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Fusion Rules for the Lattice Vertex Operator Algebra V_L

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FUSION RULES FOR THE LATTICE VERTEX OPERATOR ALGEBRA V_L

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Danquynh Thien Nguyen

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Abstract

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In this thesis, we compute the fusion rules among the irreducible modules of V_L - the vertex operator algebra associated with a positive-definite even lattice L, and then use them to determine the irreducible decomposition of fusion products of irreducible V_L-modules. Specifically, we establish the following results: the fusion product of an untwisted V_L-module and another one of twisted type is a V_L-module of twisted type while the fusion product of two twisted V_L-modules is a sum of untwisted modules satisfying a certain relation.

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Chapter 1

Introduction

The theory of vertex operator algebras is relatively new compared to other branches of mathematics and has developed quite rapidly since its inception in the late 1980s. Motivated by the representation theory of affine Lie algebras and the "moonshine module" (constructed by Igor Frenkel, James Lepowsky, and Arne Meurman in [FLM1]), Richard Borcherds introduced the mathematical formulation of *"vertex algebras"* in 1986 [B]. A couple of years later, with a few extra requirements, Frenkel, Lepowsky, and Meurman modified Borcherds's definition and introduced *"vertex operator algebras"* in their foundational work [FLM2] on the subject. And an active field of mathematical research took off from there. The theory of vertex operator algebras was motivated by and has applications in many areas of mathematics, such as number theory, group theory, the theory of modular functions, etc. Vertex (operator) algebras are the mathematical counterpart of what theoretical physicists call "chiral algebras" in two-dimensional conformal field theory, which plays an important role in string theory.

In his original paper [B], Borcherds developed a new abstract theory of what he called *"vertex operators"* by using the concrete structure of an even lattice L. Specifically, for any such lattice, he constructed a space on which the *vertex operators* corresponding to the elements in L act. These actions were shown to satisfy infinitely many relations, which then formed the axioms in the definition of a vertex algebra. In other words, the vertex algebra of an even lattice is the original example of vertex algebras. In this thesis, we study the lattice vertex operator algebra V_L associated with a positive-definite even lattice and completely determine its fusion rules. For a vertex operator algebra V with irreducible modules M^1, M^2, and M^3, the ***fusion rule*** of type $\binom{M^3}{M^1\ M^2}$ is defined to be the dimension of the vector space formed by all intertwining operators of this type. In conformal field theory, these numbers are intimately related to the *fusion coefficients* N_{ij}^k in the operator product expansion of two conformal families $[\phi_i]$ and $[\phi_j]$:

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}^k [\phi_k]$$

Roughly speaking, the fusion coefficients N_{ij}^k give the scattering amplitudes of the outgoing primary fields ϕ_k when two primary fields ϕ_i and ϕ_j come into contact. We shall see that the above equation is exactly the physical counterpart of what is called a *fusion product* in mathematics literature.

Let us now give a brief overview of this thesis. We consider a positive-definite, even, integral lattice L of rank d and denote by L° its dual lattice. Since L is even, $L \subseteq L^\circ$; we set $S := \{\lambda_1, \cdots, \lambda_k\}$ to be the complete set of representatives of equivalence classes of L in L°. It is well known that $\{V_{L+\lambda} \,|\, \lambda \in S\}$ is the complete list of (inequivalent) irreducible *untwisted* V_L-modules (see [FLM] and [D1]). There are also V_L-modules of *twisted* type, whose construction is outlined as follows. First, denote by $\hat{L}$ be the central extension of L by the cyclic group $\mathbb{Z}_2 = \big\langle \kappa \,|\, \kappa^2 = 1 \big\rangle = \langle -1 \rangle$. Let $\theta \in \mathrm{Aut}(\hat{L})$ be an automorphism of $\hat{L}$ such that $\theta^2 = Id_{\hat{L}}$ and $\theta(\kappa) = \kappa$. Let T_χ be the irreducible $\hat{L}/K$-module, where $K = \{a^{-1}\theta(a) \,|\, a \in \hat{L}\}$, associated to a central character $\chi : Z(\hat{L}/K) \to \mathbb{C}^\times$ which sends $\kappa K = (-1)K$ to -1; that is, T_χ is an irreducible $\hat{L}/K$-module on which $\kappa K = (-1)K$ acts as -1. Lastly, set $V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$; then $\{V_L^{T_\chi} \,|\, T_\chi = \text{irreducible } \hat{L}/K\text{-module associated to central character } \chi\}$ is the complete list of irreducible V_L-modules of twisted type (see [D2]).

For any vertex operator algebra V, the fusion product of two irreducible V-modules M^1 and M^2 is defined via the universal property. The pair $(M, \mathcal{Y})$ is called the ***fusion product*** of M^1 and M^2 if M is a V-module and $\mathcal{Y}$ is an intertwining operator of type $\binom{M}{M^1\ M^2}$ such that for any V-module W and any intertwining operator $\mathcal{Y}_W$ of type $\binom{W}{M^1\ M^2}$, there exists a unique V-module homomorphism $f : M \to W$ such that $\mathcal{Y}_W = f \circ \mathcal{Y}$. The fusion product of M^1 and M^2 is typically denoted by $M^1 \boxtimes_V M^2$. If V is a rational vertex operator algebra, then the fusion product of any two irreducible V-modules exists [HL], in which case we use the following definition:

$$M^1 \boxtimes_V M^2 := \sum_{M^i} N_V \binom{M^i}{M^1 M^2} M^i$$

where M^i runs over the set of equivalence classes of irreducible V-modules and the symbol $N_V\binom{M^i}{M^1 M^2}$ denotes the dimension of the space formed by all intertwining operators of type $\begin{pmatrix} M^i \\ M^1 M^2 \end{pmatrix}$, i.e. the fusion rule of this type.

Our main object of interest, the lattice VOA V_L, is known to be rational, and thus the fusion products of its modules exist. The fusion product of two untwisted irreducible V_L-modules is a well-known result, namely $V_{L+\lambda} \boxtimes_{V_L} V_{L+\mu} = V_{L+\lambda+\mu}$ (see [DL], Proposition 12.9). In this thesis, we determine the other two fusion products: $V_{L+\lambda} \boxtimes_{V_L} V_L^{T_\chi}$ and $V_L^{T_{\chi_1}} \boxtimes_{V_L} V_L^{T_{\chi_2}}$ by a method of computation briefly outlined here. We shall invoke a result from [A2], which says that the fusion rule of type $\begin{pmatrix} M^1 \\ M^2 \ M^3 \end{pmatrix}$ for V_L is either 0 or 1 for any irreducible module M^i for V_L. For $V_{L+\lambda} \boxtimes_{V_L} V_L^{T_\chi}$, we show that it is equal to $V_L^{T_{\chi^{(\lambda)}}}$ (a twisted V_L-module determined by λ and χ) by showing that the fusion rule $N_{V_L}\begin{pmatrix} V_L^{T_{\chi^{(\lambda)}}} \\ V_{L+\lambda} \ V_L^{T_\chi} \end{pmatrix} = 1$ and all other fusion rules $N_{V_L}\begin{pmatrix} M \\ V_{L+\lambda} \ V_L^{T_\chi} \end{pmatrix} = 0$ where M is any other irreducible V_L-module. This assertion is proved by an explicit construction of a non-trivial intertwining operator of this type. In almost exactly the same way, we can determine the fusion product $V_L^{T_{\chi_1}} \boxtimes_{V_L} V_L^{T_{\chi_2}}$.

This thesis is organized as follows. In Chapter 2, we start with reviewing some basic concepts in formal calculus, which is the underlying language of the theory of vertex operator algebras, and then proceed with the definition of a vertex operator algebra and its modules. This chapter also discusses some important examples of vertex operator algebras, specifically the Virasoro VOA and affine VOAs. Chapter 3 is devoted to the study of lattice vertex operator algebras. In the first two sections of this chapter, we recall the definitions of intertwining operators and fusion rules, followed by the construction of the vertex operator algebra V_L and its modules. The third short section recalls a well-known result by Chongying Dong and James Lepowsky [DL]. The last two sections of Chapter 3 are the heart of this thesis, where we give detailed computations of the two aforementioned fusion products.

Chapter 2

Basics

2.1 Formal Calculus

Formal calculus is an important tool in the study of vertex operator algebras as it allows one to make sense of operations on infinite sums. In this section, we recall some fundamental definitions and concepts in formal calculus. We shall use $z_0, z_1, z_2, z_3, ...$ to denote mutually commuting formal variables. While the underlying field throughout this thesis is the complex numbers $\mathbb{C}$, all results should remain valid over any algebraically closed field of characteristic 0. In addition, as a quick note on notations, the symbol $\mathbb{N}$ denotes the non-negative integers, $\mathbb{Z}$ the integers, and $\mathbb{Z}_+$ the positive integers. Letting V be a vector space, we start with a few related spaces of formal series that are prevalent in our discussion. The vector space of formal Laurent series, which we shall soon encounter in the definition of a vertex operator algebra, is:

$$V[[z, z^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n \mid v_n \in V \right\}$$

The space of formal Laurent polynomials:

$$V[z, z^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n \mid v_n \in V, \text{all but finitely many } v_n = 0 \right\}$$

The space of V-valued polynomials:

$$V[z] = \left\{ \sum_{n \in \mathbb{N}} v_n z^n \mid v_n \in V, \text{all but finitely many } v_n = 0 \right\}$$

The space of formal power series:

$$V[[z]] = \left\{ \sum_{n \in \mathbb{N}} v_n z^n \mid v_n \in V \right\}$$

And the space of formal power series with complex powers of z:

$$V\{z\} = \left\{ \sum_{n \in \mathbb{C}} v_n z^n \mid v_n \in V \right\}$$

An important formal series is the delta function:

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{Z}[[z, z^{-1}]]$$

Chapter 2 of [LL] contains a detailed discussion on many fundamental properties of $\delta(z)$, one of which is stated here for a reason that shall become apparent as we continue to the definition of a vertex operator algebra:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)$$

Here we use the binomial expansion convention: any expression of the form $(z_1 + z_2)^n$, $\forall n \in \mathbb{Z}$, is *always* to be expanded so that the second variable has non-negative powers:

$$(z_1 + z_2)^n = \sum_{k \in \mathbb{N}} \binom{n}{k} z_1^{n-k} z_2^k$$

where $\binom{n}{k} = \dfrac{n \cdot (n-1) \cdots (n-k+1)}{k!}$. It is clear from this notational convention that $(z_1 + z_2)^n \neq (z_2 + z_1)^n$ unless $n \geq 0$.

For $v(z) = \sum_{n \in \mathbb{Z}} v_n z^n \in V[[z, z^{-1}]]$, we define the *formal residue* Res_z and the *formal derivative* $\dfrac{d}{dz}$ as follows:

$$\mathrm{Res}_z v(z) = v_{-1} \quad \text{and} \quad \frac{d}{dz} v(z) = \sum_{n \in \mathbb{Z}} n v_n z^{n-1}$$

The third familiar concept, the *formal exponential*, is defined for $f(z) \in V[z]$:

$$e^{f(z)} = \sum_{n \in \mathbb{N}} \frac{1}{n!} f(z)^n$$

We also have the *formal logarithmic* power series:

$$\log(1 + af(z)) = -\sum_{k \in \mathbb{Z}_+} \frac{(-a)^k}{k} f(z)^k$$

for any $a \in \mathbb{C}$ and suitable $f(z)$. These formal power series obey the familiar standard rules: for any $a, b, c \in \mathbb{C}$,

$$\log(\exp f(z)) = f(z)$$

$$\exp(\log(1 + af(z))) = 1 + af(z)$$

$$\log((1 + af(z))(1 + bg(z))) = \log(1 + af(z)) + \log(1 + bg(z))$$

$$\log(1 + af(z))^c = c \log(1 + af(z))$$

2.2 Vertex Operator Algebras and Modules

In this section, we introduce the precise mathematical formulation of vertex operator algebras and their modules. Roughly speaking, a VOA is an infinite dimensional $\mathbb{Z}$-graded vector space in which between any two elements u and v, there are infinitely many "products" $u_n v$ where n runs over the integers (hence "infinitely many".)

