /\!\!/ \mathfrak{a} \mid x \in \mathcal{T}^c\rangle = \mathrm{Loc}_{\otimes}\langle \mathbb{1}/\!\!/\mathfrak{a}\rangle = \mathrm{Loc}_{\otimes}\langle \bigotimes_{1\le i\le n} \mathrm{cone}(x_i)\rangle$$

where the first equality is due to Proposition 3.1.10(a) and the second equality is explained in the proof of Proposition 4.2.3. It follows that the BIK colocalization functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$ is the finite colocalization functor associated to the Thomason subset

$$\bigcap_{1\le i\le n} \mathrm{supp}(\mathrm{cone}(x_i)) = \bigcap_{1\le i\le n} \rho^{-1}(\mathcal{V}(x_i)) = \rho^{-1}(\mathcal{V}(\mathfrak{a})).$$ $\square$

4.2.7 Corollary. *We have*

$$\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} t \simeq e_{\rho^{-1}(\mathcal{V}(\mathfrak{a}))} \otimes f_{\rho^{-1}(\mathcal{Z}(\mathfrak{p}))} \otimes t$$

for every $t \in \mathcal{T}$ and every finitely generated homogeneous ideal $\mathfrak{a}$ and every prime ideal $\mathfrak{p}$ of $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$.

Proof. Combine Lemma 4.2.5 and Lemma 4.2.6. $\square$

4.2.8 Theorem. *Consider the comparison map $\rho : \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$. We have:*

(a) $\rho(\mathrm{Supp}(t)) \subseteq \mathrm{Supp}_{\mathrm{BIK}}(t)$ *for every $t \in \mathcal{T}$.*

(b) If ρ is a homeomorphism then $\rho(\mathrm{Supp}(t)) = \mathrm{Supp}_{\mathrm{BIK}}(t)$ for every $t \in \mathcal{T}$.

Proof. Let $\mathcal{P} \in \mathrm{Supp}(t)$. If $\mathfrak{p} := \rho(\mathcal{P}) \notin \mathrm{Supp}_{\mathrm{BIK}}(t)$ then by Corollary 4.2.7 there exists a finitely generated homogeneous ideal $\mathfrak{a} \subseteq \mathfrak{p}$ such that

$$0 = \mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})} L_{\mathcal{Z}(\mathfrak{p})} t = e_{\rho^{-1}(\mathcal{V}(\mathfrak{a}))} \otimes f_{\rho^{-1}(\mathcal{Z}(\mathfrak{p}))} \otimes t.$$

Since the point $\mathcal{P}$ is contained in the weakly visible subset $\rho^{-1}(\mathcal{V}(\mathfrak{a})) \cap \rho^{-1}(\mathcal{Z}(\mathfrak{p}))^c$, we have $\mathcal{P} \notin \mathrm{Supp}(t)$, which establishes (a). To show (b), let $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ correspond to some $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$. Note that $\rho^{-1}(\mathcal{Z}(\mathfrak{p})) = Y_{\mathcal{P}}$ because ρ is a homeomorphism. Then observe that

$$\begin{aligned} \mathfrak{p} \notin \mathrm{Supp}_{\mathrm{BIK}}(t) &\iff \exists \text{ a Thomason closed } \mathcal{V} \ni \mathfrak{p} : e_{\rho^{-1}(\mathcal{V})} \otimes f_{\rho^{-1}(\mathcal{Z}(\mathfrak{p}))} \otimes t = 0 \\ &\iff \exists \text{ a Thomason closed } V \ni \mathcal{P} : e_V \otimes f_{Y_{\mathcal{P}}} \otimes t = 0 \\ &\iff \mathcal{P} \notin \mathrm{Supp}(t). \end{aligned}$$

□

4.2.9 Corollary. *For any compact object $x \in \mathcal{T}^c$ we have*

$$\rho(\mathrm{supp}(x)) \subseteq \mathrm{Supp}_{\mathrm{BIK}}(x) \subseteq \mathrm{Supp}_{\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})} \mathrm{End}^*_{\mathcal{T}}(x).$$

Moreover, if ρ is a homeomorphism then these inclusions are equalities.

Proof. The first inclusion is a special case of Theorem 4.2.8(a). For the second inclusion, let $\mathfrak{p} \notin \mathrm{Supp}_{\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})} \mathrm{End}^*_{\mathcal{T}}(x)$. By [Lau23, (2.2)] we have $L_{\mathcal{Z}(\mathfrak{p})} x = 0$ and thus $\mathfrak{p} \notin \mathrm{Supp}_{\mathrm{BIK}}(x)$. This establishes the second inclusion. The equalities follow from [Lau23, Proposition 2.10]. □

4.2.10 Remark. We now give a notion of stratification with respect to the canonical BIK support function which takes values in $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T}) \subseteq \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$:

4.2.11 Definition. We say that $\mathcal{T}$ is *canonically BIK-stratified* if the map

$$\Big\{\text{localizing ideals of } \mathcal{T}\Big\} \to \Big\{\text{closed subsets of } \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T})\Big\}$$
$$\mathcal{L} \mapsto \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{L})$$

is a bijection, where $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T})$ is equipped with the subspace topology of the localizing topology on $\mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$.

4.2.12 Remark. The map above is well-defined by Proposition 3.1.25(b). Moreover, if the comparison map $\rho\colon \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ is surjective then we have $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T}) = \mathrm{Supp}_{\mathrm{BIK}}(\mathbb{1}) = \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ by part (a) of Theorem 4.2.8.

4.2.13 Example. If $\mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ is Noetherian then ρ is surjective by [Bal10, Theorem 7.3] and hence $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T}) = \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$.

4.2.14 Example. For a commutative ring A, the unbounded derived category $\mathrm{D}(A)$ is rigidly-compactly generated and the derived category $\mathrm{D}^{\mathrm{perf}}(A)$ of perfect complexes can be identified with its subcategory of rigid-compact objects. The associated comparison map $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(A)) \to \mathrm{Spec}(A)$ is a homeomorphism by [Bal10, Proposition 8.1], so we have $\mathrm{Supp}_{\mathrm{BIK}}(\mathrm{D}(A)) = \mathrm{Spec}(A)$.

4.2.15 Corollary. *If the comparison map ρ is a homeomorphism then $\mathcal{T}$ is canonically BIK-stratified if and only if it is stratified.*

Proof. By Remark 4.2.12 the BIK space of supports $\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{T})$ is $\mathrm{Spec}^{\mathrm{h}}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$. The result thus follows from Theorem 4.2.8(b). □

4.2.16 Example. Let A be a commutative ring. By Example 4.2.14 we can identify $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(A))$ with $\mathrm{Spec}(A)$ via the comparison map, under which the BFS support and the canonical BIK support coincide, according to Theorem 4.2.8. For any prime $\mathfrak{p} = \rho(\mathcal{P}) \in \mathrm{Spec}(A)$ and any finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{p}$, it follows from [San17, Lemma 5.1] that

$$f_{\rho^{-1}(\mathcal{Z}(\mathfrak{p}))} = f_{Y_{\mathcal{P}}} \simeq A_{\mathfrak{p}} \text{ and } e_{\rho^{-1}(\mathcal{V}(\mathfrak{a}))} = e_{\mathrm{supp}(\mathbb{1}/\!\!/\mathfrak{a})} \simeq K^{\infty}(\mathfrak{a})$$

where $K^{\infty}(\mathfrak{a})$ is the stable Koszul complex of $\mathfrak{a}$. Hence for a complex $X \in \mathrm{D}(A)$ we have

$$\mathrm{Supp}(X) = \left\{ \mathfrak{p} \in \mathrm{Spec}(A) \,\middle|\, \begin{array}{l} K^{\infty}(\mathfrak{a}) \otimes X_{\mathfrak{p}} \neq 0 \text{ for any finitely} \\ \text{generated ideal } \mathfrak{a} \text{ contained in } \mathfrak{p} \end{array} \right\}.$$

This notion of support for complexes over a (non-Noetherian) commutative ring was first proposed and studied in [San17]. The condition $K^{\infty}(\mathfrak{a}) \otimes X_{\mathfrak{p}} \neq 0$ is equivalent to $A/\mathfrak{a} \otimes X_{\mathfrak{p}} \neq 0$ since $\mathrm{Loc}_{\otimes}\langle K^{\infty}(\mathfrak{a})\rangle = \mathrm{Loc}_{\otimes}\langle A/\mathfrak{a}\rangle$ by [Gre01, Proposition 5.6]. If the ideal $\mathfrak{p}$ itself is finitely generated then this condition amounts to $\kappa(\mathfrak{p}) \otimes X \neq 0$. Therefore, when A is Noetherian the BFS support recovers the support defined in [Fox79].

