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Author

Kisslinger, L.S.

Publication Date

1965-08-01

University of California

Ernest O. Lawrence Radiation Laboratory

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Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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L. S. Kisslinger

August 1965

A NOTE ON COUPLING SCHEMES IN ODD-MASS NUCLEI*

L. S. Kisslinger

Lawrence Radiation Laboratory
University of California
Berkeley, California

and

Western Reserve University
Cleveland, Ohio

August 1965

ABSTRACT

A representation which utilizes second-quantized operators for the three quasi-particle states is introduced. In this representation the pairing plus quadrupole interaction is approximately diagonalized in a j -shell. The main prediction is that associated with the opposite parity state of large spin in the major shells of medium and heavy nuclei there will appear extra low-lying states which do not arise from particle-phonon coupling. The states are characterized by forbidden $M1$ transition rates.

* This work was supported in part by the United States Atomic Energy Commission and by the National Science Foundation.

I. INTRODUCTION

In the regions of large deformation the strong coupling model,¹ which predicts rotational bands associated with particle excitations, is successful in accounting for the low-lying states of odd-mass nuclei. In the spherical regions there has been some success in studying the states of odd-mass nuclei in terms of particle modes, vibrational modes, and their interactions.* The most systematic studies with such an intermediate coupling have been carried out using a pairing plus a quadrupole interaction.^{2,3} The states correspond to systems of quasi-particles and phonons, with the quasi-particle-phonon interaction arising from a quadrupole force.

Although it is possible to systematically fit the trends of the energy levels, transition rates, and other properties of many of the levels, there sometimes appear rather low-lying levels which cannot be accounted for in this

*There is now such a large body of work on such coupling schemes that it is not possible to attempt a complete review here. References to the earliest work are given in Refs. 2 and 3, but there have been many calculations recently. In addition to the early work in which the interaction is linear in the collective variables, and therefore off-diagonal in the phonon operators, there are calculations based on the picture used by R. D. Lawson and J. L. Uretsky, Phys. Rev. 108, 1300 (1957) in which the interaction is diagonal in the phonon interaction (and therefore should involve higher order corrections to the linear terms). This later model was used, e.g., in the calculations of A. de-Shalit, Phys. Rev. 122, 1530 (1961). More recently, both the diagonal and off-diagonal terms in the phonons have been included in a calculation of the states of Cu⁶³ by V. K. Thankappan and W. W. True, Phys. Rev. 137, B793 (1965).

coupling scheme. It is the purpose of this note to point out that there might be a simple treatment which can qualitatively account for what appears to be the most notorious intruding states.

The Hamiltonian including a pairing plus a quadrupole interaction in a single j-shell is used to demonstrate the effect. Single j-shell calculations with this Hamiltonian have been carried out previously for the 7/2 shell² and the 9/2 shell,⁴ so we shall not attempt completeness but rather study the main effects on the overall coupling scheme. A new representation is introduced which utilizes the methods of second quantization to allow one to calculate easily the approximate energy spectrum of a pairing plus a quadrupole interaction in a j-shell and to show the relevance for realistic nuclei. The conclusion which is reached is that although well-developed phonons can appear in even a single j-shell, several j-shells must be important for the success of the intermediate coupling scheme in odd nuclei, and even in that situation additional low lying states will occur which arise from a different mechanism. The relation of the present work to previous work is discussed, for such intruding states were recognized to be important to the early work on the development of both the shell model and collective model.

II. SINGLE j-LEVEL

A. Phonon-Quasi-Particle Interactions

The Hamiltonian being considered consists of a pairing and a quadrupole interaction ($J = 0$ particle-particle plus $J = 2$ particle-hole force), which approximates traditional effective forces in a major shell,

$$H = H_{\text{pairing}} - \frac{1}{2} \chi Q \cdot Q \quad (1)$$

We now consider the special case of a single j-level. In the quasi-particle second-quantized representation, which approximately diagonalizes the pairing force,^{5,6,7}

$$\alpha_{jm}^+ = U_j b_{jm}^+ + V_j (-1)^{l+j+m} b_{j-m} \quad (2)$$

the Hamiltonian is, ^{*} in a single j-level,

$$\begin{aligned} H \cong & \sum_m E_j \alpha_{jm}^+ \alpha_{jm} - \frac{\chi}{2} \frac{\langle 2 || Q || 2 \rangle}{\sqrt{5}} \left\{ (U_j V_j)^2 [(A_{jj}^+ + \hat{A}_{jj}^2)(A_{jj}^+ + \hat{A}_{jj}^2)]_0 \right. \\ & - U_j V_j (U_j^2 - V_j^2) [(A_{jj}^+ + \hat{A}_{jj}^2) \cdot n_{jj}^2]_0 \\ & \left. + (U_j^2 - V_j^2)^2 [n_{jj}^2 n_{jj}^2]_0 \right\} \quad (3) \end{aligned}$$

