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The Role of Visual Representations in College Students' Understanding of Mathematical Notation

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Abstract

Developing understanding of fractions involves connections between non-symbolic visual representations and symbolic representations. Initially, teachers introduce fraction concepts with visual representations before moving to symbolic representations. Once the focus is shifted to symbolic representations, the connections between visual representations and symbolic notation are considered to be less useful, and students are rarely asked to connect symbolic notation back to visual representations. In two experiments, we ask whether visual representations affect understanding of symbolic notation for adults who understand symbolic notation. In a conceptual fraction comparison task (e.g., Which is larger, $\frac{5}{a}$ or $\frac{8}{a}$?), participants were given comparisons paired with accurate, helpful visual representations, misleading visual representations, or no visual representations. The results show that even college students perform significantly better when accurate visuals are provided over misleading or no visuals. Further, eye-tracking data suggest that these visual representations may affect performance even when only briefly looked at. Implications for theories of fraction understanding and education are discussed.

Keywords

mathematical reasoning; fractions; visual representations; symbolic representations

Early on, students are introduced to many mathematical concepts using visual representations. Such visual representations are typically non-symbolic representations that do not contain literal numbers and are thought to be more intuitive for students (Opfer & Siegler, 2012). For example, the very beginnings of fraction "concepts" are introduced to students as young as kindergarten age informally through visual representations (e.g., "pie" or "circle" representation; Common Core State Standards Initiative, 2014; Scott-Foresman & Addison-Wesley, 2011). By introducing mathematical concepts with these visual representations, students can avoid the confusion that often comes when learning the conventions of the symbolic number system (e.g., learning the words that map to the numbers). Thus, using visual representations and other non-symbolic systems can facilitate and provide bootstrapping for the later-developing symbolic number system (e.g., Condry & Spelke, 2008; LeCorre & Carey, 2007; Opfer & Siegler, 2012).

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With the use of visual representations, students are able to display basic understandings of division and partitioning from an early age. Preschoolers are able to evenly divide or partition a set of items among two or three people by using distributive counting (Frydman & Bryant, 1988). Additionally, when given the chance to visually compare scenarios where the same number of items are shared between a larger or smaller number of people, early elementary school students understand that sharing between a greater number of people (higher denominator) would result in a smaller share for each person (Sophian, Garyantes, & Change, 1997). Thus, non-symbolic representations such as sharing processes and visual cues allow young students to demonstrate conceptual understanding of fractions and the relation between numerators and denominators (Empson, 1999; Sophian et al., 1997). Additionally, visual representations even allow students to show understanding of basic fraction arithmetic (Mix, Levine, & Huttenlocher, 1999). Visual representations may thus play an important role in students' conceptual understanding of fractions.

Fractions present both a symbolic and a conceptual challenge for students. Fractions are symbolically notated with a bipartite structure with a separate numerator and denominator—rather than a unitary symbol—and fractions are the only number type that simultaneously represents a magnitude and a division relationship between the numerator and denominator. Indeed, a large body of research has pointed to misconceptions and errors that children and adults make with symbolic fraction notation, despite a seemingly well-developed intuitive understanding of fractions when using visual representations (e.g., Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; Stigler, Givvin, & Thompson, 2010; Vamvakoussi & Vosniadou, 2010). However, studies have also shown that young students are able to successfully divide a certain number of items among people only when provided with visual cues and fail to do so when the same problem is presented only with symbolic notation and no visual cues (Squire & Bryant, 2002; 2003). Therefore, though students have difficulty transferring their understanding of division and fraction concepts from the intuitive visual representations to the literal symbols of fraction notation, being provided with visual representations can facilitate the transfer of division and fraction concepts to the literal symbolic fraction notation.

Despite the seeming usefulness of visual representations, once students do learn the symbolic notation system of fractions, teachers rarely go back to the visual representations, assuming that such representations have outlived their usefulness. Instead, more complex fraction concepts and algorithms, as well as extensions into algebra, are typically developed using the more precise system of mathematical notation alone, without reference to visual representations (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Chao, Stigler, & Woodward, 2000; Kieran, 1992; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). It is unclear, however, whether the connections between visual representations and symbolic notation are no longer helping students after they learn the symbolic notation of fractions.

There is some evidence that even among college educated adults, visual representations of fractions may facilitate representations of symbolically notated magnitudes. For example, students have difficulty representing magnitudes of fractions when presented symbolically and without visual cues (DeWolf, Grounds, Bassok, & Holyoak, 2014). However, though adult students may have difficulty interpreting magnitudes of fractions, the bipartite symbolic notation of fractions have been found to be useful for more relationally rich tasks that require students to interpret visually represented ratios, especially when represented with discrete visual representations (DeWolf, Bassok, & Holyoak, 2015a; Rapp, Bassok, DeWolf, & Holyoak, 2015). Therefore, students may find a visually represented context for fractions to be helpful in interpreting what fractions are meant to represent.

It is also possible that when students move beyond a basic understanding of fractions as representations of magnitude, extensions of the concepts could still benefit from connections to visual representations. Stigler et al. (2010) asked community college developmental mathematics students to judge which of two fractions is larger, assuming that *a* is a positive whole number: $\frac{\mathrm{a}}{5}$ or $\frac{\mathrm{a}}{8}$. Students performed at chance on the task, indicating an inability to extend their basic understanding of fractions as magnitudes to a more conceptual situation. However, students who were able explain their answers by referencing a non-symbolic representation (e.g., referring to some quantity, *a*, being divided into different numbers of pieces), were always led to the correct answer. Thus, although connections to visual representations might be ignored as students progress through the mathematics curriculum, such connections may still be activated when students are asked to make more conceptual judgments and to explain such judgments.

