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THEORY OF COLLECTIVE VIBRATIONS OF
EVEN-EVEN SPHEROIDAL NUCLEI

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UC-34 Physics
TID-4500(17th Ed.)

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Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

THEORY OF COLLECTIVE VIBRATIONS
OF EVEN-EVEN SPHEROIDAL NUCLEI

Eugene R. Marshalek

(Ph. D. Thesis)

March 1962

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ERRATUM

TO: All recipients of UCRL-10046 UC-34 Physics

FROM: Technical Information Division

Subject: UCRL-10046, "Theory of Collective Vibrations of Even-Even Spheroidal Nuclei," Eugene R. Marshalek (Ph. D. Thesis), March 1962.

Please make the following correction on subject report.

Page 109, Eq. (IV-19b)

The factor $\left(1 - \frac{3\hbar^2}{\mathfrak{J}/\hbar\omega_\gamma}\right)^2$ should read: $\left(1 - \frac{2\hbar^2}{\mathfrak{J}/\hbar\omega_\gamma}\right)^2$

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Please make the following corrections on subject report.

Page 187 Place footnote on last sentence and statement at bottom of page:
"In general, the inertial parameters are functions of the $\alpha_{\lambda\mu}$, and are restricted by the requirement that the Hamiltonian be rotationally invariant. For spherical nuclei, this restriction makes the inertial parameters independent of μ . In addition, the parameters are evaluated at the equilibrium distortions in the lowest approximation."

Page 192 Eq. (B-29)(c): first term on right-hand side should read

$$\frac{-\hbar^2}{2B_\gamma} \left(\frac{1}{\gamma} \frac{\partial}{\partial \gamma} \gamma \frac{\partial}{\partial \gamma} - \frac{\hat{I}_3^2}{4\gamma^2} \right).$$

Page 209 Eq. (B-51) (a); Term in parenthesis should be

$$(D_{M,2+\Omega}^I \chi_\Omega + (-1)^{I+\Omega} D_{M,-(2+\Omega)}^I R_1 \chi_\Omega).$$

Eq. (B-51) (b); Term in parenthesis should be

$$(D_{M,2-\Omega}^I \chi_{-\Omega} + (-1)^{I+\Omega} D_{M-(2-\Omega)}^I R_1 \chi_{-\Omega}).$$

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ERRATA

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<u>Page</u>	<u>Line</u>	<u>Correction</u>
2	Title	Should read "Theory of Collective Vibrations of Even-Even Spheroidal Nuclei"
2	last	Add page number: 241
29	15	Change "doubtful" to "possible in the Sawada method"
36	3 from bottom	Change "therefore used" to "suggested"
59	3	Add footnote: "Equation (II-78) is valid for the λ th mode, corresponding to a λ pole- λ pole force by replacing $\hat{Q}_{20}$ by $\hat{Q}_{\lambda 0}$."
82	Eq. (III-2)	Delete superscript + from $a_{-\mu}^+$ (two places)
128	6 from bottom	Delete $\frac{1}{2} \frac{\partial^2 E_c}{\partial r^2}$
133	3 from bottom	Change "percentage" to "fractional."
134	2	Change "percentage" to "fractional."
141c	Diagram	Change $\frac{1}{2} - [705]$ to $\frac{1}{2} [750]$
178	4 from bottom	Delete sentence "Because each... in the variation)."
241	5	Change "Huss" to "Huus"
245	at bottom	Add 92. a: A. K. Kerman, Ann. Phys. <u>12</u> , 300 (1961).
246	10	Page number should be 610 (not 620)

THEORY OF SURFACE VIBRATIONS OF EVEN-EVEN DEFORMED NUCLEI

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THEORY OF COLLECTIVE VIBRATIONS OF EVEN-EVEN SPHEROIDAL NUCLEI

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March 1962

ABSTRACT

A version of the Inglis "cranking model" is presented which describes nuclear shape oscillations without the adiabatic approximation. The collective frequencies of a system of particles moving in a time-dependent shell-model potential are obtained as solutions of a dispersion equation usually associated with the random phase approximation (RPA). The method used is equivalent to an extension of the cranking model first devised by Araujo. It is shown that the form of the dispersion equation follows from a suitable definition of "vibrational spectrum" in a system with coupled collective and intrinsic degrees of freedom. It is also shown that the generalized cranking model is equivalent to the time-dependent self-consistent field method applied to an appropriate long-range two-body force.

Extensive numerical applications are made to quadrupole vibrations of even-even deformed nuclei, using Nilsson's single particle eigenvalues and asymptotic wave functions, and taking residual pairing interactions into account by means of the Bogolyubov canonical transformation method. Vibrational frequencies, mass

parameters, force constants, reduced electric quadrupole transition probabilities, and vibration-rotation interaction coefficients of the " β " and " γ " quadrupole modes are theoretically estimated for rare earth and actinide isotopes, and compared with experimental values.

THEORY OF COLLECTIVE VIBRATIONS OF EVEN-EVEN SPHEROIDAL NUCLEI

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March 1962

I. INTRODUCTION*

In the strong-coupling limit, applicable to nuclei with large equilibrium deformations, the Bohr-Mottelson model predicts, in addition to a ground state rotational band with spin sequence $I = 0^+, 2^+, 4^+, \dots$, rotational bands based on one-phonon surface vibrations.¹ In the case of quadrupole vibrations, one expects a " β -vibrational" band with spin sequence $I = 0^+, 2^+, 4^+, \dots$ and angular momentum projection on the intrinsic Z axis $K = 0$, and a " γ -vibrational" band with $I = 2^+, 3^+, 4^+, \dots$ and $K = 2$. In addition, odd-parity octupole vibrations have been experimentally discovered, but these will not be specifically discussed in the present paper. The β mode is one which preserves axial symmetry of the nucleus, while the γ mode corresponds to an instantaneous deviation from an axially symmetric equilibrium shape. The existence of bands with quadrupole properties has been experimentally verified in numerous deformed

*A summary of basic concepts underlying the present work is given in Appendix B for the convenience of the nonspecialist.

nuclei in the rare earth and actinide regions;^{2, 3, 4, 5} a typical case is shown in Fig. I-1. Furthermore, in some of the cases, reduced electric quadrupole transition probabilities $B(E2; 0 \rightarrow 2^+)$ for exciting the 2^+ member of vibrational bands have been measured, and indicate by their enhancement over a single-particle estimate that the bands are based on collective rather than intrinsic excitations.^{6, 7, 8}

Recently, however, Davydov and Filipov and co-workers have suggested that the observed states can be classified in a different way if certain approximations made by Bohr and Mottelson are avoided.^{9, 10} In particular, if the assumption that nuclei have axially symmetric equilibrium shapes is removed, the $2^+, 3^+, 4^+ \dots$ sequence of a γ -vibrational band of an axial nucleus can instead be interpreted as a ground state rotational sequence of a non-axial nucleus. Since there is no conclusive experimental evidence, and since no realistic calculations have been made to indicate that the nuclei in question do in fact possess axially symmetric equilibrium shapes, the Davydov generalizations warrant full consideration as contenders to the more venerable classifications. This is especially so since this model and its variants, such as the asymmetric rotor model developed by Malmann and co-workers,¹¹ have been surprisingly successful in correlating nuclear data over a wide range.

Unfortunately, neither the Bohr-Mottelson model nor the Davydov-Filipov model can predict apriori absolute energy spacings and transition probabilities of collective states, although both have achieved considerable success in predicting the corresponding ratios and spin sequences. Indeed, it may be possible to distinguish

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Fig. I-1. Level scheme of Sm^{152} . The bands are, from left to right, ground-state rotational band, β -vibrational band, γ -vibrational band, and an octupole vibrational state. The energies are in kev.

between the models on the basis of absolute properties of collective spectra. These properties depend on the inertial parameters that enter into the collective equations of motion; i.e., on the effective rotational moments of inertia, the vibrational mass parameters and restoring force constants. And these, in turn, depend on the internal motion of nucleons coherently correlated by long-range residual forces to produce collective properties. It is well known that the classical irrotational hydrodynamics cannot account for these inertial parameters, usually underestimating them, often by an order of magnitude.⁵

It is one of the chief purposes of this work to test roughly the assumptions of the axial symmetry model of Bohr and Mottelson by calculating quadrupole vibrational inertial parameters. The internal motion of nucleons is described by a variant of the Inglis¹² cranking model. In addition, residual short-range correlations producing a superfluid state are taken into account by means of the Bardeen, Cooper, Schrieffer-Bogolyubov theory.*^{13, 14}

Recently, Griffen and Rich¹⁵ and Nilsson and Prior¹⁶ have applied the cranking model to the calculation of rotational moments of inertia of deformed nuclei, achieving remarkable success with the aid of the BCS-Bogolyubov theory. Kissinger and Sorensen¹⁷ have applied the cranking model in the adiabatic approximation to collective vibrations of spherical single closed-shell nuclei, also

*This will be referred to as the BCS-Bogolyubov theory.

including pairing correlations, with fair success. A similar technique has very recently been applied to γ vibrations of spheroidal rare earth isotopes by Bes.¹⁸

While the present work was in progress, new developments in the nuclear application of theories of collective excitations of many-body systems were published by Arvieu and Veneroni,¹⁹ Baranger,²⁰ Thouless,²¹ Brown et al.,²² Marumori,²³ and Tamura and Udagawa,^{24, 25} using methods essentially equivalent to the random-phase approximation (RPA),^{26, 27} or the method of Sawada^{*28, 29, 30} previously applied to the electron gas. These applications, mainly limited to spherical single closed-shell nuclei, show that a vibrational spectrum can arise as a consequence of the long-range quadrupole-quadrupole component of the residual internucleon force. The cranking model, on the other hand, does not prove that a vibrational spectrum exists in the presence of a long-range force, but rather assumes the existence of such a spectrum and pursues the consequences. Hence, the method used here is not as fundamental as the others, although it is a convenient calculational tool. However, it is a somewhat surprising fact that the Inglis cranking model, with its quasi-classical assumption, can reproduce some of the main results of the more rigorous methods, as will be shown.

*Also referred to as the method of linearized equations of motion.

The theoretical description of collective motion is given in Ch. II, and the application of the BCS-Bogolyubov theory is discussed in Ch. III. Applications to quadrupole vibrations of deformed nuclei are made in Ch. IV, and results discussed in Ch. V.

II. INTERACTION BETWEEN COLLECTIVE AND INTRINSIC MOTION

A. The Adiabatic Cranking Model

The calculation of collective inertial parameters by means of the Inglis "cranking model" in the adiabatic approximation has been discussed in detail by Inglis,^{12, 31-33} Bohr and Mottelson,³⁴ Villars³⁵ and Moszkowski.^{36, 37} The essential idea of the method is the description of the interaction between the intrinsic structure and collective vibrations as a classical time dependence of the deformation parameters appearing in the deformed shell model field, which is an approximation to a self-consistent field in the Hartree-Fock sense. The explicit time dependence, treated quantum-mechanically, gives rise to a perturbative increment corresponding to the excitation energy of a collective state. Assuming that the collective frequencies are much smaller than the intrinsic frequencies, the above authors use the adiabatic perturbation theory.

Araujo first extended the Inglis model to the case in which the collective motion is not necessarily adiabatic.^{38, 39} However, his formalism was specialized to treat a three-dimensional time-dependent square-well potential, and the discussion centered about vibrations of closed-shell nuclei. The time-dependent method to be presented here is equivalent in principle to that of Araujo, but the resulting formulas will be presented in a more general, closed form. A simple time-independent approach leading to the same results will also be presented. However, this approach differs from the time-independent derivation of the rotational moment of inertia, where a transformation

to a rotating coordinate system fixed in the nucleus eliminates the explicit time dependence from the equations of motion. In contrast to the vibrational moments, the rotational moment of inertia appears to be unchanged by deviations from adiabaticity, as long as perturbation theory is valid.

According to Inglis,¹² the mass parameter in the adiabatic approximation is given by

$$B_{\lambda\mu} = 2\hbar^2 \sum_i \frac{|\langle 0 | \frac{\partial}{\partial \alpha_{\lambda\mu}} | 0 \rangle|^2}{E_i - E_0} \quad (\text{II-1})$$

where $\alpha_{\lambda\mu}$ is a collective coordinate corresponding to the μ^{th} normal mode of multipolarity λ .^{*} The energy denominators and matrix elements refer to eigenstates of the shell model field when the $\alpha_{\lambda\mu}$ have their equilibrium values. The classical kinetic energy of vibrations is given by

$$T = \frac{1}{2} \sum_{\lambda\mu} B_{\lambda\mu} |\dot{\alpha}_{\lambda\mu}|^2 \quad (\text{II-2})$$

and the collective potential energy by

^{*}The $\alpha_{\lambda\mu}$ may be taken as the expansion coefficients in Eq. (B-15), or any set of parameters proportional to these.

$$V = \frac{1}{2} \sum_{\lambda\mu} C_{\lambda\mu} |\alpha_{\lambda\mu} - \alpha_{\lambda\mu}(0)|^2. \quad (\text{II-3})$$

The restoring force constant $C_{\lambda\mu}$ in the adiabatic model is defined as the "surface energy." This is obtained by calculating the total energy of nucleons in the deformed shell-model field $E(\alpha_{\lambda\mu})$, as a function of deformation and then finding $C_{\lambda\mu} = \frac{\partial^2 E}{\partial \alpha_{\lambda\mu} \partial \alpha_{\lambda\mu}^*}$ at the equilibrium point $\alpha_{\lambda\mu} = \alpha_{\lambda\mu}(0)$.

Inglis considered the case of particles moving in a pure harmonic-oscillator potential well.¹² In this case the magnitude of the mass parameter (II-1) for quadrupole vibrations is only 1/4 of the irrotational liquid drop model value, whereas experiment indicates that quadrupole mass parameters are of the order of 10 times the irrotational value.¹² The corresponding frequency of collective oscillation according to Inglis exceeds the single-particle frequency by a factor of 2, thus contradicting the adiabatic assumption on which the calculation is based.¹²

Moszkowski has shown that this contradiction can be averted if one uses suitable linear combinations of the basis wave functions chosen to satisfy a "self-consistency" criterion.³⁶ Specifically, he requires that the anisotropy of the shell-model potential, measured by the distortion parameters $\alpha_{\lambda\mu}$, coincide with the anisotropy of the density distribution of the particles moving in the potential. In the framework of the adiabatic model, this constraint can be

satisfied by introducing a Lagrangian multiplier as has been done by Kisslinger and Sorensen,¹⁷ and Bès.¹⁸ According to Moszkowski, the mass parameter must then have at least the irrotational value, and the collective frequency cannot exceed the intrinsic frequency of the assumed shell-model potential.

One of the results of this paper is that this self-consistency condition can always be applied without recourse to an extra parameter in the form of a Lagrangian multiplier. Indeed, the self-consistency condition determines the collective frequency, if properly applied, the frequency being the only parameter introduced.

It is of interest to describe briefly the physical picture of adiabatic vibrations as discussed by Moszkowski,^{36, 37} for future reference.* It is assumed that as the deformation parameters change with time, not only are the single-particle wave functions transformed, but also the configuration of nucleons continuously changes so that the nucleus is always in its lowest energy state. One may consider the abscissa of a Nilsson⁴¹ diagram, for example, as a function of time. (Fig. B-1). At each instant there exists a configuration that minimizes the energy of the nucleus, which largely consists of a sum of the energies of the occupied shell-model orbitals given by the Nilsson correlation diagram. If one plots the sum of single-particle energies as a function of a deformation parameter, the result is a scalloped curve as shown in Fig. II-1. The discontinuities in the curve are due to changes in occupation as cross-overs between levels occur. The

*See also Hill and Wheeler.⁴⁰

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Fig. II-1. Nuclear deformation energy. The total energy of nucleons moving in a shell model field as a function of a deformation coordinate Q is shown schematically.

equilibrium deformation is given by the absolute minimum in the curve and the adiabatic restoring force constant for the mode in question by the quantity $\frac{\partial^2 E}{\partial \alpha^2}$ evaluated at the minimum.

In the presence of residual short-range pairing interactions, the discontinuities in the curve are smoothed out, since the occupation changes continuously due to blurring of the Fermi distribution. Of course, the nucleus cannot be exactly in a stationary state because of the assumed explicit time dependence, but rather the nuclear wave function describing the collective vibrational mode is a packet centered about the configuration which minimizes the energy at a given instant. Moreover, the wave packet must describe the density distribution in such a way that the self-consistency condition of Moszkowski is satisfied for each value of the deformation (i.e., at each instant).

Since experiment indicates that the collective excitations identified as quadrupole vibrations in the regions of interest do not generally satisfy the adiabatic condition, it is desirable to seek a method that does not depend on it, as will be done in the next section. Even when the error made by estimating a collective frequency with the adiabatic approximation is only of the order of 10%, the corresponding error in the force constant and mass parameter may be of the order of 20-40%. The sensitivity of these parameters to non-adiabatic effects is important in calculating transition probabilities which are proportional to $(B C_{\mu})^{-\frac{1}{2}}$.

B. The Cranking Model Without the Adiabatic Approximation

The starting point of the cranking model is a model potential which is considered to be an approximation to a deformed self-consistent field generated by lower harmonics of a two-body internucleon interaction. The model Hamiltonian has the form of a sum of single-particle operators describing the shell-model field, and in addition, residual short-range two-body interactions corresponding to higher harmonics of the internucleon force may be included. It is usually assumed that such a residual interaction is independent of deformation. The Hamiltonian is

$$H_{crank} \equiv H(\vec{r}_1, \vec{p}_1; \vec{r}_2, \vec{p}_2; \dots; \vec{\sigma}_1, \vec{\sigma}_2; \dots; \alpha(t)), \quad (\text{II-4})$$

H_{crank} is a many-body Hamiltonian depending on $3N$ position coordinates and conjugate momenta, and corresponding spin coordinates,* and $\alpha(t)$ is a deformation parameter which, in a more rigorous treatment, would appear as a quantum-mechanical dynamic variable, but which appears as an explicit function of time in the "quasi-classical" method.[†] Attention will be focused on a single vibrational mode although the treatment is easily generalized to several independent modes. It is furthermore assumed that the volumes enclosed by equipotential surfaces of the single-particle shell-model field are

*The "isotopic spin" formalism will not be used in this paper, since only heavy nuclei will be discussed.

[†]For convenience, α will be considered real here. No loss in generality of the final results ensues.

independent of α . This condition describes approximately the incompressibility of nuclear matter.

In order to treat harmonic vibrations, expand the Hamiltonian (II-4) to second order about the equilibrium deformation :

$$H_{\text{crank}} \cong H(\alpha_0) + \left. \frac{\partial H}{\partial \alpha} \right|_{\alpha=\alpha_0} (\alpha - \alpha_0) + \frac{1}{2} \left. \frac{\partial^2 H}{\partial \alpha^2} \right|_{\alpha=\alpha_0} (\alpha - \alpha_0)^2. \quad (\text{II-5})$$

The time dependence of α is assumed to be

$$\alpha - \alpha_0 = A \sin(\omega t + \delta), \quad (\text{II-6})$$

where A is the amplitude and ω is the frequency of collective oscillation. The choice of phase is irrelevant, so it is assumed that $\delta = 0$.

The problem is to solve the time-dependent Schrödinger equation to order A^2 , subject to the self-consistency condition of Moszkowski previously discussed. It is easier to impose instead the condition that the perturbative increment in energy* be independent of time, which reduces to the Moszkowski condition under certain circumstances. Moreover, the criterion of time independence is a partial

* Strictly speaking, $\langle \psi_{\text{crank}}(\alpha) | H_{\text{crank}}(\alpha) | \psi_{\text{crank}}(\alpha) \rangle$ is not the energy. If the deformed self-consistent field is generated by two-body interactions, the energy is not $\langle T + V_{\text{SCF}} \rangle$, but $\langle T + \frac{1}{2} V_{\text{SCF}} \rangle$. However, in dealing with energy differences it can be shown that corrections for 2-body interactions are not important.

compensation for using a non-conservative description of truly stationary states. It is required that

$$\frac{\partial}{\partial t} \langle \psi_{\text{crank}}(t) | H_{\text{crank}} | \psi_{\text{crank}}(t) \rangle = \langle \psi_{\text{crank}}(t) | \frac{\partial}{\partial t} H_{\text{crank}} | \psi_{\text{crank}}(t) \rangle \equiv 0,$$

(II-7)*

where $\psi_{\text{crank}}(t)$ is the solution of the Schrödinger equation corresponding to the Hamiltonian, H_{crank} . Expand H_{crank} in powers of α to obtain the condition:

$$\langle \psi_{\text{crank}}(t) | \frac{\partial H}{\partial \alpha} | \psi_{\text{crank}}(t) \rangle_{\alpha=0} + \alpha(t) \langle \psi_{\text{crank}}(t) | \frac{\partial^2 H}{\partial \alpha^2} | \psi_{\text{crank}}(t) \rangle_{\alpha=0} \equiv 0.$$

(II-8)

It is easy to show that if the deformation-dependent part of the shell-model potential is a volume-conserving harmonic oscillator as in the Nilsson model,⁴¹ (II-8) coincides with Moszkowski's self-consistency criterion. For example, for β deformation of a harmonic oscillator shell-model field, one gets, from Eqs. (B-21) and (B-48) (Appendix B),

* The condition (II-6) is necessary to satisfy (II-7).

$$\left. \frac{\partial H}{\partial \beta} \right|_{\beta=0} = -m\omega_0^2 \sum_i r_i^2 Y_{20}(\theta_i), \quad (a) * \quad (II-9)$$

$$\left. \frac{\partial^2 H}{\partial \beta^2} \right|_{\beta=0} = \frac{5}{4\pi} m\omega_0^2 \sum_i r_i^2 \quad (b)$$

and (II-8) leads to

$$\begin{aligned} & \langle \psi_{crank}(t) | \sum_i r_i^2 Y_{20} | \psi_{crank}(t) \rangle \\ &= \frac{5}{4\pi} \beta(t) \langle \psi_{crank}(t) | \sum_i r_i^2 | \psi_{crank}(t) \rangle \\ &\approx \frac{5}{4\pi} \beta(t) \langle 0 | \sum_i r_i^2 | 0 \rangle + O(\beta^2) + \dots, \end{aligned} \quad (II-10)$$

which coincides exactly with the condition given in references (36) and (37), and guarantees that the quadrupole moment of the shell-model potential at each instant coincides with the mass quadrupole

* Y_{20} is the normalized spherical harmonic.

$\dagger |0\rangle$ is the unperturbed ground state.

moment described by $\psi_{\text{crank}}(t)$.^{*} Other choices of single-particle potential lead to different radial dependences of $\frac{\partial H}{\partial \beta}$, but (II-10) holds to a good approximation if volume conservation is satisfied.

The Schrödinger equation to be solved is

$$H_{\text{crank}}(\alpha(t)) \psi_{\text{crank}}(t) = i\hbar \frac{\partial}{\partial t} \psi_{\text{crank}}(t). \quad (\text{II-11})$$

*Assuming a uniform density distribution within the boundaries of a prolate spheroidal nucleus, the mass quadrupole moment is related to the eccentricity parameter β by

$$\langle \sum_i r_i^2 Y_{20} \rangle = \frac{3}{4\pi} A R_0^2 \beta + \dots,$$

where R_0 is the nuclear radius.

The parameter ω_0 of the harmonic oscillator well is chosen to satisfy the condition

$$\frac{\langle \sum_i r_i^2 \rangle}{A} = \frac{3}{5} R_0^2,$$

So that Eq. (II-10) implies that the parameter β appearing in the deformed potential bears the same relation to the mass quadrupole moment as in the uniform density model.

Expand $\psi_{crank}(t)$ in terms of the complete set of eigenfunctions of the unperturbed Hamiltonian $H(\alpha_0)$:

$$\psi_{crank}(t) = \sum_j a_j(t) \Phi_j(\alpha_0) e^{-i E_j(\alpha_0) t / \hbar}, \quad (II-12)$$

where

$$H(\alpha_0) \Phi_j(\alpha_0) = E_j(\alpha_0) \Phi_j(\alpha_0). \quad (II-13)$$

The a_j rigorously satisfy the equations

$$\frac{da_j}{dt} = \dot{a}_j = -\frac{i}{\hbar} \sum_k a_k \langle j | H_{crank}(t) - H(\alpha_0) | k \rangle e^{i(E_j - E_k)t/\hbar} \quad (II-14)$$

where the matrix elements are taken between the $\Phi_j \equiv |j\rangle$.

The usual Rayleigh-Schrödinger time-dependent perturbation theory is then applied.⁴² The expansion (II-5) is used for H_{crank} . Expand the a_j as follows:

$$a_j = a_j^{(0)} + a_j^{(1)} A + a_j^{(2)} A^2 + \dots, \quad (a)$$

(II-15)

$$\dot{a}_j = \dot{a}_j^{(0)} + \dot{a}_j^{(1)} A + \dot{a}_j^{(2)} A^2 + \dots \quad (b)$$

The expansion is valid for small amplitudes, A. Substituting Eqs. (II-15) into Eqs. (II-14) and equating coefficients of like powers of A leads to the set of decoupled equations which can be solved successively:

$$\dot{a}_j^{(0)} = 0, \quad (a)$$

$$\dot{a}_j^{(1)} = -\frac{i}{\hbar} \sum_{\kappa} a_{\kappa}^{(0)} \langle j | H'(t) | \kappa \rangle e^{i\omega_{j\kappa} t}, \quad (b)$$

(II-16)

$$\begin{aligned} \dot{a}_j^{(2)} = & -\frac{i}{\hbar} \sum_{\kappa} a_{\kappa}^{(1)} \langle j | H'(t) | \kappa \rangle e^{i\omega_{j\kappa} t} \\ & -\frac{i}{\hbar} \sum_{\kappa} a_{\kappa}^{(0)} \langle j | H''(t) | \kappa \rangle e^{i\omega_{j\kappa} t}, \end{aligned} \quad (c)$$

where $H'(t) \equiv \frac{1}{A} \frac{\partial H}{\partial \alpha} \Big|_{\alpha=\alpha_0} (\alpha - \alpha_0)$
 and $H''(t) \equiv \frac{1}{2A^2} \frac{\partial^2 H}{\partial \alpha^2} \Big|_{\alpha=\alpha_0} (\alpha - \alpha_0)^2$
 and $\omega_{j\kappa} \equiv (E_j - E_{\kappa})/\hbar$.

Integrating Eqs. (II-16) gives:

$$a_j^{(0)} = \text{constant}, \quad (a)$$

$$a_j^{(1)} = -\frac{i}{\hbar} \sum_k a_k^{(0)} \int \langle j | H'(t) | k \rangle e^{i\omega_{jk}t} dt + C_j^{(1)}, \quad (b)$$

(II-17)

$$a_j^{(2)} = -\frac{i}{\hbar} \sum_k \int a_k^{(1)} \langle j | H'(t) | k \rangle e^{i\omega_{jk}t} dt - \frac{i}{\hbar} \sum_k a_k^{(0)} \int \langle j | H''(t) | k \rangle e^{i\omega_{jk}t} dt + C_j^{(2)}. \quad (c)$$

It is necessary to impose the condition that as the amplitude of the vibration, A , vanishes, Ψ_{crank} reduces to the unperturbed state.

Hence it is required that

$$a_i^{(0)} = \delta_{i0}. \quad (II-18)$$

In addition, it is necessary that the first-order perturbation energy vanish, since the collective excitation is proportional to A^2 . Hence it is necessary that

$$\langle 0 | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0} = 0, \quad (II-19)$$

where $|0\rangle \equiv \Phi_0$, the unperturbed state.

According to Moszkowski,³⁶ this condition is satisfied at equilibrium. It is just the self-consistency requirement (II-7) applied to static deformations. Hence, if α_0 is chosen as the equilibrium deformation so that (II-19) is satisfied, then it is possible to satisfy the dynamic condition (II-7), as will be shown.

The requirement that all first-order terms vanish, which follows from the assumption (II-7), can be shown to necessitate that the constants of integration $C_j^{(1)}$ vanish:

$$C_j^{(1)} = 0. \quad (\text{II-20})$$

The only second-order coefficient needed in the calculation is $a_0^{(2)}$, obtained from the requirement that $\psi_{\text{crank}}(t)$ be normalized to second order:

$$\sum_j |a_j|^2 = 1 + A(a_0^{(1)} + a_0^{(1)*}) + A^2(a_0^{(2)} + a_0^{(2)*} + \sum_{j \neq 0} |a_j^{(1)}|^2) + \dots \quad (\text{II-21})$$

$$\equiv 1.$$

The coefficient of the A term vanishes by Eqs. (II-17b), (II-18), and (II-19), and the requirement that the coefficient of A^2 vanish gives

$$a_0^{(2)} + a_0^{(2)*} = - \sum_{j \neq 0} |a_j^{(1)}|^2, \quad (\text{II-22})$$

which is all that is needed.

The integrated mixing amplitudes are:

$$a_j = - \frac{\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}}{(E_j - E_0)^2 - (\hbar\omega)^2} \left[(\alpha - \alpha_0)(E_j - E_0) + i\hbar\dot{\alpha} \right] e^{i\omega_{j0}t} \quad (\text{II-23})$$

$j \neq 0$

$$= \frac{i}{2} A \langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0} \cdot \left[\frac{e^{i\omega t}}{E_j - E_0 + \hbar\omega} - \frac{e^{-i\omega t}}{E_j - E_0 - \hbar\omega} \right] \cdot e^{i\omega_{j0}t},$$

$$\text{Re } a_0 = \frac{1}{2} (a_0 + a_0^*) = 1 - \frac{1}{2} \sum_{j \neq 0} \frac{| \langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}^2 \left[(E_j - E_0)^2 (\alpha - \alpha_0)^2 + \hbar^2 \dot{\alpha}^2 \right]}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2}. \quad (\text{II-24})$$

Calculating the expectation value of H_{crank} gives an expression of the form:*

$$\begin{aligned} \langle \Psi_{\text{crank}}(t) | H_{\text{crank}} | \Psi_{\text{crank}}(t) \rangle &= E_0 + \frac{B(\omega)}{2} \dot{\alpha}^2 + \frac{C(\omega)}{2} (\alpha - \alpha_0)^2 \\ &= E_0 + \frac{B(\omega)}{2} \omega^2 A^2 \cos^2 \omega t + \frac{C(\omega)}{2} A^2 \sin^2 \omega t. \end{aligned} \quad (\text{II-25})$$

The requirement that (II-25) be independent of time is equivalent to requiring that

$$\omega^2 B(\omega) = C(\omega), \quad (\text{II-26})$$

leading to the dispersion formula

$$\sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|_{\alpha=\alpha_0}^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} = \frac{1}{2} \langle 0 | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \quad (\text{II-27})$$

*The evaluation of $\langle \Psi_{\text{crank}}(t) | H_{\text{crank}} | \Psi_{\text{crank}}(t) \rangle$ is quite straightforward and details are given in Appendix A.

determining the possible collective frequencies . The mass parameter is given by

$$B(\omega) = 2\hbar^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|_{\alpha=\alpha_0}^2 (E_j - E_0)}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2} \quad (\text{II-28})$$

The restoring force constant $C(\omega)$ is determined trivially by Eq. (II-26) once Eq. (II-27) is solved for ω and $B(\omega)$ evaluated from Eq. (II-28).

In case of several modes of vibration, one expands the crank-ing-model Hamiltonian as follows:

$$\begin{aligned} H_{\text{crank}}(\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)) &= H(\alpha_1(0), \alpha_2(0), \dots, \alpha_n(0)) \\ &+ \sum_{\mu} \left. \frac{\partial H}{\partial \alpha_{\mu}} \right|_{\substack{\alpha_1=\alpha_1(0) \\ \dots \\ \alpha_n=\alpha_n(0)}} (\alpha_{\mu}(t) - \alpha_{\mu}(0)) \\ &+ \frac{1}{2} \sum_{\mu, \nu} \left. \frac{\partial^2 H}{\partial \alpha_{\mu} \partial \alpha_{\nu}} \right|_{\substack{\alpha_1=\alpha_1(0) \\ \dots \\ \alpha_n=\alpha_n(0)}} (\alpha_{\mu}(t) - \alpha_{\mu}(0)) (\alpha_{\nu}(t) - \alpha_{\nu}(0)) \\ &+ \dots \end{aligned} \quad (\text{II-29})$$

If the modes are all independent, the time dependence is

$$\alpha_{\mu} - \alpha_{\mu}(0) = A_{\mu} \sin \omega_{\mu} t \quad (\text{II-30})$$

The perturbation treatment gives a set of equations of the form (II-27) and (II-28) for the independent frequencies ω_μ and the mass parameters B_μ . For two modes to be independent, it is necessary that $\langle 0 | \frac{\partial H}{\partial a_\mu} | j \rangle \langle j | \frac{\partial H}{\partial a_\nu} | 0 \rangle = 0$ for all j . If H_{crank} is expressed initially in terms of coupled collective coordinates, one can introduce suitable normal coordinates to find the frequencies of the normal modes by this method.

Finally it should be mentioned that the time-dependent method can be generalized to anharmonic vibrations by expanding H_{crank} beyond second order and suitably modifying the time dependence in (II-30). Needless to say, the extension beyond second-order perturbation theory is tedious, and will not be treated in the present work. However, it would be interesting to see whether the correspondence between the Sawada method and cranking model persists to higher order. This seems doubtful as long as exchange terms are neglected.

Equation (II-27) has solutions $\hbar\omega$ sandwiched between successive values of $E_j - E_0$. The behavior of the sum in (II-27) as a function of ω is shown in Fig. II-2. Denoting the true ground state by $|\tilde{0}\rangle$, the lowest solution $\hbar\omega$ is a collective excitation built on $|\tilde{0}\rangle$.* The lowest solution corresponds to a depression of the excitation energy of the lowest intrinsic state $|j\rangle_{\text{min}}$ by the collective correlations.

*The true ground state includes zero-point oscillations.

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Fig. II-2. Graphical solution of the dispersion formula.

If $\hbar\omega \ll (E_j - E_0)_{\min}$, where $(E_j - E_0)_{\min}$ is the smallest energy difference, Equation (II-27) may be expanded to give

$$\sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{E_j - E_0} \alpha = \alpha_0 + (\hbar\omega)^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{(E_j - E_0)^3} + \dots = \frac{1}{2} \langle 0 | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \quad (\text{II-31})$$

The adiabatic approximation consists in dropping terms in the expansion higher than ω^2 . The frequency in the adiabatic method is

$$\hbar\omega_{(\text{adiabatic})} = \sqrt{\frac{\frac{1}{2} \langle 0 | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} - \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{E_j - E_0}}{\sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{(E_j - E_0)^3}}} \quad (\text{II-32})$$

It is clear from (II-31) that

$$\hbar\omega \leq \hbar\omega_{\text{adiabatic}}. \quad (\text{II-33})$$

The mass parameter can be expanded to give

$$B(\omega) = 2\hbar^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{(E_j - E_0)^3} \Big|_{\alpha=\alpha_0} + (\hbar\omega)^2 4\hbar^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{(E_j - E_0)^5} \Big|_{\alpha=\alpha_0} \dots \quad (\text{II-34})$$

The first term is just the mass parameter of the adiabatic approximation, and higher order terms represent an increase in B due to deviation from adiabaticity.

$$B_{(\text{adiabatic})} \equiv B(0) = 2\hbar^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2}{(E_j - E_0)^3} \Big|_{\alpha=\alpha_0}. \quad (\text{II-35})$$

The inequality holds that

$$B(\omega) \geq B(0) \quad (\text{II-36})$$

Equation (II-35) agrees with (II-1) because of the relation*

$$\langle i | \frac{\partial}{\partial \alpha} | j \rangle = \frac{\langle i | \frac{\partial H}{\partial \alpha} | j \rangle}{E_j - E_i} \quad \text{for } i \neq j \quad (\text{II-37})$$

From Eqs. (II-32), (II-35) and the relation $B^2 = C$, one gets the adiabatic result

$$C_{\text{adiabatic}} \equiv C(\omega) = \langle 0 | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} - 2 \sum_j \frac{|\langle i | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2}{E_j - E_0}. \quad (\text{II-38})$$

Also, it can be shown that

$$C(\omega) \geq C(0) \quad (\text{II-39})$$

*This can be seen as follows:

$$\frac{\partial}{\partial \alpha} H \Phi_j = \frac{\partial H}{\partial \alpha} \Phi_j + H \frac{\partial \Phi_j}{\partial \alpha} = E_j \frac{\partial \Phi_j}{\partial \alpha} + \frac{\partial E_j}{\partial \alpha} \Phi_j,$$

where the Φ_j are eigenfunctions of H with eigenvalues E_j and form a

complete orthogonal set. Hence $\langle \Phi_i | \frac{\partial H}{\partial \alpha} | \Phi_j \rangle = E_j \langle \Phi_i | \frac{\partial}{\partial \alpha} | \Phi_j \rangle + \langle \Phi_i | H | \frac{\partial \Phi_j}{\partial \alpha} \rangle$.

. Applying the expansion $\frac{\partial \Phi_j}{\partial \alpha} = \sum_k \langle \Phi_k | \frac{\partial}{\partial \alpha} | \Phi_j \rangle \Phi_k$,

and substituting in the last term above, gives (II-37).

Equation (II-38) can also be obtained by applying time-independent perturbation theory to H_{crank} , treating $\alpha - \alpha_0$ as a static deformation. The second-order energy is then given by

$$\frac{C(0)(\alpha - \alpha_0)^2}{2}.$$

In order for the frequency formula (II-27) to have a real solution ω , it is necessary that

$$\sum_j \frac{| \langle j | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle |^2}{E_j - E_0} \leq \frac{1}{2} \langle 0 | \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha = \alpha_0} \quad (\text{II-40})$$

If the equality sign holds, $\omega = \omega_{(\text{adiabatic})} = 0$, and $C = C(0) = 0$. When (II-40) is not satisfied, so that ω is imaginary, vibrations about $\alpha = \alpha_0$ are not stable, as can be seen from the fact that the exponentials in Eq. (II-23) become real so that the coefficients grow with time.

Nuclei in the weak-coupling limit are expected to have real solutions corresponding to a spherical equilibrium shape, while those in the strong-coupling limit should have solutions corresponding to deformed equilibrium shapes. In the intermediate coupling region, a perturbation treatment of the particle-surface coupling is expected to break down and the frequency formula (II-27) is probably invalid, leading most likely to imaginary solutions. In this case, it is best to perform an exact diagonalization of the appropriate multipole component of the internucleon force.

In case real solutions ω are very low-lying compared to the lowest intrinsic excitation, i.e., in the case when the adiabatic

approximation is very good, it may be necessary to consider anharmonicity. If the amplitude A is chosen as proportional to the quantum-mechanical root mean square value $\sqrt{\frac{\hbar}{B\omega}} = \sqrt{\frac{\hbar}{B\omega}}$, then as $\omega \rightarrow 0$, $A \rightarrow \infty$. Hence, the frequency formula may not be accurate for small solutions.

On the other hand, as $\omega \rightarrow (E_j - E_0)_{\min}$, A remains finite.

$$\omega \rightarrow (E_j - E_0)_{\min}; A \rightarrow \text{small number} \neq 0. \quad (\text{II-41})$$

If the limits are taken properly, the wave function $\psi_{\text{crank}}(t)$ (and the energy), remain finite in both cases, providing that $A \propto \sqrt{\frac{\hbar}{B\omega}}$. To take the limit properly, it is necessary to consider that ω is a function of the magnitude of the right-hand side of Eq. (II-27), and of the coefficient of $\frac{\partial H}{\partial \alpha}$.

In fact, one considers ω to be a function of a strength parameter χ , which is introduced in Section D of this chapter, in taking the limits.

As a final point in this section, an alternative representation for the perturbation treatment will be discussed. The alternative consists of an expansion of the time-dependent wave function $\psi_{\text{crank}}(t)$, not in terms of eigenfunctions of the equilibrium Hamiltonian $H(\alpha_0)$ as in (II-12), but in terms of instantaneous eigenfunctions of $H_{\text{crank}}(\alpha(t))$, where t is considered as a parameter:

$$\psi_{\text{crank}}(t) = \sum_n C_n(t) u_n(\alpha(t)) e^{-i \int_0^t \frac{E_n(\alpha(t'))}{\hbar} dt'}$$

(II-42)

and

$$H_{\text{crank}}(\alpha(t)) u_n(\alpha(t)) = E_n(\alpha(t)) u_n(\alpha(t)) .$$

(II-43)

This is the representation usually used in the adiabatic perturbation theory,⁴³ and leads directly to the form (II-1) for $B(0)$. As the adiabatic limit is approached, one term in the expansion (II-42) becomes dominant, namely the instantaneous "ground state" $u_0(\alpha(t))$. The wave function (II-42) is thus a wave packet of the sort described in the Part A of this chapter.

The expansion (II-42) is particularly convenient for describing time-dependent square wells, since one must satisfy finite boundary conditions at each instant of time. An expansion in terms of functions which already satisfy the boundary conditions thus facilitates the solution. This method was therefore used by Araujo in treating time-dependent square wells or box potentials,^{38, 39} and is equivalent to the method presented in this work.

C. A Time-Independent Approach

It will now be shown that the form of Eq. (II-27) does not depend on the quasiclassical description, but follows from a plausible definition of "vibrational spectrum." In addition, a prescription will be given for deriving the results of the previous section from a time-independent Hamiltonian with augmented degrees of freedom.

Assume that a many-body Hamiltonian can be written in the following form by means of a suitable contact transformation:

$$H = H_{intr} + H_{vibr} + H_{coupl}, \quad (II-44)$$

where $H_{intr} = H_{intr}(\pi_1, \xi_1; \pi_2, \xi_2; \dots \pi_n, \xi_n)$ (a)

describes the intrinsic motion in terms of coordinates $\xi_1, \xi_2, \xi_3, \dots$, which are not necessarily the usual coordinates in real space, and, in fact, should rigorously be fewer than the total degrees of freedom because of the presence of collective coordinates.*

and $H_{vibr} = \frac{|\pi_\alpha|^2}{2B} + \frac{C}{2}(\alpha - \alpha_0)^2$ is a Hamiltonian (b)

for collective vibrations, characterized by a collective coordinate α and conjugate momentum π_α .

and $H_{coupl} = H_{coupl}(\xi_1, \xi_2, \dots \xi_n, \alpha)$ is a (c)

term that couples the intrinsic and collective degrees of freedom.

*The π_n are momenta conjugate to the corresponding ξ_n .

It is customary to say that (II-44) will give rise to an approximately vibrational collective spectrum if H_{coupl} is small; in particular, if it vanishes, the spectrum is exactly harmonic. The question arises whether it is possible to have energy spacings of $\hbar\omega$ even in the presence of a nonvanishing H_{coupl} . The problem, then, is to find the condition under which H_{coupl} produces no shift in the energy spacings to a given order of perturbation theory. If such a condition can be satisfied, one may then say that the spectrum is vibrational.

It is probably too stringent to carry the requirement beyond 2nd-order perturbation theory because anharmonic effects should be allowed. It will be required then that energy shifts proportional to A^2 should vanish. Any deviations from the harmonic spectrum then would be due to anharmonicity.

