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Floer Homology Theories for Knots in Lens Spaces

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Allen David Boozer

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ABSTRACT OF THE DISSERTATION

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Professor Ciprian Manolescu, Chair

We describe two projects involving the construction of Floer homology theories for knots in lens spaces. In the first project, we propose definitions of complex manifolds $\mathcal{P}_M(X, m, n)$ that could potentially be used to construct the symplectic Khovanov homology of n-stranded knots in lens spaces. The manifolds $\mathcal{P}_M(X, m, n)$ are defined as moduli spaces of Hecke modifications of rank 2 parabolic bundles over an elliptic curve X. To characterize these spaces, we describe all possible Hecke modifications of all possible rank 2 vector bundles over X, and we use these results to define a canonical open embedding of $\mathcal{P}_M(X, m, n)$ into $M^s(X, m+n)$, the moduli space of stable rank 2 parabolic bundles over X with trivial determinant bundle and $m+n$ marked points. We explicitly compute $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$. For comparison, we present analogous results for the case of rational curves, for which a corresponding complex manifold $\mathcal{P}_M(\mathbb{CP}^1, 3, n)$ is isomorphic for n even to a space $\mathcal{Y}(S^2, n)$ defined by Seidel and Smith that can be used to compute the symplectic Khovanov homology of n-stranded knots in S^3.

In the second project, we describe a scheme for constructing generating sets for Kronheimer and Mrowka's singular instanton knot homology for the case of knots in lens spaces. The scheme involves Heegaard-splitting a lens space containing a knot into two solid tori. One solid torus contains a portion of the knot consisting of an unknotted arc, as well as holonomy perturbations of the Chern-Simons functional used to define the homology theory. The other solid torus contains the remainder of the knot. The Heegaard splitting yields a pair of Lagrangians in the traceless $SU(2)$-character variety of the twice-punctured torus, and the intersection points of these Lagrangians comprise the generating set that we seek. We illustrate the scheme by constructing generating sets for several example knots. Our scheme is a direct generalization of a scheme introduced by Hedden, Herald, and Kirk for describing generating sets for knots in S^3 in terms of Lagrangian intersections in the traceless $SU(2)$-character variety for the 2-sphere with four punctures.

The dissertation of Allen David Boozer is approved.

Ko Honda

Raphael Alexis Rouquier

Sucharit Sarkar

Ciprian Manolescu, Committee Chair

University of California, Los Angeles

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VITA

1996	B.S. (Mathematics and Physics), University of Virginia.
2005	Ph.D. (Physics), California Institute of Technology.
2005–2006	Postdoctoral Scholar, Physics Department, California Institute of Technology.
2006–2009	Staff Scientist, Physics Department, California Institute of Technology.
2010–2011	Postdoctoral Scholar, Physics Department, University of New Mexico.

PUBLICATIONS

Moduli spaces of Hecke modifications for rational and elliptic curves, *arXiv:1805.11184*, (2018).

Holonomy perturbations of the Chern-Simons functional for lens spaces, *arXiv:1811.01536*, (2018).

Computer bounds for Kronheimer-Mrowka foam evaluation, *arXiv:1908.07133*, (2019).

CHAPTER 1

Introduction

Knots and links play a key role in the study of 3-manifolds. Indeed, any 3-manifold can be obtained by performing Dehn surgery on a suitable link in S^3. One can consider knots in any 3-manifold, but so far most work on knot invariants has focused on knots in S^3. Little is known about invariants of knots in arbitrary 3-manifolds, but, partly because of the close connection between knots and 3-manifold topology, such results would be of great interest. If one quantifies the complexity of a 3-manifold by its Heegaard genus, then by this measure S^3, with Heegaard genus zero, is the simplest 3-manifold. The next simplest class of 3-manifolds are those with Heegaard genus one, i.e., lens spaces. We describe two projects involving the construction of Floer homology theories for knots in lens spaces.

1.1 Symplectic Khovanov homology for knots in lens spaces

The first project we consider describes an initial step towards the construction of symplectic Khovanov homology for knots in lens spaces. Khovanov homology is a powerful invariant for distinguishing links in S^3 [Kho00]. Khovanov homology can be viewed as a categorification of the Jones polynomial [Jon85]: one can recover the Jones polynomial of a link from its Khovanov homology, but the Khovanov homology generally contains more information. For example, Khovanov homology can sometimes distinguish distinct links that have the same Jones polynomial, and Khovanov homology detects the unknot [KM11a], but it is not known whether the Jones polynomial has this property. The Khovanov homology of a link can be obtained in a purely algebraic fashion by computing the homology of a chain complex constructed from a generic planar projection of the link. The Khovanov homology can also be obtained in a geometric fashion by Heegaard-splitting S^3 into two 3-balls in such a way that the intersection of the link with each 3-ball consists of r unknotted arcs. Each 3-ball determines a Lagrangian in a symplectic manifold $\mathcal{Y}(S^2, 2r)$ known as the Seidel-Smith space, and the Lagrangian Floer homology of the pair of Lagrangians yields the Khovanov homology of the link (modulo grading) [AM17, SS06]. Recently Witten has outlined gauge theory interpretations of Khovanov homology and the Jones polynomial in which the Seidel-Smith space is viewed as a moduli space of solutions to the Bogomolny equations [Wit12, Wit11, Wit16].

We consider here the problem of generalizing symplectic Khovanov homology to knots in lens spaces. In analogy with the Seidel-Smith approach to Khovanov homology, one could Heegaard-split a lens space into two solid tori, each containing r unknotted arcs, and compute the Lagrangian Floer homology of a pair of Lagrangians intersecting in a symplectic manifold $\mathcal{Y}(T^2, 2r)$ that generalizes the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$. To perform the construction, one first needs to determine a suitable symplectic manifold $\mathcal{Y}(T^2, 2r)$, and we propose two natural candidates for this space.

In outline, our approach is as follows. First, using a result due to Kamnitzer [Kam11], we reinterpret the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ as a moduli space $\mathcal{H}(\mathbb{CP}^1, 2r)$ of equivalence classes of sequences of Hecke modifications of rank 2 holomorphic vector bundles over a rational curve. Roughly speaking, a *Hecke modification* is a way of locally modifying a holomorphic vector bundle near a point to obtain a new vector bundle. We show that there is a close relationship between Hecke modifications and parabolic bundles, and we use this relationship to reinterpret the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ as a moduli space $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ of isomorphism classes of parabolic bundles with marking data. The space $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ has two natural generalizations $\mathcal{P}_M(X, 1, 2r)$ and $\mathcal{P}_M(X, 3, 2r)$ to the case of an elliptic curve X, and we propose these spaces as candidates for $\mathcal{Y}(T^2, 2r)$.

1.2 Singular instanton homology for knots in lens spaces

The second project we consider describes a scheme for constructing generating sets for Kronheimer and Mrowka's singular instanton homology for the case of knots in lens spaces. Singular instanton homology was introduced by Kronheimer and Mrowka to describe knots in 3-manifolds [KM11b, KM11a, KM14]. Singular instanton homology is defined in terms of gauge theory, but has important implications for Khovanov homology. Specifically, given a knot K in S^3, Kronheimer and Mrowka showed that there is a spectral sequence whose E_2 page is the reduced Khovanov homology of the mirror knot $\bar{K}$, and that converges to the singular instanton homology of K. Using this spectral sequence, Kronheimer and Mrowka proved that Khovanov homology detects the unknot. This result is obtained by proving the analogous result for singular instanton homology and then using the rank inequality implied by the spectral sequence.

Calculations of singular instanton homology are generally difficult to carry out, though some results are known. For example, Kronheimer and Mrowka showed that the singular instanton homology of an alternating knot in S^3 is isomorphic to the reduced Khovanov homology of its mirror (modulo grading), since for such knots the spectral sequence collapses at the E_2 page. In recent work, Hedden, Herald, and Kirk have described a scheme for producing generating sets for singular instanton homology for a variety of knots in S^3, which can sometimes be used to compute the singular instanton homology itself [HHK14]. Their scheme works as follows.

Singular instanton homology is defined in terms of the Morse complex of a perturbed Chern-Simons functional. The unperturbed Chern-Simons functional is typically not Morse, so to obtain a well-defined homology theory it is necessary to include a small perturbation term. For the case of knots in S^3, Hedden, Herald, and Kirk show how a suitable perturbation can be constructed. Their scheme involves Heegaard-splitting S^3 into a pair of solid 3-balls B_1 and B_2. The ball B_1 contains a portion of the knot K consisting of two unknotted arcs, together with a specific holonomy perturbation of the Chern-Simons functional. The ball B_2 contains the remainder of the knot. The Heegaard splitting of S^3 yields a pair of

Lagrangians L_1 and L_2 in the traceless $SU(2)$-character variety of the 2-sphere with four punctures $R(S^2, 4)$, a symplectic manifold known as the *pillowcase* that is homeomorphic to the 2-sphere. Specifically, the Lagrangians L_1 and L_2 describe conjugacy classes of $SU(2)$-representations of the fundamental group of the 2-sphere with four punctures that extend to $B_1 - K$ and $B_2 - K$, respectively. In many cases, the points of intersection of L_1 and L_2 constitute a generating set for singular instanton homology. The essential idea of the scheme is to confine all of the perturbation data to a standard ball B_1 corresponding to a Lagrangian L_1 that can be described explicitly. The problem of constructing a generating set for a particular knot thus reduces to describing the Lagrangian L_2, a task that is facilitated by the fact that the Chern-Simons functional is unperturbed on the ball B_2. In further work, Hedden, Herald, and Kirk define *pillowcase homology* to be the Lagrangian Floer homology of the pair (L_1, L_2) [HHK18]. They conjecture that pillowcase homology is isomorphic to singular instanton homology and compute some examples that support this conjecture.

We generalize the scheme of Hedden, Herald, and Kirk to the case of knots in lens spaces. We Heegaard-split a lens space Y containing a knot K into two solid tori U_1 and U_2. The solid torus U_1 contains a portion of the knot consisting of an unknotted arc, together with a specific holonomy perturbation. The solid torus U_2 contains the remainder of the knot. From the Heegaard splitting of Y we obtain a pair of Lagrangians L_1 and L_2 in the traceless $SU(2)$-character variety of the twice-punctured torus $R(T^2, 2)$, and in many cases the points of intersection of L_1 and L_2 constitute a generating set for the (reduced) singular instanton homology $I^\natural(Y, K)$.

