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PERTURBATION SOLUTIONS FOR THE PARTICLE TRAJECTORIES OF A GAS-SOLID MIXTURE ENTERING A CURVED DUCT

Woon-Shing Yeung*

ABSTRACT

A matched asymptotic expansion solution was found for the particle trajectories in a curved duct when the non-dimensional momentum equilibration length $\tilde{L}_m$ is small. The motion of the carrier fluid was assumed to be uniform. The approximate solution was compared with the exact numerical solution and good agreements were found.

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Perturbation Solutions for the Particle Trajectories of a Gas-Solid Mixture Entering a Curved Duct

Woon-Shing Yeung

Introduction

In a previous study[1], Yeung has obtained numerical solutions for the particle trajectories in a circular curved tube. Two dimensionless quantities were identified: τ_m, the nondimensional momentum equilibration length, and δ, the ratio of the mean radius of the elbow to the radius of the cross section of the pipe. It is thus natural to investigate the perturbation solution when τ_m or δ, or both, are small (or large). The main advantage of perturbation solutions is that the variables of the governing equations can be represented approximately in closed forms. These are considerably better than the numerical solutions. For example, the calculation of particle density in the region of interest can be more easily carried out using the approximate closed form solution than the numerical solution. It also saves a lot of computer storage spaces when momentum coupling between the gas and the particle phase is accounted for.

Theory

Consider a gas-particle mixture flowing into a 2-D curved duct with mean radius of curvature R as shown.

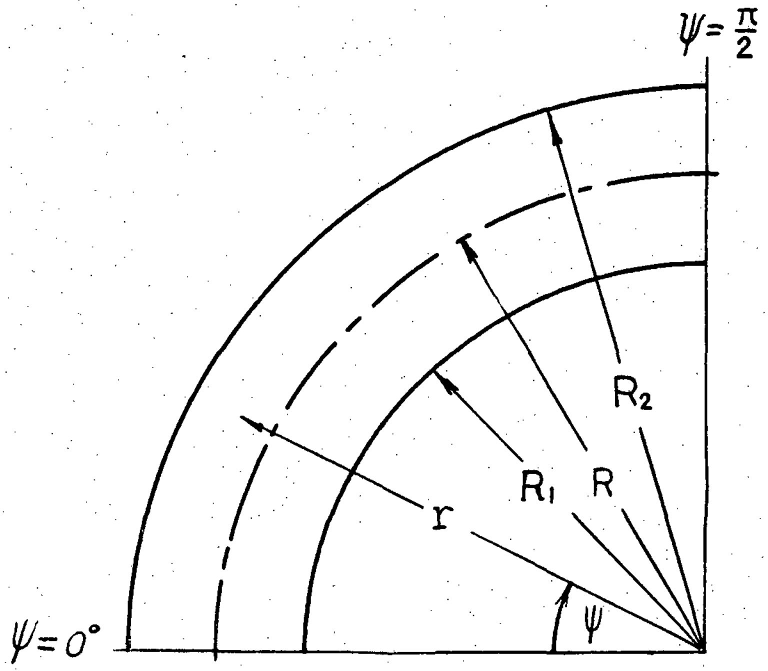

Fig. 1 Coordinate system of the geometry used.

A 2-D curved duct is chosen for simplification, so that δ is absent from the governing equations. The main assumptions are those used in the previous study[1], which are listed below:

(1) Particle-particle interaction is small compared to particle-fluid interaction.

(2) The presence of particles does not influence the gas flow field.

(3) Only the drag force exerted by the fluid phase on the particle phase is considered.

(4) The aerodynamic drag force is assumed to be given by Stokes law throughout the region of interest.

(5) The fluid moves with uniform speed along the curved pipe.

Some justifications of the above assumptions can be found in the previous investigation[1]. If we denote u as the velocity component in increasing r-direction and v as the velocity component in the increasing ψ-direction, we can write down the motion of the fluid phase in view of assumption (5):

$$u_f = 0 \quad , \tag{1}$$

$$v_f = W \quad , \tag{2}$$

where W is the initial entry velocity of the dilute suspension and subscript f denotes values pertaining to the fluid phase. For the motion of the particle phase, we shall use the Lagrangian description. As mentioned by Yeung[2], collisions among particles cannot be neglected if one tries to formulate the motion of an individual particle, which presents great difficulty in the solution process. However, when the particle phase is considered as a continuum, the particle-particle interaction becomes negligible compared with the particle-fluid interaction, as long as the suspension is dilute[2]. Thus, if the particle phase behaves as, or at least approximately, a continuum, the momentum equations of the particle phase can be expressed as

$$\frac{\partial u_p}{\partial t} - u_p \frac{\partial u_p}{\partial r} + \frac{v_p}{r} \frac{\partial u_p}{\partial \psi} - \frac{v_p^2}{r} = - \frac{u_f - u_p}{\tau_m} \quad , \tag{3}$$

$$\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial r} + \frac{v_p}{r} \frac{\partial v_p}{\partial \psi} + \frac{u_p v_p}{r} = - \frac{v_f - v_p}{\tau_m} \quad , \tag{4}$$

where we have neglected the diffusional stress tensor of the particle phase due to its interaction with the fluid phase. τ_m is the momentum equilibration time and the subscript p denotes the particle phase. By virtue of assumption (4), τ_m is given by

$$\tau_m = \frac{2}{9} \frac{\sigma^2 \bar{\rho}_p}{\mu_f} \tag{5}$$

where σ denotes the particle diameter, $\bar{\rho}_p$ the particle material density and μ_f the viscosity of the fluid phase. It is of particular importance to realize that equations (3) and (4) describe the mean motion of the particulate phase, but not the motion of individual particles. Thus, imagine a small volume surrounding a point in space, which is large enough to contain a great number of molecules, while still possessing dimensions small compared with the characteristic dimension of the physical system. The quantities u_p, v_p in equations (3) and (4) then represent the average value of the motion of all the particles contained in that small volume. Since only mean values are of main concern in most practical situations, the Lagrangian formulation of individual particle motion appears to be more than adequate. Moreover, the collisions among particles render the Lagrangian formulation impractical. To circumvent this difficulty, the notion of emsemble average‡ is used with the Lagrangian formulation of the particle motion. To begin with, we write down the equation of motion of a particle

$$M \frac{d\vec{v}}{dt} = F(t) \quad , \tag{6}$$

where M is the particle mass, $\vec{v}$ the particle velocity and F(t) the force acting upon the particle by virtue of collisions from the fluid molecules and other particles. Following Langevin[3] on the theory of Brownian motion, we assume F(t) can be written as a sum of two parts: (i) an "averaged-out" part, which represents the viscous drag experienced by the particle and (ii) a "fluctuating" part f(t). Equation (6) becomes

$$M \frac{d\vec{u}_p}{dt} = M \frac{\vec{v}_f - \vec{v}_p}{\tau_m} + f(t) \quad . \tag{7}$$

We now consider an ensemble of a large number of systems and take the ensemble average of (7),