For a microscopic treatment of odd-mass nuclei in terms of phonons and quasi-particles one introduces the quasi-bosons or phonons of energy $\hbar\omega$.

$$B_\omega^+ = \sum_{jj'} (a_{jj'}^\omega A_{jj'}^+ + b_{jj'}^\omega \hat{A}_{jj'}^2) \quad (4)$$

in terms of which the interaction becomes

$$\begin{aligned} H_{odd} \cong & \sum_m E_j \alpha_{jm}^+ \alpha_{jm} + \sum_\omega \hbar\omega B_\omega^+ B_\omega + \frac{\langle 2 || Q || 2 \rangle (2 U_j V_j (U_j^2 - V_j^2) \sqrt{E_j}}{2 \sqrt{8\omega} (E_j^2 - \omega^2)} \\ & \times \chi [(B_\omega^+ + \hat{B}_\omega) n_{jj}^2]_0 - \frac{\chi}{2} \frac{\langle 2 || Q || 2 \rangle}{\sqrt{5}} (U_j^2 - V_j^2)^2 [n_{jj}^2 n_{jj}^2]_0. \quad (5) \end{aligned}$$

The last term is generally neglected.[†] With this Hamiltonian (generalized to many neutron and proton j-levels) it has been possible to predict approximately many of the properties of low-lying levels in spherical nuclei. In

^{*} See Appendix I for notation.

[†] See Ref. 3 for a discussion of the approximations involved in the QRPA which gives rise to this Hamiltonian and for references on the application of the random phase approximation to nuclear structure physics.

the present case this consists of diagonalizing the Hamiltonian (5) in the set of states

$$\Psi_{j\alpha} = (c_0 \alpha_j^\dagger + c_1 [B^\dagger \alpha_j^\dagger]_{j\alpha} + \sum_{j_1} c_{2j_1} [(B^\dagger B^\dagger)^{j_1} \alpha_{j_1}^\dagger]_{j\alpha}) \quad (6)$$

Even without attempting detailed fits the fact that the general distribution of the levels is qualitatively predicted in this model is quite striking, with the increased density of levels near the phonon energy of the neighboring even-even nuclei strongly suggesting the quasi-particle phonon coupling scheme. Moreover, the tendency for the q.-p.'s at the Fermi surface to be more weakly coupled than the higher lying q.-p.'s as predicted by the (UU-VV) factor in the second term of the Hamiltonian (5) is systematically observed.

Consider first the Hamiltonian in the intermediate coupling form (5) for a single j-shell with j large. Figure 1 shows a typical case of rather weak coupling in which only one phonon is included. Only the level of spin j is affected. The effect of coupling with more phonons will change the results quantitatively but not qualitatively under conditions in which the approximation can be expected to be useful. Note that for the nucleus at approximately the middle of the shell $U_j \approx V_j$ and the interactions vanish. Thus in the cases where the j-level being considered the most important in the major shell the coupling is weakest, leading to a compressed level scheme as discussed in Ref. 3. In particular, the level of spin j-1 is never singled out.

B. Three Quasi-Particle States

Let us now return to the Hamiltonian (3). For the situation in which the j-shell is approximately half filled, in which case that j-level will be most important in the low-lying states, the Hamiltonian can be taken to a good approximation as

$$H \cong \sum_m E_j \alpha_{jm}^+ \alpha_{jm} - X [(A_{jj}^{+2} + A_{jj}^2)(A_{jj}^{+2} + A_{jj}^2)]_0 \quad (7)$$

with

$$X = \frac{\chi}{2} \langle j || Q || j \rangle (\mathcal{O}_j)^2 \approx \frac{\chi}{8} \langle j || Q || j \rangle \quad (7a)$$

We wish to diagonalize the Hamiltonian (7) in the one and three quasi-particle states

$$\alpha_j^+ \Phi_0, \quad \sum_j C_j [A_{jj}^{+2} \alpha_j^+]_2 \Phi_0,$$

neglecting the higher quasi-particle states. Since the Hamiltonian (7) does not mix one and three quasi-particle states, and since the diagonal element in the one quasi-particle state can be absorbed in the gap energy $= 2E_j$ by a re-normalization of the pairing force strength, the main consideration is the diagonalization of the Hamiltonian in three quasi-particle states.