Our scheme is particularly well-suited for the case of $(1,1)$-knots. By definition, a $(1,1)$-knot is a knot K in a lens space Y that has a Heegaard splitting into a pair of solid tori $U_1, U_2 \subset Y$ such that the components $U_1 \cap K$ and $U_2 \cap K$ of the knot in each solid torus are unknotted arcs. It is known that $(1,1)$-knots include all torus knots and 2-bridge knots. We illustrate our scheme by calculating generating sets for several example $(1,1)$-knots. We first rederive known results for knots in S^3: we produce generating sets for the unknot (one generator) and trefoil (three generators), which allow us to compute the singular instanton homology for these knots. Next we consider knots in lens spaces $L(p, 1)$. We prove:

Theorem 1.2.1. *If p is not a multiple of 4, then the rank of $I^\natural(L(p,1),U)$ for the unknot U in the lens space $L(p,1)$ is at most p.*

A knot K in a lens space $L(p,q)$ is said to be *simple* if the lens space can be Heegaard-split into solid tori U_1 and U_2 with meridian disks D_1 and D_2 such that D_1 intersects D_2 in p points and $K \cap U_i$ is an unknotted arc in disk D_i for $i = 1, 2$ (see [Hed11]). Up to isotopy, there is exactly one simple knot in each nonzero homology class of $H_1(L(p,q);\mathbb{Z}) = \mathbb{Z}_p$. We prove:

Theorem 1.2.2. *If K is the unique simple knot representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$ of the lens space $L(p,1)$, then the rank of $I^\natural(L(p,1),K)$ is at most p.*

To our knowledge, Theorems 1.2.1 and 1.2.2 give the first rank bounds on instanton homology for knots in 3-manifolds other than S^3. For a simple knot K in the lens space $Y = L(p,q)$, the knot Floer homology $\widehat{HFK}(Y,K)$ has rank p (see [Hed11]). Thus, Theorem 1.2.2 is consistent with Kronheimer and Mrowka's conjecture that for a knot K in a 3-manifold Y, the ranks of $I^\natural(Y,K)$ and $\widehat{HFK}(Y,K)$ are the same (see [KM10] Section 7.9).

CHAPTER 2

Symplectic Khovanov homology for knots in lens spaces

2.1 Introduction

Our goal is to generalize the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ to the case of elliptic curves by exploiting Kamnitzer's reinterpretation of this space as a moduli space of Hecke modifications $\mathcal{H}(\mathbb{CP}^1, 2r)$. Our approach is as follows. Given a rank 2 holomorphic vector bundle E over a curve C, we define a set $\mathcal{H}^{tot}(C, E, n)$ of equivalence classes of sequences of n Hecke modifications of E. As is well-known, the set $\mathcal{H}^{tot}(C, E, n)$ has the structure of a complex manifold that is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$, where each factor of $\mathbb{CP}^1$ corresponds to a single Hecke modification. The Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ is then defined to be the open submanifold of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ consisting of equivalence classes of sequences of Hecke modifications for which the terminal vector is semistable.

Example 2.1.1. For $r = 1$, we have that

$$\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2) = (\mathbb{CP}^1)^2, \qquad \mathcal{H}(\mathbb{CP}^1, 2) = (\mathbb{CP}^1)^2 - \{(a, a) \mid a \in \mathbb{CP}^1\}.$$

To generalize the Kamnitzer space to curves of higher genus, we want to define moduli spaces of sequences of Hecke modifications in which the initial vector bundle in the sequence is allowed to vary. We define such spaces via the use of parabolic bundles. For any curve C and any natural numbers m and n, we define a moduli space $\mathcal{P}_M^{tot}(C, m, n)$ of marked parabolic bundles. We prove:

Theorem 2.1.2. *The moduli space $\mathcal{P}_M^{tot}(C, m, n)$ naturally has the structure of a complex manifold isomorphic to a $(\mathbb{CP}^1)^n$-bundle over $M^s(C, m)$.*

Here the complex manifold $M^s(C,m)$ is the moduli space of stable rank 2 parabolic bundles over a curve C with trivial determinant bundle and m marked points. Roughly speaking, the space $\mathcal{P}_M^{tot}(C,m,n)$ describes isomorphism classes of sequences of n Hecke modifications in which the initial vector bundle in the sequence is allowed to range over isomorphism classes that are parameterized by $M^s(C,m)$. By imposing a condition analogous to the semistability condition used to define the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1,2r)$, we identify an open submanifold $\mathcal{P}_M(C,m,n)$ of $\mathcal{P}_M^{tot}(C,m,n)$. We prove that $\mathcal{P}_M^{tot}(\mathbb{CP}^1,n) := \mathcal{P}_M^{tot}(\mathbb{CP}^1,3,n)$ is isomorphic to $\mathcal{H}^{tot}(\mathbb{CP}^1,\mathcal{O}\oplus\mathcal{O},n)$ and $\mathcal{P}_M(\mathbb{CP}^1,2r) := \mathcal{P}_M(\mathbb{CP}^1,3,2r)$ is isomorphic to to $\mathcal{H}(\mathbb{CP}^1,2r)$. Thus the Seidel-Smith space $\mathcal{Y}(S^2,2r)$, the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1,2r)$, and the space of marked parabolic bundles $\mathcal{P}_M(\mathbb{CP}^1,2r)$ are all isomorphic. However, unlike $\mathcal{Y}(S^2,2r)$ or $\mathcal{H}(\mathbb{CP}^1,2r)$, the space $\mathcal{P}_M(\mathbb{CP}^1,2r)$ naturally generalizes to case of elliptic curves.

Although our primary motivation for introducing the spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1,n)$ and $\mathcal{P}_M(\mathbb{CP}^1,n)$ is to facilitate generalization, they are also useful for proving canonical versions of certain results for $\mathbb{CP}^1$. For example, we prove a canonical version of the noncanonical isomorphism $\mathcal{H}^{tot}(\mathbb{CP}^1,\mathcal{O}\oplus\mathcal{O},n)\to(\mathbb{CP}^1)^n$ for $\mathcal{P}_M^{tot}(\mathbb{CP}^1,n)$:

Theorem 2.1.3. *There is a canonical isomorphism* $\mathcal{P}_M^{tot}(\mathbb{CP}^1,n)\to(M^{ss}(\mathbb{CP}^1,4))^n$.

Here the complex manifold $M^{ss}(\mathbb{CP}^1,4)\cong\mathbb{CP}^1$ is the moduli space of semistable rank 2 parabolic bundles over $\mathbb{CP}^1$ with trivial determinant bundle and 4 marked points. We also prove:

Theorem 2.1.4. *There is a canonical open embedding* $\mathcal{P}_M(\mathbb{CP}^1,m,n)\to M^s(\mathbb{CP}^1,m+n)$.

Corollary 2.1.5. *There is a canonical open embedding* $\mathcal{P}_M(\mathbb{CP}^1,2r)\to M^s(\mathbb{CP}^1,2r+3)$.

Here the complex manifold $M^s(\mathbb{CP}^1,m+n)$ is the moduli space of stable rank 2 parabolic bundles over $\mathbb{CP}^1$ with trivial determinant bundle and $m+n$ marked points. For $r=1,2$ we have verified that the embedding of $\mathcal{P}_M(\mathbb{CP}^1,2r)$ into $M^s(\mathbb{CP}^1,2r+3)$ agrees with a (noncanonical) embedding due to Woodward of the Seidel-Smith space $\mathcal{Y}(S^2,2r)$ into $M^s(\mathbb{CP}^1,2r+3)$, and we conjecture that the agreement holds for all r.

We next proceed to the case of elliptic curves. We show that our reinterpretation $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ of the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ has two natural generalizations to the case of an elliptic curve X, namely $\mathcal{P}_M(X, 1, 2r)$ and $\mathcal{P}_M(X, 3, 2r)$. We prove an elliptic-curve analog to Theorem 2.1.3:

Theorem 2.1.6. *There is a canonical isomorphism* $\mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$.

Here the complex manifold $M^{ss}(X) \cong \mathbb{CP}^1$ is the moduli space of semistable rank 2 vector bundles over an elliptic curve X with trivial determinant bundle. Comparing Theorems 2.1.3 and 2.1.6, we see that $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$, whereas $\mathcal{P}_M^{tot}(X, 1, n)$ is (noncanonically) isomorphic to $(\mathbb{CP}^1)^{n+1}$. The extra factor of $\mathbb{CP}^1$ for $\mathcal{P}_M^{tot}(X, 1, n)$ can be understood from Theorem 2.1.2, which states that $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n) = \mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(\mathbb{CP}^1, 3)$ and $\mathcal{P}_M^{tot}(X, 1, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(X, 1)$. But $M^s(\mathbb{CP}^1, 3)$ is a single point, whereas $M^s(X, 1)$ is isomorphic to $\mathbb{CP}^1$. We use Theorem 2.1.6 to explicitly compute $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$:

Theorem 2.1.7. *The space* $\mathcal{P}_M(X, 1, n)$ *for* $n = 0, 1, 2$ *is given by*

$$\mathcal{P}_M(X,1,0) = \mathbb{CP}^1, \quad \mathcal{P}_M(X,1,1) = (\mathbb{CP}^1)^2 - g(X), \quad \mathcal{P}_M(X,1,2) = (\mathbb{CP}^1)^3 - f(X),$$

where $g : X \to (\mathbb{CP}^1)^2$ *and* $f : X \to (\mathbb{CP}^1)^3$ *are holomorphic embeddings defined in Sections 2.6.5.2 and 2.6.5.3.*

We also generalize the embedding result of Theorem 2.1.4 to the case of elliptic curves:

Theorem 2.1.8. *There is a canonical open embedding* $\mathcal{P}_M(X, m, n) \to M^s(X, m + n)$.

Here the complex manifold $M^s(X, m+n)$ is the moduli space of stable rank 2 parabolic bundles on X with trivial determinant bundle and $m + n$ marked points. In order to prove Theorems 2.1.6, 2.1.7, and 2.1.8, we construct a list of all possible Hecke modifications of all possible rank 2 vector bundles on an elliptic curve X.

In Section 2.7, we discuss possible applications of our results to the problem of generalizing symplectic Khovanov homology to lens spaces. We observe that the embedding results of

Theorems 2.1.4 and 2.1.8 could be related to a conjectural spectral sequence from symplectic Khovanov homology to symplectic instanton homology, which would generalize the spectral sequence due to Kronheimer and Mrowka from Khovanov homology to singular instanton homology [KM14]. Based on such considerations, we make the following conjectures:

Conjecture 2.1.9. *The space $\mathcal{P}_M(C, 3, 2r)$ is the correct generalization of the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ to a curve C of arbitrary genus.*

Conjecture 2.1.10. *Given a curve C of arbitrary genus, there is a canonical open embedding $\mathcal{P}_M(C, m, n) \to M^s(C, m+n)$.*

2.2 Vector bundles

Here we briefly review some results on holomorphic vector bundles and their moduli spaces that we will use throughout the dissertation. Useful references on vector bundles are [Le 97, Sch15, Big, Tu93].

Definition 2.2.1. The *slope* of a holomorphic vector bundle E over a curve C is $\operatorname{slope} E := (\deg E)/(\operatorname{rank} E) \in \mathbb{Q}$.

Definition 2.2.2. A holomorphic vector bundle E over a curve C is *stable* if $\operatorname{slope} F < \operatorname{slope} E$ for any proper subbundle $F \subset E$, *semistable* if $\operatorname{slope} F \leq \operatorname{slope} E$ for any proper subbundle $F \subset E$, *strictly semistable* if it is semistable but not stable, and *unstable* if there is a proper subbundle $F \subset E$ such that $\operatorname{slope} F > \operatorname{slope} E$.

If E is a stable vector bundle, then $\operatorname{Aut}(E) = \mathbb{C}^\times$ consists only of trivial automorphisms that scale the fibers by a constant factor.

Definition 2.2.3. Given a semistable vector bundle E, a *Jordan-Hölder filtration* of E is a filtration

$$F_0 = 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$$

of E by subbundles $F_i \subset E$ for $i = 0, \cdots, n$ such that the composition factors F_i/F_{i-1} are stable and $\operatorname{slope} F_i/F_{i-1} = \operatorname{slope} E$ for $i = 1, \cdots, n$.

Every semistable vector bundle E admits a Jordan-Hölder filtration. The filtration is not unique, but the composition factors F_i/F_{i-1} for $i = 1, \cdots, n$ are independent (up to permutation) of the choice of filtration.