‡An ensemble is a collection of a large number of systems similar to the one originally under consideration. The ensemble average is one which is taken over the ensemble at any instant of time. It was introduced by Gibb and is used extensively in statistical mechanics.

$$M \frac{d\langle\vec{u}_p\rangle}{dt} = M \frac{\langle\vec{v}_f\rangle - \langle\vec{u}_p\rangle}{\tau_m} + \langle f(t)\rangle \quad . \tag{8}$$

The important assumption to be made here is

$$\langle f(t)\rangle = 0 \tag{9}$$

and equation (8) is reduced to

$$\frac{d\langle v_p\rangle}{dt} = \frac{\langle\vec{v}_f\rangle - \langle\vec{v}_p\rangle}{\tau_m} \quad . \tag{10}$$

The particle may collide with other particles and thereby lose momentum or change direction. This may be looked upon as a drag force acting on the particle by all other particles. However, since the collision frequency of the particle with the gas molecules is much larger than with the other particles, the averaged drag force is represented solely by the viscous drag due to the fluid. Assumption (9) is justified by the random nature of $f(t)$†. Equation (10) is the necessary equation of motion of a particle in terms of the ensemble average value of the velocity. Since the ensemble average value is the one most likely to be observed macroscopically, it can be looked upon as the mean motion of the particle phase. From here on, we drop the use of brackets in equation (10). It should be clear that only mean values are formulated and solved for. Hence

$$\frac{d\vec{v}_p}{dt} = \frac{\vec{v}_f - \vec{v}_p}{\tau_m} \quad . \tag{11}$$

In polar coordinates,

$$\vec{v}_p = u_p \vec{e}_r + v_p \vec{e}_\psi \quad , \tag{12}$$

where $\vec{e}_r$ and $\vec{e}_\psi$ are the unit vectors in the increasing r and ψ directions, respectively. Substituting (12), (1) and (2) into equation (11) and remembering

†Strictly speaking, a special proof is required. It may be considered as a pure assumption in the present work.

that $\vec{e}_r$ and $\vec{e}_\psi$ are not constant vectors, we obtain

$$\frac{du_p}{dt} - \frac{v_p^2}{r} = -\frac{u_p}{\tau_m} \tag{13}$$

in the r-direction and

$$\frac{dv_p}{dt} + \frac{u_p v_p}{r} = \frac{W - v_p}{\tau_m} \tag{14}$$

in the ψ direction. We have used the dynamic relations

$$\frac{dr}{dt} = u_p \tag{15}$$

and

$$\frac{d\psi}{dt} = \frac{v_p}{r} \quad . \tag{16}$$

Equations (13) to (16) can be solved for the particle trajectories under some prescribed initial conditions. One may have obtained the same set of governing equations by considering the motion of an individual particle under the influence of only the viscous drag of the fluid phase. In that case, f(t) would have been identically zero at any instant, which is a highly improbable situation even when there is absolutely no collision among particles[†]. Indeed, the autocorrelation function of f(t) is important in some other quantities of interest in the theory of Brownian motion[4]. Thus, the derivation given above of equations (13) to (16) is fundamentally more correct and rigorous. Furthermore, the use of ensemble average values of macroscopic quantities is more appropriate for most physical systems in practice.

For the initial conditions, we assume the gas and particle phases are in dynamic equilibrium and that the particles are uniformly distributed at the entry. Hence the ensembled averages of the initial conditions are

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[†]Recall that f(t) is the fluctuating part of the force due to particle-molecule and particle-particle collisions as well.

$$u_p = 0 \quad , \quad v_p = W \quad . \quad r = r_o \quad , \quad \psi = 0 \tag{17}$$

at $t = 0$, and $(r_o, 0)$ being the initial position of the particle.

Matched Asymptotic Solution

As mentioned before, we are seeking analytical solutions for the particle trajectories when $\tilde{L}_m$, the nondimensional momentum equilibration length, is small. It can be readily shown, by nondimensionalizing the particle equations, that $\tilde{L}_m$ is given by

$$\tilde{L}_m = \frac{W\tau_m}{R} = \frac{\tau_m}{R/W} = \frac{\tau_m}{t^*} \quad , \tag{18}$$

where

$$t^* = R/W \quad . \tag{19}$$

Hence small $\tilde{L}_m$ implies that

$$t^* \gg \tau_m \quad . \tag{20}$$

It is well known in perturbation theory that if the small parameter, in our case $\tilde{L}_m$, is a ratio of two time scales, in our case τ_m and t^*, it will be a singular perturbation problem[5]. One cannot generate series solutions, which are uniformly valid in the domain of interest, by straightforward expansions. Several methods exist to deal with singular perturbation problems and the method of matched asymptotic expansions is most suitable for the present problem.

We begin by identifying the outer variable and the inner variable of the problem, which are t^* and τ_m, respectively. Then the outer problem is formulated by introducing the following dimensionless quantities:

$$\bar{U} = \frac{u_p}{W} \quad , \quad \bar{V} = \frac{v_p}{W} \quad , \quad \bar{R} = \frac{r}{R} \quad , \quad \bar{\psi} = \psi \quad , \quad \bar{t} = \frac{t}{t^*} \quad . \tag{21}$$

Equations (13) to (16) become

$$\tilde{L}_m \frac{d\bar{U}}{d\bar{t}} = -\bar{U} + \tilde{L}_m \frac{\bar{V}^2}{\bar{R}} \quad , \tag{22}$$

$$\tilde{L}_m \frac{d\bar{V}}{d\bar{t}} = 1 - \bar{V} - \tilde{L}_m \frac{\bar{U}\,\bar{V}}{\bar{R}} \quad , \tag{23}$$

$$\frac{d\bar{R}}{d\bar{t}} = \bar{U} \quad , \tag{24}$$

$$\frac{d\bar{\psi}}{d\bar{t}} = \frac{\bar{V}}{\bar{R}} \quad . \tag{25}$$

Examination of equations (22) and (23) shows that when $\tilde{L}_m \to 0$, they become algebraic equations instead of first order ordinary differential equations. The given initial conditions (17) cannot be satisfied consequently. The failure to satisfy all the prescribed boundary and initial conditions of the outer solution is typical in all singular perturbation problems. Hence, instead of using (17), one must obtain the general solutions of (23) to (25) and evaluate any constants in the general solution by matching with the inner solution. Pertinent solutions of (22) to (25) are written in the forms

$$\bar{U} = \bar{U}_o + \tilde{L}_m \bar{U}_1 + \tilde{L}_m^2 \bar{U}_2 + \cdots \quad , \tag{26}$$

$$\bar{V} = \bar{V}_o + \tilde{L}_m \bar{V}_1 + \tilde{L}_m^2 \bar{V}_2 + \cdots \quad , \tag{27}$$

$$\bar{R} = \bar{R}_o + \tilde{L}_m \bar{R}_1 + \tilde{L}_m^2 \bar{R}_2 + \cdots \quad , \tag{28}$$

$$\bar{\psi} = \bar{\psi}_o + \tilde{L}_m \bar{\psi}_1 + \tilde{L}_m^2 \bar{\psi}_2 + \cdots \quad . \tag{29}$$