Expand H_{coupl} about the equilibrium deformation q_0 :

$$H_{\text{coupl}}(\xi_1, \xi_2, \dots, \xi_n, q) = H_{\text{coupl}}(\xi_1, \xi_2, \dots, \xi_n, q_0) + \frac{\partial H_{\text{coupl}}}{\partial q} \bigg|_{q=q_0} (q - q_0) + \frac{1}{2} \frac{\partial^2 H_{\text{coupl}}}{\partial q^2} \bigg|_{q=q_0} (q - q_0)^2 + \dots \quad (\text{II-45})$$

The first term on the right can be included in H_{intr} . The basis wave functions have the form $\chi(\xi_1, \xi_2, \dots, \xi_n, q_0) \phi_n(q - q_0)$, where

$$H_{intr.} \chi_i = E_i \chi_i, \quad (a)$$

and

$$H_{vibr.} \varphi_n = (n + \frac{1}{2}) \hbar \omega \varphi_n, \quad (b) \quad (II-46)$$

and

$$\hbar \omega = \hbar \sqrt{\frac{C}{B}} \quad (c)$$

The φ_n are just the familiar harmonic-oscillator wave functions having the form $e^{-\frac{B\omega}{2\hbar}(\alpha-\alpha_0)^2} H_n(\alpha-\alpha_0)\sqrt{\frac{B\omega}{\hbar}}$, where H_n is an Hermite polynomial of n^{th} degree. The ground state is denoted by $\chi_0 \varphi_0$; The first-order contribution of H_{coupl} vanishes since $\langle \varphi_n | \alpha - \alpha_0 | \varphi_n \rangle$ vanishes. The following harmonic-oscillator matrix elements are needed:

$$\langle \varphi_n | \alpha - \alpha_0 | \varphi_{n-1} \rangle = \sqrt{\frac{n}{2}} \frac{\hbar \omega}{C} , \quad (a)$$

$$\langle \varphi_n | \alpha - \alpha_0 | \varphi_{n+1} \rangle = \sqrt{\frac{(n+1)}{2}} \frac{\hbar \omega}{C} , \quad (b) \quad (II-47)$$

$$\langle \varphi_n | (\alpha - \alpha_0)^2 | \varphi_n \rangle = \left(n + \frac{1}{2}\right) \frac{\hbar \omega}{C} . \quad (c)$$

The second-order energy shift of the ground state is given by

$$\Delta \mathcal{E}_0 = \frac{1}{2} \frac{\hbar \omega}{C} \left\{ \sum_j \frac{\left| \langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle \right|^2}{(E_0 - E_j - \hbar \omega)} + \frac{1}{2} \langle 0 | \frac{\partial^2 H_{\text{coupl}}}{\partial \alpha^2} | 0 \rangle \right\} . \quad (II-48)$$

The matrix elements in Eq. (II-48) are taken between intrinsic states χ_0 and χ_j . The ground state has an admixture of the

one-phonon state. The perturbation admixes the two-phonon and ground states to the one-phonon state. The second-order energy shift of an n-phonon state is

$$\begin{aligned} \Delta \mathcal{E}_{(n)} = \frac{\hbar\omega}{C} \left\{ \frac{(n+1)}{2} \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle|^2_{\alpha=\alpha_0}}{E_0 - E_j - \hbar\omega} \right. \\ + \frac{n}{2} \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle|^2_{\alpha=\alpha_0}}{E_0 - E_j + \hbar\omega} \\ \left. + (n + \frac{1}{2}) \langle 0 | \frac{1}{2} \frac{\partial^2 H_{\text{coupl}}}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \right\}. \end{aligned} \quad (\text{II-49})$$

The excitation energy of an n-phonon state relative to the ground state is

$$\begin{aligned} \mathcal{E}_{(n)} - \mathcal{E}_{(0)} = n\hbar\omega + \frac{n\hbar\omega}{C} \left\{ \langle 0 | \frac{1}{2} \frac{\partial^2 H_{\text{coupl}}}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \right. \\ \left. - \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle|^2_{\alpha=\alpha_0} (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} \right\}. \end{aligned} \quad (\text{II-50})$$

The condition that H_{coupl} must not disturb the spectrum to second order requires that the term in brackets on the right must vanish, leading to

$$\sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle|^2_{\alpha=\alpha_0} (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} = \frac{1}{2} \langle 0 | \frac{\partial^2 H_{\text{coupl}}}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0}. \quad (\text{II-51})$$

Thus one arrives at a dispersion formula of the same form as (II-27), although the formulation does not specify H_{coupl} nor the intrinsic states χ_j . The formulation merely states that if a Hamiltonian can be transformed into the form (II-44), it will have a harmonic spectrum if and only if the intrinsic states, the coupling term, and vibrational parameters satisfy certain conditions:

- (1) The coupling term can be treated by perturbation theory
- (2) To a given order of perturbation theory, and for a particular selection of H_{coupl} and H_{intr} , ω is chosen to satisfy a dispersion relation, which, to second order, is given by (II-51).

Of course, if H_{coupl} vanishes, (II-51) is trivially satisfied for any ω . On the other hand, any approximate transformation method which leads to a Hamiltonian of the form (II-44), but does not satisfy at least the second-order condition (II-51), cannot be considered as a transformation leading to a vibrational spectrum. Whenever H_{coupl} does not vanish, (II-51) represents a non-trivial consistency test.

It is interesting to note that this result can be obtained in still another way. Suppose that for a given system H_{intr} and H_{coupl} are known, but that the inertial parameters B and C are not known. Treat B and C , or, alternatively, ω and C independently as variational parameters, requiring that

$$\frac{\partial}{\partial C} (\epsilon_n - \epsilon_0) = 0 \quad (a)$$

and

(II-52)

$$\frac{\partial}{\partial \omega} (\epsilon_n - \epsilon_0) = 0 \quad (b)$$

The first condition leads immediately to (II-51). The second gives

$$\left\{ 1 + \frac{1}{C} \left[\langle 0 | \frac{1}{2} \frac{\partial^2 H_{\text{coupl}}}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} - \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} \right] \right. \\ \left. - 2 \frac{(\hbar\omega)^2}{C} \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2 (E_j - E_0)}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2} \right\} = 0. \quad (II-53)$$

Since the first term in square brackets on the right vanishes and $C = B^2$, one gets

$$B = 2\hbar^2 \sum_j \frac{|\langle j | \frac{\partial H_{\text{coupl}}}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2 (E_j - E_0)}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2} \quad (II-54)$$

This has the same form as the expression for the mass parameter (II-28). The variational derivation given here differs from the standard technique in that the varied parameters appear in the Hamiltonian as well as in the trial wave function. However, the technique is not physically implausible since the parameters B and C are determined self-consistently by the internal state of the system. In fact, in the BSC method of treating superconductivity, one also varies a set of parameters introduced formally into the Hamiltonian, as will be shown later.

Comparison with the time-dependent cranking method suggests a description of a many-body system by means of a Hamiltonian with superfluous degrees of freedom. If one chooses for H_{intr} the unperturbed shell-model Hamiltonian $H(\alpha_0)$ of the previous section, and for H_{coupl} the deformation dependent part of H_{crank} , then the results (II-51) and (II-54) will coincide with the cranking model results (II-27) and (II-28) respectively. Thus, form the extended Hamiltonian.

$$H_{ext} = H_{crank} + H_{vibr}, \quad (II-55)$$

where $H_{crank} = H(\alpha_0) + \frac{\partial H}{\partial \alpha}(\alpha - \alpha_0) + \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2}(\alpha - \alpha_0)^2$; however, this time $(\alpha - \alpha_0)$ is not considered as explicitly time-dependent but as a quantum-mechanical dynamic variable;

$$\text{and } H_{vibr} = \frac{\pi \dot{\alpha}^2}{2B} + \frac{C}{2}(\alpha - \alpha_0)^2.$$

Since H_{crank} has all the degrees of freedom of the system, H_{ext} has an extra degree of collective freedom. Nevertheless,

formulations with superfluous coordinates have frequently been used to describe collective motion. Although application of conditions (II-52) to H_{ext} leads to the same frequency equation and mass parameter as the cranking model, the extra degree of freedom engenders spurious states. To every eigenstate $|\kappa\rangle$ of $H(\alpha_0)$ corresponds an excitation with energy $\hbar\omega_\kappa$,* which represents a depression of the intrinsic state by the long-range force. There are no other non-spurious states. However, H_{ext} allows all the eigenstates of $H_{\text{intr}}(\alpha_0)$ and all the eigenstates of H_{vibr} .

In case of several normal modes of vibration, H_{crank} depends on several collective coordinates as in Eq. (II-29); for H_{vibr} , one uses the many-dimensional oscillator

$$H_{\text{vibr}} = \sum_{\mu} \left[\frac{|\pi_{\alpha_{\mu}}|^2}{2B_{\mu}} + \frac{C_{\mu}}{2} (\alpha_{\mu} - \alpha_{\mu}^{(0)})^2 \right]. \quad (\text{II-56})$$

One chooses the ω_{μ} and C_{μ} as variational parameters leading to the appropriate frequency equation for each mode. To treat anharmonicity, where the anharmonic corrections are small, H_{crank} is expanded beyond second order, and H_{vibr} is assumed to be an anharmonic oscillator, as for example,

$$H_{\text{vibr}} = \sum_{\mu} \left[\frac{|\pi_{\alpha_{\mu}}|^2}{2B_{\mu}} + \frac{C_{\mu}}{2} (\alpha_{\mu} - \alpha_{\mu}^{(0)})^2 + D_{\mu} (\alpha_{\mu} - \alpha_{\mu}^{(0)})^3 + \dots \right] \quad (\text{II-57})$$

* $\hbar\omega_{\kappa}$ is the κ^{th} solution of Eq. (II-51).

The eigenfunctions of H_{vibr} can be obtained if the anharmonic terms can be treated as perturbations, as well as the higher order contributions of H_{coupl} . The anharmonicity parameters, such as D_μ , are to be considered as additional variational parameters, so that the number of variational parameters increases with the order of the perturbation.

The prescription given in this section, involving an unconventional variational technique and the use of extra degrees of freedom is, admittedly, ad hoc. It is interesting to note that it can give the same secular equation as more established methods, but additional justification would be needed to permit the prescription to stand on its own. At this time, no attempt will be made to provide a justification starting from basic principles, except qualitative plausibility arguments.

As previously mentioned, variation of parameters which are formally introduced into the Hamiltonian is not a strange concept, if the parameters are not determined externally but depend on the internal configuration of a many-body system. Such a variation could be justified as follows. One might start with a many-body Hamiltonian, depending on ordinary space and spin coordinates, and independent of the variational parameters in question. One could choose some variational wave function, containing the parameters in question, in order to carry out a conventional variational calculation. However, it may be that the calculation would be facilitated by performing a canonical transformation of the Hamiltonian and the trial wave function, which would not alter the expectation value of the Hamiltonian, the transformation containing the variational parameters. In Ch. III, for example,

the Bogolyubov transformation facilitates the use of the BCS trial wave function, and introduces variational parameters into the Hamiltonian.

The use of extra degrees of freedom poses greater difficulties. However, various methods are available making use of such "auxiliary" or "redundant" variables.^{27, 35, 44} Some of the methods employ additional constraints to compensate for the extra variables, and some do not. The constraints affect mainly the intrinsic spectrum. Some methods use a combination of canonical transformation, extra degrees of freedom, and variational requirements. Perhaps justification for the prescription of this section might be provided by application of one of these methods.

One could also present some physical justification from the point of view of the Bohr-Mottelson model, which employs a Hamiltonian of the form (II-44), but with a different interpretation. In the Bohr-Mottelson model, the collective degrees of freedom are attributed to an even-even core, and the intrinsic degrees of freedom to outer "valence" nucleons, which are coupled to the core. Thus, the Bohr-Mottelson Hamiltonian has too few degrees of freedom, in that the intrinsic degrees of freedom of the core are neglected. In the case of spherical nuclei, the core is composed of closed shells and the outer nucleons are those outside of closed shells. In the case of deformed nuclei, the shell structure breaks down, and such a division is artificial. However, in the case of odd-A deformed nuclei, the odd nucleon can be considered as coupled to a rotating, vibrating deformed even-even core (see Eqs. (B-42) through (B-46) of Appendix B). The effect of the coupling terms to lowest order is to renormalize the collective inertial parameters.

A plausible extension of the Bohr-Mottelson picture is to consider every particle of the nucleus as moving in a shell-model potential and coupled to a core provided by all other particles, or, neglecting correction for self-interaction, consider every particle as coupled to the same core, which is composed of all the particles. Such a model has been exploited by Inglis, who considered the interaction of a single nucleon, moving in a deformed well, coupled to a quantum-mechanical rotor.³³ The effect of the coupling, which formally corresponds to a Coriolis interaction, is to renormalize the moment of inertia of the rotor by an amount equal to the contribution of a single nucleon to the moment of inertia in the cranking model. The total moment of inertia is then just a sum over single nucleon contributions and is thus identical to the original cranking model result.³¹

An analogous approach could be used to obtain the vibrational parameters. One could consider, let us say, the i^{th} particle moving in a potential well, coupled to a quantum-mechanical collective oscillator corresponding to a core composed of all particles but the i^{th} . If the coupling can be treated as a perturbation, one obtains an energy shift which can be interpreted as due to renormalization of the mass parameter and force constant by the addition of the i^{th} particle. The problem is more complicated than the rotational one since two parameters are involved. A possible approach is to use variational wave functions of the form

$$\psi = \sum_{nj} a_{nj} \varphi_{(n)}(\alpha) \psi_j(\xi), \quad (\text{II-58})$$

where ψ_j are eigenfunctions of the single-particle Hamiltonian and the $\varphi_{(n)}(\alpha)$ are harmonic oscillator wave functions, corresponding to a mass parameter B_{eff} and force constant C_{eff} which are to be considered as independent variational parameters. The form of the coefficients a_{nj} in (II-58) can be obtained from perturbation theory. The renormalization of the inertial coefficient by the coupling of the i th particle can then be defined as

$$\begin{aligned} \delta C_i &= C_{\text{eff}} - C_{\text{core}}, \\ \delta B_i &= B_{\text{eff}} - B_{\text{core}}. \end{aligned} \quad (\text{II-59})$$

The total inertial coefficients would then be the sum of the renormalizations

$$\begin{aligned} C &= \sum_i \delta C_i, \\ B &= \sum_i \delta B_i, \\ \omega &= \sqrt{\frac{C}{B}}. \end{aligned} \quad (\text{II-60})$$

Actually, such a calculation would be somewhat messy, especially if one insisted that each particle be coupled to a different core, and will not be attempted here. The method previously suggested in this chapter is more straightforward and not very different physically. Each particle is considered to be coupled to the same vibrating core, which already is characterized by the total renormalized inertial coefficients. Thus, the constraints given by (II-52) require that the renormalization effects should not be counted twice.

D. Relation of the Cranking Model to Other Methods

Time-Dependent Self-Consistent Field Method

It was previously pointed out that the constraint given by Eq. (II-8) of the cranking model amounts to a self-consistency condition given by Moszkowski.³⁶ It will now be shown in detail that the cranking model procedure using a volume-conserving single-particle model potential is equivalent to applying the time-dependent self-consistent field (TDSCF) method, also known as the time-dependent Hartree method,⁴⁵ to an appropriate residual two-body interaction. The TDSCF method has been applied previously to density fluctuations of an electron gas by Zyrianov and Eleonski,⁴⁶ Ehrenreich and Cohen,⁴⁷ and Goldstone and Gottfried,⁴⁸ who showed that the method can reproduce many of the results of the more rigorous Sawada technique,^{28, 29, 30} including the dispersion formula for the frequencies of the fluctuations. It has also been applied to nuclear breathing mode vibrations (expansion and contraction of the nuclear volume) by Ferrell,⁴⁹ and to nuclear collective rotation by Shono and Tanaka.⁵⁰

The starting point of the TDSCF method is the following variational principle⁵¹

$$\delta \langle \psi | H - i\hbar \frac{\partial}{\partial t} | \psi \rangle = 0 .$$

(II-61)

If Ψ is varied, the Euler-Lagrange equation corresponding to Eq. (II-61) is just the time-dependent Schrödinger equation for a Hamiltonian H . Thus, the variational principle holds exactly for the true wave function Ψ . However, one can use a trial wave function composed of several independent variational parameters to satisfy the above principle. Insofar as the choice of trial wave function is judicious, it may approximate the true wave function. Suppose H is a many-body Hamiltonian having the form

$$H = \sum_i (T(i) + V(i)) + \sum_{\substack{i,j \\ i > j}} v(i,j), \quad (\text{II-62})$$

where the first sum represents independent motion of particles in a shell-model potential well, and the second sum is a residual long-range two-body interaction. Moreover, it is assumed that $v(i,j)$ is a "well-behaved" function so that it is amenable to a Hartree type of approach. The single-particle well may also be of two-body origin, calculated, perhaps, by Brueckner theory from a realistic two-nucleon force, or it may be a phenomenological shell-model well. At any rate, it will first be assumed that the two-body origin of the single-particle well need not be explicitly considered. The residual force might be interpreted as a pseudo potential in Brueckner theory (see Appendix B, section 1), but this point will be of no further concern here.

A possible trial wave function satisfying Eq. (II-61) is a Slater determinant composed of single-particle orbitals generated by treating $v(i,j)$ by the time-independent Hartree method (see Appendix B, Section 4). However, it is well known that this method does not

take into account all the correlations introduced by the force $v(i,j)$. The additional correlations may be treated by considering fluctuations about the time-independent Hartree solutions.⁵¹ In the TDSCF method one uses as a trial wave function a superposition of Slater determinants composed of single-particle orbitals which are solutions of the time-independent Hartree (or Hartree-Fock) method:

$$\Psi(t) = \sum_n a_n(t) \Phi_n, \quad (\text{II-63})$$

where Φ_0 is the configuration with lowest energy, and the others are excited states obtained by promoting any number of particles above the Fermi level. This is equivalent to using as trial wave function a single Slater determinant composed of time-dependent single-particle orbitals $u_n(i,t)$, where i refers to the coordinates of the i^{th} particle, and each of the u_n is varied independently. The variational principle leads to the time-dependent Hartree-Fock equations for the u_n :

$$H_{\text{TDSCF}}(i,t) u_n(i,t) = i\hbar \frac{\partial}{\partial t} u_n(i,t), \quad (\text{II-64})$$

where

$$H_{\text{TDSCF}}(i,t) \equiv T(i) + V(i) + \sum_m \int u_m^*(j,t) v(i,j) u_m(j,t) d\tau_j - \sum_m \frac{\int u_n^*(i,t) u_m^*(j,t) v(i,j) u_m(i,t) u_n(j,t) d\tau_j}{u_n^*(i,t) u_n(j,t)} \quad (\text{II-65})$$

The last term includes "exchange terms" arising from anti-symmetry of the trial wave function, and also subtracts out the self-energy term from the average field given by the next-to-the-last term ($m = n$). The last term will be neglected.* A more general treatment including the exchange terms is given in the monograph by Thouless.⁵¹ With the neglect of fermion exchange, no special difficulty accrues to the presence of two kinds of fermions. The average time-dependent field in Eq. (II-65) is just a sum of neutron and proton contributions, assuming charge independence of $v(i,j)$. Later the case will be considered in which the neutron-neutron, proton-proton, and neutron-proton interactions all have the same form but different strengths.

The relation of the time-dependent Hartree problem to the vibrational cranking model will be illustrated by means of a specific example. The two-body force $v(i,j)$ will be chosen as the Elliott quadrupole-quadrupole force, which has often been used to explain nuclear collective motion.⁵²

$$v(i,j) = -\chi \sum_{\mu=-2}^2 r_i^2 r_j^2 Y_{2\mu}(\theta_i, \varphi_i) Y_{2\mu}^*(\theta_j, \varphi_j), \quad (\text{II-66})$$

where the constant χ is called the "quadrupole strength parameter" or "quadrupole coupling constant," and is assumed to be positive.

The Hamiltonian (II-65) takes the form (neglecting the last term)

$$H_{\text{TDSCF}}(i,t) = T(i) + V(i) - \chi \sum_{\mu=-2}^2 \hat{q}_{2\mu}(i) \langle \Psi(t) | \hat{Q}_{2\mu}^* | \Psi(t) \rangle, \quad (\text{II-67})$$

*With this approximation, the single-particle orbitals are orthogonal to each other.

where

$$\hat{q}_{2\mu}(i) \equiv r_i^2 Y_{2\mu}(\theta_i, \phi_i), \quad (a)$$

(II-63)

$$\hat{Q}_{2\mu}^* \equiv \sum_j \hat{q}_{2\mu}^*(j), \quad (b)$$

and $\Psi(t)$ is a determinant composed of the $u_n(i, t)$.

For simplicity, it will be assumed that the fluctuations occur about a spherical equilibrium shape. The problem can be solved self-consistently by guessing some time dependence for the average mass quadrupole moments $\langle \Psi(t) | \hat{Q}_{2\mu}^* | \Psi(t) \rangle$, solving the resulting time-dependent Schrödinger equation for the $u_n(i, t)$, forming a Slater determinant from these, and calculating the time-dependent average quadrupole moments, hoping to get back the original guess. A plausible guess (which works), since stable fluctuations are sought, is to assume an oscillatory time dependence. That is, make the following replacement in the Hamiltonian (II-67):

$$\langle \Psi(t) | \hat{Q}_{2\mu}^* | \Psi(t) \rangle \rightarrow A_{\mu}^* \sin \omega t \equiv Q_{2\mu}^*(t) \quad (II-69)$$

for each μ .

It is convenient to consider the Schrödinger equation

$$\mathcal{H}_{\text{TDSF}} \Psi(t) = i\hbar \frac{\partial}{\partial t} \Psi(t), \quad (\text{a})$$

where

(II-70)

$$\mathcal{H}_{\text{TDSF}} \equiv \sum_i H_{\text{TDSF}}^{(i),t} = \sum_i (T(i) + V(i)) - \chi \sum_{\mu=-2}^2 \hat{Q}_{2\mu} \hat{Q}_{2\mu}^*(t). \quad (\text{b})$$

$\mathcal{H}_{\text{TDSF}}$ has the form of H_{crank} (Eq. II-29) to first order in the $\hat{Q}_{2\mu}^*(t)$, which correspond to the $d_{\mu}(t)$, and the $\hat{Q}_{2\mu}$ correspond to the $\frac{\partial H}{\partial d_{\mu}}$. One can solve Eq. (II-70) by perturbation theory for small-amplitude fluctuations, performing the expansion (II-12) for $\Psi(t)$ in terms of configurations Φ_j , which are eigenfunctions of the spherically symmetric independent-particle Hamiltonian. To first order, the expansion coefficients can be obtained from Eq. (II-23). The total wave function is

$$\Psi(t) = \left(\Phi_0 + \sum_{j \neq 0} b_j(t) \Phi_j \right) e^{-\frac{i}{\hbar} E_0 t} \quad (\text{II-71})$$

with

$$b_j(t) = \chi \sum_{\mu} \frac{\langle j | \hat{Q}_{2\mu} | 0 \rangle}{(E_j - E_0)^2 - (\hbar\omega)^2} \left(\hat{Q}_{2\mu}^*(t) (E_j - E_0) + i\hbar \dot{\hat{Q}}_{2\mu}^*(t) \right). \quad (\text{II-72})$$

To get self-consistency with the replacement (II-69), one must try to satisfy the conditions

$$\langle \Psi(t) | \hat{Q}_{2\mu} | \Psi(t) \rangle \equiv Q_{2\mu}(t) \quad (\text{II-73})$$

for each μ .

One then obtains

$$\langle \Psi(t) | \hat{Q}_{2\mu} | \Psi(t) \rangle = \sum_{j \neq 0} \left(\langle 0 | \hat{Q}_{2\mu} | j \rangle b_j + \text{c.c.} \right) \equiv Q_{2\mu}(t), \quad (\text{II-74})$$

where "c.c." denotes the complex conjugate. It has been assumed that

$$\langle 0 | \hat{Q}_{2\mu} | 0 \rangle = 0 \quad (\text{II-75})$$

for each μ , corresponding to a spherical equilibrium shape. Eqs.

(II-74) may be written as follows:

$$\sum_j \sum_{\nu} \left[\frac{\langle j | \hat{Q}_{2\nu} | 0 \rangle \langle 0 | \hat{Q}_{2\mu} | j \rangle}{(E_j - E_0)^2 - (\hbar\omega)^2} \left(Q_{2\nu}^*(t) (E_j - E_0) + i\hbar \dot{Q}_{2\nu}^*(t) \right) + \text{c.c.} \right] = Q_{2\mu}(t) \quad (\text{II-76})$$

for each μ . Because of the time dependence given by Eq. (II-69), one separately equates to zero the coefficients of $\cos\omega t$ and of $\sin\omega t$ to yield the set of linear homogeneous equations in the amplitudes A_ν and A_ν^* .

$$x \sum_j \sum_\nu \left[\frac{\langle j | \hat{Q}_{2\nu} | 0 \rangle \langle 0 | \hat{Q}_{2\mu} | j \rangle (E_j - E_0) A_\nu^*}{(E_j - E_0)^2 - (\hbar\omega)^2} + \text{c.c.} \right] = A_\mu, \quad (\text{a})$$

(II-77)†

$$x \sum_j \sum_\nu \left[\frac{\langle j | \hat{Q}_{2\nu} | 0 \rangle \langle 0 | \hat{Q}_{2\mu} | j \rangle A_\nu^*}{(E_j - E_0)^2 - (\hbar\omega)^2} - \text{c.c.} \right] = 0 \quad (\text{b})$$

for each μ .

A great simplification comes about because of the assumed spherical symmetry of the independent-particle well. Then, the Φ_j are eigenfunctions of the total angular momentum, and the ground state of an even-even nucleus has zero total angular momentum. Application of the Wigner-Eckhart theorem for tensor operators shows that only the tensor component $\hat{Q}_{20}$ has non-zero matrix elements. Applying Eqs. (II-77) to $\mu = 0$, and noting that Eq. (II-77b) merely insures that A_0 is real, one immediately gets the frequency formula

†It is assumed that ω is real, since imaginary values of ω correspond to unstable modes.

$$\sum_j \frac{|\langle j | \hat{Q}_{20} | 0 \rangle|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} = \frac{1}{2\chi}, \quad (\text{II-78})$$

which has been obtained from the Sawada method, and has the form of (Eq. II-27) of the cranking model.

The vanishing of matrix elements of tensor components with $\mu \neq 0$ corresponds to a degeneracy of the 5 quadrupole modes as in the Bohr-Mottelson model. If the independent-particle Hamiltonian were assumed to have prolate-spheroidal instead of spherical symmetry, this degeneracy would be split. One would then have, in addition to a $\mu = 0$ mode, two two-fold degenerate modes: a $|\mu| = 2$ mode, corresponding to the γ vibration, and a $|\mu| = 1$ mode, corresponding to rotation.

The dispersion formula (II-78) can also be obtained by requiring that the energy be independent of time; that is, that

$$\frac{\partial}{\partial t} \langle \psi(t) | \mathcal{H}_{\text{TOSCF}} + \frac{\chi}{2} \sum_{\mu=-2}^2 \hat{Q}_{2\mu} Q_{2\mu}^*(t) | \psi(t) \rangle = 0, \quad (\text{II-79})$$

where 1/2 of the two-body quadrupole-quadrupole interactions are subtracted to avoid counting them twice. This should be contrasted with Eq. (II-7) of the cranking model, which does not explicitly correct for two-body interactions. However, the cranking Hamiltonian contains a term, second-order in deformation, due to the condition of volume

conservation, which contributes to the right side of the dispersion formula (II-27) and effectively takes the place of the second-order contribution coming from the two-body correction in the TDSCF method.

For any simple volume-conserving ellipsoidal well, such as a spheroidal box, or harmonic oscillator, that might be used in the cranking model of section B, one can always find a factorizable quadrupole-quadrupole force to use in the TDSCF method,* which will lead to the same frequency formula, even matching on the right-hand side. The total wave functions will also agree, of course, to first order. For example, if one chooses a deformed harmonic-oscillator well in the cranking model, the corresponding two-body force would be the Elliott force (II-66). The expansion of an ellipsoidal oscillator about a spherical equilibrium value is given by Eq. (B-48). Take as an example the $\mu = 0$ mode, previously considered in connection with the cranking model. Then, from Eqs. (II-9a), (II-9b), and (II-10), the cranking-model potential to first order is

$$V_{\text{crank}} = \sum_i \frac{m\omega_0^2}{2} r_i^2 - \chi \langle \psi_{\text{crank}}(t) | \hat{Q}_{20} | \psi_{\text{crank}}(t) \rangle \sum_i \hat{q}_{20}(i), \quad (\text{a})$$

with

(II-30)

$$\chi = \frac{4\pi}{5} \frac{m\omega_0^2}{\langle 0 | \sum_i r_i^2 | 0 \rangle} \quad (\text{b})$$

in correspondence with the TDSCF Hamiltonian (II-67).

*By factorizable force one means a force of the form: $v(i,j) = f(i)f(j)$

The cranking-model dispersion relation (II-27) for this case becomes

$$\sum_j \frac{|\langle j | \hat{Q}_{20} | 0 \rangle|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} = \frac{\langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \beta^2} | 0 \rangle_{\beta=0}}{(m\omega_0^2)^2} \quad (\text{II-31})$$

Assuming volume conservation to second order, then from Eq. (II-9c)

$$\frac{\langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \beta^2} | 0 \rangle_{\beta=0}}{(m\omega_0^2)^2} = \frac{5}{8\pi} \frac{\langle 0 | \sum_i r_i^2 | 0 \rangle}{m\omega_0^2} = \frac{1}{2\chi} \quad (\text{II-32})$$

Hence the agreement with Eq. (II-78) is complete. This would not be the case if the term proportional to β^2 in the cranking-model Hamiltonian were chosen in a different way (say by requiring that the density should be independent of volume, rather than the equipotential surfaces). An analogous situation holds for other modes, and for different choices of volume-conserving potentials.

It must be emphasized that a simple dispersion formula is obtained from the TDSCF method only if the quadrupole force is factorizable, although there is no general reason why it should be. However, it is usually maintained that the main features of nuclear vibrations do not depend strongly on the radial dependence of the long-range force, but rather on the angular dependence of the spherical harmonics, so that one may as well avail himself of simple forces.

It should also be mentioned that the TDSCF method treatment given here neglects a possible monopole component of the two-body interaction. It has been assumed tacitly that such a component contributes to the spherical well. However, if it can be treated by the Hartree method, such an additional interaction could contribute to the fluctuating average field. For example, if one adds a monopole component to the Elliott force of the form

$$v_0(c_{ij}) \propto r_i^2 r_j^2, \quad (\text{II-83})$$

this would not only renormalize a spherical harmonic-oscillator well, $V(r) = \frac{m\omega_0^2}{2} r^2$, but would also add to the total time-dependent field in Eq. (II-70) a term of the form

$$\langle \Psi(t) | \sum_i r_i^2 - \langle 0 | \sum_i r_i^2 | 0 \rangle | \Psi(t) \rangle \sum_i r_i^2, \quad (\text{II-84})$$

which would complicate matters by raising the possibility of coupling of monopole (breathing mode) vibrations with $\mu = 0$ quadrupole vibrations, and might also cast some doubt on the conventional method of simulating nuclear incompressibility in the deformed shell model. However, in the region of heavy elements, with which this paper is concerned, monopole modes are not found in low-energy spectra, so that the neglect of such effects should be a good approximation.

The Elliott force can be applied to vibrations about a non-spherical equilibrium shape. For example, consider only the $\mu = 0$ mode. The Hamiltonian may be rewritten as follows

$$\mathcal{H}_{\text{TDSCEF}} = \sum_i \left(T(i) + V(i) - \chi \hat{Q}_{20}^{(i)} \bar{Q}_{20}^* \right) - \chi \hat{Q}_{20} (Q_{20}^{(i)} - \bar{Q}_{20}^*) \quad (\text{II-85}),$$

where $\bar{Q}_{20}$ is the equilibrium quadrupole moment, which is proportional to the parameter β_0 . The basis functions for the perturbation calculation are eigenfunctions corresponding to the deformed well $V_Q = V(r) - \chi \hat{Q}_{20} \bar{Q}_{20}$. Using a similar treatment as before, and noting that $\langle \psi(t) | \hat{Q}_{20} | \psi(t) \rangle = \bar{Q}_{20} + A \sin \omega t$ one gets a dispersion formula of the same form as (II-78) except that the matrix elements and energy denominators refer to eigenstates of a deformed field with equilibrium deformation

The cranking model, however, based on a time-dependent volume-conserving harmonic-oscillator well, does not exactly agree with the TDSCEF method applied to the Elliott force for vibrations about a non-spherical equilibrium shape. One difference is that the deformed equilibrium Hamiltonian in Eq. (II-85) disagrees with the corresponding harmonic-oscillator Hamiltonian in the second-order deformation terms. This difference is important only if the equilibrium deformation is large. Another difference in the frequency formula is that in the cranking-model case one calculates matrix elements of the operator

$$\left. \frac{\partial H}{\partial \beta} \right|_{\beta=\beta_0} \propto (1 - \sqrt{\frac{5}{4\pi}} \beta_0) \hat{Q}_{20} - \frac{5}{4\pi} \beta_0 \sum_i r_i^2 \quad (\text{II-86})$$

to lowest order in β_0 , rather than just matrix elements of $\hat{Q}_{20}$ as in the TDSCF method using the Elliott force (see Eq. IV-15). The two differences do not compensate each other. In fact, it appears that the TDSCF formulation would give lower solutions of the frequency formula.

Of course, one can always concoct a two-body force to give the same result as the cranking model, although such a force might deviate from the quadrupole-quadrupole form in the case under discussion. The only justification for this form is the multipole expansion given by Eq. (B-2). However, there is nothing sacred about this expansion, since the residual shell-model force may not be a function of $\vec{r}_1 - \vec{r}_j$, which the expansion presupposes. If one considers the force responsible for collective vibrations to be some sort of "pseudopotential," as in the Brueckner theory, then, relative to our present state of knowledge, almost any form for the residual interaction that leads to results in agreement with experiment would be admissible.

Finally, it should be pointed out that the additional terms in Eq. (II-86) might be justified by considering monopole components in the residual force. In the calculations to be later presented, a

strict cranking formula, however, will be used to test the validity of the volume-conserving model, rather than a simple quadrupole-quadrupole force.*

The usual TDSCF method applies only when there are no short-range pairing correlations. A plausible extension of this method to

*A residual force which gives rise to a deformed harmonic oscillator potential, conserving the volume through 2nd order, and having the form of the expansion (B-2) is

$$v(i,j) = \chi \left\{ \left(\frac{5}{8\pi} + \frac{25}{18AR_0^2} \langle \psi | \hat{Q}_{20} | \psi \rangle^2 \right) r_i^2 r_j^2 - \left(1 - \frac{1}{3} \frac{\sqrt{5}\pi}{AR_0^2} \langle \psi | \hat{Q}_{20} | \psi \rangle \right) r_i^2 r_j^2 Y_{20}(\theta_i) Y_{20}(\theta_j) \right\}$$

The coefficients of the multipole components depend on the total nuclear wave function ψ . This is not implausible since the force is an effective interaction and not the real two-nucleon interaction. If ψ is chosen as a solution of the time-independent Hartree equation, the average field will be a volume-conserving (to 2nd order) harmonic oscillator. If ψ is chosen as a solution of the time-dependent Hartree equation, the coefficients become time-dependent, and one can obtain the dispersion formula (IV-15).

include such correlations, as well as long-range correlations, would be to use as a trial wave function an expansion in terms of Bardeen-Cooper-Schrieffer states, described in Ch. III, instead of Slater determinants. Such an expansion will be used within the framework of the cranking model. Since the cranking model derivation of Eqs. (II-27) and (II-28) did not specify that the basis functions must be Slater configurations, these equations can be evaluated further assuming that BCS states are a good approximation to the intrinsic spectrum. The effect of pairing correlations on these equations is then found to agree with the more rigorous derivation using the Savada method.

Explicit Treatment of Neutron-Proton Interaction in the TDSCF Method

Because of neglect of exchange terms, special provision for presence of two kinds of fermions was not made, assuming charge independence of the residual interaction. In that case, the fluctuating quantities, $\langle \hat{Q}_{2\mu} \rangle$, are total mass quadrupole moment components, sums of neutron and proton contributions, and Eq. (II-78) is also a sum of neutron and proton contributions. The Pauli exclusion principle only plays a role in the filling of single-particle orbitals, and in the use of determinantal basis functions for the time-dependent perturbation calculation. The basis functions may be chosen as products of neutron and proton configurations. The previous treatment is easily generalized to the case in which all nucleon-nucleon interactions have the same form but different strengths. The generalization of the Hamiltonian (II-70b) is

$$\begin{aligned}
 \mathcal{H}_{\text{TDSCE}} = & \sum_i H_0^{(n)}(i) + \sum_k H_0^{(p)}(k) \\
 & - (\chi_{nn} \hat{Q}^{(n)} + \chi_{np} \hat{Q}^{(p)}) Q^{(n)}(t) \\
 & - (\chi_{np} \hat{Q}^{(n)} + \chi_{pp} \hat{Q}^{(p)}) Q^{(p)}(t), \quad (\text{II-87})
 \end{aligned}$$

where the superscripts (n) or (p) identify an operator which consists only of neutron or proton coordinates, respectively. The same superscripts will also be used to denote expectation values of such operators. The H_0 's are single-particle Hamiltonians. The $\hat{Q}$ and $Q(t)$ have the same meanings as previously, but refer only to the $\mu = 0$ mode, for spherical nuclei, so that the subscripts have been dropped.*

The χ 's are strength parameters; χ_{nn} is the strength parameter for neutron-neutron interaction, χ_{np} for neutron-proton interaction, etc. In the general case in which the χ 's are not all equal, one must consider the average quadrupole field generated by neutrons, $Q^{(n)}(t)$, and that generated by protons, $Q^{(p)}(t)$, to be two distinct, but coupled, degrees of freedom. It will be seen that in the case that all the χ 's are equal, one gets the same dispersion formula as by assuming that there is only one degree of freedom (for each μ), namely, the total mass quadrupole moment, $Q^{(t)} = Q^{(n)}(t) + Q^{(p)}(t)$. Following the treatment of the last section, one assumes

*The other modes have vanishing matrix elements in spherical nuclei, as noted previously.

$$Q^{(n)}(t) = A^{(n)} \sin \omega t, \quad (a)$$

(II-88)

$$Q^{(p)}(t) = A^{(p)} \sin \omega t. \quad (b)$$

The total wave function has the form given by Eq. (II-71), and the admixed states Φ_j have the form

$$\Phi_j = \varphi_0^{(n)} \psi_\pi^{(p)}, \quad (II-89)$$

or

$$\Phi_{j'} = \varphi_i^{(n)} \psi_0^{(p)}.$$

That is, the basis functions are products of neutron and proton configurations; one may excite either neutrons or protons but not both simultaneously.

It is necessary to satisfy the conditions

$$\begin{aligned} \langle \psi(t) | \hat{Q}^{(n)} | \psi(t) \rangle &= \sum_{j \neq 0} \left(\langle 0 | \hat{Q}^{(n)} | j \rangle b_j(t) + c.c. \right) \quad (a) \\ &= Q^{(n)}(t), \end{aligned} \quad (II-90)$$

$$\begin{aligned} \langle \psi(t) | \hat{Q}^{(p)} | \psi(t) \rangle &= \sum_{j \neq 0} \left(\langle 0 | \hat{Q}^{(p)} | j \rangle b_j(t) + c.c. \right) \quad (b) \\ &= Q^{(p)}(t), \end{aligned}$$

where it has been assumed that

$$\langle 0 | \hat{Q}^{(n)} | 0 \rangle = \langle 0 | \hat{Q}^{(p)} | 0 \rangle = 0. \quad (II-91)$$

The mixing amplitudes are given by

$$\begin{aligned} b_j(t) &= \frac{\langle j | \chi_{nn} \hat{Q}^{(n)} + \chi_{np} \hat{Q}^{(p)} | 0 \rangle}{(E_j - E_0)^2 - (\hbar\omega)^2} \left(\dot{Q}^{(n)*}(t) (E_j - E_0) + i\hbar \ddot{Q}^{(n)*}(t) \right) \\ &+ \frac{\langle j | \chi_{np} \hat{Q}^{(n)} + \chi_{pp} \hat{Q}^{(p)} | 0 \rangle}{(E_j - E_0)^2 - (\hbar\omega)^2} \left(\dot{Q}^{(p)*}(t) (E_j - E_0) + i\hbar \ddot{Q}^{(p)*}(t) \right) \quad (II-92) \end{aligned}$$

It is clear from the form of the basis functions (II-89) that the b_j correspond either to a proton or to a neutron excitation. The corresponding energy denominators will be labeled by $E^{(p)}$ or $E^{(n)}$ and the matrix elements by $Q_{ko}^{(p)}$ or $Q_{io}^{(n)}$.

One easily obtains the linear homogeneous equations as in the previous section:

$$A^{(n)} \left[2\chi_{nn} \sum_i \frac{|Q_{io}^{(n)}|^2 (E_i - E_0)}{(E_i^{(n)} - E_0)^2 - (\hbar\omega)^2} - 1 \right] + A^{(p)} \left[2\chi_{np} \sum_i \frac{|Q_{io}^{(n)}|^2 (E_i - E_0)}{(E_i^{(n)} - E_0)^2 - (\hbar\omega)^2} \right] = 0,$$

$$A^{(n)} \left[2\chi_{np} \sum_k \frac{|Q_{ko}^{(p)}|^2 (E_k - E_0)}{(E_k^{(p)} - E_0)^2 - (\hbar\omega)^2} \right] + A^{(p)} \left[2\chi_{pp} \sum_k \frac{|Q_{ko}^{(p)}|^2 (E_k - E_0)}{(E_k^{(p)} - E_0)^2 - (\hbar\omega)^2} - 1 \right] = 0 \quad (\text{II-93})$$

The vanishing of the corresponding secular determinant leads, upon expansion, to the dispersion formula;

$$2\chi_{pp} \sum^{(p)} + 2\chi_{nn} \sum^{(n)} + 4(\chi_{np}^2 - \chi_{nn}\chi_{pp}) \sum^{(n)} \sum^{(p)} = 1, (a)$$

where $\sum^{(p)} \equiv \sum_k \frac{|Q_{ko}^{(p)}|^2 (E_k - E_0)}{(E_k^{(p)} - E_0)^2 - (\hbar\omega)^2}, \quad (b) \quad (\text{II-94})$

and $\sum^{(n)} \equiv \sum_i \frac{|Q_{io}^{(n)}|^2 (E_i - E_0)}{(E_i^{(n)} - E_0)^2 - (\hbar\omega)^2} \quad (c)$

If $\chi_{nn} = \chi_{pp} = \chi_{np} = \chi$, the last term vanishes and one obtains

$$\sum^{(p)} + \sum^{(n)} = \frac{1}{2\chi} \quad (\text{II-95})$$

If $\chi_{np} = 0$, one gets two dispersion formulae for independent neutron and proton vibrations

$$\sum^{(n)} = \frac{1}{2\chi_{nn}}, \quad (\text{a})$$

and

$$\sum^{(p)} = \frac{1}{2\chi_{pp}} \quad (\text{b}) \quad (\text{II-96})$$

The Eq. (II-94) was first derived by Tamura and Udegawa^{24,25}, for the case $\chi_{nn} = \chi_{pp} \neq \chi_{np}$, treating an Elliott force by means of the Sawada method. These authors also included pairing correlations, so that the matrix elements and energies in Eq. (II-94) refer to BCS states.

The Sawada Method

It is of interest to point out that a close connection exists between the TDSCF method, which describes collective motion "quasi-classically," and the more rigorously quantum-theoretical method of Sawada, also known as the method of linearized equations of motion, or the random phase approximation (RPA). This method, which is based

on an analogy with scalar meson pair theory,⁵³ and was designed to describe collective vibrations in an electron gas, has been applied to nuclear collective motion by Baranger, Marumori and others (see references in Ch. I). The nuclear application has been made to a Hamiltonian of the form

$$H = H_{SM} + H_{Q-Q}, \quad (\text{II-97})$$

where H_{SM} is a many-particle shell-model Hamiltonian consisting of a spherically symmetric independent-particle Hamiltonian plus possible short-range pairing forces, and H_{Q-Q} is a residual quadrupole-quadrupole force, such as the Elliott force. The philosophy of the Sawada technique will be briefly sketched.