Definition 2.2.1 A ***vertex operator algebra*** V is a vector space equipped with a linear map:

$$\begin{aligned} Y = Y(\cdot, z) : V &\to (\mathrm{End}V)[[z, z^{-1}]] \\ v \mapsto Y(v,z) &= \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (\text{where } v_n \in \mathrm{End}V) \end{aligned}$$

such that for any $u, v \in V$ and $n \in \mathbb{Z}$:

1. $u_n v = 0$ if $n \gg 0$,

2. V has an element often denoted by $\mathbf{1}$, called the *vacuum vector*, such that:

   (a) $Y(\mathbf{1}, z) = \mathrm{Id}_V$ and

   (b) $Y(u,z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0} Y(u,z)\mathbf{1} = u$,

3. The Jacobi identity is satisfied:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2)$$

If V satisfies conditions (1) - (3), then it is called a ***vertex algebra***. $Y(v,z)$ is called the ***vertex operator*** associated with v.

4. V is $\mathbb{Z}$-graded by weights :

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \text{ where } \dim V_n < \infty, \forall n \in \mathbb{Z} \text{ and } V_n = 0 \text{ if } n \ll 0$$

   If $v \in V_n$, then we call v a *homogeneous* element of weight n and write $\mathrm{wt}(v) = n$.

5. V has a distinguished homogeneous vector ω, called the *Virasoro* (or *conformal*) vector, which satisfies:

   (a) the Virasoro relation:

   $$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_V$$

   where $L(n) = \omega_{n+1}, \forall n \in \mathbb{Z}$, and $c_V \in \mathbb{C}$ (the *central charge* of V).

   (b) $L(0)|_{V_n} = n$

   (c) $Y(L(-1)v, z) = \dfrac{d}{dz}Y(v, z)$

A vertex operator algebra is denoted by a quadruple $(V, Y, \mathbf{1}, \omega)$ or simply V.

Remark 2.2.1(a): It can be easily shown that $\mathrm{wt}(\mathbf{1}) = 0$ and $\mathrm{wt}(\omega) = 2$. The "products" $u_n v$ respect the grading of V; that is, for any homogeneous vectors $u \in V_i, v \in V_j$, we have $u_n v \in V_{i+j-n-1}$.

Remark 2.2.1(b): As we shall see in Subsection 2.3.1, the Virasoro algebra $\mathfrak{Vir} = \langle\{L_n \mid n \in \mathbb{Z}\} \cup \{C\}\rangle$ is equipped with a Lie bracket that resembles the relation 5(a) above. This suggests that, under the correspondence $L(n) \leftrightarrow L_n, c_V \leftrightarrow C$, the vector space V is a representation of the Virasoro algebra $\mathfrak{Vir}$.

Definition 2.2.2 For a vertex operator algebra V, a linear map $g \in GL(V)$ is called an ***automorphism*** of V if $g(\omega) = \omega$ and the actions of g and $Y(u, z)$ on V are compatible in the sense that $gY(u, z)g^{-1} = Y(g(u), z)$, $\forall u \in V$.

It follows from the definition that $g(\mathbf{1}) = \mathbf{1}$ and $gV_n \subseteq V_n, \forall n \in \mathbb{Z}$. As usual, we use $Aut(V)$ to denote the group of automorphisms of V.

For the following definitions, we assume that $g \in Aut(V)$ is of finite order T, in which case V is decomposed into eigenspaces with respect to the action of g as:

$$V = \bigoplus_{r=0}^{T-1} V^r, \text{ where } V^r = \{v \in V \mid gv = e^{2\pi i r/T} v\}$$

Definition 2.2.3 A ***weak g-twisted V-module M*** is a vector space equipped with a linear map:

$$Y_M = Y_M(\cdot, z) : V \to (\mathrm{End} M)\{z\}$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \text{ (where } v_n \in \mathrm{End} M)$$

such that for any $u \in V^r, v \in V, w \in M$, and $0 \le r \le T-1$:

1. $u_n w = 0$ if $n \gg 0$,
2. $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$,
3. $Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}$,
4. the twisted Jacobi identity is satisfied:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{r}{T}} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2)$$

Where clarification is necessary, we use (M, Y_M), instead of just M, to denote a weak g-twisted V-module.

Definition 2.2.4 Let M be a weak g-twisted V-module and $N \subseteq M$ its subspace. If $v_n N \subseteq N, \forall v \in V, n \in \mathbb{Q}$, then N is called a ***weak g-twisted V-submodule*** of M. If the only weak g-twisted V-submodules of M are 0 and M itself, then M is said to be ***irreducible.***

Definition 2.2.5 An ***admissible g-twisted V-module*** is a weak g-twisted V-module M such that:

1. $M = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} M(n)$
2. $v_m M(n) \subseteq M(n + \mathrm{wt} v - m - 1)$ for any homogeneous $v \in V$ and for $n \in \frac{1}{T}\mathbb{N}, m \in \mathbb{Q}$.

Definition 2.2.6 Let M be an admissible g-twisted V-module and N a weak g-twisted V-submodule of M.

1. If $N = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} N \cap M(n)$, then N is called an ***admissible g-twisted V-submodule*** of M.

2. Similar to the concept of irreducibility in weak g-twisted V-modules, an ***irreducible*** admissible g-twisted V-module is one which has no nontrivial admissible submodules.

3. On the other hand, an admissible g-twisted V-module is said to be ***completely reducible*** if it is a direct sum of irreducible admissible g-twisted V-submodules.

Definition 2.2.7 An ***ordinary g-twisted V-module*** is a weak g-twisted V-module M such that:

1. $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$,
2. dim $M_\lambda < \infty$, $\forall \lambda \in \mathbb{C}$, and
3. for any fixed $\lambda \in \mathbb{C}, M_{\lambda + \frac{n}{T}} = 0$ if $n \ll 0, n \in \mathbb{Z}$.

Definition 2.2.8 A vertex operator algebra V is said to be ***g-rational*** if any admissible g-twisted V-module is completely reducible, or equivalently, if the category of admissible g-twisted V-modules is semisimple.

Remark: When $g = Id_V$, the phrase *"g-twisted"* is dropped from the above definitions of different types of modules and a g-rational VOA is simply called *rational*.

Definition 2.2.9 Let $M = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} M(n)$ be an admissible g-twisted V-module, then the ***contragredient V-module*** $(M', Y_{M'})$ is:

$$M' = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} M(n)^* = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$$

and for any $v \in V$, any $f \in M', w \in M$:

$$\langle Y_{M'}(v, z)f, w \rangle = \left\langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w \right\rangle$$

where $\langle \cdot, \cdot \rangle : M' \times M \to \mathbb{C}$ denotes the natural pairing $\langle f, w \rangle = f(w), \forall f \in M', w \in M$.

Remark: The idea of contragredient modules is in essence that of *dual* modules. However, it is troublesome to take M' to be $(\bigoplus_{n \in \frac{1}{T}\mathbb{N}} M(n))'$ since it would be too large (as M is infinite-dimensional).

2.3 Some Examples of VOAs

Unlike other algebraic structures such as vector spaces or groups, examples of VOAs are difficult to find because it is difficult to construct such a structure and then prove that the constructed structure satisfies the axioms of a vertex operator algebra. The Moonshine module $V^\natural$ is the most famous VOA whose automorphism group is the Monster $\mathbb{M}$. In this section, we present the construction of two of the most important VOAs, the Virasoro VOA and the affine VOAs; detailed proofs of VOA axioms may be found in [LL]. Another example, the VOA associated with a non-degenerate even lattice, is the main focus of this thesis and its construction is delayed until Chapter 3.

2.3.1 Virasoro VOA

The Virasoro algebra, denoted by $\mathfrak{Vir}$, is an infinite-dimensional Lie algebra with basis $\{L_n \mid n \in \mathbb{Z}\} \cup \{C\}$ with Lie brackets:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C \\
[L_m, C] &= 0
\end{aligned}$$

Set $\mathfrak{Vir}_+ = \bigoplus_{n\in\mathbb{Z}_+} \mathbb{C}L_n$, $\mathfrak{Vir}_- = \bigoplus_{n\in\mathbb{Z}_+} \mathbb{C}L_{-n}$, and $\mathfrak{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C$. For $c, h \in \mathbb{C}$, let $\mathbb{C}(c,h) = \mathbb{C}$ be the one-dimensional $\mathfrak{Vir}_+ \oplus \mathfrak{Vir}_0$-module on which $\mathfrak{Vir}_+$ acts trivially while $L_0 \cdot 1 = h$ and $C \cdot 1 = c$. Now consider the induced $\mathfrak{Vir}$-module:

$$M_{\mathfrak{Vir}(c,h)} := \mathrm{Ind}_{\mathfrak{Vir}_+\oplus\mathfrak{Vir}_0}^{\mathfrak{Vir}} \mathbb{C}(c,h) = U(\mathfrak{Vir}) \otimes_{U(\mathfrak{Vir}_+\oplus\mathfrak{Vir}_0)} \mathbb{C}(c,h)$$

where $U(\cdot)$ denotes the universal enveloping algebra, and form the quotient module:

$$V_{\mathfrak{Vir}}(c,0) = {M_{\mathfrak{Vir}}(c,0)} \Big/ {U(\mathfrak{Vir})L_{-1}(1\otimes 1)} = {M_{\mathfrak{Vir}}(c,0)} \Big/ {\langle L_{-1}(1\otimes 1)\rangle}$$

Set $\mathbb{1} := (1\otimes 1) + U(\mathfrak{Vir})L_{-1}(1\otimes 1)$ and $\omega := L_{-2}\mathbb{1}$ and define:

$$Y(\cdot,z) : V_{\mathfrak{Vir}}(c,0) \to (\mathrm{End}(V_{\mathfrak{Vir}}(c,0)))[[z,z^{-1}]]$$

$$Y(L_{-2}\mathbb{1},z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$Y(L_{-n-2}\mathbb{1},z) = \frac{1}{n!}\left(\frac{d}{dz}\right)^n L(z)$$

Then $(V_{\mathfrak{Vir}}(c,0), Y, \mathbb{1}, \omega)$ is a VOA, called the *Virasoro VOA*.