This is most easily accomplished by noting that we can find a representation in which the Hamiltonian (7) is approximately diagonal. Let us first consider normalized states

$$\Phi_{j,2,M}^2 \equiv N_{j,2}^{-1/2} [A_{jj}^{+2} \alpha_j^+]_{2,M}^2 \Phi_0, \quad (8)$$

where Ψ_0 is the q.-p- vacuum and $N_{J_1 j_a}^j$ is the normalization constant

$$(N_{J_1 j_a}^j)^2 = \frac{1}{2} [1 - 2(2J_1 + 1)W(J_1, 2, J_1; j, j_a)]^{-1} \quad (9)$$

From the properties of the second quantized operators from which the states are constructed the states are also antisymmetric in the three Fermions. If there is a unique state with spin j_a then for any J_1 (satisfying the triangle relation with j and j_a) one has the proper antisymmetrized normalized state in which to calculate the matrix elements. When there are several states with the same j_a then one must take suitable orthogonal linear combinations of the states $\phi_{J_1 j_a M_a}^j$. We would like to suggest that there is one representation which is most convenient: Define

$$\begin{aligned} \Phi_{j_a M_a}^{(\mu)} &= \phi_{2 j_a M_a}^j \\ \Phi_{j_a M_a}^{(\beta)} &= a^{(\beta)} \phi_{2 j_a M_a}^j + b^{(\beta)} \phi_{4 j_a M_a}^j \\ \Phi_{j_a M_a}^{(\gamma)} &= a^{(\gamma)} \phi_{2 j_a M_a}^j + b^{(\gamma)} \phi_{4 j_a M_a}^j + c^{(\gamma)} \phi_{6 j_a M_a}^j \end{aligned} \quad (10)$$

and choose the constants $a^{(\mu)}$, $b^{(\mu)}$, ... so that

$$\langle \Phi_{j_a M_a}^{(\mu)} | \Phi_{j_a M_a}^{(\nu)} \rangle = \delta_{\mu\nu}. \quad (11)$$

I.e., carry out a Schmidt orthogonalization process for the states $\phi_{J_1 j_a M_a}^j$, which are not orthogonal, starting with $J_1 = 2$. In the representation (10) one must find the matrix elements

$$\langle j_a M_a | H | j_a M_a \rangle = 3E_j - V_{\mu\nu}^{j_a} \quad (12a)$$

with

$$V_{\mu\nu}^{ja} = \langle j_a(\mu) | X[(A_{jj}^{\dagger 2} + \hat{A}_{jj}^2)(A_{jj}^{\dagger} + \hat{A}_{jj}^2)] | j_a(\nu) \rangle \quad (12b)$$

In Appendix II it is shown that (except for an energy shift which is the same for all of the levels) all matrix elements of the quadrupole interaction in the representation (12) vanish except for the diagonal matrix element in the state $\Psi^{(a)}$, i.e.,

$$V_{\mu\nu}^{ja} = 0 \quad \text{if } \mu \text{ or } \nu \neq \lambda$$

$$V_{\lambda\lambda}^{ja} = \frac{2}{5} X \langle j_a || Q || j_a \rangle^2 (C_{jj}^2)^2 [1 - 10W(2jj2; jjj_a)] \quad (13)$$

For spin $j \geq 7/2$ there will be at least one three quasi-particle state of spin j_a such that either $j_a < j-2$ or $j_a > j+2$, so that the state $\phi_{2j_a M_a}^j$ does not exist and the representation must be extended. The quadrupole interaction is weak in these states (See Appendix II).