Motivated by experiment, one may inquire whether the spectrum of (II-97) may be a vibrational spectrum to a good approximation. If this is so, one may look for a set of operators, in the framework of second quantization, which obey boson commutation rules, and may be considered as creation and annihilation operators for the vibrational quanta or phonons.⁵⁴ However, since the collective excitations are due to correlations between fermions, the phonon operators would have to be superpositions of products of fermion operators; in particular, products of even numbers of fermion operators to produce anything like a boson. In the Sawada approximation, one constructs "phonon" operators composed of pairs of fermion operators, neglecting quadruples etc. Such operators cannot exactly satisfy all Bose commutation rules unless certain terms appearing in the evaluation of the commutators are

neglected. Such terms can be called "exchange terms" arising from the Pauli exclusion principle.* It has been said that "a pair of fermions is like a boson, except that the fermions get in each other's way." The "phonon" operators have therefore been referred to as "quasi-bosons." In the Sawada approximation the exchange terms in question are in fact neglected. The problems involved in taking the neglected terms into account are discussed by Sawicki.⁵⁵

Let us define an operator $A_i^\dagger$ composed of a pair of fermion operators, which, operating on the ground state H_{SM} , creates an excited state of H_{SM} . If H_{SM} does not contain the pairing interaction, $A_i^\dagger$ is just a particle-hole excitation of the form

$$A_i^\dagger = a_\nu^\dagger a_{\nu'} \quad (\text{II-98})$$

If H_{SM} contains a pairing interaction, it may be treated by the BCS method (Ch. III), in which case the $A_i^\dagger$ become two-quasiparticle excitations, such as

$$A_i^\dagger = \alpha_\nu^\dagger \beta_{\nu'}^\dagger \quad (\text{II-99})$$

*Some authors use the expression "exchange term" in a different way than here; they classify the neglected terms into "exchange," "self-energy" and "random-phase terms"; all of these have been lumped together under "exchange terms" here.

For quadrupole vibrations one seeks five pairs of quasi-boson creation and annihilation operators $C_{2\mu}^\dagger$, and $C_{2\mu}$, corresponding to five independent modes, each of which carries total angular momentum $I=2$, and projection on a space-fixed axis $M=\mu$. It is convenient to formulate the commutation rules in terms of the equations of motion

$$[C_{2\mu}, H] = \hbar\omega C_{2\mu} \quad , \quad (a)$$

(II-100)

$$[C_{2\mu}^\dagger, H] = -\hbar\omega C_{2\mu}^\dagger \quad . \quad (b)$$

Instead of expressing the $C_{2\mu}^\dagger$ directly as linear combinations of the $A_i^\dagger$ and A_i , it is convenient first to form all possible linear combinations $B_{2\mu j}^\dagger$, of the $A_i^\dagger$, vector-coupled to $I=2$, and $M=\mu$, and then to express the $C_{2\mu}^\dagger$ as linear combinations of the $B_{2\mu j}^\dagger$ and $B_{2\mu j}$:

$$\begin{aligned} C_{2\mu}^\dagger &= \sum_j X_{2\mu j}^\dagger B_{2\mu j}^\dagger + \sum_j Y_{2\mu j}^* B_{2\mu j} \quad , \\ C_{2\mu} &= \sum_j X_{2\mu j} B_{2\mu j} + \sum_j Y_{2\mu j} B_{2\mu j}^\dagger \quad . \end{aligned} \quad (II-101)$$

One then proceeds to evaluate the commutators (II-100), neglecting certain terms (the afore-mentioned exchange terms) so as to obtain

a set of linear homogeneous equations in the amplitudes $X_{2\mu j}$ and $Y_{2\mu j}$, and their complex conjugates, which leads to a secular equation[†] determining the energies of the collective excitations. If the A_i are quasi-particle excitations, the neglected terms in the commutators include products of four quasi-particle operators and, in addition, terms involving products of one quasi-particle creation and one annihilation operator. According to Baranger,²⁰ these approximations are valid as long as the number of available shell-model states is large compared to the number of fermions; otherwise, the Pauli exclusion principle becomes important and the quasi-boson approximations break down. The order of magnitude of the errors involved in the Sawada approximations coincides with that of the BCS approximations.²⁰ The resulting secular equation leads to the dispersion formula given by the TDSCF method if H_{Q-Q} is a factorizable quadrupole-quadrupole force.

The vacuum wave function, ψ_{vac} , for the $C_{2\mu}^\dagger$ and $C_{2\mu}$ is obtained in the usual way by requiring that

$$C_{2\mu} \psi_{vac} = 0 \quad (\text{II-102})$$

for each μ . ψ_{vac} represents the ground state of H , (II-97), including zero-point oscillations. The vacuum is given explicitly by Sorensen.⁵⁶ Excited collective states are then obtained by operating on ψ_{vac} with the $C_{2\mu}^\dagger$.

Ehrenreich and Cohen,⁴⁷ and Marumori²³ point out that the agreement between the TDSCF method and the Sawada method in the absence of pairing correlations is due to a correspondence in the approximations made in each case. The neglect of terms leading to linearized equations of motion in the Sawada method corresponds to the neglect of Hartree-Fock exchange and self-energy terms and to the use of perturbation theory, which, to first order, linearizes the problem, in the TDSCF method. In fact, they point out that if the equations of motion of the Sawada method (II-100) are averaged, one obtains a set of equations which can also be obtained from the TDSCF method; for the Elliott force, the set of equations is equivalent to (II-76). Thus the two methods must lead to the same secular equation. A detailed comparison of the two methods is given in references (47) for the electron gas and (23) for spherical nuclei.

Lagrangian Multiplier Method

Finally, the description of collective motion given in section B will be compared to a prescription used by Belyaev,⁵⁷ Kisslinger and Sorensen,¹⁷ and Bes,¹⁸ which is a variant of the cranking model in the adiabatic limit. These authors consider a Hamiltonian of the form

$$H_Q = H_{SM} - \chi \sum_{\mu=-2}^2 \hat{Q}_{2\mu} Q_{2\mu}^*, \quad (\text{II-103})$$

where all the quantities have been previously defined. The authors wish to describe vibrations about spherical symmetry. They assume

that the collective frequencies are slow compared to the intrinsic frequencies, that is, that the adiabatic approximation is valid.

In that case, one can estimate the vibrational force constants by assuming that the $Q_{2\mu}^*$ are time-independent, finding the energy

$$E_Q = \langle \Psi_Q | H_Q + \chi \sum_{\mu=-2}^2 \hat{Q}_{2\mu} Q_{2\mu}^* | \Psi_Q \rangle \quad (\text{II-104})$$

as a function of the $Q_{2\mu}^*$, and calculating the force constants according to the formula

$$C_{2\mu} = \frac{\partial^2 E_Q}{\partial Q_{2\mu} \partial Q_{2\mu}^*} \quad (\text{II-105})$$

The mass parameters can be found from the adiabatic cranking formula*

$$B_{2\mu} = 2\hbar^2 \sum_j \frac{|\langle 0 | \frac{\partial}{\partial Q_{2\mu}^*} | j \rangle|^2}{E_j - E_0}, \quad (\text{II-106})$$

*The collective energy is $H_{\text{coll}} = \frac{1}{2} \sum_{\mu} B_{2\mu} |\dot{Q}_{2\mu}|^2 + \frac{1}{2} \sum_{\mu} C_{2\mu} |Q_{2\mu}|^2$.

where the matrix elements and energy denominators refer to eigenstates of H_Q , with the $Q_{2\mu}^*$ considered as time-independent parameters. The frequencies are then obtained from the relation

$$\omega_{2\mu}^2 = \frac{C_{2\mu}}{B_{2\mu}} \quad (\text{II-107})$$

The authors also require that a set of self-consistency conditions be fulfilled :

$$\langle \psi_a | \hat{Q}_{2\mu} | \psi_a \rangle = Q_{2\mu} . \quad (\text{II-108})$$

However, they do not use these conditions to obtain the frequencies directly as has been in this work. Instead, they require that ψ_a be the eigenfunction of H_Q with the lowest energy obtained by time-independent perturbation theory satisfying (II-108). To satisfy the constraints (II-108), they use a set of Lagrangian multipliers. Therefore they consider the auxiliary Hamiltonian

$$H_a' = H_a - \sum_{\mu} \eta_{\mu} \hat{Q}_{2\mu} = H_{SM} - \sum_{\mu} \bar{\eta}_{\mu} \hat{Q}_{2\mu} , \quad (\text{a})$$

(II-109)

where
$$\bar{\eta}_{\mu} \equiv \eta_{\mu} + \chi Q_{2\mu}^* . \quad (\text{b})$$

Once Ψ_0 is found as an eigenfunction of (II-109a), the mass quadrupole moments can be calculated, and the Lagrangian multipliers obtained from the constraints (II-108). One then eliminates the multipliers from Ψ_0 and proceeds to evaluate the $B_{2\mu}$, $C_{2\mu}$, and $\omega_{2\mu}$. The resulting expressions look superficially different from those that are obtained in the adiabatic limit as defined in this paper (see Eqs. (II-32), (II-35), (II-38)). The Lagrangian multiplier method (LM) gives the following expression for the mass parameters

$$B_{2\mu}(\text{LM}) = \frac{\hbar^2}{2} \left(\sum_j \frac{|\langle 0 | \hat{Q}_{2\mu} | j \rangle|^2}{E_j - E_0} \right)^2 \sum_j \frac{|\langle 0 | \hat{Q}_{2\mu} | j \rangle|^2}{(E_j - E_0)^3}, \quad (\text{II-110})$$

whereas the corresponding mass parameters in the adiabatic limit as defined in this paper are given by

$$B_{2\mu} = 2\hbar^2 \chi^2 \sum_j \frac{|\langle 0 | \hat{Q}_{2\mu} | j \rangle|^2}{(E_j - E_0)^3}. \quad (\text{II-111})$$

The LM method gives a frequency formula

$$(\hbar\omega_{2\mu})^2(\text{LM}) = \frac{\sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{E_j - E_0} \left(1 - 2\chi \sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{E_j - E_0} \right)}{\sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{(E_j - E_0)^3}}, \quad (\text{II-112})$$

whereas the corresponding frequency in the present formulation is

$$(\hbar\omega_{2\mu})^2 = \frac{1 - 2\chi \sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{E_j - E_0}}{2\chi \sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{(E_j - E_0)^3}} \quad (\text{II-113})$$

Closer inspection shows that the LM equations can be obtained from the adiabatic formulas (II-111) and (II-113) by means of the substitution

$$\frac{1}{2\chi} = \sum_j \frac{|\langle j | \hat{Q}_{2\mu} | 0 \rangle|^2}{E_j - E_0} \quad (\text{II-114})$$

Only one of the χ 's is eliminated in Eq. (II-113); if both were eliminated one would get $\omega_{2\mu} = 0$. It is clear that Eq. (II-114) results from (II-78) by setting $\omega_{2\mu} = 0$. In the adiabatic limit, this is approximately correct. The frequencies obtained by the LM method lie between the exact solutions of (II-78) and the adiabatic limit as given by Eq. (II-113). However, one may be critical of the devious route taken in introducing ad hoc extra parameters and then eliminating them.

III. RESIDUAL PAIRING INTERACTION AND NUCLEAR SUPERFLUIDITY

The cranking model Hamiltonian (II-4) will be chosen as follows:

$$H_{crank}(\alpha(t)) = \sum_i \left[\frac{p_i^2}{2m} + V_{sp}(x_i, y_i, z_i, \sigma_i, \alpha(t)) \right] - \sum_{\substack{i,j \\ i>j}} G_{ij} \quad (\text{III-1})^*$$

The first sum is taken over the single-particle shell model Hamiltonian, with the deformation dependence in the potential V_{sp} . It is assumed that the single-particle potential V_{sp} is velocity-independent in the actual calculations, although such a dependence would not alter the general conclusions of the arguments thus far presented. The second sum in (III-1) corresponds to a two-body deformation-independent residual pairing interaction, the chief effects of which cannot be incorporated into the self-consistent field potential. In particular, it will be assumed that G_{ij} is the "pairing force" discussed by Mottelson.⁵⁸ This is a strong, short-ranged force operative only between nucleons in time-reversed orbitals. Moreover, Mottelson assumes that all the matrix elements of G_{ij} are equal, at least in the neighborhood of the Fermi level.

*The minus sign before the last term indicates that the pairing force is attractive.

Expanding the single-particle potential V_{sp} to second order, and introducing Fermion creation and annihilation operators in a representation which diagonalizes the unperturbed shell model Hamiltonian, one obtains the second quantized expression:

$$H_{crank}(\alpha(t)) = H(\alpha_0) + \left. \frac{\partial H}{\partial \alpha} \right|_{\alpha=\alpha_0} (\alpha - \alpha_0) + \frac{1}{2} \left. \frac{\partial^2 H}{\partial \alpha^2} \right|_{\alpha=\alpha_0} (\alpha - \alpha_0)^2,$$

where

$$H(\alpha_0) = \sum_{\nu} \epsilon_{\nu}(\alpha_0) (a_{\nu}^{\dagger} a_{\nu} + a_{-\nu}^{\dagger} a_{-\nu}) - G \sum_{\nu\nu'} a_{\nu}^{\dagger} a_{-\nu'}^{\dagger} a_{-\nu} a_{\nu},$$

$$\left. \frac{\partial H}{\partial \alpha} \right|_{\alpha=\alpha_0} = \sum_{\nu\nu'} \langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle_{\alpha=\alpha_0} (a_{\nu}^{\dagger} a_{\nu'} \pm a_{-\nu'}^{\dagger} a_{-\nu}) + \sum_{\mu\mu'} \langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | -\mu' \rangle_{\alpha=\alpha_0} (a_{\mu}^{\dagger} a_{\mu'}^{\dagger} \mp a_{-\mu}^{\dagger} a_{-\mu'}), \quad (\text{III-2})$$

$$\left. \frac{\partial^2 H}{\partial \alpha^2} \right|_{\alpha=\alpha_0} = \sum_{\nu\nu'} \langle \nu | \frac{\partial^2 V_{sp}}{\partial \alpha^2} | \nu' \rangle_{\alpha=\alpha_0} (a_{\nu}^{\dagger} a_{\nu'} \pm a_{-\nu'}^{\dagger} a_{-\nu}) + \sum_{\mu\mu'} \langle \mu | \frac{\partial^2 V_{sp}}{\partial \alpha^2} | -\mu' \rangle_{\alpha=\alpha_0} (a_{\mu}^{\dagger} a_{\mu'}^{\dagger} \mp a_{-\mu}^{\dagger} a_{-\mu'}).$$

$$(\nu > 0, \mu > 0)$$

The quantum numbers of the single particle states are denoted collectively by the subscript ν . Time-reversed states are differentiated by subscripts ν and $-\nu$; i.e., $T|\nu\rangle = |-\nu\rangle$, where T is the time-reversal operator. The single-particle creation and annihilation operators are $a_{\nu}^{\dagger}$ and a_{ν} , respectively, and obey the usual Fermion anti-commutation relations. The matrix elements are taken between the single-particle eigenstates of the unperturbed Hamiltonian $H(\alpha_0)$. Matrix elements of the two-body operator G_{ij} , which are all equal by hypothesis, are denoted by G .* The ϵ_{ν} are the shell-model eigenvalues corresponding to the deformation α_0 .

*Specifically, $G = \langle \nu' - \nu' | G_{ij} | -\nu \nu \rangle = \langle \nu' - \nu' | G_{ij} | \nu - \nu \rangle$.

The behavior of the matrix elements under time reversal is indicated by the $\pm$ signs, the top sign referring to even-time-reversal transformation and the bottom sign to odd-time-reversal. The definition of even-and-odd-time-reversal behavior used here is as follows:

Even Case

$$\langle \nu | \frac{\partial V_{SP}}{\partial \alpha} | \nu' \rangle = \langle -\nu | \frac{\partial V_{SP}}{\partial \alpha} | -\nu' \rangle ,$$

(III-3)

$$\langle \mu | \frac{\partial V_{SP}}{\partial \alpha} | -\mu' \rangle = - \langle -\mu | \frac{\partial V_{SP}}{\partial \alpha} | \mu' \rangle .$$

Odd Case

$$\langle \nu | \frac{\partial V_{SP}}{\partial \alpha} | \nu' \rangle = - \langle -\nu | \frac{\partial V_{SP}}{\partial \alpha} | -\nu' \rangle ,$$

$$\langle \mu | \frac{\partial V_{SP}}{\partial \alpha} | -\mu' \rangle = \langle -\mu | \frac{\partial V_{SP}}{\partial \alpha} | \mu' \rangle . \quad \text{(III-4)}$$

The index μ is used for convenience to differentiate transitions occurring between states with opposite senses of time, from those transitions between states with the same sense of time denoted by ν . The odd behavior occurs, for example, when describing rotations by the cranking model, since $\frac{\partial V_{SP}}{\partial \alpha}$ is proportional to the angular

momentum operator $\hat{p}_x$. In describing β and γ vibrations, only the even case will be needed, so that lower signs will be dropped hereafter. Moreover, an operator $\frac{\partial^2 H}{\partial \alpha^2}$ for quadrupole vibrations has no transitions of the " μ -type," so that the index will be suppressed for this operator.

In order to evaluate Eqs. (II-27) and (II-28) for the given choice of H_{crank} , it is necessary to find eigenfunctions of $H(\alpha_0)$, which includes the pairing interaction. In order to handle this interaction, which is intractable to perturbation and Hartree-Fock methods, the approximation method of Bogolyubov,¹⁴ equivalent to that of Bardeen, Cooper, Schrieffer,¹³ will be used. The nuclear application of this technique, which was originally devised to handle the phenomenon of superconductivity, has been extensively discussed by Mottelson⁵⁸ and Belyaev,⁵⁷ and will only be sketched here.

The BCS-Bogolyubov method is one of a class of approximations to the solution of the many-body problem in quantum mechanics, a class which is characterized by premeditated violation of some conservation law as the price for obtaining a solution. This point has been aptly discussed by Lipkin.⁵⁹ The cranking model or TDSCF are examples of such a method, since the energy is not taken as a constant of motion, although partial compensation is afforded by satisfying the condition that $\langle T + V \rangle$ be independent of time.

An approximate solution to the superconductivity problem is obtained by using a Hamiltonian that does not commute with the particle number operator

$$\mathcal{N} = \sum_{\nu} (a_{\nu}^{\dagger} a_{\nu} + a_{-\nu}^{\dagger} a_{-\nu}) ; \nu > 0 . \quad (\text{III-5})$$

Partial compensation for flouting the conservation law is obtained by satisfying the condition that the average particle number equal the correct eigenvalue of (III-5), although fluctuation about $\mathcal{N}$ is permitted. The essence of the BCS-Bogolyubov method is the neglect of certain terms in the original Hamiltonian, leaving a mutilated Hamiltonian, which no longer commutes with $\mathcal{N}$, but which has the form of an independent-particle operator,

It is convenient to perform the Bogolyubov transformation to the "quasi-particle" operators

$$\begin{aligned} \alpha_{\nu} &= U_{\nu} a_{\nu} - V_{\nu} a_{-\nu}^{\dagger} , \\ \alpha_{-\nu} \equiv \beta_{\nu} &= U_{\nu} a_{-\nu} + V_{\nu} a_{\nu}^{\dagger} ; \nu > 0 . \end{aligned} \quad (\text{III-6})$$

The transformation is canonical, provided that

$$U_{\nu}^2 + V_{\nu}^2 = 1 . \quad (\text{III-7})$$

The quasi-particle operators therefore obey the Fermion anticommutation rules. The U_{ν}^2 and the V_{ν}^2 , which play the role of variational parameters, can be physically interpreted as the probability that the single particle state $|\nu\rangle$ is unoccupied or occupied, respectively. Also, $U_{\nu}^2 = U_{\nu}^2$. In the absence of the pairing force, which blurs the Fermi distribution by scattering pairs across the Fermi level ϵ_F , $U_{\nu}^2 = 0$ or 1 and, correspondingly, $V_{\nu}^2 = 1$ or 0, so that the quasi-particle operators become particle or hole operators. The modification of the Fermi distribution by the pairing interaction is illustrated in Fig. III-1.

Instead of using the Hamiltonian $H(\alpha_0)$, (III-2a), one uses the auxiliary Hamiltonian

$$H'(\alpha_0) = H(\alpha_0) - \lambda \mathcal{N}, \quad (\text{III-8})$$

where λ is a Lagrangian multiplier. Performing the transformation (III-6), $H'(\alpha_0)$ takes the form:

$$H'(\alpha_0) = U + H_{11} + H_{20} + H_{int} \quad (\text{III-9})$$

Here H_{int} contains products of four quasi-particle operators, some of the terms corresponding to residual interactions between quasi-particles, and is neglected. (Further discussion of the importance of this omission is given by Belyaev,⁵⁷ and Nilsson and Prior.¹⁶) The term U is a constant and is the energy of the quasi-particle vacuum, while H_{11} has the form of an independent-particle operator. H_{20} is given by

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Fig. III-1. Modification of Fermi distribution by a strong pairing interaction. The probability that a single-particle state with energy ϵ_ν is occupied is V_ν^2 . For simplicity, a continuous spectrum of single-particle states is shown.

$$H_{20} = \sum_{\nu} \left[(\epsilon_{\nu} - \lambda) 2U_{\nu}V_{\nu} - \Delta (U_{\nu}^2 - V_{\nu}^2) \right] (\alpha_{\nu}^{\dagger} \beta_{\nu}^{\dagger} + \beta_{\nu} \alpha_{\nu}) , \quad (\text{III-10})$$

where

$$\Delta = G \sum_{\nu} U_{\nu} V_{\nu} . \quad (\text{III-11})$$

An independent-particle Hamiltonian is obtained by choosing the U_{ν} and V_{ν} so that (III-10) vanishes.

$$H_{20} = 0 . \quad (\text{III-12})$$

Taking into account Eq. (III-7), the U_{ν} and V_{ν} are

$$U_{\nu}^2 = \frac{1}{2} \left(1 + \frac{\epsilon_{\nu} - \lambda}{\sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2}} \right) , \quad (\text{a})$$

(III-13)

$$V_{\nu}^2 = \frac{1}{2} \left(1 - \frac{\epsilon_{\nu} - \lambda}{\sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2}} \right) . \quad (\text{b})$$

The truncated Hamiltonian is^{*}

$$\mathcal{H}(\alpha_0) = \mathcal{U} + H_{II} ,$$

$$\text{where } \mathcal{U} = \sum_{\nu} (\epsilon_{\nu} - \lambda) \lambda V_{\nu}^2 - \frac{\Delta^2}{G} , \quad (a)$$

$$H_{II} = \sum_{\nu} E_{\nu} (\alpha_{\nu}^{\dagger} \alpha_{\nu} + \beta_{\nu}^{\dagger} \beta_{\nu}) , \quad (b) \quad (\text{III-14})$$

$$\text{and } E_{\nu} = \sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2} . \quad (c)$$

In terms of the quasi-particle operators, the number operator is given by

*Certain terms have been omitted which can be considered as re-normalizing the E_{ν} .

[†]The use of the word truncated in the present context is not to be confused with its use in solid-state applications of BCS theory; there, the word has been applied to the method whereby the electron-phonon interaction is replaced by an effective pair interaction.

$$\mathcal{N} = \sum_{\nu} 2V_{\nu}^2 + \sum_{\nu} (2U_{\nu}^2 - V_{\nu}^2) (\alpha_{\nu}^{\dagger} \alpha_{\nu} + \beta_{\nu}^{\dagger} \beta_{\nu}) + \sum_{\nu} 2U_{\nu} V_{\nu} (\alpha_{\nu}^{\dagger} \beta_{\nu}^{\dagger} + \beta_{\nu} \alpha_{\nu}) . \quad (\text{III-15})$$

In order to obtain the proper number of particles, since $\mathcal{H}(\alpha_0)$ does not commute with $\mathcal{N}$, it is required that

$$\langle \psi_0 | \mathcal{N} | \psi_0 \rangle = n , \quad (\text{III-16})$$

where ψ_0 is the vacuum wave function of the quasi-particle operators, and n is the number of interacting particles scattered by the pairing force.

Eqs. (III-11) and (III-16) can be written as follows:

$$\frac{G}{\lambda} \sum_{\nu} \frac{1}{\sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2}} = 1 , \quad (\text{III-17})$$

$$n = \sum_{\nu} 2V_{\nu}^2 = \sum_{\nu} \left(1 - \frac{\epsilon_{\nu} - \lambda}{\sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2}} \right) . \quad (\text{III-18})$$

These two equations can be solved simultaneously for λ and Δ . In the limit $G \rightarrow 0$, $\Delta \rightarrow 0$, and $\lambda \rightarrow \epsilon_F$, where ϵ_F is the Fermi level.* For

*Even in the presence of superfluidity, $\lambda \approx \epsilon_f$. The term "Fermi level" will be also applied to λ .

sufficiently strong G , when the condition $\frac{G}{2} \sum_{\nu} \frac{1}{|\epsilon_{\nu} - \lambda|} > 1$ is satisfied, there exists, in addition, a non-trivial solution corresponding to a super conducting (superfluid) state.

The quasi-particle vacuum, defined by the conditions

$$\begin{aligned} \alpha_{\nu} \psi_0 &= 0, \\ \beta_{\nu} \psi_0 &= 0, \end{aligned} \tag{III-19}$$

can readily be shown to have the structure

$$\psi_0 = \prod_{\nu} (u_{\nu} + v_{\nu} a_{\nu}^{\dagger} a_{-\nu}^{\dagger}) |0\rangle, \tag{III-20}$$

where $\prod$ stands for "product," and $|0\rangle$ is the particle vacuum.

The quasi-particle vacuum, on expanding the product, is readily seen to be a packet composed of states with different numbers of particles, but having the correct average number. It describes a superfluid ground state of condensed pairs.

The ground-state energy is given by

$$W = \mathcal{U} + \lambda n = \sum_{\nu} \epsilon_{\nu} 2 V_{\nu}^2 - \frac{\Delta^2}{G}. \tag{III-21}$$

Excited states are obtained by operating on Ψ_0 with the quasi-particle operators. An excited two quasi-particle state, with excitation energy $E_\nu + E_{\nu'} = \sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2} + \sqrt{(\epsilon_{\nu'} - \lambda)^2 + \Delta^2}$, is written as

$$\begin{aligned} \Psi_{(2)} &= \alpha_\nu^\dagger \beta_{\nu'}^\dagger \Psi_0 \quad (\alpha_{\nu'}^\dagger \beta_\nu^\dagger \Psi_0, \text{ also}), \\ \text{or} \quad \Psi'_{(2)} &= \alpha_\nu^\dagger \alpha_{\nu'}^\dagger \Psi_0 \quad (\nu \neq \nu'), \\ \text{or} \quad \Psi''_{(2)} &= \beta_\nu^\dagger \beta_{\nu'}^\dagger \Psi_0 \quad (\nu \neq \nu'). \end{aligned} \quad (\text{III-22})$$

Since the excitation energy of a two-quasi-particle state must be at least 2Δ , there is an energy gap in the spectrum of an even particle system, providing that Δ exceeds the average single-particle level spacing. It can be shown that a state with n quasi-particles is also a state of seniority n (when seniority is a good quantum number). Hence, a two quasi-particle state is spoken of as a state with a "broken pair." The ground state of a system with an odd number of particles is represented as

$$\Psi_{(odd)} = \alpha_{\nu_0}^\dagger \Psi_0 \quad \text{or} \quad \beta_{\nu_0}^\dagger \Psi_0. \quad (\text{III-23})$$

Low-lying excited states of an odd system are obtained by changing the orbital $\nu_0 \rightarrow \nu$, $\epsilon_\nu > \epsilon_{\nu_0}$ in (III-23). The excitation energy is thus

$$\Delta E = \sqrt{(\epsilon_{\nu_0} - \lambda)^2 + \Delta^2} - \sqrt{(\epsilon_{\nu} - \lambda)^2 + \Delta^2} \quad (\text{III-24})$$

in which case there is no energy gap.

Since particle number is not conserved, there is no upper limit to the number of pairs that can be (theoretically) excited. Hence, the vacuum Ψ_0 can be thought of as an infinite reservoir of condensed pairs. Such an approximation is valid only when the number of particles is large compared to the fluctuation in $\mathcal{N}$ and is expected to be fairly good for the heavy nuclei treated in this paper. Further discussion of the accuracy of the BCS-Bogolyubov method is given by Belyaev,⁵⁷ Nilsson and Prior,¹⁶ Kerman et. al.⁶⁰

At this point, it should be mentioned that the treatment of the pairing interaction given here is equivalent to a variational calculation using the vacuum Ψ_0 as trial wave function. One minimizes $\langle \Psi_0 | H(d_0) | \Psi_0 \rangle$ with respect to the U_ν , V_ν , subject to the constraints (III-7) and (III-16). This method was originally used by Bardeen, Cooper, Schrieffer to describe the phenomenon of superconductivity.

It is of importance to note that the particle number condition (III-16) is not automatically satisfied for excited states. One can, however, adjust λ and Δ for each excited state so that the correct average particle number is obtained. The chief effect of such a correction is to decrease Δ , since the broken pairs "block" the coherent scattering which gives rise to the superfluidity. In an odd system, the blocking effect of the odd particle may reduce Δ by 10-20%

compared to adjacent even systems with the same pairing strength G .¹⁶ Although the blocking effect is believed physically authentic, and calculations indicate some improvement in taking it into account,⁶¹ the method described here has the disadvantage that the excited states are not exactly orthogonal to each other nor to the ground state, since each state has a different λ and Δ . It seems also that such corrections vitiate the simplicity that the BCS approximation was designed to provide. A more natural way to treat the blocking effect is by means of the Green's function technique.⁶² In the calculations to be later described, the effect of blocking on intrinsic excitations has been omitted.

In order to evaluate the expressions for the mass parameter and the frequency, the quasi-particle transformation is performed on $\frac{\partial H}{\partial \alpha}$ and $\frac{\partial^2 H}{\partial \alpha^2}$ to give:

$$\begin{aligned} \frac{\partial H}{\partial \alpha} \Big|_{\alpha=\alpha_0} &= \sum_{\nu} \langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu \rangle_{\alpha=\alpha_0} 2V_{\nu}^2 + \sum_{\nu\nu'} \langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle (U_{\nu}U_{\nu'} - V_{\nu}V_{\nu'}) (\alpha_{\nu}^{\dagger}\alpha_{\nu'}^{\dagger} + \beta_{\nu}^{\dagger}\beta_{\nu'}) \\ &+ \sum_{\nu\nu'} \langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle_{\alpha=\alpha_0} (U_{\nu}V_{\nu'} + V_{\nu}U_{\nu'}) (\alpha_{\nu}^{\dagger}\beta_{\nu'}^{\dagger} + \beta_{\nu}\alpha_{\nu'}) \\ &+ \sum_{\mu\mu'} \langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle (U_{\mu}U_{\mu'} - V_{\mu}V_{\mu'}) (\alpha_{\mu}^{\dagger}\beta_{\mu'}^{\dagger} - \beta_{\mu}^{\dagger}\alpha_{\mu'}) \\ &+ \sum_{\mu\mu'} \langle -\mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle U_{\mu}V_{\mu'} (\alpha_{\mu}^{\dagger}\alpha_{\mu'}^{\dagger} + \beta_{\mu}^{\dagger}\beta_{\mu'}^{\dagger} \\ &\quad + \alpha_{\mu}^{\dagger}\alpha_{\mu} + \beta_{\mu'}^{\dagger}\beta_{\mu'}) , \end{aligned} \tag{III-25}$$

$$\begin{aligned} \frac{\partial^2 H}{\partial \alpha^2} \Big|_{\alpha=\alpha_0} &= \sum_{\nu} \left\langle \nu \left| \frac{\partial^2 V_{sp}}{\partial \alpha^2} \right| \nu \right\rangle_{\alpha=\alpha_0} V_{\nu}^2 + \sum_{\nu \nu'} \left\langle \nu \left| \frac{\partial^2 V_{sp}}{\partial \alpha^2} \right| \nu' \right\rangle_{\alpha=\alpha_0} (U_{\nu} U_{\nu'} - V_{\nu} V_{\nu'}) (\alpha_{\nu}^{\dagger} \alpha_{\nu'}^{\dagger} + \beta_{\nu'}^{\dagger} \beta_{\nu}) \\ &+ \sum_{\nu \nu'} \left\langle \nu \left| \frac{\partial^2 V_{sp}}{\partial \alpha^2} \right| \nu' \right\rangle_{\alpha=\alpha_0} (U_{\nu} V_{\nu'} + V_{\nu} U_{\nu'}) (\alpha_{\nu}^{\dagger} \beta_{\nu'}^{\dagger} + \beta_{\nu}^{\dagger} \alpha_{\nu'}) . \quad (\text{III-26}) \end{aligned}$$

The operators $\frac{\partial V_{sp}}{\partial \alpha}$, $\frac{\partial^2 V_{sp}}{\partial \alpha^2}$ are assumed even under time reversal.

Also, the matrix elements are assumed real so that

$$\left\langle \mu \left| \frac{\partial V_{sp}}{\partial \alpha} \right| -\mu' \right\rangle = - \left\langle -\mu \left| \frac{\partial V_{sp}}{\partial \alpha} \right| \mu' \right\rangle = - \left\langle \mu' \left| \frac{\partial V_{sp}}{\partial \alpha} \right| -\mu \right\rangle . \quad (\text{III-27})$$

The operator $\frac{\partial H}{\partial \alpha} \Big|_{\alpha=\alpha_0}$ connects the ground state Ψ_0 with two-quasi-particle states of the form $\alpha_{\nu}^{\dagger} \beta_{\nu'}^{\dagger} \Psi_0$ or $\alpha_{\mu}^{\dagger} \alpha_{\mu'}^{\dagger} \Psi_0$ and $\beta_{\mu}^{\dagger} \beta_{\mu'}^{\dagger} \Psi_0$.

Eq. (II-27) takes the form:

$$\begin{aligned} \sum_{\eta \eta'} \frac{\left| \left\langle \eta \left| \frac{\partial V_{sp}}{\partial \alpha} \right| \eta' \right\rangle \right|_{\alpha=\alpha_0}^2 (U_{\eta} V_{\eta'} + U_{\eta'} V_{\eta})^2 (E_{\eta} + E_{\eta'})}{(E_{\eta} + E_{\eta'})^2 - (\hbar \omega)^2} \\ = \frac{1}{2} \left\langle \Psi_0 \left| \frac{\partial^2 H}{\partial \alpha^2} \right| \Psi_0 \right\rangle_{\alpha=\alpha_0} = \sum_{\nu} \left\langle \nu \left| \frac{\partial^2 V_{sp}}{\partial \alpha^2} \right| \nu \right\rangle_{\alpha=\alpha_0} V_{\nu}^2 . \quad (\text{III-28}) \end{aligned}$$

and Eq. (II-28) becomes

$$B(\omega) = 2\hbar^2 \sum_{\eta\eta'} \frac{\left| \langle \eta | \frac{\partial V_{sp}}{\partial \alpha} | \eta' \rangle \right|_{\alpha=\alpha_0}^2 (U_\eta V_{\eta'} + U_{\eta'} V_\eta)^2 (E_\eta + E_{\eta'})}{\left[(E_\eta + E_{\eta'})^2 - (\hbar\omega)^2 \right]^2}, \quad (\text{III-29})$$

where $E_\eta + E_{\eta'} = \sqrt{(\epsilon_\eta - \lambda)^2 + \Delta^2} + \sqrt{(\epsilon_{\eta'} - \lambda)^2 + \Delta^2}$. The index η has been used to subsume both matrix elements of the type $\langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle$ and $\langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle$. Note that to every transition $\langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle$ corresponds a transition $\langle -\nu | \frac{\partial V_{sp}}{\partial \alpha} | -\nu' \rangle$ which contributes equally. Both contributions are included by interchange of indices $\eta \eta'$. Likewise, to every transition $\langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle$ corresponds the transition $\langle -\mu | \frac{\partial V_{sp}}{\partial \alpha} | -\mu' \rangle$ which contributes equally and is included by interchange of indices.

In the presence of superfluidity, Eqs. (III-28) and (III-29) include transitions between levels with single-particle energies ϵ_ν and $\epsilon_{\nu'}$, respectively, corresponding to a pair excitation of energy $E_\nu + E_{\nu'}$, such that $\epsilon_\nu < \lambda$ and $\epsilon_{\nu'} < \lambda$, or $\epsilon_\nu > \lambda$ and $\epsilon_{\nu'} > \lambda$, as well as transitions with $\epsilon_\nu < \lambda$ and $\epsilon_{\nu'} > \lambda$ which alone are present in the absence of superfluidity. In addition, if $\langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu \rangle \neq 0$ one also gets diagonal contributions corresponding to a spin-zero pair excitation $\nu_\nu^+ \beta_\nu^+ \Psi_0$. The pairing effect reduces the magnitude of contributions across the Fermi level λ by decreasing matrix elements through the factor $(U_\nu V_{\nu'} + U_{\nu'} V_\nu)$, and increasing energy denominators. However, the new contributions arising from superfluidity partially compensate. In the case of moments of inertia, the net effect of superfluidity is a reduction by a factor of about 2. Belyaev has shown, however, that in the case of axially symmetric quadrupole vibrations of spherical nuclei,

the diagonal contributions far outweigh all others leading to an amplification of the mass parameter by a factor of about 10^{57} . In the limit of vanishing superfluidity, $\Delta \rightarrow 0$, $(U_\eta V_{\eta'} + U_{\eta'} V_\eta) \rightarrow 1$ for $\epsilon_\eta < \lambda$, $\epsilon_{\eta'} > \lambda$ and approaches zero otherwise, while $E_\eta + E_{\eta'} \rightarrow \epsilon_{\eta'} - \epsilon_\eta$ for $\epsilon_\eta < \lambda$ and $\epsilon_{\eta'} > \lambda$, and $\lambda \rightarrow \epsilon_F$, so that (III-28) becomes

$$\sum_{\substack{\eta, \eta' \\ \epsilon_\eta \leq \epsilon_F \\ \epsilon_{\eta'} > \epsilon_F}} \frac{2 \left| \langle \eta | \frac{\partial V_{sp}}{\partial \alpha} | \eta' \rangle \right|^2 (\epsilon_{\eta'} - \epsilon_\eta)}{(\epsilon_{\eta'} - \epsilon_\eta)^2 - (\hbar \omega)^2} = \sum_{\substack{\eta \\ \epsilon_\eta \leq \epsilon_F}} \langle \eta | \frac{\partial^2 V_{sp}}{\partial \alpha^2} | \eta \rangle. \quad (\text{III-30})^*$$

A few criticisms about the derivation of Eqs. (III-28) and (III-29) are in order. These have been obtained by expanding the deformation-dependent Hamiltonian $H_{\text{crank}}(\alpha)$ in powers of $(\alpha - \alpha_0)$ and treating the unperturbed Hamiltonian $H(\alpha_0)$ by means of the BCS approximation, thus replacing $H(\alpha_0)$ by the mutilated Hamiltonian $\mathcal{H}(\alpha_0)$, and using the eigenfunctions of $\mathcal{H}(\alpha_0)$ as the unperturbed basis functions for evaluating Eqs. (II-27) and (II-28). However, an alternative procedure could have been used. The effect of the deformation (time-dependence) could be treated as a transformation of the single particle state, i.e., $\phi_\nu(\alpha) = \phi_\nu(\alpha_0 + \delta\alpha)$. One could then introduce the quasi-particle transformation (III-6) with the $U_\nu = U_\nu(\alpha)$ etc.; that is, introduce quasi-particle operators appropriate to an instantaneous value of α , and neglect terms with the same structure as previously, leading to

*The factor of 2 takes the place of interchange of indices, and is needed to include transitions $\langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle$ and $\langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle$, since the index η denotes only transitions $\langle \nu | \frac{\partial V_{sp}}{\partial \alpha} | \nu' \rangle$ and $\langle \mu | \frac{\partial V_{sp}}{\partial \alpha} | \mu' \rangle$ as previously explained.

$$\mathcal{H}_{\text{crank}}(\alpha(t)) = \sum_{\nu} (\epsilon_{\nu}(\alpha) - \lambda(\alpha)) \lambda V_{\nu}^2(\alpha) - \frac{\Delta^2(\alpha)}{G} \\ + \sum_{\nu} E_{\nu}(\alpha) (\alpha_{\nu}^{\dagger}(\alpha) \alpha_{\nu}(\alpha) + \beta_{\nu}^{\dagger}(\alpha) \beta_{\nu}(\alpha)). \quad (\text{III-31})$$

The instantaneous quasi-particle operators $\alpha_{\nu}^{\dagger}(\alpha(t))$, $\beta_{\nu}^{\dagger}(\alpha(t))$ are related to those at equilibrium $\alpha_{\nu}^{\dagger}(\alpha_0)$, by

$$\alpha_{\nu}^{\dagger}(\alpha) = \alpha_{\nu}^{\dagger}(\alpha_0) + \delta\alpha \left\{ \frac{\partial U_{\nu}}{\partial \alpha} \frac{1}{V_{\nu}} \beta_{\nu} + \sum_{\nu'} \frac{\langle \nu' | \frac{\partial V_{sp}}{\partial \alpha} | \nu \rangle}{\epsilon_{\nu} - \epsilon_{\nu'}} \left[(U_{\nu} U_{\nu'} + V_{\nu} V_{\nu'}) \alpha_{\nu'}^{\dagger} + (U_{\nu} V_{\nu'} - U_{\nu'} V_{\nu}) \beta_{\nu'} \right] \right\}_{\alpha=\alpha_0},$$

(III-32)

$$\beta_{\nu}^{\dagger}(\alpha) = \beta_{\nu}^{\dagger}(\alpha_0) + \delta\alpha \left\{ \frac{\partial U_{\nu}}{\partial \alpha} \frac{1}{V_{\nu}} \alpha_{\nu} + \sum_{\nu'} \frac{\langle \nu' | \frac{\partial V_{sp}}{\partial \alpha} | \nu \rangle}{\epsilon_{\nu} - \epsilon_{\nu'}} \left[(U_{\nu} U_{\nu'} + V_{\nu} V_{\nu'}) \beta_{\nu'}^{\dagger} - (U_{\nu} V_{\nu'} - U_{\nu'} V_{\nu}) \alpha_{\nu'} \right] \right\}_{\alpha=\alpha_0}.$$

Finally, one can expand (III-31) and evaluate Eqs. (II-27) and (II-28). However, one would find that $\frac{\partial \mathcal{H}_{\text{crank}}}{\partial \alpha} \Big|_{\alpha=\alpha_0}$ differs from $\frac{\partial \mathcal{H}_{\text{crank}}}{\partial \alpha} \Big|_{\alpha=\alpha_0}$ by terms involving $\frac{d\Delta}{d\alpha}$ and $\frac{d\lambda}{d\alpha}$, which can be evaluated assuming that (III-17) and (III-18) hold at each value of α . Hence,

one would arrive in general at somewhat different expressions for the mass parameter and frequency equation if, to state the matter crudely, one pairs first and then expands, than if one expands first and then pairs. The reason for the discrepancy is that the truncated terms are different in each case although having the same form. In the first case, one throws away $H_{\text{TRUNC}}(\alpha)$ and in the second $H_{\text{TRUNC}}(\alpha_0)$. The second procedure has been followed here because the additional terms arising from the first are most likely largely spurious. These terms only occur for the β vibration and not the γ vibration. Moreover, the assumption that Eqs. (III-17) and (III-18) hold at each value of α is not likely to be good if the adiabatic condition is not satisfied.

One can also obtain additional terms by adding to the total energy the quantity $\lambda \langle \psi(e) | \mathcal{H} | \psi(e) \rangle$, where the average is taken using the perturbed cranking model wave function rather than the unperturbed ground state. Moreover, since the collective excitation is composed of quasi-particle pairs, one should consider the blocking effect of the superposed pairs just as one should consider the blocking in excited intrinsic states. That is, one could readjust λ and Δ for the collective state so as to give the right average number of particles. In particular, as the solution of the dispersion formula approaches the lowest intrinsic pair excitation energy, the blocking effect should approach the blocking due to an intrinsic pair excitation. Such corrections might be of considerable importance when the adiabatic condition is not fulfilled.

As a final and closely related point, it is emphasized that the U_λ and V_λ introduced into the problem are variational parameters.