Indeed, asking students to generate explanations for their thinking has proven to be an important tool in better understanding how students think differently about fractions depending on the context (Fazio, DeWolf & Siegler, 2016; Stigler et al., 2010). For example, Fazio et al. (2016) found that adults spontaneously use a variety of different types of visual representations or cues to help think about the magnitudes of fractions (e.g., a $\frac{1}{4}$ measuring cup is smaller than a $\frac{1}{3}$ measuring cup). Similarly, Vosniadou and colleagues (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004) have looked at middle school and high school student explanations of fractions. These analyses revealed important misconceptions about fractions and important insights about how these change over time. Such explanations help to identify the types of strategies people use for thinking about fraction magnitudes abstractly and conceptually.

The current study thus examines the extent to which visual representations of fractions influence college students who already have an understanding of the symbolic notation of fractions. When offered the chance to view visual representations, do such students use them at all, or focus only on the more precise symbolic notation of fractions? Many previous studies have tested fraction understanding with traditional magnitude comparison tasks (Schneider & Siegler, 2010; Bonato, Fabbri & Umilta, 2007; DeWolf, et al., 2014). A primary goal of the current study was to test whether a conceptual understanding of fractions relates to their role in a division relationship. Therefore, in two experiments, students were given fraction comparisons in the form of algebraic expressions. Students were asked to compare abstract fraction expressions (e.g., “Which fraction is larger, $\frac{5}{a}$ or $\frac{8}{a}$?”) when paired with accurate, helpful visual representations that matched the expression or misleading, unhelpful visual representations that did not match the expression. Importantly, in this task, students must have a conceptual understanding of the relative sizes of fraction components (numerators and denominators), division, and variables. Though a simple strategy like “plugging in” a number for *a* is certainly possible, students must still conceptualize the division relationship between the numerator and denominator. The hypothesis is that students will perform better when accurate visual representations are provided than when misleading visual representations are provided. We also make use of eye tracking technology to assess the extent to which students’ visual attention to the visual representations during the decision-making process may influence their performance on the fraction comparison problems. Additionally, though the main focus of the current study was on college level students’ performance on a fraction comparison task, we also gave participants a traditional magnitude comparison task to test whether students’ understanding of magnitudes—when expressed solely with symbolic fraction notation—is related to their performance on the fraction comparison task.

Experiment 1

The primary goal of Experiment 1 was to examine whether accurate visual representations of fractions may help students’ performance on a fraction comparison task. If visual representations—and visual attention to visual representations—indeed affect students’ performance on fraction comparison problems, students should have higher accuracy on problems that are presented with accurate, helpful visual representations and have lower accuracy on problems that are presented with misleading, unhelpful visual representations. Additionally, a secondary goal was to characterize students’ strategies in solving fraction comparison problems and examine whether students could discriminate between accurate visual representations and misleading visual representations.

Method

Participants—Thirty-six undergraduate students participated (n_{female}=18). Participants were between the ages of 18.46 and 28.04 years (*M*=21.08, *SD*=2.03) and were students enrolled at a selective American university. Four participants’ data were excluded from analyses for the following reasons: poor eye-tracking calibration (n=1), reaction times that were more than two standard deviations from the mean (n=1), lack of responses during one entire task (n=1), and experimenter error (n=1). The final sample included 32 participants (n_{female}=16). None of the participants were majoring or minoring in mathematics or a math-related field.

Apparatus—Stimuli were presented on a ViewSonic VX2268wm monitor with a 47.4 cm by 29.6 cm display (resolution: 1680×1050 pixels). Participants were seated approximately 60 cm from the display. Eye movement data were collected via an SR Eyelink 1000 eye tracker, and eye movements were recorded at 500 Hz with spatial accuracy of approximately 0.5-1°. Using Experiment Builder software, each participant’s point of gaze was calibrated with a series of dynamic circular stimuli shown at five points on the screen (top middle, bottom middle, left center, right center, center).

Materials and Procedure—Four tasks were used in this experiment, in the following order: (1) Traditional Magnitude Comparisons, (2) Fraction Comparisons, (3) Fraction Comparison Explanations, and (4) Visual Representation Comparisons. Each task was introduced by an instruction slide that detailed what the participant was to do in that particular task. All study stimuli were created using Adobe Photoshop.

Traditional Magnitude Comparisons: Participants' understanding of magnitudes when expressed solely with symbolic fraction notation was tested using a Traditional Magnitude Comparison task. Participants saw a series of 10 different fractions (e.g., $\frac{35}{54}$, $\frac{20}{97}$, $\frac{5}{9}$) and were asked to compare each fraction to $\frac{3}{5}$ (DeWolf et al., 2014). The 10 magnitude comparison problems were displayed in the middle of the screen, and the order in which participants were presented with the 10 fractions was randomized for each participant. Participants were given up to 120 seconds to respond to each problem and responded via mouse click: a left mouse click to indicate that the fraction was less than $\frac{3}{5}$, and a right mouse click to indicate that the fraction was greater than $\frac{3}{5}$. Participants' performance on these traditional magnitude comparison problems were used to evaluate whether understanding of magnitudes expressed with symbolic fraction notation was related to performance on the more conceptual Fraction Comparisons task (below).