2.3.2 Affine VOAs

Let $\mathfrak{g}$ be a d-dimensional Lie algebra equipped with a symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ and consider its associated affine Lie algebra $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} := \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{k}$$

whose Lie bracket is defined by:

$$[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + m\,\langle a,b\rangle\,\delta_{m+n,0}\mathbf{k}$$

$$[a\otimes t^m, \mathbf{k}] = 0$$

for any $a,b\in\mathfrak{g}$ and any $m,n\in\mathbb{Z}$. Then $\hat{\mathfrak{g}}$ has the following subalgebras:

$$\hat{\mathfrak{g}}_{(\leq 0)} = \left(\coprod_{n>0}\mathfrak{g}\otimes t^n\right)\oplus\mathfrak{g}\oplus\mathbb{C}\mathbf{k}$$

Denote by $\mathbb{C}_l = \mathbb{C}$ the $\hat{\mathfrak{g}}_{(\leq 0)}$-module on which $\coprod_{n>0} \mathfrak{g} \otimes t^n$ and $\mathfrak{g}$ act trivially while $\mathbf{k}$ acts as multiplication by a scalar $l \in \mathbb{C}$. We then form the induced module:

$$V_{\hat{\mathfrak{g}}}(l,0) := \mathrm{Ind}_{\hat{\mathfrak{g}}_{(\leq 0)}}^{\hat{\mathfrak{g}}} \mathbb{C}_l = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} \mathbb{C}_l$$

Suppose that $\{u_1, u_2, \cdots, u_d\}$ is an orthonormal basis of $\hat{\mathfrak{g}}$ with respect to the form $\langle \cdot, \cdot \rangle$. For $u \in \mathfrak{g}$, we define the generating function:

$$u(z) := \sum_{n \in \mathbb{Z}} (u \otimes t^n) z^{-n-1} = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}$$

Setting $\mathbb{1} = 1 \in \mathbb{C}$ and $Y(\mathbb{1}, z) = \mathrm{Id}_{V_{\hat{\mathfrak{g}}}(l,0)}$, we inductively define:

$$Y(u(n)v, z) = \mathrm{Res}_{z_0} \{ (z_0 - z)^n u(z_0) Y(v, z) - (-z + z_0)^n Y(v, z) u(z_0) \}$$

Lastly, take

$$\omega = \frac{1}{2(l + h^\vee)} \sum_{i=1}^{d} u_i(-1) u_i(-1) \mathbb{1}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Then, if $l \neq -h^\vee$, $(V_{\hat{\mathfrak{g}}}(l,0), Y, \mathbb{1}, \omega)$ is a vertex operator algebra (see [LL]).

2.3.3 Lattice VOAs

Associated with a positive-definite even lattice L is the vertex operator algebra V_L. The detailed construction of this VOA is carried out in Section 3.2.

Chapter 3

Intertwining Operators, Fusion Rules, and V_L

3.1 Intertwining Operators and Fusion Rules

Definition 3.1.1 Let $(M^i, Y_{M^i}), i \in \{1,2,3\}$, be weak V-modules. An ***intertwining operator*** of type $\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}$ is a linear map:

$$\mathcal{Y} = \mathcal{Y}(\cdot, z) : M^1 \to (\mathrm{Hom}(M^2, M^3))\{z\}$$

$$u \mapsto \mathcal{Y}(u,z) = \sum_{n \in \mathbb{C}} u_n z^{-n-1} \quad (u_n \in \mathrm{Hom}(M^2, M^3))$$

satisfying the following properties:

1. For any $u \in M^1, v \in M^2$, and $\lambda \in \mathbb{C}$, $u_{m+\lambda}v = 0$ if $m \gg 0, m \in \mathbb{Z}$,

2. For any $a \in V, u \in M^1$, the Jacobi identity is satisfied:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(a, z_1)\mathcal{Y}(u, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(u, z_2)Y_{M^2}(a, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}(Y_{M^1}(a, z_0)u, z_2)$$

3. For $u \in M^1$, the $L(-1)$-derivative property is satisfied:

$$\mathcal{Y}(L(-1)u, z) = \frac{d}{dz}\mathcal{Y}(u, z)$$

In the language of conformal field theory, an intertwining operator of type $\binom{M^3}{M^1\ M^2}$ is called a *chiral vertex operator* of this type. We actually have seen an example of an intertwining operator of type $\binom{M}{V\ M}$, where (M, Y_M) is an irreducible V-module, and it is precisely the map Y_M. In fact, this map spans the 1-dimensional vector space of all intertwining operators of type $\binom{M}{V\ M}$. If $(V, Y, \mathbf{1}, \omega)$ is a simple VOA, then Y spans the the 1-dimensional vector space of all intertwining operators of type $\binom{V}{V\ V}$ (see [L]).

Denoting by $\mathcal{I}_V\binom{M^3}{M^1\ M^2}$ the vector space formed by all intertwining operators of type $\binom{M^3}{M^1\ M^2}$, we have the following definition:

Definition 3.1.2 The ***fusion rule*** of type $\binom{M^3}{M^1\ M^2}$ for V is:

$$N_V\binom{M^3}{M^1\ M^2} := \dim\mathcal{I}_V\binom{M^3}{M^1\ M^2}$$

Fusion rules have the following well-known symmetries (see [FHL], Props. 5.4.7 and 5.5.2):

Proposition 3.1.3 *Let $M_i (i = 1, 2, 3)$ be V-modules and M_i' the corresponding contragredient modules, then:*

$$N_V\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix} = N_V\begin{pmatrix} M_3 \\ M_2\ M_1 \end{pmatrix} = N_V\begin{pmatrix} M_2' \\ M_1\ M_3' \end{pmatrix}$$

We also quote here a very useful result from [ADL] (Prop. 2.9), which shall be invoked repeatedly in the derivation of our main results:

Proposition 3.1.4 *Let V be a vertex operator algebra and let M^1, M^2, M^3 be V-modules among which M^1 and M^2 are irreducible. Suppose that U is a vertex operator subalgebra of V (with the same Virasoro element) and that N^1 and N^2 are irreducible U-submodules of M^1 and M^2, respectively. Then the restriction map from $\mathcal{I}_V\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}$ to $\mathcal{I}_U\begin{pmatrix} M^3 \\ N^1\ N^2 \end{pmatrix}$ is injective. In particular,*

$$\dim \mathcal{I}_V\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix} \leq \dim \mathcal{I}_U\begin{pmatrix} M^3 \\ N^1\ N^2 \end{pmatrix}$$

Definition 3.1.5 Let V be a vertex operator algebra and M^1, M^2 its modules. The ***fusion product*** of M^1 and M^2 is a V-module $M^1 \boxtimes_V M^2$ together with an intertwining operator $\mathcal{Y} \in \mathcal{I}_V\begin{pmatrix} M^1 \boxtimes_V M^2 \\ M^1\ M^2 \end{pmatrix}$ that satisfies the following *universal property*: for any V-module W and $\mathcal{Y}_W \in \mathcal{I}_V\begin{pmatrix} W \\ M^1\ M^2 \end{pmatrix}$, there exists a unique V-module homomorphism $f : M^1 \boxtimes_V M^2 \to W$ such that $\mathcal{Y}_W = f \circ \mathcal{Y}$.

Remark: a fusion product may not exist; but when it does, it is unique up to isomorphism as a consequence of the universal property.

If V is a rational vertex operator algebra, then the fusion product of any two irreducible V-modules exists (Proposition 4.13 in [HL]). Motivated by the concept of a *fusion algebra* in conformal field theory (Equation (2.130) in [BP]), we shall define the fusion product, if it exists, as follows:

$$M^1 \boxtimes_V M^2 := \sum_{M^i} N_V \binom{M^i}{M^1 M^2} M^i$$

where M^i runs over the set of equivalence classes of irreducible V-modules. When the context is clear, we may drop the subscript V in $M^1 \boxtimes_V M^2$ and simply write $M^1 \boxtimes M^2$.

3.2 The VOA V_L and its Modules

Let L denote a positive-definite even lattice of rank d; that is, L is a rank-d free abelian group equipped with a $\mathbb{Z}$-valued non-degenerate, positive-definite symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$:

$$\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$$

$$\langle \alpha, \alpha \rangle \in 2\mathbb{Z},\ \forall \alpha \in L\ (\textit{even})$$

$$\langle \alpha, L \rangle = \{0\} \implies \alpha = 0\ (\textit{non-degenerate})$$

$$\langle \alpha, \alpha \rangle > 0,\ \forall \alpha \in L\ (\textit{positive-definite})$$

Our main interest is V_L, whatever this symbol means at this point, and its irreducible modules. As a preview, $V_L = M(1) \otimes \mathbb{C}[L]$, so we first recall the construction of $M(1)$.

3.2.1 $M(1)$ and Its Modules

Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ be the complexification of L, then $\mathfrak{h}$ is a d-dimensional vector space which naturally inherits the bilinear form $\langle \cdot, \cdot \rangle$ as the extension of the form on L. L is identified with $L \otimes_{\mathbb{Z}} 1$ as a subspace of $\mathfrak{h}$. Viewing $\mathfrak{h}$ as an abelian Lie algebra, we form the following affine Lie algebra:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \ \ (C \neq 0)$$

with the commutation relations:

$$\begin{aligned} [\alpha_1 \otimes t^m, \alpha_2 \otimes t^n] &= m \langle \alpha_1, \alpha_2 \rangle \delta_{m+n,0} C, \ \forall \alpha_1, \alpha_2 \in \mathfrak{h}, \ \forall m, n \in \mathbb{Z} \\ [C, \hat{\mathfrak{h}}] &= 0 \end{aligned}$$

$\hat{\mathfrak{h}}$ has an abelian Lie subalgebra:

$$\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}C$$

For any $\lambda \in \mathfrak{h}$, let $\mathbb{C}e^\lambda$ denote the 1-dimensional $\hat{\mathfrak{h}}^+$-module with module actions:

$$\begin{aligned} \hat{\mathfrak{h}}^+ \ \times \ \mathbb{C}e^\lambda &\to \mathbb{C}e^\lambda \\ \mathfrak{h} \otimes t\mathbb{C}[t] \cdot e^\lambda &= 0 \\ h \otimes t^0 \cdot e^\lambda &= \langle \lambda, h \rangle e^\lambda, \ \ \forall h \in \mathfrak{h} \\ C \cdot e^\lambda &= e^\lambda \end{aligned}$$

Now we consider the induced $\hat{\mathfrak{h}}$-module:

$$M(1, \lambda) := \mathrm{Ind}_{\hat{\mathfrak{h}}^+}^{\hat{\mathfrak{h}}} \mathbb{C}e^\lambda = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathfrak{h}$$

where $U(\cdot)$ denotes the universal enveloping algebra and $S(\cdot)$ the symmetric algebra. We follow the convention for $\hat{\mathfrak{h}}$-module actions: on any $\hat{\mathfrak{h}}$-module, the action of $h \otimes t^n \in \hat{\mathfrak{h}}$ is denoted by $h(n), \forall h \in \mathfrak{h}$ and $\forall n \in \mathbb{Z}$. Any $v \in M(1,0)$ has the form $v = h_1(-n_1) \cdots h_k(-n_k) \otimes e^0$ where $h_i \in \mathfrak{h}$ and $n_i \geq 1$. To give $M(1,0)$ the structure of a vertex operator algebra, we define a linear map:

$$Y = Y(\cdot, z) : M(1,0) \to (\mathrm{End} M(1,\lambda))[[z, z^{-1}]]$$

$$Y(v,z) := \mathop{\circ}\limits_{\circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dz}\right)^{n_1-1} h_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dz}\right)^{n_k-1} h_k(z) \right) \mathop{\circ}\limits_{\circ}$$

where $h_i(z) = \sum_{n\in\mathbb{Z}} h_i(n) z^{-n-1}$.