In the present work, these parameters have been chosen to minimize the (intrinsic) energy of the unperturbed Hamiltonian $H(\alpha'_0)$. However, it seems plausible that improved accuracy may result by requiring that the parameters minimize the sum of the collective and intrinsic energy. Such a treatment would also describe the change in superfluidity resulting from excitation of a collective state and should be considered together with the blocking effect. This problem has recently been discussed in its application to rotational motion by Mottelson and Valatin,⁶³ Prange,⁶⁴ Katz and Blatt,⁶⁵ Belyaev,⁶⁶ and Katz.⁶⁷ These authors take into account the decrease in superfluidity with increasing angular momentum, and also show that these considerations lead to additional contributions to the moment of inertia. They indicate a need for taking into account the neglected interactions between quasi-particles, and for generalizing the Bogolyubov transformation to take into account the additional effects. In addition, the recent considerations emphasize the importance of including residual interactions between nucleons which are not in time-reversed states, as well as the simple pairing interaction between pairs with zero angular momentum.

Even within the framework of the cranking model, one could introduce time-dependent parameters $V_\lambda(t)$, $U_\lambda(t)$ by using a suitable representation for the perturbation expansion, and requiring that at each instant the U_λ , V_λ be chosen to minimize $\langle \psi(t) | T + V | \psi(t) \rangle$, i.e., the intrinsic energy plus the perturbative increment. Such a requirement would correspond closely to the physical picture described in the first section of Ch. II, since the occupation of the single-particle

states described by $V_N^2(t)$ would continuously change with deformation in such a way that the nucleus is always in its lowest energy state.

All the above-mentioned esoteric corrections have been neglected in the calculations since the time-dependent cranking model and the simple BCS-Bogolyubov method seem to be dubious vehicles for making these corrections; one could not be sure that results are not spurious as long as conservation laws are violated. Moreover, other uncertainties to be later discussed may be more important. Perhaps extensions of the Sawada method or the Green's function method⁶² could be used to treat the special effects more rigorously. So far, applications of the usual Sawada method to nuclear vibrations have not taken these effects into account.

IV. QUADRUPOLE VIBRATIONS OF SPHEROIDAL

EVEN-EVEN NUCLEI

A. Beta-and Gamma-Vibrational Hamiltonian

Corresponding to the Bohr-Mottelson model Hamiltonian, (B-28), one can write a cranking-model Hamiltonian which will permit evaluation of the mass parameters B_β and B_γ , the force constants C_β and C_γ , and the frequencies $\hbar\omega_\beta$ and $\hbar\omega_\gamma$. The single-particle potential is chosen, as usual, so that the equipotential surfaces have the same symmetry as the nuclear density distribution and the volume enclosed by an equipotential is independent of deformation. It is convenient to use as the deformation-dependent part of the single-particle potential an anisotropic harmonic oscillator expressed in terms of the shape parameters β and γ :^{63*}

$$V_{osc.}(\beta, r) = \frac{m\omega_0^2}{2} \left(e^{-2\sqrt{\frac{5}{4\pi}}\beta \cos(r - 2\pi/3)} \cdot x^2 + e^{-2\sqrt{\frac{5}{4\pi}}\beta \cos(r - 4\pi/3)} \cdot y^2 + e^{-2\sqrt{\frac{5}{4\pi}}\beta \cos r} \cdot z^2 \right) \quad (IV-1)$$

*Eq. (IV-1) may be expanded as in Eq. (B-48).

The total single-particle potential is obtained by embellishing (IV-1) with the deformation-independent spin-orbit and centrifugal terms of the Nilsson model⁴¹

$$V_{sp} = V_{osc}(\beta, r) - \chi \hbar \omega_0 (2\vec{l} \cdot \vec{\sigma} + \mu l^2). \quad (IV-2)$$

The dimensionless parameters χ and μ were chosen by Nilsson to reproduce the proper spin sequence in deformed odd-mass nuclei. The parameter $\hbar \omega_0 \approx 41 A^{-1/3}$ is estimated by requiring a root mean-square nuclear radius of $1.2 \sqrt{5/3} A^{1/3}$ fermis.

Assuming an axial equilibrium shape, so that $\gamma_0 = 0$ and defining

$$\alpha_0 \equiv \sqrt{\frac{5}{4\pi}} \beta_0, \quad (IV-3)$$

the derivatives of V_{sp} are:

$$\begin{aligned} \left. \frac{\partial V_{sp}}{\partial \beta} \right|_{\substack{\beta=\beta_0 \\ r=0}} &= \frac{m\omega_0^2}{\lambda} \sqrt{\frac{5}{4\pi}} \left[e^{\alpha_0} (x^2+y^2) - 2e^{-2\alpha_0} z^2 \right] \\ &= -\frac{m\omega_0^2}{3} \left[(e^{\alpha_0} + 2e^{-2\alpha_0}) r^2 Y_{20} - \frac{1}{\lambda} \sqrt{\frac{5}{\pi}} (e^{\alpha_0} - 2e^{-2\alpha_0}) r^2 \right] \quad (\text{IV-4}) \end{aligned}$$

$$\begin{aligned} \left. \frac{1}{\lambda} \frac{\partial^2 V_{sp}}{\partial \beta^2} \right|_{\substack{\beta=\beta_0 \\ r=0}} &= \frac{5}{16\pi} m\omega_0^2 \left[e^{\alpha_0} (x^2+y^2) + 4e^{-2\alpha_0} z^2 \right] \\ &= \frac{1}{12} \sqrt{\frac{5}{\pi}} m\omega_0^2 \left[(4e^{-2\alpha_0} - e^{\alpha_0}) r^2 Y_{20} + \frac{1}{\lambda} \sqrt{\frac{5}{\pi}} (e^{\alpha_0} + 2e^{-2\alpha_0}) r^2 \right] \quad (\text{IV-5})^* \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial V_{sp}}{\partial r} \right|_{\substack{\beta=\beta_0 \\ r=0}} &= -m\omega_0^2 \frac{\sqrt{3}}{2} \alpha_0 e^{\alpha_0} (x^2 - y^2) \\ &= -m\omega_0^2 \sqrt{\frac{2\pi}{5}} \alpha_0 e^{\alpha_0} r^2 (Y_{22} + Y_{2-2}) \quad (\text{IV-6}) \end{aligned}$$

*This expression differs slightly from Eq. (II-9b) because of a difference in the definition of β to second order. However, in calculating $\left\langle \frac{\partial^2 H}{\partial \beta^2} \right\rangle$, the result is the same in both cases.

$$\begin{aligned}
 \left. \frac{1}{2} \frac{\partial^2 V_{sp}}{\partial r^2} \right|_{\beta=\beta_0, r=0} &= \frac{m\omega_0^2}{4} \alpha_0 \left[e^{\alpha_0} (3\alpha_0 - 1)(x^2 + y^2) + 2e^{-2\alpha_0} z^2 \right] \\
 &= \frac{m\omega_0^2}{3} \alpha_0 \sqrt{\frac{\pi}{5}} \left\{ \left[2e^{-2\alpha_0} - e^{\alpha_0} (3\alpha_0 - 1) \right] r^2 Y_{20} \right. \\
 &\quad \left. + \frac{1}{2} \sqrt{\frac{5}{\pi}} \left[e^{-2\alpha_0} + e^{\alpha_0} (3\alpha_0 - 1) \right] r^2 \right\}, \tag{IV-7}
 \end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$, and the $Y_{2\mu}$ are spherical harmonics,*

According to Eq. (II-19), at equilibrium

$$\left\langle \frac{\partial H}{\partial \beta} \right\rangle_{\beta=\beta_0} = 0, \tag{IV-8}$$

where the average is taken with the ground-state intrinsic wave function. Or, using (IV-4)

$$\begin{aligned}
 \left\langle \sum_i r_i^2 Y_{20}(\theta_i) \right\rangle &= \frac{1}{2} \sqrt{\frac{5}{\pi}} \frac{(e^{\alpha_0} - e^{-2\alpha_0})}{e^{\alpha_0} + 2e^{-2\alpha_0}} \left\langle \sum_i r_i^2 \right\rangle \\
 &\approx \sqrt{\frac{5}{4\pi}} \alpha_0 \left\langle \sum_i r_i^2 \right\rangle. \tag{IV-9}^4
 \end{aligned}$$

Combining this result with (IV-5) and (IV-7) gives

*Condon and Shortley phases are used.

^4The sums are over all nucleons.

$$\left\langle \frac{1}{2} \sum_i \frac{\partial^2 V_{sp}(i)}{\partial \beta^2} \right\rangle_{\substack{\beta=\beta_0 \\ r=0}} = \frac{15}{8\pi} m \omega_0^2 \frac{e^{-\alpha_0}}{e^{\alpha_0} + 2e^{-2\alpha_0}} \left\langle \sum_i r_i^2 \right\rangle, \quad (\text{IV-10})$$

and

$$\left\langle \frac{1}{2} \sum_i \frac{\partial^2 V_{sp}}{\partial r^2} \right\rangle_{\substack{\beta=\beta_0 \\ r=0}} = \frac{3}{2} m \omega_0^2 \alpha_0^2 \frac{e^{-\alpha_0}}{e^{\alpha_0} + 2e^{-2\alpha_0}} \left\langle \sum_i r_i^2 \right\rangle, \quad (\text{IV-11})$$

where the averages again are taken with the ground-state nuclear wave function.

Ideally, the self-consistent shell-model potential should allow the following consistency conditions to hold:

$$\left\langle \sum_i \frac{x_i^2}{A} \right\rangle = \left\langle \sum_i \frac{y_i^2}{A} \right\rangle = e^{-\alpha_0} \frac{R_0^2}{5}, \quad (\text{a})$$

(IV-12)

$$\left\langle \sum_i \frac{z_i^2}{A} \right\rangle = e^{2\alpha_0} \frac{R_0^2}{5}, \quad (\text{b})$$

where $R_0 = 1.2 A^{1/3}$ fermis. The conditions (IV-12) agree with (IV-9) in lowest order but are more restrictive. In fact, there is no guarantee that the choice of potential made here will allow conditions (IV-12) to hold exactly, although (IV-9) should hold if the theoretical (as distinguished from experimental values deduced from measured quadrupole

moments) equilibrium values are used. However, assuming that (IV-12) is approximately valid, (IV-10) and (IV-11) become

$$\left\langle \frac{1}{2} \sum_i \frac{\partial^2 V_{sp}(i)}{\partial \beta^2} \right\rangle_{\substack{\beta=\beta_0 \\ r=0}} = \frac{3}{8\pi} m \omega_0^2 A R_0^2, \quad (\text{IV-13})$$

and

$$\left\langle \frac{1}{2} \sum_i \frac{\partial^2 V_{sp}}{\partial r^2} \right\rangle_{\substack{\beta=\beta_0 \\ r=0}} = \frac{3}{8\pi} m \omega_0^2 A R_0^2 \beta_0^2. \quad (\text{IV-14})$$

However, even if (IV-12) is not exactly true for the choice of potential, the value $\hbar \omega_0 = 41 A^{-1/3}$ insures that approximately $\langle \sum_i r_i^2 \rangle \approx \frac{3}{5} A R_0^2$. Hence (IV-13) and (IV-14) should hold at least to lowest order in β_0 , and these estimates will be used in the numerical calculations.

It is also pointed out that $\frac{\partial V_{sp}}{\partial \beta}$ and $\frac{\partial V_{sp}}{\partial r}$ have different selection rules so that β and r can be considered as normal coordinates in agreement with the Bohr-Mottelson Hamiltonian. As a consequence the frequencies ω_β and ω_r can be obtained from two independent dispersion formulas of the form (III-28) and the corresponding mass parameters from (III-29). The restoring force constants are then found from the condition $C = B \omega^2$.

Using the estimates (IV-13) and (IV-14) for the right-hand side of Eq. (III-28), and using Eqs. (IV-4) and (IV-6), the dispersion

relations for β and γ vibrations can be written in a form analogous to that obtained from the Sawada method

$$\sum_{\eta\eta'} \frac{|\langle \eta | (1 - d_0 + \dots) r^2 Y_{20} - (\frac{\sqrt{5}}{4\pi} d_0 + \dots) r^2 | \eta' \rangle|^2 (U_\eta V_{\eta'} + U_{\eta'} V_\eta)^2 (E_\eta + E_{\eta'})}{(E_\eta + E_{\eta'})^2 - (\hbar\omega_\beta)^2} = \frac{1}{2\chi}, \quad (a)$$

$$\sum_{\eta\eta'} \frac{|\langle \eta | \frac{e}{\sqrt{2}} r^2 (Y_{22} + Y_{2-2}) | \eta' \rangle|^2 (U_\eta V_{\eta'} + U_{\eta'} V_\eta)^2 (E_\eta + E_{\eta'})}{(E_\eta + E_{\eta'})^2 - (\hbar\omega_\gamma)^2} = \frac{1}{2\chi}, \quad (b) \quad (IV-15)^*$$

where the coupling constant is given by

$$\chi = \frac{4\pi}{3} \frac{m\omega_0^2}{AR_0^2}. \quad (IV-16)$$

Once the quantities $\hbar\omega$, B and C are calculated, several other quantities of interest may easily be obtained. From the Bohr-Mottelson Hamiltonian (B-28), one may obtain expressions for the reduced electric quadrupole transition probabilities from an initial state with angular momentum I_i to a final state with angular momentum I_f , defined by

$$B(E2; I_i \rightarrow I_f) = \sum_{\mu} |\langle i | M_{2\mu} | f \rangle|^2, \quad (IV-17)$$

* These equations do not include a small correction for coulombic repulsion of protons, discussed later in the section.

where the $M_{2\mu}$ are components of the electric quadrupole tensor operator. In particular, the transition probabilities from the ground state to the $2+$ state of a one-phonon β - or γ -vibrational band, which have been measured in some cases by coulomb excitation, will be theoretically calculated. Neglecting vibration-rotation interaction perturbations, the transition probabilities to the β and γ bands, respectively, are⁶⁹

$$B^{\beta}(E2; 0 \rightarrow 2') = \left(\frac{3}{4\pi} Z e R_0^2 \right)^2 \frac{1}{2} \frac{\hbar \omega_{\beta}}{C_{\beta}}, \quad (a)$$

(IV-18)

$$B^{\gamma}(E2; 0 \rightarrow 2') = \left(\frac{3}{4\pi} Z e R_0^2 \right)^2 \beta_0^2 \frac{\hbar \omega_{\gamma}}{C_{\gamma}}. \quad (b)$$

If mixing of the collective model wave function due to the vibration-rotation terms (B-29d) and (B-29e) is included, the corresponding transition probabilities, to lowest order in perturbation theory are:

$$B'^{\beta}(E2; 0 \rightarrow 2') = B^{\beta}(E2; 0 \rightarrow 2') \left(1 - \frac{6\hbar^2}{\mathcal{I} \hbar \omega_{\beta}} \right)^2_{(a)},$$

(IV-19) *

$$B'^{\gamma}(E2; 0 \rightarrow 2') = B^{\gamma}(E2; 0 \rightarrow 2') \left(1 - \frac{2\hbar^2}{\mathcal{I} \hbar \omega_{\gamma}} \right)^2_{(b)}.$$

* $\mathcal{I}$ is the moment of inertia of the ground-state band.

Eqs. (IV-18) and (IV-19) agree with those obtainable by means of the Sawada method, if the dynamic force constants $C(\omega) = B(\omega)\omega^2$ rather than the corresponding adiabatic quantities are used.²⁰ Specifically, the force constants to be used for β and γ vibrations, respectively, are:

$$C_{\beta}(\omega) = 2(\hbar\omega_{\beta})^2 \sum_{\eta\eta'} \frac{|\langle \eta | \frac{\partial V_{SP}}{\partial \beta} | \eta' \rangle|^2}{\left[(E_{\eta} + E_{\eta'})^2 - (\hbar\omega_{\beta})^2 \right]^2} (U_{\eta}V_{\eta'} + U_{\eta'}V_{\eta})^2 (E_{\eta} + E_{\eta'})^2 \quad (a),$$

(IV-20)

$$C_{\gamma}(\omega) = 2(\hbar\omega_{\gamma})^2 \sum_{\eta\eta'} \frac{|\langle \eta | \frac{\partial V_{SP}}{\partial \gamma} | \eta' \rangle|^2}{\left[(E_{\eta} + E_{\eta'})^2 - (\hbar\omega_{\gamma})^2 \right]^2} (U_{\eta}V_{\eta'} + U_{\eta'}V_{\eta})^2 (E_{\eta} + E_{\eta'})^2 \quad (b),$$

where $\hbar\omega_{\beta}$ and $\hbar\omega_{\gamma}$ are obtained by solving Eqs. (IV-15a) and (IV-15b).

With a knowledge of the force constants C_{β} and C_{γ} , one could attempt a calculation of the vibration-rotation interaction coefficient F in Eq. (B-39). An expression for F may be obtained by applying second-order perturbation theory to the terms (B-29d) and (B-29e) of the Bohr-Mottelson Hamiltonian.

The vibration-rotation coefficient may also be obtained from the cranking model, assuming that the deformation of the cranked potential changes until the centrifugal force balances the restoring force. In either case, assuming that the moments of inertia about the principal axes have the deformation dependence

$$\mathcal{F}_r \propto f(\beta) \sin^2(\gamma - 2\pi/3 r), \quad r=1,2,3, \quad (\text{IV-21})$$

one obtains for F ⁶⁹

$$F = \frac{1}{6C_r} \left(\frac{\hbar^2}{\mathcal{F}_1} \right)_{\substack{\beta=\beta_0 \\ \gamma=0}}^2 + \frac{1}{2C_\beta} \left(\frac{\partial}{\partial \beta} \frac{\hbar^2}{2\mathcal{F}_1} \right)_{\substack{\beta=\beta_0 \\ \gamma=0}}^2. \quad (\text{IV-22})$$

The first term on the right is contributed by the γ band and the second by the β band.

In the Bohr-Mottelson model, the β dependence of the moment of inertia on deformation is given by

$$\mathcal{F}_r \propto \beta^2. \quad (\text{IV-23})$$

In the liquid-drop model, the moments of inertia are proportional to the quadrupole vibrational mass parameters, $B_{2\mu}$, which are all equal. However, Kerman has presented arguments that in general the proportionality constants $B_{2\mu}$ are not all equal, although the quadratic β dependence* still holds.⁷⁰ However, Migdal, applying

*See also Eq. (B-23) based on this point of view.

the Green's function method to superfluid nuclei, gets the following β dependence: ⁶²

$$\tilde{\mathcal{S}}_{\text{Migdal}} = \frac{2}{5} M A R_0^2 (2.33\beta - .10) . \quad (\text{IV-24})$$

Anyhow, the proportionality constant is not needed if the derivative of $\tilde{\mathcal{S}}_1$ is evaluated for a β^n dependence since

$$\left. \frac{\partial}{\partial \beta} \frac{1}{\tilde{\mathcal{S}}_1} \right|_{\substack{\beta=\beta_0 \\ r=0}} = \left. \frac{\partial}{\partial \beta} \frac{1}{\kappa \beta^n} \right|_{\substack{\beta=\beta_0 \\ r=0}} = -\frac{n}{\kappa} \frac{1}{\beta_0^{n+1}} = \frac{-n}{\beta_0 \tilde{\mathcal{S}}_1(\beta_0)} . \quad (\text{IV-25})$$

Since $\tilde{\mathcal{S}}_1(\beta_0)$ is known from experiment (and can be calculated, theoretically in good agreement),^{15,16} one can compare Eq. (IV-22) with experiment using different β dependences and the calculated force constants.

There is a further difficulty in the interpretation of (IV-22). Within the framework of the quantum-mechanical Hamiltonian (B-28) as interpreted in this paper, the force constants C_β and C_γ are the dynamic values $C(\omega)$ appearing in the vibrational Hamiltonian. However, the cranking-model derivation refers to the adiabatic (static) force constants $C(0)$. Of course, if the collective vibration is adiabatic, $C(\omega)$ approaches $C(0)$ and there is no ambiguity. The Bohr-Mottelson Hamiltonian was originally intended to apply to the adiabatic limit. On physical grounds, it seems more plausible that the adiabatic force constants should be used in (IV-22) even though $C(\omega)$ deviates from $C(0)$. In the nuclear reference frame, the Coriolis force has the

effect of an external deforming force, the resistance to which would, ostensibly, involve the static restoring force, as described by the cranking model. In order to settle the question, more rigorous methods may have to be used. Perhaps the method of linearized equations of motion (Sawada method) could be applied to the case where simultaneous vibration and rotation are allowed to occur as consequences of a quadrupole-quadrupole force. This has not yet been done.

In this paper, a phenomenological approach to the evaluation of F is used. Eq. (IV-22) is evaluated using both the $C(\omega)$ and $C(0)$ for both a linear and quadratic β dependence of the moment of inertia, and results will be compared with experiment in a later chapter.

B. Details of Calculations

Single Particle Matrix Elements

The shell-model states used in the calculation are eigenstates of the axially symmetric Nilsson potential,⁴¹ obtained from (IV-1) by setting $\gamma = 0$ and $\beta = \beta_0$:

$$V_{\text{Nilsson}} = \frac{m\omega_0^2}{2} \left[e^{\alpha_0} (\chi^2 + \eta^2) + e^{-2\alpha_0} z^2 \right] - \kappa \hbar \omega_0 (2\vec{l} \cdot \vec{\sigma} + \mu l^2), \quad (\text{IV-26})$$

where the deformation parameter α_0 is given by Eq. (IV-3).*

The Nilsson levels account satisfactorily for the spin sequence of odd-mass deformed nuclei,⁷¹ and have been used in calculations of moments of inertia of rotating nuclei,^{15,16} and in the calculation of equilibrium deformations and electric quadrupole moments,^{72,73} giving reasonably good agreement with experiment.

In the case of strongly deformed nuclei, it is convenient to use the asymptotic representation suggested by Nilsson.⁴¹ The Hamiltonian corresponding to (IV-26) is written in terms of cylindrical coordinates r, ϕ, z , where z lies along the axis of cylindrical symmetry. This representation is convenient because the spin and orbital motion become decoupled in the limit of increasing deformation, and the eigenfunctions of H_{Nilsson} tend toward those of a pure harmonic

* α_0 is related to Nilsson's parameter δ , by $\alpha_0 \approx \frac{2}{3}\delta$.

oscillator in cylindrical coordinates. Exact diagonalizations by Rassey⁷⁴ indicate that the asymptotic components comprise more than 90% of the exact state vector for many states with deformations of the observed magnitude, although for a few states with high oscillator quantum numbers the percentage is as low as 50%.

The approximation of using pure asymptotic single-particle wave functions in the numerical calculations was made, since some compensation of errors occurs in performing the various summations. Moreover, the advantage is gained of having matrix elements and selection rules in closed form.

The asymptotic solutions may be written in the form^{41,75}

$$\begin{aligned}
 |N n_1 \Lambda \Sigma\rangle &\equiv \\
 |n_1 n_2 \Lambda \Sigma\rangle &= f_{\Lambda \Sigma} \left\{ \left(\frac{2m\omega_0}{\hbar} \right)^{3/2} \cdot \frac{1}{2^{n_2} n_2!} \cdot \left(\frac{n_1 - |\Lambda|}{2} \right)! \cdot \left[\left(\frac{n_1 + |\Lambda|}{2} \right)! \right]^3 \right\}^{1/2} \\
 &\cdot e^{i\Lambda\phi} e^{-\frac{m\omega_0}{2\hbar} z^2} H_{n_2} \left(\sqrt{\frac{m\omega_0}{\hbar}} e^{-\frac{d_0}{2}} z \right) \cdot \quad (IV-27) \\
 &\cdot e^{-\frac{m\omega_0}{2\hbar} r^2} \left(\frac{m\omega_0}{\hbar} e^{d_0/2} r^2 \right)^{\frac{|\Lambda|}{2}} L_{\frac{n_1 + |\Lambda|}{2}} \left(\frac{m\omega_0}{\hbar} e^{d_0/2} r^2 \right),
 \end{aligned}$$

where n_1 is the number of oscillator quanta perpendicular to the z-axis;

where n_2 is the number of oscillator quanta parallel to the z-axis;

where $N = n_1 + n_2$ is the major oscillator quantum number;

where Λ represents the units of orbital angular momentum projected on the z-axis, and for a given n_1 takes on the values of $n_1, n_1 - 2, \dots, -n_1$;

where Σ is the projection of the intrinsic nucleon spin on the Z-axis and takes on the value $\pm \frac{1}{2}$, with χ_{Σ} the corresponding spin vector;

where H_{n_z} is a Hermite polynomial and $L_{\frac{n_z+|\Lambda|}{2}}^{|\Lambda|}$ is an associated Laguerre function;

where $f_{\Lambda\Sigma}$ is a phase factor chosen to satisfy transformation under the time-reversal operator $T=UK$, where K is the operation of complex conjugation and $U = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Time-reversed states are related by $T|Nn_z\Lambda\Sigma\rangle = (-1)^{\Sigma+\frac{1}{2}}\langle Nn_z\Lambda\Sigma| = |Nn_z-\Lambda-\Sigma\rangle$. The following conventions are chosen for a vector $|Nn_z\Lambda\Sigma\rangle$,

$$\text{if } \Omega \equiv \Lambda + \Sigma > 0, \quad f_{\Lambda\Sigma} = 1,$$

$$\text{if } \Omega \equiv \Lambda + \Sigma < 0, \quad f_{\Lambda\Sigma} = (-1)^{\Sigma-\frac{1}{2}},$$

$$\text{or } f_{\Lambda\Sigma} = \left(\frac{\Omega}{|\Omega|}\right)^{\Sigma-\frac{1}{2}}.$$

Because of the spin-orbit coupling, Λ, Σ are not strictly good quantum numbers individually although Ω is a good quantum number. For large deformations, however, the Nilsson states may be labeled by the quantum numbers of the dominant asymptotic component. In referring later to single-particle states, the notation of Nilsson will be used in which each state is labeled by $N, n_z, \Lambda, \Omega, \pi$, where π is the parity and is even or odd when N is even or odd. Corresponding to each Nilsson eigenvalue there are two degenerate states differing in the sign of Ω , which are therefore time-reversed pairs.

The single-particle matrix elements of $\frac{\partial V_{sp}}{\partial \beta}$ and $\frac{\partial V_{sp}}{\partial r}$ can be obtained with the aid of Shaffer's tables⁷⁵ and are given below:

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial \beta} | n_1+2 n_2 \Lambda \Sigma \rangle = -\frac{1}{4} \sqrt{\frac{5}{4\pi}} e^{\alpha_0/2} \hbar \omega_0 \sqrt{(n_1+1+2)(n_1+2)} , \quad (a)$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial \beta} | n_1-2 n_2 \Lambda \Sigma \rangle = -\frac{1}{4} \sqrt{\frac{5}{4\pi}} e^{\alpha_0/2} \hbar \omega_0 \sqrt{(n_1+1)(n_1-1)} , \quad (b)$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial \beta} | n_1 n_2+2 \Lambda \Sigma \rangle = -\frac{1}{2} \sqrt{\frac{5}{4\pi}} e^{-\alpha_0} \hbar \omega_0 \sqrt{(n_2+1)(n_2+2)} \quad (c), \quad (IV-28)$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial \beta} | n_1 n_2-2 \Lambda \Sigma \rangle = -\frac{1}{2} \sqrt{\frac{5}{4\pi}} e^{-\alpha_0} \hbar \omega_0 \sqrt{n_2(n_2-1)} , \quad (d)$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial \beta} | n_1 n_2 \Lambda \Sigma \rangle = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \hbar \omega_0 \left[e^{\alpha_0/2} (n_1+1) - e^{-\alpha_0} (2n_2+1) \right]_{(e)} ,$$

*It is understood that the matrix elements are evaluated at

$\beta = \beta_0$, $r = 0$.

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1+2 n_2 \Lambda+2 \Sigma \rangle = -(\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1+\Lambda+2)(n_1+\Lambda+4)}_{(a)},$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1-2 n_2 \Lambda-2 \Sigma \rangle = -(\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1+\Lambda-2)(n_1+\Lambda)}_{(b)},$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1-2 n_2 \Lambda+2 \Sigma \rangle = -(\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1-\Lambda-2)(n_1-\Lambda)}_{(c)},$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1+2 n_2 \Lambda-2 \Sigma \rangle = -(\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1-\Lambda+2)(n_1-\Lambda+4)}_{(d)}, \quad (IV-29)$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1 n_2 \Lambda+2 \Sigma \rangle = (\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1+\Lambda+2)(n_1-\Lambda)}_{(e)},$$

$$\langle n_1 n_2 \Lambda \Sigma | \frac{\partial V_{sp}}{\partial r} | n_1 n_2 \Lambda-2 \Sigma \rangle = (\pm)^{\Sigma-\frac{1}{2}} \hbar \omega_0 \frac{\sqrt{3}}{8} d_0 e^{\frac{d_0}{2}} \sqrt{(n_1-\Lambda+2)(n_1+\Lambda)}_{(f)},$$

where the factor $(\pm)^{\Sigma - \frac{1}{2}}$ takes into account time-reversal, transformation; the upper sign is used for transition of the type $\langle \nu | \frac{\partial V_{sp}}{\partial r} | \nu' \rangle$ and $\langle -\nu | \frac{\partial V_{sp}}{\partial r} | -\nu' \rangle$, that is $|\Omega| \pm |\Omega'|$ or $-|\Omega| \pm -|\Omega'|$, while the lower sign is used for transitions $\langle -\mu | \frac{\partial V_{sp}}{\partial r} | \mu' \rangle$ and $\langle \mu | \frac{\partial V_{sp}}{\partial r} | -\mu' \rangle$, that is $-|\Omega| \pm |\Omega'|$. It is easily verified that this choice of phase agrees with the transformation properties given by Eq. (III-3).

The deformation parameter $\alpha_0 = \sqrt{\frac{5}{4\pi}} \beta_0$ as previously. The operator $\frac{\partial H}{\partial \beta}$ (see Eq. III-25) has matrix elements only between major shells with $\Delta N = 2$ in the absence of pairing as can be seen from (IV-28). In the presence of superfluidity there are the diagonal contributions, (IV-28e), due to excitation of states of the form $d_{\nu}^{\dagger} \beta_{\nu}^{\dagger} \Psi_0$. The pair excitations $d_{\nu}^{\dagger} \beta_{\nu}^{\dagger} \Psi_0$ connected to the ground state by $\frac{\partial H}{\partial \beta}$ have $\Omega = 0$ as can be seen from the selection rules (IV-28). The operator $\frac{\partial H}{\partial r}$, on the other hand, has transitions within a major shell even in the absence of superfluidity, in addition to the $\Delta N = 2$ transitions. The pair excitations $d_{\nu}^{\dagger} \beta_{\nu}^{\dagger} \Psi_0$, $d_{\mu}^{\dagger} d_{\mu}^{\dagger} \Psi_0$ and $\beta_{\mu}^{\dagger} \beta_{\mu}^{\dagger} \Psi_0$ connected to the ground state by $\frac{\partial H}{\partial r}$ have $\Omega = \pm 2$. The β -vibrational operator does not have matrix elements of the type previously denoted by $\langle \mu | \frac{\partial V}{\partial \alpha} | \mu' \rangle$, while the γ -vibrational operator has such transitions of the type $\Omega = \frac{1}{2} \rightleftharpoons \Omega = -\frac{3}{2}$ and $\Omega = \frac{3}{2} \rightleftharpoons \Omega = -\frac{1}{2}$.

The Pairing Interaction

The inclusion of the pairing interaction generally requires the simultaneous solution of the non-linear equations (III-17) and (III-18) for λ and Δ if the pairing strength G is known. Mottelson has estimated a value of $G \approx \frac{25-30}{A}$ Mev from nucleon-nucleon scattering

data.¹⁶ However, Δ can be estimated independently from empirical pairing energies, beta decay energies, and in a few cases directly from observed gaps in the spectra of even-even isotopes. The values of Δ for neutrons and protons estimated in this way by Nilsson and Prior¹⁶ and used in the moment of inertia calculations have also been used in the present calculation. These are listed in Table IV-1. Consequently, it was necessary to solve only Eq. (III-18) for λ separately for neutrons and protons. The corresponding G was obtained from Eq. (III-17). The procedure chosen saves considerable computer time as compared to simultaneous solution of the equations for a fixed G .

The corresponding value of G depends on the number of single-particle levels included in the region where the Fermi distribution is modified. In the case of a spherical nucleus, only the states outside of closed shells are included, as long as the energy separation between shells exceeds the energy gap, since nucleons composing the closed-shell core cannot scatter coherently when all the single-particle states are occupied. In the case of strongly deformed nuclei, the shell structure breaks down, so that a cutoff on the region of coherent scattering is rather arbitrary. The arbitrariness can be removed by using a less idealized pairing force, in which case, all the matrix elements of G_{ij} would not be equal, and Δ would be a function of ϵ_j , gradually diminishing away from the Fermi level λ . The choice of a constant two-body matrix element G corresponds to an average value in a region near the Fermi level where all the matrix elements have the same sign. However, considering all the uncertainties of the theory, it is more profitable to study the qualitative effects of the idealized

pairing force than to solve a large number of coupled nonlinear equations as would otherwise be necessary.

In the numerical calculations, all the Nilsson levels were included within a region of energy width E_{pair} symmetric about λ , where E_{pair} was taken as $1.5 \hbar\omega_0$ for neutrons and protons in the rare earths and $1.2 \hbar\omega_0$ in the actinides. It was found that inclusion of more states does not significantly alter the calculated results. Moreover, the estimate of Δ from pairing energies is not valid very far from λ . The number of nucleons within the region of width E_{pair} is the same as the number included with a sharp Fermi surface. With these conventions, the region corresponds approximately to a half-full shell. However, transitions were allowed between states outside of the region and those within; so that the cutoff does not introduce any serious artifacts.

The values of GxA corresponding to the chosen Δ 's are listed for each isotope in Table IV-1. It is seen that this quantity is not strictly constant; the values at the extremes of a region of deformed isotopes tend to be larger than in the middle. This may possibly be due to overestimates of Δ because of failure to properly account for the rearrangement energy involved in the transition from deformed to spherical shape. However, it is doubtful that the results of the calculations would be significantly better if a constant value of GxA were required. Besides, there is no rigorous justification for a strict A^{-1} rule.

Equilibrium Deformations

For the sake of consistency, the equilibrium deformations chosen for the calculation should be the theoretical values corresponding to

Table IV-1 Proton and neutron values of Δ and strength parameters G^a

Nucleus	$\Delta_n / \mu\omega_0$	$\Delta_p / \mu\omega_0$	Case (a)		Case (b)	
			$G_n \times A$	$G_p \times A$	$G_n \times A$	$G_p \times A$
			(Mev)	(Mev)	(Mev)	(Mev)
Sm ¹⁵²	.1627	.1751	28.1	31.6	24.5	Same as case (a) for all nuclides
Sm ¹⁵⁴	.1360	.1665	26.5	31.1	23.4	
Gd ¹⁵⁴	.1635	.1665	27.6	30.8	24.6	
Gd ¹⁵⁶	.1366	.1443	26.7	29.3	23.7	
Gd ¹⁵⁸	.1240	.1353	24.5	28.8	22.3	
Gd ¹⁶⁰	.1160	.1326	23.4	28.8	21.8	
Dy ¹⁶⁰	.1245	.1326	24.8	29.4	23.0	
Dy ¹⁶²	.1165	.1287	23.5	29.3	22.4	
Dy ¹⁶⁴	.1088	.1251	23.5	29.4	22.3	
Er ¹⁶⁴	.1170	.1251	23.6	28.9	22.5	
Er ¹⁶⁶	.1093	.1224	23.2	28.7	22.2	
Er ¹⁶⁸	.1023	.1202	22.3	28.7	21.5	
Er ¹⁷⁰	.09525	.1179	21.8	28.6	21.1	
Yb ¹⁷⁰	.1027	.1179	22.4	28.9	21.6	
Yb ¹⁷²	.09565	.1145	21.7	28.6	21.0	
Yb ¹⁷⁴	.09030	.1112	21.1	28.4	20.4	
Yb ¹⁷⁶	.08940	.1095	20.9	28.0	20.4	
Hf ¹⁷⁶	.09060	.1095	21.6	28.9	20.9	
Hf ¹⁷⁸	.08975	.1125	21.2	28.7	20.6	
Hf ¹⁸⁰	.09985	.1169	21.1	29.2	20.7	
W ¹⁸²	.1002	.1228	21.1	31.3	20.6	
W ¹⁸⁴	.1179	.1279	21.7	32.4	21.7	
W ¹⁸⁶	.1330	.1326	22.3	32.4	22.3	
Average Value in Rare Earth Region			23.2	29.6	22.0	

^a All values of Δ are taken from reference (16), unless otherwise indicated.

Table IV-1 Proton and neutron values of Δ and strength parameters G^a (cont)

Nucleus	$\Delta_n/k\omega_o$	$\Delta_p/k\omega_o$	Case (a)		Case (b)	
			$G_n \times A$ (Mev)	$G_p \times A$ (Mev)	$G_n \times A$ (Mev)	$G_p \times A$ (Mev)
Th ²²⁸	.1177	.1337	23.3	29.6	21.8	28.0
Th ²³⁰	.1141	.1341	23.5	30.6	22.2	28.3
Th ²³²	.1067	.1345	23.1	29.7	21.7	28.3
Th ²³⁴	.09950	.1348	23.0	29.7	21.2	28.5
U ²³⁰	.1141	.1337	23.0	31.3	21.7	29.0
U ²³²	.1067	.1340	23.4	31.6	21.7	29.2
U ²³⁴	.09950	.1344	22.6	33.7	20.7	31.1
U ²³⁶	.09300	.1348	22.5	32.1	20.7	29.7
U ²³⁸	.08705	.1352	22.2	32.1	20.1	30.0
Pu ²³⁶	.09300	.1126	22.2	31.2	20.8	28.8
Pu ²³⁸	.08705	.1129	22.0	31.4	20.3	29.0
Pu ²⁴⁰	.08260	.1132	22.0	31.7	19.8	29.3
Pu ²⁴²	.08210	.1136	22.9	31.7	20.5	29.5
Cm ²⁴²	.08210	.1130	21.9	33.4	19.9	31.2
Cm ²⁴⁴	.08490	.1133	23.4	33.7	20.3	31.2
Cm ²⁴⁶	.09020	.1136	24.6	33.9	21.9	31.5
Cm ²⁴⁸	.09455	.1139	25.0	34.0	22.3	31.7
Cf ²⁴⁸	.09455	.1166	25.0	34.0	22.3	32.5
Cf ²⁵⁰	.09610	.1170	25.3	34.3	22.5	32.8
Cf ²⁵²	(.09850) ^a	(.1172)				
Cf ²⁵⁴	(.1006)	(.1206)				
Fm ²⁵²	(.09857)	(.1177)				
Fm ²⁵⁴	(.1005)	(.1206)				
Fm ²⁵⁶	(.1055)	(.1226)				
Average Value in Actinide Region			23.2	32.1	21.2	30.0

^a Values in parentheses are extrapolated from preceding values.

the chosen potential, rather than the experimental values obtained from electric quadrupole moment measurements. The theoretical deformations of Bes and Szymanski⁷² for the rare earths and those of Szymanski⁷³ for the actinides were used. Their values were obtained by minimizing Eq. (III-21) using Nilsson levels, and in general are in excellent agreement with experiment, except for a few cases near the beginning of a region of deformation. There are slight differences in the conditions of calculation between their work and the present calculations, in choice of gaps, positions of a few single-particle orbitals, and treatment of the Coulomb correction, but it is believed that the differences are generally too small to cause any inconsistencies.

Modification of Nilsson Scheme

The positions of some single-particle levels have been slightly modified from those originally given by Nilsson.⁴¹ According to Mottelson and Nilsson,⁷¹ somewhat different values of the parameter μ (see Eq. IV-26) are necessary for certain shells in order to account extensively for observed levels in deformed odd-mass nuclei. The effect of a change in μ is chiefly to translate uniformly the levels of an affected subshell along the energy coordinate in the Nilsson diagram. In addition to the modifications suggested by Mottelson and Nilsson, one must also shift some levels from lower occupied shells and from upper unoccupied levels which are not observed in the spectra of

*The author is indebted to S. G. Nilsson for information about this work in advance of publication.

deformed nuclei. Some of these levels must be shifted in order to prevent interference with the observed order of filling of levels between two magic numbers, although the magnitudes of the shifts are uncertain.

It was found that the results of the calculations are quite sensitive to the positions of such levels at the extremes of a region of deformation. Consequently, in both the rare earth and actinide regions, complete calculations were carried out for two sets of shifts denoted by (a) and (b) in each case. The actual shifts that were made are given in Table IV-2.

The γ -vibrational results in the rare earth region are especially sensitive to the position of the neutron $N = 4\frac{1}{2}g_{9/2}$ levels. In case (a), the eigenvalues were lowered by $.37 \hbar\omega_0$. The calculations of Bes and Szymanski indicate that this large shift is necessary to prevent nuclei with neutron number less than 90 from acquiring anomalously large deformations. Even with the shift, the transition from small to large deformations occurs theoretically at neutron number 88 rather than 90 as indicated by experiment. Unfortunately, this large shift makes the vibrational energies at the beginning of the rare earth region too high.

It is possible that the locations of the single-particle levels may change significantly in going from isotope to isotope at the beginning of the region, since there is no a priori reason for accepting a universal level scheme parametrized by the quantity $\hbar\omega_0 \sim A^{-1/3}$. Also, it is possible that neglect of intershell couplings by Nilsson affects matters at the beginning of the region, and that if these couplings were taken into account, such large shifts would not be

needed.* It is also likely that estimates of the shifts would be different if the spin-orbit strength K were allowed to vary from shell to shell; K , however, has been kept constant at $K = .05$ throughout.

At any rate, it seemed profitable to try the smaller shift of $-.14 \hbar\omega_0$ for the $N = 4$ levels in case (b). The other modifications are in line with ref. (71).

In the actinide region, in addition to shifts recommended in ref. (71), the proton $N = 4 \frac{7}{2} g_{9/2}$ and neutron $N = 5 \frac{7}{2} h_{11/2}$ levels were modified as shown in Table IV-2. The results were found to be quite sensitive to the positions of these levels. The shifts in case (a) are close to those used by Szymanski⁷³ and by Nilsson and Prior.¹⁶ Those in case (b) are the smallest changes consistent with experimental data on odd-A nuclei. It is necessary, however, to assume slightly smaller equilibrium deformations at the beginning of the region in case (b) than in case (a) to obtain the correct single-particle levels for odd-A nuclei, although the theoretical deformations seem to increase in case (b) as indicated by imaginary solutions of the dispersion formula. In this case, calculations were performed for several deformations for nuclei at the beginning of the region, and the best were plotted.

The Nilsson eigenvalues, given for values of the deformation $\eta = 2, 4, 6$ (where $\eta \approx \delta/R$) were parabolically interpolated.

*According to Nilsson, the neglected couplings can approximately be accounted for by a change in scale as explained on p. 39 of reference (41). This correction was made in the present calculations.

Table IV-2 Shifts of Nilsson eigenvalues ^a

Region		Shells used	Shells shifted	Amount shifted (units of $\hbar\omega_0$)	
			<u>N</u> <u>subshell</u>	<u>case(a)</u>	<u>case(b)</u>
Rare earths	Neutrons	$2 \leq N^b \leq 7$ for case (a) and (b)	4 $\neq g_{9/2}^c$	-.37	-.14
			5 = $p_{3/2}$	+.04	+.04
			6 $\neq i_{13/2}$	+.15	+.15
			7 all	+.15	+.15
	Protons	$2 \leq N \leq 7$ for case (a) and (b)	5 = $h_{11/2}$	-.20	-.20
			5 $\neq h_{11/2}$	+.15	+.15
			6 all	+.15	+.15
			7 all	+.15	+.15
Actinides	Neutrons	$3 \leq N \leq 7$ for case (a)	5 $\neq h_{11/2}$	-.25	-.18
			7 = $j_{15/2}$	-.06	-.06
			880 $1/2$	∞ position corresponds to $\mu=.35$	
			871 $1/2$		
			871 $3/2$		
			860 $1/2$		
	Protons	$4 \leq N \leq 8$ for case (b)	862 $3/2$		
			862 $5/2$		
			4 $\neq g_{9/2}$	-.25	-.15
			6 = $i_{13/2}$	-.35	-.35
			7 = $j_{15/2}$	-.06	-.06

^a The original eigenvalues are taken from references (41) and (71).