Fraction Comparisons with Visual Representations: Participants were presented with a series of 40 fraction comparison problems, paired with visual representations of the fractions in the problem (Figure 1). Each fraction comparison problem contained two fractions with an unknown variable (*a, b, c, x,* or *y*); for each problem, participants were asked to identify which fraction was larger (e.g., "Which is larger, $\frac{a}{5}$ or $\frac{a}{8}$?"). For every trial, a visual representation was provided for each of the fractions in the comparison. Half of the visual representations accurately represented the fractions ("accurate visual representations;" Figures 1a and 1b) and half of the visual representations represented the fractions in a misleading way ("misleading visual representations;" Figure 1c and 1d).

Each visual representation consisted of a simple bar representation composed of discrete parts. Each bar representation had the same number of discrete parts as the number in the fraction. Bar representations with discrete parts were used because adults show a preference for discrete visual representations for fractions over continuous representations such as circle graphs or pie charts (Rapp et al., 2015). For example, the problem "Which is larger, $\frac{a}{5}$ or $\frac{a}{8}$?" would be shown with one bar representation composed of five discrete parts (corresponding to $\frac{a}{5}$) and another bar representation composed of eight discrete parts (corresponding to $\frac{a}{8}$). When this fraction comparison problem was paired with two bar representations of different lengths (i.e., a misleading representation for a common denominator problem), each of the discrete parts in both bar representations were 100×100 pixels and had 6 pixel borders. When this fraction comparison problem was paired with two bar representations of the same length (i.e., an accurate representation for a common denominator problem), the bar representation with more discrete parts (i.e., $\frac{a}{8}$ in this example problem) had 100×100 pixel discrete parts and 6 pixel borders; the bar representation with fewer discrete parts (i.e., $\frac{a}{5}$ in this example problem) had discrete parts that were stretched evenly to make the entire bar representation match the length of the bar representation with more discrete parts, but each discrete part still had 6 pixel borders around it. On an instruction slide, participants were told, "You will be presented with math questions that will ask you to compare two fractions. Please answer these fraction comparison problems. In all problems, the letters (*x, y, a, b, c*) represent positive, whole numbers (e.g., 1, 2, 3, 4, 5,…). You will also see visual representations of these fractions. Please look at each visual representation *before* you answer the fraction comparison problem." Participants were given up to 120 seconds to respond to each fraction comparison problem and responded to each problem via mouse click, clicking the left button to select the first fraction (e.g., $\frac{a}{5}$) or the right button to select the second fraction (e.g., $\frac{a}{8}$).

There were four conditions in this task, and all participants were presented with 10 trials of each condition. Two different fraction types (common numerator fractions: $\frac{a}{5}, \frac{a}{8}$; common denominator fractions: $\frac{5}{a}, \frac{8}{a}$) and two different visual representation types (accurate visual representation, misleading visual representation) were combined to create the four different conditions: (1) common numerator fractions with accurate visual representations, (2) common numerator fractions with misleading visual representations, (3) common denominator fractions with accurate visual representations, and (4) common denominator fractions with misleading visual representations. Trials were randomized for each participant such that no two participants were presented with the same order of problems.

Fraction Comparison Explanations: Participants were presented with one trial of each condition from Fraction Comparisons (for a total of four trials in this task). In this task, participants were again presented with fraction comparison problems and asked to identify which of the two fractions was larger; participants solved each problem and responded with mouse clicks in the same way as in the Fraction Comparisons task. After solving the fraction comparison problem, however, participants were also asked to verbally explain why they believed their answer to be correct. Verbal explanations were audio-recorded using Experiment Builder software and later transcribed for analysis. The order of trials was fixed for all participants, and all participants were asked to solve the following problems in the following order: (1) $\frac{a}{7}$ versus $\frac{a}{4}$ with an accurate visual representation, (2) $\frac{a}{3}$ versus $\frac{a}{5}$ with a misleading visual representation, (3) $\frac{6}{a}$ versus $\frac{9}{a}$ with an accurate visual representation, and (4) $\frac{5}{a}$ versus $\frac{4}{a}$ with a misleading visual representation.

Visual Representation Comparisons: Participants were presented with a fraction comparison problem and both the accurate and misleading visual representations (Figure 2). Participants were asked to identify—using mouse clicks—the visual representation most helpful for solving the fraction comparison problem; left mouse clicks corresponded to the visual representations on the left side of the screen and right mouse clicks corresponded to the visual representations on the right side of the screen. Unlike previous tasks in this study, this task instructed participants to look at and compare the two visual representations and select the visual representation that they felt was more useful in solving the fraction comparison problem. Participants were then instructed to verbally explain why they thought the visual representation they selected was more useful for solving the fraction comparison problem. Verbal explanations were again audio-recorded using Experiment Builder software and later transcribed for analysis. The same two problems—a common numerator problem ($\frac{a}{7}$ versus $\frac{a}{10}$), followed by a common denominator problem ($\frac{3}{a}$ versus $\frac{2}{a}$)—were presented in the same order to all participants. The side of the screen on which the accurate and misleading visual representations were presented was counterbalanced across participants.

