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May 8, 1957

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ABSTRACT

Maxwell's equations in a relativistic, rotating reference frame are discussed by use of a covariant formalism.

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It has previously been recognized¹ that a rotating co-ordinate system is a simple yet nontrivial example of a reference frame that is curvilinear in space-time. The study of the geometry of such a co-ordinate system provides a convenient pedagogical device for illustrating the physical concepts that underlie many of the formal manipulations in the exposition of the theory of general relativity. Simultaneously, one may utilize the framework of the general theory in order to study a problem that has a certain intrinsic interest--the formulation of the equations of electrodynamics in a rotating frame of reference.

The basis of the discussion lies in the choice of a co-ordinate system in the rotating system. Following the procedure of Reference 1, we first choose a set of cylindrical co-ordinates, R, θ, Z, T in an inertial system, I_0 , whose origin coincides with the axis of rotation of the rotating system I . In I we have the corresponding set r, ϕ, z, t defined by

$$r = R, \quad \phi = \theta - \omega T, \quad z = Z, \quad t = T, \quad (1)$$

where ω is the angular velocity of I .

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

¹ C. Møller, The Theory of Relativity (Oxford, 1955).

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It should be recognized that the choice of a co-ordinate system for the rotating frame is quite arbitrary. In fact the only limitations imposed by the general theory of relativity are those of continuity and differentiability.² Consequently the particular form of the transformation equations (1) is chosen for its mathematical and conceptual simplicity. On the other hand, the transformation equations have in themselves no information concerning the rate of a clock or the length of a meter stick at some arbitrary point of the rotating system. Such quantities as these must be determined from the principle of covariance. On the basis of this principle one finds just the foreshortening and time dilation effects that would follow from the naive application of the special theory.³

We now proceed to construct the equation for the invariant space-time interval in I and to inspect the formal geometry of this reference frame. We denote the co-ordinates in the two frames, I_0 and I , respectively, by

$$X^\mu = R, \theta, Z, T \quad \text{and} \quad x^\mu = r, \phi, z, t,$$

which are related by⁴

$$dX^\mu = A_\nu^\mu dx^\nu, \quad A_1^1 = A_2^2 = A_3^3 = A_4^4 = 1, \quad A_4^2 = -\frac{i\omega}{c}. \quad (2a)$$

² Peter Bergmann, Introduction to the Theory of Relativity (Prentice-Hall, 1942, p. 158 ff.

³ Møller, op. cit., p. 240 ff.

⁴ We use summation convention for a repeated index that occurs once covariantly and once contravariantly. Greek indices run from 1 to 4, Latin indices from 1 to 3.

The inverse transformation has nonvanishing elements,

$$\overset{\vee}{A}_1^1 = \overset{\vee}{A}_2^2 = \overset{\vee}{A}_3^3 = \overset{\vee}{A}_4^4 = 1, \quad \overset{\vee}{A}_4^2 = +\frac{i\omega}{c}. \quad (2b)$$

Then

$$\begin{aligned} ds^2 &= g_{\mu\nu} dX^\mu dX^\nu \\ &= dR^2 + R^2 d\theta^2 + dz^2 - c^2 dt^2 \\ &= dr^2 + r^2 d\phi^2 + dz^2 - c^2 \left(1 - \frac{\omega^2 r^2}{c^2}\right) dt^2 + 2 \frac{r^2 \omega}{c} d\phi c dt, \end{aligned} \quad (3)$$

from which it follows that the metric tensor in I has nonvanishing components,

$$\begin{aligned} g_{11} = g_{22}/r^2 = g_{33} = 1, \quad g_{44} = \left(1 - \frac{\omega^2 r^2}{c^2}\right), \\ g_{24} = g_{42} = -\frac{i\omega r^2}{c}. \end{aligned} \quad (4a)$$

The determinant is found to be

$$g = \det g_{\mu\nu} = r^2, \quad (4b)$$

from which it follows that the inverse matrix, $g^{\mu\nu}$, is

$$\begin{aligned} g^{11} = g^{33} = g^{44} = 1, \quad g^{22} = (1/r^2) \left(1 - \frac{r^2 \omega^2}{c^2}\right), \\ g^{24} = g^{42} = i\omega/c. \end{aligned} \quad (4c)$$

For the sake of completeness we give the values of the Christoffel symbols. The nonvanishing elements are

$$\Gamma_{1,22} = -\Gamma_{2,21} = -r, \quad \Gamma_{1,44} = -\Gamma_{4,41} = r\omega^2/c^2, \quad (5a)$$

$$\Gamma_{1,24} = -\Gamma_{2,41} = -\Gamma_{4,21} = ir\omega/c,$$

$$\Gamma_{22}^1 = -r, \quad \Gamma_{44}^1 = r\omega^2/c^2, \quad \Gamma_{24}^1 = ir\omega/c, \quad (5b)$$

$$\Gamma_{21}^2 = 1/r, \quad \Gamma_{41}^2 = -i\omega/rc,$$

plus those obtainable from the above by the symmetry of the last two covariant indices.