Thus it has been demonstrated that the pairing plus quadrupole interaction for a single j -level is approximately diagonalized when the quasi-particle corresponding to that j is low-lying and that for each state j_a there is one state $\Psi_{j_a}^{(a)}$ which can be significantly affected by the quadrupole force. From Eq. (13) one can see that for a given nucleus it is the Racah coefficient $W(2jj2; jjj_a)$ which determined the relative positions of the various levels j_a . Finally we note that

$$W(2jj2; jjj_a) \geq 0 \quad \text{for } j_a \neq j-1$$

$$W(2jj2; jjj-1) < 0 \quad (14)$$

Thus we see that there is one level, the one of spin $j_a = j-1$, which is affected much more than the other levels. The results for the levels $j = 7/2, 9/2, 11/2$ and $13/2$ are given in Fig. 2. The results are schematic but roughly indicate the results obtained from the parameters of Ref. 1. Examples of the general conclusions which have been drawn here can be observed in exact diagonalization of the pairing plus quadrupole force Hamiltonian in the three particle states of the $7/2$ level² and the $9/2$ level.⁴

III. REALISTIC NUCLEI-TRANSITION RATES

A. Several j-Levels

First let us note that the matrix element $V_{\alpha\alpha}^{j_a = j-1}$ of Eq. (13) is roughly proportional to the spin j of the j -level. This indicates that this effect of lowering of three quasi-particle states by a large diagonal matrix element rather than the phonon particle interaction will only be important for the large-spin, opposite-parity level which appears in each major harmonic oscillator shell of medium and heavy nuclei.

Moreover, the opposite parity level not only tends to make a fairly small contribution to the phonon, but usually is extremely weakly coupled to the phonon in the odd-mass nuclei* and the corresponding one quasi-particle state and the states of this quasi-particle coupled to the phonon tend to be quite pure. The results of the preceding section suggest that there will be one additional opposite parity state of spin $j_a = j-1$ which can be quite low-lying and yet not mix very much into the other states. Of course there will be some mixing. In fact the $j-1$ level of the same parity found from the one quasi-particle plus one phonon is not orthogonal to this state and could mix quite strongly. But to the extent that the phonon is formed from many levels, this new state can be relatively pure.

*This is apparent from the small coefficients $C_{j,12}^j$ in Ref. 3.

B. Electromagnetic Transition States

The characteristic feature of this new "intruder" state is to be found in the transition rates. The main transition which will be seen from this intruder state leads to the one quasi-particle state. The transition is a $\Delta J = 1$, no parity change. Therefore one expects M1 or E2 transitions to dominate. For example, if the $j-1$ level were the spin orbit partner, which is a natural assumption when one too many of these states is found in experiment than can arise out of the phonon-q.-p. coupling scheme, then the M1 dominates.

The M1 and E2 transition operators are:

$$B(M1) = - \sum_{j,j'} \langle 2' || B(M1) || 2 \rangle_{s.p.} \frac{1}{\sqrt{3}} (\vec{D}_j \cdot \vec{V}_j + \vec{V}_j \cdot \vec{V}_j) + \frac{(\vec{D}_j \cdot \vec{V}_j - \vec{V}_j \cdot \vec{V}_j)}{2} (A_{jj'}^{1+} + \hat{A}_{jj'}^{1-}) \quad (15a)$$

$$B(E2) = \sum_{j,j'} \langle 2' || B(E2) || 2 \rangle_{s.p.} \frac{1}{\sqrt{5}} (\vec{D}_j \cdot \vec{V}_j + \vec{V}_j \cdot \vec{V}_j) + \frac{(\vec{D}_j \cdot \vec{V}_j - \vec{V}_j \cdot \vec{V}_j)}{2} (A_{jj'}^{2+} + \hat{A}_{jj'}^{2-}) \quad (15b)$$

The transition matrix element which are pertinent are

$$\begin{aligned} \langle 2_m | B_\mu(E2) | \Psi_{j-1}^\alpha \rangle &= \sqrt{\frac{2}{5}} \sqrt{2j-1} \vec{D}_j \cdot \vec{V}_j (-1)^L \frac{\langle 2 || B(E2) || 2 \rangle}{\langle 2 || B(E2) || j-1 \rangle} \frac{1}{\sqrt{2}} [-10 W(2j) 2; j j-1] \\ &\times \langle 2_m | B(E2) | j-1 \rangle_{s.p.} \end{aligned} \quad (16a)$$

$$\langle 2_m | B_\mu(M1) | \Psi_{j-1}^\alpha \rangle = 0 \quad (16b)$$

The M1 matrix element vanishes while the E2 is given in Eq. (16a). Typical results for the $B(E2)$'s are given in the table. In that table $B(E2)_{s.p.}$ is calculated by assuming that the transition is between the spin orbit doublet states $j, j-1$ with the orbital angular momentum l given by the quasi-particle j .