Results and Discussion

The primary goal of this experiment was to examine how visual representations of fractions might facilitate college students' performance on a fraction comparison task. We first examined the distribution of students' accuracy on the traditional magnitude comparison task and fraction comparison task. Students' accuracy on the traditional magnitude comparison task was normally distributed, and students correctly answered an average of 59.7% of trials (SD=1.18%, range=40-80%). On the fraction comparison task, students correctly answered an average of 86.40% of trials (SD=18.4%, range=50-100%). However, a histogram of students' accuracy on the fraction comparison task revealed a bimodal distribution: 7 students responded correctly on 50-60% of trials, and 25 students responded correctly on 80-100% of trials. Because most effects did not differ when only high-performing students' data were examined, all analyses include all 32 participants.

To examine whether visual representations of fractions facilitated students' accuracy on the fraction comparison task, a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA was conducted. A significant main effect of visual type was found, such that students had higher accuracy when the fraction comparison problems were paired with accurate visuals (M=92.65%, SD=1.32%) than with misleading visuals (M=80.15%, SD=2.85%), $F(1, 31)=8.23$, p=.007, η^2=.210. No other significant main effects or interactions were found. These results suggest that visual representations of fractions can influence students' accuracy on fraction comparison problems, such that accurate visual representations can improve students' accuracy.

Response times (RTs) on trials in which students responded correctly to the fraction comparison problem were examined as well. Log transformed RTs were entered into a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA, which revealed no significant main effects or interactions (all ps>.05). These results suggest that the amount of time students took to respond to fraction comparison problems did not differ by the fraction type or visual type of the problem.

We measured students' visual attention to further examine how visual representations of fractions might facilitate college students' performance on fraction comparisons. We defined two areas of interest (AOIs) surrounding each of the two fractions and each of the corresponding visual representations on the screen. AOIs of fractions were 72×102 pixels, and AOIs of visual representations were the size of the entire visual representation with an extra 6 pixels around the entire visual representation (e.g., if a visual representation had three 100×100 pixel discrete parts with 6 pixel borders, then the entire visual representation was 324×100 pixels and the AOI was 330×106 pixels). A paired-samples t-test comparing students' proportion of time spent looking to the visual representation versus fractions revealed students looked significantly longer at fractions (M=.37, SD=.58) than visual representations (M=.12, SD=.14), t(31)=2.27, p=.03. However, a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA examining students' proportion of time spent looking to visual representations on different trial types revealed no significant main effects or interactions (all ps>.05), suggesting that attention to the visual representations did not differ as a function of trial type. Additionally, analyses of the number of fixations students made to each AOI revealed similar results: students made significantly more fixations to fractions (M=3.69, SD=2.25) than visual representations (M=2.71, SD=2.65), t(31)=2.11, p=.04, and students' fixations to visual representations did not differ as a function of trial type (all ps>.05). Altogether, these results—in combination with the results from students' accuracy on fraction comparison problems—suggest that though students did not visually attend to the visual representation as much as they did to the fractions, students' accuracy on the fraction comparison problems was still affected by the accuracy of the visual representations. These results are further considered in the discussion.

Qualitative Analyses of Student Explanations—To further understand students' conceptual understanding of fractions, we examined students' verbal explanations on the Fraction Comparison Explanations and Visual Representation Comparisons tasks.

Student explanations from the Fraction Comparison Explanations task: We examined students' verbal explanations for why they felt their answers to each trial type of fraction comparison problem were correct. Table 1 shows the different types of explanations students provided and the percentages for each type. For all four trial types, students most often cited using substitution strategies to explain why they felt their answers were correct (40-43% of all explanations). The second most common type of explanation cited the relation between the numerator and denominator (29-35% of all explanations). Other types of explanations students provided were ones that cited division strategies, parts of a whole, and factual information about numbers. Overall, these results demonstrate that regardless of the type of fraction problem or visual representation shown, the majority of students explained their answers to fraction problems using substitution strategies.

Student explanations from the Visual Representation Comparisons task: We also examined (1) whether students could identify which visual representations were accurate or misleading for a common numerator and common denominator problem and (2) why they felt a visual representation was more useful than the other. Table 2 shows the different types of explanations students provided and the percentages for each type. For both the common numerator and common denominator problems, 62.5% of students correctly identified the accurate visual representation. When asked why one visual representation was more useful than the other, students most often cited the size of the discrete parts or length of the entire visual representation in their explanations (57-65% of all explanations). For the common *numerator* problem, the second most common explanation type (15%) was one that cited the relation between the numerator and denominator; for the common *denominator* problem, the second most common explanation type (14%) was one that simply described factual information about numbers or the visual representation. Other types of explanations cited division strategies, substitution strategies, and cross-multiplication. These results demonstrate that most students could indeed identify which visual representation was accurate for a given problem and did so by attending to the size of the discrete parts or whole length of the visual representation.

Students' Magnitude Understanding and Conceptual Understanding of Fractions—A secondary goal of the present study was to examine how students' understanding of magnitudes when expressed solely with symbolic fraction notation is related to their more conceptual understanding of fractions. Bivariate correlations revealed significant relations between students' accuracy on (1) the traditional magnitude comparison task and fraction comparison task (r=.45, p=.006) and (2) the traditional magnitude comparison task and trials of the fraction comparison task that were paired with accurate visual representations (r=.41, p=.01). Thus, students with a better understanding of symbolically-notated magnitudes performed better on the fraction comparison task. This suggests that understanding fraction magnitudes may be related to students' conceptual understanding of fractions as well.