Substituting (26), (27), (28), (29) into equations (22) to (25) and equating equal order of $\tilde{L}_m$ we obtain

$$0 = -\bar{U}_o \quad ,$$

$$0 = 1 - \bar{V}_o \quad ,$$

$$\dot{\bar{R}}_o = U_o \quad , \qquad (30)$$

$$\dot{\bar{\psi}}_o = \frac{\bar{V}_o}{\bar{R}_o}$$

for the zeroth order and

$$\dot{\bar{U}}_o = -\bar{U}_1 + \bar{V}_o^2/\bar{R}_o \quad ,$$

$$\dot{\bar{V}}_o = -\bar{V}_1 - \frac{\bar{V}_o\bar{U}_o}{\bar{R}_o} \quad ,$$

$$\dot{\bar{R}}_1 = U_1 \quad , \qquad (31)$$

$$\dot{\bar{\psi}}_1 = \frac{1}{\bar{R}_o}\left(\bar{V}_1 - \frac{\bar{R}_1}{\bar{R}_o}\bar{V}_o\right)$$

for the first order of $\tilde{L}_m$. The dot denotes differentiation with respect to $\bar{t}$. The governing equations for higher order terms are complicated algebraically and are not recorded here.

The general solution to the outer problem is found, to order of $\tilde{L}_m^3$, as follows:

$$\bar{U} = \frac{1}{A}\tilde{L}_m - \frac{1}{A^2}\left(\frac{\bar{t}}{A} + C\right)\tilde{L}_m^2 + \frac{1}{A}\left(-\frac{1}{A^2} - \frac{E}{A} + \frac{C^2}{A^2} + \frac{3C}{A^2}\bar{t} + \frac{3}{2A^4}\bar{t}^2\right)\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \qquad (32)^*$$

$$\bar{V} = 1 - \frac{1}{A^2}\tilde{L}_m^2 + \frac{1}{A}\left[\frac{2}{A^2}\left(\frac{\bar{t}}{A} + C\right)\right]\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \qquad (33)$$

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*In view of equation (32), u_p should have been nondimensionalized according to $\bar{U} = u_p/W\tilde{L}_m$ so as to render $\bar{U}$ of order 1. However, the final results were found to be identical with the present results when the new form of $\bar{U}$ was used.

$$\bar{R} = A + (C + \frac{\bar{t}}{A})\tilde{L}_m + [E - \frac{1}{A^2}(C\bar{t} + \frac{\bar{t}^2}{2A})]\tilde{L}_m^2$$
$$+ \{\frac{1}{A}[\bar{t}(-\frac{1}{A^2} - \frac{E}{A} + \frac{C^2}{A^2}) + \frac{3C}{2A^3}\bar{t}^2 + \frac{1}{2A^4}\bar{t}^3] + G\}\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \tag{34}$$

$$\bar{\psi} = (\frac{\bar{t}}{A} + B) + [D - \frac{1}{A^2}(C\bar{t} + \frac{\bar{t}^2}{2A})]\tilde{L}_m + \{F + \frac{1}{A}[\bar{t}(\frac{-1}{A^2} - \frac{E}{A}) + \frac{C}{2A^3}\bar{t}^2 + \frac{1}{6A^4}t^3]\}\tilde{L}_m^2$$
$$+ \{\frac{1}{A}[(\frac{3C}{A^3} + \frac{2EC}{A^2} - \frac{G}{A} - \frac{C^3}{A^3})\bar{t} + \frac{\bar{t}^2}{2}(\frac{4}{A^4} + \frac{3E}{A^3} - \frac{6C^2}{A^4}) + \frac{\bar{t}^3}{3}(-\frac{15}{2}\frac{C}{A^3})$$
$$+ \frac{\bar{t}^4}{4}(-\frac{5}{2A^6})] + H\}\tilde{L}_m^3 \quad , \tag{35}$$

where A,B,C,D,E,F,G,H are constants to be determined by matching conditions.

For the inner problem, the following nondimensional quantities are used:

$$U = \frac{u}{W} \quad , \quad V = \frac{v}{W} \quad , \quad R = \frac{r}{R} \quad , \quad \Psi = \psi \quad , \quad T = \frac{t}{\tau_m} \quad , \tag{36}$$

where we have nondimensionalized t with the inner variable τ_m instead of t^*. T and $\bar{t}$ are related by the following

$$T = \frac{\bar{t}}{\tilde{L}_m} \quad . \tag{37}$$

Substitute (36) into equations (13) to (16) to obtain

$$\frac{dU}{dT} = -U + \tilde{L}_m \frac{V^2}{R} \quad , \tag{38}$$

$$\frac{dV}{dT} = 1 - V - \frac{VU}{R}\tilde{L}_m \quad , \tag{39}$$

$$\frac{dR}{dT} = U\tilde{L}_m \quad , \tag{40}$$

$$\frac{d\Psi}{dT} = \frac{V}{R}\tilde{L}_m \quad . \tag{41}$$

The main difference between the outer and the inner problem is that the latter does not alter the order of the original system of differential equations, as seen from (38) to (41). Thus, the initial conditions (17) can be satisfied by the inner solution. As before, U,V,R and Ψ are assumed to be of the following forms

$$U = U_o + \tilde{L}_m U_1 + \tilde{L}_m^2 U_2 + \cdots \quad , \tag{42}$$

$$V = V_o + \tilde{L}_m V_1 + \tilde{L}_m^2 V_2 + \cdots \quad , \tag{43}$$

$$R = R_o + \tilde{L}_m R_1 + \tilde{L}_m^2 R_2 + \cdots \quad , \tag{44}$$

$$\Psi = \Psi_o + \tilde{L}_m \Psi_1 + \tilde{L}_m^2 \Psi_2 + \cdots \quad . \tag{45}$$