The symbol $\mathop{\circ}\limits_{\circ} \cdot \mathop{\circ}\limits_{\circ}$ denotes a *normal-ordered product* (also called *normal ordering*) which reorders the items enclosed between the colons so that the operators $h_i(n)$, for $n < 0$, are to be placed to the left of the operators $h_i(n)$, for $n > 0$, *before* the multiplication is performed. The motivation behind normal ordering is this: the formal expression enclosed between the colons may be a product of infinite expressions and may not converge, and thus may not be an operator on $M(1,\lambda)$. Normal ordering ensures that it is a well-defined operator.

When $\lambda = 0$, we simply write:

$$M(1) := M(1,0)$$

Suppose that $\{\beta_1, \cdots \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$ $(= L \otimes_{\mathbb{Z}} \mathbb{C})$ with respect to the form $\langle \cdot, \cdot \rangle$ associated with it. We use $\mathbf{1}$ and ω to denote the following two distinguished elements of $M(1)$:

$$\mathbf{1} := 1 \otimes e^0 \in M(1)$$

$$\omega := \frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)\beta_i(-1) \otimes e^0 \in M(1)$$

Then, as shown in [FLM], $(M(1), Y(\cdot,z), \mathbf{1}, \omega)$ is a simple vertex operator algebra and $\{M(1,\lambda) \mid \lambda \in \mathfrak{h}\}$ are the irreducible $M(1)$-modules.

3.2.2 V_L and Its Modules

Let $(\hat{L}, -)$ be the central extension of L by the cyclic group $\langle \kappa \rangle = \left\langle \kappa \mid \kappa^2 = 1 \right\rangle$. This means that we have the following exact sequence:

$$1 \longrightarrow \langle \kappa \rangle = \langle -1 \rangle \longrightarrow \hat{L} \xrightarrow{-} L \longrightarrow 0$$

Associated with this extension is a commutator map:

$$c : L \times L \to \mathbb{C}^\times$$

$$c(\alpha, \beta) = \kappa^{\langle \alpha, \beta \rangle} = (-1)^{\langle \alpha, \beta \rangle}, \ \forall \alpha, \beta \in L$$

Let $e : L \to \hat{L}, \alpha \mapsto e_\alpha$ be a section such that $0 \mapsto e_0 = 1$. Then

$$\hat{L} = \{\kappa^i e_\alpha \mid \alpha \in L, i = 0, 1\}$$

This section has a corresponding 2-cocycle given by:

$$\epsilon : L \times L \to \mathbb{C}^\times$$

$$e_\alpha e_\beta = \epsilon(\alpha, \beta) e_{\alpha+\beta}$$

By [FLM], the following properties of ϵ are known for any $\alpha, \beta, \gamma \in L$:

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma)$$

$$\epsilon(\alpha, \beta)(\epsilon(\beta, \alpha))^{-1} = c(\alpha, \beta)$$

$$\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$$

We next consider the group algebra $\mathbb{C}[L] = \bigoplus_{\lambda \in L} \mathbb{C}e^{\lambda}$, which is an $\hat{L}$-module under the actions:

$$\begin{aligned} &\hat{L} \times \mathbb{C}[L] \to \mathbb{C}[L] \\ &\quad e_{\alpha} \cdot e^{\lambda} = \epsilon(\alpha, \lambda) e^{\alpha+\lambda}, \ \ \forall \alpha, \lambda \in L \\ &\quad \kappa \cdot e^{\lambda} = -e^{\lambda}, \ \ \forall \lambda \in L \end{aligned}$$

We are now ready to define V_L:

$$V_L := M(1) \otimes \mathbb{C}[L]$$

The $\hat{\mathfrak{h}}$-module structure of $M(1)$ extends naturally to the $\hat{\mathfrak{h}}$-module structure of V_L:

$$\begin{aligned} &\hat{\mathfrak{h}} \, (= \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C) \ \times \ V_L \to V_L \\ &\qquad h(n) \cdot (u \otimes e^{\lambda}) = (h(n) \cdot u) \otimes e^{\lambda}, \ \ \forall n \neq 0 \\ &\qquad h(0) \cdot (u \otimes e^{\lambda}) = \langle h, \lambda \rangle \, (u \otimes e^{\lambda}) \\ &\qquad C \cdot (u \otimes e^{\lambda}) = u \otimes e^{\lambda} \end{aligned}$$

for all $h \in \mathfrak{h}, u \in M(1)$, and $\lambda \in L$.

Next, we explain that V_L has the structure of a vertex operator algebra. For each $v \in V_L, v = h_1(-n_1) \cdots h_k(-n_k) \otimes e^{\lambda}$, for some $\lambda \in L$ and $h_i \in \mathfrak{h}, n_i \geq 1$. We start by defining the vertex operator associated to e^{λ}:

$$Y(e^{\lambda}, z) := \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(-n)}{n} z^{n} \right) \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} z^{-n} \right) e_{\lambda} z^{\lambda}$$

Note that $\mathbb{C}[L]$ is an $\hat{L}$-module as described above, so e_{λ} is the left action of $e_{\lambda} \in \hat{L}$ on $\mathbb{C}[L]$; and z^{λ} is the operator on $\mathbb{C}[L]$ defined by:

$$z^{\lambda} \cdot e^{\mu} = z^{\langle \lambda, \mu \rangle} e^{\mu}$$

Using this, we then define the vertex operator associated to $v \in V_L$:

$$Y = Y(\cdot, z) : V_L \to (\mathrm{End} V_L)\{z\}$$

$$v \mapsto Y(v, z)$$

$$Y(v,z) := {\circ \atop \circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dz} \right)^{n_1-1} h_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dz} \right)^{n_k-1} h_k(z) \right) Y(e^{\lambda}, z) {\circ \atop \circ}$$

With $\mathbf{1} = 1 \otimes e^0 \in M(1) \subseteq V_L$ and $\omega = \dfrac{1}{2} \sum_{i=1}^{d} \beta_i(-1)\beta_i(-1) \otimes e^0 \in M(1) \subseteq V_L$, the structure $(V_L, Y, \mathbf{1}, \omega)$ has been shown (in [FLM], [LL]) to be a simple vertex operator algebra.

To classify V_L-modules, we first introduce the dual lattice of L:

$$L^{\circ} = \{\beta \in \mathfrak{h} |\, \langle \alpha, \beta \rangle \in \mathbb{Z}, \forall \alpha \in L\}$$

Since L is an even lattice, it follows that $L \subseteq L^{\circ}$. Let $S := \{\lambda_1, \cdots, \lambda_k\}$ be the complete set of representatives of equivalence classes of L in its dual lattice L°. Then

$$\mathbb{C}[L^{\circ}] = \mathbb{C}[L + \lambda_1] \oplus \cdots \oplus \mathbb{C}[L + \lambda_k]$$

$$V_{L^{\circ}} = V_{L+\lambda_1} \oplus \cdots \oplus V_{L+\lambda_k}$$

By the work of [FLM2] and [D1], $\{V_{L+\lambda} \,|\, \lambda \in S\}$ is the complete list of (inequivalent) irreducible *untwisted* V_L-modules. The classification of irreducible *twisted* modules for V_L was done in [D2] and is recalled below.

Let $\theta \in \mathrm{Aut}(\hat{L})$ be an automorphism of $\hat{L}$ such that $\theta^2 = Id_{\hat{L}}$ and $\theta(\kappa) = \kappa$ (in other words, θ preserves -1). Recall that $\hat{L} = \{\kappa^i e_{\alpha} \,|\, \alpha \in L, i = 0, 1\}$, so the action of

θ on $\hat{L}$ can be viewed as:

$$\theta(\kappa^i e_\alpha) = \kappa^i e_{-\alpha}$$

It can be easily observed that θ induces an automorphism $\bar{\theta}$ on L such that $\bar{\theta}^2 = Id_L$ and $\bar{\theta}(\alpha) = -\alpha,\ \forall \alpha \in L$.

Now we define the action of θ on V_L $(= M(1) \otimes \mathbb{C}[L])$ by:

$$\theta : V_L \to V_L$$

$$(h_1(-n_1) \cdots h_k(-n_k)) \otimes e^\alpha \mapsto (-1)^k (h_1(-n_1) \cdots h_k(-n_k)) \otimes e^{-\alpha}$$

for $h_i \in \mathfrak{h}, n_i \geq 1$, and $\alpha \in L$. In fact, θ turns out to be an automorphism of V_L which has two important eigensubspaces of eigenvalues 1 and -1, respectively:

$$V_L^{\pm} = \{v \in V_L \,|\, \theta(v) = \pm v\}$$

A thorough treatment of the fusion rules for V_L^+ has been done by Abe, Dong, and Li [ADL], which lays the foundation for our results here.

We now consider a θ-twisted affine Lie algebra:

$$\hat{\mathfrak{h}}[\theta] := \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$$

with the following Lie brackets for all $\alpha_1, \alpha_2 \in \mathfrak{h}$ and $m, n \in \mathbb{Z} + \frac{1}{2}$:

$$[\alpha_1 \otimes t^m, \alpha_2 \otimes t^n] = m \langle \alpha_1, \alpha_2 \rangle \delta_{m+n,0} C$$

$$[C, \hat{\mathfrak{h}}[\theta]] = 0$$

$\hat{\mathfrak{h}}[\theta]$ has the following subspaces:

$$\hat{\mathfrak{h}}[\theta]^+ = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t] \qquad \text{and} \qquad \hat{\mathfrak{h}}[\theta]^- = \mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]$$

Viewing $\mathbb{C}$ as a module for $\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C$ on which $\hat{\mathfrak{h}}[\theta]^+$ acts trivially and C acts as a multiplication by 1, we consider the induced module:

$$\begin{aligned} M(1)(\theta) &:= \mathrm{Ind}_{\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C}^{\hat{\mathfrak{h}}[\theta]} \mathbb{C} \\ &= U(\hat{\mathfrak{h}}[\theta]) \otimes_{U(\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C)} \mathbb{C} \\ &\cong S(t^{-1/2}\mathbb{C}[t^{-1}]) \otimes \mathfrak{h} \end{aligned}$$

Finally, we define:

$$K := \{a^{-1}\theta(a) \mid a \in \hat{L}\,\}$$

And let T_χ be the irreducible $\hat{L}/K$-module associated to a central character χ:

$$\begin{aligned} \chi : Z(\hat{L}/K) &\to \mathbb{C}^\times \\ (-1)K &\mapsto -1 \end{aligned}$$

(that is, T_χ is an irreducible $\hat{L}/K$-module on which $(-1)K$ acts as -1). For each such T_χ, define a twisted space:

$$V_L^{T_\chi} := M(1)(\theta) \otimes T_\chi$$

Then $\{V_L^{T_\chi} \mid T_\chi = \text{irreducible } \hat{L}/K\text{-module as described above}\}$ exhausts all the irreducible θ-twisted V_L modules. These are also called V_L-modules of *twisted* type, to distinguish from the $V_{L+\lambda}$ mentioned earlier, which are of untwisted type. The action of θ on $M(1)(\theta)$ extends to an action on $V_L^{T_\chi}$:

$$\theta : V_L^{T_\chi} \to V_L^{T_\chi}$$

$$(h_1(-n_1)\cdots h_k(-n_k)) \otimes t \mapsto (-1)^k (h_1(-n_1)\cdots h_k(-n_k)) \otimes t \tag{3.2.2.1}$$

for $h_i \in \mathfrak{h}, n_i \in \mathbb{Z} + \frac{1}{2}$, and $t \in T_\chi$. As before, we denote by $V_L^{T_\chi,+}$ and $V_L^{T_\chi,-}$ the eigensubspaces of $V_L^{T_\chi}$ of eigenvalues 1 and -1, respectively.

