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A Global Game with Strategic Substitutes and Complements

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Abstract

We study a global game in which actions are strategic complements over some region and strategic substitutes over another region. An agent's payoff depends on a market fundamental and the actions of other agents. If the degree of congestion is sufficiently large, agents' strategies are non-monotonic in their signal about the market fundamental. In this case, a signal that makes them believe that the market fundamental is more favorable for an action may make them less likely to take the action, because of the risk of overcrowding.

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"Nobody goes there anymore because it is too crowded" — Yogi Berra

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1 Introduction

In many settings, the benefit to an agent of taking a particular action depends on the number of other agents who take the same action. In some cases, this relationship is non-monotonic. For example, positive network effects may make it more attractive for a firm to enter a new market if a few other firms also enter. However, if a large number of firms enter, the market becomes too crowded and further entry is unattractive. Agents may also have different information about a random variable that affects all of their payoffs, i.e., about a 'market fundamental'. For example, the market fundamental may be the true size of the market demand, or the true cost of production. The resulting incomplete information game is a coordination problem with congestion effects.

This game contains two widely studied limiting cases: where the relative payoff is always decreasing or always increasing with the actions of others. In the simplest situation, the relative payoff always decreases with the number of other agents who take the same action. In this case of strategic substitutes, there is generically a unique equilibrium when the market fundamental is common knowledge.

When actions are strategic complements, the relative payoff to taking the action increases with the number of others who take this action. With strategic complements and common knowledge of the market fundamental, there may be multiple equilibria. For example, in Matsuyama (1991) and Krugman (1991), workers' productivity in the manufacturing sector of a two-sector model is higher, the greater the number of workers employed in the manufacturing sector. In Katz and Shapiro (1985) and the literature on networks, the utility from consumption of a networked good increases when more agents consume the good. In Diamond and Dybvig (1983), investors must decide whether to withdraw or roll-over a deposit to a bank; the payoff from withdrawal (rolling-over) increases with the number of other investors who withdraw (roll-over). In all of these models, there can be multiple equilibria: if all agents expect one outcome (all to work in the manufacturing sector, join a network or roll-over), then it is optimal for each agent to act in the same way; if all expect another outcome (all to work in agriculture, not join a network or withdraw), then it is optimal for each agent to act in that way.

A small amount of uncertainty can lead to a unique equilibrium in these settings. Carlsson and van Damme (1993) analyzed *global games*, in which the game selected by nature is observed by players with some noise. They show that, in the limit as the noise becomes small, iterated deletion of dominated strategies leads to a unique equilibrium. The key to the uniqueness result is that the support of any agent's higher order beliefs about other agents' signals becomes arbitrarily large for a sufficiently high order of beliefs. ("Higher order beliefs" are the beliefs that an agent has about other's beliefs about other's beliefs, and so on.) Further, the limit equilibrium is noise independent – it is the risk dominant equilibrium of the complete information game. As in Rubinstein (1989), iterated deletion eliminates equilibria that rely on expectations, leading to a unique equilibrium in which agents' strategies depend on their signals.

Morris and Shin (1998) use this idea to analyze currency attacks and Morris and Shin (1999) use it to analyze the pricing of debt. Karp (1999) applies it to Krugman's model.[1] These papers emphasize the fact that even a small amount of incomplete information about the market fundamental leads to a unique equilibrium. This equilibrium is, typically, monotonic: in a binary action game, as the market fundamental becomes more favorable for a particular action, the fraction of agents who take that action increases (more exactly, never decreases).

We derive qualitative properties of the equilibrium set of an incomplete information coordination game with regions of congestion. (Uniqueness is a more difficult issue, and is not addressed here.) In the case where the market fundamental is common knowledge, the equilibrium set in our game is easy to characterize. Over an intermediate range of values of the fundamental, there are multiple equilibria. When unstable equilibria (defined below) are eliminated, the remaining (stable) equilibria are monotonic in the fundamental. When the congestion effect is extremely weak, our model resembles the pure coordination games mentioned above. Not surprisingly, we find that in this case our model contains the kind of monotonic pure strategy equilibrium that the previous models had identified as being unique.

If the congestion effect is sufficiently strong, however, only non-monotonic equilibria exist. This result means that over some interval, as agents receive signals that suggest that the market fundamental is more favorable to a particular action, they become less likely to take that action. The explanation is that a more favorable signal informs the agent that others are also receiving more favorable signals, leading to the possibility that congestion will be severe. In some equilibria, there are disjoint intervals of signal space with the following characteristic: agents who

[1] Other approaches to equilibrium selection have been developed. Herrendorf, Valentinyi, and Waldmann (2000) show how heterogeneity in the manufacturing productivity (rather than the information) of agents in the Matsuyama model can remove indeterminacy and multiplicity of equilibrium. Frankel and Pauzner (2000), following Burdzy, Frankel, and Pauzner (2001), show that exogenous shocks can lead to a unique equilibrium in the Matsuyama setting.

receive signals in a particular interval take the action with probability 1, whereas agents who receive signals the reflect a better market fundamental refrain from the action with probability 1. In this case, the fraction of agents who actually take the action, as a function of the true value of the market fundamental, is non-monotonic.

Although economic situations motivate our model, it is applicable to more general social situations, and possibly to natural environments.[2] For example, agents' utility from going to a bar may depend on the quality of the music (the market fundamental) and the size of the crowd.[3] They prefer the bar when it is neither empty nor extremely crowded. Yogi Berra observed that "Nobody goes there anymore because it is too crowded". He may have exaggerated, but he had the right idea.