Experiment 2

Experiment 1 demonstrated that students had higher accuracy on fraction comparison problems that were paired with accurate visual representations than on fraction comparison problems that were paired with misleading visual representations. To further examine the effects of visual representations on students' performance on fraction comparison problems, we added a control condition—a fraction comparison task without visual representations—in Experiment 2. If helpful or accurate visual representations indeed improve students' accuracy on fraction comparison problems, then students should be more accurate when problems are presented with accurate visual representations than when problems are presented with misleading visual representations or no visual representations. On the other hand, if helpful or accurate visual representations do not affect students' accuracy on fraction comparison problems, then students' accuracy on fraction comparison problems should not differ by the presence or absence of visual representations.

Method

Participants—Thirty-four undergraduate students participated (n_{female}=26). Participants were between the ages of 18.25 and 22.94 years (*M*=20.58, *SD*=1.32) and were students enrolled at a selective American university. Two participants' data were excluded from analyses for the following reasons: poor eye-tracking calibration (n=1) and experimenter error (n=1). The final sample included 32 participants (n_{female}=24). None of the participants were majoring or minoring in mathematics or a math-related field.

Apparatus—The same apparatus as Experiment 1 were used.

Materials and Procedure—Three tasks were used in this experiment: (1) Fraction Comparisons with Visual Representations, (2) Fraction Comparisons *without* Visual Representations, and (3) Visual Representation Comparisons. Presentation of the first two tasks—Fraction Comparisons with Visual Representations task and Fraction Comparisons without Visual Representations task—were counterbalanced across participants; the Visual Representation Comparisons task was presented after those first two tasks for all participants. Each task was introduced by an instruction slide that detailed what the participant was to do in that particular task. All study stimuli were created using Adobe Photoshop. Additionally, to assess participants' general mathematical abilities, participants also completed a paper-and-pencil math assessment after completing the three eye-tracking tasks.

Fraction Comparisons with Visual Representations: This task was identical to the Fraction Comparisons task used in Experiment 1.

Fraction Comparisons without Visual Representations: This task was a combination of (1) Fraction Comparisons problems used in Experiment 1, except that no visual representations of the fractions were given with the fraction comparison problems—that is, participants only saw fraction comparison problems (e.g., "Which is larger, $\frac{a}{5}$ or $\frac{a}{8}$?")—and (2) Traditional Magnitude Comparisons problems used in Experiment 1 (e.g., "Which is larger, $\frac{3}{5}$ or $\frac{26}{71}$?"). Twenty traditional magnitude comparison problems and 20 fraction comparison problems were randomly ordered within the task for each participant. The purpose of this task was to measure participants' (1) performance purely on fraction comparison problems—without the aid of visual representations—and (2) understanding of magnitudes expressed with symbolic fraction notation.

Visual Representation Comparisons: This task was identical to the Visual Representation Comparisons task in Experiment 1, except that participants were not asked to verbally explain their answers.

Algebra Assessment: Because previous work has found relational understanding of fractions to be related to algebra understanding (DeWolf et al., 2015b), a measure of algebra understanding was added in Experiment 2. A 27-question paper-and-pencil assessment provided a baseline measure of participants' algebra understanding (DeWolf, Son, Bassok,

& Holyoak, 2015; adapted from DeWolf et al., 2015b). This assessment included algebra problems that were either taken from the California State Standards for Grade 8 or adapted from Booth, Newton, and Twiss-Garrity (2014).

Results and Discussion

The goal of this experiment was to examine whether accurate, helpful visual representations of fractions indeed improved college students' performance on a fraction comparison task. We first examined the distribution of students' accuracy on the algebra assessment, traditional magnitude comparison task, fraction comparison task without visual representations, and fraction comparison task with visual representations. On the algebra assessment, students correctly answered an average of 84.78% of questions (SD=8.14%, range=60-96%). Students correctly answered an average of 53.73% of trials (SD=12.06%, range=15-70%) on the traditional magnitude comparison task, whereas they correctly answered an average of 78.75% of trials (SD=10.97%, range=45-95%) on the fraction comparison task without visual representations. Finally, on the fraction comparison task with visual representations, students correctly answered an average of 84.33% of trials (SD=18.33%, range=40-100%). Unlike Experiment 1, histograms of students' accuracy on all tasks revealed unimodal distributions; thus, all 32 participants' data were included in all analyses.

To examine whether visual representations of fractions facilitated students' accuracy on fraction comparison problems, a 2 (fraction type: common numerator vs. common denominator) × 3 (visual type: accurate vs. misleading vs. none) repeated-measures ANOVA was conducted. A significant main effect of visual type was found, such that students had higher accuracy when the fraction comparison problems were paired with accurate visuals (M=90.27%, SD=13.41%) than with misleading visuals (M=78.39%, SD=27.33%) or no visuals (M=78.75%, SD=10.97%), $F(2, 62)=8.11$, p=.001, η^2=.207. No other significant main effects or interactions were found. Consistent with our findings in Experiment 1, these results demonstrate that accurate, helpful visual representations of fractions improve students' accuracy on fraction comparison problems. Moreover, these results suggest that accurate visual representations can boost students' accuracy more than when misleading visual representations or no visual representations are provided.