^b $\underline{N}$ is the major oscillator quantum number.

^c " $N = 4 \neq g_{9/2}$ " means all levels with $N = 4$, except those of the $g_{9/2}$ subshell.

Eigenvalues beyond $\eta = 6$ were obtained by a perturbation technique for $\eta = 10$, and a parabola fitted to the eigenvalues for $\eta = 4, 6, 10$.

Coulomb Correction

The calculations should also include a correction for the coulombic repulsion of protons. The coulomb force, being of long range, contributes to the quadrupole field. Thus, it could be taken into account by using a larger coupling constant for the coupling of protons than for the coupling of neutrons to the deformed field. The Nilsson model, however, assumes the same coupling constants (or an average of the two). In the present calculations, the electrostatic repulsion has been treated crudely by adding the coulomb energy of a uniformly charged spheroid to the total deformation energy. For deformations, this is given by

$$E_c = \frac{3}{5} \frac{(Ze)^2}{R_0} \left(1 - \frac{\beta^2}{4\pi} - \dots \right). \quad (\text{IV-30})$$

The quantity $\frac{1}{2} \frac{\partial^2 E_c}{\partial \beta^2} \Big|_{\beta=\beta_0}$ is added to the right hand side of the dispersion formula (III-28) for β vibrations. For γ vibrations, $\frac{1}{2} \frac{\partial^2 E_c}{\partial \gamma^2} = \frac{1}{2} \frac{\partial^2 E_c}{\partial \beta^2} \beta_0^2$ is used. Bés and Szymanski treated the coulomb force in a similar manner in calculating the equilibrium deformations, except for the use of a trapezoidal instead of uniform charge distribution.

Neutron and Proton Contributions

The sums in (III-28) and (III-29) break up into separate neutron and proton contributions, $\sum(\text{neutrons}) + \sum(\text{protons})$, since

H_{crank} has the form $H(\text{neutrons}) + H(\text{protons})$ with unperturbed basis wave functions $\psi(\text{neutrons}) \cdot \psi(\text{protons})$. The long-range neutron-proton interaction has been included in the average quadrupole field. The pairing interaction between neutrons and protons, being of short range, is assumed negligible as long as neutrons and protons near the top of the respective Fermi seas occupy different orbitals, as is the case with heavy nuclei. The fluctuating quantities involved in the collective vibrations are assumed to be the total mass quadrupole tensor components generated by the sum of the proton and neutron density distributions.

As previously mentioned, Tamura and Udagawa,^{24,25} however, give a dispersion formula for quadrupole vibrations obtained by the Sawada method, in which the long-range neutron-proton interaction is explicitly included. This formula has also been obtained by the TDSCF method in this paper (Eq. II-94). It is assumed in the calculations that the strengths of the neutron-proton, neutron-neutron, and proton-proton interactions are all equal so that, according to Eq. (II-95), the dispersion formula breaks up into a sum of separate neutron and proton contributions.

Calculated Quantities

The frequencies $\hbar\omega_\beta, \hbar\omega_\gamma$, the corresponding mass parameters B_β, B_γ , and force constants C_β and C_γ were calculated with and without the adiabatic approximation for isotopes in the rare earth and actinide regions on the I.B.M. 650 computer. In addition, the reduced transition probabilities $B(E2; 0 \rightarrow 2')$, and the vibration-rotation interaction coefficients were evaluated. The results are presented in tabular and graphical form in Appendix C, and discussed in Ch. V.

C. Summary of Approximations and Uncertainties
in the Numerical Calculations

The most important neglects, approximations, and uncertainties of the calculations are summarized below.

Choice of Single-Particle Eigenfunctions and Eigenvalues

The most significant shortcoming in the present calculations is the use of asymptotic single-particle wave functions. This approximation is expected to be less accurate in the actinide region, where the equilibrium deformations are smaller, than in the rare earth region. If the exact Nilsson⁴¹ or Rasmussen⁷⁴ wave functions were used, the breakdown of the asymptotic selection rules would permit many additional transitions. Although the matrix elements of the new transitions are generally an order of magnitude smaller than the asymptotic ones, the new transitions within a major shell may be more important than asymptotic transitions between different shells. Although the additional contributions are partially compensated for by some attenuation of the asymptotic transitions, the net effect, in the strongly deformed isotopes, is expected to be a lowering of the solution ~~has~~ when the exact eigenfunctions are used.

The use of an infinitely deep well, such as the harmonic oscillator well, is also questionable, since the magnitude of the "surface energy" depends strongly on the binding of nucleons near the top of the Fermi sea. However, it is possible that the empirically motivated embellishments of the harmonic oscillator well and the additional shifts of levels, bring the final level scheme in line with

those of a more realistic finite well. Comparisons of the Nilsson scheme with several other deformed wells reveal considerable similarity. Moreover, if a finite well is used, one must also take into account the effect of the continuum, which is difficult to do.

In the actual calculations, all Nilsson states up to $N = 7$ were included. No attempt was made to introduce a cutoff corresponding to a finite well since it was felt that a complete set of states should be used. Moreover, calculations indicate that such a cutoff does not significantly affect the results. A much more important factor is the location of levels belonging to lower shells, as previously discussed. The locations of these depend on the rather arbitrary choice of the parameter μ for each shell. The use of a realistic deformed well, such as Woods-Saxon potential,⁷⁶ would remove this arbitrariness, since the well parameters can be estimated from experiment for this case.

Finally, an additional neglected factor is the velocity dependence of the single-particle well. While it has been argued that the velocity dependence should not affect rotational moments of inertia,³³ the question has not been considered for collective vibrations. If the only effects of a velocity dependence were to replace the nucleon mass by an effective mass $m^* \approx m/2$, and the oscillator well frequency by an effective value $\omega_0^* = 2\omega_0$, then it is easy to see from the dispersion formula (IV-15) that there would be no effect.

Magnitude of Energy Gap

As previously discussed, the pairing force has been treated approximately by assuming that all matrix elements are constant. Such a procedure corresponds to averaging the set of coupled nonlinear equations defining the correlation functions Δ , and replacing them by an average value Δ .⁵⁷ The error involved in estimating Δ from empirical pairing energies is discussed by Nilsson and Prior.¹⁶ Their estimates of Δ are believed to be reasonably good for pair excitations near the Fermi level.

A more important factor, however, may be the change in superfluidity due to excitation of a collective state, as previously discussed in Ch. III. Such a change might alter Δ and λ from the ground state values and also give rise to new terms in Eqs. (III-28) and (III-29), especially if the motion is not adiabatic.

Form of Dispersion Relation

The work of Tamura and Udagawa^{24,25} indicates the possibility of corrections to the frequency equation (III-28) when there are neutrons and protons outside of closed shells as discussed in the previous section. The treatment given here presupposes that the neutron-neutron, proton-proton, neutron-proton coupling strengths are equal.

Sensitivity to χ

The quadrupole coupling constant estimate is given by Eq. (IV-16). The estimate $\hbar\omega_0 \approx 41 A^{-1/3}$, used in the calculations,

has been obtained by requiring that ω_0 be chosen to satisfy the condition

$$\langle \sum_i r_i^2 \rangle = \frac{3}{5} A R_0^2, \quad (\text{IV-31})$$

where the average is taken over a configuration of nucleons moving in a harmonic oscillator well with $R_0 = 1.2A^{1/3}$ fermis.

However, this estimate is rigorously correct only for a closed shell nucleus with equal numbers of protons and neutrons.³⁷ Corrections to ω_0 are expected because of embellishments of the harmonic oscillator well, pairing interaction, and different numbers of neutrons and protons outside of closed shells. Such a change in ω_0 would affect the coupling constant χ and the single-particle eigenvalues. Some rough estimates indicate, however, that the additional factors only result in a change of about 10%. There is also some uncertainty in the nuclear radius of this order of magnitude. Finally, some inconsistency may be present if the chosen potential does not exactly satisfy the self-consistency condition (IV-12).

The solution ω of the dispersion relation is quite sensitive to χ , especially in the adiabatic limit. The percentage uncertainty in ω is related to that in χ by

$$\frac{\delta \omega}{\omega} \approx - \frac{B_{\text{irrot}}}{B(\omega)} \left(\frac{\omega_0}{\omega} \right)^2 \frac{\delta \chi}{\chi}, \quad (\text{IV-32})$$

where B_{irrot} is the irrotational liquid drop mass parameter. For a typical rare earth isotope $\frac{B_{\text{irrot}}}{B} \approx \frac{1}{20}$, $\frac{\omega_0}{\omega} \approx 8$, so that the percentage error in ω is more than 3 times the error in γ .

Use of BSC Wave Functions

The accuracy of the BSC approximation in describing the state has been discussed by Mottelson,⁵⁸ Nilsson and Prior,¹⁶ and Kerman et. al.⁶⁰ In the last reference, it was found, by comparison with exact matrix diagonalizations, that the excited two-quasi-particle states are given with fair accuracy, although the ground-state energy is not given too accurately by the BSC method.

As described in Ch. III, the time-dependent wave function describing the collective motion does not exactly satisfy the requirement that $\langle \underbrace{\psi(\omega)}_{\text{crank}} | \mathcal{N} | \underbrace{\psi(\omega)}_{\text{crank}} \rangle = n$. It seems desirable, however, not only to satisfy this condition, but in addition at least the condition

$$\langle \mathcal{N}^2 \rangle = n^2 \quad (\text{IV-33})$$

when describing the effect of long-range forces, which give rise to a term proportional to n^2 in the binding energy. At the same time, it would be desirable to take into account the four-quasi-particle terms of H_{int} , which are of the same order of magnitude. In the rotational problem, it has been shown that such terms are necessary to give the irrotational moment of inertia in the limit of infinitely strong pair interactions.^{66,67}

Interaction Between Beta and Gamma Bands

Coupling between β and γ bands can be quite important when the band heads lie close together. This is the case experimentally in the actinide region. Although to second order β and γ may be normal coordinates, higher-order terms in the potential energy of collective vibrations cannot be treated by perturbation theory if $\hbar\omega_\beta \approx \hbar\omega_\gamma$. In addition, the bands can be coupled through a Coriolis interaction. The positions of the band heads are not expected to be drastically shifted by the coupling, although the exact order may be. The reduced transition probabilities are, however, quite sensitive to such corrections, which have been neglected in the present work.

Anharmonicity

The effects of anharmonicity arising from terms beyond second-order perturbation theory in the cranking model have been neglected. The γ -vibrational theoretical results in the rare earth region indicate that anharmonic corrections should be considered at the extremes of the region.

Conclusion

In consideration of the uncertainties in the calculations, the opinion will be ventured that agreement with experiment better than 25% would be fortuitous (although gratifying). It is anticipated that the approximations made would have the overall effect of increasing the solutions $^+\omega$ of the frequency formula. The chief interest here lies

in the general orders of magnitude and in the trends of the calculated quantities. However, experience has shown that extreme laxity and overoptimistic "hand-waving" approximations can lead to deceptively "good" results. Therefore, it is felt that what pains were taken in the numerical work were not out of proportion to the modest aims, but just sufficient to permit a sober evaluation of the present model of collective vibrations.

V. DISCUSSION OF RESULTS

Rare Earth Trends

The most salient features of the numerical results will be pointed out.

Phonon Energies of Beta Vibrations

In case(a), the solutions $\hbar\omega_\beta$ lie very close to the corresponding energy gaps. The 0^- states have some collective character in Sm^{152} , Gd^{154} , W^{182} , W^{184} , but in the other isotopes these states are, theoretically, largely intrinsic. The solutions at the beginning of the region are much too high, whereas experimentally they are low-lying adiabatic vibrations. Part of the difficulty here may be due to overestimates of Δ because of failure to properly take into account the rearrangement energy in the transition from spherical to deformed shapes. However, the experimental 0^- states of Er^{166} , Hf^{178} , W^{182*} do in fact lie close to the gaps and thus do not disagree badly with the calculated values.

The trend of $\hbar\omega_\beta$ generally follows that of the gap because of the proximity, and is incorrect at the beginning of the region, where the calculation shows a decrease. The adiabatic solutions, on

*Only the 2^- member of the β band is observed in W^{182} , but the 0^- state can be estimated.

the other hand, while much too high, do show a more reasonable trend, with an increase in $\hbar\omega_\beta$ at the beginning of the region and a slight decrease near the end.

The drop in $\hbar\omega_\beta$ at neutron number 98 is probably due to the (505)11/2, (521) 1/2, (633) 7/2* levels which all lie near the Fermi level λ .

The results for case (b) do not differ significantly from case (a). The smaller shift of the $N = 4$ neutron levels in case (b) decreases $\hbar\omega_\beta$ slightly because of smaller energy denominators involving transitions from these levels. However, the effect is not dramatic since the shifted levels do not have large values of n_z , which favor low-lying β vibrations.

Force Constants (Surface Energies of Beta Vibrations)

The dynamic force constants or "surface energies," $C_\beta(\omega)$ in both case (a) and case (b) tend to become very large since the frequencies approach the intrinsic values. The adiabatic values, $C_\beta(0)$, which are considerably smaller than the $C_\beta(\omega)$, still exceed the surface energies estimated from the semi-empirical mass formula by about a factor of 10.

*Nilsson states will be indicated by $N n_z \Lambda \Omega$.

It is something of a misnomer, perhaps, to use the term "surface" energy, since in a deformed nucleus contributions to C_β and C_γ come from states well below the Fermi level, although the most important contribution comes from states near the top of the Fermi sea.

Mass Parameters of Beta Vibrations

The dynamic mass parameters $B_\beta(\omega)$ also get very large as expected in cases (a) and (b), while the adiabatic values are surprisingly small, exceeding the irrotational hydrodynamic values on the average by a factor of only 4. The formula $\tilde{S} = 3B_\beta^2$, used in the hydrodynamic model, holds roughly if the $B_\beta(0)$ are used.

Calculations were also performed without pairing and it was found that $B_\beta(0)/B_{\text{irrot}}^{\frac{1}{2}}$, which is the same order of magnitude as obtained with a pure harmonic oscillator well. This is so, since asymptotic selection rules were used and these do not allow transitions within a major shell in the absence of pairing for β vibrations, although such transitions could occur if the exact Nilsson wave functions are used. In the presence of pairing, there are the "diagonal contributions" due to excitation of $\alpha_\nu^\dagger \beta_\nu^\dagger \psi_0$ states, which contribute most of the mass parameter. The diagonal matrix elements corresponding to the lowest intrinsic 0^- excitation are plotted as a function of neutron and proton number in Fig. V-1 and Fig. V-2. These figures illustrate that when $n_1 \approx 2n_2$, as in the middle of deformed regions, the diagonal matrix elements are small, and the solutions $\hbar\omega_\beta$ lie close to the gap.

Reduced Transition Probabilities - to the Beta Band

The transition probabilities, $B(E2; 0 \rightarrow 2^+)$, for exciting the β band are generally much smaller than the Weisskopf single-particle estimate, $2.97 \times 10^{-53} A^{4/3} e^2 \text{cm}^4$, in both cases (a) and (b), and tend to zero as $\hbar\omega_\beta$ approaches 2Δ . However, Eqs. (IV-18) and (IV-19),

MU-27213

Fig. V-1. Diagonal single-particle matrix elements corresponding to the lowest 0+ intrinsic excitation in the rare earth region. The ordinate is proportional to

$$\left| \left\langle N n_z \Lambda \Sigma \left| \frac{\partial V_{sp}}{\partial \beta} \right| N n_z \Lambda \Sigma \right\rangle \right|^2,$$

which is plotted against neutron number in (a), and proton number in (b). The matrix element corresponds to the important diagonal contribution nearest the Fermi level for a given neutron or proton number.

MU-27214

Fig. V-2. Diagonal single-particle matrix elements corresponding to the lowest 0+ intrinsic excitation in the actinide region. The ordinate is proportional to $\left| \langle N n_z \Lambda \Sigma \left| \frac{\partial V}{\partial \beta} \right| N n_z \Lambda \Sigma \rangle \right|^2$, which is plotted against neutron number in (a), and proton number in (b).

used to calculate the $B(E2)$'s, probably do not properly take into account the admixture of intrinsic states as the collective description breaks down. In calculating the $B(E2)$'s, no use was made of an effective charge, since all the nucleons were included. The effect of vibration-rotation coupling on the calculated value is unreasonably small due to the large theoretical value of $\hbar\omega_\beta$.

Only at the extremes of the rare earth region do the $B^\beta(E2)$ approach the single-particle unit, indicating some collective enhancement. The only experimental value is that given by Sheline et. al. for Sm^{152} , which is 3 ± 1 single-particle units.⁶ Although the calculated value is a little less than 2 single-particle units, one is not impressed if one quantity has a reasonable value, while other related quantities are not so reasonable as is the case for Sm^{152} .

Phonon Energies of Gamma Vibrations

The γ vibrations are generally adiabatic, in contrast to the β vibrations in this region according to calculations. In the Yb isotopes, however, $\hbar\omega_\gamma(\text{adiab})$ somewhat exceeds the lowest 2^- intrinsic excitation, although the solution $\hbar\omega_\gamma$ indicates considerable collective character.

The results for case (a) and case (b) differ significantly only at the beginning of the region. Case (a) leads to values of $\hbar\omega_\gamma$ which are much too high, and which have the wrong trend at the beginning, although the trends become more reasonable beyond Gd^{156} . In case (b) one obtains the right general trend throughout the region and surprisingly reasonable magnitudes, especially at the beginning. The smaller

shift of the $N = 4$ neutron levels in case (b) accentuates the contributions of the transition between the close-lying $(402)3/2$ and $(400)1/2$ orbitals, causing a drop in the calculated $\hbar\omega_\gamma$. Also of some importance at the beginning of the region is the neutron $(530)1/2 \rightarrow (532)3/2$ transition.

The calculations predict a minimum in $\hbar\omega_\gamma$ at Dy^{162} , but the γ band has not been observed as yet in this isotope. Experimentally, the γ vibrations are quite low in Er^{166} and Er^{168} , while the calculated values are somewhat too high, although having the right trend. The relatively low γ vibrations calculated in the Dy isotopes are due to $(411)3/2 \rightarrow (411)1/2$, $(413)5/2 \rightarrow (411)1/2$ proton transitions, and $(523)5/2 \rightarrow (521)1/2$, $(521)3/2 \rightarrow (521)1/2$ neutron transitions. If these levels actually lie higher than in the Nilsson scheme, one could get a minimum in $\hbar\omega_\gamma$ in the Er isotopes.

It is interesting to note that the calculated $\hbar\omega_\gamma$ show a conspicuous peak at Yb^{172} , as apparently corroborated by experiment.* The peak is due to an absence of transitions between close-lying states near the Fermi level as can easily be seen from the Nilsson diagram. The decline in $\hbar\omega_\gamma$ in Yb^{174} is due to the $(512)5/2 \rightarrow (510)1/2$ neutron transition. The downgrade continues through the Hf isotopes because of the $(404)7/2 \rightarrow (402)3/2$ proton transition and the $(514)7/2 \rightarrow (512)3/2$ neutron transition. The calculations also predict the downward trend in the W isotopes, which are primarily due to the $(510)1/2 \rightarrow (512)3/2$ neutron transition; of lesser importance is the $(402)5/2 \rightarrow (400)1/2$ proton transition.

*The experimental assignments in the Yb isotopes have not been fully confirmed.

MU-27215

Fig. V-3. Important asymptotic neutron transitions in the rare earth region. The transitions corresponding to the matrix element

$$\left| \langle N n_z \Lambda \Sigma \left| \frac{\partial V}{\partial \gamma} \right| N n_z \Lambda + 2 \Sigma \right\rangle \right|^2 \sim (n_z + \Lambda + 2)(n_z - \Lambda)$$

are shown by arrows at approximately the deformation when they become important. The number next to each arrow indicates the relative strength of the matrix element.

MU-27216

Fig. V-4. Important asymptotic proton transitions in the rare earth region. The transitions corresponding to the matrix element

$$\left| \left\langle N n_z \Lambda \Sigma \left| \frac{\partial V}{\partial \gamma} \right| N n_z \Lambda + 2 \Sigma \right\rangle \right|^2 \sim (n_{\perp} + \Lambda + 2)(n_{\perp} - \Lambda)$$

are shown by arrows at approximately the deformation when they become important. The number next to each arrow indicates the relative strength of the matrix element.

MU-27217

Fig. V-5. Important asymptotic neutron transitions in the actinide region. The transitions corresponding to the matrix element

$$\left| \left\langle N n_z \Lambda \Sigma \left| \frac{\partial V}{\partial \gamma} \right| N n_z \Lambda + 2 \Sigma \right\rangle \right|^2 \sim (n_{\perp} + \Lambda + 2)(n_{\perp} - \Lambda)$$

are shown by arrows at approximately the deformation when they become important. The number next to each arrow indicates the relative strength of the matrix element.

MU-27218

Fig. V-6. Important asymptotic proton transitions in the actinide region. The transitions corresponding to the matrix element

$$\left| \left\langle N n_z \Lambda \Sigma \left| \frac{\partial V}{\partial \gamma} \right| N n_z \Lambda + 2 \Sigma \right\rangle \right|^2 \sim (n_{\perp} + \Lambda + 2)(n_{\perp} - \Lambda)$$

are shown by arrows at approximately the deformation when they become important. The number next to each arrow indicates the relative strength of the matrix element.

Results are also given for γ vibrations neglecting the pairing interaction. The rough trends are still present, since these can be explained in terms of transitions that persist in the absence of pairing. In the case of β -vibrations, the trends depend critically on the "diagonal" contributions which are only present in the superfluid state. The effect of pairing on γ vibrations is mainly to increase energy denominators and to attenuate matrix elements through the $(U_\gamma V_\gamma + U_\gamma' V_\gamma')$ factor, thus tending to raise $\hbar\omega_\gamma$. The additional transitions between orbitals lying on either side of λ , are not sufficient in most cases to outweigh the attenuation of the transitions across λ , unless these are between remote orbitals, as in Sm^{152} and Yb^{172} . It should be noted that there are only imaginary solutions, $\hbar\omega_\gamma$, in the absence of pairing for several isotopes of Gd, Dy, Er, indicating γ -instability. The effect of the superfluidity is to stabilize the $\gamma = 0$ equilibrium shape for these isotopes, giving real solutions for $\hbar\omega_\gamma$. In contrast, $\hbar\omega_\beta$ is much greater in the absence of pairing, being of the order of $\hbar\omega_0 \sim 7$ Mev, if only asymptotic selection rules are used, although the breakdown of these selection rules would permit transitions within a major shell even in the absence of pairing. Thus, it may be that the error involved in using the asymptotic wave functions is more serious for the β vibrations, than for the γ -vibrations. Important neutron and proton transitions are shown on Nilsson diagrams to illustrate the relation between the γ -vibrational trends and these transitions in Figs. V-3 through V-6. These figures show that low-lying γ -vibrations are favored by transitions between states with large η_1 and small λ , in the neighborhood of the Fermi level.

Force Constants (Surface Energies) of Gamma Vibrations

In the irrotational hydrodynamic model, the force constant for γ vibrations is related to that for β vibrations by $C_\beta = C_\gamma / \beta_0^2$. According to the present calculations, C_γ / β_0^2 is generally smaller than the corresponding C_β , although exceeding the estimate of the semi-empirical mass formula by a factor of 2-8. The adiabatic and dynamic values of C_γ / β_0^2 follow essentially the same trend, which also conforms to the trend of $\hbar\omega_\gamma$ in both cases (a) and (b).

The experimental values of the force constant C and of the mass parameter B are obtained from measurements of the reduced transition probabilities $B(E2; 0 \rightarrow 2^+_\gamma) \sim \frac{\hbar^2}{C}$, together with knowledge of the phonon energy $\hbar\omega = \hbar\sqrt{\frac{C}{B}}$ obtained from observed spectra. The accuracy is usually of the order of 25%. The $B(E2)$ data of Nathan and Popov⁷⁷ for Gd and Er isotopes are guaranteed only to within a factor of 2. Their measurements for W^{184} , W^{186} differ significantly from those of Alkhazov et. al.⁷⁸ and of McGowan and Stelson.⁸ The data of Alkhazov et. al. for the W^{182} , W^{184} , W^{186} isotopes were used for the plots. The order of magnitude of the theoretical C_γ is reasonable, and in the W isotopes is within experimental uncertainty using Alkhazov's data, in case (a), and in marginal agreement in case (b). Of course, the measured $B(E2)$'s are quite sensitive to many factors such as anharmonicity, coupling between β and γ bands etc., which are not taken into account by Eqs. (IV-18) and (IV-19) from which the "experimental" mass parameters and force constants are extracted.

Mass Parameters of Gamma Vibrations

The mass parameters are also smaller generally than the experimental values. The adiabatic mass parameters are rather small, averaging about 4 times the hydrodynamic estimate, as does the adiabatic β -vibrational mass parameter. Thus, the liquid drop relation $B_\beta = B_r / \beta_0^2$ holds approximately if the adiabatic values are used. The dynamic values $B_r(\omega)$ deviate significantly from the adiabatic estimate in the Yb isotopes, reaching a maximum of about 12 times the hydrodynamic value. According to experiment, however, $B_r / \beta_0^2 B_{\text{irrot}} \approx 15-30$.

The only calculated value in good agreement with experiment is that of W^{182} when compared with Alkhazov's data.

In the absence of pairing, B_r is considerably larger. The use of smaller Δ 's would therefore increase B_r and C_r and consequently decrease the calculated $B_r(E2)$, thus bringing the values into closer agreement with experiment. According to previous discussion, it is possible that the excitation of a collective state would modify the energy gap, the change being different for different modes. Moreover, the excitation of a vibration necessitates simultaneous rotation, since $I \geq 2$. As discussed in Ch. III, the effect of rotation is to decrease Δ .

Reduced Transition Probabilities to the Gamma Band

The calculated values of $B^\gamma(E2; 0 \rightarrow 2^+)$ exceed the single-particle estimate by a factor of 4-19, indicating that the states can

safely be called collective. However, the theoretical values are generally 2-4 times the experimental values. The disagreement is more pronounced in case (b) where the theoretical $\hbar\omega_\gamma$ are in best agreement with experiment. The anomalously large calculated values correspond to large values of the vibrational amplitude, suggesting a need for including anharmonic corrections. As previously discussed, the use of smaller Δ 's in the calculation would also decrease the $B(E2)$'s. The only case of agreement with experiment occurs in W^{132} compared to value of Alkhazov. It is interesting to note, however, that experimentally, the reduced transition probabilities to the γ -vibrational band usually exceed the transition probabilities to the β -vibrational band, which qualitatively agrees with the few cases in which both $B^\gamma(E2)$ and $B^\beta(E2)$ are known. The effect of the vibration-rotation coupling on the calculated $B^\gamma(E2; 0 \rightarrow 2^+)$ is a reduction in most cases of only 10% or less.

Vibration-Rotation Interaction Coefficients

It has been conjectured in Ch. IV that the restoring-force constants involved in the vibration-rotation interaction coefficients F of Eq. (IV-22) should be the adiabatic values $C(0)$ rather than the dynamic $C(\omega)$. One can find experimental evidence to test this speculation. In Th^{232} and Sm^{152} , estimates of both C_β and C_γ may be obtained from experimental $B(E2)$ data; the force constants should be the dynamic values, according to Eqs. (IV-18) and (IV-19), as also confirmed by the Sawada method. If the values of $C(\omega) > C(0)$ are then substituted in Eq. (IV-22), the calculated value of F should be

smaller than the value of F obtained directly from the ground-state rotational band if the conjecture is correct. Assuming a quadratic dependence of the moment of inertia on deformation, which gives a larger value of F than a linear dependence, one finds that the value of F estimated from $B(E2)$ data are of the right magnitude, provided that vibration-rotation coupling is included in the $B(E2)$ formula. For Sm^{152} , one gets $F = 140 \text{ ev}$, exactly in agreement and for Th^{232} one gets $F = 10 \text{ ev}$, from Stelson and McGowan's data, which agrees within experimental limits with the value of 12 ev .

Of course, Eq. (IV-22) may not represent the full contribution to the $I^2(I+1)^2$ term, since other effects, such as a fourth-order Inglis cranking-model contribution, have been neglected.

The calculated values of F using the $C(\omega)$, however, are generally significantly smaller than the experimental values obtained from ground-state rotational bands, except near the end of the region. The theoretical values using the $C(0)$ are also much too small at the beginning of the region, because of the excessively large $C(0)$ for Sm^{152} , Gd^{154} , and too large near the end of the region, although the experimental values for Hf^{180} and W^{186} are based on somewhat inaccurate ground-state bands.

In the middle of the rare earth region, the experimental values for the adiabatic calculation fall near the $\mathcal{F} \sim \beta$ or the $\mathcal{F} \sim \beta^2$ curve, although neither dependence is clearly favored. The quadratic cases give values of F about 1.5 times those of the linear case, the difference being due to the contribution of the β vibration. Because the calculated C_β are quite large, the γ vibration contributes a large proportion. When, however, the C_β are small, the two curves are more widely separated. Case (b) gives slightly better results than case (a).

A calculation involving the exact Nilsson wave functions would probably lead to smaller C's and larger F's. In that case, a linear dependence of $\mathcal{E}$ on β might be favored. It is also possible that the dynamic C's would then give a better fit.

Although the present calculation somewhat favors the $C(0)$ over the $C(\omega)$ in calculating the vibration-rotation coefficient, the conjecture is certainly not answered by these calculations. Also, one probably could get a good fit by assuming that $\mathcal{E} \sim \beta^n$ with $n > 2$, if nothing else works. In this case, it would probably be more profitable to investigate the problem with a more rigorous theory, than with semi-empirical adjustments.

Actinide Trends

Phonon Energies of Beta Vibrations

The equilibrium deformation calculations of Szymanski⁷³ indicate possible inconsistencies in the application of the Nilsson scheme at the beginning of the deformed actinide region because the calculated deformations, using the Nilsson levels, are significantly smaller than the experimental values. The Ra isotopes are nearly spherical theoretically, and even Th²²⁶ has a calculated deformation of $\delta = .12$ as compared with $\delta = .20$ experimentally.

Because of these difficulties, the calculations were started with Th²²⁸, which has a calculated value of $\delta = .16$, as compared with the experimental value of $\delta = .20$. For this case, however, no real solution, $\hbar\omega_\beta$, exists, indicating that this mode of oscillation is unstable. Because of slight differences in the conditions of the calculation, it is possible that the appropriate equilibrium deformations for the present set of conditions differ from those appropriate to Szymanski's conditions, especially at the beginning of the region where the shallow potential energy curves are sensitive to small variations in Δ , and in the coulomb correction. Nevertheless, no solution $\hbar\omega_\beta$ could be found for Th²²⁸ at different values of δ within a reasonable range. This was true in both cases (a) and (b). It is interesting to note that no β -vibrational bands have been found in Th²²⁶ and Th²²⁸, in spite of careful experimental search.

Both Th^{230} and U^{230} have low-lying β vibrations theoretically in case (a). The calculated value $\hbar\omega_\beta = 589$ kev for Th^{230} agrees well with the observed 0_2^- state at 634 kev. Beyond $A = 230$, the calculated $\hbar\omega_\beta$ rapidly approach gap excitations, especially in case (a). The low-lying 0_2^- states at the beginning of the region are due, in part at least, to a concentration of Nilsson states with large n_z quantum numbers from the $N = 6$ proton shells and the more important $N = 6, 7$ neutron shells.

In case (b), where the shifts of lower levels are much smaller, the solutions $\hbar\omega_\beta$ are lower. The shifts in case (b) differ significantly from those used by Szymanski. Since the potential energy curves are very shallow and the collective frequencies very low at the beginning of the region, it is expected that the equilibrium deformations will differ from those calculated by Szymanski; in particular, they should exceed Szymanski's values. Instead of recalculating the deformations, separate calculations of the vibrational parameters were performed at different values of δ and the "best" results plotted.

In case (b), no real β -vibrational solutions could be obtained for Th^{230} and U^{230} , as well as for Th^{228} , using the original deformations of Szymanski. However, solutions were found for Th^{230} and U^{230} at larger values of δ . Because of the extreme sensitivity of the solutions to deformation at the beginning of the region, the results chosen as "best" should not be taken too seriously, since the corresponding deformations may not be the correct equilibrium values.

At neutron number 146, the 0_2^- state becomes an intrinsic state theoretically. At this point, the effect of the minor shell

at neutron number 152 begins to be felt. This gap in the Nilsson scheme leads to large energy denominators for transitions which cross it, and hence favors high values of $\hbar\omega_\beta$.^{*} Beyond neutron number 152 there is a drop in $\hbar\omega_\beta$. The observed 0^+ states in U^{238} , Pu^{238} , and Pu^{240} are fairly close to the energy gap, although they have more collective character than the calculation allows.

In case (b), the trends and magnitudes at the beginning of the region are in better agreement with experiment than in case (a), although the case (b) results are vitiated by the uncertain equilibrium deformations.

Force Constants (Surface Energies) of Beta Vibrations

The calculated $C_\beta(0)$ follow the expected trend of increasing sharply at the beginning of the region, and decreasing just before neutron number 152. The $C_\beta(0)$ exceed the corresponding values obtained from the semi-empirical mass formula by a factor of more than 20 in some cases. The dynamic $C_\beta(\omega)$ deviate from the adiabatic values and blow up as the 0^+ states approach intrinsic excitations.

Experimental values of C_β exist for Th^{232} and U^{238} , although there is a significant discrepancy between the values deduced from the B (E2) data of McGowan and Stelson⁷ and that of Durham et. al.⁷⁹ In case (b), where $\hbar\omega_\beta$ agrees best with experiment, the former experimental value lies closer. The calculated $C_\beta(\omega)$ for U^{238} "diverges" in both cases (a) and (b), thus disagreeing badly with experiment. The mass parameters and force constants obtained from the B(E2) data

^{*}Small magnitudes of the diagonal matrix elements near the Fermi level also favor high values of $\hbar\omega_\beta$.

in this region cannot be taken seriously, since the effect of coupling of the close-lying β - and γ - vibrational bands has not been taken into account.

Mass Parameters of Beta Vibrations

The $B_{\beta}(0)$ range from about 12 times the hydrodynamic value at the beginning of the region, to only twice the hydrodynamic value as the neutron 152 subshell is filled. The corresponding $B_{\beta}(\omega)$ tend to diverge as this subshell is approached.

In case (b), $B_{\beta}(\omega)$ agrees marginally with the experimental values of McGowan and Stelson⁷ for Th²³². The mass parameter $B_{\beta}(\omega)$ diverges badly for U²³⁸ in cases (a) and (b).

Reduced Transition Probabilities

In case (b), the $B^{\beta}(E2; 0 \rightarrow 2^{+})$ are generally somewhat more reasonable than in case (a), although the theoretical values drop below the single-particles estimate beyond $A = 236$, whereas according to experiment, they should exceed the single-particle unit at U²³⁸. Near the end of the region, the $B^{\beta}(E2)$ approach 1 single-particle unit again in case (b). This is partly due to inclusion of some $N = 8$ levels, not originally calculated by Nilsson.

The theoretical results do not clearly favor the conflicting experimental results of either McGowan and Stelson⁷ or Durham et.al.⁷⁹ In case (a), the theoretical $B^{\beta}(E2)$ for Th²³² lies close to that of Durham et. al., although the corresponding theoretical $B^{\gamma}(E2)$ lies closer to that of McGowan and Stelson. In case (b) the situation is

reversed; the theoretical $B(E2)$ is closer to that of McGowan and Stelson, while the $B(E2)$ favors Durham et. al. In U^{238} , the calculated $B(E2)$ is much smaller than the experimental results of either investigator.

Phonon Energies of Gamma Vibrations

The gamma vibrational phonon energies $\hbar\omega_\gamma$ are somewhat more sensitive to the shifts of the proton $N_p = 4$ and neutron $N_n = 5$ levels than the $\hbar\omega_\beta$, since the levels in question have large n_1 quantum numbers which favor low-lying γ vibrations. Thus, the smaller shifts of case (b) lead to lower solutions $\hbar\omega_\gamma$ due to the following transitions near the Fermi level: $(501)3/2 \rightarrow (501)1/2$, $(503)5/2 \rightarrow (501)1/2$ neutron, and $(402)3/2 \rightarrow (400)1/2$ proton.

In case (a), the solutions deviate considerably from adiabaticity throughout, and exceed experimental values by as much as a factor of 2. In case (b), the solutions are closer to the adiabatic values at the beginning of the region but deviate beyond U^{234} , although they have collective properties. In case (b), the agreement with experiment is fair at the beginning of the region for certain choices of deformations, but the results are quite sensitive to this choice; hence, an exact recalculation of the equilibrium deformations would be desirable.

The downward trend in $\hbar\omega_\gamma$, especially in case (a), is promoted by the $(631)3/2 \rightarrow (631)1/2$ neutron transition; a minimum in $\hbar\omega_\gamma$ at neutron number 140 is reached with ^{the} $(633)5/2 \rightarrow (631)1/2$ neutron transition. Both cases (a) and (b) predict an increase in $\hbar\omega_\gamma$ at neutron number 142 with a maximum being reached at neutron number 148 and a

decline thereafter. The decrease, occurring before the filling of the 152 neutron subshell may be attributed to the $(624)7/2 \rightarrow (622)3/2$ and $(622)5/2 \rightarrow (620)1/2$ neutron transitions; and $(523)5/2 \rightarrow (521)1/2$, and $(521)3/2 \rightarrow (521)1/2$ proton transitions in the Cm and Cf isotopes.

Experimentally, a dramatic decrease in $\hbar\omega_\gamma$ occurs just beyond the 152 subshell in Fm^{254} . This sharp decline may be due to the $(620)1/2 \rightarrow (622)3/2$ neutron transition between the close-lying orbitals. Although the calculations indicate a downward trend beyond the 152 subshell, the decrease is not exceptionally pronounced, and the calculated values are nearly twice as large as the experimental. In order to reproduce the experimental results, it would be necessary to slightly shift the $(620)1/2$, $(622)3/2$ neutron orbitals down. In addition, one could also move down a pair of crossing $(752)3/2$ and $(750)1/2$ levels which lie a little higher up. It is entirely possible that the states in question should be located lower than in the Nilsson scheme. The inclusion of a few $N = 8$ levels, not given by Nilsson, in case (b) does not affect the results very much.

Force Constants (Surface Energies) of Gamma Vibrations

In case (a), the adiabatic C_γ/β_0^2 vary but little throughout the region, averaging about 1000 Mev, while the dynamic values tend to "diverge." In case (b), the adiabatic values are low at the beginning, gradually increase and slowly decrease near the end of the region; the dynamic values diverge badly only for Cm^{242} .

The theoretical $C_\gamma(\omega)$ in case (b) for Th^{232} agrees best with that obtained from the data of Durham et al.⁷⁹ rather than with

McGowan and Stelson,⁷ although as previously discussed, the reverse is true for $C_\beta(\omega)$. In case (a), the theoretical value is not in good agreement with either experimenter.

Mass Parameters of Gamma Vibrations

The theoretical $B_r(0)$ are quite small, being of the order of 3-5 times the hydrodynamic estimate. In case (a), the $B_r(\omega)$ tend to blow up, while in case (b) they average about 25 times the hydrodynamic estimate. The theoretical B_r is notably smaller than the experimental values for Th^{232} .

Reduced Transition Probabilities to the Gamma Band

In case (a), the calculated $B^r(E2; 0 \rightarrow 2^+)$ usually exceed the single-particle estimate, although there are some notable exceptions. In case (b), the theoretical values exceed the single-particle value by a comfortable margin in all cases, but are somewhat too large at the beginning of the region for the deformations which give the best values of $\hbar\omega_\gamma$. The effect of vibration-rotation coupling on the transition probabilities to the γ band is only a 5 percent effect in this region. The effect on the β -band transition probabilities is quite important at the beginning of the region where $\hbar\omega_\beta$ is small and results in a decrease by almost 1/2.

In general, the qualitative rule seems to hold that $B^r(E2; 0 \rightarrow 2^+) > B^\beta(E2; 0 \rightarrow 2^+)$, with few exceptions,

occurring at the extremes of the region. According to Eqs. (IV-18)

$$\frac{B^r(E2; 0 \rightarrow 2^+)}{B^B(E2; 0 \rightarrow 2^+)} = \frac{2 \hbar \omega_r}{\hbar \omega_\beta} \frac{C_\beta(\omega)}{C_r(\omega)} \beta_0^2. \quad (\text{IV-1})$$

As long as $\hbar \omega_r \approx \hbar \omega_\beta$, and $C_\beta \geq C_r/\beta_0^2$ as is generally the case, the above ratio is usually at least 2. The above formula, however, may not properly take into account the coupling between intrinsic and collective modes.

Vibration-Rotation Interaction Coefficients

Experimental data on the vibration-rotation interaction coefficients are still sparse in the actinide region. The theoretical values of F using the $C(\omega)$ are generally smaller than experimental values, which fall close to either the $\mathcal{F}_{\alpha\beta}$ or $\mathcal{F}_{\alpha\beta}^2$ curves, using the $C(0)$, although neither curve is clearly favored. The experimental values fall somewhat closer to the case (b) theoretical curves.

Conclusions

The results of the numerical calculations are commensurate with the many uncertainties and approximations previously outlined in Ch. IV. Although the quantitative agreement with experiment leaves much to be desired, a number of satisfactory qualitative features can be pointed out. The calculations indicate the non-adiabaticity of most collective vibrations in the deformed regions, and therefore justify theoretical treatments which do not make use of the adiabatic approximation. In particular, the results indicate that the γ vibrations are generally more adiabatic than the corresponding β vibrations, except possibly at the beginning of a region of deformation. And in connection with this, the theory predicts that usually the $B^{\gamma}(E2; 0 \rightarrow 2'_{\gamma}) > B^{\beta}(E2; 0 \rightarrow 2'_{\beta})$, for which there is some experimental support. According to the calculations, the mass parameters and force constants should exceed the corresponding liquid-drop model parameters by a large amount, especially because of non-adiabatic effects, in qualitative agreement with observation in the deformed regions. The general orders of magnitude of these parameters are reasonable in view of the many uncertainties in the theory.

The results are quite sensitive to the location of those Nilsson levels which cannot be experimentally fixed. For not unreasonable shifts of these levels, many features of the observed trends can be reproduced. A particularly striking example is the trend of $\hbar\omega_{\gamma}$ in the rare earths. On the other hand, the γ -vibrational results in the actinide region are not nearly as gratifying. However, these results are quite sensitive to the location of levels at the

extremes of the region. If one chooses the locations given by case (b), the magnitudes are reasonable as well as the trends at the beginning of the region.

The results for β vibrations are somewhat disappointing in the rare earth region, in contrast to the γ vibrations. There is a little experimental evidence, however, indicating that 0_2^+ states lie rather high in the middle and end of the rare earth region, in qualitative agreement with the calculations. However, the calculations cannot give low-lying 0_2^+ states at the beginning of the region, as indicated experimentally by the 685 kev state in Sm^{152} and the 681 kev state in Gd^{154} . Reasonable shifts in levels, small changes in the equilibrium deformations, energy gaps, quadrupole coupling constants etc. would not do much good here. It has been pointed out in section II-D that the cranking model applied to a volume-conserving shell-model potential yields somewhat different results for the 0_2^+ vibrations of deformed nuclei than the TDSCF method applied to the Elliott force, and that the latter would give lower collective frequencies. It was also pointed out that the TDSCF method would agree with the cranking-model formulation, provided that a residual two-body force is chosen which deviates somewhat from the simple Elliott form.* It is possible that the volume-conserving formulation has been adhered to too closely and that use of the Elliott force would improve the situation at the beginning of the region. On the other hand, relaxation

*See the footnote at the bottom of page 65. Also, see reference (50) for discussion of a similar force.

of the volume conservation condition would raise considerably the γ -vibrational energies by replacing the factor e^{α_0} in Eq. (IV-15b) by unity. The β -vibrational frequencies calculated at the beginning of the actinide region are quite low in agreement with experiment despite the volume conservation. However, in the middle of the region, the solutions probably lie too close to the corresponding energy gaps.