2 The Model

An individual is considering which of two actions to take, $a \in \{0, 1\}$. To make the situation concrete, we speak of a visit to a bar and associate the action $a = 0$ with "don't go to the bar" and the action $a = 1$ with "go to the bar". The individual is small and therefore behaves non-strategically, and the mass of individuals is 1. The utility that each individual receives from choosing 'go' depends on two factors: an underlying state, denoted $\theta \in \mathbf{R}$; and the fraction of individuals who undertake the action, $\alpha \in [0, 1]$. The payoff function $U(\theta, \alpha)$ from choosing 'go' satisfies:

Assumption 1 $U(\theta, \alpha) = \theta + f(\alpha)$, *where* $f(\alpha) : [0, 1] \to \mathbf{R}$.

Hence the payoff from going to the bar increases with the state: $\partial U/\partial\theta > 0$. Additive separability ($\partial^2 U/\partial\theta\partial\alpha = 0$) does not affect qualitative aspects of the main results but simplifies the analysis. The individual receives zero utility from not going to the bar. In discussing pure strategies, we assume that an individual who is indifferent between the two actions stays home; this assumption plays no substantial role.

The interaction term $f(\alpha)$ satisfies:

[2]As an example of a natural environment, consider a herd that is grazing in a particular location. Individuals can move to another, possibly superior, location. If only a small number of individuals move, they become exposed to predators, but if many move the new field becomes too crowded.

[3]Arthur (1994)'s bar analogy is based on a common knowledge game with strict substitutes (pure congestion effects).

Assumption 2 *(i). $f(\alpha)$ is continuously differentiable in α (and hence bounded over $[0,1]$);*

(ii). $f(0) = 0$;

(iii). $f'(0) > 0$;

(iv). $f(\alpha)$ is quasi-concave.

Part (i) of the assumption could be generalized; for the purpose of this paper, the generalization would not be interesting. Part (ii) is simply a normalization. Part (iii) ensures that there is a region over which individuals' actions are strategic complements (see Bulow, Geanakoplos, and Klemperer (1985)): that is, the payoff to each individual of choosing 'go' increases with the proportion of others who choose 'go', when that proportion is sufficiently small. Part (iv) ensures that the interaction function is uni-modal; this assumption could be relaxed at the cost of additional notation, but would not produce additional insights.

We use the following definitions:

Definition 1 *(i). $\overline{f} \equiv \max_{\alpha\in[0,1]} f(\alpha) > 0$;*

(ii). $\underline{\theta} \equiv -\overline{f} < 0$;

(iii). $\overline{\theta} \equiv -\min[0, f(1)] \geq \underline{\theta}$;

(iv). $\hat{\theta} \equiv -f(1)$; hence $\underline{\theta} \leq \hat{\theta} \leq \overline{\theta}$.

We say that there is *weak congestion* when $f'(\alpha) < 0$ for some α and $f(1) \geq 0$. In this case, for $\theta \geq 0$ an agent's utility remains positive, even though a more crowded bar decreases utility. There is *strong congestion* when $f(1) < 0$.

The values $\underline{\theta}$ and $\overline{\theta}$ define *dominance regions*: for $\theta < \underline{\theta}$, it is a strictly dominant action for each individual to choose 'don't go', while for $\theta > \overline{\theta}$, it is a strictly dominant action for each individual to choose 'go'. For $\theta \in [\underline{\theta}, \overline{\theta}]$, there is a coordination problem: there is no strictly dominant action for any individual, and the optimal action depends on the proportion of individuals taking the action. For $\theta > \hat{\theta}$, it is an equilibrium for all individuals to choose 'go' when θ is common knowledge.

Figure 1 illustrates the model when θ is common knowledge when $\hat{\theta} = \hat{\theta}$ ($f(1) < 0$). Panel (a) graphs the utility function for $\theta < 0$ (the vertical intercept). In this case there is a pure strategy equilibrium 'don't go', and two mixed strategy equilibria indicated by points s and u.

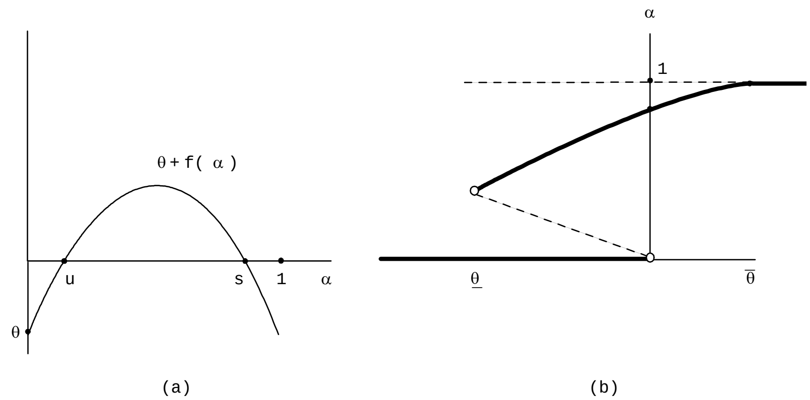

Figure 1: The payoff function (a) and the equilibrium α correspondence (b).

The former is stable, in the sense that any perturbation to the proportion of agents choosing 'go' is self-correcting. (If for some reason, a greater proportion than s of agents chooses 'go', then those agents who now choose 'go' receive negative expected utility and so will change their action.) By the same reasoning, the point u is unstable. Panel (b) graphs the equilibrium α correspondence for different values of the publicly observed parameter θ. For $\theta \leq \underline{\theta}$, the unique pure strategy equilibrium is 'don't go' (i.e., $\alpha = 0$); for $\underline{\theta} < \theta < 0$, there are two equilibria: a mixed strategy and the pure equilibrium 'don't go'. For $0 \leq \theta$, there is a unique equilibrium, either a mixed strategy or the pure strategy 'go', depending on the magnitude of θ. The mixed strategy equilibrium is increasing in θ.