* It is apparent from the small coefficients $C_{j,j-1}^{\pm}$ in eq. (16a) that

C. Discussion and Conclusion

In the intermediate coupling picture for spherical nuclei, associated with the large-spin opposite parity j -level in the shell are predicted levels of the same parity and spins j , $j\pm 1$, and $j\pm 2$ at an excitation energy of the vibrational phonon energy, $\hbar\omega$, above the quasi-particle j . It is observed in the present work that one state of three α_j^+ quasi-particles of spin $j-1$ is very much lowered in energy by the quadrupole interaction. Although the state of spin $j-1$ formed from a phonon and the quasi-particle α_j^+ is not orthogonal to this level, the result is an extra level of that spin which is much lower-lying than predicted in the conventional treatment. This should be a characteristic and systematic feature of odd-mass nuclei. The low-lying $7/2^+$ states in the odd-mass Ag isotopes and the $5/2^-$ state in V^{51} are examples. This $9/2^-$ state which appears in the diagonalization of one and three quasi-particle states by Baranger and Kuo,⁸ which is the most dramatic difference from the calculation of Ref. 3, is also an example of this state associated with the $11/2^-$ shell model level. Recently, a low-lying $9/2^-$ state in Te^{125} has been observed⁹ with a very retarded M1 transition rate, in addition to a second $9/2^-$ state which can be identified with the state in Ref. 3.

A characteristic feature of this intruder state is the complete absence of M1 transition, so the E2 will dominate. The M1 transition from the $j-1$ state arising from one q.-p. and one phonon also will tend to be reduced, although not so much. As a result one will see an extra state of the spin $j-1$ corresponding to the large opposite parity j -level, with the amount of M1 depending upon the mixing of the two states but in any case much smaller than single-particle. An estimate of the $B(E2)$ is given for the most important cases.

The empirical fact that in odd-mass nuclei in which a level of spin j is presumably being filled there is a competition between a spin j and spin $j-1$ level for the ground state was recognized as important during the earliest days of the $j-j$ shell model. The calculations with a zero-range force¹⁰ showed that the level j is lowest, in agreement with the usual rule in spherical nuclei. Although the level $j-1$ can become the ground state for a finite range force, in the single j -level calculation the range must be more than twice as large as the currently accepted range of the effective interaction in order for the spin $j-1$ level to cross below the level of spin j .^{11,12}

In the present work it is the quadrupole force which reproduces this effect of a long range force. However, the results are quite different, as might be expected since the quadrupole force is a long range force in the angular space and there is no exact correspondence between the range of the two body force and the effect of the quadrupole component. For example, the relative positions of the levels j and $j-1$ are much more sensitive to the relative amount of pairing and quadrupole force in the treatment used here than to the range of the effective interaction in the conventional shell model treatment. This can be seen by comparing the results of this note with those of Kurath.¹¹

The possible connection between the appearance of the $j-1$ level as ground state and the onset of deformation has been pointed out by Bohr and Mottelson.¹³ They have stressed the difficulty in predicting the ground states of $j-1$ type in a conventional spherical shell model with traditional forces. More recently there has been both experimental and theoretical³ evidence for instability toward deformation of some of these nuclei in the regions where levels which must be identified as this $j-1$ character are the ground state or very low-lying. On the other hand, it seems that especially for $A \geq 80$ there are regions of nuclei in which the coupling scheme is best described as spherical, but in

which there appear these low-lying intruder states. It will be extremely interesting to have systematic experimental information on these states, which are associated with the deforming quadrupole interaction and yet are low-lying in spherical nuclei.

The author would like to thank Drs. N. Stone and D. Shirley for discussion of their experimental results on Te^{125} . He would also like to acknowledge helpful discussions with a number of theorists at Berkeley during the summer, 1965. He is grateful for the hospitality extended by the Lawrence Radiation Laboratory during this period.