4.2.17 Remark. Neeman proved that $\mathrm{D}(A)$ is (canonically BIK-)stratified whenever A is Noetherian; see [Nee92, Theorem 2.8]. This result was extended to the absolutely flat approximations of topologically Noetherian commutative rings by Stevenson; see [Ste14, Theorem 4.23]. On the other hand, Neeman [Nee00] gave an example of a non-Noetherian commutative ring such that the stratification fails. It remains an open question to determine for which commutative rings stratification holds. However, our Corollary 4.2.15 and Theorem 4.1.14 show that for any commutative ring A, if $\mathrm{D}(A)$ is stratified then $\mathrm{Spec}(A)$ is necessarily weakly Noetherian.

4.2.18 Remark. In [San17, Theorem 5.5(3)] it was shown that if the prime ideals of a commutative ring A satisfy the descending chain condition then $\mathrm{D}(A)$ has the detection property. This can also be deduced from Theorem 4.2.8 and Corollary 3.1.20. In fact, our Corollary 3.1.20 generalizes [San17, Theorem 5.5(2)].

4.2.19 Example. For a Noetherian scheme X, the derived category $\mathrm{D}_{\mathrm{qc}}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology is stratified; see [BHS23b, Corollary 5.10]. However, $\mathrm{D}_{\mathrm{qc}}(X)$ is not canonically BIK-stratified in general, since the graded endomorphism ring of the unit object in this category, i.e., the sheaf cohomology ring $H(X, \mathcal{O}_X)$, may not have enough prime ideals when X is nonaffine; see [Bal10, Remark 8.2], for example.

Chapter 5

Applications

5.1 The stable module category

As an application of Theorem 4.2.8, we show that the stable module category is canonically BIK-stratified by the Tate cohomology ring.

5.1.1 Remark. Let G be a finite group and k a field of characteristic $p > 0$ such that p divides the order of G. The big stable module category $\mathrm{StMod}(kG)$ is BIK-stratified by $H^*(G, k)$ ([BIK11a, Theorem 10.3]). The associated Balmer spectrum $\mathrm{Spc}(\mathrm{stmod}(kG))$ is homeomorphic to $\mathrm{Proj}(H^*(G, k))$, which is a Noetherian space since $H^*(G, k)$ is a Noetherian ring by the Evens-Venkov theorem [Ben98, II(3.10)]. Note that this BIK-stratification for $\mathrm{StMod}(kG)$ is not canonical since the graded endomorphism ring of the unit for $\mathrm{StMod}(kG)$ is not $H^*(G, k)$ but rather the Tate cohomology ring $\hat{H}^*(G, k)$; see [BK02, page 26]. The original statement of Benson–Iyengar–Krause cannot be applied to the canonical action of $\hat{H}^*(G, k)$ on $\mathrm{StMod}(kG)$ since $\hat{H}^*(G, k)$ is rarely Noetherian. In fact, $\hat{H}^*(G, k)$ is Noetherian if and only if the p-rank of G is 1 if and only if $\hat{H}^*(G, k)$ is periodic; see [BIK08, Lemma 10.1].

On the other hand, $\mathrm{StMod}(kG)$ is stratified in the sense of Definition 4.1.2 by

[BHS23b, Example 7.12]. In the following we will show that $\operatorname{StMod}(kG)$ is in fact canonically BIK-stratified in the sense of Definition 4.2.11. In other words, it is canonically stratified by the Tate cohomology ring $\hat{H}^*(G,k)$. Beware that this does not follow directly from Theorem 4.2.8(b) because in this example the comparison map ρ is not a homeomorphism in general, as we shall see below.

5.1.2 Theorem. *Let G be a finite group and k a field of characteristic $p > 0$ such that p divides the order of G. The stable module category* $\operatorname{StMod}(kG)$ *is canonically BIK-stratified.*

Proof. By Rickard [Ric89] the small stable module category $\operatorname{stmod}(kG)$ is equivalent to the quotient $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}(kG))/\,\mathrm{D}^{\mathrm{perf}}(kG)$. Note that the graded endomorphism ring of the unit object of $\mathcal{K} := \mathrm{D}^{\mathrm{b}}(\operatorname{mod}(kG))$ is isomorphic to the group cohomology ring. Therefore, we can identify $H^*(G,k)$ with $\operatorname{End}^*_{\mathcal{K}}(\mathbb{1})$ and $\hat{H}^*(G,k)$ with $\operatorname{End}^*_{\operatorname{stmod}(kG)}(\mathbb{1})$; see Remark 5.1.1.

Now consider the functor $q\colon \mathcal{K} \to \mathcal{K}/\,\mathrm{D}^{\mathrm{perf}}(kG) \simeq \operatorname{stmod}(kG)$. The naturality of the comparision map gives us the following commutative diagram:

$$\begin{tikzcd} \operatorname{Spc}(\operatorname{stmod}(kG)) \arrow[r, hook, "\operatorname{Spc}(q)"] \arrow[d, hook, "\rho"] & \operatorname{Spc}(\mathcal{K}) \arrow[d, "\rho_{\mathcal{K}}", "\wr"'] \\ \operatorname{Spec}^{\mathrm{h}}(\hat{H}^*(G,k)) \arrow[r, "\operatorname{Spec}^{\mathrm{h}}(\iota)"] & \operatorname{Spec}^{\mathrm{h}}(H^*(G,k)) \end{tikzcd} \tag{5.1.3}$$

where $\operatorname{Spc}(q)$ is an open embedding and $\rho_{\mathcal{K}}$ is a homeomorphism by [Bal10, Proposition 8.5]. Moreover, $\iota\colon H^*(G,k) \hookrightarrow \hat{H}^*(G,k)$ is the first map that appears in [BIK11a, (10.2)], which views $H^*(G,k)$ as a subring of $\hat{H}^*(G,k)$; see also [BK02, (2.1)]. It follows that ρ is injective. On the other hand, we have the following commutative diagram from the proof of [Bal10, Proposition 8.5]:

$$\begin{tikzcd} \mathrm{Spc}(\mathrm{stmod}(kG)) \arrow[r, hook, "\mathrm{Spc}(q)"] \arrow[d, "\varphi^{-1}", "\wr"'] & \mathrm{Spc}(\mathcal{K}) \arrow[d, "\rho_{\mathcal{K}}", "\wr"'] \\ \mathrm{Proj}(H^*(G,k)) \arrow[r, hook] & \mathrm{Spec}^{\mathrm{h}}(H^*(G,k)) \end{tikzcd} \tag{5.1.4}$$

in which $\varphi\colon \mathrm{Proj}(H^*(G,k)) \xrightarrow{\sim} \mathrm{Spc}(\mathrm{stmod}(kG))$ is the homeomorphism described in [Bal05, Corollary 5.10] and $\mathrm{Proj}(H^*(G,k)) \hookrightarrow \mathrm{Spec}^{\mathrm{h}}(H^*(G,k))$ is the canonical open embedding which misses the unique closed point $H^+(G,k)$ in $\mathrm{Spec}^{\mathrm{h}}(H^*(G,k))$. Combining (5.1.3) and (5.1.4) we obtain a commutative diagram:

$$\begin{tikzcd} \mathrm{Spc}(\mathrm{stmod}(kG)) \arrow[r, "\varphi^{-1}", "\simeq"'] \arrow[d, hook, "\rho"] & \mathrm{Proj}(H^*(G,k)) \arrow[d, hook] \\ \mathrm{Spec}^{\mathrm{h}}(\hat{H}^*(G,k)) \arrow[r, "\mathrm{Spec}^{\mathrm{h}}(\iota)"] & \mathrm{Spec}^{\mathrm{h}}(H^*(G,k)). \end{tikzcd} \tag{5.1.5}$$