Definition 2.2.4. Given a semistable holomorphic vector bundle E over a curve C, the *associated graded* vector bundle $\operatorname{gr} E$ is defined to be

$$\operatorname{gr} E = \bigoplus_{i=1}^{n} F_i/F_{i-1},$$

where $F_0 = 0 \subset F_1 \subset \cdots \subset F_n = E$ is a Jordan-Hölder filtration of E.

The bundle $\operatorname{gr} E$ is independent (up to isomorphism) of the choice of filtration, and $\operatorname{slope}(\operatorname{gr} E) = \operatorname{slope} E$.

Definition 2.2.5. Two semistable vector bundles are said to be *S-equivalent* if their associated graded bundles are isomorphic.

Example 2.2.6. In Section 2.6.1 we define a strictly semistable rank 2 vector bundle F_2 and a stable rank 2 vector bundle $G_2(p)$ over an elliptic curve X. A Jordan-Hölder filtration of F_2 is $\mathcal{O} \subset F_2$, and the associated graded bundle is $\operatorname{gr} F_2 = \mathcal{O} \oplus \mathcal{O}$. It follows that F_2 and $\mathcal{O} \oplus \mathcal{O}$ are S-equivalent. A Jordan-Hölder filtration of $G_2(p)$ is just $G_2(p)$, and the associated graded bundle is $\operatorname{gr} G_2(p) = G_2(p)$.

Isomorphic bundles are S-equivalent. For rational curves, S-equivalent bundles are isomorphic, but this is not true in general. For example, on an elliptic curve the bundles F_2 and $\mathcal{O} \oplus \mathcal{O}$ are S-equivalent but not isomorphic.

For many applications we will want to quantify the degree to which a vector bundle is unstable. To do so, we introduce some terminology specific to this dissertation:

Definition 2.2.7. Given a rank 2 holomorphic vector bundle E over a curve C, we define the *instability degree* of E to be $\deg L - \deg E/L$, where $L \subset E$ is a line subbundle of maximal degree.

The degree of the proper subbundles of a vector bundle E on a curve C is bounded above (see for example [Sch15] Lemma 3.21), so the notion of instability degree is well-defined. The instability degree is positive for unstable bundles, 0 for strictly semistable bundles, and negative for stable bundles.

Definition 2.2.8. We define $M^{ss}(C)$ (respectively $M^{s}(C)$) to be the moduli space of semistable (respectively stable) rank 2 holomorphic vector bundles over curve C with trivial determinant bundle, mod S-equivalence. This space is defined in [Ses67]; see also [MO12].

Remark 2.2.9. An alternative way of interpreting $M^{ss}(C)$ is as the space of flat $SU(2)$-connections on a trivial rank 2 complex vector bundle $E \to C$, mod gauge transformations. Yet another way of interpreting the space $M^{ss}(C)$ is as the character variety $R(C)$ of conjugacy classes of group homomorphisms $\pi_1(C) \to SU(2)$.

The moduli space $M^{s}(C)$ has the structure of a complex manifold of dimension $3(g-1)$, where g is the genus of the curve C. The space $M^{s}(C)$ carries a canonical symplectic form, which is obtained by interpreting $M^{s}(C)$ as a Hamiltonian reduction of a space of $SU(2)$-connections.

Example 2.2.10. For rational curves, the bundle $\mathcal{O} \oplus \mathcal{O}$ is the unique semistable rank 2 bundle with trivial determinant bundle, and there are no stable rank 2 bundles, so

$$M^{ss}(\mathbb{CP}^1) = \{pt\} = \{[\mathcal{O} \oplus \mathcal{O}]\}, \qquad\qquad M^{s}(\mathbb{CP}^1) = \varnothing.$$

Example 2.2.11. For an elliptic curve X, semistable rank 2 bundles with trivial determinant bundle have the form $L \oplus L^{-1}$, where L is a degree 0 line bundle, or $F_2 \otimes L_i$, where L_i for $i = 1, \cdots, 4$ are the four 2-torsion line bundles. The bundles $L_i \oplus L_i$ and $F_2 \otimes L_i$ are S-equivalent. The bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ are isomorphic, hence S-equivalent. There are no stable rank 2 bundles with trivial determinant bundle. As shown in [Tu93], we have that

$$M^{ss}(X) = \{[L \oplus L^{-1}] \mid [L] \in \mathrm{Jac}(X)\} = \mathbb{CP}^1, \qquad\qquad M^{s}(X) = \varnothing.$$

2.3 Parabolic bundles

Here we briefly review some results on parabolic bundles and their moduli spaces that we will use throughout the dissertation. Useful references on parabolic bundles are [MS80, Nag17].

2.3.1 Definition of a parabolic bundle

The concept of a parabolic bundle was introduced in [MS80]:

Definition 2.3.1. A *parabolic bundle* of rank r on a curve C consists of following data:

1. A rank r holomorphic vector bundle $\pi_E : E \to C$.

2. Distinct marked points $(p_1, \cdots, p_n) \in C^n$.

3. For each marked point p_i, a flag of vector spaces $E^j_{p_i}$ in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point p_i:

$$E^0_{p_i} = 0 \subset E^1_{p_i} \subset E^2_{p_i} \subset \cdots \subset E^{s_i}_{p_i} = E_{p_i}.$$

4. For each marked point p_i, a strictly decreasing list of *weights* $\lambda^j_{p_i} \in \mathbb{R}$:

$$\lambda^1_{p_i} > \lambda^2_{p_i} > \cdots > \lambda^{s_i}_{p_i}.$$

We refer the data of the marked points, the flags, and the weights as a *parabolic structure* on E. We refer to the data of just the marked points and flags, without the weights, as a *quasi-parabolic structure* on E. We define the *multiplicity* of the weight $\lambda^j_{p_i}$ to be $m^j_{p_i} := \dim(E^j_{p_i}) - \dim(E^{j-1}_{p_i})$. The definition of a parabolic bundle given in [MS80] differs slightly from our definition, in that the marked points are unordered and the weights are required to lie in the range $[0, 1)$.

Definition 2.3.2. Two parabolic bundles with underlying vector bundles E and F are *isomorphic* if the marked points and weights for the two bundles are the same, and there is a bundle isomorphism $\alpha : E \to F$ that carries each flag of E to the corresponding flag of F; that is, $\alpha(E^j_{p_i}) = F^j_{p_i}$ for $j = 1, \cdots, s_i$ and $i = 1, \cdots, n$.

Definition 2.3.3. The *rank* of a parabolic bundle is the rank of its underlying vector bundle.

Definition 2.3.4. The *parabolic degree* and *parabolic slope* of a parabolic bundle $\mathcal{E}$ with underlying vector bundle E are defined to be

$$\operatorname{pdeg}(\mathcal{E}) = \deg E + \sum_{i=1}^{n} \sum_{j=1}^{s_i} m_{p_i}^j \lambda_{p_i}^j \in \mathbb{R}, \qquad \operatorname{pslope}(\mathcal{E}) = (\operatorname{pdeg} \mathcal{E})/(\operatorname{rank} \mathcal{E}) \in \mathbb{Q}.$$

We will not need the full generality of the concept of a parabolic bundle. Rather, we will consider only parabolic line bundles and rank 2 parabolic bundles of a certain restricted form.

First we consider parabolic line bundles. For such bundles there is no flag data, so the parabolic structure is specified by a list of marked points $p_1, \cdots, p_n$ and a list of weights $\lambda_{p_1}^1, \cdots, \lambda_{p_n}^1$. We fix a parameter $\mu > 0$ and restrict to the case $\lambda_{p_i}^1 \in \{\pm\mu\}$ for $i = 1, \cdots, n$. A parabolic line bundle of this form thus consists of the data $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$, where $\pi_L : L \to C$ is a holomorphic line bundle and $\sigma_{p_i} \in \{\pm 1\}$. The parabolic degree and parabolic slope of a parabolic line bundle $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ are given by

$$\operatorname{pdeg}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) = \operatorname{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) = \deg L + \mu \sum_{i=1}^{n} \sigma_{p_i}.$$

Next we consider rank 2 parabolic bundles. We fix a parameter $\mu > 0$ and restrict to the case $s_i = 2$, $m_{p_i}^1 = m_{p_i}^2 = 1$, and $\lambda_{p_i}^1 = -\lambda_{p_i}^2 = \mu$ for $i = 1, \cdots, n$. A rank 2 parabolic bundle of this form thus consists of the data $(E, \ell_{p_1}, \cdots, \ell_{p_n})$, where $\pi_E : E \to C$ is a rank 2 holomorphic vector bundle and $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ is a line in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point p_i for $i = 1, \cdots, n$. The parabolic slope and parabolic degree of a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ are given by

$$\operatorname{pdeg}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \deg E, \qquad \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} E.$$

2.3.2 Stable, semistable, and unstable parabolic bundles

Consider a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and a line subbundle $L \subset E$. There are induced parabolic structures on the line bundles L and E/L given by $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and

$(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$, where

$$\sigma_{p_i} = \begin{cases} +1 & \text{if } L_{p_i} = \ell_{p_i}, \\ -1 & \text{if } L_{p_i} \neq \ell_{p_i}. \end{cases}$$

Definition 2.3.5. Given a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and a line subbundle $L \subset E$, we say that the induced parabolic bundle $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ is a *parabolic subbundle* of $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and the induced parabolic bundle $(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$ is a *parabolic quotient bundle* of $(E, \ell_{p_1}, \cdots, \ell_{p_n})$.

Definition 2.3.6. A rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is said to be *decomposable* if there exists a decomposition $E = L \oplus L'$ for line bundles L and L' such that $\ell_{p_i} \in \{L_{p_i}, L'_{p_i}\}$ for $i = 1, \cdots, n$. For a rank 2 decomposable parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ we write

$$(E, \ell_{p_1}, \cdots, \ell_{p_n}) = (L, \sigma_{p_1}, \cdots, \sigma_{p_n}) \oplus (L', \sigma'_{p_1}, \cdots, \sigma'_{p_n}),$$

where $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and $(L', \sigma'_{p_1}, \cdots, \sigma'_{p_n})$ are the induced parabolic structures on L and L'.

Definition 2.3.7. A rank 2 parabolic bundle is *stable* if its parabolic slope is strictly greater than the parabolic slope of any of its proper parabolic subbundles, *semistable* if its parabolic slope is greater than or equal than the parabolic slope of any of its proper parabolic subbundles, *strictly semistable* if it is semistable but not stable, and *unstable* if it has a proper parabolic subbundle of strictly greater slope.

If $\mathcal{E}$ is a stable parabolic bundle, then $\mathrm{Aut}(\mathcal{E}) = \mathbb{C}^\times$ consists only of trivial automorphisms that scale the fibers of the underlying vector bundle by a constant factor.