It is emphasized that no attempts have been made to vary the quadrupole coupling constants to fit experimental data. This would be a perfectly legitimate thing to do since the estimate given by Eq. (IV-16) obtained from the Nilsson model is uncertain by at least 10 - 20% as explained in section (IV-C). Moreover, the possibility of different coupling strengths for neutron-neutron, neutron-proton, and proton-proton interactions has not been taken into account. It might appear that variation of so many parameters would cast doubt on the physical meaning of the calculations; however, the freedom is not so great, since for each isotope one must fit five observables: $\hbar\omega_\beta, \hbar\omega_\gamma, B^\beta(E2), B^\gamma(E2), F$. Of course, one could attempt to use different coupling constants for β and γ vibrations, but this would contradict the multipole expansion given by Eq. (B-2). In addition, the coupling strengths should not deviate greatly from the estimate (IV-16), since this would contradict the idea that the same residual interaction is responsible for the deformed independent-particle potential and for the quadrupole vibrations, upon which the calculations are based.

Of the approximations discussed in section IV-C, the poorest is probably the use of the asymptotic single-particle eigenfunctions. As mentioned in that section, the use of exact single-particle eigenfunctions might lead to somewhat lower collective solutions. Bes¹⁸ has

calculated γ -vibrational frequencies for some rare earth isotopes using the more exact Rassey single-particle eigenfunctions.⁷⁴ A detailed comparison of Bes' work with the present work is difficult to make because of differences in details of the calculations, which are not fully specified by him. He used the Lagrangian multiplier method (section II-D), invoking the adiabatic approximation, and apparently also used somewhat different energy gaps, level shifts, and equilibrium deformations. Moreover, he examined the effects of changing the quadrupole coupling constants χ_{nn} , χ_{pp} , χ_{np} . However, comparison of his results for the case that all the constants are equal to the value given (to a good approximation) by Eq. (IV-16) with the present results for cases (a) and (b) provides no definite evidence for a significant improvement from the use of Rassey eigenfunctions in the rare earth region. In fact, values of $\hbar\omega_\gamma$ in the adiabatic limit, as given in this paper, are in many cases a little closer to the experimental values than the corresponding values in Bes' work (especially in the first half of the region), which, however, gives more reasonable results for the B(E2) values, although he makes use of an effective charge.

It should be noted that in the present calculations a coulomb correction, and also the factor e^{α_0} in Eq. (IV-15b) are included, which tend to decrease $\hbar\omega_\gamma$. The coulomb effect is at most a 20% decrease, but the exponential amplifying the matrix elements can decrease the frequency by as much as a factor of 2. Neither of the two effects is included by Bes. It is possible that the lowering effects in the present calculations slightly outweigh the lowering effects from the

use of exact single-particle eigenfunctions in Bés' work. On the other hand, no mention is made of any shift of the $N = 4$ neutron levels by Bés. In the present work, it was found that if these are not depressed, the result would be a considerable decrease in the values of $\hbar\omega_\gamma$, with γ -instability at the beginning of the region. If it is the case that Bés did not shift the $N = 4$ neutron levels, then one can conclude that the use of Rassey eigenfunctions produces little, if any, improvement over asymptotic eigenfunctions in the first half of the rare earth region, where the equilibrium deformations are large.

In order to reach definite conclusions about the validity of the Sawada formula for deformed nuclei, it would be necessary to carry out extensive calculations using the more exact Nilsson or Rassey single-particle eigenfunctions. This would seem especially desirable in the actinide region, where the small equilibrium deformations make the use of asymptotic eigenfunctions dubious. Unfortunately, use of the ^{exact} eigenfunctions precludes the convenience of matrix elements written in closed form and necessitates the inclusion of many more transitions, making even a machine calculation rather long-winded (many more single-particle levels are needed for deformed than for spherical nuclei, for which such extensive calculations have been carried out).

As a final point, it is emphasized that many challenging theoretical problems of collective motion in deformed nuclei are still to be solved. For example, it would be nice to obtain simultaneously vibrations and rotations of deformed nuclei, as a consequence of a residual two-body interaction in the framework of the Sawada theory. The rotations could be described, perhaps, by some sort of "roton" operators, which are linear combinations of phonon operators.

Of course, one would like to know the exact nature of the residual two-body force in the shell model, as derived from fundamental considerations, such as the Brueckner theory applied to finite systems. It has already been indicated that collective solutions of the Sawada frequency formula may depend critically on whether or not the residual interaction is one which actually gives rise to a volume-conserving single-particle deformed potential, at least in the case of nuclei with large equilibrium deformations.

There is also the challenging question of whether nuclei have unaxial equilibrium shapes, as suggested by the work of Davydov and Filipov and others.⁹⁻¹¹ As mentioned earlier in this chapter, the pairing interaction seems to stabilize an axial shape but this does not eliminate the possibility of more stable strongly unaxial equilibrium shapes.

The effect of excitation of a collective state on superfluidity poses another problem, as discussed in Ch. III. It is the opinion of this author that the particle non-conservation of the BCS states may lead to spurious results in the context of such considerations. It seems desirable to use wave functions that satisfy at least the condition $\langle \psi^\dagger \psi \rangle = n$, as well as the condition $\langle \psi \psi^\dagger \rangle = n$, in order to properly take into account the effect of collective correlations superimposed on pairing correlations.

Then, there remains the problem of investigating the importance of the neglected terms in the Sawada theory, corresponding to linearization and omission of self-energy and exchange corrections. A start in this direction has been made by Sawicki for the case of vibrations of spherical nuclei.⁵⁵

Finally, there is the problem of accounting for the so-called octupole vibrations that have been found abundantly in deformed nuclei, especially in the actinide region. This can be done by a straight-forward extension of the methods discussed in this paper and is presently under investigation by the author.

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APPENDICES

A. Evaluation of $\langle H_{\text{crank}} \rangle$

The quantity $\langle \psi_{\text{crank}}(t) | H_{\text{crank}} | \psi_{\text{crank}}(t) \rangle$ of Ch. II may be evaluated as follows:

$$H_{\text{crank}} \psi_{\text{crank}}(t) = i\hbar \frac{\partial}{\partial t} \psi_{\text{crank}}(t) = \sum_j \left(i\hbar \dot{q}_j + E_j q_j \right) \Phi_j e^{-\frac{i}{\hbar} E_j t} \quad (\text{A-1})$$

because of the expansion (II-12). Since the Φ_j form an orthogonal set, one easily gets

$$\langle \psi_{\text{crank}}(t) | H_{\text{crank}} | \psi_{\text{crank}}(t) \rangle = \langle H_{\text{crank}} \rangle = \sum_j q_j^* (i\hbar \dot{q}_j + q_j E_j) \quad (\text{A-2})$$

Expanding the q_j according to (II-15) and using (II-18), one gets to 2nd order

$$\langle H_{\text{crank}} \rangle = E_0 + A^2 \left[\sum_{j \neq 0} |q_j^{(1)}|^2 E_j + (q_0^{(2)} + q_0^{(2)*}) E_0 + i\hbar \sum_{j \neq 0} q_j^{(1)*} \dot{q}_j^{(1)} + i\hbar \dot{q}_0^{(2)} \right] \quad (\text{A-3})$$

or, according to Eq. (II-22)

$$\langle H_{\text{crank}} - E_0 \rangle = \delta E = A^2 \left[\sum_j |q_j^{(1)}|^2 (E_j - E_0) + i\hbar \sum_{j \neq 0} q_j^{(1)*} \dot{q}_j^{(1)} + i\hbar \dot{q}_0^{(2)} \right] \quad (\text{A-4})$$

From Eqs. (II-16b) and (II-18) one gets

$$i\hbar \dot{q}_j^{(1)} = \langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0} \sin \omega t e^{i\omega_{j0}t}, \quad (A-5)$$

and from Eq. (II-16c) and (II-18) one gets

$$i\hbar \dot{q}_0^{(2)} = \sum_j q_j^{(1)} \langle 0 | \frac{\partial H}{\partial \alpha} | j \rangle_{\alpha=\alpha_0} e^{i\omega_{j0}t} \sin \omega t + \langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \sin^2 \omega t. \quad (A-6)$$

Thus, the perturbation increment takes the form

$$\begin{aligned} \delta E = A^2 \left[\sum_j |q_j^{(1)}|^2 (E_j - E_0) + \sum_{j \neq 0} (q_j^{(1)} \langle 0 | \frac{\partial H}{\partial \alpha} | j \rangle_{\alpha=\alpha_0} e^{i\omega_{j0}t} \sin \omega t + \text{c.c.}) \right. \\ \left. + \langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle_{\alpha=\alpha_0} \sin^2 \omega t \right], \quad (A-7) \end{aligned}$$

where "c.c." denote the complex conjugate.

The quantity $q_j^{(1)}$ is given by Eq. (II-23), from which

$$\begin{aligned} \sum_j |q_j^{(1)}|^2 (E_j - E_0) = \frac{1}{2} \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2 (E_j - E_0) [(E_j - E_0)^2 + (\hbar\omega)^2]}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2} \\ - \frac{1}{2} (1 - 2 \sin^2 \omega t) \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle_{\alpha=\alpha_0}|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} \quad (A-8) \end{aligned}$$

Finally one obtains

$$\begin{aligned} \langle H \rangle &= E_0 \\ \delta E &= A^2 \left\{ (\hbar\omega)^2 \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2 (E_j - E_0)}{[(E_j - E_0)^2 - (\hbar\omega)^2]^2} \right. \\ &\quad \left. + \sin^2 \omega t \left[\langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle - \sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} \right] \right\}. \quad (A-9) \end{aligned}$$

The condition that δE be independent of time requires that ω be chosen so that the time-dependent term vanishes leading to Eq. (II-27).

By correspondence with the classical oscillator energy $\omega A^2/2$, the time-independent part of (A-9) gives an expression for C, and using (II-26), gives Eq. (II-28) for B. Of course, Eq. (A-9) can be written in the form $\delta E = \frac{B(\omega)}{2} \dot{\alpha}^2 + \frac{C(\omega)}{2} (\alpha - \alpha_0)^2$. The condition of time independence which is consistent with Eq. (II-6) then leads to Eqs. (II-26) through (II-28).

It is interesting to consider the probability amplitude Q_0 , in particular, the imaginary part obtained from Eq. (II-17c), which, apart from an arbitrary constant is

$$\text{Im } Q_0 = \frac{A^2}{2\hbar} \left[\sum_j \frac{|\langle j | \frac{\partial H}{\partial \alpha} | 0 \rangle|^2 (E_j - E_0)}{(E_j - E_0)^2 - (\hbar\omega)^2} - \langle 0 | \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} | 0 \rangle \right] \quad (A-10)$$

$$\cdot \left(t - \frac{\sin 2\omega t}{2\omega} \right)$$

It is clear that this term would grow with time unless the dispersion formula (II-27) is obeyed. The vanishing of the imaginary part could have been used as a requirement to provide a shorter route to the result (II-27).*

*The author is grateful to Dr. John O. Rasmussen for pointing this out.

B. Basic Concepts of Several Nuclear Models

Because of the complexity of the many-body problem and lack of knowledge of the exact interactions between nucleons in nuclear matter, low-energy nuclear spectroscopy has advanced largely through a phenomenological approach. The history of low-energy nuclear physics has been beset by a welter of different models, each of which emphasizes a distinct nuclear property to the exclusion or de-emphasis of other, equally important, properties. Recent developments, however, point to a unification of these diverse approaches. It is the purpose of this section to summarize qualitatively concepts of the more important models and their interrelationships as relevant to an understanding of the present work. More detailed and complete accounts are available in several review articles.^{5,35,37,70,80,81}

1. The Shell Model

One of the earliest models capable of correlating a wide range of nuclear data was the "independent particle" model, which assumed that each nucleon moves independently in an average, spherically symmetric potential well, generated by its interaction with all other nucleons.⁸² Because of extreme difficulties involved in calculating such a Hartree-Fock type potential starting with a phenomenological internucleon force, simple, i.e. soluble potential wells were chosen, having the form:

$$V_{sp} = V(r) - W(r) \vec{l} \cdot \vec{\sigma}, \quad (B-1)$$

where $\underline{r}$ is the distance from the center of force, coinciding with the center of mass, and the last term is a negative spin-orbit coupling, which was found necessary to give closed shells corresponding to the "magic numbers" characterizing exceptional isotopic stability. The closing of a shell is analogous to the completion of a rare gas configuration in atomic physics. The stability of a closed-shell nucleus in comparison with an open-shell nucleus arises from the much greater energetic difference between levels in different shells than those in the same shell.

For light nuclei, a harmonic oscillator form for $V(r)$ was found effective, while for heavier nuclei, a square well is somewhat better.⁸³

In the extreme independent particle model, nucleons fill the single-particle orbitals according to the Pauli exclusion principle, with nucleons in time-reversed orbitals paired off to zero angular momentum. The ground-state spin of an odd-A nucleus is taken to be the angular momentum assigned to the single-particle orbital occupied by the odd nucleon.

With more than one particle or hole outside of closed shells, it was found necessary, however, to augment the model by assuming residual interparticle correlations not included by a Hartree-Fock field in order to explain nuclear spectra. If the number of particles

outside of closed shells is not too large (say no more than four), the dynamics are amenable to methods familiar in atomic physics, such as "configuration interaction."

In this way, spin sequences, energy splittings, magnetic moments, electric quadrupole moments, and transition probabilities for β and γ radiation of many nuclei can be accounted for. Moreover, binding energy calculations have given fair agreement with experiment, providing that the central potential, $V(r)$, is also assumed to be momentum-dependent.⁸⁵

It must be pointed out that nuclear forces inferred from high-energy (100-300 Mev) nucleon-nucleon scattering data cannot be treated by the usual Hartree-Fock self-consistent field method because of the presence of a repulsive or "hard" core in the interaction. However, a many-body theory developed by Brueckner,⁸⁴ Goldstone⁸⁵ and others (see ref. 85 for other references), based on an analogy with multiple scattering theory,^{86,87} is capable of taking into account such singular interactions, and provides a basis for a new self-consistent field method. This work indicates that the hard core is not very important for low-energy nuclear properties, so that one may meaningfully describe such properties by means of a shell structure, which arises mainly from the non-singular part of the nuclear force. The Brueckner theory also provides some basis for use of well-behaved "residual" forces in shell-model calculations which are not identical with two-nucleon forces obtained from scattering data.⁸⁵ Such a procedure would correspond to use of "pseudopotentials." The reader is referred to ref. (84) for the details of this complex theory.

2. Evidence for Collective Residual Correlations

The greatest disagreement with the "spherical shell model" occurs in regions with many particles outside of closed shells. Various "collective models" have been introduced to explain the following experimental results.

a. Nuclei in the light region, $A \sim 24$, in the rare earth region, $150 < A < 190$, and in the actinide region, $A > 220$, have permanent electric quadrupole moments exceeding the magnitudes reasonably expected from the independent particle model (spherical potential well) by as much as a factor of 30.

b. Nuclear spectra in these regions display enhanced low-lying electric quadrupole excitations. The E2 transitions for nuclei far away from closed shells are about 100 times more intense than would be expected for the promotion of a single nucleon from one orbital to another. The states in question have spin sequences and energy ratios corresponding approximately to those expected for the rotation of a quantized top.

c. Nuclei in regions intermediate between closed shells and the "rotational regions" display E2 transitions enhanced by a factor of 5 or 10. The energy ratios and spins of these states in many cases correspond roughly to those expected for harmonic surface oscillations of a quantized liquid drop.

The experimental facts suggest some kind of cooperative behavior of nucleons, that is a deviation from the independent particle

model just described. However, the large number of nucleons outside of closed shells make the problem intractable to the configuration interaction version of the shell model.

One is reminded of the old "liquid drop model," which has been used to qualitatively describe fission.⁸³ This model allows for collective excitations in the form of rotations and surface vibrations (traveling tidal waves), assuming an irrotational flow hydrodynamics. The comparison of a nucleus to a liquid drop is only qualitative. The extreme collective behavior implied by this model requires that the nucleonic mean free path be short relative to nuclear dimensions and that each nucleon interact strongly only with its nearest neighbors. Thus the irrotational model contradicts the assumptions of the independent particle model, in which each nucleon effectively interacts with every other nucleon through an average field, presupposing a relatively long mean free path.

The liquid drop and shell models correspond to a thesis and an antithesis. The Bohr-Mottelson model of collective nuclear behavior,¹ to be later described, can crudely be characterized as a synthesis of the apparently contradictory independent particle and collective properties of nuclear motion. However, its conceptual foundation, to be described in the next section, places it in the classification of a shell model which takes into account residual correlations.

3. Effects of a Long-Range Residual Interaction in the Shell Model

Recent concepts, which provide a justification for the assumptions of the Bohr-Mottelson model, will be briefly discussed. A residual two-body interaction, of the sort used in shell-model calculations, can be expanded in spherical harmonics

$$\begin{aligned} v(\vec{r}_i - \vec{r}_j) &= \sum_{\lambda} \sqrt{\frac{4\pi}{2\lambda+1}} k_{\lambda}(r_i, r_j) \sum_{\mu} Y_{\lambda\mu}(\theta_i, \phi_i) Y_{\lambda\mu}^*(\theta_j, \phi_j) \\ &= \sum_{\lambda} k_{\lambda}(r_i, r_j) P_{\lambda}(\cos \theta_{ij}) \end{aligned} \quad (\text{B-2})$$

It can be shown that the low multipoles, λ , correspond to long-range components of the interaction.⁵⁸ The component with $\lambda = 0$ is responsible for contributions to a spherically symmetric shell-model field. The quadrupole component, $\lambda = 2$, produces both static and dynamic effects. The static effect is an anisotropic contribution to the average single-particle field, and is responsible for the large permanent electric quadrupole moments previously described. The static effects will be described in more detail in the next section.

The dynamic effect of a long-range quadrupole force, which coherently correlates many nucleons, is to produce collective vibrations and rotations of the nuclear density distribution. The collective dynamic variables characterizing the motion are proportional to components of the nuclear mass quadrupole tensor, which fluctuate about the permanent (equilibrium) values. Arguments, ostensibly from

fundamental principles, have thus far been advanced to show that the quadrupole component of a force of the form (B-2) can give rise to collective rotations in permanently deformed nuclei,⁵² and to collective vibrations in spherical nuclei.¹⁹⁻²³ In demonstrating the possible existence of vibrations, methods have recently been borrowed from the theory of oscillations of an electron gas. Some discussion of these methods is given in section II-D.

The higher order multipoles of (B-2) correspond to shorter range components and include a "pairing interaction" between nucleons in time-reversed orbitals. Attributing different nuclear properties to different multipole components of a residual force has been termed "specificity" by the Copenhagen school.⁸⁹

4. The "Deformed Shell Model"

First, the static effects of the long-range quadrupole component of the force (B-2) will be described. The large experimental quadrupole moments found in nuclei with many nucleons outside of closed shells suggest a nonspherical density distribution. Since the shell-model potential is conceived of as an approximation to a self-consistent field, it can be argued that the symmetry and spatial orientation of the potential must be the same as that of the density distribution.³⁶ Based on this reasoning, one could try using an anisotropic independent-particle potential of the form

$$V_{sp} = V(r) - W(r) \vec{l} \cdot \vec{\sigma} - K(r) \sum_{\mu} a_{2\mu} Y_{2\mu}(\theta, \varphi), \quad (B-3)$$

as first proposed by Rainwater.⁹⁰ The $a_{2\mu}$ are arbitrary quadrupole deformation parameters determined by a variational procedure to be described. The splitting of the degeneracies of eigenstates of a spherical potential by the deformation is shown in Fig. (B-1), assuming a single deformation parameter, which preserves axial symmetry of the equipotential surfaces. For each value of the deformation, the single-particle states are occupied according to the Pauli principle. The equilibrium values of the $a_{2\mu}$ are chosen so as to give an absolute minimum in the total energy of the nucleus. It can be shown that this choice of the $a_{2\mu}$ guarantees that the shape of the potential coincides with that of the density distribution.³⁶

If the number of nucleons is "magic," a spherical shape (the $a_{2\mu} = 0$) gives the lowest energy in accordance with experiment. Nucleons outside of closed shells favor a non-spherical density distribution. The equilibrium deformations are determined by the competition between closed shells, which favor a spherical shape, and the polarizing tendency of nucleons outside of closed shells. In the case of most anisotropic independent-particle potentials, one particle outside of closed shells may be sufficient to produce some degree of polarization. Experimentally, however, isotopes with several nucleons outside of closed shells may retain a spherical equilibrium shape until

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Fig. B-1. Splitting of single-particle degenerate levels of a spherical shell model field by a deformation preserving axial symmetry. Specifically, the single particle levels are those of the Nilsson potential. The ordinate is the energy in units of $\hbar\omega_0(\delta) \propto A^{-1/3}$. The abscissa shows the relation of several commonly used deformation parameters: $\alpha \approx \sqrt{\frac{5}{4\pi}} \beta \approx 2/3 \delta$.

In the spherical limit, j is a good quantum number. The appropriate quantum numbers for large deformations as shown are Ω , π , N , n_z , Λ .

a critical number of nucleons is reached. The extra stability against deformation has been attributed to a short-range residual pairing interaction between nucleons in time-reversed orbitals.⁵⁷

The motion of nucleons in the orbitals of a deformed well, including, at most, short-range residual forces, is often referred to as "intrinsic motion," in contrast to collective motion, in which many nucleons are correlated by a long-range force. However, it should be remembered that the use of an anisotropic well really takes into account, approximately, a static collective effect, namely, the cooperation of many nucleons to produce a large permanent quadrupole moment.* The deformed-well eigenfunctions, which are not eigenfunctions of the total angular momentum (since the potential does not have spherical symmetry), can, however, be expanded in terms of spherically symmetric eigenfunctions which are eigenfunctions of the angular momentum. Thus, the deformed-well wave functions effectively take into account configuration interaction. As will later be seen, by supplementing the present description of intrinsic motion with collective rotation, one can obtain eigenfunctions of the total angular momentum.

*Recent theoretical calculations of electric quadrupole moments of rare earth and actinide isotopes by Bés and Szymanski, using a deformed shell-model potential (Nilsson potential) and including short-range residual correlations,^{72,73} give excellent agreement with the observed moments.

It is instructive to consider the relation of the model potential (B-3) to a Hartree-Fock potential arising from a quadrupole-quadrupole force. Consider a many-body Hamiltonian of the form

$$H = \sum_i (T(i) + V(i)) + \sum_{\substack{i,j \\ i > j}} v(i,j) , \quad (B-4)$$

where the first term is the kinetic energy, the second an effective spherically symmetric single-particle potential, and the last a residual two-body interaction. It will be assumed that the single-particle potential has already been obtained (say by means of the Breuckner method) from a phenomenological two-nucleon force, and that $v(i,j)$ is a force which introduces correlations not included in the spherical field (it could be a pseudopotential in the Breuckner theory).

It will be assumed that $v(i,j)$ is a "well-behaved" quadrupole-quadrupole force, which can be treated by the familiar Hartree-Fock method, which uses as a trial variational wave function for a Hamiltonian of the form (B-4) a Slater determinant Ψ composed of single-particle orbitals, $u_n(i)$, which are varied. This treatment assumes that one is dealing with a set of identical fermions. The variational principle

$$\delta \langle \Psi | H | \Psi \rangle = 0 , \quad (B-5)$$

subject to the constraint that ψ remain normalized during the variation, then leads to the famous Hartree-Fock equations determining the single-particle orbitals $u_n(i)$ ⁹¹

$$H_{scf}(i) u_n(i) = \epsilon_n u_n(i) , \quad (B-6)$$

where

$$H_{scf}(i) = T(i) + V(i) + \sum_m \int u_m^*(j) v(i,j) u_m(j) d\tau_j - \sum_m \frac{\int u_n^*(i) u_m^*(j) v(i,j) u_m(i) u_n(j) d\tau_j}{u_n^*(i) u_n(i)} \quad (B-7)$$

The symbol $u_n(i)$ refers to the coordinates of the i^{th} particle moving in a single-particle orbital with quantum numbers denoted collectively by $\underline{n}$. The volume element corresponding to the coordinates of the i^{th} particle is $d\tau_i$. The last term in (B-7) includes "exchange terms" coming from the assumption of antisymmetry embodied in the use of Slater determinants; it also subtracts out a self-energy contribution included in the average field given by the next-to-the-last term ($m = n$). The last term makes the self-consistent eigenvalue problem (B-6) very difficult, since it represents a non-local potential. Because each of the $u_n(i)$ correspond to a different Hamiltonian $H_{scf}(i)$, they also would not be orthogonal (although one could make them so by employing additional constraints in the variation). These difficulties

can be evaded by simply neglecting the last term which corresponds to the Hartree approximation, and also ^{the assumption} permits that self-energy corrections are small. With this approximation, Eq. (B-7) can also be applied to the case of two kinds of interacting fermions, that is, neutrons and protons, assuming charge-independent interactions. A treatment taking into account exchange terms when there is an interaction between two different kinds of fermions would be more complicated than the usual Hartree-Fock treatment.

The exclusion principle can still be partially taken into account by using Slater determinants composed of solutions of Eq. (B-7) with the last term, however, neglected. The total wave function for a nucleus would then be a product of a Slater determinant for protons and one for neutrons.* In addition, one could include a Coulomb force in $v(ij)$ when solving (B-6) for proton orbitals. Finally, small adjustments in parameters of the neutron and proton orbitals can be made as dictated by experiment, which might take into account some of the neglected effects of the exclusion principle.

It has been argued that the specific form of the quadrupole force $v(ij)$ is not very important in explaining nuclear deformation. Hence, let us consider a simple force first used by Elliott in explaining collective rotation:⁵²

*The i-spin formalism will not be used in this paper.

$$v(i,j) = -\chi \sum_{\mu=-2}^2 r_i^2 r_j^2 Y_{2\mu}(\theta_i, \varphi_i) Y_{2\mu}^*(\theta_j, \varphi_j) \quad (B-8)$$

This is a special case of Eq. (B-2), and has the property of factorizability, that is, it can be written in the form, $f(i)f(j)$. A monopole component should be added to (B-8), but it is assumed that this merely has the effect of renormalizing the spherical potential $V(i)$. The coefficient χ is just a coupling constant, assumed positive.

The self-consistent field Hamiltonian (B-7) can be written (with the approximations included) in the form

$$H_{scf} = T(i) + V(i) - \chi \sum_{\mu=-2}^2 \hat{q}_{2\mu}(i) \langle \psi | \hat{Q}_{2\mu}^* | \psi \rangle, \quad (B-9)$$

where

$$\hat{q}_{2\mu}(i) \equiv r_i^2 Y_{2\mu}(\theta_i, \varphi_i), \quad (a)$$

(B-10)

$$\hat{Q}_{2\mu}^* \equiv \sum_j \hat{q}_{2\mu}^*(j). \quad (b)$$

The expectation value in (B-9) is taken with the total nuclear wave function Ψ . The eigenvalue problem (B-6) can be solved self-consistently as follows. Choose a definite value for $\langle \Psi | \hat{Q}_{2\mu}^* | \Psi \rangle$, say $Q_{2\mu}^*$, so that (B-9) has the form of an anisotropic independent-particle potential such as (B-3); diagonalize (B-9) using basis functions corresponding to $V(i)$; find the $u_n(i)$ and thus Ψ as a function of the $Q_{2\mu}^*$; finally calculate the corresponding values of $\langle \Psi | \hat{Q}_{2\mu}^* | \Psi \rangle$. The self-consistent solution thus corresponds to that set of values $Q_{2\mu}^*$ for which

$$\langle \Psi(Q_{20}^*, \dots) | \hat{Q}_{2\mu}^* | \Psi(Q_{20}^*, \dots) \rangle = Q_{2\mu}^* \quad (\text{B-11})$$

for each value of μ . Eq. (B-11) justifies the previous statement that the anisotropy of the potential must coincide with that of the density distribution. It has been shown by Kerman that the values of $Q_{2\mu}^*$ satisfying Eq. (B-11) make the total energy

$$E_a = \langle \Psi(Q_{20}^*, \dots) | H | \Psi(Q_{20}^*, \dots) \rangle \quad (\text{B-12})$$

stationary, where H is the original Hamiltonian (B-4).^{92a} This is equivalent to the stationarity of

$$\langle \Psi(Q_{20}^*, \dots) | \sum_i H_{scf}(i) + \chi/2 \sum_{\mu=-2}^2 \sum_i \hat{q}_{2\mu}(i) Q_{2\mu}^* | \Psi(Q_{20}^*, \dots) \rangle \quad (B-13)$$

to small changes in the $Q_{2\mu}^*$, where $1/2$ of the self-consistent potential energy is subtracted in order to avoid counting two-body interactions twice. This correction provides terms proportional to $|Q_{2\mu}|^2$ in the energy, corresponding to a "restoring force" necessary to provide minima.

The approach to nuclear shapes given here has been generalized by Baranger to take into account short-range pairing correlations which are not included by the Hartree method.⁹²

The above discussion is meant to show qualitatively what approximations are necessary to justify the empirically motivated introduction of anisotropic independent-particle potential wells. The line of reasoning is easily generalized to other factorizable quadrupole-quadrupole forces, although the multipole expansion (B-2) does not generally require the factorizability of the radial components. The point is, however, that simple anisotropic wells that have proved useful, such as ellipsoidal harmonic oscillators and box potentials, can be associated with factorizable quadrupole-quadrupole forces.

An additional assumption is often made that the volumes enclosed by anisotropic equipotential surfaces are independent of the deformation parameters. This restriction is supposed to simulate the incompressibility of nuclear matter. It can be satisfied to a given

order of accuracy by adding terms of higher than first order in the deformation parameters to the deformed shell-model potential (B-3). It can be shown that eigenfunctions of such a modified potential satisfy self-consistency conditions such as (B-11) provided that the deformation parameters are chosen to minimize, not the total energy (B-13), but rather the sum of the eigenvalues of occupied single-particle orbitals, and that the values of the deformation parameters so obtained are approximately equal to those obtained by minimizing the total energy using solutions of the unmodified Hartree potential.⁷² For small values of the deformation parameters, the eigenfunctions in the two cases differ by very little. The use of volume-conserving potentials in finding equilibrium deformations, however, effectively corrects for counting two-body interactions twice, since it also gives rise to a "restoring force." A specific example of such a volume-conserving potential will be discussed later.

The "deformed shell model" is an example of how an approximate independent-particle description can be still used even in the presence of collective effects. As Lipkin has pointed out, independent-particle models are retained with great tenacity because the physicist has considerable difficulty in handling any other kind.⁵⁹ Even short-range pairing correlations, which are not included in a Hartree field can be described approximately by means of a new independent-particle model as discussed in Ch. III. Collective nuclear vibrations about the equilibrium Hartree shapes can be described approximately in independent-particle language; in this case, the independent "particle" is a phonon or oscillator quantum, which, unlike the other cases, is

a boson.* One could apply to these "particles" a phrase used by Bohm: "collective individual particles."²⁷

5. Separation of Collective and Intrinsic Motion

The Bohr-Mottelson collective model presupposes that the variations in the shape and orientation of nuclei, called collective motion, can be separated from what has been referred to as intrinsic motion, in a manner analogous to the separation of center-of-mass motion from the relative motion of constituents of a many-body system. The center-of-mass coordinates may be said to constitute a set of collective dynamic variables, as do the shape and orientation parameters of the Bohr-Mottelson model, in the sense that they depend on the positions of all the other particles. On the other hand, the analogy is perhaps misleading because center-of-mass motion occurs whether or not the particles of a system interact, while the nuclear collective motion depends on an interparticle force.

The assumption of the collective model is that rotations and vibrations of the density distribution are approximately independent modes which can be extracted from a shell-model Hamiltonian with long-range residual interactions by means of appropriate canonical transformations.

Indeed, considerable early work in the field was aimed at showing that such a separation in a many-body system is feasible,

*Actually, "quasi-boson" would be a better name; see Ch. II, section D on the Sawada method.

that is, that the nuclear Hamiltonian can be written in the form^{35,44}

$$H = \sum_i H_i(p_i, q_i) + \sum_n H_n(\pi_n, \xi_n) + H_{\text{coupl}}(p_i, q_i; \dots \pi_n, \xi_n), \quad (\text{B-14})$$

where the p_i, q_i are canonically conjugate "intrinsic" variables, the π_n, ξ_n are canonical collective variables, and H_{coupl} is a term which couples the intrinsic and collective variables. In order to obtain a separation of collective and intrinsic modes, the coupling term must be at most a weak perturbation, and in some cases vanish exactly as in the case of center-of-mass motion. It is usually stated that an adiabatic situation should prevail, in which the collective frequencies are slow compared to intrinsic frequencies, in order to obtain a physical separation. An analogy is often drawn to the Born-Oppenheimer separation in molecules, in which the relatively rapid electronic motion corresponds to intrinsic motion and the slower motion of constituent atoms corresponds to collective motion. In case the coupling term vanishes identically, no special relation between collective and intrinsic motion need hold, of course, as in the center-of-mass problem.

6. The Bohr-Mottelson Collective Model

The early attempts at describing collective motion by means of canonical transformations provided some support for the approximate separation of collective and intrinsic modes postulated at the outset by Bohr and Mottelson, who adopted the form of the liquid-drop model Hamiltonian as a means of classifying low-energy nuclear spectra with collective properties.¹ However, the irrotational flow assumption of the earlier model, which fixed the inertial parameters, had to be dropped, and the parameters fitted to empirical data, or estimated by other assumptions to be described later.

The collective dynamic variables, $\alpha_{\lambda\mu}$, are defined by an expansion of the nuclear surface in spherical harmonics

$$R(\theta, \varphi) = R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \varphi) \right], \quad (\text{B-15})$$

where $\alpha_{\lambda\mu} = (-1)^\mu \alpha_{\lambda, -\mu}^*$, to insure that $R(\theta, \varphi)$ is real (assuming Condon and Shortley phases for the $Y_{\lambda\mu}$), and R_0 is the equilibrium radius.

The collective Hamiltonian is

$$H_{\text{coll}} = T + V. \quad (\text{B-16})$$

The kinetic energy of surface vibrations is given by

$$T = \frac{1}{2} \sum_{\lambda\mu} B_{\lambda\mu} |\dot{\alpha}_{\lambda\mu}|^2 = -\frac{\hbar^2}{2} \sum_{\lambda\mu} \frac{1}{B_{\lambda\mu}} \frac{\partial}{\partial \alpha_{\lambda\mu}} \frac{\partial}{\partial \alpha_{\lambda\mu}^*}, \quad (\text{B-17})$$

and the potential energy by

$$V = \frac{1}{2} \sum_{\lambda\mu} C_{\lambda\mu} |\alpha_{\lambda\mu} - \alpha_{\lambda\mu}(0)|^2, \quad (\text{B-18})$$

where the $\alpha_{\lambda\mu}(0)$ are equilibrium distortions and the coefficients $B_{\lambda\mu}$ and $C_{\lambda\mu}$ depend on details of the nucleonic configuration. In the irrotational flow model, the coefficients depend only on λ , with C_λ , a "surface" energy, obtained from the semi-empirical binding energy formula, and B_λ , a "mass parameter," given by (according to hydrodynamic assumptions)

$$B_\lambda = \frac{1}{\lambda} \frac{3}{4\pi} A M R_0^2, \quad (\text{B-19})$$

where M is the nucleon mass. The independence of the inertial parameters from μ can be shown to be strictly valid only for spherical nuclei. Under these conditions, the excitation quanta, or phonons, are bose "particles" with total angular momentum $\sqrt{\lambda(\lambda+1)}\hbar$ and projection $\mu\hbar$ on a space-fixed axis.

Interest in this paper will be focused on quadrupole deformations, $\lambda = 2$, although considerable evidence for low-lying collective octupole excitations, $\lambda = 3$, now exists. In particular, the concern will be with spheroidal equilibrium shapes. It is then convenient to choose a new set of collective coordinates, namely, a set of Eulerian angles (θ, φ, ψ) , characterizing the orientation of the ellipsoid in space, and two shape parameters (β, γ) , to be defined. The 5 deformation parameters $a_{2\mu}$, transform as follows under rotation of the ellipsoid in space:

$$a_{2\nu} = \sum_{\mu=-2}^2 a_{2\mu} D_{\mu\nu}^{(2)}(\theta, \varphi, \psi), \quad (\text{B-20})$$

where the $D_{\mu\nu}^{(2)}$ are the transformation functions for spherical harmonics of order 2.

For a transformation from a space-fixed coordinate system to one whose axes coincide with the principal axes of the ellipsoidal nucleus, $a_1 = a_{-1} = 0$. The coordinates β and γ are defined in terms of the non-vanishing parameters by

$$a_{20} \equiv \beta \cos \gamma = \frac{4\pi}{3} \frac{Q_{20}}{AR_0^2}, \quad (\text{B-21})$$

$$a_{22} = a_{2-2} \equiv \frac{1}{\sqrt{2}} \beta \sin \gamma = \frac{4\pi}{3} \frac{(Q_{22} + Q_{2-2})}{AR_0^2},$$

where the $Q_{2\mu}$ are components of the mass quadrupole tensor assuming a uniform density distribution within the spheroidal boundary defined by Eq. (B-15). Thus, β is a measure of the total deformation, and γ of the deviation from rotational symmetry. The coordinate γ will be restricted to the interval

$$0 \leq \gamma \leq \pi/3 \quad (\text{B-22})$$

For β positive and $\gamma = 0$, the nucleus has a prolate spheroidal shape. Other values of γ which are integral multiples of $\pi/3$ would give the same shape but with the principal axes of the spheroid interchanged; hence the restriction (B-22). With this convention, oblate spheroidal shapes would correspond to negative β . Measurements of nuclear electric quadrupole moments indicate that most equilibrium shapes are prolate spheroidal.

The transformed collective Hamiltonian becomes

$$\begin{aligned} H_{\text{coll}} = & -\frac{\hbar^2}{2B_{20}} \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{2} C_{20} (\beta - \beta_0)^2 \\ & - \frac{\hbar^2}{2B_{22}} \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \frac{1}{2} C_{22} \beta^2 \sin^2 \gamma \quad (\text{B-23}) \\ & + \frac{\hbar^2}{8\beta^2} \left[\frac{1}{B_{21}} \left(\frac{\hat{I}_1^2}{\sin^2(\gamma - \pi/3)} + \frac{\hat{I}_2^2}{\sin^2(\gamma - 4\pi/3)} \right) + \frac{\hat{I}_3^2}{B_{22} \sin^2(\gamma - 2\pi)} \right], \end{aligned}$$

where the relations $C_{2-2} = C_{22}$, $B_{2-2} = B_{22}$, $B_{2-1} = B_{21}$ have been used. The above Hamiltonian is a generalization of that due to Bohr and Mottelson, proposed by Kerman.^{92a} In the Bohr-Mottelson Hamiltonian, as previously mentioned, all the $C_{2\mu}$ are equal and all the $B_{2\mu}$ are equal, which is only appropriate for spherical nuclei. The last terms in the first and second lines are potential energy terms, assuming that the equilibrium shape is prolate spheroidal, with $\beta = \beta_0$, $\gamma = 0$ at equilibrium. The operators, $\hat{I}_p$, $r = 1, 2, 3$, are projections of the total nuclear angular momentum on the principal axes, and are thus functions of the Eulerian angles (θ, ϕ, ψ) . Eigenfunctions of (B-23) are normalized in a space with volume element

$$d\Omega = \beta^4 d\beta |\sin 3\gamma| d\gamma d\Omega \quad (a)$$

(B-24)

$$d\Omega = \sin \theta d\theta d\phi d\psi \quad (b)$$

Recently, Davydov and Filipov, and others have studied the theoretical consequences of assuming that nuclei deviate from axial equilibrium shapes.⁹⁻¹¹ The interest in this paper will be concentrated, however, on the axial symmetry assumption of Bohr and Mottelson.

It is convenient to make the substitution

$$\psi_{coll} = \frac{1}{\beta^2} \Phi_{coll} , \quad (B-25)$$

where ψ_{coll} is the eigenfunction of (B-23). This leads to the eigenvalue problem

$$H'_{coll} \Phi_{coll} = E_{coll} \Phi_{coll} , \quad (B-26)$$

where E_{coll} is the total collective energy and H'_{coll} is given by

$$\begin{aligned} H'_{coll} = & \frac{-\hbar^2}{2B_{20}} \left(\frac{\partial^2}{\partial \beta^2} - \frac{2}{\beta^2} \right) + \frac{C_{20}}{2} (\beta - \beta_0)^2 \\ & - \frac{\hbar^2}{2B_{20}} \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial \sin 3\gamma}{\partial \gamma} \frac{\partial}{\partial \gamma} + \frac{C_{22}}{2} \beta^2 \sin^2 \gamma \\ & + \frac{\hbar^2}{8\beta^2} \left[\frac{1}{B_{21}} \left(\frac{\hat{I}_1^2}{\sin^2(\gamma - 2\pi/3)} + \frac{\hat{I}_2^2}{\sin^2(\gamma - 4\pi/3)} \right) + \frac{\hat{I}_3^2}{B_{22} \sin^2(\gamma - 2\pi)} \right] , \end{aligned} \quad (B-27)$$

which differs from (B-23) in the first term on the right. The normal modes corresponding to (B-27) can easily be obtained by making the usual assumption of small oscillations; i.e., assume that the

amplitudes of β and γ are small compared to the equilibrium distortion β_0 . Hence (B-27) can be linearized by replacing β by β_0 , $\sin 3\gamma$ by 3γ , $\cos \gamma$ by 1, etc. in the appropriate places. The effective Hamiltonian (B-27) takes the form

$$H'_{\text{coll}} \approx H_{\text{rot}} + H_{\beta} + H_{\gamma} + U + U', \quad (\text{B-28})$$

where

$$H_{\text{rot}}(\theta, \varphi, \psi) \equiv \frac{\hbar^2}{2\mathcal{I}} (\hat{I}^2 - \hat{I}_3^2); \quad \mathcal{I} = 3B_2\beta_0^2 \quad (\text{a})$$

$$H_{\beta}(\beta) \equiv \frac{-\hbar^2}{2B_{\beta}} \frac{\partial^2}{\partial \beta^2} + \frac{C_{\beta}}{2} (\beta - \beta_0)^2 + \frac{\hbar^2}{B_{\beta}\beta_0^2}; \quad B_{\beta} \equiv B_{20}, \quad C_{\beta} \equiv C_{20} \quad (\text{b})$$

$$H_{\gamma}(\gamma) \equiv \frac{-\hbar^2}{2B_{\gamma}} \left(\frac{1}{\gamma} \frac{\partial}{\partial \gamma} \gamma \frac{\partial}{\partial \gamma} - \frac{\hat{I}_3^2}{4\gamma^2} \right) + \frac{C_{\gamma}}{2} \gamma^2; \quad B_{\gamma} \equiv \beta_0^2 B_{22}, \quad C_{\gamma} \equiv \beta_0^2 C_{22} \quad (\text{B-29})$$

$$U \equiv -\frac{\gamma}{\sqrt{3}} \frac{\hbar^2}{\mathcal{I}} (\hat{I}_1^2 - \hat{I}_2^2); \quad (\text{d})$$

$$U' \equiv \frac{\partial}{\partial \beta} \left(\frac{\hbar^2}{2\mathcal{I}} \right)_{\beta=\beta_0} (\hat{I}^2 - \hat{I}_3^2) (\beta - \beta_0). \quad (\text{e})$$

The terms (B-28d) and (B-28e) represent vibration-rotation interaction terms, which only have a small perturbing effect on spectra of strongly deformed nuclei and may be neglected in the first approximation. In that case, the eigenvalue equation (B-26) becomes separable. The eigenfunction may be written in the form

$$\Phi_{coll} = f(\beta) g(r) D(\theta, \varphi, \psi), \quad (B-30)$$

which is normalized in a space with volume element

$$d\mathcal{V} = d\beta r dr d\Omega. \quad (B-31)$$

The rotational Hamiltonian (B-29a) corresponds to a symmetric top constrained so as not to rotate about the 3-axis (body Z-axis), which has been chosen as the axis of symmetry. The moment of inertia $\mathcal{I}$ is not the geometrical (rigid) moment but depends on the potential flow of nuclear matter during rotation. Theoretical values of $\mathcal{I}$ based on the assumption of irrotational flow underestimate considerably the values obtained from nuclear spectra. Because of the assumption of axial symmetry, not only is the total angular momentum a good quantum number but also its projection on the 3-axis. Thus, one may replace $\hat{I}^2$ by $I(I+1)$ and $\hat{I}_3^2$ by K^2 , where I and K are the angular momentum quantum numbers. The quantum number K will be used to designate the absolute value of $\hat{I}_3$.