If $\hat{H}^*(G,k)$ is Noetherian then [Bal10, Theorem 7.3] implies that ρ is surjective and hence a bijection. By [BIK08, Lemma 10.1] $\hat{H}^*(G,k)$ being Noetherian is equivalent to that the p-rank of G equals 1, which implies that the Krull dimension of $\mathrm{Spec}^{\mathrm{h}}(H^*(G,k))$ is equal to 1 by Quillen stratification theorem [Qui71]. It follows that $\mathrm{Spc}(\mathrm{stmod}(kG)) \cong \mathrm{Proj}(H^*(G,k))$ has zero Krull dimension, that is, $\mathrm{Spc}(\mathrm{stmod}(kG))$ is a discrete space. Moreover, since the trivial representation is indecomposable, $\mathrm{Spc}(\mathrm{stmod}(kG))$ is a singleton by [Bal07, Theorem 2.11], which forces ρ to be a homeomorphism. Therefore, $\mathrm{StMod}(kG)$ is canonically BIK-stratified by Corollary 4.2.15.

If $\hat{H}^*(G,k)$ is not Noetherian (i.e., the p-rank of G is at least 2) then the negative part $\hat{H}^-(G,k)$ is nilpotent by [BK02, Proposition 2.4]. It follows that

$$\mathrm{Spec}^{\mathrm{h}}(\iota)\colon\ \mathrm{Spec}^{\mathrm{h}}(\hat{H}^*(G,k)) \to \mathrm{Spec}^{\mathrm{h}}(H^*(G,k))$$
$$\mathfrak{p} = \hat{H}^-(G,k) \oplus \mathfrak{p}^{\geq 0} \mapsto \mathfrak{p}^{\geq 0}$$

is a homeomorphism where $\mathfrak{p}^{\geq 0}$ denotes the nonnegative part of a graded prime $\mathfrak{p}$. On the other hand, by (5.1.5) the map $\rho\colon \operatorname{Spc}(\operatorname{stmod}(kG)) \hookrightarrow \operatorname{Spec}^{\mathrm{h}}(\hat{H}^*(G,k))$ is an open embedding which misses the unique closed point $\mathfrak{m} := \hat{H}^{i\neq 0}(G,k)$. Since $\operatorname{Spec}^{\mathrm{h}}(\hat{H}^*(G,k)) \cong \operatorname{Spec}^{\mathrm{h}}(H^*(G,k))$ is Noetherian, $\mathcal{V}(\mathfrak{m}) = \{\mathfrak{m}\}$ is Thomason closed. We thus have $e_{\rho^{-1}(\mathcal{V}(\mathfrak{m}))} \otimes f_{\rho^{-1}(\mathcal{Z}(\mathfrak{m}))} = e_{\varnothing} \otimes f_{\operatorname{Spc}(\operatorname{stmod}(kG))} \simeq 0 \otimes \mathbb{1} = 0$ and hence $\mathfrak{m} \notin \operatorname{Supp}_{\mathrm{BIK}}(t)$ for every $t \in \operatorname{StMod}(kG)$. It then follows from Theorem 4.2.8(a) that

$$\operatorname{Supp}_{\mathrm{BIK}}(\mathcal{T}) = \operatorname{Supp}_{\mathrm{BIK}}(\mathbb{1}) = \operatorname{im}\rho. \tag{5.1.6}$$

Now suppose that $\mathfrak{p} = \rho(\mathcal{P}) \in \operatorname{im}\rho$ is a nonclosed point in $\operatorname{Spec}^{\mathrm{h}}(\hat{H}^*(G,k))$. Note that $\{\mathfrak{p}\} = \mathcal{V}(\mathfrak{p}) \cap \mathcal{Z}(\mathfrak{p})^c$, where $\mathcal{V}(\mathfrak{p})$ is Thomason closed since $\operatorname{Spec}^{\mathrm{h}}(\hat{H}^*(G,k))$ is Noetherian. It follows that for every $t \in \operatorname{StMod}(kG)$ we have

$$\mathfrak{p} \in \operatorname{Supp}_{\mathrm{BIK}}(t) \iff e_{\rho^{-1}(\mathcal{V}(\mathfrak{p}))} \otimes f_{\rho^{-1}(\mathcal{Z}(\mathfrak{p}))} \otimes t \neq 0 \iff \mathcal{P} \in \operatorname{Supp}(t).$$

Therefore $\rho(\operatorname{Supp}(t)) = \operatorname{Supp}_{\mathrm{BIK}}(t)$ for all $t \in \operatorname{StMod}(kG)$. From (5.1.6) and the fact that $\operatorname{StMod}(kG)$ is stratified, we conclude that $\operatorname{StMod}(kG)$ is canonically BIK-stratified. □

5.1.7 Remark. Our definition of cohomological stratification (Definition 4.2.11) generalizes the one given by [BCH+25, Definition 2.21] which requires $\mathcal{T}$ to be Noetherian ([BCH+25, Definition 2.9]) and the comparison map ρ to be a homeomorphism. Indeed, if $\mathcal{T}$ is Noetherian then $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$ is Noetherian and thus every subset of $\operatorname{Supp}_{\mathrm{BIK}}(\mathcal{T})$ is localizing closed. Moreover, if ρ is a homeomorphism then $\operatorname{Supp}_{\mathrm{BIK}}(\mathcal{T}) = \operatorname{Spec}^{\mathrm{h}}(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$ by Remark 4.2.12. Note, however, that our Definition 4.2.11 does not put any restriction on ρ. As Theorem 5.1.2 shows, requiring ρ to be a homeomorphism would eliminate interesting examples of canonically BIK-stratified categories in the non-Noetherian context.

5.2 Convexity in tensor triangular geometry

In this section, we further explore the consequence of the comparison of the two support theories in Section 4.2. In particular, we show that for nicely stratified big tt-∞-categories $\mathcal{C}$, the dualizable localizing ideals correspond exactly to the convex subsets of the Balmer spectrum $\mathrm{Spc}(\mathcal{C}^c)$, generalizing a theorem of Efimov [Efi24, Theorem 3.10] to tensor triangular geometry.

5.2.1 Hypothesis. Throughout this section, $\mathcal{T}$ will denote a rigidly-compactly generated tt-category whose spectrum $\mathrm{Spc}(\mathcal{T}^c)$ is weakly Noetherian. We will occasionally assume that there is an underlying (rigidly-compactly generated) tt-∞-category $\mathcal{C}$ and write $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$ to indicate this; see Section 2.5.

5.2.2 Remark. Since $\mathrm{Spc}(\mathcal{T}^c)$ is weakly Noetherian, the BFS support recovers the Balmer–Favi support and is given by

$$\mathrm{Supp}(t) = \left\{ \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \,\middle|\, t \otimes g_{\mathcal{P}} \neq 0 \right\}$$

for all $t \in \mathcal{T}$. In this case, $\mathcal{T}$ is stratified if the map

$$\left\{\text{localizing ideals of } \mathcal{T}\right\} \to \left\{\text{subsets of } \mathrm{Spc}(\mathcal{T}^c)\right\}$$

sending $\mathcal{L}$ to $\mathrm{Supp}(\mathcal{L})$ is a bijection. The inverse is given by sending S to

$$\mathcal{T}_S := \left\{ t \in \mathcal{T} \,\middle|\, \mathrm{Supp}(t) \subseteq S \right\}.$$

5.2.3 Remark. Assume that $\mathcal{T}$ is stratified. For any subset $S \subseteq \mathrm{Spc}(\mathcal{T}^c)$, the localizing ideal $\mathcal{T}_S$ is generated by the set of objects $\left\{ g_{\mathcal{P}} \in \mathcal{T} \,\middle|\, \mathcal{P} \in S \right\}$, since we have $S = \bigcup_{\mathcal{P} \in S} \mathrm{Supp}(g_{\mathcal{P}})$. Hence, $\mathcal{T}_S$ is well generated; see [DM24, Remark A.9].