The initial conditions (17) become

$$U_o = 0 \quad , \quad V_o = 1 \quad , \quad R_o = \bar{r}_o \quad , \quad \Psi_o = 0$$

and

$$U_i = 0 \quad , \quad V_i = 0 \quad , \quad R_i = 0 \quad , \quad \Psi_i = 0 \quad ; \quad \forall i \geqq 1 \tag{46}$$

at

$$T = 0 \quad ,$$

where

$$\bar{r}_o = \frac{r_o}{R} \quad . \tag{47}$$

Substituting equations (42) to (45) into (38) to (41) and equating equal orders of $\tilde{L}_m$ as before, we obtain systems of differential equations, subject to the appropriate initial conditions given by (46). The solution procedure is straightforward and the inner expansions of U,V,R and Ψ are found to be, to order $O(\tilde{L}_m^5)$, the following:

$$U = \frac{1}{\bar{r}_o}(1 - e^{-T})\tilde{L}_m + \frac{1}{\bar{r}_o^3}(Te^{-T} + T^2e^{-T} - T)\tilde{L}_m^3$$
$$+ \frac{1}{\bar{r}_o^5}\left(\frac{3}{2}T^2 + 8e^{-T} + 13Te^{-T} + \frac{3T^2}{2}e^{-T} - \frac{4T^3}{3}e^{-T} - \frac{1}{6}T^4e^{-T} + T - 8 - 2Te^{-2T}\right)\tilde{L}_m^5$$
$$+ O(\tilde{L}_m^6) \quad , \tag{48}$$

$$V = 1 + \frac{1}{\bar{r}_o^2}(e^{-T} + Te^{-T} - 1)\tilde{L}_m^2 + \frac{1}{\bar{r}_o^4}(2T - 2 + 3e^{-T} - \frac{3}{2}T^2e^{-T} - \frac{T^3}{3}e^{-T}$$

$$- Te^{-2T} - e^{-2T})\tilde{L}_m^4 + 0 + O(\tilde{L}_m^6) \quad , \tag{49}$$

$$R = \bar{r}_o + \frac{1}{\bar{r}_o}(T + e^{-T} - 1)\tilde{L}_m^2 + \frac{1}{\bar{r}_o^3}(3 - 3e^{-T} - 3Te^{-T} - \frac{T^2}{2} - T^2e^{-T})\tilde{L}_m^4$$

$$+ O(\tilde{L}_m^6) \quad , \tag{50}$$

$$\Psi = \frac{T}{\bar{r}_o}\tilde{L}_m + \frac{1}{\bar{r}_o^3}(1 - e^{-T} - Te^{-T} - \frac{T^2}{2})\tilde{L}_m^3$$

$$+ \frac{1}{\bar{r}_o^5}\{9 - 5T + \frac{T^2}{2} + \frac{T^3}{6} - 10e^{-T} + \frac{5}{2}T^2e^{-T} + \frac{1}{3}T^3e^{-T} + Te^{-2T} + e^{-2T}\}\tilde{L}_m^5$$

$$+ O(\tilde{L}_m^6) \quad . \tag{51}$$

Applying Van Dyke's matching principle[5], the constants appear in the outer solution (32) to (35) are found as

$$A = \bar{r}_o \ , \ B = 0 \ , \ C = 0 \ , \ D = 0 \ , \ E = -1/\bar{r}_o \ , \ F = 0 \ , \ G = 0 \ , \ H = \frac{1}{\bar{r}_o^3} \quad . \tag{52}$$

The outer solution then becomes

$$\bar{U} = \frac{1}{\bar{r}_o}\tilde{L}_m - \frac{\bar{t}}{\bar{r}_o^3}\tilde{L}_m^2 + \frac{3}{2\bar{r}_o^5}\bar{t}^2\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \tag{53}$$

$$\bar{V} = 1 - \frac{1}{\bar{r}_o^2}\tilde{L}_m^2 + \frac{2\bar{t}}{\bar{r}_o^4}\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \tag{54}$$

$$\bar{R} = \bar{r}_o + \frac{\bar{t}}{\bar{r}_o}\tilde{L}_m + (-\frac{1}{\bar{r}_o} - \frac{\bar{t}^2}{2\bar{r}_o^3})\tilde{L}_m^2 + (\frac{\bar{t}^3}{2\bar{r}_o^5})\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad , \tag{55}$$

$$\bar{\Psi} = \frac{\bar{t}}{\bar{r}_o} - \frac{\bar{t}^2}{2\bar{r}_o^3}\tilde{L}_m + \frac{\bar{t}^3}{6\bar{r}_o^5}\tilde{L}_m^2 + (\frac{\bar{t}^2}{2\bar{r}_o^5} - \frac{5}{8}\frac{\bar{t}^4}{\bar{r}_o^7} + \frac{1}{\bar{r}_o^3})\tilde{L}_m^3 + O(\tilde{L}_m^4) \quad . \tag{56}$$

Equations (48) to (51) and equations (53) to (56) then constitute the solution of the particle motion governed by equations (13) to (16) when $\tilde{L}_m$ is small.

For time of order τ_m, the inner solution applies and for time of order t^*, the outer solution applies. However, it is quite inconvenient to use two separate expansions from a practical viewpoint and a composite expansion valid for all time is desirable. One such composite expansion is given by Erdelyi[6]

$$f_c^{(\delta,\Delta)} = f_i^{(\delta)} + f_o^{(\Delta)} - [f_o^{(\Delta)}]_i^{(\delta)} \quad , \tag{57}$$

where

$f_c^{(\delta,\Delta)}$ is the composite expansion

$f_i^{(\delta)}$ is the inner expansion to order δ

$f_o^{(\Delta)}$ is the outer expansion to order Δ

and $[f_o^{(\Delta)}]_i^{(\delta)}$ denotes the inner expansion to order δ of the outer expansion $f_o^{(\Delta)}$.

Using (57), the composite expansions for U,V,R and Ψ are

$$\begin{aligned} U = {} & \frac{\tilde{L}_m}{\bar{r}_o} - \frac{\bar{t}}{\bar{r}_o^3}\tilde{L}_m^2 + \frac{3}{2}\frac{\bar{t}^2}{\bar{r}_o^5}\tilde{L}_m^3 + \frac{\bar{t}}{\bar{r}_o^5}\tilde{L}_m^4 - \frac{8}{\bar{r}_o^5}\tilde{L}_m^5 \\ & + e^{-\bar{t}/\tilde{L}_m}\{(-\frac{1}{\bar{r}_o} + \frac{\bar{t}^2}{\bar{r}_o^3} - \frac{1}{6}\frac{\bar{t}^4}{\bar{r}_o^5})\tilde{L}_m + (\frac{\bar{t}}{\bar{r}_o^3} - \frac{4}{3}\frac{\bar{t}^3}{\bar{r}_o^5})\tilde{L}_m^2 + (\frac{3}{2}\frac{\bar{t}^2}{\bar{r}_o^5})\tilde{L}_m^3 \\ & + \frac{13\bar{t}}{\bar{r}_o^5}\tilde{L}_m^4 + \frac{8}{\bar{r}_o^5}\tilde{L}_m^5\} + e^{-2\bar{t}/\tilde{L}_m}(-\frac{2\bar{t}}{\bar{r}_o^5}\tilde{L}_m^4) \quad , \end{aligned} \tag{58}$$

$$\begin{aligned} V = {} & 1 - \frac{1}{\bar{r}_o^2}\tilde{L}_m^2 + \frac{2\bar{t}}{\bar{r}_o^4}\tilde{L}_m^3 - \frac{2}{\bar{r}_o^4}\tilde{L}_m^4 \\ & + e^{-\bar{t}/\tilde{L}_m}\{(\frac{\bar{t}}{\bar{r}_o^2} - \frac{\bar{t}^3}{3\bar{r}_o^4})\tilde{L}_m + (\frac{1}{\bar{r}_o^2} - \frac{3}{2}\frac{\bar{t}^2}{\bar{r}_o^4})\tilde{L}_m^2 + (\frac{3}{\bar{r}_o^4})\tilde{L}_m^4\} \\ & + e^{-2\bar{t}/\tilde{L}_m}\{-\frac{\bar{t}}{\bar{r}_o^4}\tilde{L}_m^3 - \frac{1}{\bar{r}_o^4}\tilde{L}_m^4\} \quad , \end{aligned} \tag{59}$$