We mention now two results from [ADL] and [A2] concerning V_L^+:

Proposition 3.2.1 *([ADL], Theorem 3.4) Let L be a positive-definite even lattice and let $\{\lambda_i\}$ be a set of representatives of L°/L. Then any irreducible V_L^+-module is isomorphic to one of the irreducible modules $V_L^\pm, V_{\lambda_i+L}$ with $2\lambda_i \notin L, V_{\lambda_i+L}^\pm$ with $2\lambda_i \in L$ or $V_L^{T_\chi,\pm}$ for a central character χ of $\hat{L}/K$ with $\chi(\kappa) = -1$.*

Proposition 3.2.2 *([A2], Proposition 3.3) Let W^1, W^2, and W^3 be irreducible V_L^+-modules. Then the following hold:*

(1) The fusion rules $N\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is zero or one.

(2) If all $W^i (i = 1, 2, 3)$ are twisted type modules, then the fusion rule $N\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is zero.

(3) If one of $W^i (i = 1, 2, 3)$ is a twisted type module and the others are of untwisted type, then the fusion rule $N\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is zero.

The next three sections discuss the three different fusion products of V_L-modules. The first one, Section 3.3, is a result directly obtained from [DL] concerning modules of untwisted type, while the other two delve into the cases when at least one module of twisted type is involved in the fusion product.

3.3 The Fusion Product $V_{L+\lambda} \boxtimes V_{L+\mu}$

For the remaining three sections, we shall drop the subscript V_L in the fusion rule N_{V_L} and fusion product $\boxtimes_{V_L}$ notations and simply write N and $\boxtimes$, respectively. Recall that $S = \{\lambda_1, \cdots, \lambda_k\}$ is the complete set of representatives of equivalence classes of L in its dual lattice L°. As mentioned above, the following proposition is an immediate consequence of Proposition 12.9 in [DL].

Proposition 3.3.1 *For any* $\lambda, \mu \in S$:

$$V_{L+\lambda} \boxtimes V_{L+\mu} = V_{L+\lambda+\mu}$$

Proof. Let M^i run over the equivalence classes of irreducible V_L-modules. By definition, we have:

$$\begin{aligned} V_{L+\lambda} \boxtimes V_{L+\mu} &= \sum_i N\binom{M^i}{V_{L+\lambda}\ V_{L+\mu}} M^i \\ &= \sum_{\nu \in S} N\binom{V_{L+\nu}}{V_{L+\lambda}\ V_{L+\mu}} V_{L+\nu} + \sum_{V_L^{T_\chi}} N\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} V_L^{T_\chi} \end{aligned}$$

where $V_L^{T_\chi}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules. Now by [DL],

$$N\binom{V_{L+\nu}}{V_{L+\lambda}\ V_{L+\mu}} = 1 \ \text{ iff } \ \nu = \lambda + \mu$$

Recall that V_L^+ is a vertex operator subalgebra of V_L and that $\{V_{L+\lambda} \mid \lambda \in S\}$ are the θ-untwisted modules and $\{V_L^{T_\chi}\}$ are the θ-twisted V_L^+-modules. Applying Prop. 3.1.4 of Section 3.1, we have:

$$N_{V_L}\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} \le N_{V_L^+}\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} = 0$$

The last equality follows immediately from Prop. 3.2.2 (3) mentioned above. Thus,

$$V_{L+\lambda} \boxtimes V_{L+\mu} = N\binom{V_{L+\lambda+\mu}}{V_{L+\lambda}\ V_{L+\mu}} V_{L+\lambda+\mu} = V_{L+\lambda+\mu}$$

□

3.4 The Fusion Product $V_{L+\lambda} \boxtimes V_L^{T_\chi}$

Let M^k denote the irreducible V_L-modules, then:

$$\begin{aligned} V_{L+\lambda} \boxtimes V_L^{T_\chi} &= \sum_k N\binom{M^k}{V_{L+\lambda}\ V_L^{T_\chi}} M^k \\ &= \sum_{\mu \in S} N\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} V_{L+\mu} + \sum_{V_L^{T_{\chi_2}}} N\binom{V_L^{T_{\chi_2}}}{V_{L+\lambda}\ V_L^{T_\chi}} V_L^{T_{\chi_2}} \end{aligned}$$

where $V_L^{T_{\chi_2}}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules.

Lemma 3.4.1 *For any $\lambda, \mu \in L^\circ$ and central character χ of $\hat{L}/K$ such that $\chi(\kappa) = -1$:*

$$N\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} = 0$$

Proof. This is an immediate consequence of Prop. 3.1.4. For any $\mu \in L^\circ$, $V_{L+\mu}$ is a V_L-module and thus is also a V_L^+-module. (Note that the fact that it is also irreducible is not needed here.) Recall that $V_L^{T_\chi}$ is a twisted irreducible V_L-module while its submodule $V_L^{T_\chi,+}$ is an irreducible V_L^+-module of twisted type by Prop. 3.2.1 above.

Case 1: If $2\lambda \notin L$, then $V_{L+\lambda}$ is an untwisted irreducible V_L^+-module by Prop. 3.2.1. So, by Prop. 3.1.4 and Prop. 3.2.2(3), we have:

$$N_{V_L}\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} \leq N_{V_L^+}\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi,+}} = 0$$

Case 2: If $2\lambda \in L$, then $V_{L+\lambda}^{\pm}$ are untwisted irreducible V_L^+-modules, also by Prop. 3.2.1. Then:

$$N_{V_L}\begin{pmatrix} V_{L+\mu} \\ V_{L+\lambda}\ V_L^{T_\chi} \end{pmatrix} \leq N_{V_L^+}\begin{pmatrix} V_{L+\mu} \\ V_{L+\lambda}^+\ V_L^{T_\chi,+} \end{pmatrix} = 0$$

□

We now show that there exists an intertwining operator of type $\begin{pmatrix} V_L^{T_{\chi_1}} \\ V_{L+\lambda}\ V_L^{T_\chi} \end{pmatrix}$ for V_L by explicitly constructing it. We should point out that χ_1 is actually determined by both χ and λ by a formula to be given in the discussion below.

Let χ be any central character of $\hat{L}/K$ such that $\chi(\kappa) = -1$. That is,

$$\chi : Z(\hat{L}/K) \to \mathbb{C}^\times$$

$$\kappa \mapsto -1$$

and T_χ the corresponding irreducible $\hat{L}/K$-module under the action:

$$\hat{L}/K \quad \times \quad T_\chi \to T_\chi$$

$$\kappa \cdot v = -v$$

As done in Subsection 3.3.2, $V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$, which is a θ-twisted V_L-module. Let $\lambda \in L^\circ$ and define an automorphism σ_λ of $\hat{L}$:

$$\sigma_\lambda : \hat{L} \to \hat{L}$$

$$a \mapsto \sigma_\lambda(a) := \kappa^{\langle \lambda, \bar{a} \rangle} a = (-1)^{\langle \lambda, \bar{a} \rangle} a$$

Let $a \in \hat{L}$, then $\sigma_\lambda(\theta(a)) = \kappa^{\langle \lambda, \overline{\theta(a)} \rangle}\theta(a)$, while $\theta(\sigma_\lambda(a)) = \theta(\kappa^{\langle \lambda, \bar{a} \rangle} a) = \kappa^{\langle \lambda, \bar{a} \rangle}\theta(a)$. So, $\sigma_\lambda(\theta(a)) = \theta(\sigma_\lambda(a))$. For $a^{-1}\theta(a) \in K$, σ_λ sends it back to K because:

$$\sigma_\lambda(a^{-1}\theta(a)) = \sigma_\lambda(a^{-1})\sigma_\lambda(\theta(a)) = (\sigma_\lambda(a))^{-1}\theta(\sigma_\lambda(a)) \in K$$

And thus, σ_λ stabilizes K and consequently induces an automorphism on $\hat{L}/K$:

$$\sigma_\lambda : \hat{L}/K \to \hat{L}/K$$

$$aK \mapsto \sigma_\lambda(aK) = \sigma_\lambda(a)K = \kappa^{\langle \lambda, \bar{a} \rangle} aK = (-1)^{\langle \lambda, \bar{a} \rangle} aK$$

For any $\hat{L}/K$-module T, we denote by $T \circ \sigma_\lambda$ the $\hat{L}/K$-*module twisted by* σ_λ. This means that $T \circ \sigma_\lambda \cong T$ as vector spaces but there is an additional action of $\hat{L}/K$ on $T \circ \sigma_\lambda$ which is determined by σ_λ:

$$\hat{L}/K \quad \times \quad T \circ \sigma_\lambda \, (= T) \to T \circ \sigma_\lambda \, (= T)$$

$$a \cdot t = \sigma_\lambda(a)t$$

When $T = T_\chi$, we have:

$$\hat{L}/K \quad \times \quad T_\chi \circ \sigma_\lambda \, (= T_\chi) \to T_\chi \circ \sigma_\lambda \, (= T_\chi)$$

$$\kappa \cdot t = -t, \ \forall t \in T_\chi$$

$$a \cdot t = \sigma_\lambda(a)t, \ \forall a \in \hat{L}/K, \ t \in T_\chi$$

Since T_χ is irreducible, so is $T_\chi \circ \sigma_\lambda$. With the number of central characters of $\hat{L}/K$ which send κ to -1 being finite ([FLM], Prop. 7.4.8), there must exist a unique central character χ_1 of $\hat{L}/K$ such that the corresponding $\hat{L}/K$-module T_{χ_1} satisfies $T_{\chi_1} \cong T_\chi \circ \sigma_\lambda$.

To emphasize the fact that χ_1 is dependent upon χ and λ, we use $\chi^{(\lambda)}$ instead of χ_1, and so $T_{\chi^{(\lambda)}} \cong T_\chi \circ \sigma_\lambda$. Let f denote this isomorphism:

$$f : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}} \qquad (\hat{L}/K\textit{-module isomorphism})$$

$$T_\chi \to T_{\chi^{(\lambda)}} \qquad (\textit{linear isomorphism})$$

$$f(\sigma_\lambda(a)t) = af(t),\ \forall a \in \hat{L}/K, t \in T_\chi$$

Following [ADL], we consider $\lambda \in L^\circ$ and $\alpha \in L$ and define another linear isomorphism:

$$\eta_{\lambda+\alpha} : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}}$$

$$\eta_{\lambda+\alpha} = \epsilon(-\alpha, \lambda) e_\alpha \circ f = (-1)^{\langle -\alpha, \lambda \rangle} e_\alpha \circ f$$

Recall that e_α is the left action of $e_\alpha \in \hat{L}$ on $\mathbb{C}[L]$ with the following properties:

Lemma 3.4.2 *For any* $\alpha, \beta \in L, e_\alpha e_\beta = (-1)^{\langle \alpha, \beta \rangle} e_\beta e_\alpha$ *as operators on* $\mathbb{C}[L]$.