Therefore, if $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$, then the ∞-category $\mathcal{C}_S$ underlying $\mathcal{T}_S$ is presentable, in view of Remark 2.5.14.

5.2.4 Remark. Suppose that $\mathcal{T}$ is stratified and let $S_2 \subseteq S_1$ be subsets of $\mathrm{Spc}(\mathcal{T}^c)$. Since $\mathcal{T}_{S_1}$ and $\mathcal{T}_{S_2}$ are well generated, the Verdier quotient $q\colon \mathcal{T}_{S_1} \to \mathcal{T}_{S_1}/\mathcal{T}_{S_2}$ is a Bousfield localization. This follows from the results of [Nee01]; see also [BCHS25, Theorem 2.13]. Therefore, we have

$$\mathcal{T}_{S_1}/\mathcal{T}_{S_2} \simeq \left\{ t \in \mathcal{T}_{S_1} \;\middle|\; \mathrm{Hom}_{\mathcal{T}}(x,t) = 0 \text{ for all } x \in \mathcal{T}_{S_2} \right\} = \mathcal{T}_{S_1} \cap \mathcal{T}_{S_2}^{\perp}. \tag{5.2.5}$$

If $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$ then the Verdier quotient $\mathcal{C}_{S_1}/\mathcal{C}_{S_2}$ can be defined as the cofiber of the inclusion $\mathcal{C}_{S_2} \hookrightarrow \mathcal{C}_{S_1}$ in $\mathsf{Pr}^L_{\mathrm{st}}$. Hence $\mathcal{T}_{S_1}/\mathcal{T}_{S_2} \simeq \mathrm{Ho}(\mathcal{C}_{S_1}/\mathcal{C}_{S_2})$. See [BGT13, Section 5] for further details.

5.2.6 Lemma. *Suppose that $\mathcal{T}$ is stratified. For any subsets $S_2 \subseteq S_1$ of $\mathrm{Spc}(\mathcal{T}^c)$ with S_2 Thomason, we have*

$$\mathcal{T}_{S_1}/\mathcal{T}_{S_2} \simeq \mathcal{T}_{S_1 \setminus S_2}.$$

Proof. Observe that

$$\mathcal{T}_{S_1}/\mathcal{T}_{S_2} \simeq \mathcal{T}_{S_1} \cap \mathcal{T}_{S_2}^{\perp} = \mathcal{T}_{S_1} \cap (\mathcal{T} \cap \mathcal{T}_{S_2}^{\perp}) \simeq \mathcal{T}_{S_1} \cap \mathcal{T}/\mathcal{T}_{S_2} \simeq \mathcal{T}_{S_1} \cap \mathcal{T}_{S_2^c} = \mathcal{T}_{S_1 \setminus S_2}$$

where the first and second equivalence is due to (5.2.5) and the third equivalence follows from (2.4.13) and stratification. $\square$

5.2.7 Proposition. *Suppose that $\mathcal{T}$ is stratified with $\mathrm{Spc}(\mathcal{T}^c)$ Noetherian. If S is a convex subset of $\mathrm{Spc}(\mathcal{T}^c)$ then $\mathcal{T}_S$ is compactly generated.*

Proof. By Lemma 2.1.6 there exist some specialization closed subsets S_1, S_2 such that $S = S_1 \setminus S_2$. Since $\mathrm{Spc}(\mathcal{T}^c)$ is Noetherian, every specialization closed subset of $\mathrm{Spc}(\mathcal{T}^c)$ is Thomason. We thus have $\mathcal{T}_S \simeq \mathcal{T}_{S_1}/\mathcal{T}_{S_2}$ by Lemma 5.2.6. Moreover, since both S_1 and S_2 are Thomason, both $\mathcal{T}_{S_1}$ and $\mathcal{T}_{S_2}$ are generated by compact objects of $\mathcal{T}$. Hence, $\mathcal{T}_{S_1}$ is itself compactly generated and $\mathcal{T}_{S_2}$ is generated by compact objects of $\mathcal{T}_{S_1}$. Therefore, $\mathcal{T}$ is compactly generated by [Nee96, Theorem 2.1]. □

5.2.8 Remark. A subtle detail to keep in mind is that a localizing ideal $\mathcal{T}_S$ can be compactly generated (as a triangulated category) without being generated by compact objects in the ambient category $\mathcal{T}$. In other words, the inclusion $\mathcal{T}_S \hookrightarrow \mathcal{T}$ need not preserve compact objects. For this reason, we will avoid the term "compactly generated localizing ideal" used in [BHS23b] which means "generated by compact objects in $\mathcal{T}$".

5.2.9 Definition. We say that $\mathcal{T}$ is *cohomologically stratified* if:

(a) $\mathcal{T}$ is stratified;

(b) $\mathcal{T}$ is *Noetherian*, meaning that the ring $R_{\mathcal{T}} = \mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ is Noetherian and the $R_{\mathcal{T}}$-module $\mathrm{Hom}^*_{\mathcal{T}}(c, d)$ is finitely generated for any compact objects $c, d \in \mathcal{T}^c$;

(c) $\rho_{\mathcal{T}}\colon \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^{\mathrm{h}}(R_{\mathcal{T}})$ is a homeomorphism.

5.2.10 Remark. By Corollary 4.2.15, $\mathcal{T}$ is cohomologically stratified if and only if $\mathcal{T}$ is Noetherian and canonically BIK-stratified. Therefore, for all $t \in \mathcal{T}$, we have $\rho_{\mathcal{T}}(\mathrm{Supp}(t)) = \mathrm{Supp}_{\mathrm{BIK}}(t)$.

5.2.11 Example. The derived category $\mathrm{D}(A)$ of a commutative Noetherian ring is cohomologically stratified.

5.2.12 Definition. We say that $\mathcal{T}$ is *locally cohomologically stratified* if the local category $\mathcal{T}_{\mathcal{P}}$ is cohomologically stratified for every $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$.

5.2.13 Example. Let X be a Noetherian scheme and $\mathrm{D}_{\mathrm{qc}}(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Although $\mathrm{D}_{\mathrm{qc}}(X)$ is rarely cohomologically stratified if X is nonaffine, it is always locally cohomologically stratified. Indeed, for any $\mathcal{P} \in \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$ we can choose an affine open neighborhood $U \cong \mathrm{Spec}(A)$ of $\mathcal{P}$ so A is a commutative Noetherian ring. It then follows from [BHS23b, Remark 5.9, Proposition 1.32, and Example 1.36] that we have an equivalence

$$\mathrm{D}_{\mathrm{qc}}(X)_{\mathcal{P}} \simeq \mathrm{D}(A_{\mathfrak{p}})$$

where $\mathfrak{p} \in \mathrm{Spec}(A) \cong U$ is the prime corresponding to $\mathcal{P}$. Therefore, $\mathrm{D}_{\mathrm{qc}}(X)_{\mathcal{P}}$ is cohomologically stratified since $\mathrm{D}(A_{\mathfrak{p}})$ is by Example 5.2.11.