$$R = \bar{r}_o + \frac{\bar{t}}{\bar{r}_o}\tilde{L}_m + \left(-\frac{1}{\bar{r}_o} - \frac{\bar{t}^2}{2\bar{r}_o^3}\right)\tilde{L}_m^2 + \frac{1}{2}\frac{\bar{t}^3}{\bar{r}_o^5}\tilde{L}_m^3 + \frac{3}{\bar{r}_o^3}\tilde{L}_m^4$$

$$+ e^{-\bar{t}/\tilde{L}_m}\left\{\left(\frac{1}{\bar{r}_o} - \frac{\bar{t}^2}{\bar{r}_o^3}\right)\tilde{L}_m^2 - \frac{3\bar{t}}{\bar{r}_o^3}\tilde{L}_m^3 - \frac{3}{\bar{r}_o^3}\tilde{L}_m^4\right\} \quad , \tag{60}$$

$$\Psi = \frac{\bar{t}}{\bar{r}_o} - \frac{\bar{t}^2}{2\bar{r}_o^3}\tilde{L}_m + \frac{\bar{t}^3}{6\bar{r}_o^5}\tilde{L}_m^2 + \left(\frac{1}{\bar{r}_o^3} + \frac{\bar{t}^2}{2\bar{r}_o^5} - \frac{5}{8}\frac{\bar{t}^4}{\bar{r}_o^7}\right)\tilde{L}_m^3 - \frac{5\bar{t}}{\bar{r}_o^5}\tilde{L}_m^4 + \frac{9}{\bar{r}_o^5}\tilde{L}_m^5$$

$$+ e^{-\bar{t}/\tilde{L}_m}\left\{\left(-\frac{\bar{t}}{\bar{r}_o^3} + \frac{\bar{t}^3}{3\bar{r}_o^5}\right)\tilde{L}_m^2 + \left(-\frac{1}{\bar{r}_o^3} + \frac{5}{2}\frac{\bar{t}^2}{\bar{r}_o^5}\right)\tilde{L}_m^3 - \frac{10}{\bar{r}_o^5}\tilde{L}_m^5\right\}$$

$$+ e^{-2\bar{t}/\tilde{L}_m}\left\{\frac{\bar{t}}{\bar{r}_o^5}\tilde{L}_m^4 + \frac{1}{\bar{r}_o^5}\tilde{L}_m^5\right\} \quad . \tag{61}$$

Results and Discussion

The exact solutions can be found by numerically integrating equations (22) to (25), subject to the initial conditions (17) nondimensionalized as follows

$$\text{at } \bar{t} = 0 \ , \ \bar{U} = 0 \ , \ \bar{V} = 1 \ , \ \bar{R} = r_o \ , \ \bar{\Psi} = 0 \quad . \tag{62}$$

Figures (2) to (5) show the comparison between the exact numerical solution and the approximate solution given by the composite expansions (58) to (61) for various values of $\tilde{L}_m$. Excellent agreements between the numerical exact solutions and the matched asymptotic solutions are indicated for $\tilde{L}_m$ as large as 0.4. For large values of $\tilde{L}_m$ (but still less than 1), the approximate solutions tend to over predict the exact values in general. The comparisons are made for particle trajectories initiated at $\bar{r}_o = 1$, i.e., center of the curve duct. Also, the results are presented up to the time when the particle hits the pipe wall or exits the pipe. Having obtained the approximate solution for the particle motion, one can calculate, for example, the erosion rate on the pipe wall due to particle impacts[1].

The present results also apply to the mid-plane of a circular curved pipe, as long as the fluid motion is uniform over the cross-section and along the pipe[1]. To show this analytically, we notice that the particle velocity is

$$\vec{v}_p = u_p\vec{e}_r + v_p\vec{e}_\psi + w_p\vec{e}_z \tag{63}$$

written in polar coordinates with the z-axis pointing into the paper in Fig. 1. w_p is the velocity component in the increasing z-direction. Carrying out the same analysis as before, we arrive at the following governing equation for the ensemble average of the velocity components:

$$\frac{du_p}{dt} - \frac{v_p^2}{r} = -\frac{u_p}{\tau_m} \quad , \tag{64}$$

$$\frac{dv_p}{dt} + \frac{u_p v_p}{r} = \frac{W-V_p}{\tau_m} \quad , \tag{65}$$

$$\frac{dw_p}{dt} = -\frac{w_p}{\tau_m} \quad , \tag{66}$$

$$\frac{du}{dt} = u_p \quad , \tag{67}$$

$$\frac{d\psi}{dt} = \frac{v_p}{r} \quad , \tag{68}$$

$$\frac{dz}{dt} = w_p \quad , \tag{69}$$

with initial conditions

$$u_p = 0 \;,\; v_p = w \;,\; w_p = 0 \;,\; r = r_o \;,\; \phi = 0 \;,\; z = z_o \tag{70}$$

at $t = 0$. The initial position of the particle is now $(r_o, 0, z_o)$. The general solution to (66) is

$$w_p = \text{const.}\; e^{-t/\tau_m} \quad , \tag{71}$$

which gives, by virtue of the initial condition on w_p,

$$w_p = 0 \quad , \quad \forall t \quad . \tag{72}$$

Hence equation (69) becomes

$$z = \bar{z}_o \quad , \quad \forall t \quad . \tag{73}$$

The remaining equations (64), (65), (67) and (68) and the initial conditions are seen to be identical with equations (13)-(16) and (17). Hence equations (58) to (61) together with (72) and (73) constitute the matched asymptotic solutions to equations (64)-(69). The particle trajectory will hit the pipe wall when the following is satisfied

$$z^2 + (R-r)^2 = a^2 \quad , \tag{74}$$

where R is the mean radius of the elbow as before and a is the radius of the circular cross-section in this case. By virtue of (73), (74) can be solved for r to obtain

$$r = R - \sqrt{a^2 - \bar{z}_o^2} \quad \text{for} \quad R > r \tag{75}$$

and

$$r = R + \sqrt{a^2 - \bar{z}_o^2} \quad \text{for} \quad R < r \tag{76}$$

as the conditions that the particle hits the wall.

Conclusion

The method of matched asymptotic expansion has been applied successfully to obtain series solutions for the particle trajectories when $\tilde{L}_m$ assumes small values. A simple system was chosen, and it is straightforward to extend the analysis to, say, three-dimensional geometries, such as a circular curved pipe. We have already mentioned the advantages of having close form solutions for the particle phase, especially when the motion of the fluid phase cannot be simplified as in the present case. Finally, we mention that the same analysis can be applied to systems where $\tilde{L}_m$ assumes a large value.