Proof. Consider $e^\mu \in \mathbb{C}[L]$ for $\mu \in L$:

$$\begin{aligned} e_\alpha e_\beta \cdot e^\mu &= e_\alpha (e_\beta \cdot e^\mu) = e_\alpha(\epsilon(\beta, \mu) e^{\beta+\mu}) \\ &= \epsilon(\beta, \mu) e_\alpha \cdot e^{\beta+\mu} = \epsilon(\beta, \mu)\epsilon(\alpha, \beta+\mu) e^{\alpha+(\beta+\mu)} \\ &= \epsilon(\beta, \mu)\epsilon(\alpha, \beta)\epsilon(\alpha, \mu) e^{\alpha+\beta+\mu} \end{aligned}$$

On the other hand:

$$\begin{aligned} e_\beta e_\alpha \cdot e^\mu &= e_\beta(\epsilon(\alpha, \mu) e^{\alpha+\mu}) \\ &= \epsilon(\alpha, \mu) e_{\beta, \alpha+\mu} e^{\beta+(\alpha+\mu)} \\ &= \epsilon(\alpha, \mu)\epsilon(\beta, \alpha)\epsilon(\beta, \mu) e^{\beta+\alpha+\mu} \end{aligned}$$

Multiplying both sides by $(-1)^{\langle \alpha,\beta \rangle}$ yields:

$$
\begin{aligned}
(-1)^{\langle \alpha,\beta \rangle} e_\beta e_\alpha \cdot e^\mu &= (-1)^{\langle \alpha,\beta \rangle} \epsilon(\beta,\alpha)\epsilon(\alpha,\mu)\epsilon(\beta,\mu) e^{\beta+\alpha+\mu} \\
&= \epsilon(\alpha,\beta)\epsilon(\alpha,\mu)\epsilon(\beta,\mu) e^{\beta+\alpha+\mu} \\
&= e_\alpha e_\beta \cdot e^\mu
\end{aligned}
$$

The second-to-last equality follows from the fact that $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{\langle \alpha,\beta \rangle}$.

$\square$

Lemma 3.4.3 *For the $\hat{L}/K$-module isomorphism $f : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}}$ defined earlier and any $\alpha \in L$, e_α satisfies $e_\alpha \circ f = (-1)^{\langle \alpha,\lambda \rangle} f \circ e_\alpha$ as operators on $\mathbb{C}[L]$.*

Proof. Consider $e^\mu \in \mathbb{C}[L]$ for $\mu \in L$:

$$
\begin{aligned}
(-1)^{\langle \alpha,\lambda \rangle} f \circ e_\alpha \cdot e^\mu &= (-1)^{\langle \alpha,\lambda \rangle} f(\epsilon(\alpha,\mu) e^{\alpha+\mu} \\
&= (-1)^{\langle \alpha,\lambda \rangle} \epsilon(\alpha,\mu) f(e^{\alpha+\mu})
\end{aligned}
$$

On the other hand, recall that for $a \in \hat{L}$, $f(\sigma_\lambda(a)t) = af(t)$. Thus:

$$
\begin{aligned}
e_\alpha \circ f \cdot e^\mu &= f(\sigma_\lambda(e_\alpha) e^\mu) \\
&= f(\kappa^{\langle \lambda, \overline{e_\alpha} \rangle} e_\alpha e^\mu) \\
&= \kappa^{\langle \lambda, \overline{e_\alpha} \rangle} f(e_\alpha e^\mu) \\
&= \kappa^{\langle \lambda, \bar{\alpha} \rangle} f(\epsilon(\alpha,\mu) e^{\alpha+\mu}) \\
&= (-1)^{\langle \alpha,\lambda \rangle} \epsilon(\alpha,\mu) f(e^{\alpha+\mu}) \\
&= (-1)^{\langle \alpha,\lambda \rangle} f \circ e_\alpha \cdot e^\mu
\end{aligned}
$$

$\square$

We have the following facts about η_γ ([ADL]):

Lemma 3.4.4 *([ADL] Lemma 5.8) For any* $\gamma \in L + \lambda$ *and* $\alpha \in L$*:*

$$e_\alpha \circ \eta_\gamma = (-1)^{\langle \alpha, \gamma \rangle} \eta_\gamma \circ e_\alpha$$

$$e_\alpha \circ \eta_\gamma = \epsilon(\alpha, \gamma)\eta_{\gamma+\alpha} = \epsilon(-\alpha, \gamma)\eta_{\gamma-\alpha}$$

Proof. Let $\gamma \in L + \lambda$, then $\gamma = \beta + \lambda$ for some $\beta \in L$. For any $e^\mu \in \mathbb{C}[L]$ (where $\mu \in L$):

$$\begin{aligned}
(e_\alpha \circ \eta_\gamma) \cdot e^\mu &= e_\alpha \circ \eta_{\beta+\lambda} \cdot e^\mu \\
&= e_\alpha(\epsilon(-\beta, \lambda) e_\beta \circ f)(e^\mu) && \text{(by definition of } \eta_{\lambda+\alpha}) \\
&= \epsilon(-\beta, \lambda) e_\alpha e_\beta \circ f(e^\mu) \\
&= \epsilon(-\beta, \lambda)(-1)^{\langle \alpha, \beta \rangle} e_\beta e_\alpha \circ f(e^\mu) && \text{(by Prop. 3.4.2)} \\
&= \epsilon(-\beta, \lambda)(-1)^{\langle \alpha, \beta \rangle} e_\beta (-1)^{\langle \alpha, \lambda \rangle} f \circ e_\alpha(e^\mu) && \text{(by Prop. 3.4.3)} \\
&= \epsilon(-\beta, \lambda)(-1)^{\langle \alpha, \beta \rangle}(-1)^{\langle \alpha, \lambda \rangle} e_\beta f(\epsilon(\alpha, \mu) e^{\alpha+\mu}) \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \mu)(-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \lambda \rangle} e_\beta f(e^{\alpha+\mu}) \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \mu)(-1)^{\langle \alpha, \beta+\lambda \rangle} e_\beta f(e^{\alpha+\mu}) \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \mu)(-1)^{\langle \alpha, \gamma \rangle} e_\beta f(e^{\alpha+\mu}) \\
&= (-1)^{\langle \alpha, \gamma \rangle}\epsilon(\alpha, \mu)\epsilon(-\beta, \lambda) e_\beta f(e^{\alpha+\mu}) && \text{(rearranging terms)} \\
&= (-1)^{\langle \alpha, \gamma \rangle}\epsilon(\alpha, \mu)\eta_{\beta+\lambda}(e^{\alpha+\mu}) && \text{(by definition of } \eta) \\
&= (-1)^{\langle \alpha, \gamma \rangle}\eta_{\beta+\lambda}(\epsilon(\alpha, \mu) e^{\alpha+\mu}) \\
&= (-1)^{\langle \alpha, \gamma \rangle}(\eta_{\beta+\lambda} \circ e_\alpha) \cdot e^\mu && \text{(action of } e_\alpha \text{ on } \mathbb{C}[L]) \\
&= ((-1)^{\langle \alpha, \gamma \rangle}\eta_\gamma \circ e_\alpha) \cdot e^\mu
\end{aligned}$$

Thus we have shown the first equality. To show $e_\alpha \circ \eta_\gamma = \epsilon(\alpha, \gamma)\eta_{\alpha+\gamma}$ in the second one:

$$\begin{aligned}
e_\alpha \circ \eta_\gamma &= e_\alpha \circ \eta_{\beta+\lambda} = e_\alpha \circ \epsilon(-\beta, \lambda)e_\beta \circ f \\
&= \epsilon(-\beta, \lambda)e_\alpha \circ e_\beta \circ f \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)e_{\alpha+\beta} \circ f \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)\epsilon(0, \lambda)e_{\alpha+\beta} \circ f \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)\epsilon(\alpha+\beta-\alpha-\beta, \lambda)e_{\alpha+\beta} \circ f \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)\epsilon(\alpha+\beta, \lambda)\epsilon(-\alpha-\beta, \lambda)e_{\alpha+\beta} \circ f \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)\epsilon(\alpha+\beta, \lambda)\eta_{\lambda+\alpha+\beta} \\
&= \epsilon(-\beta, \lambda)\epsilon(\alpha, \beta)\epsilon(\alpha+\beta, \lambda)\eta_{\gamma+\alpha} \\
&= \epsilon(\alpha, \lambda)\epsilon(\alpha, \beta)\eta_{\gamma+\alpha} \\
&= \epsilon(\alpha, \lambda+\beta)\eta_{\gamma+\alpha} \\
&= \epsilon(\alpha, \gamma)\eta_{\gamma+\alpha}
\end{aligned}$$

It follows immediately that $e_{-\alpha} \circ \eta_\gamma = \epsilon(-\alpha, \gamma)\eta_{-\alpha+\gamma}$. Now recall from Subsection 3.2.2 that the action of θ on $M(1)(\theta)$ extends to an action on $V_L^{T_\chi}(= M(1)(\theta) \otimes T_\chi)$:

$$\begin{aligned}
&\theta : V_L^{T_\chi} \to V_L^{T_\chi} \\
&(h_1(-n_1)\cdots h_k(-n_k)) \otimes t \mapsto (-1)^k(h_1(-n_1)\cdots h_k(-n_k)) \otimes t \qquad (3.2.2.1)
\end{aligned}$$

for $h_i \in \mathfrak{h}, n_i \in \mathbb{Z} + \frac{1}{2}$, and $t \in T_\chi$. In other words, T_χ is compatible with θ in the sense that $\theta(a) = a, \forall a \in \hat{L}$, as operators on T_χ (see [FLM] (7.4.14)). Hence, $\theta(e_\alpha) = e_\alpha$ as operators on T_χ since $e_\alpha \in \hat{L}$. On the other hand, recall that $\theta(\kappa^i e_\alpha) = \kappa^i e_{-\alpha}$, which implies $\theta(e_\alpha) = e_{-\alpha}$. And therefore, $e_{-\alpha} = e_\alpha$ as operators on T_χ. With this, we have $\epsilon(\alpha, \gamma)\eta_{\alpha+\gamma} = \epsilon(-\alpha, \gamma)\eta_{-\alpha+\gamma}$ as desired.