The eigenvalues of H_{rot} are

$$E_{\text{rot}} = \frac{\hbar^2}{2\mathcal{I}} [I(I+1) - K^2] \quad (\text{B-32})$$

The eigenfunctions of H_{rot} are the familiar transformation functions of spherical harmonics under finite rotation, $D_{MK}^I(\theta, \varphi, \psi)$, where M is the projection of the angular momentum on the space Z -axis. The relation between the vectors I , K , and M is shown in figure (B-2).

Eq. (B-29b) is the β -vibrational Hamiltonian, which has the form of a one-dimensional harmonic oscillator, with eigenvalues (neglecting the meaningless constant $\frac{\hbar^2}{2B_\beta \beta_0^2}$) given by

$$E_\beta = \left(n_\beta + \frac{1}{2} \right) \hbar \omega_\beta, \quad (\text{a})$$

$$\hbar \omega_\beta = \hbar \sqrt{\frac{C_\beta}{B_\beta}}, \quad (\text{b}) \quad (\text{B-33})$$

$$n_\beta = 0, 1, 2, \dots \quad (\text{c})$$

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Fig. B-2. Angular momentum coupling scheme of strongly deformed nuclei. The total nuclear angular momentum is $\vec{I}$. Its projection on a space-fixed axis is M , and on the nuclear symmetry axis is K . The angular momentum due only to collective rotation is $\vec{R}$.

The normalized β -vibrational wave function is

$$f_{n_\beta}(\beta) = \left(\left(\frac{B_\beta \omega_\beta}{\hbar \pi} \right)^{1/2} \right)^{1/2} e^{-\frac{B_\beta \omega_\beta}{2\hbar} \beta^2} H_{n_\beta} \left(\sqrt{\frac{B_\beta \omega_\beta}{\hbar}} \beta \right). \quad (\text{B-34})$$

The γ -vibrational Hamiltonian (B-28c) has the form of the well-known radial equation of a two-dimensional harmonic oscillator in polar coordinates,⁷⁵ with one exception. In the familiar harmonic oscillator problem, the radius vector has the range $0 \leq r < \infty$, while γ has the range $0 \leq \gamma \leq \pi/3$. However, the wave function corresponding to the former boundary condition should drop off sufficiently fast to be applicable to the latter case. Difficulties would only arise if the amplitude of γ approaches $\pi/3$, which would be contrary to the small amplitude approximation. Thus, the eigenvalues of H_γ are given by

$$E_\gamma = (n_\gamma + 1) \hbar \omega_\gamma, \quad (\text{a})$$

$$\hbar \omega_\gamma = \hbar \sqrt{\frac{C_\gamma}{B_\gamma}}, \quad (\text{b}) \quad (\text{B-35})$$

$$n_\gamma = 0, 1, 2, \dots \quad (\text{c})$$

For a given n_r , K takes on the values

$$\frac{K}{2} = n_r, n_r - 2, n_r - 4, \dots \geq 0 ; \quad (a)$$

$$n_r = 0 ; K = 0$$

(B-36)

$$n_r = 1 ; K = 2$$

(b)

$$n_r = 2 ; K = 0, 4$$

.....

Since H_r is invariant to replacing K by $-K$, the r vibration is a doubly-degenerate mode. The normalized eigenfunction is

$$g_{K, n_r}(r) = \left\{ \frac{2 B_r \omega_r}{\hbar} \frac{\left(\frac{n_r - K/2}{2} \right)!}{\left[\left(\frac{n_r + K/2}{2} \right)! \right]} \right\}^{\frac{1}{2}} e^{-\frac{B_r \omega_r r^2}{2\hbar}} \left(\frac{B_r \omega_r r^2}{\hbar} \right)^{\frac{K}{4}} L_{\frac{n_r + K/2}{2}}^{\frac{K/2}{2}} \left(\frac{B_r \omega_r r^2}{\hbar} \right) \quad (B-37)$$

where the last factor is a Laguerre polynomial.

The total collective energy is then

$$E_{coll} = \frac{\hbar^2}{2\mathcal{I}} [I(I+1) - K^2] + (n_\beta + \frac{1}{2})\hbar\omega_\beta + (n_\gamma + 1)\hbar\omega_\gamma. \quad (B-38)$$

The predicted pure collective quadrupole spectrum consists of rotational bands based on zero-point and one-phonon β and γ vibrations.* The one-phonon ($n_\gamma = 1$) γ vibration projects 2 units of angular momentum on the z-axis ($K = 2$), while the one-phonon ($n_\beta = 1$) β vibration has $K = 0$. The γ -vibrational wave can be pictured as an instantaneous deformation of the nuclear cross section perpendicular to the z-axis and revolving around it so as to project 2 units of angular momentum. The β vibration, on the other hand, corresponds to a distortion parallel to the z-axis with preservation of axial symmetry. These modes are depicted in Fig. (B-3).

The spin sequence of a rotational band, characterized by the band constant K , is $I = K, K+1, K+2, \dots$ for $K \neq 0$, and $I = 0, 2, 4, \dots$ for $K = 0$. These restrictions on I are consequences of imposing on the total collective-model wave function the symmetry requirements appropriate to a prolate spheroid, e.g., invariance to arbitrary rotations about the symmetry axis and to rotations of 180° about any axis perpendicular to the symmetry axis. The additional constraints are

*Vibrations with two or more phonons are generally not observed in low-energy spectra of strongly deformed even-even nuclei.

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Fig. B-3. Normal quadrupole vibrational modes of a spheroidal nucleus. The β mode is depicted in (a) and the γ mode in (b). The β mode preserves rotational symmetry about the nuclear z-axis while the γ mode does not, as shown in (a') and (b') respectively. If $K \neq 0$, the γ -mode involves a rotation of the surface wave about the z-axis as shown in (b').

necessary because the transformation given by Eq. (B-20) is not a one-to-one transformation. The energy spacings between members of a band are given by the first term of Eq. (B-38), if vibration-rotation coupling is neglected. The chief effect of this coupling is to add a negative term to Eq. (B-38) of the form

$$- F \left[I(I+1) - K^2 \right]^2, \quad (B-39)$$

as in the spectra of diatomic molecules.* The parity of quadrupole vibrations can be shown to be even, and, as a consequence of the requirement of inversion symmetry, the parity of each member of the band is also even.

Hence, a ground-state band, with $K = 0$, has a spin sequence $I = 0, 2, 4$ and even parity, as does a one-phonon β -vibrational band, while a γ band, with $K = 2$, has a spin sequence $I = 2, 3, 4, \dots$ with even parity. A typical low-energy spectrum of a deformed even-even nucleus is shown in Fig. (I-1).

According to (B-32), the energy ratios of members of a ground-state band are given by

*The rotation-vibration perturbations also renormalize the moment of inertia very slightly.

$$E_4/E_2 = 10/3 ,$$

(B-40)

$$E_6/E_2 = 7 ,$$

$$E_8/E_2 = 12 ,$$

which agree remarkably well with experiment in the regions of giant electric quadrupole moments, where ground-state band spins as high as $I = 12$ have been observed in some cases by means of multiple Coulomb excitation.⁹³ The agreement is even better if the vibration-rotation term (B-39) is included, where F may be obtained from the first two members of the ground-state band and used to predict higher members. Moments of inertia of bands based on one-phonon vibrations may differ slightly from ground-state band moments.

The total symmetrized wave function is

$$\Phi_{\omega II} = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{K0})}} \int_{n_\beta}(\beta) g_{n_\kappa, n_r}(\gamma) \left(D_{M\kappa}^I + (-1)^I D_{M-\kappa}^I \right) . \quad (B-41)$$

The collective Hamiltonian that has been described applies to even-even nuclei. In the simple version of the Bohr-Mottelson model, the intrinsic degrees of freedom of an even-even nucleus are not explicitly taken into account. The justification for this omission is that short-range residual interactions between pairs of nucleons in time-reversed orbitals produce an energy gap in the intrinsic spectrum, a gap exceeding the excitation energy of low-lying collective states, which are a result of coherent motion of pairs correlated by the long-range component of the residual interaction. That is to say, in order to observe an intrinsic excitation in an even-even nucleus, sufficient energy must be provided to "break a pair," i.e., to scatter two nucleons into different single-particle orbitals. The energy required to do this is considerably greater than that needed to excite a rotational state, although, in some cases, it is comparable to vibrational phonon energies. A typical spectrum of an even-even spheroidal nucleus is compared with the spectrum of a typical adjacent odd-A nucleus in Fig. (B-4).

In the case of odd-even nuclei, however, one must also take into account the intrinsic motion of the odd nucleon, which is coupled to the collective motion of the even-even core.*

*This is also true for odd-odd nuclei, but the picture is further complicated by the need for including an interaction between the odd neutron and proton.

Fig. B-4. Comparison of adjacent even-even and even-odd isotopes.

Thus, one must add to the collective Hamiltonian a term describing the intrinsic motion of the odd nucleon. To obtain the correct collective Hamiltonian for this case, it is necessary to replace $\vec{I}$ by $\vec{I} - \vec{j}$ ($\vec{j}$ being the particle angular momentum) everywhere in Eq. (B-29). That is, the collective angular momentum is the difference between the total angular momentum and the particle angular momentum. The following modifications of (B-29) result for an odd-A nucleus.

(1) The rotational energy becomes

$$E_{\text{rot}}(\text{odd}) = \frac{\hbar^2}{2\mathcal{I}} \left[I(I+1) - K^2 - \Omega^2 + \hat{j}^2 \right], \quad (\text{B-42})$$

where $\Omega = |\hat{j}_3|$, a good quantum number. The last term, proportional to $\hat{j}^2$, can be included in the intrinsic Hamiltonian to be described. Its effect is quite negligible.

(2) A new term is added, representing a Coriolis interaction, which may, under some circumstances, be regarded as a small perturbation

$$H_{\text{coriolis}} = -\frac{\hbar^2}{\mathcal{I}} \left(\hat{I}_1 \hat{j}_1 + \hat{I}_2 \hat{j}_2 \right). \quad (\text{B-43})$$

(3) The γ -vibrational Hamiltonian takes the form

$$H_{\gamma}(\text{odd}) = \frac{\hbar^2}{2B_{\gamma}} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(\hat{I}_3 - \hat{j}_3)^2}{4r^2} \right] + \frac{C_{\gamma}}{2} r^2. \quad (\text{B-44})$$

Defining $K' = |\hat{I}_3 - \hat{j}_3|$, one gets from Eq. (B-36),

$$\frac{K'}{2} = n_r, n_r-2, n_r-4, \dots \quad (a)$$

or

$$\frac{K}{2} = |n_r \pm \Omega/2|, |n_r-2 \pm \Omega/2|, \dots, \quad (b) \quad (B-45)$$

for

$$\begin{aligned} n_r = 0, K = \Omega \\ n_r = 1, K = |2 \pm \Omega| \end{aligned} \quad (c)$$

(4) The vibration-rotation term U and U' are modified by replacing $\vec{I}$ by $\vec{I} - \vec{j}$.

(5) A new term is also added describing the coupling of the odd particle to the β and γ vibrations:

$$H_{vpc} \equiv -\beta k(r_p) \left[\cos \gamma Y_{20}(\theta_p) + \frac{\sin \gamma}{\sqrt{2}} \left(Y_{22}(\theta_p, \varphi_p) + Y_{2-2}(\theta_p, \varphi_p) \right) \right], \quad (B-46)$$

where r_p, θ_p, φ_p are intrinsic coordinates of the odd particle. This term can often be treated as a perturbation.

(6) A new term, $H_{\text{intr}}(\vec{r}_p, \beta_0)$ is added to account for the intrinsic motion of the odd particle.

The intrinsic Hamiltonian, $H_{\text{intr}}(\vec{r}_p, \beta_0)$, describing the motion of the odd particle only, is chosen as a deformed independent particle well of the type (B-3), previously discussed, with eigenstates corresponding to an equilibrium shape, $\beta = \beta_0$ and $\gamma = 0$. The lowest state of the odd particle is determined according to the exclusion principle by filling in pairs the time-reversed orbitals corresponding to a double degeneracy of each eigenvalue of the spheroidal potential. The doubly degenerate states are labeled by the projection of the particle angular momentum $j_z = \pm \Omega$. According to (B-45c), $K = \Omega$, for bands based on zero-point γ vibrations ($n_\gamma = 0$).

Experimental spectra of many odd-mass deformed nuclei agree with the picture of rotational bands based on various states of an odd nucleon in a deformed well, coupled to zero point vibrations.* Each band is characterized by $K = \Omega$, with spin sequence $I = K, K+1, \dots$, where K is half-integral. There is also some evidence for bands in which an odd particle is coupled to a one-phonon γ vibration.⁹⁴ Since the vibration projects 2 units of angular momentum on the z-axis, one can get two band heads with $K = |2+\Omega|$ and $K = |2-\Omega|$ according to Eq. (B-45c). The latter band head, in which an odd particle "swims" against a γ -vibrational wave is expected to generally lie lower than the former, in which particle and wave motion are in phase, according to Eq. (B-42). This appears to be the case experimentally.

*Exceptions to this rule can usually be attributed to the Coriolis perturbation.

A deformed axially symmetric, independent particle potential, which has proved remarkably successful in accounting for observed spins, magnetic moments, etc. of deformed odd-mass nuclei, is the Nilsson potential.⁷¹ This has the form of an anisotropic harmonic oscillator with embellishments⁴¹

$$V_{sp} = \frac{m}{2} \left[\omega_{12}^2 (x_1^2 + x_2^2) + \omega_3^2 x_3^2 \right] - C \vec{l} \cdot \vec{\sigma} - D l^2, \quad (B-47)$$

where the last term is added ad hoc to impart some of the effects of a square well for heavy nuclei. The coefficients C and D are chosen to give the proper magic numbers and spin sequences in the limit of spherical symmetry, $\omega_3 = \omega_{12}$.^{*} These frequencies, which depend on deformation, are chosen so that the volume within an equipotential is independent of deformation, i.e., $\omega_{12}^2 \omega_3 = \text{constant}$. This requirement satisfies approximately the condition of incompressibility of nuclear matter as previously discussed.

The harmonic-oscillator part of (B-47) may be expanded in terms of the $a_{2\mu}$ variables defined by Eq. (B-20)³⁷

$$V_{osc} = \frac{m\omega_0^2}{2} r^2 \left(1 + \frac{5}{4\pi} \sum_{\mu} |a_{2\mu}|^2 - 2 \sum_{\mu} a_{2\mu} Y_{2\mu} \right). \quad (B-48)$$

^{*}Slightly different parameters are used for neutrons and protons.

The equipotential surfaces are given by

$$r = \left(\frac{2V_{osc}}{m\omega_0^2} \right)^{1/2} \left(1 - \frac{1}{4\pi} \sum_{\mu} |a_{2\mu}|^2 + \sum_{\mu} a_{2\mu}^* Y_{2\mu} \right) \quad (B-49)$$

The terms in $|a_{2\mu}|^2$ conserve the volume to second order. For an arbitrary orientation of the potential in space, the $a_{2\mu}$ are replaced by the $d_{2\mu}$.

The symmetrized wave function for a member of the ground state band of an odd-A nucleus is given by

$$\Phi_{coll}^I(I, M, K=\Omega) = \sqrt{\frac{2I+1}{16\pi^2}} \int_{\eta_p=0}^{\eta_p=\pi} f(\beta) \phi(r) \left(D_{\mu K}^I \chi_K + (-1)^{\mu-K} D_{\mu-K}^I R_1 \chi_K \right), \quad (B-50)$$

where χ_K is the intrinsic wave function for the odd particle, $\Omega = K$, and R_1 is the operation of rotation about an axis perpendicular to the symmetry axis.⁷⁰ The wave functions corresponding to the two bands of one-phonon⁷ vibrations are

$$\Phi_{\text{coll}}(I, M, K=2+\Omega) = \sqrt{\frac{2I+1}{16\pi^2}} f(\beta) g(r) \left(D_{M, 2+\Omega}^I \chi_{\Omega}^{(a)} + (-1)^I D_{M, -(2+\Omega)}^I R_1 \chi_{\Omega} \right), \quad (\text{B-51})$$

$$\Phi_{\text{coll}}(I, M, K=|2-\Omega|) = \sqrt{\frac{2I+1}{16\pi^2}} f(\beta) g(r) \left(D_{M, |2-\Omega|}^I \chi_{-\Omega}^{(b)} + (-1)^I D_{M, -|2-\Omega|}^I R_1 \chi_{-\Omega} \right).$$

As a final point in this section, it should be noted that if the motion of all the nucleons in the deformed field were included in the term $H_{\text{intr}}(\vec{r}, \beta_0)$, then the resulting Hamiltonian would have too many degrees of freedom since $\sum_i H_{\text{intr.}}(\vec{r}_i, \vec{\sigma}_i)$ would already exhaust all degrees of freedom. Nevertheless, collective models have been formulated for systems with many degrees of freedom, in which so-called "superfluous" and "redundant" coordinates were introduced.^{27,44} A wave function for a nucleus in the Bohr-Mottelson model, including both intrinsic motion of all nucleons in a deformed well and collective motion, could be written in the form $\psi = \chi(\vec{r}_1, \vec{\sigma}_1; \vec{r}_2, \vec{\sigma}_2; \dots d_0) \varphi(\alpha)$, where α denotes all collective variables. This has the form of a wave function in the Born-Oppenheimer approximation, where α would correspond to internuclear distances, and the $\vec{r}_i$ would be electron coordinates relative to a set of axes defined by the equilibrium configuration of atoms. An important difference, however, is that a molecular wave

function has just the right number of degrees of freedom in the Born-Oppenheimer approximation, while a nuclear wave function conforming to the same idea would have too many. If $\varphi(\alpha)$ is a rotational function, one can write a wave function with an extra degree of freedom as

$$\psi = \chi_K(\vec{r}_1, \vec{\sigma}_1, \dots, \vec{r}_n, \vec{\sigma}_n, \theta_i) D_{MK}^I(\theta_i), \quad (\text{B-52})$$

which can physically be interpreted as follows. The intrinsic function, χ_K describes the state of all nucleons in a deformed field with a fixed orientation defined by a set of Eulerian angles denoted by θ_i , while the rotational function represents the probability amplitude that a particular orientation occurs. An improved nuclear wave function for a state with angular momentum I can be obtained by projecting out of the deformed-well wave function the component with the appropriate value of I

$$\psi_M^I(\vec{r}) = \int \chi_K(\vec{r}, \theta_i) D_{MK}^I(\theta_i) d\theta_i. \quad (\text{B-53})$$

The integration removes the extra degree of freedom and introduces additional correlations. Such projection methods for describing collective motion have been given by Peierls and Yoccoz⁹⁵ and Gross.⁹⁶

*Neglecting the symmetrization, which is not essential to the argument.

7. The Cranking Model

Although the collective model just described has been remarkably successful in explaining spin sequences and energy ratios of low-lying spectra of deformed nuclei, it cannot account a priori for observed energy spacings. Further assumptions are needed to estimate moments of inertia, vibrational mass parameters, and restoring force constants. Experimental moments of inertia are 2-5 times larger than those based on the irrotational flow model. Experimental mass parameters and force constants, which can be deduced from a knowledge of phonon energies $\hbar\omega = \hbar\sqrt{\frac{C}{B}}$, and transition probabilities proportional to $(BC)^{-\frac{1}{2}}$ are 10-30 times the liquid-drop values (although the liquid-drop model gives orders of magnitude of $\hbar\omega$ which are not unreasonable).⁵

The problem of determining rotational moments of inertia, which depend on dynamics of the intrinsic motion, has been attacked by a variety of methods, only one of which, immediately relevant to the present paper, will be described. This is the so-called "cranking model," first applied to nuclei by Inglis.³¹ Although non-rigorous in its description of collective motion, it leads to a formal expression for the rotational moment of inertia which has been corroborated by more rigorous approaches.⁹⁷ The method had been previously used to calculate electronic contributions to moments of inertia of rotating molecules.⁹⁸

The Inglis model visualizes collective rotation as a slow end-over-end revolution of equipotential surfaces of a spheroidal independent particle field, which is adiabatically followed by the motion

of individual nucleons. That is, particles are imagined to move in an explicitly time-dependent anisotropic potential well, $V(x,y,z, \beta_0, \theta(t))$,* where the orientation of the nucleus is characterized by an angle θ . Since the energy is not a constant of motion, the fictitious rotation is generated, so to speak, by an external crank (driven by little demons?). This description is sometimes regarded as "semi-classical," although the mathematical analysis employs quantum mechanics.

The explicit time dependence may be conveniently removed by transforming from a space coordinate system characterized by displacement vector $\vec{r}$, to a rotating coordinate system coinciding with the principal axes of the spheroidal potential, characterized by vector $\vec{r}'$. Assuming axial symmetry about the z' axis, one need only crank about the $x = x'$ axis with constant angular velocity Ω ; the transformation is

$$\begin{aligned} y' &= y \cos \Omega t + z \sin \Omega t \\ z' &= -y \sin \Omega t + z \cos \Omega t. \end{aligned} \quad (B-54)$$

*If the well is isotropic, the time dependence described vanishes, since rotation of a spherical potential carries it into itself.

The transformation removes the time dependence of V , while giving rise to a Coriolis term from the kinetic energy. The Schrödinger equation in the primed frame of reference is

$$\left[\frac{-\hbar^2}{2m} \sum_i \left(\frac{\partial^2}{\partial x_i'^2} + \frac{\partial^2}{\partial y_i'^2} + \frac{\partial^2}{\partial z_i'^2} \right) + \sum_i V(x_i', y_i', z_i', \beta_0) - \Omega \hat{J}_x \right] \chi' = E(\Omega) \chi' \quad (\text{B-55})$$

For slow rotation, the Coriolis term, $-\Omega \hat{J}_x$, can be treated by second-order time-independent perturbation theory, to give the perturbation increment

$$\Delta E = - \frac{\mathcal{S}_{\text{eff}}}{2} \Omega^2 \quad (\text{B-56})$$

with

$$\mathcal{S}_{\text{eff}} = 2\hbar^2 \sum_n \frac{|\langle n | \hat{J}_x | 0 \rangle|^2}{E_n - E_0}, \quad (\text{B-57})$$

where the eigenvalues E_n and vectors $|n\rangle$ refer to Slater configurations of nucleons moving in the field $V = V(x', y', z', \beta_0)$.

The average energy of the corresponding Hamiltonian in the space system, $H(t) = \frac{1}{2m} \sum_i p_i^2 + \sum_i V(x_i, y_i, z_i, \beta_0, \theta(t))$, is related to the eigenvalues E' in the body system by

$$\langle \chi(t) | H(t) | \chi(t) \rangle = E' + \mathcal{I}_{\text{eff}} \Omega^2 = E_0 + \frac{\mathcal{I}_{\text{eff}}}{2} \Omega^2, \quad (\text{B-58})$$

where $\chi(t)$ is the perturbed wave function in the space coordinate system. The perturbation energy is the extra energy needed to make the particles follow the revolving potential, and allows the evaluation of the effective moment of inertia, $\mathcal{I}_{\text{eff}}$, in terms of the nucleonic configuration in a deformed field.

It is a surprising result that the application of Eq. (B-57) to the case of particles moving in a pure anisotropic harmonic oscillator well gives a value of $\mathcal{I}_{\text{eff}}$ exactly equal to the value for rotation of a rigid body, $\mathcal{I}_{\text{rig}} = \langle 0 | m \sum_i (z_i^2 + y_i^2) | 0 \rangle$, in spite of the existence of surface currents, provided that the deformation of the potential is chosen to be the equilibrium value. The choice of other independent particle wells gives nearly rigid-body values.

Since the rigid-body movements of inertia are on the average about twice the observed moments, the cranking model first appeared to be unsatisfactory. However, Bohr and Mottelson proposed an argument that the moments would be reduced by including (rotationally invariant) residual pairing forces.³⁴ Extensive calculations by Griffen and Rich,¹⁵ and by Nilsson and Prior,¹⁶ taking into account

a pairing force by means of the "superconductor model" described in detail in Ch. III, verified this conjecture and gave good agreement with empirical moments of inertia.

The general idea of the cranking model has been extended to other problems. There exists a linear analogue, sometimes known as the "pushing model,"⁶⁵ and also vibrational analogues, which, for want of a better name, will be subsumed under "cranking models."^{12,38,39}

In the vibrational analogues, nucleons move in deformed shell-model wells $V(x, y, z, Q_\mu(t))$, where $Q_\mu(t)$ is a shape (rather than orientation) parameter, which is an explicit harmonic function of time. Since further details on vibrations are given in Ch. II of this paper, it will suffice to say that in the adiabatic perturbation theory, one obtains an expression for vibrational mass parameters analogous to (B-57), with J_x replaced by the operator $\frac{\partial}{\partial Q_\mu}$.

8. The Superconductor Model of the Nucleus

The nuclear "superconductor model" consists of application of the Bardeen, Cooper, and Schrieffer - Bogolyubov (BCS-Bogolyubov) formalism,^{13,14} originally devised to describe the phenomenon of superconductivity, to approximately take into account residual short-range pairing correlations. Since the formalism is described at some length in Ch. III, only the justification for its use will be given here.

The following nuclear phenomena have been cited as providing evidence for the existence of pairing correlations, in that they can be explained if such an interaction is assumed.

a. Existence of gaps in intrinsic spectra of even-even nuclei. It is meaningful to speak of gaps when these are considerably greater than the average spacing of single-particle states. In most even-even nuclei, intrinsic excitations are not seen below 1 Mev., while in adjacent odd-mass nuclei, close-lying transitions of an odd nucleon between single-particle orbitals can be identified below 1 Mev. The gap is most easily seen near closed shells, since away from closed shells, collective excitations fill the gap.⁵⁸

b. Stability of the spherical shape near closed shells. In addition, the transition from spherical to deformed nuclei is rather abrupt, which can be explained by means of pairing correlations.⁵⁷

c. Empirical moments of inertia ^{that} are considerably smaller than the corresponding rigid-body values, although larger than irrotational values. Their rotational value can be shown to occur in the limit of infinitely strong short-range forces.⁶⁶

d. The necessity for a pairing energy correction in the semi-empirical binding energy formula, corresponding to an even-odd mass difference.

e. The fact that low-energy nuclear spectra do not become appreciably more complex away from closed shells has been attributed to short-range correlations.⁵⁸

Bohr, Mottleson, and Pines first suggested a possible analogy between nuclei and superconductors on the basis of energy gaps in both cases.⁹⁹ A significant difference is that finite size effects can be

neglected in superconductors, where the gap occurs in an otherwise continuous spectrum; not so in nuclei.

However, there exists a physical and formal analogy between the effective correlation of pairs of electrons with opposite momenta in superconductors, and the residual correlation of pairs of nucleons with opposite angular momentum in nuclei. In both cases, one can apply the same mathematical formalism, which depends on a canonical transformation from the original particles to new "quasi-particles," allowing the original Hamiltonian with correlation terms to be replaced by an approximate independent quasi-particle Hamiltonian.

The importance of testing the superconductor model on nuclei lies in its consequence that even "small" interactions between Fermion pairs near the Fermi level may produce fundamental changes in the properties of a system, provided that the interactions introduce a coherent correlation of the pairs. In superconductors, the pairing interaction between electrons effectively takes into account the electron-phonon interaction, which modifies the Fermi sea. The net effect resembles a Bose-Einstein condensation of Fermion pairs to form a superconducting (superfluid) state.

C. Results of Calculations

The results of the calculations are presented in numerical tables and in corresponding figures, in the rare earth and actinide regions. For each region, the following types of tables are given.

1. Phonon Energies of Beta and Gamma Vibrations

The 1st column identifies the nuclides, and does so in all ensuing tables. The 2nd column identifies the corresponding equilibrium deformations ϵ , which is approximately equal to Nilsson's parameter δ , and is related to the parameter $\alpha = \sqrt{\frac{5}{4\pi}} \beta$ by $\alpha = \frac{2}{3} \epsilon + \epsilon^2/4 + \dots$. In some cases, calculations were performed for several values of ϵ for a given nuclide, and corresponding results are given. The equilibrium deformations are not indicated in ensuing tables, but when several values of a quantity are given for a single nuclide, it is understood that these correspond to different values of ϵ as given in the first table.

The 3rd column gives the lowest 0⁺ intrinsic excitation or energy gap (approximately equal to, but not identical to Δ), below which the β -vibrational excitations occur. The 4th column gives the theoretical value of $\hbar\omega_\beta$ in the adiabatic approximation, while the 5th column gives the more exact theoretical solution of the dispersion formula. The experimental values of $\hbar\omega_\beta$ are given in the 6th column. Bibliographic references for the experimental data are indicated by superscripts in parentheses. This convention is used in all ensuing tables. The 7th column gives the lowest 2⁺ intrinsic excitation, below which $\hbar\omega_\gamma$ must lie. Columns 8-10 give the theoretical and

experimental values of $\hbar\omega_\gamma$ in correspondence with $\hbar\omega_\beta$. All energies in this table are in Kev.

2. Force Constants(Surface Energies) of Beta and Gamma Vibrations

The adiabatic and dynamic (theoretical) and experimental values of C_β are given in columns 2, 3, 4 respectively. Correspondingly, values of C_γ/β_0^2 are listed in columns 5, 6, 7. The quantity C_γ/β_0^2 rather than C_γ is listed, since in the hydrodynamic model $C_\gamma/\beta_0^2 = C_\beta$. In the present model, this relation does not hold in most cases. All quantities are in Mev.

3. Mass Parameters of Beta and Gamma Vibrations

The adiabatic and dynamic (theoretical) and experimental values of B_β are given in columns 2, 3, 4 respectively. Correspondingly, values of B_γ/β_0^2 are listed in columns 5, 6, 7. The quantity B_γ/β_0^2 rather than B_γ is listed, since in the hydrodynamic model $B_\gamma/\beta_0^2 = B_\beta$. The mass parameters are given in units of the irrotational flow mass parameter $B_{2irrot} = \frac{5}{8\pi} m \langle \sum_i r_i^2 \rangle$.

4. Reduced Electric Quadrupole Transition Probabilities

This table gives the reduced electric quadrupole transition probabilities for exciting the 2_+ member of a one-phonon β - or γ -vibrational band, $B(E2; 0 \rightarrow 2_+)$, in Weisskopf single-particle units⁵ $B(E2)_{s.p.} = 2.97 \times 10^{-53} A^{4/3} e^2 \text{ cm}^4$. Columns 2, 3, 4 refer to the β band. Column 2 gives the theoretical value

neglecting a vibration-rotation interaction correction, while Column 3 gives the theoretical value including this correction. Column 4 gives the experimental results. The corresponding transition probabilities for the γ band are given in columns 5, 6, 7.

5. Vibration-Rotation Interaction Coefficients

This table gives values of the vibration-rotation interaction coefficients, F , calculated with adiabatic and with dynamic force constants, for the two cases when the moment of inertia has a linear or quadratic deformation dependence. Columns 2 and 3 refer to the case $\mathcal{I} \propto \beta$; column 2 gives the theoretical values corresponding to adiabatic force constants, while column 3 gives theoretical values corresponding to dynamic force constants. Corresponding theoretical values for the case $\mathcal{I} \propto \beta^2$ as given in columns 4 and 5. The experimental values of F as obtained from ground state rotational bands are given in column 6. All values are in units of ev.

Results corresponding to different shifts of single-particle levels, previously denoted by case (a) and case (b) are given. All calculated quantities in the tables are based on the assumptions of the superfluid model. Some results for gamma vibrations, neglecting the superfluidity are shown only in figures marked by the statement "no pairing." All other figures correspond to tabulated material. The correspondence is indicated by using a Roman numeral for tables and the equivalent decimal for figures. Thus, the data in Fig. C.3 come from Table C.III. If two figures refer to the same table, "A" or "B" is added to the figure number.

Table C.I. Phonon energies of beta and gamma vibrations in the rare earth region - Case (a).

Nuclide	ϵ^a	Lowest intrinsic 0+ state (keV)	$\hbar\omega_\beta$ (keV)			Lowest intrinsic 2+ state (keV)	$\hbar\omega_\gamma$ (keV)		
			theory		experiment		theory		experiment
			adiabatic approx.	dispersion formula			adiabatic approx.	dispersion formula	
Sm ¹⁵²	.34	2503	3113	2075	685 ⁽⁶⁾	3488	3334	2607	1034 ⁽⁶⁾
Sm ¹⁵⁴	(.34)	2142	3303	1945		3042	2950	2280	
Gd ¹⁵⁴	.33	2505	3048	2055	681 ⁽⁴⁾	2928	2595	2061	949 ⁽⁴⁾
Gd ¹⁵⁶	.33	2137	3175	1881	[~812] ^{(3) b}	2615	2155	1745	1126 ⁽⁷⁷⁾
Gd ¹⁵⁸	.34	1913	3525	1872		2485	1574	1357	1156 ⁽⁷⁷⁾
Gd ¹⁶⁰	.34	1851	3521	1833		2070	1193	1067	
Dy ¹⁶⁰	.31	1906	3948	1901		2239	1316	1164	937 ⁽¹⁰⁰⁾
Dy ¹⁶²	.31	1846	3651	1841		2093	986	907	
Dy ¹⁶⁴	.30	1636	2435	1392		1967	1029	933	
Er ¹⁶⁴	.30	1837	3311	1747		2104	1265	1115	
Er ¹⁶⁶	.29	1634	2466	1395	1460 ⁽³⁾	1954	1271	1108	763 ⁽³⁾
Er ¹⁶⁸	.28	1528	3391	1518		2021	1527	1275	797 ⁽⁴⁾
Er ¹⁷⁰	.27	1503	3609	1446		2151	1871	1485	
Yb ¹⁷⁰	.27	1531	3153	1509		2007	2025	1542	1204 ⁽⁴⁾
Yb ¹⁷²	.26	1472	2997	1404		2271	2394	1812	1441 ⁽⁴⁾
Yb ¹⁷⁴	.25	1368	2569	1267		2348	2178	1693	
Yb ¹⁷⁶	.24	1366	2781	1330		2187	1795	1451	
Hf ¹⁷⁶	.24	1351	1886	1180		2334	1977	1597	
Hf ¹⁷⁸	.22	1366	2135	1289	1197 ⁽⁴⁾	2186	1626	1365	
Hf ¹⁸⁰	.21	1512	1959	1328		2220	1439	1251	
W ¹⁸²	.20	1504	2181	1425	1180 ⁽⁴⁾	2225	1214	1094	1188 ⁽⁴⁾
W ¹⁸⁴	.18	1701	1612	1225		2042	1053	972	869 ⁽⁴⁾
W ¹⁸⁶	.17	1911	1687	1312		2026	1018	944	689 ⁽⁸⁾

^a The deformations are taken from reference (72) except those in parentheses which have been interpolated from values of reference (72).

^b This assignment is not certain.

Table C.II. Phonon energies of beta and gamma vibrations in the rare earth region - Case (b).

Nuclide	ϵ^a	Lowest intrinsic 0+ state (keV)	$\hbar\omega_\beta$ (keV)			Lowest intrinsic 2+ state (keV)	$\hbar\omega_\gamma$ (keV)		
			theory		experiment		theory		experiment
			adiabatic approx.	dispersion formula			adiabatic approx.	dispersion formula	
Sm^{152}	.34	2512	2420	1846	685 ⁽⁶⁾	2905	1256	1187	1034 ⁽⁶⁾
Sm^{154}	(.34)	2094	2955	1944		2905	1755	1565	
Gd^{154}	.33	2503	2505	1884	681 ⁽⁴⁾	2928	1023	981	949 ⁽⁴⁾
Gd^{156}	.33	2096	2927	1872	[~812] ⁽³⁾ , b	2615	1322	1214	1126 ⁽⁷⁷⁾
Gd^{158}	.34	1891	3311	1855		2418	1061	988	1156 ⁽⁷⁷⁾
Gd^{160}	.34	1882	3345	1832		2057	942	877	
Dy^{160}	.31	1892	3800	1888		2239	996	926	937 ⁽¹⁰⁰⁾
Dy^{162}	.31	1865	3489	1852		2084	784	743	
Dy^{164}	.30	1633	2377	1381		1971	911	844	
Er^{164}	.30	1853	3177	1721		2095	1101	999	
Er^{166}	.29	1637	2428	1390	1460 ⁽³⁾	1966	1168	1039	763 ⁽³⁾
Er^{168}	.28	1526	3367	1519		2028	1452	1232	797 ⁽⁴⁾
Er^{170}	.27	1496	3550	1440		2151	1799	1450	
Yb^{170}	.27	1533	3118	1511		2014	1953	1516	1204 ⁽⁴⁾
Yb^{172}	.26	1469	2951	1398		2279	2319	1785	1441 ⁽⁴⁾
Yb^{174}	.25	1365	2538	1264		2345	2103	1658	
Yb^{176}	.24	1363	2752	1329		2186	1730	1417	
Hf^{176}	.24	1349	1861	1175		2331	1904	1559	
Hf^{178}	.22	1364	2109	1285	1197 ⁽⁴⁾	2185	1561	1326	
Hf^{180}	.21	1514	1926	1319		2215	1373	1207	
W^{182}	.20	1505	2144	1416	1180 ⁽⁴⁾	2220	1144	1042	1188 ⁽⁴⁾
W^{184}	.18	1701	1584	1214		2042	981	915	869 ⁽⁴⁾
W^{186}	.17	1911	1658	1299		2026	945	886	689 ⁽⁸⁾

^a The deformations are taken from reference (72) except those in parentheses which have been interpolated from the values of reference (72).

^b This assignment is not certain.

Table C.III. Force constants (surface energies) of beta and gamma vibrations in the rare earth region - Case (a)

Nuclide ^a	Theory (Mev)		Experiment (Mev)	Theory (Mev)		Experiment (Mev)
	$c_{\beta}(0)$	$c_{\beta}(\omega)$	c_{β}	$\frac{c_{\gamma}(0)}{\beta_0^2}$	$\frac{c_{\gamma}(\omega)}{\beta_0^2}$	$\frac{c_{\gamma}}{\beta_0^2}$
Sm ¹⁵²	534	1350	72±24 ⁽⁶⁾	348	615	270±50 ⁽⁶⁾
Sm ¹⁵⁴	640	2250		347	624	
Gd ¹⁵⁴	529	1300		292	493	
Gd ¹⁵⁶	648	2140		274	435	~540 ⁽⁷⁷⁾
Gd ¹⁵⁸	739	4490		208	283	~550 ⁽⁷⁷⁾
Gd ¹⁶⁰	764	7960		161	202	
Dy ¹⁶⁰	759	63000		175	226	
Dy ¹⁶²	781	39400		126	150	
Dy ¹⁶⁴	711	2400		147	180	
Er ¹⁶⁴	758	3470		184	239	
Er ¹⁶⁶	719	2470		201	268	~300 ⁽⁷⁷⁾
Er ¹⁶⁸	840	20200		263	384	~320 ⁽⁷⁷⁾
Er ¹⁷⁰	905	10300		340	562	
Yb ¹⁷⁰	795	17800		354	666	
Yb ¹⁷²	835	7090		416	795	
Yb ¹⁷⁴	830	4690		405	705	
Yb ¹⁷⁶	854	7810		357	563	
Hf ¹⁷⁶	699	2030		366	581	
Hf ¹⁷⁸	741	2580		315	457	
Hf ¹⁸⁰	656	1520		271	363	
W ¹⁸²	688	2060		215	266	{ 290±70 ⁽⁷⁸⁾ ~600 ⁽⁷⁷⁾
W ¹⁸⁴	511	914		169	199	
						{ 260±70 ⁽⁷⁸⁾ 280±80 ⁽⁸⁾ ~380 ⁽⁷⁷⁾
W ¹⁸⁶	493	837		160	187	{ 190±50 ⁽⁷⁸⁾ 210±40 ⁽⁸⁾ ~310 ⁽⁷⁷⁾

^a The deformations ϵ correspond to those listed in Table C.I.

Table C.IV. Force constants (surface energies) of beta and gamma vibrations in the rare earth regions - Case (b)

Nuclide ^a	Theory (Mev)		Experiment (Mev)	Theory (Mev)		Experiment (Mev)
	$C_{\beta}(0)$	$C_{\beta}(\omega)$	C_{β}	$\frac{C_{\gamma}(0)}{\beta_0^2}$	$\frac{C_{\gamma}(\omega)}{\beta_0^2}$	$\frac{C_{\gamma}}{\beta_0^2}$
Sm ¹⁵²	406	731	72±24 ⁽⁶⁾	88.7	99.5	270±50 ⁽⁶⁾
Sm ¹⁵⁴	555	1490		173	220	
Gd ¹⁵⁴	422	792		66.2	72.0	
Gd ¹⁵⁶	577	1620		125	150	~540 ⁽⁷⁷⁾
Gd ¹⁵⁸	681	3830		105	121	~550 ⁽⁷⁷⁾
Gd ¹⁶⁰	720	3590		103	119	
Dy ¹⁶⁰	719	71900		105	121	
Dy ¹⁶²	745	8180		81.0	90.3	
Dy ¹⁶⁴	685	2230		116	135	
Er ¹⁶⁴	725	3010		142	173	
Er ¹⁶⁶	697	2330		170	217	~300 ⁽⁷⁷⁾
Er ¹⁶⁸	821	23100		237	334	~320 ⁽⁷⁷⁾
Er ¹⁷⁰	885	9720		315	502	
Yb ¹⁷⁰	775	16600		328	589	
Yb ¹⁷²	815	6700		391	714	
Yb ¹⁷⁴	812	4500		380	640	
Yb ¹⁷⁶	836	7710		333	510	
Hf ¹⁷⁶	681	1940		342	526	
Hf ¹⁷⁸	723	2450		292	412	
Hf ¹⁸⁰	637	1440		247	323	
W ¹⁸²	669	1880		191	232	$\begin{cases} 290\pm70^{(78)} \\ \sim 600^{(77)} \end{cases}$
W ¹⁸⁴	494	870		147	170	$\begin{cases} 260\pm70^{(78)} \\ 280\pm80^{(8)} \\ \sim 380^{(77)} \end{cases}$
W ¹⁸⁶	477	796		139	158	$\begin{cases} 190\pm50^{(78)} \\ 210\pm40^{(8)} \\ \sim 310^{(77)} \end{cases}$

^a The deformations ϵ correspond to those listed in Table C.II.

Table C.V. Mass parameters of beta and gamma vibrations in the rare earth region - Case (a).

Region - Case (a).