This model contains two well-known limiting cases. If $f(\cdot)$ is everywhere decreasing – i.e., if actions are always strategic substitutes – there is a unique equilibrium, which either pure or mixed depending on the value of θ. If $f(\cdot)$ is everywhere increasing—i.e., if actions are always strategic complements—there are two pure strategy equilibria for θ outside the dominance regions.

Our objective is to analyze the equilibrium set when $f(\cdot)$ is non-monotonic and there is incomplete information about θ. Nature chooses the state according to a uniform distribution on the real line. (Morris and Shin (2000) provide a justification for the assumption of improper (diffuse) priors.) Each individual i receives a private signal $x_i = \theta + \eta_i$ where η_i is a random variable drawn uniformly from the interval $[-\epsilon, \epsilon]$ with $\epsilon > 0$. Conditional on θ, the signals are independently drawn from an identical distribution across individuals. The distributional assumptions imply that the agent's posterior distribution is uniform. The extensive form is that individuals choose actions simultaneously, after observing their private signal. The structure of the game is common knowledge.

In the case of incomplete information, a pure strategy for player i is a mapping from the signal space to the action set; a mixed strategy is a mapping from the signal space to the set of probability measures on the set of actions. Despite the uncountability of players together with continuous type space, Aumann (1964)'s results imply that because of the finite action space the standard definition of a mixed strategy is applicable. Hence we can represent both pure and mixed strategies by $y_i : \mathbb{R} \to [0, 1]$, where $y_i(x)$ denotes the probability that player i chooses action $a = 1$ when it receives signal x; a pure strategy is obtained when y_i equals 0 or 1.

An equilibrium is a profile of strategies (one for each agent) such that the agent's strategy maximizes its expected payoff conditional on the information available, when all other individuals follow the strategies in the profile. Since agents are identical except for the signal that they receive, we consider only symmetric equilibria. Agents play a different strategy only because they have different signals. We consider four classes of equilibria, depending on whether the equilibrium is pure or mixed, and monotonic or non-monotonic.

3 Equilibrium with Incomplete Information

Previous analysis of global games shows that when there is incomplete information and strict complementarities, there is a unique equilibrium in switching strategies (defined formally below). We show that this kind of equilibrium exists in our setting if and only if the degree of non-monotonicity of the interaction function $f(\alpha)$ is sufficiently small – that is, if the amount of congestion is small. When this condition is not satisfied, equilibria are not monotonic in the signal.

Although we cannot prove the existence of *nonmonotonic* pure strategy equilibria , we show that if they exist, then for small ϵ the intervals of signal space over which it is optimal to take one action alternate frequently with intervals for which the other action is optimal. Thus, for small ϵ and sufficiently large congestion, any pure strategy equilibrium gives rise to what appears to be erratic behavior. Next we also show that there can be no monotonic equilibrium in mixed strategies.

As explained in the next subsection, we know that an equilibrium exists for this game. Consequently, the results described above imply that if the amount of congestion is sufficiently great, the equilibrium decision rule must be non-monotonic in the signal.

3.1 Existence of equilibrium

Proving existence of equilibria for our model is not completely straightforward because the game has an uncountable agent space, an uncountable signal space and signal draws are only conditionally independent. Theorem 5.3 of Kim and Yannelis (1997) proves the existence of Bayesian Nash equilibrium when the state space is *countable* while the agent space and the signal space can be uncountable and the signal draws are conditionally independent. Their Remark 6.3 provides a further condition for existence when the state space is uncountable: the payoff function is linear in the strategy.

Using Assumption 1 we write the agent's payoff as

$$U(\theta, \alpha, y(x)) = [\theta + f(\alpha)] \cdot y(x),$$

where $y(x)$ is the probability of going to the bar as defined in the previous section. With a finite action space, the payoff is obviously linear in the mixed strategy. Thus, our model satisfies the linearity assumption that is required for the existence proof in Kim and Yannelis (1997) when the state space is uncountable.

3.2 Monotonic Pure Strategy Equilibrium

As a preliminary step for the analysis we first note that the proportion of agents going to the bar is a continuous function of the state. The proportion of individuals taking the action $a = 1$, given the symmetric strategy profile $\{y(x)\}$ and a realization of the state θ, is

$$\alpha(\theta, y) = \frac{1}{2\epsilon} \int_{\theta-\epsilon}^{\theta+\epsilon} y(z) dz.$$

The next lemma is an immediate consequence of the continuous and conditionally independent distributions of individuals' signals, and so is stated without proof.

Lemma 1 *$\alpha(\theta, y)$ is a continuous in θ for any strategy profile $\{y(\cdot)\}$.*

When the strategy profile $\{y(\cdot)\}$ is played, an individual who has a private signal of x receives an expected payoff of

$$u(x, y) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (\theta + f(\alpha(\theta, y))) \, d\theta.$$

We start by considering the switching – i.e., monotonic pure strategy – equilibrium identified in the global game papers:

$$S_k(x_i) = \begin{cases} \text{go} & \text{if} \quad x_i > k, \\ \text{don't go} & \text{if} \quad x_i \leq k. \end{cases}$$

Given the threshold $k = x^*$,

$$\alpha(\theta, S_{x^*}) = \mathbb{P}(\theta + \eta \geq x^*) = \begin{cases} 0 & \text{if} \quad \theta < x^* - \epsilon, \\ \frac{\theta - x^* + \epsilon}{2\epsilon} & \text{if} \quad x^* - \epsilon \leq \theta \leq x^* + \epsilon, \\ 1 & \text{if} \quad \theta > x^* + \epsilon. \end{cases} \tag{1}$$

If a monotonic switching equilibrium is played, then an individual who receives a signal equal to the equilibrium switch point, x^*, is indifferent between playing 'go' and 'don't go'. Lemma 2 determines the value of the unique x^* that ensures indifference.