The geometry of the system is characterized by the values of the curvature tensor, $R_{\mu\nu\lambda\sigma}$, which vanishes. This result implies the existence of a Cartesian co-ordinate system by which the rotating frame may be described. One finds that the transformation to Cartesian co-ordinates merely carries us from I back into I_0 .

The discussion of electrodynamics has its origin in the covariant form of Maxwell's equations. We first introduce the two antisymmetric tensors $H^{\mu\nu}$ and $F_{\mu\nu}$, which satisfy the equations^{5,6}

⁵ The comma denotes the ordinary partial derivative. It is interesting to note that the covariant Maxwell equations are "suprametric" in the sense that the metric tensor does not occur in them explicitly. To see this in the case of Eq. (6a) one must write it in terms of the tensor densities $g^{\frac{1}{2}} H^{\mu\nu}$ and $g^{\frac{1}{2}} j^{\nu}$.

⁶ Møller, op. cit., p. 302.

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$$g^{-\frac{1}{2}} (g^{\frac{1}{2}} H^{\mu\nu})_{,\mu} = j^{\nu} \quad (6a)$$

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0. \quad (6b)$$

In order to identify the components of these tensors with the field quantities \vec{E} , \vec{H} , \vec{D} , \vec{B} , we impose the requirement that the noncovariant Maxwell equations (i.e., $\nabla \cdot \vec{D} = \rho$, etc.) retain their usual form in the rotating system, I. This requirement will be satisfied if we have

$$H^{21} = H_z/r, \quad H^{13} = H_\phi, \quad H^{32} = H_r/r, \quad (7a)$$

$$H^{\lambda 4} = i c (D_r, D_\phi/r, D_z), \quad (7b)$$

$$j^{\nu} = \rho^0 U^{\nu}, \quad (7c)$$

where U^{λ} is the four-velocity of a small, charged region whose density of charge, measured in a local rest system of inertia, is ρ^0 . The components are

$$U^{\mu} = dx^{\mu}/d\tau = \{ \Gamma \vec{u}, i c \Gamma \}, \quad (7d)$$

$$\begin{aligned} \vec{u} &= d\vec{x}/dt \\ &= \left\{ 1 - v^2/c^2 \right\}^{-\frac{1}{2}} \end{aligned}$$

In the above expression for Γ , \vec{v} (as distinguished from $\vec{\mu}$, the "local" velocity) is the total velocity of the charge including the rotational velocity $r\omega$. Upon defining the charge density in I,

$$\rho = \Gamma \rho^0,$$

one easily verifies that Eq. (6a) takes the form of the two Maxwell equations that relate the fields \vec{D} and \vec{H} to the charge and current densities.

The components of the tensor $F_{\mu\nu}$ are now chosen to be

$$F_{21} = r B_z, \quad F_{13} = B_\phi, \quad F_{32} = r B_r, \quad (8a)$$

$$F_{\kappa\lambda} = +(i/c) \{ E_r, \quad r E_\phi, \quad E_z \}. \quad (8b)$$

Substitution into Eq. (6b) then gives the remaining two Maxwell equations in their usual forms.

The constitutive equations are obtained from the relation

$$H_{\mu\nu} \equiv \epsilon_{\mu\lambda} \epsilon_{\nu\tau} H^{\lambda\tau} = (1/\mu_0) F_{\mu\nu},$$

which is valid in I_0 , and, as a tensor equation, must also be valid in I .

Putting this relation in terms of the components gives

$$\vec{B} = \mu_0 \left[\vec{H} + (\vec{\omega} \times \vec{r}) \times \vec{D} \right], \quad (9a)$$

$$\epsilon_0 \vec{E} = \vec{D} + (1/c^2)(\vec{\omega} \times \vec{r}) \times \left[\vec{H} + (\vec{\omega} \times \vec{r}) \times \vec{D} \right]. \quad (9b)$$

Equation (9b) may be put into the esthetically more pleasing form

$$\vec{D} = \epsilon_0 \left[\vec{E} - (\vec{\omega} \times \vec{r}) \times \vec{B} \right]. \quad (9c)$$

It is well to remark at this point that the physical significance of the field quantities is not all all obvious. There was a certain arbitrariness in the way we related their components to the tensor components $F_{\mu\nu}$ and $H^{\mu\nu}$ in the system I . The only constraint that one need consider is that the definitions of \vec{E} , \vec{H} , \vec{D} , and \vec{B} reduce to the usual ones in

the limit of vanishing ω . In order to understand the significance of these fields we must consider the equations of motion for a test particle viewed from the moving frame.

In order to do this we consider the electromagnetic four-force

$$\mathcal{F}_\mu = (\gamma \vec{\mathcal{F}}, \mathcal{F}_4) = -e F_{\mu\nu} U^\nu,$$

which leads to the usual form

$$\vec{\mathcal{F}} = e \left[\vec{E} + \vec{u} \times \vec{B} \right]. \quad (10a)$$

The fourth component of the force is

$$\mathcal{F}_4 = \frac{ie}{c} \gamma \vec{u} \cdot \vec{E}. \quad (10b)$$

One sees that the \vec{E} and \vec{B} fields retain their status as electric and magnetic field intensities. It is only the relationships of these fields to the charge and current distributions by which they are produced that have undergone change.