Theorem 2.3.8. *If the rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable and $\mu < 1/2n$, then E is semistable.*

Proof. We will prove the contrapositive, so assume that E is unstable. Then there is a line subbundle $L \subset E$ such that $\operatorname{slope} L > \operatorname{slope} E$. Consider the parabolic structure $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ induced on L by $(E, \ell_{p_1}, \cdots, \ell_{p_n})$. We have that

$$\operatorname{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) - \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} L + \mu \sum_{i=1}^{n} \sigma_{p_i} - \operatorname{slope} E. \tag{2.1}$$

Since slope L is an integer, slope E is an integer or half-integer, and slope $L >$ slope E, it follows that slope $L-$slope $E \geq 1/2$. From equation (2.1) and the assumption that $\mu < 1/2n$, it follows that

$$\text{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) - \text{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) \geq 1/2 - n\mu > 0,$$

so $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. □

Throughout this dissertation we will always assume $\mu \ll 1$, by which we mean that μ is always chosen to be sufficiently small such that Theorem 2.3.8 holds under whatever circumstances we are considering. We now introduce some terminology specific to this dissertation:

Definition 2.3.9. Given a rank 2 vector bundle E over a curve C and a point $p \in C$, we say that a line $\ell_p \in \mathbb{P}(E_p)$ in the fiber E_p over p is *bad* if there is a line subbundle $L \subset E$ of maximal degree such that $L_p = \ell_p$, and *good* otherwise.

Definition 2.3.10. Given a rank 2 vector bundle E over a curve C and points $p_1, \cdots, p_n \in C$, we say that lines $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ for $i = 1, \cdots, n$ are *bad in the same direction* if there is a line subbundle $L \subset E$ of maximal degree such that $L_{p_i} = \ell_{p_i}$ for $i = 1, \cdots, n$.

Definition 2.3.11. Given a semistable rank 2 vector bundle E over a curve C and a point $p \in C$, we say that a line $\ell_p \in \mathbb{P}(E_p)$ is *destabilizing* if there is a line subbundle $L \subset E$ such that $\deg L \geq \deg E$ and $L_p = \ell_p$.

Definition 2.3.12. Given a semistable rank 2 vector bundle E over a curve C and points $p_1, \cdots, p_n \in C$, we say that lines $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ for $i = 1, \cdots, n$ are *destabilizing in the same direction* if there is a line subbundle $L \subset E$ such that $\deg L \geq \deg E$ and $L_{p_i} = \ell_{p_i}$ for $i = 1, \cdots, n$.

Example 2.3.13. For the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ over $\mathbb{CP}^1$, all lines are bad and destabilizing.

Consider a rank 2 vector bundle E over a curve C. If E is unstable, then Theorem 2.3.8 implies that the parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. If E is a semistable, then the stability of the parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ can be characterized as follows:

Theorem 2.3.14. *Consider a rank 2 parabolic bundle of the form $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ with E semistable. Let m be the maximum number of lines that are destabilizing in the same direction. Such a parabolic bundle is stable if and only if $m < n/2$, semistable if and only if $m \leq n/2$, and unstable if and only if $m > n/2$. In particular, if n is odd then stability and semistability are equivalent.*

Example 2.3.15. As a special case of Theorem 2.3.14, consider parabolic bundles $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$ with underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$. We can globally trivialize E and identify all the fibers with $\mathbb{C}^2$. All lines of E are destabilizing, and lines are destabilizing in the same direction if and only if they are equal under the global trivialization. Let m denote the maximum number of lines ℓ_{p_i} equal to any given line in $\mathbb{CP}^1$. From Theorem 2.3.14, we find that $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is stable if $m < n/2$, semistable if $m \leq n/2$, and unstable if $m > n/2$. For example, (E, ℓ_{p_1}) is unstable, $(E, \ell_{p_1}, \ell_{p_2})$ is strictly semistable if the lines are distinct and unstable otherwise, and $(E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3})$ is stable if the lines are distinct and unstable otherwise.

2.3.3 S-equivalent semistable parabolic bundles

There is Jordan-Hölder theorem for parabolic bundles, which asserts that any semistable parabolic bundle of parabolic degree 0 has a filtration in which quotients of successive parabolic bundles (composition factors) in the filtration are stable with parabolic slope 0 (see [MS80] Remark 1.16). The filtration is not unique, but the composition factors are unique up to permutation. It follows that one can define an associated graded bundle of a semistable parabolic bundle of parabolic degree 0 that is unique up to isomorphism.

We will need the concept of an associated graded parabolic bundle only for the case of semistable rank 2 parabolic bundles. If such a parabolic bundle $\mathcal{E}$ is stable, then its associated graded parabolic bundle $\operatorname{gr}\mathcal{E}$ is just $\mathcal{E}$. Now consider a strictly semistable parabolic bundle $\mathcal{E} = (E, \ell_{p_1}, \cdots, \ell_{p_n})$. Under our standard assumption that $\mu \ll 1$, it follows from Theorem 2.3.8 that E is semistable. The associated graded parabolic bundle $\operatorname{gr}\mathcal{E}$ is given by

$$\operatorname{gr}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = (L, \sigma_{p_1}, \cdots, \sigma_{p_n}) \oplus (E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n}),$$

where $L \subset E$ is a line subbundle such that $\operatorname{slope} L = \operatorname{slope} E$, and $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and $(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$ are the induced parabolic structures on L and E/L. Note that

$$\operatorname{pslope}(\operatorname{gr}(E, \ell_{p_1}, \cdots, \ell_{p_n})) = \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} E.$$

Definition 2.3.16. We say that two semistable rank 2 parabolic bundles are S-*equivalent* if their associated graded bundles are isomorphic.

Isomorphic parabolic bundles are S-equivalent. Here we give an example to show that the converse does not always hold:

Example 2.3.17. Consider parabolic bundles over $\mathbb{CP}^1$ with underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$. We can globally trivialize E and identify all the fibers with $\mathbb{C}^2$. Let A, B, and C be distinct lines in $\mathbb{CP}^1$, and consider the two parabolic bundles

$$\begin{aligned}\mathcal{E} &:= (E,\, \ell_{p_1} = A,\, \ell_{p_2} = A,\, \ell_{p_3} = B,\, \ell_{p_4} = C),\\ \mathcal{E}' &:= (E,\, \ell'_{p_1} = B,\, \ell'_{p_2} = C,\, \ell'_{p_3} = A,\, \ell'_{p_4} = A).\end{aligned}$$

Let $L \cong \mathcal{O}$ be the line subbundle of E such that $L_p = A$ for any point $p \in \mathbb{CP}^1$. The bundles $\mathcal{E}$ and $\mathcal{E}'$ are not isomorphic but are S-equivalent, since the associated graded bundles of both bundles are isomorphic to

$$(L,\, \sigma_{p_1} = 1,\, \sigma_{p_2} = 1,\, \sigma_{p_3} = -1,\, \sigma_{p_4} = -1) \oplus (E/L,\, \sigma_{p_1} = -1,\, \sigma_{p_2} = -1,\, \sigma_{p_3} = 1,\, \sigma_{p_4} = 1).$$

2.3.4 Moduli spaces of rank 2 parabolic bundles

Definition 2.3.18. We define $M^{ss}(C, n)$ (respectively $M^s(C, n)$), to be the moduli space of semistable (respectively stable) rank 2 parabolic bundles of the form $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ such that E has trivial determinant bundle, mod S-equivalence. In particular, $M^{ss}(C, 0) = M^{ss}(C)$ and $M^s(C, 0) = M^s(C)$. As always, we assume that $\mu \ll 1$. This space is defined in [MS80]; see also [Bho89].

Remark 2.3.19. An alternative way of interpreting $M^{ss}(C, n)$ is as the space of flat $SU(2)$-connections on a trivial rank 2 complex vector bundle $E \to C - \{p_1, \cdots, p_n\}$, where the holonomy around each puncture point p_i is conjugate to $\mathrm{diag}(e^{2\pi i\mu}, e^{-2\pi i\mu})$, mod $SU(2)$ gauge transformations. Yet another way of interpreting the space $M^{ss}(C,n)$ is as the character variety $R(C,n)$ of conjugacy classes of group homomorphisms $\pi_1(C-\{p_1,\cdots,p_n\}) \to SU(2)$ that take loops around the marked points to matrices conjugate to $\mathrm{diag}(e^{2\pi i\mu}, e^{-2\pi i\mu})$. Note that $\mu = 1/4$ corresponds to the traceless character variety.

For rational μ, the moduli space $M^s(C,n)$ has the structure of a complex manifold of dimension $3(g-1)+n$, where g is the genus of the curve C, and $M^s(C,n)$ is compact for n odd. The space $M^s(C,n)$ carries a canonical symplectic form, which is obtained by viewing $M^s(C,n)$ as a Hamiltonian reduction of a space of $SU(2)$-connections with prescribed singularities.

Example 2.3.20. Let $C = \mathbb{CP}^1$ be a rational curve. If we fix $n \leq 3$, then all rank 2 parabolic bundles of the form $(\mathcal{O}\oplus\mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ for which all the lines are distinct are isomorphic. From this fact, together with the results of Example 2.3.15, we find that

$$\begin{aligned} &M^{ss}(\mathbb{CP}^1,0)=\{pt\}, \quad M^{ss}(\mathbb{CP}^1,1)=\varnothing, \quad M^{ss}(\mathbb{CP}^1,2)=\{pt\}, \quad M^{ss}(\mathbb{CP}^1,3)=\{pt\},\\ &M^{s}(\mathbb{CP}^1,0)=\varnothing, \qquad M^{s}(\mathbb{CP}^1,1)=\varnothing, \quad M^{s}(\mathbb{CP}^1,2)=\varnothing, \qquad M^{s}(\mathbb{CP}^1,3)=\{pt\}. \end{aligned}$$

Using the cross-ratio and considerations of S-equivalence as described in Example 2.3.17, one can show

$$M^{ss}(\mathbb{CP}^1,4)=\mathbb{CP}^1, \qquad\qquad M^{s}(\mathbb{CP}^1,4)=\mathbb{CP}^1-\{3\text{ points}\}.$$

Example 2.3.21. Let X be an elliptic curve. From Corollary 2.6.25, Example 2.2.11, and Theorem 2.3.14, we have that

$$M^{ss}(X,0)=\mathbb{CP}^1, \qquad M^{s}(X,0)=\varnothing, \qquad M^{ss}(X,1)=\mathbb{CP}^1, \qquad M^{s}(X,1)=\mathbb{CP}^1.$$

In [Var16] it is shown that

$$M^{ss}(X,2)=(\mathbb{CP}^1)^2, \qquad\qquad M^{s}(X,2)=(\mathbb{CP}^1)^2-g(X),$$

where $g : X \to (\mathbb{CP}^1)^2$ is a holomorphic embedding. We also derive this result in Section 2.6.5.2.

Remark 2.3.22. Throughout this dissertation we assume $\mu \ll 1$, but for some applications one wants to take $\mu = 1/4$ in order to interpret $M^{ss}(C, n)$ as a traceless character variety, as described in Remark 2.3.19. In general $M^{ss}(C, n)$ depends on μ; for example, for $0 \leq n \leq 4$ the space $M^{ss}(\mathbb{CP}^1, n)$ is the same for $\mu \ll 1/4$ and $\mu = 1/4$, but, as shown in [Sei03], for $n = 5$ we have

$$M^{ss}(\mathbb{CP}^1, 5) = \begin{cases} \mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 & \text{for } \mu \ll 1, \\ \mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2 & \text{for } \mu = 1/4. \end{cases}$$

The dependence of $M^{ss}(C, n)$ on μ is discussed in [BH95].

2.4 Hecke modifications

2.4.1 Hecke modifications at a single point

A fundamental concept for us is the notion of a Hecke modification of a rank 2 holomorphic vector bundle. This notion is described in [Kam11, KW07]. Here we consider the case of a single Hecke modification.