Lemma 2 *The expected payoff $u(x^*, S_{x^*})$ of an agent with a signal equal to the equilibrium switch point, x^*, is zero if and only if $x^* = -\int_0^1 f(\alpha)d\alpha$.*

All proofs are contained in the Appendix.

The following proposition provides the necessary and sufficient conditions for the existence of a switching equilibrium.

Proposition 1 *The switching strategy around $x^* = -\int_0^1 f(\alpha)d\alpha$ is an equilibrium if and only if*

$$2\epsilon \geq \max\left\{ \max_{z\in(-1,0)} \left(\frac{1}{z} \int_{z+1}^{1} f(\alpha)d\alpha \right),\ \max_{z\in(0,1)} \left(\frac{1}{z} \int_{0}^{z} f(\alpha)d\alpha - f(1) \right) \right\}. \tag{2}$$

The following two remarks are less general but more intuitive than Proposition 1. Remark 1 shows that the sufficient condition is satisfied if the amount of congestion is small enough. Remark 2 shows that the necessary condition fails if the amount of congestion is large enough. Recall that $\overline{f}$ is defined as the maximum value of $f(\alpha)$. If Assumption 2(iv) holds, then $\overline{f} - f(1)$ is one measure of the amount of congestion, and of the importance to an agent of making correct inferences about how other agents will behave.

Remark 1 *A sufficient condition for an equilibrium in switching strategies to exist is $\overline{f} - f(1) \leq 2\epsilon$.*

Remark 2 *If $f(1) < -x^*$, then there is an $\overline{\epsilon} > 0$ such that when $\epsilon < \overline{\epsilon}$, the switching strategy around x^* is not an equilibrium.*

We now consider two examples where we can relate the existence of a switching equilibrium to the model's parameter values. In the first example, the amount of congestion is independent of the uncertainty parameter ϵ. For this example, a switching equilibrium exists for all $\epsilon > 0$ only if there is weak congestion. In the second example, the amount of congestion depends on the uncertainty parameter ϵ. In this case, a switching equilibrium can exist even if there is strong congestion.

Example 1 *Let $f(\alpha) = \alpha - b\alpha^2$, $b \geq 0$. In this case we can compute the integrals in equation* (2). *Since ϵ can be arbitrarily close to 0, Proposition 2 implies that a switching equilibrium exists for all $\epsilon \geq 0$ if and only if*

$$0 \geq \max\left\{\max_{z\in[-1,0)}\left(\frac{1}{3}bz^2 + \left(b - \frac{1}{2}\right)z + b - 1\right), \max_{z\in[0,1]}\left(\frac{1}{2}z - \frac{1}{3}bz^2\right)\right\}.$$

It is straightforward to show that this inequality holds if and only if $b \leq 0.75$.

For this example, $f'(1) = 1-2b$. Thus, a switching equilibrium exists even if there is congestion ($f'(1) < 0$) provided that the amount of congestion is small ($f(1) = 1 - b \geq 0.25 > 0$). In this case, there is weak congestion, since $f(1) > 0$.

Example 2 *Let $f(\alpha) = (\alpha - b\alpha^2)\epsilon$, $b \geq 0$, $\epsilon \geq 0$. As with example 1, we can compute the integrals in inequality* (2). *In this case, however, we can cancel ϵ from both sides, so that the inequality is independent of ϵ. After some simplification, we can rewrite inequality* (2) *as*

$$0 \geq \max\left\{\max_{z\in(-1,0)}\left(2bz^2 + (6b-3)z + 6b - 18\right), \max_{z\in(0,1)}\left(-2bz^2 + 3z + 6b - 18\right)\right\}.$$

Some straightforward calculation shows that this inequality is satisfied if and only if $b < 3.0$.

For both Examples 1 and 2, the sufficient condition in Remark 1 holds for all $\epsilon \geq 0$ only if $b \leq 0.5$. In this circumstance there is no congestion: $f'(\alpha) \geq 0$ for all α. Thus, the examples show the extent to which the sufficient condition in Remark 1 is too strong.

The results above reveal the role of signals as a coordination device in the class of games under investigation. An equilibrium determines the distribution of strategies used by other agents conditional on the private signal, since a strategy is a function of a signal whose conditional distribution is known. When the noise decreases, the distribution of signals becomes concentrated. This concentration allows agents to predict the equilibrium effect of other agents' strategies with greater precision, relative to the fundamental.

The fundamental θ has a direct effect on the payoff and an indirect (or strategic) effect via the equilibrium value of α. A decrease in the noise has no effect on $E\theta$ and therefore does not alter the direct effect of the fundamental. However, using equation (1), for interior values of α and for a given threshold strategy, $\frac{\partial \alpha}{\partial \theta} = \frac{1}{2\epsilon}$; the strategic effect $\left(\frac{\partial U}{\partial \alpha}\frac{\partial \alpha}{\partial \theta}\right)$ of the fundamental increases as the noise decreases.

The agent's expectation of θ always increases linearly with the signal, but in the region slightly above the threshold, a larger signal has a large strategic effect on the agent's expected payoff. If the distance between the threshold and the upper dominance region is large relative to ϵ, the agent expects that all other agents will go to the bar even when the signal indicates that the fundamental is not good enough to justify going under this circumstance. Hence, smaller noise tends to eliminate the monotonic pure strategy equilibrium.

Thus, with congestion, an equilibrium of the game may not be monotonic. Since monotonicity is a crucial property in establishing uniqueness in global games without congestion, we conjecture that the equilibria may not be unique when there is congestion.