5.2.14 Remark. The example above shows that a locally cohomologically stratified category need not be cohomologically stratified. Nevertheless, if $\mathrm{Spc}(\mathcal{T}^c)$ is Noetherian, then being locally cohomologically stratified implies being stratified:

5.2.15 Proposition. *Consider the following statements:*

(a) $\mathcal{T}$ *is cohomologically stratified;*

(b) $\mathcal{T}$ *is locally cohomologically stratified and* $\mathrm{Spc}(\mathcal{T}^c)$ *is Noetherian;*

(c) $\mathcal{T}$ *is stratified and* $\mathrm{Spc}(\mathcal{T}^c)$ *is Noetherian.*

We have (a) $\implies$ *(b)* $\implies$ *(c).*

Proof. For (b) $\implies$ (c), since $\mathrm{Spc}(\mathcal{T}^c)$ is Noetherian, the local-to-global principle holds by [BHS23b, Theorem 3.22]. Thus, to prove that $\mathcal{T}$ is stratified it suffices to show that $\mathcal{T}_{\mathcal{P}}$ satisfies the minimality condition at the unique closed point, by [BHS23b, Corollary 5.3]. This follows from the fact that $\mathcal{T}_{\mathcal{P}}$ is stratified since it is cohomologically stratified by our assumption.

For (a) $\implies$ (b), since $\rho_{\mathcal{T}}$ is a homeomorphism, the localization $\mathcal{T} \to \mathcal{T}_{\mathcal{P}}$ at a prime $\mathcal{P}$ coincides with the algebraic localization at the prime $\mathfrak{p} := \rho_{\mathcal{T}}(\mathcal{P})$ by Definition 3.3.2. Hence, by [Bal10, Construction 3.5 and Theorem 3.6] we have

$$\mathrm{Hom}^*_{\mathcal{T}_{\mathcal{P}}}(x_{\mathcal{P}}, y_{\mathcal{P}}) \cong \mathrm{Hom}^*_{\mathcal{T}}(x, y)_{\mathfrak{p}}$$

for any compact objects $x, y \in \mathcal{T}^c$. In particular, $R_{\mathcal{T}_{\mathcal{P}}} \cong (R_{\mathcal{T}})_{\mathfrak{p}}$ is Noetherian. To check that $\mathcal{T}_{\mathcal{P}}$ is Noetherian, suppose that c, d are compact objects of $\mathcal{T}_{\mathcal{P}}$. By the Neeman–Thomason localization theorem there exist compact objects $a, b \in \mathcal{T}^c$ such that $a_{\mathcal{P}} = c \oplus \Sigma c$ and $b_{\mathcal{P}} = d \oplus \Sigma d$. It follows that

$$\mathrm{Hom}^*_{\mathcal{T}_{\mathcal{P}}}(c \oplus \Sigma c, d \oplus \Sigma d) \cong \mathrm{Hom}^*_{\mathcal{T}}(a, b)_{\mathfrak{p}}$$

is finitely generated over $R_{\mathcal{T}_{\mathcal{P}}}$. Hence, $\mathrm{Hom}^*_{\mathcal{T}_{\mathcal{P}}}(c, d)$ is also finitely generated. Therefore, $\mathcal{T}_{\mathcal{P}}$ is Noetherian. Since $R_{\mathcal{T}_{\mathcal{P}}}$ is Noetherian, $\rho_{\mathcal{T}_{\mathcal{P}}}$ is surjective by [Bal10, Theorem 7.3]. Moreover, the following diagram commutes by the naturality of ρ ([Bal10, Theorem 5.3(c)]):

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}^c_{\mathcal{P}}) \arrow[r, two heads, "\rho_{\mathcal{T}_{\mathcal{P}}}"] \arrow[d, hook] & \mathrm{Spec}^{\mathrm{h}}(R_{\mathcal{T}_{\mathcal{P}}}) \arrow[d, hook] \\
\mathrm{Spc}(\mathcal{T}^c) \arrow[r, "\sim", "\rho_{\mathcal{T}}"'] & \mathrm{Spec}^{\mathrm{h}}(R_{\mathcal{T}}).
\end{tikzcd}$$

It follows that $\rho_{\mathcal{T}_{\mathcal{P}}}$ is a bijection and hence a homeomorphism by [Lau23, Corollary 2.8]. It remains to show that stratification passes to local categories, but this was proved in [BHS23b, Corollary 4.9]. $\square$

5.2.16 Remark. We are now ready to state the main result of this section.

5.2.17 Theorem. *If $\mathcal{C}$ is locally cohomologically stratified with Noetherian Balmer* *spectrum* $\mathrm{Spc}(\mathcal{C}^c)$ *then there is an inclusion-preserving bijection*

$$\{\textit{dualizable localizing ideals of } \mathcal{C}\} \xrightarrow{\sim} \{\textit{convex subsets of } \mathrm{Spc}(\mathcal{C}^c)\}$$

sending $\mathcal{L}$ *to* $\mathrm{Supp}(\mathcal{L})$ *with inverse sending* S *to* $\mathcal{C}_S$*. Moreover, in this case, the dualizable localizing ideals are themselves compactly generated.*

5.2.18 Remark. To prove Theorem 5.2.17, we need some preparation.

5.2.19 Notation. Let $\mathcal{T}$ be a rigidly-compactly generated tt-category. Recall that we have the graded ring $R \coloneqq R_{\mathcal{T}} = \mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ and an R-module $\mathrm{Hom}^*_{\mathcal{T}}(a, b)$ for any objects $a, b \in \mathcal{T}$. For any $n \in \mathbb{Z}$ and any R-module M, we write $M[n]$ for the degree shift by n. For example, we have

$$\mathrm{Hom}^*_{\mathcal{T}}(a, \Sigma^n b) \cong \mathrm{Hom}^*_{\mathcal{T}}(a, b)[n].$$

We will also write $H^*_a(b)$ for $\mathrm{Hom}^*_{\mathcal{T}}(a, b)$ and think of it as the cohomology of b with respect to a.

5.2.20 Lemma. *Suppose that* I *is an injective* R*-module. There exists an object* $I_{\mathbb{1}} \in \mathcal{T}$ *such that for any set* $\{n_i\}_i$ *of integers there is a natural isomorphism*

$$\alpha\colon\ \mathrm{Hom}_{\mathcal{T}}(-, \coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \to \mathrm{Hom}_R(H^*_{\mathbb{1}}(-), \bigoplus_i I[n_i]).$$

Proof. Since I is injective, Brown representability yields an object $I_{\mathbb{1}} \in \mathcal{T}$ and a natural isomorphism

$$\Phi\colon\ \mathrm{Hom}_{\mathcal{T}}(-, I_{\mathbb{1}}) \xrightarrow{\sim} \mathrm{Hom}_R(H^*_{\mathbb{1}}(-), I)$$

which, by a degree shifting argument, extends to a natural isomorphism of R-modules

$$\Phi^*\colon\ \mathrm{Hom}^*_{\mathcal{T}}(-, I_{\mathbb{1}}) \xrightarrow{\sim} \mathrm{Hom}^*_R(H^*_{\mathbb{1}}(-), I). \tag{5.2.21}$$

Observe that

$$\Phi^n_{\Sigma^n I_{\mathbb{1}}}(\mathrm{id}_{\Sigma^n I_{\mathbb{1}}}) = \Phi_{I_{\mathbb{1}}}(\mathrm{id}_{I_{\mathbb{1}}})[n] \tag{5.2.22}$$

for any $n \in \mathbb{Z}$.