Definition 2.4.1. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *Hecke modification* $E \xleftarrow[p]{\alpha} F$ of E at a point $p \in C$ is a rank 2 holomorphic vector bundle $\pi_F : F \to C$ together with a bundle map $\alpha : F \to E$ that satisfies the following two conditions:

1. The induced maps on fibers $\alpha_q : F_q \to E_q$ are isomorphisms for all points $q \in C$ such that $q \neq p$.

2. We also impose a condition on the behavior of α near p. We require that there is an open neighborhood $U \subset C$ of p, local coordinates $\xi : U \to V$ for $V \subset \mathbb{C}$ such that $\xi(p) = 0$, and local trivializations $\psi_E : \pi_E^{-1}(U) \to U \times \mathbb{C}^2$ and $\psi_F : \pi_F^{-1}(U) \to U \times \mathbb{C}^2$

of E and F over U such that the following diagram commutes:

$$\begin{tikzcd}
\pi_E^{-1}(U) \arrow[d, "\psi_E", "\cong"'] & \pi_F^{-1}(U) \arrow[l, "\alpha"'] \arrow[d, "\psi_F", "\cong"'] \\
U \times \mathbb{C}^2 & U \times \mathbb{C}^2, \arrow[l]
\end{tikzcd}$$

where the bottom horizontal arrow is

$$(\psi_E \circ \alpha \circ \psi_F^{-1})(q, v) = (q, \bar{\alpha}(\xi(q))v)$$

and $\bar{\alpha} : V \to M(2, \mathbb{C})$ has the form

$$\bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

It follows directly from Definition 2.4.1 that $\det F = (\det E) \otimes \mathcal{O}(-p)$ and $\deg F = \deg E - 1$. There is a natural notion of equivalence of Hecke modifications:

Definition 2.4.2. We say that two Hecke modifications $E \xleftarrow[p]{\alpha} F$ and $E \xleftarrow[p]{\alpha'} F'$ of E at a point $p \in C$ are *equivalent* if there is an isomorphism $\phi : F \to F'$ such that $\alpha = \alpha' \circ \phi$.

Definition 2.4.3. We define the *total space of Hecke modifications* $\mathcal{H}^{tot}(C, E; p)$ to be the set of equivalence classes of Hecke modifications of a rank 2 vector bundle $\pi_E : E \to C$ at a point $p \in C$.

As is well-known, the set $\mathcal{H}^{tot}(C, E; p)$ naturally has the structure of a complex manifold that is (noncanonically) isomorphic to $\mathbb{CP}^1$. A canonical version of this statement is:

Theorem 2.4.4. *There is a canonical isomorphism* $\mathcal{H}^{tot}(C, E; p) \to \mathbb{P}(E_p)$, $[E \xleftarrow[p]{\alpha} F] \mapsto \mathrm{im}(\alpha_p : F_p \to E_p)$.

It is also useful to think about Hecke modifications in terms of sheaves of sections. Consider a rank 2 vector bundle E and a line $\ell_p \in \mathbb{P}(E_p)$. Let $\mathcal{E}$ be the sheaf of sections of E, and define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open set $U \subset C$ is given by

$$\mathcal{F}(U) = \{s \in \mathcal{E}(U) \mid p \in U \implies s(p) \in \ell_p\}.$$

Define F to be the vector bundle whose sheaf of sections is $\mathcal{F}$, and define $\alpha : F \to E$ to be the bundle map corresponding to the inclusion of sheaves $\mathcal{F} \to \mathcal{E}$. Then $[E \xleftarrow[p]{\alpha} F] \in \mathcal{H}^{tot}(C, E; p)$ corresponds to $\ell_p \in \mathbb{P}(E_p)$ under the isomorphism described in Theorem 2.4.4. We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

where $\mathbb{C}_p$ is a skyscraper sheaf supported at the point p. It is important to note, however, that the usual notion of equivalence of extensions differs from the notion of equivalence of Hecke modifications given in Definition 2.4.2.

2.4.2 Sequences of Hecke modifications

We would now like to generalize the notion of a Hecke modification of a vector bundle at a single point $p \in C$ to the notion of a sequence of Hecke modifications at distinct points $(p_1, \cdots, p_n) \in C^n$.

Definition 2.4.5. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *sequence of Hecke modifications* $E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ of E at distinct points $(p_1, p_2, \cdots, p_n) \in C^n$ is a collection of rank 2 holomorphic vector bundles $\pi_{E_i} : E_i \to C$ and Hecke modifications $E_{i-1} \xleftarrow[p_i]{\alpha_i} E_i$ for $i = 1, 2, \cdots, n$, where $E_0 := E$.

Definition 2.4.6. Two sequences of Hecke modifications $E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ and $E \xleftarrow[p_1]{\alpha'_1} E'_1 \xleftarrow[p_2]{\alpha'_2} \cdots \xleftarrow[p_n]{\alpha'_n} E'_n$ are *equivalent* if there are isomorphisms $\phi_i : E_i \to E'_i$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[d, "="'] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\cong"', "\phi_1"] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\cong"', "\phi_2"] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\cong"', "\phi_n"] \\
E & E'_1 \arrow[l, "\alpha'_1"'] & E'_2 \arrow[l, "\alpha'_2"'] & \cdots \arrow[l, "\alpha'_3"'] & E'_n. \arrow[l, "\alpha'_n"']
\end{tikzcd}$$

Definition 2.4.7. We define the *total space of Hecke modifications* $\mathcal{H}^{tot}(C, E; p_1, \cdots, p_n)$ to be the set of equivalence classes of sequences of Hecke modifications of the rank 2 vector bundle $\pi_E : E \to C$ at points $(p_1, \cdots, p_n) \in C^n$. For simplicity, we will often suppress the dependence on $p_1, \cdots, p_n$ and denote this space as $\mathcal{H}^{tot}(C, E, n)$.

Definition 2.4.8. We say that an isomorphism of vector bundles $\phi : E \to E'$ is an *isomorphism of equivalence classes of sequences of Hecke modifications* $\phi : [E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] \to [E' \xleftarrow[p_1]{\alpha'_1} E'_1 \xleftarrow[p_2]{\alpha'_2} \cdots \xleftarrow[p_n]{\alpha'_n} E'_n]$ if

$$[E' \xleftarrow[p_1]{\beta_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] = [E' \xleftarrow[p_1]{\alpha'_1} E'_1 \xleftarrow[p_2]{\alpha'_2} \cdots \xleftarrow[p_n]{\alpha'_n} E'_n],$$

where $\beta_1 := \phi \circ \alpha_1$, or equivalently, if there are isomorphisms $\phi_i : E_i \to E'_i$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[d, "\cong"', "\phi"] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\cong"', "\phi_1"] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\cong"', "\phi_2"] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\cong"', "\phi_n"] \\
E' & E'_1 \arrow[l, "\alpha'_1"'] & E'_2 \arrow[l, "\alpha'_2"'] & \cdots \arrow[l, "\alpha'_3"'] & E'_n. \arrow[l, "\alpha'_n"']
\end{tikzcd}$$

In what follows, it will be useful to reinterpret equivalence classes of sequences of Hecke modifications in terms of parabolic bundles. The relevant background material on parabolic bundles is discussed in Section 2.3. For our purposes here, a rank 2 parabolic bundle over a curve C consists of a rank 2 holomorphic vector bundle $\pi_E : E \to C$, a parameter $\mu > 0$, called the *weight*, and a choice of line $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point $p_i \in C$ for a finite number of distinct points $(p_1, \cdots, p_n) \in C^n$. The data of just the marked points and lines, without the weight, is referred to as a *quasi-parabolic structure* on E. The additional data of the weight allows us to define the notions of stable, semistable, and unstable parabolic bundles.

Definition 2.4.9. We define $\mathcal{P}^{tot}(C, E; p_1, \cdots, p_n) = \mathbb{P}(E_{p_1}) \times \cdots \times \mathbb{P}(E_{p_n})$ to be the set of all quasi-parabolic structures with marked points $(p_1, \cdots, p_n) \in C^n$ on a rank 2 holomorphic vector bundle $\pi_E : E \to C$. For simplicity, we will often suppress the dependence on $p_1, \cdots, p_n$ and denote this space as $\mathcal{P}^{tot}(C, E, n)$.

Since $\mathbb{P}(E_{p_i})$ is (noncanonically) isomorphic to $\mathbb{CP}^1$, the set $\mathcal{P}^{tot}(C, E; p_1, \cdots, p_n)$ naturally has the structure of a complex manifold that is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$. We have the following generalization of Theorem 2.4.4, which allows us to reinterpret Hecke modifications in terms of parabolic bundles:

Theorem 2.4.10. *There is a canonical isomorphism* $\mathcal{H}^{tot}(C, E; p_1, \cdots, p_n) \to \mathcal{P}^{tot}(C, E; p_1, \cdots, p_n)$ *given by*

$$[E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] \mapsto (E, \ell_{p_1}, \cdots, \ell_{p_n}),$$

where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \cdots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.

Under our reinterpretation of Hecke modifications in terms of parabolic bundles, an isomorphism of equivalence classes of sequences of Hecke modifications corresponds to an isomorphism of parabolic bundles:

Definition 2.4.11. We say that an isomorphism of vector bundles $\phi : E \to E'$ is an *isomorphism of parabolic bundles* $\phi : (E, \ell_{p_1}, \cdots, \ell_{p_n}) \to (E', \ell'_{p_1}, \cdots, \ell'_{p_n})$ if $\phi(\ell_{p_i}) = \ell'_{p_i}$ for $i = 1, \cdots, n$.

Theorem 2.4.12. *Two equivalence classes of sequences of Hecke modifications are isomorphic if and only if their corresponding parabolic bundles are isomorphic.*

Proof. This is a direct consequence of Theorem 2.4.10 and Definitions 2.4.8 and 2.4.11. □

For many applications, given an equivalence class $[E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n]$ of sequences of Hecke modifications we will be interested only in the isomorphism class of the terminal vector bundle E_n, and it is useful to have a means of extracting this information from the corresponding parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$:

Definition 2.4.13. Let $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ be a parabolic bundle over a curve C. We define the *Hecke transform* $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ of E to be the vector bundle F that is constructed as follows. Let $\mathcal{E}$ be the sheaf of sections of E. Define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open set $U \subset C$ is given by

$$\mathcal{F}(U) = \{s \in \mathcal{E}(U) \mid p_i \in U \implies s(p_i) \in \ell_{p_i} \text{ for } i = 1, \cdots, n\}.$$

Now define F to be the vector bundle whose sheaf of sections is $\mathcal{F}$.

In particular, $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is isomorphic to E_n. We will often want to pick out an open subset of $\mathcal{P}^{tot}(C, E, n)$ by using the Hecke transform to impose a semistability condition:

Definition 2.4.14. Given a rank 2 holomorphic vector bundle E over a curve C and distinct points $(p_1, \cdots, p_n) \in C^n$, we define $\mathcal{P}(C, E, n)$ to be the subset of $\mathcal{P}^{tot}(C, E, n)$ consisting of parabolic bundles $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ such that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable:

$$\mathcal{P}(C, E, n) = \{(E, \ell_{p_1}, \cdots, \ell_{p_n}) \in \mathcal{P}^{tot}(C, E, n) \mid H(E, \ell_{p_1}, \cdots, \ell_{p_n}) \text{ is semistable}\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}(C, m, n)$ on $p_1, \cdots, p_n$ in the notation.