3.3 Non-monotonic Pure Strategy Equilibrium

When the condition given by Proposition 1 is not satisfied – i.e., if the congestion effect is sufficiently large – a switching equilibrium does not exist. In this case, any symmetric pure strategy equilibrium is non-monotonic. In this subsection, we provide a characterization of non-monotonic pure strategy equilibria under the (unproven) assumption that they exist.

In the non-dominance region, there is at least one interval such that an agent who receives a signal in the interval(s) chooses 'go' with probability 1, and an agent who receives a signal outside the interval(s) chooses 'don't go' with probability 1. We refer to these intervals of signals where agents choose 'go' with probability 1 as *islands*. As before, there are also the upper and lower dominance regions.

The 'lowest island' is the island with the smallest lower boundary. For all signals below the lowest island, agents choose 'don't go'. The distance between the upper boundary of this island and the nearest island (or the boundary of upper dominance region in the case where there is only one island) is finite. If that distance is greater than 2ϵ, we define the lowest island as *isolated*. Similarly, any other island is isolated if its boundaries are greater than 2ϵ from the boundary of the nearest island or of the upper dominance region. The following proposition states that the islands cannot be very large, and they cannot be very far apart.

Proposition 2 *In any pure strategy non-monotonic equilibrium, none of these islands is isolated, and none of these islands has length greater than* 4ϵ*.*

This proposition implies that even if a pure strategy equilibrium exists when there is significant congestion, the equilibrium behavior changes rapidly with the signal. In that sense, the behavior appears erratic.

3.4 Mixed Strategy Equilibrium

In describing the game when θ is common knowledge, we pointed out that for some range of θ there is a mixed strategy equilibrium. From Figure 1, it is obvious that in this game the stable mixed strategy equilibrium is monotonic in θ. This observation might suggest that even in cases where a monotonic pure strategy (i.e. switching) equilibrium does not exist in the game without common knowledge, there might be a monotonic mixed strategy equilibrium. This subsection shows that conjecture to be false. There never exists a monotonic mixed strategy equilibrium, regardless of whether there is a monotonic pure strategy equilibrium. In order to obtain this result with a minimum of technical complications, we impose the following regularity condition:

Assumption 3 *The mixed strategy equilibrium is continuous in the signal except for at most a finite number of points.*

This assumption implies that $\alpha(\theta)$ is differentiable except possibly at a finite number of points.

At a mixed strategy equilibrium, an agent who observes a signal x in the interval for which randomization takes place receives zero expected payoff:

$$x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\alpha(\theta, y_M))d\theta = 0. \tag{3}$$

Since we only consider mixed strategies that are almost everywhere continuous in the private signal, and since the agent must receive zero expected payoff at all signals for which randomization takes place, the derivative of the expected payoff in equation (3) must also be zero:

$$\frac{\partial}{\partial x}\left(x + \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\alpha(\theta, y_M))d\theta\right) = 0. \tag{4}$$

This equation implies

$$f(\alpha(x+\epsilon)) - f(\alpha(x-\epsilon)) = -2\epsilon. \tag{5}$$

The mixed strategy equilibrium must satisfy equations (3) and (5).

If there were two or more disjoint mixing intervals, the equilibrium strategy would necessarily be non-monotonic. Since we want to show that monotonic mixed strategy equilibria do not exist, we can therefore restrict attention to mixed strategy equilibria in which there is a single mixing interval. For any signal below that interval agents choose 'don't go' with probability 1, and for any signal above that interval they chose 'go' with probability 1.

When there is a single randomization interval $[\underline{x}, \overline{x}]$, the strategy satisfies

$$y_M(x) \begin{cases} = 0 & x < \underline{x}, \\ \in [0,1] & x \in [\underline{x}, \overline{x}], \\ = 1 & x > \overline{x}. \end{cases} \tag{6}$$

In a mixed strategy equilibrium with a single randomization interval, any individual who receives a private signal in this interval must be indifferent between choosing 'don't go' and 'go'; i.e., the conditional expected payoff function satisfies equations (3) and (5). Also the function $\alpha(\cdot)$ describing the proportion of individuals choosing 'go' is continuous in its argument (see Lemma 1).

We first show that there can be no equilibrium with a single mixing interval when $f(1) \geq 0$.

Proposition 3 *If $f(1) \geq 0$ then there is no equilibrium with a single randomization interval.*

We now show that if $f(1) < 0$ and assumption 2(iv) holds, any equilibrium with a single mixing interval must be non-monotonic.

Proposition 4 *Any equilibrium with a single randomization interval is non-monotonic.*

3.5 Numerical examples

When the condition in Proposition 1 is not satisfied we know that any equilibrium is non-monotonic, but we do not know whether the equilibrium is unique, or whether it consists of pure strategy islands or is a mixed strategy. Numerical methods provide some insight into these questions.

We use a finite-state approximation to the continuous state model. Rather than being an element of the real line, here θ can take one of N possible values with equal probability, $N << \infty$. When $\theta = \theta_i$, the signal can equal θ_i or any of the M values above and below θ_i with equal probability. The grid points are evenly spaced and we retain the assumption of uniformity. Since the support of θ, conditional on the signal θ_i, contains $2M + 1$ elements, and the prior support contains N elements, $\frac{2M+1}{N}$ is an inverse measure of the degree to which the signal is informative, as is ϵ in the continuous state space model. For a given N, a smaller value of M in this model corresponds to a small value of ϵ in the continuous model. In view of Proposition 1, the interesting case is where ϵ is small, so we emphasize $M = 1$ (although we also ran simulations for other values). We report results for values of N ranging from 50 to 60.