RTs on trials in which students responded correctly to the fraction comparison problems were examined as well. However, problems without visual representations inherently had less stimuli to be looked at and processed. Therefore, we first examined whether RTs differed between problems with visual representations and problems without visual representations. Paired-samples t-tests of the Log transformed RTs for problems with and without visual representations showed that problems without visual representations (M=3.44 seconds, SD=1.30 seconds) were solved significantly faster than problems with visual representations (M=4.72 seconds, SD=2.44 seconds), $t(31)=2.83$, p=.008. As such, we next examined RTs for only the problems that were paired with visual representations. Log transformed RTs were entered into a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA, which revealed a significant main effect of visual type, such that problems with accurate visual representations (M=4.34 seconds, SD=2.04 seconds) were solved faster than those with misleading visual representations (M=5.10 seconds, SD=3.03 seconds), $F(1, 26)=5.02$, $p=.03$, $\eta^2=.162$. No other significant main effects or interactions were found. Thus, students solved fraction comparison problems without visual representations faster than those with visual representations, and—consistent with findings regarding students' accuracy—students solved fraction comparison problems with accurate visual representations faster than they solved fraction comparison problems with misleading visual representations.

These RT findings do, however, contrast with those found in Experiment 1. In Experiment 1, the speed with which students solved fraction comparison problems did not differ by visual type. Thus, we examined RTs of only the students who completed the Fraction Comparisons with Visual Representations task—the task identical to that in Experiment 1—before completing the Fraction Comparisons without Visual Representations task (n=17). Log transformed RTs were entered into a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA, which revealed no significant main effects or interactions (all ps>.05). In contrast, the same analyses with only the students who completed the tasks in the opposite order (n=15)—that is, the Fraction Comparisons without Visual Representations task before the Fraction Comparisons with Visual Representations task—revealed a significant main effect of visual type, such that students solved problems paired with accurate visuals faster than problems paired with misleading visuals, $F(1,11)=6.35$, $p=.03$, $\eta^2=.366$. Together, these results suggest that the contrasting RT findings between Experiments 1 and 2 may be due to differences in methodology between the two experiments. In particular, solving fraction comparison problems without visual representations before being presented with problems that do have visual representations may bias participants into spending more time on problems that are paired with misleading visual representations.

Students' visual attention on fraction comparison problems with visual representations were also examined. The same two AOIs as in Experiment 1—that is, AOIs surrounding each of the two fractions and each of the corresponding visual representations—were used in Experiment 2. A paired-samples t-test comparing proportion of time spent looking to the visual representation versus fractions revealed students looked significantly longer at fractions (M=.15, SD=.11) than visual representations (M=.07, SD=.07), $t(31)=2.99$, $p=.005$. However, a 2 (fraction type: common numerator vs. common denominator) × 2 (visual type: accurate vs. misleading visual) repeated-measures ANOVA examining students' proportion of time spent looking to visual representations on different trial types revealed no significant main effects or interactions (all ps>.05), suggesting that attention to the visual representations did not differ as a function of trial type. Interestingly, analyses of the number of fixations students made to each AOI revealed students did not differ in the number of fixations made to fractions (M=3.52, SD=2.35) and visual representations (M=2.69, SD=2.66), $t(31)=1.55$, $p=.13$, but students made significantly more fixations to misleading visual representations (M=2.96, SD=3.11) than to accurate visual representations (M=2.40, SD=2.41), $F(1, 31)=7.51$, $p=.01$, $\eta^2=.195$. However, students who looked longer at the fractions also made more fixations to the fraction ($r=.69$, $p<.001$), and students who looked longer at the visuals also made more fixations to the visuals ($r=.66$, $p<.001$). Altogether, these results suggest that students did not attend to the visual representations for very long or with many fixations and may have found looking to the fractions to be more useful than looking to the visual representations. Additionally, students' increased fixations to misleading visuals over accurate visuals—combined with results from students' RT on problems with misleading versus accurate visuals—suggest students required more time and visual attention to process misleading visuals.

Visual Representation Comparisons task—We also examined whether students could identify which visual representations were accurate or misleading for a common numerator and common denominator problem. On average, only 64.7% students correctly identified the accurate visual representation for common numerator problems, whereas 76.5% of students correctly identified the accurate visual representation for common denominator problems. Nonparametric tests examining whether students who correctly versus incorrectly identified the accurate visual representations differed in their performance (in terms of accuracy, RT, proportion of looking time, and fixations) on the Fraction Comparison Task—with and without visual representations—revealed no significant differences between the two groups (all *p*s>.05). These results suggest that students' ability to correctly discriminate between accurate and misleading visual representations did not affect their performance on the Fraction Comparison Task.

Students' Algebra Understanding, Magnitude Understanding, and Conceptual Understanding of Fractions—A bivariate correlation revealed no reliable relation between algebra understanding and accuracy on the traditional magnitude comparison task; additionally, there was no reliable association between accuracy on the traditional magnitude comparison task and accuracy on fraction comparison problems either (all *p*s>.05). However, there were significant correlations between (1) algebra understanding and accuracy on fraction comparison problems without visual representations (r=.49, p=.005) and (2) algebra understanding and accuracy on trials of the fraction comparison task paired with accurate visual representations (r=.57, p=.001). These results are consistent with previous work showing relational—but not literal magnitude—understanding of fractions to be related to algebra understanding (DeWolf et al., 2015b) and suggest that understanding of algebra—rather than symbolically-notated magnitudes—may be related to performance on the fraction comparison task.