Setting $u \coloneqq \Phi_{I_{\mathbb{1}}}(\mathrm{id}_{I_{\mathbb{1}}})\colon H^*_{\mathbb{1}}(I_{\mathbb{1}}) \to I$, we note that by the Yoneda lemma the map

$$v\colon H^*_{\mathbb{1}}(\coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \cong \bigoplus_i H^*_{\mathbb{1}}(\Sigma^{n_i} I_{\mathbb{1}}) \xrightarrow{\bigoplus_i u[n_i]} \bigoplus_i I[n_i] \tag{5.2.23}$$

gives rise to a natural transformation

$$\alpha\colon \mathrm{Hom}_{\mathcal{T}}(-, \coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \to \mathrm{Hom}_R(H^*_{\mathbb{1}}(-), \bigoplus_i I[n_i]).$$

To prove that α is an isomorphism. It suffices to show that α_c is an isomorphism for any compact object $c \in \mathcal{T}^c$. Indeed, we will show that for any $c \in \mathcal{T}^c$, the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(c, \Sigma^{n_i} I_{\mathbb{1}}) \arrow[r, "\bigoplus_i \Phi_c^{n_i}", "\sim"'] \arrow[d, "\lambda_1"', "\wr"] & \bigoplus_i \mathrm{Hom}_R(H^*_{\mathbb{1}}(c), I[n_i]) \arrow[d, "\lambda_2", "\wr"'] \\
\mathrm{Hom}_{\mathcal{T}}(c, \coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \arrow[r, "\alpha_c"] & \mathrm{Hom}_R(H^*_{\mathbb{1}}(c), \bigoplus_i I[n_i])
\end{tikzcd}$$

where λ_1, λ_2 are the canonical isomorphisms. It suffices to check that for any $k \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(c, \Sigma^{n_k} I_{\mathbb{1}}) \arrow[r, "\Phi_c^{n_k}", "\sim"'] \arrow[d, hook, "\iota_1"'] & \mathrm{Hom}_R(H^*_{\mathbb{1}}(c), I[n_k]) \arrow[d, hook, "\iota_2"] \\
\mathrm{Hom}_{\mathcal{T}}(c, \coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \arrow[r, "\alpha_c"] & \mathrm{Hom}_R(H^*_{\mathbb{1}}(c), \bigoplus_i I[n_i])
\end{tikzcd}$$

where ι_1 and ι_2 are the natural inclusions. Indeed, consider any map $h\colon c \to \Sigma^{n_k} I_{\mathbb{1}}$. By definition, $\alpha_c = v \circ H^*_{\mathbb{1}}(-)$. Hence, $\alpha_c \iota_1$ sends h to the map

$$H^*_{\mathbb{1}}(c) \xrightarrow{H^*_{\mathbb{1}}(h)} H^*_{\mathbb{1}}(\Sigma^{n_k} I_{\mathbb{1}}) \hookrightarrow H^*_{\mathbb{1}}(\coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \xrightarrow{v} \bigoplus_i I[n_i]. \tag{5.2.24}$$

On the other hand, we have

$$\Phi_c^{n_k}(h) = \mathrm{Hom}_R(h \circ \Phi^{n_k}_{\Sigma^{n_k} I_{\mathbb{1}}}(\mathrm{id}_{\Sigma^{n_k} I_{\mathbb{1}}}), I[n_k]) = \mathrm{Hom}_R(h \circ u[n_k], I[n_k])$$

where the first equality holds by the naturality of Φ^{n_k} and the second is due to (5.2.22). Hence, $\iota_2 \Phi_c^{n_k}$ sends h to the map

$$H^*_{\mathbb{1}}(c) \xrightarrow{H^*_{\mathbb{1}}(h)} H^*_{\mathbb{1}}(\Sigma^{n_k} I_{\mathbb{1}}) \xrightarrow{u[n_k]} I[n_k] \hookrightarrow \bigoplus_i I[n_i]. \tag{5.2.25}$$

A routine diagram chase shows that (5.2.24) coincides with (5.2.25), which finishes the proof. $\square$

5.2.26 Lemma. *Suppose that $\mathcal{T}$ is cohomologically stratified and that $\mathcal{T}$ is local with unique closed point $\mathcal{M}$. If t is an object of $\mathcal{T}$ such that the functor $\mathrm{Hom}_{\mathcal{T}}(t, -)$ commutes with coproducts in $\mathcal{T}_{\{\mathcal{M}\}}$, then the R-module $H^*_{\mathbb{1}}(t)$ is finitely generated.*

Proof. By cohomological stratification the comparison map $\rho\colon \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}(R)$ is a homeomorphism. It follows that the Noetherian ring R is local with unique maximal ideal $\mathfrak{m} := \rho(\mathcal{M})$. We denote by I the injective hull of $R/\mathfrak{m}$. Therefore, the modules $\{I[n] \mid n \in \mathbb{Z}\}$ cogenerate the category of R-modules (cf. [Lam99, Theorem 19.8]). If $H^*_{\mathbb{1}}(t)$ is not finitely generated then we can choose an infinite sequence of submodules

$$0 = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq H^*_{\mathbb{1}}(t).$$

By cogeneration, for every $i \geq 1$ there exists a nonzero map $f_i\colon N_i \to I[n_i]$ for some n_i with $f_i|_{N_{i-1}} = 0$. Write $N := \bigcup_{i\geq 1} N_i$. By injectivity, we extend each f_i to a map $g_i\colon N \to I[n_i]$ to obtain a map $N \to \prod_i I[n_i]$. The image of this map is contained in $\bigoplus_i I[n_i]$. Indeed, for any nonzero element $m \in N$, there exists some k such that $m \notin N_{k-1}$ and $m \in N_k$, so g_i annihilates m whenever $i > k$.

Therefore, we obtain a map $g\colon N \to \bigoplus_i I[n_i]$ which does not factor through any finite direct sum since every g_i is nonzero. Since R is Noetherian, the target of g is also injective. Hence, g extends to a map $f\colon H^*_{\mathbb{1}}(t) \to \bigoplus_i I[n_i]$ which does not factor through any finite direct sum since g does not.

By Lemma 5.2.20 we obtain an isomorphism

$$\alpha_t\colon \operatorname{Hom}_{\mathcal{T}}(t, \coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \xrightarrow{\sim} \operatorname{Hom}_R(H^*_{\mathbb{1}}(t), \bigoplus_i I[n_i]).$$

Let $\tilde{f}\colon t \to \coprod_i \Sigma^{n_i} I_{\mathbb{1}}$ be the map corresponding to the map $f\colon H^*_{\mathbb{1}}(t) \to \bigoplus_i I[n_i]$. Hence, $f = \alpha_t(\tilde{f}) = v \circ H^*_{\mathbb{1}}(\tilde{f})$ where $v\colon H^*_{\mathbb{1}}(\coprod_i \Sigma^{n_i} I_{\mathbb{1}}) \to \bigoplus_i I[n_i]$ comes from (5.2.23). We claim that $\tilde{f}$ does not factor through any finite coproduct. Otherwise, there would exist a map $\tilde{f}'\colon t \to \coprod_j \Sigma^{n_j} I_{\mathbb{1}}$ where j ranges over a finite set of positive integers such that the following diagram commutes:

$$\begin{tikzcd} t \arrow[r, "\tilde{f}'"'] \arrow[rr, bend left, "\tilde{f}"] & \coprod_j \Sigma^{n_j} I_{\mathbb{1}} \arrow[r, hook] & \coprod_i \Sigma^{n_i} I_{\mathbb{1}}. \end{tikzcd}$$

Applying Lemma 5.2.20 to the integers $\{n_j\}_j$, we obtain an isomorphism

$$\alpha'_t\colon \operatorname{Hom}_{\mathcal{T}}(t, \coprod_j \Sigma^{n_j} I_{\mathbb{1}}) \xrightarrow{\sim} \operatorname{Hom}_R(H^*_{\mathbb{1}}(t), \bigoplus_j I[n_j])$$

such that $\alpha'_t = v' \circ H^*_{\mathbb{1}}(-)$ where $v'\colon H^*_{\mathbb{1}}(\coprod_j \Sigma^{n_j} I_{\mathbb{1}}) \to \bigoplus_j I[n_j]$ is the map (5.2.23) with i replaced by j. Setting $f' = \alpha'_t(\tilde{f}') = v' \circ H^*_{\mathbb{1}}(\tilde{f}')$ we observe that the diagram below commutes:

$$\begin{tikzcd}
H^*_{\mathbb{1}}(t) \arrow[r, "H^*_{\mathbb{1}}(\tilde{f}')"] \arrow[rr, bend left, "H^*_{\mathbb{1}}(\tilde{f})"] \arrow[rrr, bend left=50, "f"] \arrow[drr, "f'"'] & H^*_{\mathbb{1}}(\coprod_j \Sigma^{n_j} I_{\mathbb{1}}) \arrow[r, hook] \arrow[dr, "v'"] & H^*_{\mathbb{1}}(\coprod_i \Sigma^{n_i} I_{\mathbb{1}}). \arrow[r, "v"] & \bigoplus_i I[n_i]. \\
 & & \bigoplus_j I[n_j] \arrow[ur, hook] &
\end{tikzcd}$$