Theorem 2.4.15. *The set $\mathcal{P}(C, E, n)$ is an open submanifold of $\mathcal{P}^{tot}(C, E, n)$.*

Proof. This follows from the fact that semistability is an open condition. □

We have generalized the notion of a Hecke modification to the case of multiple points $p_1, \cdots, p_n$ by considering sequences of Hecke modifications, for which the points must be ordered. For most of our purposes we could equally well use an alternative generalization, described in [Kam11], for which the the points need not be ordered. Though we will not use it here, we briefly describe this alternative generalization and show how it relates to parabolic bundles:

Definition 2.4.16. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *simultaneous Hecke modification* $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha} F$ of E at a set of distinct points $\{p_1, p_2, \cdots, p_n\} \subset C$ is a rank 2 holomorphic vector bundle $\pi_F : F \to C$ and a bundle map $\alpha : F \to E$ that satisfies the following two conditions:

1. The induced map on fibers $\alpha_q : E_q \to F_q$ is an isomorphism for all points $q \notin \{p_1, \cdots, p_n\}$.
2. Condition (2) of Definition 2.4.1, which constrains the local behavior of α near a Hecke-modification point, holds at each of the points $p_1, \cdots, p_n$.

Definition 2.4.17. Two simultaneous Hecke modifications $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha} F$ and $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha'} F'$ are *equivalent* if there is an isomorphism $\phi : F \to F'$ such that $\alpha = \alpha' \circ \phi$.

Definition 2.4.18. We define the *total space of simultaneous Hecke modifications* $\bar{\mathcal{H}}^{tot}(C, E, \{p_1, \cdots, p_n\})$ to be the set of equivalence classes of simultaneous Hecke modifications of the rank 2 vector bundle $\pi_E : E \to C$ at the set of points $\{p_1, \cdots, p_n\} \subset C$. For simplicity, we will often suppress the dependence on $\{p_1, \cdots, p_n\}$ and denote this space as $\bar{\mathcal{H}}^{tot}(C, E, n)$.

We can define a set of parabolic bundles $\bar{\mathcal{P}}^{tot}(C, E, n)$ for which the marked points are unordered. We can define an isomorphism $\bar{\mathcal{H}}^{tot}(C, E, n) \to \bar{P}^{tot}(C, E, n)$ by

$$[E \xleftarrow[\{p_1,\cdots,p_n\}]{\alpha} F] \mapsto (E, \ell_{p_1}, \cdots, \ell_{p_n}),$$

where $\ell_{p_i} = \mathrm{im}(\alpha_{p_i} : F_{p_i} \to E_{p_i})$. Since the Hecke transform does not depend on the ordering of the points, it is well-defined on parabolic bundles in $\bar{\mathcal{P}}(C, E, n)$, and we have that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is isomorphic to F.

2.4.3 Moduli spaces of marked parabolic bundles

So far we have considered spaces of isomorphism classes of sequences of Hecke modifications in which the initial vector bundle in the sequence is held fixed. But in what follows we will want to generalize these spaces so the initial vector bundle is allowed to range over the isomorphism classes in a moduli space of vector bundles. Translating into the language of parabolic bundles, such spaces are equivalent to spaces of isomorphism classes of parabolic bundles in which the underlying vector bundles are allowed to range over the isomorphism classes in a moduli space of vector bundles. However, there is a problem with defining such spaces arising from the fact that vector bundles often have nontrivial automorphisms.

To illustrate the problem, consider the space $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}; p_1)$, which is (noncanonically) isomorphic to $\mathbb{CP}^1$. We might want to reinterpret this space as a moduli space of isomorphism classes of parabolic bundles of the form (E, ℓ_{p_1}) for $[E] \in M^{ss}(\mathbb{CP}^1)$, where $M^{ss}(\mathbb{CP}^1)$, the moduli space of semistable rank 2 vector bundles over $\mathbb{CP}^1$ with trivial determinant bundle, consists of the single point $[\mathcal{O} \oplus \mathcal{O}]$. But $\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = GL(2, \mathbb{C})$, and for any pair of parabolic bundles $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1})$ and $(\mathcal{O} \oplus \mathcal{O}, \ell'_{p_1})$ there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O})$ such that $\phi(\ell_{p_1}) = \ell'_{p_1}$. It follows that all parabolic bundles of the form $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1})$ are isomorphic and our proposed moduli space collapses to a point, when what we wanted was $\mathbb{CP}^1$.

To remedy the problem, we will add marking data to eliminate the nontrivial automorphisms. In particular, since stable parabolic bundles have no nontrivial automorphisms, we make the following definition:

Definition 2.4.19. Given a curve C and distinct points $(q_1, \cdots, q_m, p_1, \cdots, p_n) \in C^{m+n}$, we define the *total space of marked parabolic bundles* $\mathcal{P}_M^{tot}(C, m, n)$ to be the set of isomorphism of classes of parabolic bundles of the form $(E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n})$ such that $[E, \ell_{q_1}, \cdots, \ell_{q_m}] \in M^s(C, m)$:

$$\mathcal{P}_M^{tot}(C, m, n) = \{[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \mid [E, \ell_{q_1}, \cdots, \ell_{q_m}] \in M^s(C, m)\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}_M^{tot}(C, m, n)$ on $q_1, \cdots, q_m, p_1, \cdots, p_n$ in the notation.

Here the complex manifold $M^s(C, m)$ is the moduli space of stable rank 2 parabolic bundles over C with trivial determinant bundle and m marked points. We will refer to the lines $\ell_{q_1}, \cdots, \ell_{q_m}$ as *marking lines*, since their purpose is to add additional structure to E so as to eliminate nontrivial automorphisms. We will refer to the lines $\ell_{p_1}, \cdots, \ell_{p_n}$ as *Hecke lines*, since their purpose is to parameterize Hecke modifications at the points $p_1, \cdots, p_n$. Because we have defined $\mathcal{P}_M^{tot}(C, m, n)$ in terms of stable parabolic bundles, which have no nontrivial automorphisms, the collapsing phenomenon described above does not occur, and we have the following result:

Theorem 2.4.20. *The set $\mathcal{P}_M^{tot}(C, m, n)$ naturally has the structure of a complex manifold isomorphic to a $(\mathbb{CP}^1)^n$-bundle over $M^s(C, m)$.*

The base manifold $M^s(C, m)$ constitutes the moduli space over which the isomorphism classes of vector bundles with marking data range, and the $(\mathbb{CP}^1)^n$ fibers correspond to a space of Hecke modifications $\mathbb{CP}^1$ for each of the points $p_1, \cdots, p_n$. We will prove Theorem 2.4.20 by constructing $\mathcal{P}_M^{tot}(C, m, n)$ from a universal $\mathbb{CP}^1$-bundle, which we first describe for the case $m = 0$:

Lemma 2.4.21. *There is a universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$, which has the the property that for any complex manifold S and any $\mathbb{CP}^1$-bundle $Q \to C \times S$, the bundle Q is isomorphic to the pullback of P along $1_C \times f_Q$ for a unique map $f_Q : S \to M^s(C)$.*

Lemma 2.4.21 is proven in [BBN09]. One way to understand this result is as follows. Let $M^s_{r,d}(C)$ denote the moduli space of stable vector bundles of rank r and degree d on a curve C. As discussed in [Hof07], one can define a corresponding moduli stack $\mathrm{Bun}^s_{r,d}(C)$ and a $\mathbb{G}_m$-gerbe $\pi : \mathrm{Bun}^s_{r,d}(C) \to \mathrm{Hom}(-, M^s_{r,d}(C))$; this is a morphism of stacks for which all the fibers are isomorphic to $B\mathbb{G}_m$. The stack $\mathrm{Hom}(-, C) \times \mathrm{Bun}^s_{r,d}(C)$ carries a universal rank 2 vector bundle $\mathcal{E}$. One can show (see [Hei10] Corollary 3.12) that if $\gcd(r, d) = 1$ then $M^s_{r,d}(C)$ is a fine moduli space and $\mathcal{E}$ descends to a universal vector bundle $E \to C \times M^s_{r,d}(C)$, which can be viewed as a generalization of the Poincaré line bundle $L \to C \times \mathrm{Jac}(C)$ for the case $r = 1$, $d = 0$. By projectivizing E, we also get a universal $\mathbb{CP}^1$-bundle $\mathbb{P}(E) \to C \times M^s_{r,d}(C)$. If $\gcd(r, d) \neq 1$, then $M^s_{r,d}(C)$ is not a fine moduli space and $C \times M^s_{r,d}(C)$ does not carry a universal vector bundle. It is still possible, however, to use $\mathcal{E}$ to construct a universal $\mathbb{CP}^1$-bundle $P \to C \times M^s_{r,d}(C)$, only now this $\mathbb{CP}^1$-bundle is not the projectivization of a universal vector bundle. One way to make this result plausible is to note that whereas a stable vector bundle has automorphism group $\mathbb{C}^\times$, consisting of automorphisms that scale the fibers by a constant factor, the projectivization of a stable vector bundle has trivial automorphism group, consisting of just the identity automorphism.

Similar results hold for moduli spaces of stable vector bundles for which the determinant bundle is a fixed line bundle. In particular, the space $M^s(C)$ of stable rank 2 vector bundles with trivial determinant bundle is not a fine moduli space and $C \times M^s(C)$ does not carry a universal vector bundle; nonetheless, it does carry a universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$. We will use this universal $\mathbb{CP}^1$-bundle to construct $\mathcal{P}^{tot}_M(C, m, n)$ for the case $m = 0$:

Proof of Theorem 2.4.20. First consider the case $m = 0$, and note that $M^s(C, 0) = M^s(C)$. Given a point $p \in C$, let $P(p) \to M^s(C)$ denote the pullback of the universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$ described in Lemma 2.4.21 along the inclusion $i_p : M^s(C) \to C \times M^s(C)$, $[E] \mapsto (p, [E])$. Given distinct points $(p_1, \cdots, p_n) \in C^n$, we can pull back the $(\mathbb{CP}^1)^n$-

bundle $P(p_1) \times \cdots \times P(p_n) \to (M^s(C))^n$ along the diagonal map $M^s(C) \to (M^s(C))^n$, $[E] \mapsto ([E], \cdots, [E])$ to obtain $\mathcal{P}_M^{tot}(C, 0, n)$.

The proof for $m > 0$ is the same. One can define a moduli stack corresponding to $M^s(C, m)$ that carries a universal rank 2 parabolic bundle [Hof07]. Using a numerical condition analogous to the condition $\gcd(r, d) = 1$ for vector bundles (see [Hof07] Example 5.7 and [BH95] Proposition 3.2), one can show that $M^s(C, m)$ is a fine moduli space for $m > 0$ and the universal parabolic bundle on the moduli stack descends to a universal parabolic bundle on $C \times M^s(C, m)$. We can projectivize this latter bundle and use it to construct $\mathcal{P}_M^{tot}(C, m, n)$ in the same manner as for the $m = 0$ case. □

We will often want to pick out an open subset of $\mathcal{P}_M^{tot}(C, m, n)$ by imposing a semistability condition:

Definition 2.4.22. Given a curve C and distinct points $(q_1, \cdots, q_m, p_1, \cdots, p_n) \in C^{m+n}$, we define $\mathcal{P}_M(C, m, n)$ to be the subset of $\mathcal{P}_M^{tot}(C, m, n)$ consisting of points $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}]$ such that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable:

$$\mathcal{P}_M(C, m, n) = \\ \{[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}_M^{tot}(C, m, n) \mid H(E, \ell_{p_1}, \cdots, \ell_{p_n}) \text{ is semistable}\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}_M(C, m, n)$ on $q_1, \cdots, q_m, p_1, \cdots, p_n$ in the notation.