$\square$

We are now ready to define an non-trivial intertwining operator of type $\begin{pmatrix} & V_L^{T_{\chi(\lambda)}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix}$ for V_L where $\lambda \in L^\circ$. Following [FLM], we define:

$$\mathcal{Y}_\lambda^{tw}(\cdot, z) : M(1, \lambda) \to (\text{End } (M(1)(\theta)))\{z\}$$

$$v \mapsto \mathcal{Y}_\lambda^{tw}(v, z)$$

for $v = h_1(-n_1)h_2(-n_2)\cdots h_k(-n_k) \otimes e^\lambda$, where $h_i \in \mathfrak{h}$ and $n_i \geq 1$, by first specifying how it acts on e^λ:

$$\mathcal{Y}_\lambda^{tw}(e^\lambda, z) := 2^{-\langle\lambda,\lambda\rangle} z^{-\frac{\langle\lambda,\lambda\rangle}{2}} \exp\left(\sum_{n\in\mathbb{N}+\frac{1}{2}} \frac{\lambda(-n)}{n} z^n\right) \exp\left(-\sum_{n\in\mathbb{N}+\frac{1}{2}} \frac{\lambda(n)}{n} z^{-n}\right)$$

And then defining:

$$W(v, z) := {\scriptstyle\circ\atop\circ} \left(\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} \beta_1(z)\right) \cdots \left(\frac{1}{(n_k-1)!}\left(\frac{d}{dz}\right)^{n_k-1} \beta_k(z)\right) \mathcal{Y}_\lambda^{tw}(e^\lambda, z) {\scriptstyle\circ\atop\circ}$$

where, as before, the normal ordering places $h_i(n)$ for $n < 0$ to the left of $h_i(n)$ for $n > 0$. Finally, for $v \in M(1, \lambda)$ we define:

$$\mathcal{Y}_\lambda^{tw}(v, z) := W(e^{\Delta_z} v, z)$$

where

$$\Delta_z = \sum_{i=1}^{d} \sum_{m,n=0}^{\infty} c_{mn} \beta_i(m)\beta_i(n) z^{-m-n}$$

where $\{\beta_1, \beta_2, \cdots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$ and c_{mn} are the coefficients determined by the following expansion:

$$-\log\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right) = \sum_{m,n=0}^{\infty} c_{mn}x^m y^n$$

Now let $u \in V_{L+\lambda}$. We know that the V_L-module $V_{L+\lambda}$ has the following decomposition:

$$V_{L+\lambda} \cong \bigoplus_{\beta \in L} M(1, \beta+\lambda)$$

where $M(1, \beta+\lambda)$ are irreducible $M(1)$-modules. So, there exists some $\beta \in L$ such that $u \in M(1, \beta+\lambda)$. Using $\mathcal{Y}_{\lambda}^{tw}$, we define yet another map:

$$\tilde{\mathcal{Y}}_{\lambda}^{tw}(u, z) := \mathcal{Y}_{\lambda+\beta}^{tw}(u, z) \otimes \eta_{\lambda+\beta}$$

Recall that $\eta_{\lambda+\beta}$ is a linear isomorphism between T_χ and $T_{\chi^{(\lambda)}}$, while the components of $\tilde{\mathcal{Y}}_{\lambda}^{tw}(u, z)$ are elements of $\mathrm{End}(M(1)(\theta))\{z\}$, and $M(1)(\theta)$ can be identified with $M(1)(\theta) \otimes 1$ as a subspace of $M(1)(\theta) \otimes T_\chi = V_L^{T_\chi}$. Thus, we have the following linear map:

$$\tilde{\mathcal{Y}}_{\lambda}^{tw} : V_{L+\lambda} \to (\mathrm{Hom}(V_L^{T_\chi}, V_L^{T_{\chi^{(\lambda)}}}))\{z\}$$

$$u \mapsto \tilde{\mathcal{Y}}_{\lambda}^{tw}(u, z) = \mathcal{Y}_{\lambda+\beta}^{tw}(u, z) \otimes \eta_{\lambda+\beta}$$

The next three lemmas show that $\tilde{\mathcal{Y}}_{\lambda}^{tw}$ satisfies the three conditions stated in Definition 3.1.1 and thus is an intertwining operator of type $\begin{pmatrix} & V_L^{T_{\chi^{(\lambda)}}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix}$ for V_L. From there, we shall argue that the fusion rule $N_{V_L}\begin{pmatrix} & V_L^{T_{\chi^{(\lambda)}}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix} = 1$.

Lemma 3.4.5 *For any* $u \in V_{L+\lambda}, v \in V_L^{T_\chi}$*, and any fixed* $\alpha \in \mathbb{C}$*,* $u_{n+\alpha}v = 0$ *if* $n \gg 0$*.*

Proof. Since $v \in V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$, $v = w \otimes t$ for some $w \in M(1)(\theta)$ and $t \in T_\chi$. Then:

$$\begin{aligned}\tilde{\mathcal{Y}}_\lambda^{tw}(u,z)v &= \tilde{\mathcal{Y}}_\lambda^{tw}(u,z)(w \otimes t)\\ &= \mathcal{Y}_{\lambda+\beta}^{tw}(u,z)(w) \otimes \eta_{\lambda+\beta}(t)\end{aligned}$$

But $\mathcal{Y}_{\lambda+\beta}^{tw}$ is a nonzero intertwining operator of type $\begin{pmatrix} M(1)(\theta) \\ M(1,\lambda+\beta) \;\; M(1)(\theta)\end{pmatrix}$ for $M(1)$ (see [ADL], p. 191). So, for any $u \in M(1,\lambda+\beta) \subset V_{L+\lambda}, u_{n+\alpha}w = 0$ if $n \gg 0$.

$\square$

Lemma 3.4.6 *Let* $\alpha, \beta \in L$*. For any* $a \in M(1,\alpha), u \in M(1,\beta+\lambda)$*:*

$$\begin{aligned}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_{V_L^{T_{\chi(\lambda)}}}(a,z_1)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2)Y_{V_L^{T_\chi}}(a,z_1)\\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(Y_{V_{L+\lambda}}(a,z_0)u,z_2)\end{aligned}$$

where $Y_{V_L^{T_{\chi(\lambda)}}}(a,z_1)$ *is the vertex operator associated with* $a \in M(1,\alpha) \subseteq V_L$*:*

$$\begin{aligned}Y_{V_L^{T_{\chi(\lambda)}}}(\cdot,z_1) : M(1,\alpha) \subseteq V_L &\to (\mathrm{End}(V_L^{T_{\chi(\lambda)}}))\{z_1\}\\ a &\mapsto Y_{V_L^{T_{\chi(\lambda)}}}(a,z_1)\end{aligned}$$

Proof. Recall the map $\tilde{\mathcal{Y}}_\lambda^{tw} : M(1,\lambda+\beta) \subseteq V_{L+\lambda} \to (\mathrm{Hom}(V_L^{T_\chi}, V_L^{T_{\chi(\lambda)}}))\{z\}$. When we take $\lambda = 0$ and $\beta = \alpha$, then:

$$\tilde{\mathcal{Y}}_0^{tw} : M(1,0+\alpha) \subseteq V_L \to (\mathrm{Hom}(V_L^{T_\chi}, V_L^{T_{\chi(0)}}))\{z\}$$

That is: $$\tilde{\mathcal{Y}}_0^{tw} : \qquad M(1,\alpha) \subseteq V_L \to (\mathrm{End}(V_L^{T_\chi}))\{z\} \qquad (*)$$

For any $w \otimes t \in M(1)(\theta) \otimes T_\chi (= V_L^{T_\chi})$:

$$\begin{aligned}\tilde{\mathcal{Y}}_0^{tw}(a, z_1)(w \otimes t) &= (\mathcal{Y}_{0+\alpha}^{tw}(a, z_1) \otimes \eta_{0+\alpha})(w \otimes t)\\ &= \mathcal{Y}_\alpha^{tw}(a, z_1)(w) \otimes \eta_\alpha(t)\\ &= \mathcal{Y}_\alpha^{tw}(a, z_1)(w) \otimes e_\alpha(t)\\ &= (\mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha)(w \otimes t)\end{aligned}$$

The third equality above follows from the fact that:

$$\eta_\alpha = \eta_{0+\alpha} = \epsilon(-\alpha, 0) e_\alpha \circ f = 1 e_\alpha \circ f = e_\alpha \circ f = e_\alpha$$

since f is an isomorphism of T_χ.

But by (*) above, the map $\tilde{\mathcal{Y}}_0^{tw}(a, z_1)$ is the twisted vertex operator associated with $a \in M(1,\alpha) \subseteq V_L$. That is to say,

$$Y_{V_L^{T_{\chi^{(\lambda)}}}}(a, z_1) = \tilde{\mathcal{Y}}_0^{tw}(a, z_1) = \mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha$$

By the same argument, we have:

$$Y_{V_L^{T_\chi}}(a, z_1) = \tilde{\mathcal{Y}}_\alpha^{tw}(a, z_1) \otimes e_\alpha$$

Note 1: Recall the map:

$$\mathcal{Y}_{\alpha,\lambda+\beta}(\cdot, z_0) : M(1,\alpha) \to \left(\mathrm{Hom}(M(1,\lambda+\beta), M(1,\alpha+\lambda+\beta))\right)\{z_0\}$$

where $M(1,\alpha) \subseteq V_L, M(1,\lambda+\beta) \subseteq V_{L+\lambda}$, and $M(1,\alpha+\lambda+\beta) \subseteq V_{L+\lambda}$. This map satisfies the Jacobi identity and the $L(-1)$-derivative property. So, it is the map giving the V_L-module structure for $V_{L+\lambda}$. As a result, $\mathcal{Y}_{\alpha,\lambda+\beta}(a, z_0) = Y_{V_{L+\lambda}}(a, z_0)$.

Note 2:

$$\begin{aligned}
\tilde{\mathcal{Y}}_\lambda^{tw}\left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2\right) &= \tilde{\mathcal{Y}}_\lambda^{tw}\left(\theta Y_{V_{L+\lambda}}(a, z_0)\theta^{-1}u, z_2\right) \\
&= \tilde{\mathcal{Y}}_\lambda^{tw}\left(Y_{V_{L+\lambda}}(a, z_0)\theta\theta^{-1}u, z_2\right) \\
&= \tilde{\mathcal{Y}}_\lambda^{tw}\left(Y_{V_{L+\lambda}}(a, z_0)u, z_2\right)
\end{aligned}$$

Let us now start with the left-hand side of the Jacobi identity:

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{V_L^{T_{\chi(\lambda)}}}(a, z_1)\tilde{\mathcal{Y}}_\lambda^{tw}(u, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u, z_2)Y_{V_L^{T_\chi}}(a, z_1) \\
&= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u, z_2) \\
&\qquad\qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u, z_2)\left(\mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha\right) \\
&= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2) \otimes \eta_{\lambda+\beta}\right) \\
&\qquad\qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2) \otimes \eta_{\lambda+\beta}\right)\left(\mathcal{Y}_\alpha^{tw}(a, z_1) \otimes e_\alpha\right) \\
&= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a, z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2)\right) \otimes (e_\alpha \circ \eta_{\lambda+\beta}) \\
&\qquad\qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2)\mathcal{Y}_\alpha^{tw}(a, z_1)\right) \otimes (\eta_{\lambda+\beta} \circ e_\alpha) \\
&= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a, z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2)\right) \otimes (e_\alpha \circ \eta_{\lambda+\beta}) \\
&\qquad\qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2)\mathcal{Y}_\alpha^{tw}(a, z_1)\right) \otimes \left((-1)^{(\alpha,\lambda+\beta)} e_\alpha \circ \eta_{\lambda+\beta}\right) \\
&= \Bigg[z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{Y}_\alpha^{tw}(a, z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2) \\
&\qquad\qquad - (-1)^{(\alpha,\lambda+\beta)} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}_{\lambda+\beta}^{tw}(u, z_2)\mathcal{Y}_\alpha^{tw}(a, z_1)\Bigg] \otimes (e_\alpha \circ \eta_{\lambda+\beta})
\end{aligned}$$

$$= \left[\frac{1}{2} \sum_{p=0,1} z_2^{-1} \delta \left((-1)^p \frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+(-1)^p\alpha} \left(\mathcal{Y}_{(-1)^p\alpha, \lambda+\beta}(\theta^p(a), z_0)u, z_2 \right) \right]$$

$$\otimes (e_\alpha \circ \eta_{\lambda+\beta})$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+\alpha} \left(\mathcal{Y}_{\alpha, \lambda+\beta}(a, z_0)u, z_2 \right) \otimes (e_\alpha \circ \eta_{\lambda+\beta})$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta-\alpha} \left(\mathcal{Y}_{-\alpha, \lambda+\beta}(\theta(a), z_0)u, z_2 \right) \otimes (e_\alpha \circ \eta_{\lambda+\beta})$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+\alpha} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right) \otimes (\epsilon(\alpha, \lambda+\beta)\eta_{\lambda+\beta+\alpha})$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta-\alpha} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right) \otimes (\epsilon(-\alpha, \lambda+\beta)\eta_{\lambda+\beta-\alpha}) \quad (**)$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+(\beta+\alpha)} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right) \otimes \eta_{\lambda+(\beta+\alpha)}$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+(\beta-\alpha)} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right) \otimes \eta_{\lambda+(\beta-\alpha)}$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right)$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right)$$

$$= z_2^{-1} \frac{1}{2} \left[\delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) + \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \right] \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right) \quad (***)$$

$$= z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right)$$

Lines (**) and (***) follow from *Note 1* and *Note 2*, respectively, while the last equality follows from the fact that $\delta(z) = \frac{1}{2}\left[\delta(z^{1/2}) + \delta(-z^{1/2})\right]$. This completes the proof of the Jacobi identity.