However, this leads to a contradiction since we showed that f does not factor through any finite direct sum, so the claim follows. That is, $\tilde{f}\colon t \to \coprod_i \Sigma^{n_i} I_{\mathbb{1}}$ does not factor through any finite coproduct. This will contradict the hypothesis that $\mathrm{Hom}_{\mathcal{T}}(t, -)$ commutes with coproducts in $\mathcal{T}_{\{\mathcal{M}\}}$ if we can show $I_{\mathbb{1}} \in \mathcal{T}_{\{\mathcal{M}\}}$. Indeed, by (5.2.21) we have $H^*_c(I_{\mathbb{1}}) \cong \mathrm{Hom}^*_R(H^*_{\mathbb{1}}(c), I)$ for any $c \in \mathcal{T}^c$. The latter is $\mathfrak{m}$-torsion since I is $\mathfrak{m}$-torsion and $H^*_{\mathbb{1}}(c)$ is finitely generated by the Noetherian assumption (Definition 5.2.9). It then follows from [BIK08, Lemma 2.4(2) and Corollary 5.3] that $\mathrm{Supp}_{\mathrm{BIK}}(I_{\mathbb{1}}) \subseteq \{\mathfrak{m}\}$, which is equivalent to $\mathrm{Supp}(I_{\mathbb{1}}) \subseteq \{\mathcal{M}\}$ by Theorem 4.2.8. Therefore $I_{\mathbb{1}} \in \mathcal{T}_{\{\mathcal{M}\}}$, which completes the proof. □

5.2.27 Remark. In the case that $\mathcal{T}$ is *unigenic*, meaning that it is generated by the unit $\mathbb{1}$, the lemma above has a much shorter proof by utilizing [Hov07, Lemma 2.1]. We leave the details to the interested reader.

5.2.28 Corollary. *In the situation of Lemma 5.2.26, the support* $\mathrm{Supp}(t)$ *is a specialization closed subset of* $\mathrm{Spc}(\mathcal{T}^c)$*.*

Proof. First we claim that $H^*_c(t)$ is finitely generated for any $c \in \mathcal{T}^c$. Indeed, since we have $H^*_{c^\vee \otimes t}(-) \cong H^*_t(c \otimes -)$ by adjunction, $c^\vee \otimes t$ also satisfies the condition in Lemma 5.2.26, so $H^*_c(t) \cong H^*_{\mathbb{1}}(c^\vee \otimes t)$ is finitely generated. It then follows from

[BIK08, Corollary 5.3 and Lemma 2.2(1)] that

$$\mathrm{Supp}_{\mathrm{BIK}}(t) \subseteq \bigcup_{c \in \mathcal{T}^c} \mathcal{V}(\mathrm{Ann}_R\, H^*_c(t))$$

Moreover, by [BIK08, Theorem 5.5 and Lemma 2.2(1)] we have

$$\bigcup_{c \in \mathcal{T}^c} \mathcal{V}(\mathrm{Ann}_R\, H^*_c(t)) \subseteq \mathrm{Supp}_{\mathrm{BIK}}(t).$$

Hence, $\mathrm{Supp}_{\mathrm{BIK}}(t) = \bigcup_{c \in \mathcal{T}^c} \mathcal{V}(\mathrm{Ann}_R\, H^*_c(t))$ is specialization closed. Since the comparison map ρ is a homeomorphism by our assumption, we invoke Theorem 4.2.8 to conclude that $\mathrm{Supp}(t) = \rho^{-1}(\mathrm{Supp}_{\mathrm{BIK}}(t))$ is also specialization closed. □

5.2.29 Lemma. *Suppose that $\mathcal{C}$ is local and cohomologically stratified. Let S be a subset of $\mathrm{Spc}(\mathcal{C}^c)$ which contains the unique closed point $\mathcal{M}$ and which also contains a point $\mathcal{Q}$ such that $\overline{\{\mathcal{Q}\}} \not\subseteq S$ and $S \cap \overline{\{\mathcal{Q}\}} \setminus \{\mathcal{Q}\}$ is specialization closed. Then the localizing ideal $\mathcal{C}_{S \cap \overline{\{\mathcal{Q}\}}}$ is not dualizable.*

Proof. Write S_1 for $S \cap \overline{\{\mathcal{Q}\}}$ and S_2 for $S \cap \overline{\{\mathcal{Q}\}} \setminus \{\mathcal{Q}\}$ and consider the localization sequence

$$\mathcal{C}_{S_2} \overset{i}{\hookrightarrow} \mathcal{C}_{S_1} \overset{q}{\twoheadrightarrow} \mathcal{C}_{S_1}/\mathcal{C}_{S_2}.$$

Since S_2 is specialization closed, it is Thomason by the assumption on $\mathcal{C}$. Lemma 5.2.6 then implies

$$\operatorname{im} q^R \simeq \mathcal{C}_{S_1}/\mathcal{C}_{S_2} \simeq \mathcal{C}_{S_1 \setminus S_2} = \mathcal{C}_{\{\mathcal{Q}\}}.$$

Singletons are convex, so $\mathcal{C}_{S_1}/\mathcal{C}_{S_2}$ is compactly generated by Proposition 5.2.7. Suppose *ab absurdo* that $\mathcal{C}_{S_1}$ is dualizable. By stratification, $\mathcal{C}_{S_2} = \mathrm{Loc}_{\otimes}\langle \mathcal{C}^c_{S_2} \rangle$ is generated by a set of compact objects of $\mathcal{C}$. Hence the objects of $\mathcal{C}^c_{S_2}$ are also compact in $\mathcal{C}_{S_1}$. Therefore, we can invoke the Neeman–Thomason localization theorem ([Efi24, Proposition 1.18]): Choosing any nonzero compact object y of $\mathcal{C}_{S_1}/\mathcal{C}_{S_2}$, we find a compact object t of $\mathcal{C}_{S_1}$ such that $q(t) \cong y \oplus \Sigma y$. Now consider the cofiber sequence

$$ii^R(t) \to t \to q^R q(t).$$

Note that $\operatorname{Supp}(q^R q(t)) \subseteq \{\mathcal{Q}\}$ and $\operatorname{Supp}(ii^R(t)) \subseteq S_2$. Since $q(t)$ is nonzero, we have $\operatorname{Supp}(q^R q(t)) = \{\mathcal{Q}\}$ by stratification. It then follows from [BF11, Proposition 7.17(e)] that $\mathcal{Q} \in \operatorname{Supp}(t)$. Moreover, since t is compact in $\mathcal{C}_{S_1}$ and S_1 contains $\mathcal{M}$, we see that t satisfies the condition in Lemma 5.2.26 and hence $\operatorname{Supp}(t)$ is specialization closed by Corollary 5.2.28. This implies $\overline{\{\mathcal{Q}\}} \subseteq \operatorname{Supp}(t) \subseteq S$, which contradicts our hypothesis. □