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Figure Captions

Fig. 1. Coordinate system of the geometry used.

Fig. 2. Comparison of the approximate solution to the exact numerical solution of the radial velocity component.

Fig. 3. Comparison of the approximate solution to the exact numerical solution of the circumferential velocity component.

Fig. 4. Comparison of the approximate solution to the exact numerical solution of the radial coordinate.

Fig. 5. Comparison of the approximate solution to the exact numerical solution of the angular coordinate.

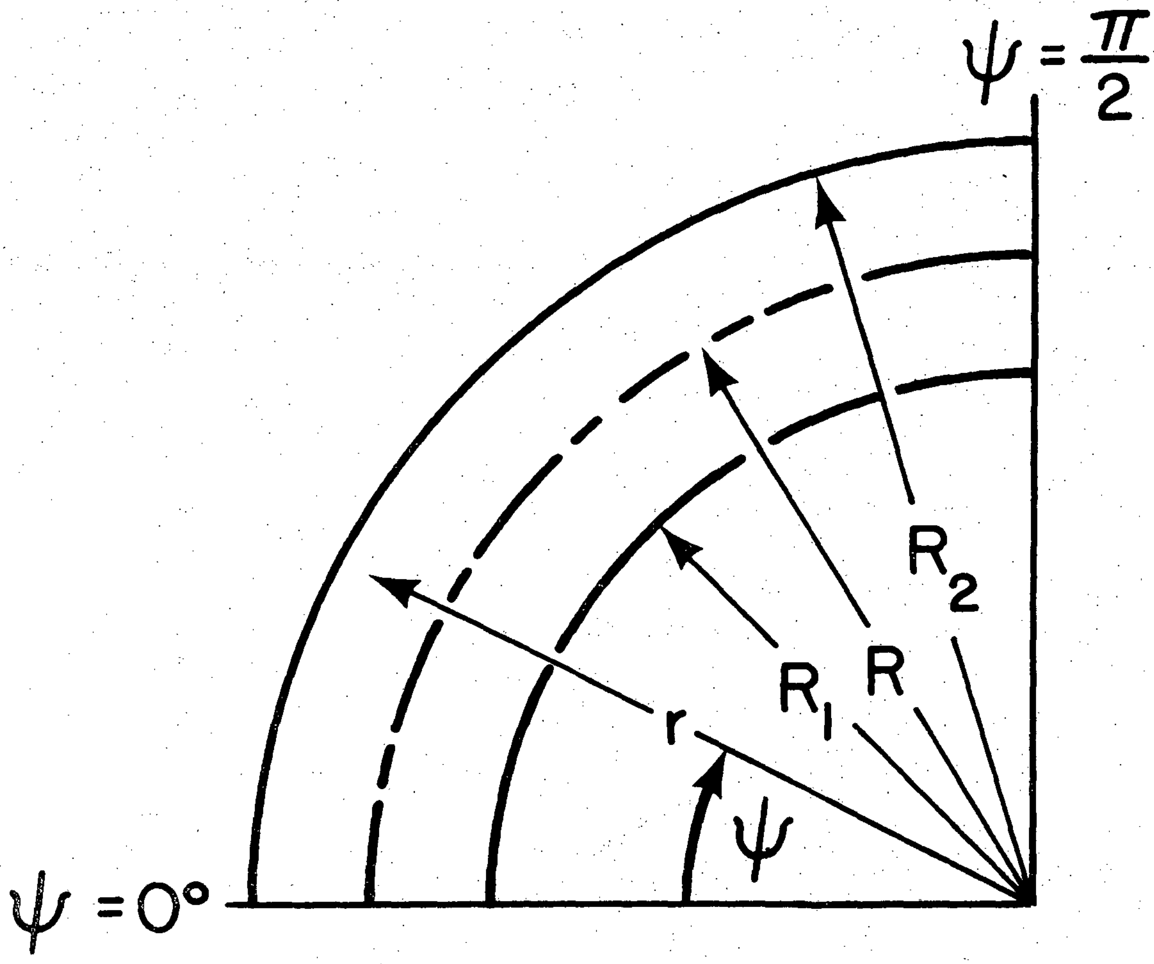

Figure 1

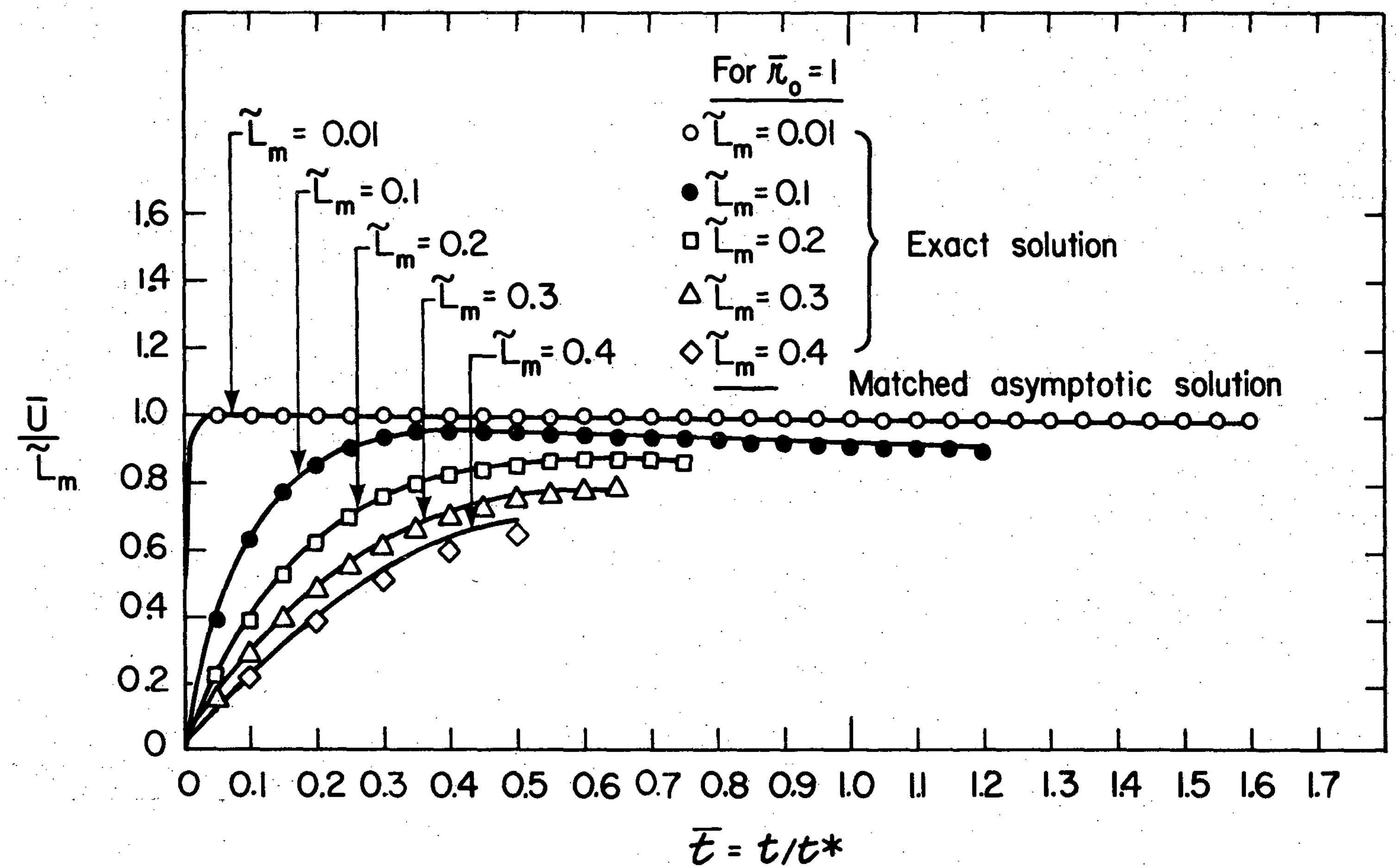

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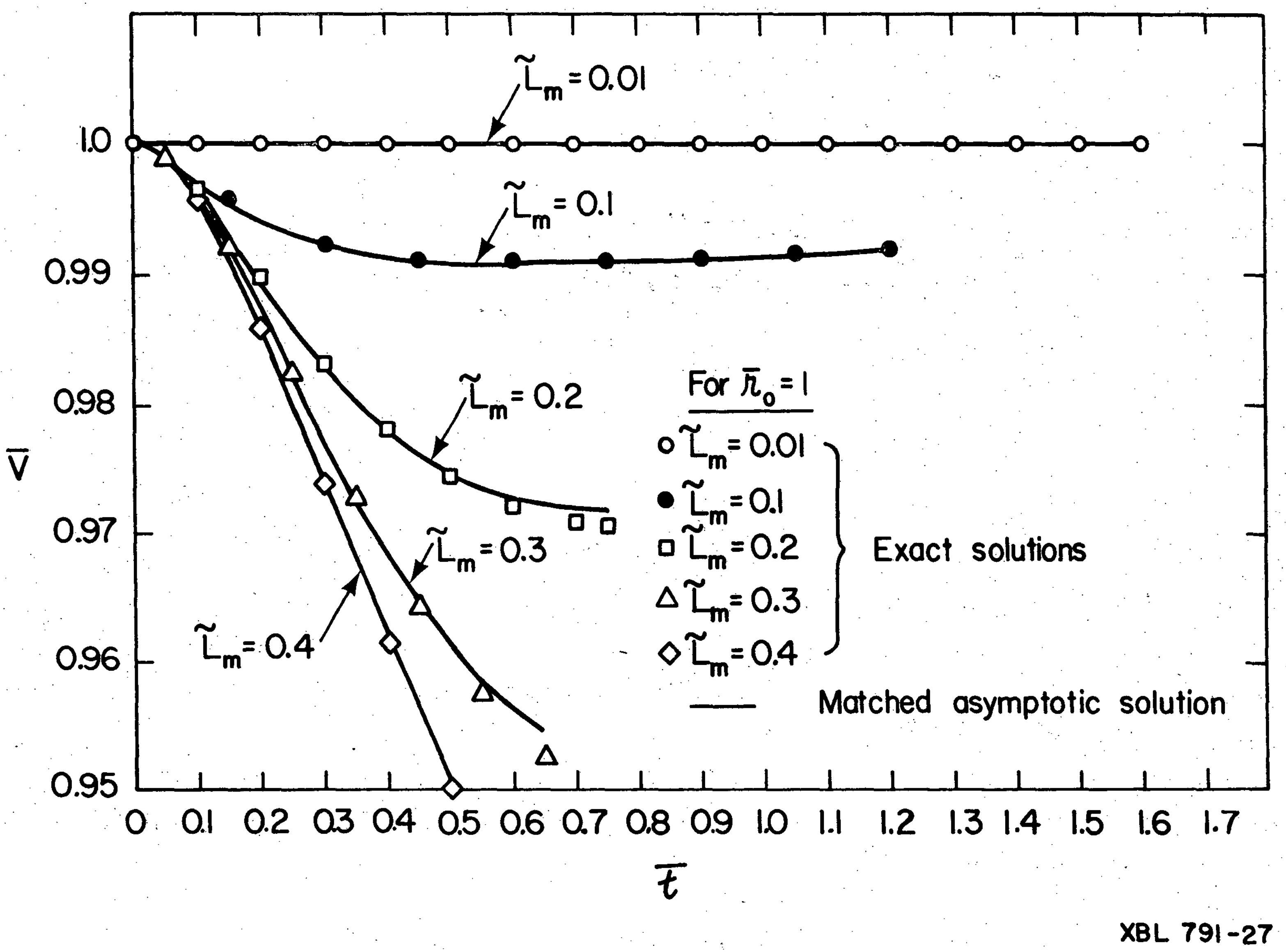
Figure 2



$\tilde{L}_m = 0.01$
$\tilde{L}_m = 0.1$
$\tilde{L}_m = 0.2$
$\tilde{L}_m = 0.3$
$\tilde{L}_m = 0.4$
1.0
0.99
0.98
0.97
0.96
0.95
$\bar{V}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
$\bar{t}$
For $\bar{r}_o = 1$
$\tilde{L}_m = 0.01$
$\tilde{L}_m = 0.1$
$\tilde{L}_m = 0.2$
$\tilde{L}_m = 0.3$
$\tilde{L}_m = 0.4$
Exact solutions
Matched asymptotic solution
XBL 791-27