We use the quadratic formulation $f(\alpha) = \alpha(1 - b\alpha)$ from Example 1 of Section 3.2. We are interested in the case where there is congestion, so we considered only $b > 0.5$. The appendix describes the numerical methods. Our results are as follows:

1. For $b \leq 0.75$ the only equilibrium we find is the switching equilibrium. For b slightly greater than 0.75 the only equilibrium we find consists of a single pure-strategy island. For values of b larger than 0.83 we find no pure strategy equilibria.

2. For b slightly smaller than 0.83 we find mixed strategy equilibria in addition to the pure strategy equilibrium. For larger values of b we find only mixed strategy equilibria. These are not unique.

3. All of the mixed strategy equilibria are non-monotonic in the signal. In some case they consist of a set of signals for which the probability of going is positive, followed by a set of signals where the probability is 0 ("mixed strategy islands"); in other cases, although the probabilities are non-monotonic there are no 0-probability signals between positive-probability signals.

Since we use a finite state space model here, and since computing constraints limit the magnitude of N, we cannot guarantee that the numerical results also apply to the continuous state space model. However, these results are consistent with our analytic results; they suggest that the equilibrium is not unique when the amount of congestion is significant. The results may be of independent interest because the discrete state-space game may be as good an approximation of the environment as is the continuous state-space game.

4 Conclusion

We characterized the equilibrium set in an incomplete information global game where players' actions can be both strategic complements and strategic substitutes. We provided a necessary and sufficient condition for existence of the switching strategy equilibrium that is identified as the unique equilibrium in models where actions are always strategic complements. This condition requires that congestion is not too severe, compared to the noise in players' signals. An alternative view of this condition is that strategic uncertainty is not too great compared to fundamental uncertainty.

We showed that when this condition is not satisfied, agents' decisions are non-monotonic in the signal. If a pure strategy equilibrium exists in this situation, the length of intervals over which a particular action is optimal is linearly related to ϵ, the amount of noise. Thus, when the noise is small, the optimal action changes frequently as the signal varies. Any mixed strategy equilibrium must also be non-monotonic. These non-monotonic equilibria capture the spirit of Yogi Berra's aphorism that we quote at the beginning of this paper. In equilibrium, agents do not go to a bar, despite getting a more favorable signal about its fundamental quality, because they anticipate that too many others will be there.

We have not dealt with the uniqueness of equilibrium in the incomplete information game. The iterated deletion argument used for uniqueness in previous analysis of global games relies heavily on the fact that the game is supermodular when actions are strategic complements; see Morris and Shin (2000), Milgrom and Roberts (1990) and Vives (1990). It seems, however, that under some situations, uniqueness can be established even when there are congestion effects—see Goldstein and Pauzner (2000) for an example. The exact conditions for this are not understood; we hope to address this in future work.

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A Appendix

The first part of the appendix proves the propositions and remarks, and the second part discusses the algorithm used to solve the discrete time model.

A.1 Proofs

Proof. (**Lemma 2**) From the definition of expected payoff,

$$u(x^*, S_{x^*}) = \frac{1}{2\epsilon} \int_{x^*-\epsilon}^{x^*+\epsilon} (\theta + f(\alpha(\theta, S_{x^*})))\, d\theta. \tag{7}$$

Conditional on receiving the signal x^*, the distribution of α is uniform on $[0, 1]$. Therefore

$$u(x^*, S_{x^*}) = x^* + \int_0^1 f(\alpha) d\alpha.$$

Hence $u(x^*, S_{x^*}) = 0$ if and only if $x^* = -\int_0^1 f(\alpha)d\alpha$. ■

Proof. (**Proposition 1**) The switching strategy around x^* is an equilibrium if and only if $u(x, S_{x^*})$ is single upward crossing: that is, (i) $u(x, S_{x^*}) < 0$ for $x < x^*$; (ii) $u(x, S_{x^*}) > 0$ for $x > x^*$. We show that these two conditions are equivalent to condition (2).

>From the definition of the expected payoff and the equilibrium proportion of individuals who choose 'go', from equation (1),

$$\begin{aligned} u(x, S_{x^*}) &= x + \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f(\alpha(\theta, S_{x^*}))d\theta \\ &= \begin{cases} x & \text{if} \quad x < x^* - 2\epsilon, \\ x + \frac{1}{2\epsilon'} \int_{x^*-\epsilon}^{x+\epsilon} f(\frac{1}{2} + \frac{1}{2\epsilon}(\theta - x^*))d\theta & \text{if} \quad x^* - 2\epsilon \leq x < x^*, \\ x + \frac{1}{2\epsilon} \int_{x-\epsilon}^{x^*+\epsilon} f(\frac{1}{2} + \frac{1}{2\epsilon}(\theta - x^*))d\theta + \frac{1}{2\epsilon} \int_{x^*+\epsilon}^{x+\epsilon} f(1)d\theta & \text{if} \quad x^* \leq x < x^* + 2\epsilon, \\ x + f(1) & \text{if} \quad x \geq x^* + 2\epsilon. \end{cases} \end{aligned} \tag{8}$$

>From Lemma 2, $x^* = -\int_0^1 f(\alpha)d\alpha$. Let $z \equiv \frac{x-x^*}{2\epsilon}$. We first consider the case where $x < x^*$ and then the case where $x > x^*$.