Proof of Theorem 5.2.17. Since $\mathcal{C}$ is locally cohomologically stratified and $\operatorname{Spc}(\mathcal{C}^c)$ is Noetherian, $\mathcal{C}$ is stratified by Proposition 5.2.15. By Proposition 5.2.7, if S is convex then $\mathcal{C}_S$ is compactly generated and hence dualizable by Example 2.5.10. It remains to show that if $\mathcal{C}_S$ is dualizable then S is convex. To this end, let S be a nonconvex subset of $\operatorname{Spc}(\mathcal{C}^c)$ such that $\mathcal{C}_S$ is dualizable. By nonconvexity, there exist two points $\mathcal{Q}, \mathcal{P} \in S$ such that $\overline{\{\mathcal{Q}\}} \cap \operatorname{gen}(\mathcal{P}) \nsubseteq S$. Consider the set $\left\{ \mathcal{R} \in S \,\middle|\, \overline{\{\mathcal{R}\}} \cap \operatorname{gen}(\mathcal{P}) \nsubseteq S \right\}$. Being a nonempty subset of the Noetherian space $\operatorname{Spc}(\mathcal{C}^c)$, it admits an element that is maximal with respect to the specialization order, which we still denote by $\mathcal{Q}$. Hence, we have

$$\overline{\{\mathcal{Q}\}} \cap \operatorname{gen}(\mathcal{P}) \nsubseteq S \text{ and } \forall \mathcal{R} \in S \cap \overline{\{\mathcal{Q}\}} \setminus \{\mathcal{Q}\} \colon \overline{\{\mathcal{R}\}} \cap \operatorname{gen}(\mathcal{P}) \subseteq S.$$

Consider the finite localization $q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{P}} =: \mathcal{D}$ at the prime $\mathcal{P}$. Recall from Example 2.4.14 that the associated map $\varphi \colon \operatorname{Spc}(\mathcal{D}^c) \hookrightarrow \operatorname{Spc}(\mathcal{C}^c)$ is an embedding with image $\operatorname{gen}(\mathcal{P})$. Writing S' for $\varphi^{-1}(S)$ and $\mathcal{Q}'$ for $\varphi^{-1}(\mathcal{Q})$, and working in $\operatorname{Spc}(\mathcal{D}^c)$, we have

$$\overline{\{\mathcal{Q}'\}} \nsubseteq S' \text{ and } \forall \mathcal{R}' \in S' \cap \overline{\{\mathcal{Q}'\}} \setminus \{\mathcal{Q}'\} \colon \overline{\{\mathcal{R}'\}} \subseteq S'.$$

Note that $\mathcal{D}$ is cohomologically stratified since $\mathcal{C}$ is locally cohomologically stratified. Hence, $\mathcal{D}$, S', and $\mathcal{Q}'$ satisfy the conditions in Lemma 5.2.29. It follows that $\mathcal{D}_{S'\cap\overline{\{\mathcal{Q}'\}}}$ is not dualizable.

On the other hand, applying the functor $-\otimes_{\mathcal{C}}\mathcal{D}$ to the inclusion $\mathcal{C}_S \hookrightarrow \mathcal{C}$, by Lemma 2.5.15 we obtain a fully faithful functor

$$\mathcal{C}_S \otimes_{\mathcal{C}} \mathcal{D} \hookrightarrow \mathcal{C} \otimes_{\mathcal{C}} \mathcal{D} \simeq \mathcal{D}$$

with essential image $q(\mathcal{C}_S)$. Note that

$$\operatorname{Supp}(q(\mathcal{C}_S)) = \varphi^{-1}(\operatorname{Supp}(\mathcal{C}_S)) = \varphi^{-1}(S) = S'$$

where the first equality follows from Corollary 3.2.20(b) and the second is due to the stratification of $\mathcal{C}$. This tells us $\mathcal{C}_S \otimes_{\mathcal{C}} \mathcal{D} \simeq q(\mathcal{C}_S) = \mathcal{D}_{S'}$ by the stratification of $\mathcal{D}$. Lemma 2.5.16 then implies that $\mathcal{D}_{S'}$ is dualizable. Now consider the colocalization $i^R\colon \mathcal{D} \to \mathcal{D}_{\overline{\{\mathcal{Q}'\}}}$. By Lemma 2.5.15 again, we obtain a fully faithful functor

$$\mathcal{D}_{S'} \otimes_{\mathcal{D}} \mathcal{D}_{\overline{\{\mathcal{Q}'\}}} \hookrightarrow \mathcal{D} \otimes_{\mathcal{D}} \mathcal{D}_{\overline{\{\mathcal{Q}'\}}} \simeq \mathcal{D}_{\overline{\{\mathcal{Q}'\}}}$$

with essential image $i^R(\mathcal{D}_{S'})$. Note that

$$\operatorname{Supp}(ii^R(\mathcal{D}_{S'})) = \overline{\{\mathcal{Q}'\}} \cap \operatorname{Supp}(\mathcal{D}_{S'}) = \overline{\{\mathcal{Q}'\}} \cap S'$$

where the first equality is by Proposition 3.2.11(f) and the second by the stratification of $\mathcal{D}$. We thus have $\mathcal{D}_{S'} \otimes_{\mathcal{D}} \mathcal{D}_{\overline{\{\mathcal{Q}'\}}} \simeq i^R(\mathcal{D}_{S'}) = \mathcal{D}_{S'\cap\overline{\{\mathcal{Q}'\}}}$ by the stratification of $\mathcal{D}$ again. Invoking Lemma 2.5.16 one more time, we see that $\mathcal{D}_{S'\cap\overline{\{\mathcal{Q}'\}}}$ is dualizable, which is absurd. $\square$

5.2.30 Remark. The proof of Theorem 5.2.17 is inspired by [Efi24, Theorem 3.10] where Efimov establishes a one-to-one correspondence between the dualizable localizing ideals of the derived category $\mathrm{D}(A)$ of any commutative Noetherian ring A and the convex subsets of the spectrum $\mathrm{Spec}(A)$. A key result in Efimov's proof is [Efi24, Lemma 3.12] which shows that in the case $\mathcal{C} = \mathrm{D}(A)$, the object t in Lemma 5.2.26 is compact in $\mathcal{C}$. This is a strictly stronger statement than our conclusion in Corollary 5.2.28 that the support of t is specialization closed in $\mathrm{Spc}(\mathcal{C}^c)$. However, the latter is sufficient to establish the desired Lemma 5.2.29 which corresponds to [Efi24, Lemma 3.13]. Another important result due to Efimov is the extension of the Neeman–Thomason localization theorem from compactly generated categories to dualizable categories ([Efi24, Proposition 1.18]). In the absence of this generalization, the proof of Theorem 5.2.17 only establishes a one-to-one correspondence between the convex subsets of $\mathrm{Spc}(\mathcal{C}^c)$ and the localizing ideals of $\mathcal{C}$ which are compactly generated.

5.2.31 Remark. As explained in [Efi24, Remark 3.14], the classification of dualizable localizing ideals via convex subsets can fail for derived categories of non-Noetherian rings. Nevertheless, it does hold in some cases. For example, consider the derived category $\mathrm{D}(A)$ of any non-Noetherian semi-artinian absolutely flat ring. It is stratified by [Ste17, Theorem 6.3]. Moreover, every subset of $\mathrm{Spec}(A)$ is specialization closed since A has Krull dimension 0. Therefore, every localizing ideal of $\mathrm{D}(A)$ is generated by a set of compact objects. In particular, all localizing ideals of $\mathrm{D}(A)$ are dualizable and all subsets of $\mathrm{Spec}(A)$ are convex.

5.2.32 Corollary. *If $\mathcal{C}$ is cohomologically stratified then we have a bijection*

$$\{\textit{dualizable localizing ideals of } \mathcal{C}\} \xrightarrow{\sim} \{\textit{convex subsets of } \mathrm{Spc}(\mathcal{C}^c)\}$$

Proof. This follows from Theorem 5.2.17 by Proposition 5.2.15. □

5.2.33 Example. Cohomologically stratified categories abound. The reader is invited to consult [BIK11a, BIK11b, DS16, BIKP18, Bar21, Bar22, Lau23, BIKP24, BCH+24, BBI+25, BCH+25] for many such examples.

5.2.34 Example. The derived category $\mathrm{D}_{\mathrm{qc}}(X)$ of any Noetherian scheme is locally cohomologically stratified (Example 5.2.13). Moreover, $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$ is Noetherian. Hence, Theorem 5.2.17 applies.

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