Theorem 2.4.23. *The set $\mathcal{P}_M(C, m, n)$ is an open submanifold of $\mathcal{P}_M^{tot}(C, m, n)$.*

Proof. This follows from the fact that semistability is an open condition. □

We can interpret marked parabolic bundles in terms of Hecke modifications as follows:

Definition 2.4.24. We define a *sequence of Hecke modifications of a parabolic bundle* $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ to be a sequence of Hecke modifications of the underlying vector bundle E.

Definition 2.4.25. We say that two sequences of Hecke modifications

$$(E, \ell_{q_1}, \cdots, \ell_{q_m}) \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n, \quad (E', \ell'_{q_1}, \cdots, \ell'_{q_m}) \xleftarrow[p_1]{\alpha'_1} E'_1 \xleftarrow[p_2]{\alpha'_2} \cdots \xleftarrow[p_n]{\alpha'_n} E'_n$$

are *equivalent* if there are isomorphisms $\phi_i : E_i \to E'_i$ for $i = 0, \cdots, n$ such that the following diagram commutes:

$$\begin{tikzcd}
(E, \ell_{q_1}, \cdots, \ell_{q_m}) \arrow[d, "\phi_0", "\cong"'] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\phi_1", "\cong"'] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\phi_2", "\cong"'] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\phi_n", "\cong"'] \\
(E', \ell'_{q_1}, \cdots, \ell'_{q_m}) & E'_1 \arrow[l, "\alpha'_1"'] & E'_2 \arrow[l, "\alpha'_2"'] & \cdots \arrow[l, "\alpha'_3"'] & E'_n. \arrow[l, "\alpha'_n"']
\end{tikzcd}$$

The space $\mathcal{P}_M^{tot}(C, m, n)$ can then be interpreted as a moduli space of equivalence classes of sequences of Hecke modifications of parabolic bundles, and the space $\mathcal{P}_M(C, m, n)$ can be interpreted as the subspace $\mathcal{P}_M^{tot}(C, m, n)$ consisting of equivalence classes of sequences for which the terminal vector bundles are semistable. We will not use these interpretations here, since it is simpler to work directly with the marked parabolic bundles.

2.5 Rational curves

2.5.1 Vector bundles on rational curves

Grothendieck showed that all rank 2 holomorphic vector bundles on (smooth projective) rational curves are decomposable [Gro57]; that is, they have the form $\mathcal{O}(n) \oplus \mathcal{O}(m)$ for integers n and m. The instability degree of $\mathcal{O}(n) \oplus \mathcal{O}(m)$ is $|n - m|$, so the bundle $\mathcal{O}(n) \oplus \mathcal{O}(m)$ is strictly semistable if $n = m$ and unstable otherwise. There are no stable rank 2 vector bundles on rational curves.

2.5.2 List of all possible single Hecke modifications

Here we present a list of all possible Hecke modifications at a point $p \in \mathbb{CP}^1$ of all possible rank 2 vector bundles on $\mathbb{CP}^1$. We will parameterize Hecke modifications of a vector bundle E at a point p in terms of lines $\ell_p \in \mathbb{P}(E_p)$, as described in Theorem 2.4.4. Since we are always free to tensor a Hecke modification with a line bundle, it suffices to consider vector bundles of nonnegative degree.

Theorem 2.5.1. *Consider the vector bundle $\mathcal{O}(n)\oplus\mathcal{O}$ for $n \geq 1$ (unstable, instability degree n). The possible Hecke modifications are*

$$\mathcal{O}(n)\oplus\mathcal{O} \leftarrow \begin{cases} \mathcal{O}(n)\oplus\mathcal{O}(-1) & \text{if } \ell_p = \mathcal{O}(n)_p \text{ (a bad line)}, \\ \mathcal{O}(n-1)\oplus\mathcal{O} & \text{otherwise (a good line)}. \end{cases}$$

Proof. (1) The case $\ell_p = \mathcal{O}(n)_p$. A Hecke modification $\alpha : \mathcal{O}(n)\oplus\mathcal{O} \to \mathcal{O}(n)\oplus\mathcal{O}(-1)$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where $f : \mathcal{O}(-1) \to \mathcal{O}$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p.

(2) The case $\ell_p \neq \mathcal{O}(n)_p$. Since $n \geq 1$, we can choose a section t of $\mathcal{O}(n)$ such that $t(p) \neq 0$. Choose a section $s = (at, b)$ of $\mathcal{O}(n)\oplus\mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s \in \ell_p$. A Hecke modification $\alpha : \mathcal{O}(n)\oplus\mathcal{O} \to \mathcal{O}(n-1)\oplus\mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} f & at \\ 0 & b \end{pmatrix},$$

where $f : \mathcal{O}(n-1) \to \mathcal{O}(n)$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p. $\square$

Theorem 2.5.2. *Consider the vector bundle $\mathcal{O}\oplus\mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$\mathcal{O}\oplus\mathcal{O} \leftarrow \mathcal{O}\oplus\mathcal{O}(-1) \qquad \text{for all } \ell_p \text{ (all lines are bad)}.$$

Proof. Define a section $s = (a, b)$ of $\mathcal{O}\oplus\mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. If $b = 0$, then a Hecke modification $\alpha : \mathcal{O}\oplus\mathcal{O}(-1) \to \mathcal{O}\oplus\mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & f \end{pmatrix},$$

and if $b \neq 0$, then a Hecke modification $\alpha : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} a & f \\ b & 0 \end{pmatrix},$$

where $f : \mathcal{O}(-1) \to \mathcal{O}$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p. □

2.5.2.1 Observations

From this list, we make the following observations:

Lemma 2.5.3. *The following results hold for Hecke modifications of a rank 2 vector bundle E over $\mathbb{CP}^1$:*

1. *A Hecke modification of E changes the instability degree by ± 1.*
2. *Hecke modification of E corresponding to a line $\ell_p \in \mathbb{P}(E_p)$ changes the instability degree by -1 if ℓ_p is a good line and $+1$ if ℓ_p is a bad line.*
3. *A generic Hecke modification of E changes the instability degree by -1 unless E has the minimum possible instability degree 0, in which case all Hecke modifications of E change the instability degree by $+1$.*

2.5.3 Moduli spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1, m, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, m, n)$

Our goal is to define a moduli space of Hecke modifications that is isomorphic to the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$. Kamnitzer showed that such a space can be defined as follows:

Definition 2.5.4 (Kamnitzer [Kam11])**.** Given distinct points $(p_1, \cdots, p_{2r}) \in (\mathbb{CP}^1)^{2r}$, define the *Kamnitzer space* $\mathcal{H}(\mathbb{CP}^1, 2r)$ to be the subset of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ consisting of equivalence classes of sequences of Hecke modifications $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_{2r}]{\alpha_n} E_{2r}$ such that E_{2r} is semistable.

In particular, the condition that E_{2r} must be semistable implies that $E_{2r} = \mathcal{O}(-r) \oplus \mathcal{O}(-r)$.

Theorem 2.5.5 (Kamnitzer [Kam11]). *The Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ has the structure of a complex manifold isomorphic to the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$.*

We will describe the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ and Kamnitzer's isomorphism in Section 2.5.6. We can use the results of Section 2.4.2 to reinterpret the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ in terms of parabolic bundles:

Definition 2.5.6. Define $\mathcal{P}^{tot}(\mathbb{CP}^1, n) := \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r) := \mathcal{P}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$.

Theorem 2.5.7. *There is a canonical isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}(\mathbb{CP}^1, 2r)$.*

Proof. This follows from restricting the domain and range of the canonical isomorphism $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r) \to \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ described in Theorem 2.4.10. □

We can also reinterpret the spaces $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r)$ in terms of the moduli spaces of marked parabolic bundles that we defined in Section 2.4.3. In what follows, we will choose a global trivialization $\mathcal{O} \oplus \mathcal{O} \to \mathbb{CP}^1 \times \mathbb{C}^2$ and identify all the fibers of $\mathcal{O} \oplus \mathcal{O}$ with $\mathbb{C}^2$. We can then identify lines $\ell_p \in \mathbb{P}(E_p)$ with points in $\mathbb{CP}^1$ and speak of lines in different fibers as being equal or unequal. In Section 2.2 we define a moduli space $M^{ss}(\mathbb{CP}^1)$ of semistable rank 2 vector bundles with trivial determinant bundle, and in Section 2.3 we define a moduli space $M^s(\mathbb{CP}^1, m)$ of stable rank 2 parabolic bundles with trivial determinant bundle and m marked points. From the fact that $M^{ss}(\mathbb{CP}^1) = \{[\mathcal{O} \oplus \mathcal{O}]\}$ and $\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = GL(2, \mathbb{C})$, we obtain the following results:

Theorem 2.5.8. *The moduli space $M^s(\mathbb{CP}^1, 3)$ consists of the single point $[\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}]$, where $\ell_{q_1}, \ell_{q_2}, \ell_{q_3}$ are any three distinct lines. Given any two stable parabolic bundles of the form $(\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ and $(\mathcal{O} \oplus \mathcal{O}, \ell'_{q_1}, \ell'_{q_2}, \ell'_{q_3})$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O})$ such that $\phi(\ell_{q_i}) = \ell'_{q_i}$ for $i = 1, 2, 3$.*

Corollary 2.5.9. *There is an isomorphism $M^s(\mathbb{CP}^1, 3) \to M^{ss}(\mathbb{CP}^1)$, $[\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}] \mapsto [\mathcal{O} \oplus \mathcal{O}]$.*

These results motivate the following definitions of "marked" versions of $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r)$:

Definition 2.5.10. Define $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n) := \mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, n) := \mathcal{P}_M(\mathbb{CP}^1, 3, n)$.

The marked and unmarked versions of these spaces are easily seen to be isomorphic:

Theorem 2.5.11. *The spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ are (noncanonically) isomorphic.*

Proof. Choose three distinct lines ℓ_{q_1}, ℓ_{q_2}, ℓ_{q_3}, and define an isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) \to \mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ by

$$[\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}] \mapsto [\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}].$$

The isomorphism is not canonical, since it depends on the choice of lines ℓ_{q_1}, ℓ_{q_2}, ℓ_{q_3} $\square$

Theorem 2.5.12. *The spaces $\mathcal{P}_M(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, n)$ are (noncanonically) isomorphic*

Proof. This follows from restricting the domain and range of the isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) \to \mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ described in Theorem 2.5.11 $\square$

Our primary motivation for defining $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ is to draw a parallel with the case of elliptic curves, which we consider in Section 2.6. But the space $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ also has an advantage over $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ in that we can use the marking lines to render certain constructions canonical. For example, we can define a canonical version of the noncanonical isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) = \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n) \to (\mathbb{CP}^1)^n$:

Lemma 2.5.13. *Fix distinct points $q_1, q_2, q_3, p \in \mathbb{CP}^1$ and a parabolic bundle $(E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ such that $[E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}] \in M^s(\mathbb{CP}^1, 3)$. There is a canonical isomorphism $\mathbb{P}(E_p) \to M^{ss}(\mathbb{CP}^1, 4) \cong \mathbb{CP}^1$ given by*

$$\ell_p \mapsto [E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_p].$$

Proof. This follows from the fact that $(E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ is stable, so the lines $\ell_{q_1}, \ell_{q_2}, \ell_{q_3}$ are all distinct under the global trivialization of E. $\square$

Theorem 2.5.14. *There is a canonical isomorphism $h : \mathcal{P}_M^{tot}(\mathbb{CP}^1, n) \to (M^{ss}(\mathbb{CP}^1, 4))^n$.*

Proof. Define maps $h_i : \mathcal{P}_M^{tot}(\mathbb{CP}^1, n) \to M^{ss}(\mathbb{CP}^1, 4)$ for $i = 1, \cdots, n$ by

$$h_i([E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_i}].$$

Then $h := (h_1, \cdots, h_n)$ is an isomorphism by Theorem 2.4.20 and Lemma 2.5.13. $\square$

Remark 2.5.15. Definition 2.4.22 for $\mathcal{P}_M(C, m, n)$ implies that $\mathcal{P}_M(\mathbb{CP}^1, m, n) = \varnothing$ for odd n, since there are no semistable rank 2 vector bundles of odd degree on $\mathbb{CP}^1$. We could alternatively define $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ by requiring that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ have the minimal possible instability degree, which is 0 for n even and 1 for n odd. This condition is equivalent to semistability for n even, but is a distinct condition for n odd, and gives a nonempty space.