This last point is readily illustrated by a simple example. Consider a circular cylinder of infinite length with surface charge density and axis coincident with the axis of rotation. If the cylinder appears to be at rest with respect to the moving frame we may apply Gauss's law and Ampere's rule, both of which hold by virtue of the invariance of the noncovariant equations. Introducing the unit vectors \vec{a}_1 , we find

$$\vec{D} = \vec{a}_r \frac{\sigma}{2\pi r}, \quad \vec{H} = \vec{H}_0$$

outside the cylinder, and

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$$\vec{D} = 0, \quad \vec{H} = \vec{H}_0$$

on the inside. Consideration of Eq. (9) and the boundary condition that the \vec{B} and \vec{E} fields must tend to zero for large r then leads to

$$\vec{B} = 0, \quad E = \frac{\sigma}{2\pi \epsilon_0 r} \vec{a}_r \quad (A1)$$

on the outside and

$$\vec{B} = \mu_0 \frac{\sigma \omega}{2} \vec{a}_z, \quad \vec{E} = (\vec{\omega} \times \vec{r}) \times \vec{B} \quad (A2)$$

on the inside.

To proceed with the discussion of particle dynamics we must resort to the covariant form of Newton's law:⁸

$$\mathcal{F}_\mu = d/d\tau P_\mu - \frac{1}{2} g_{\mu\lambda,\nu} U^\nu P^\lambda. \quad (11)$$

The four-momentum is defined by

$$P^\lambda = m_0 U^\lambda,$$

with m_0 the rest mass of a charged particle. After appropriate algebraic manipulations Eq. (11) results in

$$\vec{\mathcal{F}} = m_0 \gamma^2 \left\{ \vec{a} + \frac{1}{2} \vec{v} \gamma^2 \frac{d}{dt} \frac{v^2}{c^2} \right\}. \quad (12a)$$

$$\mathcal{F}_4 = i(m_0/2c) \frac{d}{dt} v^2. \quad (12b)$$

⁸ Møller, op. cit., p. 295.

The quantity \vec{a} is defined by

$$\vec{a} = \vec{a}_r [\ddot{r} - r(\omega + \dot{\phi})^2] + \vec{a}_\phi [r\ddot{\phi} + 2\dot{r}(\omega + \dot{\phi})] + \vec{a}_z \ddot{z} . \quad (12c)$$

Combining Eqs. (10) and (12) gives the final result,

$$\vec{a} = (e/m_0) \Gamma^{-1} \left\{ \vec{E} + \vec{u} \times \vec{B} - \Gamma^4 (\mu \cdot \vec{E}) \vec{v}/c^2 \right\} . \quad (13)$$

We now use Eq. (13) to obtain the equation of motion in terms of the fields measured in the rest system, I_0 . Given a charge and current density (\vec{j}' , ρ') in the rest system, together with their \vec{D}' and \vec{H}' fields, we may use the transformation coefficients (2b) to obtain

$$\vec{j} = \vec{j}' - (\vec{\omega} \times \vec{r}) \rho' ; \quad \rho = \rho' , \quad (14)$$

and

$$\vec{H} = \vec{H}' - (\vec{\omega} \times \vec{r}) \times \vec{D}' , \quad \vec{D} = \vec{D}' . \quad (15)$$

Then in consequence of Eqs. (9) the field intensities become

$$\vec{B} = \mu_0 \vec{H}' = B' , \quad (16a)$$

$$\epsilon_0 \vec{E} = \vec{D}' + (1/c^2)(\vec{\omega} \times \vec{r}) \times \vec{H}' , \quad (16b)$$

or

$$\vec{E} = \vec{E}' + (\vec{\omega} \times \vec{r}) \times \vec{B}' . \quad (16c)$$

Of course Eq. (16) could instead have been obtained by a direct transformation.

The motion of a charge as measured in the rotating system is now given by substituting Eq. (16) into the force law Eq. (13a), which gives

$$\vec{a} = (e/m_0) \gamma^{-1} \left\{ \vec{E}' + \vec{v} \times \vec{B}' - \frac{1}{2} \gamma^4 \frac{\vec{u} \cdot [\vec{E}' + (\vec{\omega} \times \vec{r}) \times \vec{B}']}{c^2} \vec{v} \right\} \quad (17)$$

It seems worth while to emphasize that the simple form of the result, Eq. (17), is an immediate consequence of the simple form of the transformation equations (1). In fact, the equations derived in this paper might even be thought of as being nonphysical in the sense that they do not give, directly, the results of measurements carried out with local yardsticks and clocks. On the other hand, the equations do provide a convenient mathematical framework for solving problems in a rotating frame. Comparison with experiment may then be achieved by transforming theoretical and experimental results into a common reference system.

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