Figure 3

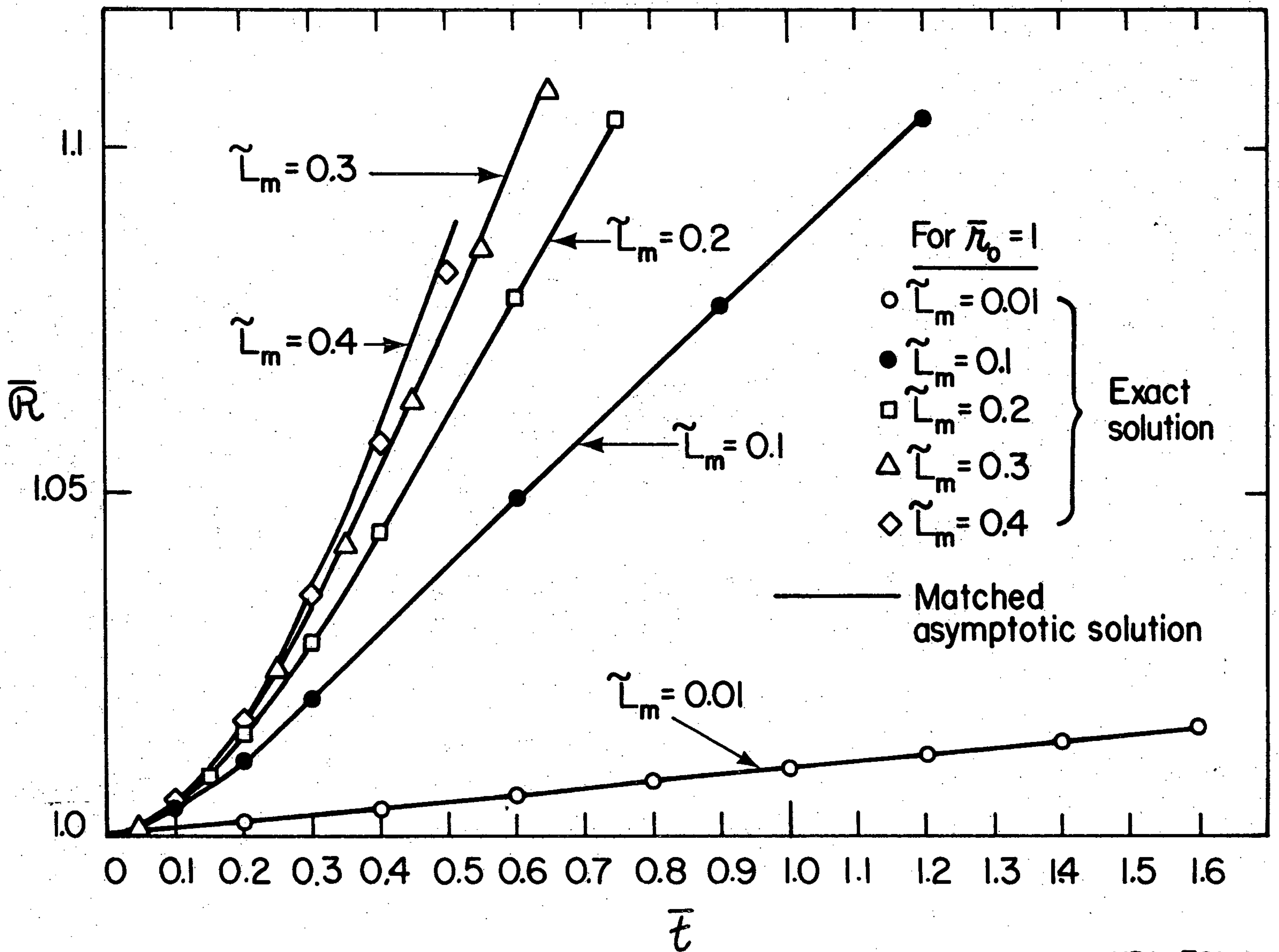


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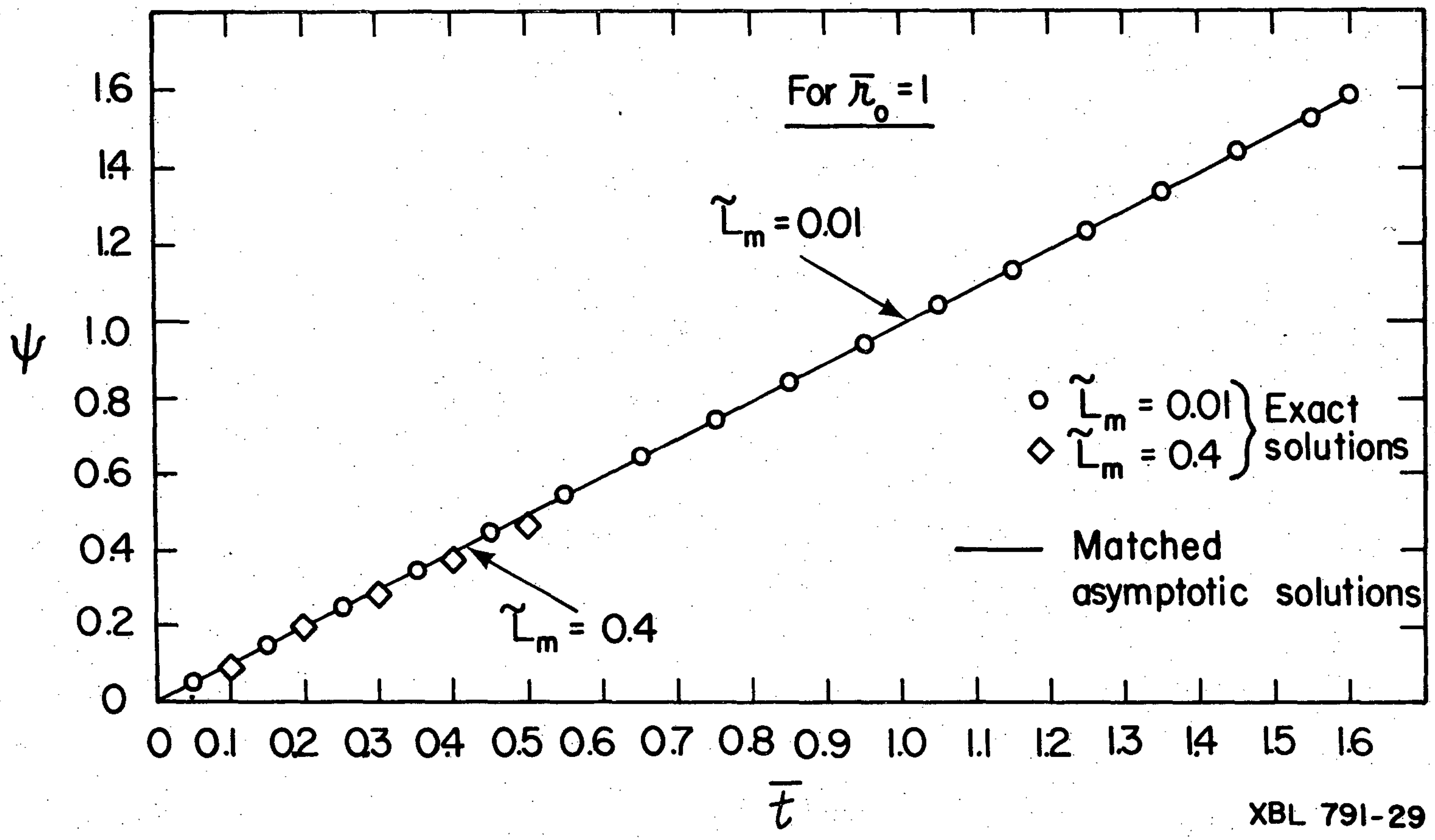
Figure 4



For $\bar{r}_o = 1$
$\tilde{L}_m = 0.01$
$\tilde{L}_m = 0.4$
ψ
$\bar{t}$
○ $\tilde{L}_m = 0.01$
◇ $\tilde{L}_m = 0.4$
Exact solutions
Matched asymptotic solutions
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4 1.5 1.6
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Figure 5

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