Nuclide ^a	Theory		Experiment	Theory		Experiment
	B _β (0)	B _β (ω)	B _β	B _γ (0)	B _γ (ω)	B _γ
	B ₂ irrot	B ₂ irrot	B ₂ irrot	β ₀ ² B ₂ irrot	β ₀ ² B ₂ irrot	β ₀ ² B ₂ irrot
Sm ¹⁵²	2.88	16.5	8±3 ⁽⁶⁾	1.64	4.74	13±2 ⁽⁶⁾
Sm ¹⁵⁴	3.00	30.5		2.04	6.15	
Gd ¹⁵⁴	2.93	15.8		2.23	5.96	
Gd ¹⁵⁶	3.23	30.4		2.97	7.19	~22 ⁽⁷⁷⁾
Gd ¹⁵⁸	2.92	62.8		4.12	7.55	~21 ⁽⁷⁷⁾
Gd ¹⁶⁰	2.96	114		5.42	8.53	
Dy ¹⁶⁰	2.37	847		4.91	8.09	
Dy ¹⁶²	2.79	553		6.18	8.66	
Dy ¹⁶⁴	5.61	57.9		6.52	9.66	
Er ¹⁶⁴	3.23	53.2		5.39	9.00	
Er ¹⁶⁶	5.44	58.4		5.73	10.0	~23 ⁽⁷⁷⁾
Er ¹⁶⁸	3.31	397		5.10	10.7	~23 ⁽⁷⁷⁾
Er ¹⁷⁰	3.09	219		4.32	11.3	
Yb ¹⁷⁰	3.56	347		3.84	12.5	
Yb ¹⁷²	4.07	158		3.17	10.6	
Yb ¹⁷⁴	5.41	126		3.67	10.6	
Yb ¹⁷⁶	4.68	187		4.69	11.3	
Hf ¹⁷⁶	8.33	61.7		3.97	9.65	
Hf ¹⁷⁸	6.79	64.8		4.98	10.3	
Hf ¹⁸⁰	7.03	35.4		5.37	9.52	
W ¹⁸²	5.85	41.0		5.90	9.00	{ 8.0±2.0 ⁽⁷⁸⁾ ~17 ⁽⁷⁷⁾ 13±3 ⁽⁷⁸⁾ 15±4 ⁽⁸⁾ ~19 ⁽⁷⁷⁾
W ¹⁸⁴	7.84	24.3		6.09	8.42	
W ¹⁸⁶	6.80	19.1		6.07	8.24	

^a The deformations ϵ correspond to those listed in Table C.I.

Table C.VI. Mass parameters for beta and gamma vibrations in the rare earth region - Case (b)

Nuclide ^a	Theory		Experiment	Theory		Experiment
	$B_{\beta}(0)$	$B_{\beta}(\omega)$	B_{β}	$B_{\gamma}(0)$	$B_{\gamma}(\omega)$	B_{γ}
	B_2 irrot	B_2 irrot	B_2 irrot	$\beta_0^2 B_2$ irrot	$\beta_0^2 B_2$ irrot	$\beta_0^2 B_2$ irrot
Sm ¹⁵²	3.63	11.2	8±3 ⁽⁶⁾	2.94	3.69	13±2 ⁽⁶⁾
Sm ¹⁵⁴	3.26	20.2		2.88	4.60	
Gd ¹⁵⁴	3.46	11.5		3.25	3.85	
Gd ¹⁵⁶	3.39	23.3		3.61	5.10	~22 ⁽⁷⁷⁾
Gd ¹⁵⁸	3.05	54.7		4.56	6.09	~21 ⁽⁷⁷⁾
Gd ¹⁶⁰	3.09	51.4		5.56	7.42	
Dy ¹⁶⁰	2.42	981		5.13	6.89	
Dy ¹⁶²	2.92	113		6.26	7.79	
Dy ¹⁶⁴	5.68	54.8		6.52	8.89	
Er ¹⁶⁴	3.36	47.7		5.47	8.12	
Er ¹⁶⁶	5.44	55.5		5.73	9.23	~23 ⁽⁷⁷⁾
Er ¹⁶⁸	3.28	454		5.08	9.96	~23 ⁽⁷⁷⁾
Er ¹⁷⁰	3.13	209		4.33	10.6	
Yb ¹⁷⁰	3.55	323		3.82	11.4	
Yb ¹⁷²	4.10	150		3.18	9.81	
Yb ¹⁷⁴	5.43	121		3.70	10.0	
Yb ¹⁷⁶	4.68	185		4.71	10.7	
Hf ¹⁷⁶	8.33	59.4		3.99	9.16	
Hf ¹⁷⁸	6.79	62.0		5.00	9.77	
Hf ¹⁸⁰	7.06	34.1		5.39	9.11	
W ¹⁸²	5.88	38.0		5.92	8.64	8.0±2.0 ⁽⁷⁸⁾ ~17 ⁽⁷⁷⁾
W ¹⁸⁴	7.85	23.5		6.10	8.09	13±3 ⁽⁷⁸⁾ 15±4 ⁽⁸⁾ ~19 ⁽⁷⁷⁾
W ¹⁸⁶	6.81	18.5		6.08	7.92	15±4 ⁽⁷⁸⁾ 17±3 ⁽⁸⁾ ~25 ⁽⁷⁷⁾

^a The deformations ϵ correspond to those listed in Table C.II.

Table C.VII. Reduced electric quadrupole transition probabilities $B(E2; 0 \rightarrow 2^+)$ for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the rare earth region. - Case (a)

Nuclide ^a	Beta band (single-particle units)			Gamma band (single-particle units)		
	theory		experiment	theory		experiment
	no vibration-rotation interaction correction	including vibration-rotation interaction correction		no vibration-rotation interaction correction	including vibration-rotation interaction correction	
Sm ¹⁵²			$3 \pm 1^{(6)}$			$5 \pm 1^{(6)}$
Sm ¹⁵⁴	1.17	.92		6.48	6.10	
Gd ¹⁵⁴	.66	.54		5.59	5.30	
Gd ¹⁵⁶	1.29	1.02		6.82	6.32	
Gd ¹⁵⁸	.72	.59		6.54	6.10	$\sim 3^{(77)}$
Gd ¹⁶⁰	.34	.29		7.81	7.22	$\sim 3^{(77)}$
Dy ¹⁶⁰	.19	.16		8.61	7.85	
Dy ¹⁶²	.026	.022		8.94	7.21	
Dy ¹⁶⁴	.040	.03		10.5	9.33	
Er ¹⁶⁴	.50	.41		9.00	8.10	
Er ¹⁶⁶	.46	.39		8.58	7.86	
Er ¹⁶⁸	.52	.42		7.62	6.98	$\sim 4.1^{(77)}$
Er ¹⁷⁰	.069	.060		6.10	5.68	$\sim 4.0^{(77)}$
Yb ¹⁷⁰	.13	.11		4.86	4.57	
Yb ¹⁷²	.083	.070		4.51	4.25	
Yb ¹⁷⁴	.19	.16		4.45	4.24	
Yb ¹⁷⁶	.26	.21		4.68	4.44	
Hf ¹⁷⁶	.17	.14		5.03	4.73	
Hf ¹⁷⁸	.60	.47		5.67	5.36	
Hf ¹⁸⁰	.52	.42		6.16	5.77	
W ¹⁸²	.90	.71		7.12	6.59	
W ¹⁸⁴	.75	.59		8.95	8.11	$\begin{cases} 8.2 \pm 2.0^{(78)} \\ \sim 3.9^{(77)} \end{cases}$
W ¹⁸⁶	1.46	1.1		10.6	9.40	$\begin{cases} 6.1 \pm 1.5^{(78)} \\ 5.5 \pm 1.6^{(8)} \\ \sim 4.2^{(77)} \end{cases}$
						$\begin{cases} 6.3 \pm 1.6^{(78)} \\ 5.5 \pm 1.0^{(8)} \\ \sim 3.8^{(77)} \end{cases}$

^a The deformations ϵ correspond to those in Table C.I.

Table C.VIII. Reduced electric quadrupole transition probabilities $B(E2; 0 \rightarrow 2^+)$ for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the rare earth region - Case (b).

Nuclide ^a	Beta band (single-particle units)			Gamma band (single-particle units)		
	theory		experiment	theory		experiment
	no vibration - rotation interaction correction	including vibration - rotation interaction correction		no vibration - rotation interaction correction	including vibration - rotation interaction correction	
Sm ¹⁵²	1.93	1.5	3±1 ⁽⁶⁾	18.3	16.0	5±1 ⁽⁶⁾
Sm ¹⁵⁴	.997	.82		10.9	10.0	
Gd ¹⁵⁴	1.94	1.5		22.2	18.9	
Gd ¹⁵⁶	.94	.77		13.2	11.9	~3 ⁽⁷⁷⁾
Gd ¹⁵⁸	.39	.33		13.3	11.9	~3 ⁽⁷⁷⁾
Gd ¹⁶⁰	.42	.36		12.0	10.7	
Dy ¹⁶⁰	.023	.019		13.2	11.7	
Dy ¹⁶²	.20	.17		14.3	12.4	
Dy ¹⁶⁴	.54	.44		10.8	9.60	
Er ¹⁶⁴	.53	.44		10.6	9.60	
Er ¹⁶⁶	.55	.45		8.83	8.04	~4.1 ⁽⁷⁷⁾
Er ¹⁶⁸	.060	.050		6.79	6.31	~4.0 ⁽⁷⁷⁾
Er ¹⁷⁰	.14	.12		5.32	5.01	
Yb ¹⁷⁰	.089	.075		5.02	4.73	
Yb ¹⁷²	.20	.16		4.88	4.64	
Yb ¹⁷⁴	.27	.22		5.05	4.78	
Yb ¹⁷⁶	.17	.14		5.43	5.10	
Hf ¹⁷⁶	.63	.50		6.12	5.78	
Hf ¹⁷⁸	.54	.44		6.65	6.22	
Hf ¹⁸⁰	.94	.74		7.72	7.12	
W ¹⁸²	.82	.65		9.80	8.85	{ 8.2±2.0 ⁽⁷⁸⁾ ~3.9 ⁽⁷⁷⁾
W ¹⁸⁴	1.52	1.1		11.7	10.3	{ 6.1±1.5 ⁽⁷⁸⁾ 5.5±1.6 ⁽⁸⁾ ~4.2 ⁽⁷⁷⁾
W ¹⁸⁶	1.78	1.2		12.2	10.3	{ 6.3±1.6 ⁽⁷⁸⁾ 5.5±1.0 ⁽⁸⁾ ~3.8 ⁽⁷⁷⁾

^a The deformations ϵ correspond to those listed in Table C.II.

Table C.IX. Vibration-rotation interaction coefficients in the rare earth region - Case (a)

Nuclide ^a	F(ev) from theory				F(ev) ^b from ground- state band
	using $\mathfrak{S} \propto \beta$		using $\mathfrak{S} \propto \beta^2$		
	using adiabatic force constants	using dynamic force constants	using adiabatic force constants	using dynamic force constants	
Sm ¹⁵²	7.2	3.7	14	6.4	141
Sm ¹⁵⁴	4.0	1.9	7.5	2.9	
Gd ¹⁵⁴	8.9	4.6	16	7.7	140
Gd ¹⁵⁶	5.2	2.9	9.0	4.0	29.9
Gd ¹⁵⁸	4.4	2.8	6.7	3.2	11.8
Gd ¹⁶⁰	4.7	3.3	6.6	3.5	
Dy ¹⁶⁰	6.9	4.6	9.7	4.7	20
Dy ¹⁶²	7.7	5.8	10	5.8	11.9
Dy ¹⁶⁴	6.3	4.8	8.9	5.5	
Er ¹⁶⁴	5.9	4.1	8.6	4.6	
Er ¹⁶⁶	5.3	3.6	8.1	4.3	13.0
Er ¹⁶⁸	4.1	2.3	6.5	2.4	5.7
Er ¹⁷⁰	3.2	1.6	5.3	1.8	
Yb ¹⁷⁰	3.7	1.5	6.4	1.6	10.8
Yb ¹⁷²	3.1	1.3	5.6	1.5	7.6
Yb ¹⁷⁴	3.3	1.5	5.9	2.0	
Yb ¹⁷⁶	4.0	2.0	6.8	2.4	
Hf ¹⁷⁶	4.3	2.4	8.0	3.6	14.4
Hf ¹⁷⁸	6.0	3.6	10	4.8	13.1
Hf ¹⁸⁰	9.0	6.0	15	8.8	6.0
W ¹⁸²	13	9.4	21	12.0	15.1
W ¹⁸⁴	27	21	43	30	23.3
W ¹⁸⁶	46	37	74	53	8 (8)

^a The deformations ϵ correspond to those listed in Table C.I..

^b These are all taken from references (3), unless otherwise indicated.

Table C.X. Vibration-rotation interaction coefficients in the rare earth region - Case (b).

Nuclide ^a	F(ev) from theory				F(ev) ^b from ground - state band
	using $\mathfrak{S} \propto \beta$		using $\mathfrak{S} \propto \beta^2$		
	using adiabatic force constants	using dynamic force constants	using adiabatic force constants	using dynamic force constants	
Sm ¹⁵²	22	19	31	24	141
Sm ¹⁵⁴	7.0	5.0	11	6.4	
Gd ¹⁵⁴	30	26	40	32	140
Gd ¹⁵⁶	10	7.8	14	9.0	29.9
Gd ¹⁵⁸	8.1	6.4	11	6.9	11.8
Gd ¹⁶⁰	7.1	5.6	9.1	6.1	
Dy ¹⁶⁰	11	8.5	14	8.5	20
Dy ¹⁶²	11	9.5	14	9.9	11.9
Dy ¹⁶⁴	7.9	6.3	10	7.1	
Er ¹⁶⁴	7.4	5.5	10	6.2	
Er ¹⁶⁶	6.2	4.3	9.1	5.1	13.0
Er ¹⁶⁸	4.5	2.7	6.9	2.7	5.7
Er ¹⁷⁰	3.4	1.8	5.6	2.0	
Yb ¹⁷⁰	3.9	1.7	6.6	1.8	10.8
Yb ¹⁷²	3.2	1.4	5.8	1.7	7.6
Yb ¹⁷⁴	3.5	1.7	6.1	2.2	
Yb ¹⁷⁶	4.3	2.3	7.4	2.6	
Hf ¹⁷⁶	4.6	2.6	8.3	4.0	14.4
Hf ¹⁷⁸	6.2	4.0	11	5.2	13.1
Hf ¹⁸⁰	9.8	6.7	16	9.8	6.0
W ¹⁸²	15	11	22	14	15.1
W ¹⁸⁴	31	25	47	34	23.3
W ¹⁸⁶	53	44	81	61	8(8)

^a The deformations ϵ correspond to those in Table C.II.

^b These are all taken from reference (3) unless otherwise marked.

Table C.XI. Phonon energies of beta and gamma vibrations in the actinide region - Case (a).

Nuclide	ϵ^a	Lowest intrinsic 0+ state (keV)	$\hbar\omega_\beta$ (keV)			Lowest intrinsic 2+ state (keV)	$\hbar\omega_\gamma$ (keV)		
			theory		experiment		theory		experiment
			adiabatic approx.	dispersion formula			adiabatic approx.	dispersion formula	
Th ²²⁸	.16			imaginary		1995	3924	1913	950 ⁽¹⁰¹⁾
Th ²³⁰	.20	1576	619	589	634 ⁽³⁾	2022	3167	1807	765 ⁽³⁾
Th ²³²	.23	1441	1344	1056	730 ^(103, 79, 7)	1756	2531	1566	768 ^(103, 79, 7)
Th ²³⁴	.24	1359	1637	1144		1748	2579	1582	
U ²³⁰	.21	1527	629	593		2128	3141	1897	
U ²³²	.24	1474	1512	1124	693 ⁽¹⁰²⁾	1756	2318	1527	852 ⁽¹⁰²⁾
U ²³⁴	.25	1324	1875	1130	812 ⁽³⁾	1656	2467	1491	908 ⁽³⁾
U ²³⁶	.24	1274	1872	1141		1672	2779	1548	
U ²³⁸	.24	1156	2401	1141	997 ^(103, 79, 7)	1798	3104	1701	1047 ⁽¹⁰³⁾
Pu ²³⁶	(.23)	1258	1788	1090		1602	2701	1471	
Pu ²³⁸	.24	1196	2025	1102	935 ⁽¹⁰⁵⁾	1604	2847	1496	1015 ⁽¹⁰⁵⁾
Pu ²⁴⁰	.24	1094	2691	1083	858 ⁽¹⁰⁶⁾	1746	3221	1663	[928] ⁽¹⁰⁶⁾ , [1006] ⁽³⁾ , b
Pu ²⁴²	(.24)	1093	3375	1089		2086	3408	1932	
Cm ²⁴²	.24	1084	3281	1076		1736	3160	1652	
Cm ²⁴⁴	.24	1125	4460	1122		2107	3349	1932	
Cm ²⁴⁶	(.24)	1186	5121	1185		2066	3215	1854	
Cm ²⁴⁸	(.24)	1386	4400	1385		1982	2992	1765	
Cf ²⁴⁸	.24	1238	5742	1238		2096	3055	1821	
Cf ²⁵⁰	.24	1399	4829	1398		1989	2844	1733	1018 ⁽¹⁰⁷⁾
Cf ²⁵²	(.24)	1346	3025	1299		1650	2509	1494	
Cf ²⁵⁴	(.23)	1324	2515	1246		1421	2189	1263	
Fm ²⁵²	(.24)	1423	4791	1423		1982	2747	1694	
Fm ²⁵⁴	(.23)	1333	3427	1294		1688	2508	1520	678 ⁽¹⁰⁸⁾
Fm ²⁵⁶	(.23)	1382	2662	1304		1484	2219	1315	

^a The deformations are taken from reference (73), except/in parentheses which have been interpolated or extrapolated from the values of reference (73).
^b Assignments are uncertain.

Table C.XII. Phonon energies of beta and gamma vibrations in the actinide region - Case (b).

Nuclide	ϵ^a	Lowest intrinsic 0+ state (keV)	$\hbar\omega_\beta$ (keV)			Lowest intrinsic 2+ state (keV)	$\hbar\omega_\gamma$ (keV)		
			theory		experiment		theory		experiment
			adiabatic approx.	dispersion formula			adiabatic approx.	dispersion formula	
Th ²²⁸	(.20)			imaginary		2080	2046	1552	950 ⁽¹⁰¹⁾
"	(.21)			imaginary		2043	1631	1326	
"	(.22) ^b			imaginary		1947	1236	1074	
"	(.23)			imaginary		1869	825	766	
Th ²³⁰	(.21)			imaginary	634 ⁽³⁾	1956	1891	1445	765 ⁽³⁾
"	(.23)	1528	418	407		1862	1210	1050	
"	(.24) ^b	1543	589	560		1828	877	807	
Th ²³²	.23	1432	873	783	730 ^(103, 79, 7)	1758	1523	1228	768 ^(103, 79, 7)
"	(.24) ^b	1425	861	766		1752	1279	1086	
Th ²³⁴	.24	1351	1192	970		1759	1623	1275	
U ²³⁰	(.22)			imaginary		1946	1473	1220	
"	(.23) ^b	1551	444	431		1819	1054	938	
U ²³²	.24	1443	897	789	693 ⁽¹⁰²⁾	1742	1068	943	852 ⁽¹⁰²⁾
U ²³⁴	.25	1327	1297	969	812 ⁽³⁾	1671	1397	1147	908 ⁽³⁾
U ²³⁶	.24	1265	1462	1052		1683	1878	1377	
U ²³⁸	.24	1158	1954	1129	997 ^(103, 79, 7)	1807	2211	1540	1047 ⁽¹⁰³⁾
Pu ²³⁵	(.23)	1247	1418	1008		1603	1919	1346	
"	(.24) ^b	1235	1266	929		1594	1643	1244	
Pu ²³⁸	.24	1187	1662	1055	935 ⁽¹⁰⁵⁾	1614	2056	1398	1015 ⁽¹⁰⁵⁾
Pu ²⁴⁰	.24	1096	2265	1079	858 ⁽¹⁰⁶⁾	1754	2434	1575	[928] ⁽¹⁰⁶⁾ , [1006] ⁽³⁾ , c
Pu ²⁴²	(.24)	1093	2833	1087		1960	2627	1702	
Cm ²⁴²	.24	1086	2795	1075		1744	2647	1619	
Cm ²⁴⁴	.24	1125	3718	1121		2108	2868	1843	
Cm ²⁴⁵	(.24)	1189	4077	1188		2074	2806	1786	
Cm ²⁴³	(.24)	1346	3337	1345		1988	2656	1711	
Cf ²⁴³	.24	1242	4534	1241		2106	2785	1771	
Cf ²⁵⁰	.24	1360	3651	1359		1996	2632	1695	1018 ⁽¹⁰⁷⁾
Cf ²⁵²	(.24)	1399	2164	1297		1747	2396	1534	
Cf ²⁵⁴	(.23)	1305	1536	1091		1490	2131	1311	
Fm ²⁵²	(.24)	1385	3665	1384		1979	2596	1667	
Fm ²⁵⁴	(.23)	1367	2509	1300		1775	2430	1559	678 ⁽¹⁰⁸⁾
Fm ²⁵⁶	(.23)	1365	1658	1156		1555	2178	1364	

^a The deformations are taken from reference (73), except those in parentheses which have been interpolated, extrapolated, or slightly changed to take into account shifts of single-particle levels, from those of reference (73).

^b This deformation corresponds to the values plotted on the graphs.

^c Assignments are uncertain.

Table C.XIII. Force constants of beta and gamma vibrations in the actinide region - Case (a)

Nuclide ^a	Theory (MeV)		Experiment (MeV)	Theory (MeV)		Experiment (MeV)
	$C_{\beta}(0)$	$C_{\beta}(\omega)$	C_{β}	$\frac{C_{\gamma}(0)}{\beta_0^2}$	$\frac{C_{\gamma}(\omega)}{\beta_0^2}$	$\frac{C_{\gamma}}{\beta_0^2}$
Th ²²⁸	< 0	x		1090	9340	
Th ²³⁰	179	198		965	3790	
Th ²³²	686	1160	$\begin{cases} 310 \pm 60^{(7)} \\ 630^{(79)} \end{cases}$	846	2900	$\begin{cases} 1100 \pm 220^{(7)} \\ 600^{(79)} \end{cases}$
Th ²³⁴	924	2120		872	3240	
U ²³⁰	194	218		926	3230	
U ²³²	797	1540		775	2200	
U ²³⁴	1140	3570		847	3240	
U ²³⁶	1130	3680		953	4850	
U ²³⁸	1330	27800	$\begin{cases} 370 \pm 70^{(7)} \\ 530^{(79)} \end{cases}$	1030	6470	
Pu ²³⁶	1140	3510		976	4730	
Pu ²³⁸	1340	5630		1020	5920	
Pu ²⁴⁰	1550	48200		1090	8250	
Pu ²⁴²	1700	181000		1140	4600	
Cm ²⁴²	1730	77800		1102	8010	
Cm ²⁴⁴	1870	260000		1150	4260	
Cm ²⁴⁶	1930	1670000		1150	4270	
Cm ²⁴⁸	1890	1980000		1130	3930	
Cf ²⁴⁸	2000	1750000		1130	3650	
Cf ²⁵⁰	1970	2000000		1110	3400	
Cf ²⁵²	1800	19200		1060	4140	
Cf ²⁵⁴	1690	9610		1020	4060	
Fm ²⁵²	2000	2070000		1100	3260	
Fm ²⁵⁴	1880	27200		1060	3880	
Fm ²⁵⁶	1730	10100		1020	3860	

^a The deformations ϵ correspond to those listed in Table C.XI.

Table C.XIV. Force constants (surface energies) of beta and gamma vibrations in the actinide region - Case (b)

Nuclide ^a	Theory (Mev)		Experiment (Mev)	Theory (Mev)		Experiment (Mev)
	$c_{\beta}(0)$	$c_{\beta}(\omega)$	c_{β}	$\frac{c_{\gamma}(0)}{\beta_o^2}$	$\frac{c_{\gamma}(\omega)}{\beta_o^2}$	$\frac{c_{\gamma}}{\beta_o^2}$
Th ²²⁸	< 0	x		608	1100	
"	< 0	x		477	739	
"	< 0	x		339	453	
"	< 0	x		185	216	
Th ²³⁰	< 0	x		582	1050	
"	80.0	84.2		339	455	
"	158	175		210	249	
Th ²³²	312	392	$\left\{ \begin{array}{l} 310 \pm 60^{(7)} \\ 630^{(79)} \end{array} \right.$	478	764	$\left\{ \begin{array}{l} 1100 \pm 220^{(7)} \\ 600^{(79)} \end{array} \right.$
"	327	417		382	538	
Th ²³⁴	538	839		528	900	
U ²³⁰	< 0			440	651	
"	89.7	95.1		283	360	
U ²³²	351	459		303	393	
U ²³⁴	677	1280		445	680	
U ²³⁶	743	1600		635	1320	
U ²³⁸	903	9210	$\left\{ \begin{array}{l} 370 \pm 70^{(7)} \\ 530^{(79)} \end{array} \right.$	748	1790	
Pu ²³⁶	766	1600		685	1620	
"	721	1420		586	1100	
Pu ²³⁸	955	2800		738	1960	
Pu ²⁴⁰	1180	22000		849	2650	
Pu ²⁴²	1320	94300		912	2580	
Cm ²⁴²	1370	42200		915	3970	
Cm ²⁴⁴	1500	156000		973	2670	
Cm ²⁴⁶	1530	937000		983	2790	
Cm ²⁴⁸	1430	771000		968	2680	
Cf ²⁴⁸	1600	1170000		989	2770	
Cf ²⁵⁰	1510	920000		976	2640	
Cf ²⁵²	1210	4940		940	2820	
Cf ²⁵⁴	952	2070		904	3160	
Fm ²⁵²	1550	948000		984	2680	
Fm ²⁵⁴	1330	9690		963	2840	
Fm ²⁵⁶	1010	2300		918	3060	

^a The deformations ϵ correspond to those listed in Table C.XII.

Table C.XV. Mass Parameters for beta and gamma vibrations in the actinide region - Case (a).

Nuclide ^a	Theory		Experiment	Theory		Experiment
	$B_{\beta}(0)$	$B_{\beta}(\omega)$	B_{β}	$B_{\gamma}(0)$	$B_{\gamma}(\omega)$	B_{γ}
	$\frac{B_{\beta}(0)}{B_2 \text{ irrot}}$	$\frac{B_{\beta}(\omega)}{B_2 \text{ irrot}}$	$\frac{B_{\beta}}{B_2 \text{ irrot}}$	$\frac{B_{\gamma}(0)}{\beta_0^2 B_2 \text{ irrot}}$	$\frac{B_{\gamma}(\omega)}{\beta_0^2 B_2 \text{ irrot}}$	$\frac{B_{\gamma}}{\beta_0^2 B_2 \text{ irrot}}$
Th ²²⁸	x	x		1.97	71.4	
Th ²³⁰	12.8	15.6		2.63	31.7	
Th ²³²	10.2	28.0	16±3(7)	3.54	31.6	48±11(7)
Th ²³⁴	9.09	42.6	31(79)	3.45	34.1	27(79)
U ²³⁰	13.4	16.9		2.56	24.5	
U ²³²	9.32	32.6		3.84	25.3	
U ²³⁴	8.50	73.4		3.64	38.2	
U ²³⁶	8.36	73.4		3.19	52.6	
U ²³⁸	5.93	547	10±2(7) 14(79)	2.73	57.3	
Pu ²⁸⁶	9.30	77.0		3.48	56.9	
Pu ²³⁸	8.34	119		3.22	67.8	
Pu ²⁴⁰	5.40	1040		2.66	75.3	
Pu ²⁴²	3.71	3800		2.45	30.7	
Cm ²⁴²	4.01	1670		2.75	73.1	
Cm ²⁴⁴	2.31	5090		2.51	28.0	
Cm ²⁴⁶	1.79	28800		2.70	30.1	
Cm ²⁴⁸	2.34	24700		3.02	30.2	
Cf ²⁴⁸	1.45	27300		2.89	26.3	
Cf ²⁵⁰	1.99	24300		3.23	26.8	
Cf ²⁵²	4.57	265		3.91	43.2	
Cf ²⁵⁴	6.14	143		4.92	58.7	
Fm ²⁵²	2.03	23900		3.38	26.4	
Fm ²⁵⁴	3.70	375		3.90	38.7	
Fm ²⁵⁶	5.56	135		4.73	50.8	

^a The deformations ϵ correspond to those given in Table C.XI.

Table C.XVI. Mass parameters of beta and gamma vibrations in the actinide region - Case (b).

Nuclide ^a	Theory		Experiment	Theory		Experiment
	$B_{\beta}(0)$	$B_{\beta}(\omega)$	B_{β}	$B_{\gamma}(0)$	$B_{\gamma}(\omega)$	B_{γ}
	B_2 irrot	B_2 irrot	B_2 irrot	$\beta_0^2 B_2$ irrot	$\beta_0^2 B_2$ irrot	$\beta_0^2 B_2$ irrot
Th ²²⁸	x	x	x	4.02	12.7	
"	x	x	x	4.98	11.7	
"	x	x	x	6.13	10.9	
"	x	x	x	7.50	10.1	
Th ²³⁰	x	x	x	4.45	13.7	
"	x	x	x	6.30	11.2	
"	12.3	15.2	{ 16±3(7) 31(79)	7.42	10.4	{ 48±11(7) 27(79)
Th ²³²	11.0	17.1		5.53	13.6	
"	11.8	19.0		6.24	12.2	
Th ²³⁴	9.99	23.5		5.27	14.6	
U ²³⁰	x	x	x	5.54	12.0	
"	12.4	13.9		6.94	11.1	
U ²³²	11.7	19.7		7.10	11.8	
U ²³⁴	10.6	35.7		6.00	13.6	
U ²³⁶	9.03	37.5	{ 10±2(7) 14(79)	4.68	18.1	
U ²³⁸	6.46	185		3.92	19.3	
Pu ²³⁶	9.92	42.4		4.84	23.3	
"	11.7	42.8		5.63	18.4	
Pu ²³⁸	8.86	64.4		4.47	25.8	
Pu ²⁴⁰	5.81	478		3.62	27.0	
Pu ²⁴²	4.11	1990		3.29	22.2	
Cm ²⁴²	4.37	910		3.25	37.7	
Cm ²⁴⁴	2.67	3060		2.90	19.3	
Cm ²⁴⁶	2.23	16100		3.03	21.2	
Cm ²⁴⁸	3.07	10200		3.28	21.9	
Cf ²⁴⁸	1.86	18100		3.05	21.1	
Cf ²⁵⁰	2.68	11800		3.32	21.7	
Cf ²⁵²	6.03	68.4		3.81	27.7	
Cf ²⁵⁴	9.30	40.0		4.58	42.4	
Fm ²⁵²	2.68	11500		3.40	22.5	
Fm ²⁵⁴	4.89	132		3.76	26.9	
Fm ²⁵⁶	8.39	39.1		4.40	37.4	

^a The deformations ϵ correspond to those given in Table C.XII.

Table C.XVII. Reduced electric quadrupole transition probabilities $B(E2; 0 \rightarrow 2^+)$ for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the actinide region - Case (a).

Nuclide ^a	Beta band (single-particle units)			Gamma band (single-particle units)		
	theory		experiment	theory		experiment
	no vibration-rotation interaction correction	including vibration-rotation interaction correction		no vibration-rotation interaction correction	including vibration-rotation interaction correction	
Th ²²⁸	x	x		.66	.63	
Th ²³⁰	4.80	3.23		1.54	1.48	
Th ²³²	1.46	1.20	{ 2.8±.6 ⁽⁷⁾ 1.4 ⁽⁷⁹⁾	1.74	1.67	{ 2.1±.5 ⁽⁷⁾ 3.8 ⁽⁷⁹⁾
Th ²³⁴	.87	.73		1.58	1.52	
U ²³⁰	4.58	3.13		1.98	1.91	
U ²³²	1.23	1.03		2.33	2.23	
U ²³⁴	.53	.45		1.55	1.49	
U ²³⁶	.52	.44	{ 3.7±.7 ⁽⁷⁾ 2.6 ⁽⁷⁹⁾	1.07	1.03	
U ²³⁸	.069	.059		.89	.86	
Pu ²³⁶	.55	.46		1.09	1.05	
Pu ²³⁸	.34	.29		.89	.85	
Pu ²⁴⁰	.040	.034		.71	.69	
Pu ²⁴²	.011	.009		1.48	1.44	
Cm ²⁴²	.025	.021		.76	.73	
Cm ²⁴⁴	.008	.007		1.67	1.62	
Cm ²⁴⁶	.001	.001		1.59	1.54	
Cm ²⁴⁸	.001	.001		1.65	1.60	
Cf ²⁴⁸	.001	.001		1.91	1.85	
Cf ²⁵⁰	.001	.001		1.95	1.89	
Cf ²⁵²	.13	.11		1.38	1.33	
Cf ²⁵⁴	.25	.22		1.19	1.14	
Fm ²⁵²	.001	.001		2.07	2.00	
Fm ²⁵⁴	.095	.083		1.56	1.50	
Fm ²⁵⁶	.26	.23		1.36	1.30	

^a The deformations ϵ correspond to those listed in Table C.XI.

Table C.XVIII. Reduced electric quadrupole transition probabilities $B(E2; 0 \rightarrow 2^+)$ for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the actinide region - Case (b).

Nuclide ^a	Beta band (simple-particle units)			Gamma band (single-particle units)		
	theory		experiment	theory		experiment
	no vibration - rotation interaction correction	including vibration - rotation interaction correction		no vibration - rotation interaction correction	including vibration - rotation interaction correction	
²²⁸ Th	x	x		5.78 ^b	5.45	
"	x	x		7.64 ^c	7.08	
²³⁰ Th	x	x		4.45	4.24	
"	7.79	4.27		7.44	6.94	
"	5.15	3.39		10.4	9.51	
²³² Th	3.22	2.45	{ 2.8±.6(7) 1.4(79)	5.18	4.91	{ 2.1±.5(7) 3.8(79)
"	2.96	2.24		6.51	6.11	
²³⁴ Th	1.86	1.51		4.57	4.35	
²³⁰ U	x	x		6.31	5.96	
"	7.63	4.42		8.77	8.13	
²³² U	2.90	2.24		8.09	7.55	
²³⁴ U	1.28	1.06		5.68	5.40	
²³⁶ U	1.11	.93	{ 3.7±.7(7) 2.6(79)	3.50	3.35	
²³⁸ U	.21	.18		2.90	2.79	
²³⁶ Pu	1.07	.89		2.92	2.79	
"	1.15	.94		3.98	3.79	
²³⁸ Pu	.66	.55		2.50	2.40	
²⁴⁰ Pu	.086	.073		2.09	2.01	
²⁴² Pu	.020	.017		2.32	2.24	
²⁴² Cm	.047	.040		1.50	1.45	
²⁴⁴ Cm	.013	.011		2.53	2.45	
²⁴⁶ Cm	.002	.002		2.35	2.27	
²⁴⁸ Cm	.003	.003		2.34	2.26	
²⁴⁸ Cf	.002	.002		2.44	2.36	
²⁵⁰ Cf	.003	.003		2.45	2.37	
²⁵² Cf	.50	.44		2.09	2.01	
²⁵⁴ Cf	1.01	.86		1.58	1.51	
²⁵² Fm	.003	.003		2.47	2.39	
²⁵⁴ Fm	.27	.24		2.19	2.11	
²⁵⁶ Fm	1.00	.86		1.78	1.71	

^a The deformations ϵ correspond to those listed in Table C.XII.

^b These values correspond to $\epsilon = .21$.

^c These values correspond to $\epsilon = .22$.

Table C. XIX. Vibration-rotation interaction coefficients in the actinide region - Case (a).

Nuclide ^a	F(ev) from theory				F(ev) from ground - state band ^b
	using $\mathfrak{S} \propto \beta$		using $\mathfrak{S} \propto \beta^2$		
	using adiabatic force constants	using dynamic force constants	using adiabatic force constants	using dynamic force constants	
Th ²²⁸	x	x	x	x	18
Th ²³⁰	5.68	4.41	19.4	16.8	7.8
Th ²³²	1.62	.71	3.96	2.08	8.5, 12 ^c , (104)
Th ²³⁴	1.19	.40	2.66	1.05	
U ²³⁰	4.60	3.49	15.4	13.1	
U ²³²	1.31	.55	2.97	1.41	3.6
U ²³⁴	.84	.24	1.74	.52	4.9
U ²³⁶	.92	.22	2.00	.55	
U ²³⁸	.81	.096	1.70	.14	2.7 ^c , (104)
Pu ²³⁶	.96	.24	2.09	.61	
Pu ²³⁸	.79	.15	1.65	.36	3.3
Pu ²⁴⁰	.68	.07	1.39	.09	4.0
Pu ²⁴²	.69	.12	1.38	.12	
Cm ²⁴²	.63	.06	1.23	.08	
Cm ²⁴⁴	.60	.11	1.17	.12	
Cm ²⁴⁶	.61	.11	1.17	.11	
Cm ²⁴⁸	.63	.13	1.22	.13	
Cf ²⁴⁸	.58	.13	1.09	.13	
Cf ²⁵⁰	.60	.14	1.14	.14	
Cf ²⁵²	.62	.13	1.20	.18	
Cf ²⁵⁴	.75	.17	1.46	.30	
Fm ²⁵²	.59	.14	1.10	.14	
Fm ²⁵⁴	.70	.15	1.34	.19	
Fm ²⁵⁶	.72	.17	1.39	.29	

^a The deformations ϵ correspond to those listed in Table C.XI.

^b These values are taken from reference (3) unless otherwise marked.

^c This value assumes that rotational band spacings are given by $E = AI(I+1) - FI^2(I+1)^2 + GI^3(I+1)^3$.

Table C.XX. Vibration-rotation interaction coefficients in the actinide region - Case(b).

Nuclide ^a	F(ev) from theory				F(ev) from ground - state band ^b
	using $\beta \propto \beta$		using $\beta \propto \beta^2$		
	using adiabatic force constants	using dynamic force constants	using adiabatic force constants	using dynamic force constants	
Th ²²⁸	x	x	x	x	18
Th ²³⁰	x	x	x	x	7.8
"	10.0	9.00	32.8	30.7	
"	7.06	6.16	17.7	15.7	
Th ²³²	3.20	2.30	8.34	6.40	8.5, 12 ^c , (104)
"	3.20	2.38	7.69	5.90	
Th ²³⁴	1.99	1.21	4.53	2.84	
U ²³⁰ ^d	9.14	8.19	28.4	26.4	
U ²³²	3.20	2.46	6.96	5.35	3.6
U ²³⁴	1.53	.94	3.05	1.75	4.9
U ²³⁶	1.39	.66	3.02	1.42	
U ²³⁸	1.12	.34	2.35	.47	2.7 ^c , (104)
Pu ²³⁶	1.40	.61	3.08	1.39	
"	1.44	.75	3.08	1.58	
Pu ²³⁸	1.09	.40	2.30	.81	3.3
Pu ²⁴⁰	.88	.20	1.81	.25	4.0
Pu ²⁴²	.87	.21	1.75	.22	
Cm ²⁴²	.77	.13	1.53	.15	
Cm ²⁴⁴	.73	.18	1.44	.19	
Cm ²⁴⁶	.73	.17	1.44	.17	
Cm ²⁴⁸	.77	.18	1.55	.19	
Cf ²⁴⁸	.68	.17	1.32	.17	
Cf ²⁵⁰	.71	.18	1.41	.18	
Cf ²⁵²	.77	.23	1.62	.44	
Cf ²⁵⁴	1.00	.36	2.25	.93	
Fm ²⁵²	.69	.17	1.35	.17	
Fm ²⁵⁴	.85	.23	1.74	.35	
Fm ²⁵⁶	.94	.34	2.08	.84	

^a The deformations ϵ correspond to those given in Table C.XII.

^b These values are all taken from reference (3) unless otherwise marked.

^c This value assumes that rotational band spacings are given by $E = A I(I+1) - B I^2(I+1)^2 + C I^3(I+1)^3$.

^d The values correspond to $\epsilon = .23$.

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Fig. C.1A. Phonon energies of beta vibrations in the rare earth region - Case (a).

MU-27224

Fig. C.2A. Phonon energies of beta vibrations in the rare earth region - Case (b).

MU-27226

Fig. C. 3A. Force constants (surface energies) of beta vibrations in the rare earth region - Case (a).

MU-27228

Fig. C. 4A. Force constants (surface energies) of beta vibrations in the rare earth region - Case (b).

MU-27230

Fig. C. 5A. Mass parameters of beta vibrations in the rare earth region - Case (a).

MU-27232

Fig. C.6A. Mass parameters of beta vibrations in the rare earth region - Case (b).

MU-27223

Fig. C.1B. Phonon energies of gamma vibrations in the rare earth region - Case (a).

MU-27225

Fig. C.2B. Phonon energies of gamma vibrations in the rare earth region - Case (b).

MU-27227

Fig. C.3B. Force constants (surface energies) of gamma vibrations in the rare earth region - Case (a).

MU-27229

Fig. C.4B. Force constants (surface energies) of gamma vibrations in the rare earth region - Case (b).

MU-27231

Fig. C. 5B. Mass parameters of gamma vibrations in the rare earth region - Case (a).

MU-27233

Fig. C.6B. Mass parameters of gamma vibrations in the rare earth region - Case (b).

MU-27254

Fig. C.7. Reduced electric quadrupole transition probabilities for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the rare earth region - Case (a).

MU-27255

Fig. C. 8. Reduced electric quadrupole transition probabilities for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the rare earth region - Case (b).

MU-27234

Fig. C.9A. Vibration-rotation interaction coefficients in the rare earth region calculated using dynamic force constants - Case (a).

MU-27235

Fig. C. 9B. Vibration-rotation interaction coefficients in the rare earth region calculated using adiabatic force constants - Case (a).

MU-27236

Fig. C.10A. Vibration-rotation interaction coefficients in the rare earth region calculated using dynamic force constants - Case (b).

MU-27237

Fig. C.10B. Vibration-rotation interaction coefficients in the rare earth region calculated using adiabatic force constants - Case (b).

MU-27238

Fig. C. 11A. Phonon energies of beta vibrations in the actinide region - Case (a).

MU-27240

Fig. C.12A. Phonon energies of beta vibrations in the actinide region - Case (b).

MU-27242

Fig. C.13A. Force constants (surface energies) of beta vibrations in the actinide region - Case (a).

MU-27244

Fig. C.14A. Force constants (surface energies) of beta vibrations in the actinide region - Case (b).

MU-27246

Fig. C. 15A. Mass parameters of beta vibrations in the actinide region - Case (a).

MU-27248

Fig. C. 16A. Mass parameters of beta vibrations in the actinide region - Case (b).

MU-27239

Fig. C.11B. Phonon energies of gamma vibrations in the actinide region - Case (a).

MU-27241

Fig. C. 12B. Phonon energies of gamma vibrations in the actinide region - Case (b).

MU-27243

Fig. C.13B. Force constants (surface energies) of gamma vibrations in the actinide region - Case (a).

MU-27245

Fig. C.14B. Force constants (surface energies) of gamma vibrations in the actinide region - Case (b).

MU-27247

Fig. C.15B. Mass parameters of gamma vibrations in the actinide region - Case (a).

MU-27249

Fig. C. 16B. Mass parameters of gamma vibrations in the actinide region - Case (b).

MU-27256

Fig. C. 17. Reduced electric quadrupole transition probabilities for exciting the 2^+ member of one-phonon beta and gamma vibrational bands in the actinide region - Case (a).

MU-27257

Fig. C. 18. Reduced electric quadrupole transition probabilities for exciting the $2+$ member of one-phonon beta and gamma vibrational bands in the actinide region - Case (b).

MU-27250

Fig. C.19A. Vibration-rotation interaction coefficients in the actinide region calculated using dynamic force constants - Case (a).

MU-27251

Fig. C. 19B. Vibration-rotation interaction coefficients in the actinide region calculated using adiabatic force constants - Case (a).

MU-27252

Fig. C. 20A. Vibration-rotation interaction coefficient in the actinide region calculated using dynamic force constants - Case (b).

MU-27253

Fig. C.20B. Vibration-rotation interaction coefficients in the actinide region calculated using adiabatic force constants - Case (b).

MU-27258

Fig. C.21. Phonon energies of gamma vibrations without pairing correlations in the rare earth region - Case (b).

MU-27259

Fig. C. 22. Force constants (surface energies) of gamma vibrations without pairing correlations in the rare earth region - Case (b).

MU-27260

Fig. C. 23. Mass parameters of gamma vibrations without pairing correlations in the rare earth region - Case (b).

MU-27261

Fig. C. 24. Reduced electric quadrupole transition probabilities for exciting the 2^+ member of the one-phonon gamma vibrational bands without pairing correlations in the rare earth region - Case (b). All values are expressed in Weisskopf single-particle units.

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