Using the first line of equation (8), $u(x, S_{x^*}) < 0$ for $x < x^* - 2\epsilon$, if and only if $x^* - 2\epsilon < 0$ i.e.,

$$2\epsilon > -\int_0^1 f(\alpha)d\alpha. \tag{9}$$

For $x \in [x^* - 2\epsilon, x^*]$, the second line of equation (8) implies,

$$u(x, S_{x^*}) = 2\epsilon z + x^* + \int_0^{z+1} f(\alpha) d\alpha. \tag{10}$$

Using equation (10) and the definition of x^*, we write the equilibrium condition $u(x, S_{x^*}) < 0$ as

$$2\epsilon z - \int_0^1 f(\alpha) d\alpha + \int_0^{z+1} f(\alpha) d\alpha = 2\epsilon z - \int_{z+1}^1 f(\alpha) d\alpha < 0$$

for all $z \in (-1, 0)$. Since z is a negative number, the last inequality is equivalent to

$$2\epsilon \geq \frac{1}{z} \int_{z+1}^1 f(\alpha) d\alpha \tag{11}$$

for all $z \in (-1, 0)$. Inequalities (9) and (11) are equivalent to

$$2\epsilon \geq \max_{z \in [-1,0)} \left(\frac{1}{z} \int_{z+1}^1 f(\alpha) d\alpha \right). \tag{12}$$

We now consider the case where $x > x^*$. From the last line of equation (8), $u(x, S_{x^*}) > 0$ for $x > x^* + 2\epsilon$, if and only if $x^* + 2\epsilon + f(1) > 0$, i.e.,

$$2\epsilon > \int_0^1 f(\alpha) d\alpha - f(1). \tag{13}$$

Using the third line of line of equation (8) and the definition of z, we have $u(x, S_{x^*}) > 0$ for $x^* \leq x < x^* + 2\epsilon$, if and only if $x^* + 2\epsilon z + \int_z^1 f(\alpha) d\alpha + z f(1) > 0$ for $z \in [0, 1)$. Using the definition of x^*, we rewrite this inequality as

$$2\epsilon > \frac{1}{z} \int_0^z f(\alpha) d\alpha - f(1) \tag{14}$$

for $z \in [0, 1)$. Inequalities (13) and (14) are equivalent to

$$2\epsilon > \max_{z \in [0,1]} \left(\frac{1}{z} \int_0^z f(\alpha) d\alpha - f(1) \right). \tag{15}$$

Inequalities (12) and (15) are both satisfied if and only if equation (2) is satisfied. ■

Proof. (**Remark 1**) If the inequality

$$\frac{\partial}{\partial x} u(x, S_{x^*}) > 0 \tag{16}$$

holds for *arbitrary* x^*, then the switching strategy with switch point x^* specified in lemma 2 is an equilibrium. At this value of x^*, agents are indifferent between the two actions, and if inequality (16) holds, they strictly prefer to go to the bar (not go to the bar) when $x > x^*$ ($x < x^*$). Thus it is sufficient to show that inequality (16) holds for arbitrary x^*.

Using equation (8), we have

$$\frac{\partial u(x, S_{x^*})}{\partial x} = \begin{cases} 1 & \text{if} \quad x < x^* - 2\epsilon, \\ 1 + \frac{1}{2\epsilon} f(\frac{1}{2} - \frac{1}{2\epsilon}(x - x^* + \epsilon)) & \text{if} \quad x^* - 2\epsilon \leq x < x^*, \\ 1 - \frac{1}{2\epsilon} f(\frac{1}{2} - \frac{1}{2\epsilon}(x - x^* - \epsilon) + \frac{1}{2\epsilon} f(1) & \text{if} \quad x^* \leq x < x^* + 2\epsilon, \\ 1 & \text{if} \quad x \geq x^* + 2\epsilon. \end{cases} \tag{17}$$

>From the third line, $\frac{\partial}{\partial x} u(x, S_{x^*}) > 0$ if $f(1) + 2\epsilon > f(\alpha)$ for all $\alpha \in [0, 1]$. This relation holds if and only if $f(1) \geq \overline{f} - 2\epsilon$. From the second line of equation (17), $\frac{\partial}{\partial x} u(x, S_{x^*}) > 0$ if $f(\alpha) > -2\epsilon$ for all $\alpha \in [0, 1]$. By assumption 2(iv), this relation holds if and only if $f(1) > -2\epsilon$, which is weaker than the condition $f(1) \geq \overline{f} - 2\epsilon$. Thus, the latter inequality is equivalent to the requirement that condition (16) holds for arbitrary x^*. ■

Proof. (**Remark 2**) If $-\hat{\theta} \equiv f(1) < \int_0^1 f(\alpha) d\alpha \equiv -x^*$, then $x^* < \hat{\theta}$. Let $\overline{\epsilon} = (\hat{\theta} - x^*)/2 > 0$, and suppose that $\epsilon < \overline{\epsilon}$. Given such an ϵ, the individual who receives a private signal of $x \in (x^* + 2\epsilon, \hat{\theta})$ knows with certainty that $\alpha = 1$, since $x > x^* + 2\epsilon$. The expected payoff from choosing 'go' is therefore $x + f(1) < 0$, since $x < \hat{\theta}$. It follows that the monotonic switching strategy around x^* cannot be an equilibrium. ■

Proof. (**Proposition 2**) In order to show that no island is isolated, suppose that at least one island is isolated. Denote the lower and upper boundaries of this island by $\underline{x}$ and $\overline{x}$. The assumption that the island is isolated implies that the distribution of α is symmetric at the two boundaries. This fact implies that $\mathbb{E}(f(\alpha) \mid \underline{x}) = \mathbb{E}(f(\alpha) \mid \overline{x})$. This equality is not consistent with the equilibrium condition that at the boundary of an island the agent is indifferent between 'go' and 'don't go', i.e., $\mathbb{E}(\theta + f(\alpha) \mid \underline{x}) = 0 = \mathbb{E}(\theta + f(\alpha) \mid \overline{x})$ and $\underline{x} < \overline{x}$. Thus, the island cannot be isolated.