APPENDIX I. NOTATION

The notation is essentially that of Ref. 3, but differs in some details. The shell model particle creation operator b_{jm}^+ and shell model vacuum $|0\rangle$ are defined as $b_{jm}^+|0\rangle = |jm\rangle$, the one particle state with the other quantum numbers n, l , etc. suppressed. Condon and Shortley phases are used throughout (b and b^+ satisfy Fermian commutation rules $b_{j'm'}^+ b_{jm} + b_{jm} b_{j'm'} = \delta_{jj'} \delta_{ll'} \delta_{mm'}$). The Bogoliubov transformation is the form of Eq. (2)

$$\alpha_{jm}^+ = U_{jm} b_{jm}^+ + V_{jm} (-1)^{l+j+m} b_{j-m}$$

is equivalent to, e.g., that of Belyaev⁶ and of Kisslinger and Sorensen.^{2,3} It results in positive values for both U_{jm} and V_{jm} . With the definition

$$\alpha_{jm} = (-1)^{l+j+m} \hat{\alpha}_{j-m} \quad (I-1)$$

where $\alpha_{jm} = (\alpha_{jm}^+)^+$, $\alpha_{jm} \alpha_{j'm'} + \alpha_{j'm'} \alpha_{jm} = \delta_{jj'} \delta_{ll'} \delta_{mm'}$, the operators $\hat{\alpha}_{j-m}$ are Condon and Shortley operators and can be coupled quite simply.¹⁰

With these we define

$$A_{j'j}^{LM} = (-1)^{l'+l} [\alpha_{j'}^+ \alpha_j^+]_M^L \quad (I-2)$$

where the notation $[]$ signifies vector coupling i.e.,

$$[\alpha_{j'}^+ \alpha_j^+]_M^L = \sum_m C_{m, m-m}^{j' j L} \alpha_{j'm}^+ \alpha_{j, m-m}$$

with $C_{m, m-m}^{j' j L}$ being a Clebsch-Gordan coefficient. In terms of these operators we define:

$$A_{jj'}^{LM} = (A_{jj'}^{LM+})^+ \quad (I-3)$$

$$\hat{A}_{jj'}^{LM} = (-1)^M A_{jj'}^{L-M} = (-1)^{L+L} [\hat{L}_{j'}^+ \hat{L}_j]^L_M \quad (I-4)$$

$$\eta_{jj'}^{LM} = [\hat{L}_{j'}^+ \hat{L}_j]^L_M \quad (I-5)$$

$$B_{\omega}^{+M} = \sum_{jj'} (a_{jj'}^{\omega} \hat{A}_{jj'}^{2M} + b_{jj'}^{\omega} \hat{A}_{jj'}^{2M}) \quad (I-6)$$

$$\hat{B}_{\omega}^M = (-1)^M \hat{B}_{\omega}^{-M} \quad (I-7)$$

The operators $A_{jj'}^{LM+}$ are the same as those used in Ref. 3. The operator $\eta_{jj'}^{LM}$ differs by a factor (-1) . The $a_{jj'}$ and $b_{jj'}$ differ in that

$$a_{jj'}^{\omega} = \pm \eta_{jj'}^{\omega} \quad (I-8)$$

$$b_{jj'}^{\omega} = -\frac{1}{2} S_{jj'}^{\omega} \quad (I-9)$$

APPENDIX II.

In this appendix we give some of the details of the calculation

A. Representation for a Single j-Shell

The states of the system which we wish to include are the one quasi-particle state and the three quasi-particle states

$$\mathcal{L}_j^+ \Psi_0 \quad (\text{II-1})$$

$$[\mathcal{L}_j^+ \mathcal{L}_j^+ \mathcal{L}_j^+]_{j_a} \Psi_0 \quad (\text{II-2})$$

For $j \leq 7/2$ there is no more than one state for each angular momentum (i.e., there is no more than one state for each j_a and $j_a \neq j$). For $j = 9/2$ there is no more than one state for each j_a , but $j_a = j = 9/2$ appears. This state $[\alpha_j^+ \alpha_j^+ \alpha_j^+]_{j_a=j} \Psi_0$ contains seniority one as well as seniority three components. For spins $j \geq 11/2$ the spin j_a does not uniquely determine the state. We would like to find a convenient representation for the complete set of states (of the subspace which is being considered).*