□

Lemma 3.4.7 *The map $\tilde{\mathcal{Y}}_{\lambda}^{tw}$ satisfies the $L(-1)$-derivative property; that is:*

$$\tilde{\mathcal{Y}}_{\lambda}^{tw}(L(-1)u, z) = \frac{d}{dz}\tilde{\mathcal{Y}}_{\lambda}^{tw}(u, z)$$

Proof. Let $u \in M(1, \lambda + \beta) \subseteq V_{L+\lambda}$:

$$\begin{aligned}
\tilde{\mathcal{Y}}_{\lambda}^{tw}(L(-1)u, z) &= \mathcal{Y}_{\lambda+\beta}^{tw}(L(-1)u, z) \otimes \eta_{\lambda+\beta} \\
&= \left(\frac{d}{dz}\mathcal{Y}_{\lambda+\beta}^{tw}(u, z)\right) \otimes \eta_{\lambda+\beta} \\
&= \frac{d}{dz}\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u, z) \otimes \eta_{\lambda+\beta}\right) \\
&= \frac{d}{dz}\tilde{\mathcal{Y}}_{\lambda}^{tw}(u, z)
\end{aligned}$$

The second equation follows from Proposition 9.4.3 of [FLM].

□

Since $\tilde{\mathcal{Y}}_{\lambda}^{tw}$ is a non-trivial intertwining operator of type $\begin{pmatrix} & V_L^{T_{\chi^{(\lambda)}}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix}$ for V_L, we now have:

$$N\begin{pmatrix} & V_L^{T_{\chi^{(\lambda)}}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix} \geq 1$$

However, Prop. 3.2.2 (1) and Prop. 3.1.4 together imply that

$$N\begin{pmatrix} & V_L^{T_{\chi^{(\lambda)}}} & \\ V_{L+\lambda} & & V_L^{T_\chi} \end{pmatrix} = 1$$

Thus, together with Lemma 3.4.1, we have shown the following:

Proposition 3.4.8 *For any $\lambda \in S$ and any irreducible $\hat{L}/K$-module T_χ,*

$$V_{L+\lambda} \boxtimes V_L^{T_\chi} = V_L^{T_{\chi^{(\lambda)}}}$$

where $T_{\chi^{(\lambda)}}$ is an irreducible $\hat{L}/K$-module such that $\chi^{(\lambda)}(a) = (-1)^{\langle \lambda, \bar{a} \rangle}\chi(a)$, $\forall a \in \hat{L}/K$.

3.5 The Fusion Product $V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}}$

We now compute the fusion product of two V_L-modules of twisted type. Again, let M^i run over the set of equivalence classes of irreducible V_L-modules, then by definition:

$$\begin{aligned} V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}} &= \sum_{M^i} N_{V_L}\binom{M^i}{V_L^{T_{\chi_1}} V_L^{T_{\chi_2}}} M^i \\ &= \sum_{\lambda \in S} N_{V_L}\binom{V_{L+\lambda}}{V_L^{T_{\chi_1}} V_L^{T_{\chi_2}}} V_{L+\lambda} + \sum_{V_L^{T_{\chi_j}}} N_{V_L}\binom{V_L^{T_{\chi_j}}}{V_L^{T_{\chi_1}} V_L^{T_{\chi_2}}} V_L^{T_{\chi_j}} \end{aligned}$$

where $S = \{\lambda_1, \cdots, \lambda_k\}$ is the set of representatives of equivalence classes of L in its dual lattice L° and $V_L^{T_{\chi_j}}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules.

We begin by quoting here only a part of an important theorem from [ADL]:

Theorem 3.5.1 ([ADL], *Theorem 5.1*) *Let L be a positive-definite even lattice. For any irreducible V_L^+-modules $M^i (i = 1, 2, 3)$, the fusion rule of type $\binom{M^3}{M^1 M^2}$ is either 0 or 1. The fusion rule of type $\binom{M^3}{M^1 M^2}$ is 1 if and only if $M^i (i = 1, 2, 3)$ satisfy one of the following conditions:*

1. $M^1 = V_L^{T_\chi,+}$ *for an irreducible $\hat{L}/K$-module T_χ and (M^2, M^3) is one of the following pairs:*
   (a) $(V_{L+\lambda}, V_L^{T_{\chi^{(\lambda)}},\pm}), ((V_L^{T_{\chi^{(\lambda)}},\pm})', (V_{L+\lambda})')$ *for $\lambda \in L^o$ such that $2\lambda \notin L$*
2. $M^1 = V_L^{T_\chi,-}$ *for an irreducible $\hat{L}/K$-module T_χ and (M^2, M^3) is one of the following pairs:*
   (a) $(V_{L+\lambda}, V_L^{T_{\chi^{(\lambda)}},\pm}), ((V_L^{T_{\chi^{(\lambda)}},\pm})', (V_{L+\lambda})')$ *for $\lambda \in L^o$ such that $2\lambda \notin L$*

We now show the first lemma of this section:

Lemma 3.5.2 *Let* $\lambda \in S$. *If* χ_1 *and* χ_2 *are central characters of* $\hat{L}/K$ *such that* $\chi_2(a) = (-1)^{\langle \bar{a}, \lambda \rangle}\chi_1(a), \forall a \in \hat{L}$, *then:*

$$N_{V_L}\binom{V_{L+\lambda}}{V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2}}} = 1$$

Proof. By Theorem 5.1.4(a) of [ADL]:

$$N_{V_L^+}\binom{(V_{L+\lambda})'}{V_L^{T_{\chi_1},+} \quad (V_L^{T_{\chi_2},+})'} = 1$$

for $\lambda \in L$ such that $2\lambda \notin L$ and $\chi_2(a) = (-1)^{\langle \bar{a}, \lambda \rangle}\chi_1(a), \forall a \in \hat{L}$. We also refer to Proposition 3.7 of [ADL] for the following contragredient modules:

$$(V_{L+\lambda})' \cong V_{L-\lambda} \text{ and } (V_L^{T_{\chi_2},+})' \cong V_L^{T_{\chi_2'},+}$$

where $\chi_2'(a) = (-1)^{\langle \bar{a}, \bar{a} \rangle/2}\chi_2(a)$ for any $a \in \hat{L}$. So:

$$N_{V_L^+}\binom{V_{L-\lambda}}{V_L^{T_{\chi_1},+} \quad V_L^{T_{\chi_2'},+}} = 1$$

By Proposition 3.1.4,

$$N_{V_L}\binom{V_{L-\lambda}}{V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2'}}} \leq N_{V_L^+}\binom{V_{L-\lambda}}{V_L^{T_{\chi_1},+} \quad V_L^{T_{\chi_2'},+}} = 1$$

Now by the well-known symmetries of fusion rules (Prop. 3.1.3):

$$\begin{aligned}
N_{V_L}\begin{pmatrix} V_{L-\lambda} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2'}} \end{pmatrix} &= N_{V_L}\begin{pmatrix} (V_L^{T_{\chi_2'}})' \\ V_L^{T_{\chi_1}} \quad (V_{L-\lambda})' \end{pmatrix} \\
&= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2''}} \\ V_L^{T_{\chi_1}} \quad V_{L+\lambda} \end{pmatrix} \\
&= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2}} \\ V_L^{T_{\chi_1}} \quad V_{L+\lambda} \end{pmatrix} \\
&= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2}} \\ V_{L+\lambda} \quad V_L^{T_{\chi_1}} \end{pmatrix} \\
&= 1
\end{aligned}$$

In the above computation, the third equation follows from:

$$\begin{aligned}
\chi_2''(a) &= (-1)^{(\bar{a},\bar{a})/2}\chi_2'(a) \\
&= (-1)^{(\bar{a},\bar{a})/2}(-1)^{(\bar{a},\bar{a})/2}\chi_2(a) \\
&= \chi_2(a)
\end{aligned}$$

while the last equation (which is $N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2}} \\ V_{L+\lambda} \quad V_L^{T_{\chi_1}} \end{pmatrix} = 1$) follows from Section 3.4.

□

Lemma 3.5.3 *Let χ_1 and χ_2 be central characters of $\hat{L}/K$ and χ_i any central character of $\hat{L}/K$ such that $\chi_i(\kappa) = -1$. Then:*

$$N_{V_L}\begin{pmatrix} V_L^{T_{\chi_i}} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2}} \end{pmatrix} = 0$$

Proof. Let $\varepsilon_i \in \{\pm\}, i = 1, 2$, then:

$$\begin{aligned}
N_{V_L}\binom{V_L^{T_{\chi_i}}}{V_L^{T_{\chi_1}}\ \ V_L^{T_{\chi_2}}} &\leq N_{V_L^+}\binom{V_L^{T_{\chi_i}}}{V_L^{T_{\chi_1},\varepsilon_1}\ \ V_L^{T_{\chi_2},\varepsilon_2}} \text{ (by Prop. 3.1.4)} \\
&= N_{V_L^+}\binom{(V_L^{T_{\chi_2},\varepsilon_2})'}{V_L^{T_{\chi_1},\varepsilon_1}\ \ (V_L^{T_{\chi_i}})'} \text{ (by symmetries of fusion rules)} \\
&= N_{V_L^+}\binom{V_L^{T_{\chi_2'},\varepsilon_2}}{V_L^{T_{\chi_1},\varepsilon_1}\ \ V_L^{T_{\chi_i'}}} \\
&\leq N_{V_L^+}\binom{V_L^{T_{\chi_2'},\varepsilon_2}}{V_L^{T_{\chi_1},\varepsilon_1}\ \ V_L^{T_{\chi_i'},\varepsilon_i}} \text{ (by Prop. 3.1.4)} \\
&= 0
\end{aligned}$$

since all three are of twisted type (by Prop. 3.3.2(2)).

□

Thus, we have shown the following:

Proposition 3.5.4 *Let $\lambda \in L^\circ/L$. If χ_1 and χ_2 are central characters of $\hat{L}/K$ such that $\chi_2(a) = (-1)^{\langle \bar{a}, \lambda \rangle}\chi_1(a), \forall a \in \hat{L}$, then:*

$$V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}} = \sum_{\lambda^*} V_{L+\lambda^*}$$

where λ^* runs over the set $\{\lambda \in L^\circ/L \mid \chi_2(a) = (-1)^{\langle \bar{a}, \lambda \rangle}\chi_1(a), \forall a \in \hat{L}\}$.

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