In order to show that the length of the island is less than 4ϵ, suppose that it is greater than 4ϵ. The agent who receives a signal near the midpoint of this island is certain that all other agents received signals on the island, and therefore is certain that $\alpha = 1$. Since the island is in the non-dominance region, it is not an equilibrium for this agent to choose 'go'. Thus, the island cannot be longer than 4ϵ. ■

Proof. (**Proposition 3**) If $f(1) \geq 0$, then, by Assumption 2.iv $f(\alpha) \geq 0$ for all $\alpha \in [0, 1]$. Suppose that there is a mixed strategy equilibrium with a single randomization interval.

The equilibrium condition (5) implies that $f(\alpha(\underline{x}) + \epsilon)) = f(\alpha(\underline{x}) - \epsilon)) - 2\epsilon$. Moreover $f(\alpha(\underline{x}) - \epsilon)) = f(0) = 0$. Hence $f(\alpha(\underline{x}) + \epsilon)) = -2\epsilon$, which is impossible if $f(\alpha) \geq 0$ for all $\alpha \in [0, 1]$. It follows that there is no mixed strategy equilibrium with a single randomization interval. ■

Proof. (**Proposition 4**) Suppose that there is a single randomization interval $[\underline{x}, \bar{x}]$. By Proposition 3, we need only consider the case where $f(1) < 0$. To show that the equilibrium randomization strategy is non-monotonic, it suffices to show that the function $\alpha(x)$ induced by the equilibrium is non-monotonic.

Define $\phi(x) \equiv f \circ \alpha$, the composition of the functions $f(\alpha)$ and $\alpha(x)$. From the definition of ϕ:

$$\phi(x) = f(\alpha(x)) \tag{18}$$

$$\phi'(x) = f'(\alpha(x))\alpha'(x) \tag{19}$$

where the derivatives exist, i.e., everywhere except for at most a finite number of points (due to Assumption 3).

By Assumption 2, $f(\alpha)$ has a single peak and it is increasing at $\alpha = 0$. Therefore: (i) $f(\alpha) > 0$ for α small, and (ii) $f(\alpha) < 0$ implies $f'(\alpha) < 0$. Since $\alpha(\underline{x}-\epsilon) = 0$, α is continuous, and is positive on the mixing interval, it must be increasing in the neighborhood to the right of $\underline{x} - \epsilon$. Therefore ϕ is increasing in the neighborhood to the right of $\underline{x} - \epsilon$; that is, $\phi(\underline{x} - \epsilon + \delta)$ is increasing in δ for small positive δ. From equation (5), $\phi(\underline{x} + \epsilon + \delta) = \phi(\underline{x} - \epsilon + \delta) - 2\epsilon$. Consequently ϕ is increasing in the neighborhood to the right of $\underline{x} + \epsilon$.

Due to the fact that $\phi(\underline{x}-\epsilon) = f(0) = 0$ and the equilibrium condition (5), $\phi(\underline{x}+\epsilon) = -2\epsilon$. Therefore $f(\alpha(\underline{x} + \epsilon)) < 0$. (This inequality is possible because we consider only the case $f(1) < 0$.) Therefore $f'(\alpha(\underline{x} + \epsilon)) < 0$. By continuity of α, $f' < 0$ in the neighborhood to the right of $\underline{x} + \epsilon$.

Since $\phi' > 0$ and $f' < 0$ in the neighborhood to the right of $\underline{x} + \epsilon$, we conclude that $\alpha'(x) < 0$ in that neighborhood (at points where the derivative exists). We noted above that α is increasing in the neighborhood to the right of $\underline{x} - \epsilon$. Thus, α is non-monotonic. ■

A.2 Details of numerical methods

We used two algorithms. The simple algorithm restricts strategies to be pure and begins with the guess of the switching strategy discussed in Section 3.2. Using this guess, we calculate the induced $\alpha(\theta)$, and then find the equilibrium response. Using this response as our next guess, we iterate. This algorithm converges quickly to a pure strategy island equilibrium, or it fails to converge—even after hundreds of thousands of iterations.

The more elaborate algorithm finds a mixed strategy equilibrium by solving a nonlinear complementarity problem. Define p as the N dimensional vector of probabilities, with p_i equal to the probability that an agent who observes signal i goes to the bar. The vector p induces a conditional distribution $\alpha(\theta_i)$ which is used to calculate $u(i,p)$, the expected benefit of going to the bar for an agent who observes signal i, when the strategy profile is p. The equilibrium condition is

$$\begin{aligned} p_i > 0 &\Longrightarrow u(p,i) \geq 0, \\ p_i < 1 &\Longrightarrow u(p,i) \leq 0. \end{aligned} \tag{20}$$

The following discussion relies on Chapter 3 of Miranda and Fackler (2002), and we use the Matlab toolbox that accompanies their book to implement the algorithm. A vector p solves the complementarity problem (20) if and only if it solves the root-finding problem

$$\hat{u}(p) = \min\left(\max\left(u(p), 0\varrho - p,\right), 1\varrho - p\right) = 0,$$

where $u(p)$ is the vector-valued function whose ith element is $u(p,i)$, and ϱ is an $N-$dimensional vector of ones.

The function $\hat{u}$ has points of non-differentiability, which may make it difficult to find a solution. This problem can be avoided by using Fischer's function, defined as

$$\tilde{u}(p) = \phi^{-}\left(\phi^{+}\left(u(p), 0\varrho - p\right), 1\varrho - p\right)$$

where

$$\phi_i^{\pm}(x,y) = x_i + y_i \pm \sqrt{x_i^2 + y_i^2}.$$

The functions $\hat{u}(p)$ and $\tilde{u}(p)$ have the same roots and the same signs, but $\tilde{u}$ is smoother. We solve the original problem by finding the roots to $\tilde{u}(p) = 0$.