Let us take as an example the case where there are two states of the spin j_a (e.g., the $j-1$ levels for $j = 11/2$ and $13/2$). Defining (see Eqs. (8) and (9) in body of paper)

$$\varphi_{J_1 j_a M_a}^j = N_{J_1 j_a}^j [A_{j j}^{J_1} \mathcal{L}_j^+]_{M_a}^{j_a} \Psi_0 \quad (8)$$

$$\langle \varphi_{J_1 j_a M_a}^j | \varphi_{J_1 j_a M_a}^j \rangle = 1$$

In particular we call the state with $J_1 = 2$ the α -state which is thus defined (assume j , j_a , and 2 form a triangle) by

(11)

* In the traditional way of forming antisymmetrical states one must sum over all of the spins of the antisymmetrized two-particle wave function, using fractional parentage coefficients. With second quantized operators one only needs the number of terms equal to the number of states of the same spin. Thereby the calculation of matrix elements as well as the states themselves is greatly facilitated. We neglect considerations of spurious states, assuming that they are spread over the states arising from several j -levels.

$$\Phi_{jj_a M_a}^{(\alpha)} = \varphi_{2j_a M_a}^2 = N_{2j_a}^2 [A_{jj}^{2+} \alpha_j^+]_{M_a}^{j_a} \quad (\text{II-3})$$

In the example being considered there is one other state of spin j_a . We look for constants a and b such that

$$\Phi_{jj_a M_a}^{(\beta)} = a \varphi_{2j_a M_a}^2 + b \varphi_{4j_a M_a}^2 \quad (\text{II-4})$$

and that

$$\langle \Phi_{jj_a M_a}^{(\alpha)} | \Phi_{jj_a M_a}^{(\beta)} \rangle = 1 \quad (\text{II-5a})$$

$$\langle \Phi_{jj_a M_a}^{(\alpha)} | \Phi_{jj_a M_a}^{(\alpha)} \rangle = 0 \quad (\text{II-5b})$$

The overlap between two states $\phi_{J_1 j_a M_a}^j$ and $\phi_{J_1' j_a M_a}^j$ is given by

$$\begin{aligned} \langle \varphi_{J_1 j_a M_a}^2 | \varphi_{J_1' j_a M_a}^2 \rangle &= N_{J_1 j_a}^2 N_{J_1' j_a}^2 \langle \Phi_0 | [A_{jj}^{J_1+} \alpha_j^+]_{M_a}^{j_a} [A_{jj}^{J_1'+} \alpha_j^+]_{M_a}^{j_a} | \Phi_0 \rangle \\ &= \frac{\delta_{J_1 J_1'} - 2\sqrt{(2J_1+1)(2J_1'+1)} W(J_1 j j J_1' j j j_a)}{\{[1-2(2J_1+1)W(J_1 j j J_1 j j j_a)][1-2(2J_1'+1)W(J_1' j j J_1 j j j_a)]\}^{1/2}} \end{aligned} \quad (\text{II-6})$$

Using Eqs. (II-6), (II-5a), and (II-5b) one finds that

$$\Phi_{jj_a M_a}^{(\beta)} = \frac{1}{\sqrt{1-\{x\}}} [\{x\} \varphi_{2j_a M_a}^2 - \varphi_{4j_a M_a}^2] \quad (\text{II-7})$$

where

$$\{x\} = - \frac{6\sqrt{5} W(2jj^4; jj_a)}{\{[1-10W(2jj^2; jj_a)][1-18W(4jj^4; jj_a)]\}^{1/2}}$$

Thus the states $\Psi^{(\alpha)}$ and $\Psi^{(\beta)}$ form the complete representation for this example.

We wish to find the energy spectrum in the Hamiltonian of Eq. (3) in the representation

$$\begin{aligned} & \alpha_j^+ \Phi_0 \\ & \Phi_{jj_a}^{(\alpha)}, \Phi_{jj_a}^{(\beta)}, \dots \end{aligned} \quad (\text{II-8})$$