General Discussion

In this study we investigated the extent to which adults utilize visual representations during a fraction comparison task. The design of the fraction comparison task required participants to solve fraction comparison problems composed of abstract symbolic fractions, either with a common denominator (e.g., $\frac{5}{a}$ vs. $\frac{7}{a}$) or a common numerator (e.g., $\frac{b}{4}$ vs. $\frac{b}{9}$); in Experiments 1 and 2, each of these fraction comparison problems was also paired with a visual representation of the abstract symbolic fractions. The visual representations were set up so that they either had equal total lengths split into different size pieces (modeling common numerators) or equal size pieces that were different in total length (modeling common denominators). This study is unique in that we also made use of eye tracking technology to verify whether participants attended to the visual representations or only the symbolic representation.

College students performed more accurately when the visual representation was helpful or accurate than when it was misleading (Experiments 1 & 2) or when no visual representation was provided (Experiment 2). Students' accuracy on fraction comparison problems were improved by accurate visual representations, but accuracy on problems with misleading visual representations was not different from problems without visual representations. Thus, though accurate visual representations improved students' performance on fraction comparison problems, misleading visual representations did not impair students' performance. Students may not have been impaired by misleading visual representations because students looked more at misleading visual representations than accurate ones (Experiment 2), and consequently, took longer to solve fraction comparison problems with misleading visual representations than those with accurate visual representations (Experiment 2). Therefore, it is possible that students may have actively examined and then disregarded the misleading visual representation when solving fraction comparison problems. Interestingly, however, our participants tended to view the visual representations only briefly, yet accurate visual representations still improved accuracy performance. Thus, minimal exposure to the accurate visual representation seems to have affected performance.

The participant explanations (Experiment 1) also provided useful insight into how students were incorporating the visual representations in their assessments of the symbolic fractions, as well as how they thought about the comparisons in general. Participants showed a strong tendency to use substitution strategies but also showed evidence of thinking more abstractly about the task by providing general rules about division and how the numerator and denominator correspond to each other. Additionally, most participants were able to discriminate between visual representations that were accurate and those that were misleading. This suggests that participants were able to incorporate their abstract understanding of how the symbolic and visual representations model the same or different types of division.

The fraction comparison task required participants not only to integrate visual and symbolic representations but also to have an abstract conceptual understanding of different aspects of division. Even though some participants substituted numbers in place of the algebraic symbols, they still had to consider many substitute cases and how those cases corresponded to each other across the two fractions, and this in turn requires some level of understanding of how the symbolic and visual representations were modeling different aspects of division. The common numerator case modeled a slightly simpler definition of division in which equal sized "wholes" were divided into different sized pieces. This is similar to how many students are introduced to the idea of fractions and map division to fractions (Empson, 1999; Wu, 2009). The common denominator case was slightly more complicated despite the high levels of performance on the common denominator problems. In this case, the visual representation was modeling the multiplicative definition (e.g., $\frac{5}{a}$ is equivalent to 5 "$\frac{1}{a}$" parts or: $\frac{1}{a}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}$). In this sense, students needed to understand fractions as units and how that corresponds to their theoretical sizes when compared to a fraction also made of equal sized units. This type of understanding is less often taught but is another critical conceptual component of understanding fractions and their relation to division (Kellman et al., 2008).

A secondary question was whether performance on our fraction comparison task—which utilized both visual representations and symbolic representations—is related to performance on a traditional fraction magnitude comparison task. Experiment 1 revealed performance on the fraction comparison task to be related to the traditional magnitude comparison task, suggesting that understanding of magnitudes may benefit conceptual understandings of fractions as well. However, Experiment 2 did not find a significant correlation between performance on the fraction comparison task and traditional magnitude comparison; instead, performance on the fraction comparison task was related to algebra understanding. In the fraction comparison task, the fractions were actually algebraic expressions (e.g., $\frac{5}{a}$); they did not represent an absolute magnitude, as the stimuli in the traditional magnitude comparison task did. The fraction comparison task required more abstract relational reasoning about the relative sizes of various expressions whereas the traditional magnitude comparison task is typically thought to measure the representations of actual magnitudes. Thus, as has been previously posited (DeWolf et al., 2015b), the relation between algebra performance and the fraction comparison task—but not the magnitude comparison task—in Experiment 2 suggests a possible dissociation between thinking more abstractly about the relation between the numerator and denominator and actually assessing the size of the fraction magnitude in the traditional task. Further studies might investigate the relation among conceptual fraction understanding, magnitude understanding, and algebra understanding.

In general, these findings suggest that even adult participants at a selective university are affected by the correspondence between visual and symbolic representations. These findings have important implications for educators in that visual representations of fractions must be considered carefully. Further, the shift from understanding visual representations to symbolic representations is not straightforward. That is, even when students have good working knowledge of symbolic representations, students still incorporate visual representations when they are provided. They do not seem to ignore unhelpful representations and go with the more precise symbolic representation to make their judgment. Thus these findings suggest that the symbolic understanding of fraction expressions is somewhat fragile and can be confused when conflicting cues, such as misleading visual representations, are provided.