When the q.-p. j is low lying, $U_j \approx V_j$ so one can simply treat the Hamiltonian (7). Therefore, the problem is to diagonalize the operator

$$[(A_{jj}^{2+} + \hat{A}_{jj}^2)(A_{jj}^{2+} + \hat{A}_{jj}^2)]_0 \quad (\text{II-9})$$

in the states (II-8). Making use of the identity

$$[\hat{A}_{jj}^2, A_{jj}^{2+}]_0 = - \frac{4\sqrt{5}}{\sqrt{2j+1}} \eta_{jj}^0 \quad (\text{II-10})$$

in the representation (II-8) the Hamiltonian can be written

$$\begin{aligned} H &= \sum_m E_j \alpha_{jm}^+ \alpha_{jm} - 2X [A_{jj}^{2+} A_{jj}]_0 + \frac{4\sqrt{5}X}{\sqrt{2j+1}} \eta_{jj}^0 \\ &= - \sqrt{2j+1} E_j' \eta_{jj}^0 - 2X [A_{jj}^{2+} A_{jj}]_0 \end{aligned} \quad (\text{II-11})$$

In writing (II-11) we make use of the fact that the η_{jj}^0 term just renormalizes the quasi-particle energy, which we indicate by replacing E_j by E_j' . Since

$$[A_{jj}^{2+} A_{jj}^{2+}]_0 \chi_j^T \Psi_0 = 0 \quad (\text{II-12})$$

only the three q.-p. states must be considered. However, from the manner from which the states $\Psi^{(\alpha)}$, $\Psi^{(\beta)}$, etc were constructed it immediately follows that:

$$[A_{jj}^{2+} A_{jj}^{2+}]_0 \Phi_{jja}^{(\mu)} = 0 \text{ for } \mu \neq \lambda \quad (\text{II-13})$$

From this follows the results of Eqs. (12) and (13).

Finally, one must consider those three quasi-particle states of spin j_a such that they cannot be formed by coupling a q.-p. α_j^+ to A_{jj}^{2+} . One must extend the representation to, e.g.,

$$\begin{aligned} \Phi_{jj_a M_a}^{(\mu)} &= \varphi_{4j_a M_a}^{\mu} \\ \Phi_{jj_a M_a}^{(\nu)} &= a^{(\nu)} \varphi_{4j_a M_a}^{\nu} + b^{(\nu)} \varphi_{6j_a M_a}^{\nu} \end{aligned} \quad (\text{II-14})$$

From Eq. (II-6) and some manipulation one can show that the interaction energy in these states involves Racah coefficients which are small. For example, the diagonal matrix element for the state $\Psi^{(\mu)}$ is

$$\Delta E^{(\mu)} \sim W(2jj4; jj_a) \quad (\text{II-15})$$

which is small. Thus the energy shift of the state $j_a = j-1$ is considerably larger than that of the other states.

B. Several j-Levels

Note that if we extend these considerations to the situation in which there are several j levels the picture is not altered much when the large-spin opposite-parity quasi-particle is low lying. From parity considerations the new unperturbed states which must be considered are

$$\varphi_{j_1 j_2 j_a}^{j_1 j_2 j} = N [A_{j_1 j_2}^{+j} L_j^{+}] j_a \quad (\text{II-16})$$

where $j_1, j_2 \neq j$. Since the terms analogous to the second term in Eq. (13b) is not present in these new matrix elements and since the j 's are smaller than j , the new matrix elements are all smaller than the large $j_a = j = 1$ matrix element discussed above and are of varying sign. Therefore the conclusions reached for the single j -shell should not be seriously modified by the inclusion of the other states.

Moreover, one cannot simply add these additional interactions, since they have already been included in part in the phonon-quasi-particle interaction.

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FIGURE CAPTIONS

Figure 1. Quasi-particle interaction with phonon for a single j -level, with one phonon included.

Figure 2. The approximation low-lying eigenstates of a pairing plus quadrupole interaction in a pure $7/2$, $9/2$, $11/2$, and $13/2$ levels, as determined from Eqns. (12) and (13) in the text. The energies roughly correspond to those obtained with the parameters of Ref. 3. The states for which j_a , j_l and 2 do not form a triangle are placed at the unperturbed three-quasi-particle energy.

TABLE

E2 transition rates compared to single-particle. See text for definitions.

j	$B(E2)/B(E2)_{s.p.}$
7/2	9.12 $(2U_j V_j)^2$
9/2	17.6 $(2U_j V_j)^2$
11/2	29.7 $(2U_j V_j)^2$
13/2	46.0 $(2U_j V_j)^2$

_____ j

phonon // // // // // // // // j - 2, j - 1, j + 1, j + 2

_____ j

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Fig. 1

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Fig. 2

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