Eye tracking data also point to another important implication for educators and designers of instructional materials. Even when students only process visual cues with a minimal number of fixations, accurate visual representations can benefit students. This indicates that educators should carefully consider how material is presented on a white board or other visual display. For example, if a problem is presented on a white board with potentially misleading information surrounding it, this information could lead to confusion in students —even if the student's attention is not drawn to the problematic information; on the other hand, presenting problems on the white board with accurate information could benefit students. Future research could determine what exactly constitutes misleading information that can cause this confusion in students, and further, what level of exposure is necessary to cause detrimental effects.

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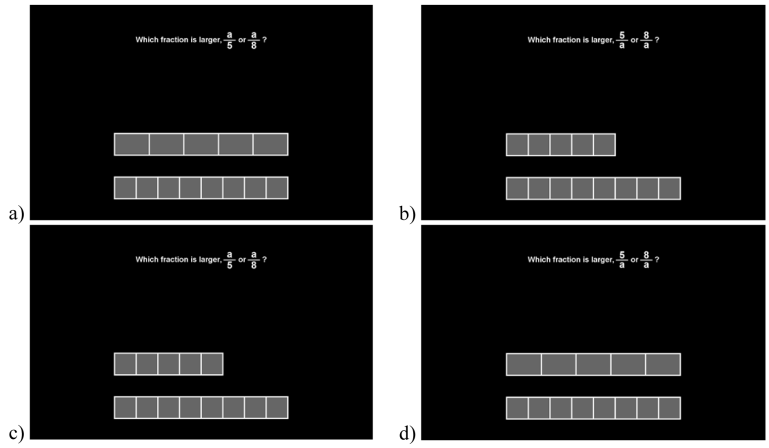

Figure 1.

Example stimuli of different trial types in the Fraction Comparison task. Accurate (a, b) and misleading (c, d) visual representations were provided with common numerator (a, c) and common denominator (b, d) problems.

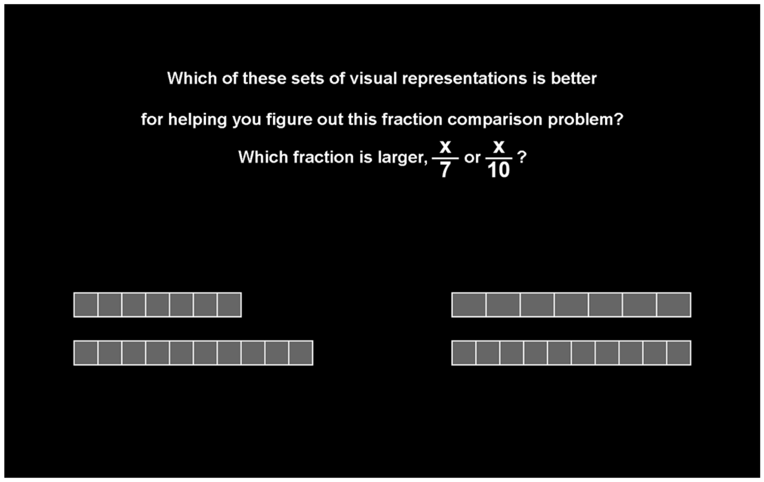

Figure 2.

Example stimulus from the Visual Representation Comparisons task. Fraction comparison problems were shown with both the accurate and misleading visual representations. Participants were asked to identify which visual representation was most helpful for solving the fraction comparison problem and explain why they felt that visual representation was most helpful.

Table 1

Students' explanations on the Fraction Comparison Explanations task

		Percent of times students provided this explanation			
		Common Numerator		Common Denominator	
	Example explanations	Accurate Visual	Misleading Visual	Accurate Visual	Misleading Visual
1. Substitution	*If x were 7, and 7 over 7 is 1, then 7 over 4 is larger than 1.*	41%	43%	43%	40%
2. Numerator-Denominator relation	*…when the numerator is the same, then the fraction that has the smaller denominator is bigger.*	29%	33%	33%	35%
3. Division	*The number is bigger whenever it's divided by a smaller number.*	12%	14%	10%	10%
4. Parts of a whole	*9 is bigger than 6, so 9 into x parts is going to be bigger than 6 into the same x parts.*	12%	5%	10%	5%
5. Fact	*Because 4 is smaller than 7.*	6%	5%	5%	10%

Table 2

Students' explanations on the Visual Representation Comparisons task

	Example explanations	Percent of times students provided this explanation: Common Numerator	Common Denominator
1. Size	*The individual boxes are bigger for x/7.*	65%	57%
2. Numerator-Denominator Relation	*x over 7 is larger because the denominator is smaller.*	15%	10%
3. Fact	*1 goes into 3 three times and 1 goes into 2 two times. The upper right side has three blocks and then the lower right side has two blocks representing the fraction.*	0%	14%
4. Substitution	*… because you can fill in like a random number, so you can make x equal maybe a 2 or 3 and you can see…how much [the visual representation] would be full …*	10%	10%
5. Division	*The left one is helping me to figure out which is larger because it is a different number divided by the same number.*	10%	5%
6. Cross-multiplication	*x is one unit and you multiply that by 3 or 2 so there's 3 units of x in 2…so 3 is larger than that.*	0%	5%