2.5.4 Embedding $\mathcal{P}_M(\mathbb{CP}^1, m, n) \to M^s(\mathbb{CP}^1, m+n)$

We will now describe a canonical open embedding of the space $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ into the space of stable parabolic bundles $M^s(\mathbb{CP}^1, m+n)$. We first need two Lemmas:

Lemma 2.5.16. *Given a parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$, if $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction then $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}) = \mathcal{O} \oplus \mathcal{O}(-n)$.*

Proof. Since $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction, we have that $\ell_{p_1} = \cdots = \ell_{p_n}$ under a global trivialization of $\mathcal{O} \oplus \mathcal{O}$ in which all the fibers are identified with $\mathbb{C}^2$. An explicit sequence of Hecke modifications with $\ell_{p_1} = \cdots = \ell_{p_n}$ is given by

$$\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O} \oplus \mathcal{O}(-n),$$

where $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1)$ is a Hecke modification corresponding to the line ℓ_{p_1} and for $i = 2, \cdots, n$ we define

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & f_i \end{pmatrix},$$

where $f_i : \mathcal{O}(-i-1) \to \mathcal{O}(-i)$ is the unique (up to rescaling by a constant) nonzero morphism such that $(f_i)_{p_i} = 0$ on the fibers over p_i. $\square$

Lemma 2.5.17. *Given a parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$, if $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable then $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable.*

Proof. We will prove the contrapositive, so assume that $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. It follows that more $n/2$ of the lines are bad in the same direction. Let s denote the number of such lines, and choose a permutation $\sigma \in \Sigma_n$ such that the first s points of $(\sigma(p_1), \cdots, \sigma(p_n))$ correspond to these lines. By Lemma 2.5.16 we have that $H(\mathcal{O} \oplus \mathcal{O}, \ell_{\sigma(p_1)}, \cdots, \ell_{\sigma(p_s)}) = \mathcal{O} \oplus \mathcal{O}(-s)$, which has instability degree s. Lemma 2.5.3 states that a single Hecke modification changes the instability degree by ± 1, so $H(\mathcal{O} \oplus \mathcal{O}, \ell_{\sigma(p_1)}, \cdots, \ell_{\sigma(p_n)}) = H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ has instability degree at least $s - (n-s) = 2s - n > 0$, and is thus unstable. $\square$

Remark 2.5.18. The converse to Lemma 2.5.17 is does not always hold; for example, consider the semistable parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}, \ell_{p_4})$ for points $p_i = [1 : \mu_i] \in \mathbb{CP}^1$, where

$$\ell_{p_1} = [1:0], \quad \ell_{p_2} = [0:1], \quad \ell_{p_3} = [1:1], \quad \ell_{p_4} = [(\mu_3 - \mu_1)(\mu_4 - \mu_2) : (\mu_3 - \mu_2)(\mu_4 - \mu_1)].$$

One can show that $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}, \ell_{p_4}) = \mathcal{O}(-3) \oplus \mathcal{O}(-1)$, which is unstable.

Theorem 2.5.19. *There is a canonical open embedding $\mathcal{P}_M(\mathbb{CP}^1, m, n) \to M^s(\mathbb{CP}^1, m+n)$.*

Proof. Take $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}_M(\mathbb{CP}^1, m, n)$; note that $E = \mathcal{O} \oplus \mathcal{O}$. Since $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ is stable, fewer than $m/2$ of the lines $\ell_{q_1}, \cdots, \ell_{q_m}$ are equal under the global trivialization of E. Since $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable, it follows from Lemma 2.5.17 that $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable, so at most $n/2$ of the lines $\ell_{p_1}, \cdots, \ell_{p_n}$ are equal. It follows that fewer than $(m+n)/2$ of the lines $\ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}$ are equal, so $(E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n})$ is stable. So $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ is a subset of $M^s(\mathbb{CP}^1, m+n)$. Specifically, the set $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ consists of points $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in M^s(\mathbb{CP}^1, m+n)$ such that $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ is stable and $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable. Since stability and semistability are open conditions, we have that $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ is an open subset of $M^s(\mathbb{CP}^1, m+n)$. $\square$

2.5.5 Examples

We can generalize the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, n)$ to allow for both even and odd n, in analogy with the generalization described in Remark 2.5.15:

Definition 2.5.20. Given distinct points $(p_1, \cdots, p_n) \in (\mathbb{CP}^1)^n$, define the *Kamnitzer space* $\mathcal{H}(\mathbb{CP}^1, n)$ to be the subset of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ consisting of equivalence classes of sequences of Hecke modifications $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ such that E_n has the minimum possible instability degree (0 for n even, 1 for n odd.)

Here we compute Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, n)$ for $n = 0, 1, 2, 3$.

2.5.5.1 Calculate $\mathcal{H}(\mathbb{CP}^1, 0)$

We have

$$\mathcal{H}(\mathbb{CP}^1, 0) = \mathcal{H}^{tot}(\mathbb{CP}^1, 0) = \{\mathcal{O} \oplus \mathcal{O}\}.$$

2.5.5.2 Calculate $\mathcal{H}(\mathbb{CP}^1, 1)$

All Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$ give $\mathcal{O} \oplus \mathcal{O}(-1)$, which has instability degree 1, so

$$\mathcal{H}(\mathbb{CP}^1, 1) = \mathcal{H}^{tot}(\mathbb{CP}^1, 1) = \mathbb{CP}^1.$$

2.5.5.3 Calculate $\mathcal{H}(\mathbb{CP}^1, 2)$

A sequence of two Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$ must have one of two forms:

$$\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1) \xleftarrow[p_2]{\alpha_2} \mathcal{O}(-1) \oplus \mathcal{O}(-1), \quad \mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1) \xleftarrow[p_2]{\alpha_2} \mathcal{O} \oplus \mathcal{O}(-2).$$

In the first case the terminal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is semistable, whereas in the second case the terminal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$ is unstable. So $\mathcal{H}(\mathbb{CP}^1, 2)$ is the complement in $\mathcal{H}^{tot}(\mathbb{CP}^1, 2)$ of sequences of Hecke modifications of the second form. As we showed in the proof of Lemma 2.5.16, the resulting space is

$$\mathcal{H}(\mathbb{CP}^1, 2) = (\mathbb{CP}^1)^2 - \{(a, a) \mid a \in \mathbb{CP}^1\}.$$

2.5.5.4 Calculate $\mathcal{H}(\mathbb{CP}^1, 3)$

Now consider a sequence of three Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$. The only sequences for which the terminal bundle does not have instability degree 1 are of the form

$$\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1) \xleftarrow[p_2]{\alpha_2} \mathcal{O} \oplus \mathcal{O}(-2) \xleftarrow[p_3]{\alpha_3} \mathcal{O} \oplus \mathcal{O}(-3).$$

So $\mathcal{H}(\mathbb{CP}^1, 3)$ is the complement in $\mathcal{H}^{tot}(\mathbb{CP}^1, 3)$ of sequences of Hecke modifications of this form. As we showed in the proof of Lemma 2.5.16, the resulting space is

$$\mathcal{H}(\mathbb{CP}^1, 3) = (\mathbb{CP}^1)^3 - \{(a, a, a) \mid a \in \mathbb{CP}^1\}.$$

2.5.6 The Seidel-Smith space $\mathcal{Y}(S^2, 2r)$

Here we compare the embedding of $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ into $M^s(\mathbb{CP}^1, 2r + 3)$ that we defined in Theorem 2.5.19 with an embedding of the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ into $M^s(\mathbb{CP}^1, 2r+3)$ due to Woodward. We begin by defining the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$.

Definition 2.5.21. We define the *Slodowy slice* S_{2r} to be the subspace of $\mathfrak{gl}(2r, \mathbb{C})$ consisting of matrices with 2×2 identity matrices I on the superdiagonal, arbitrary 2×2 matrices in the left column, and zeros everywhere else.

Example 2.5.22. Elements of S_6 have the form

$$\begin{pmatrix} Y_1 & I & 0 \\ Y_2 & 0 & I \\ Y_3 & 0 & 0 \end{pmatrix},$$

where Y_1, Y_2, and Y_3 are arbitrary 2×2 complex matrices.

Definition 2.5.23. Define a map $\chi : S_{2r} \to \mathbb{C}^{2r}/\Sigma_{2r}$ that sends a matrix to the multiset of the roots of its characteristic polynomial, where a root of multiplicity m occurs m times in the multiset.

Definition 2.5.24. Given distinct points $(\mu_1, \cdots, \mu_{2r}) \in \mathbb{C}^{2r}$, define the *Seidel-Smith space* $\mathcal{Y}(S^2, 2r)$ to be the fiber $\chi^{-1}(\{\mu_1, \cdots, \mu_{2r}\})$. For simplicity, we are suppressing the dependence of $\mathcal{Y}(S^2, 2r)$ on $\mu_1, \cdots, \mu_{2r}$ in the notation. This space was introduced in [SS06], which denotes $\mathcal{Y}(S^2, 2r)$ by $\mathcal{Y}_r$.

The Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ naturally has the structure of a complex manifold, in fact a smooth complex affine variety. In what follows, it will be useful to define local coordinates $\xi : U \to V$ on $\mathbb{CP}^1$, where $U = \{[1 : z] \mid z \in \mathbb{C}\} \subset \mathbb{CP}^1$, $V = \mathbb{C}$, and $\xi([1 : z]) = z$. We define points $p_i := \xi^{-1}(\mu_i) \in \mathbb{CP}^1$ corresponding to μ_i for $i = 1, \cdots, 2r$.

2.5.6.1 Kamnitzer isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{Y}(S^2, 2r)$

Here we describe an isomorphism due to Kamnitzer from the space of Hecke modifications $\mathcal{H}(\mathbb{CP}^1, 2r)$ to the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$.

Define global meromorphic sections s_n of $\mathcal{O}(n)$ such that $\operatorname{div} s_n = n \cdot [\infty]$. For each rank 2 vector bundle $E = \mathcal{O}(n) \oplus \mathcal{O}(m)$, define standard meromorphic sections

$$e_E^1 = (s_n, 0), \qquad e_E^2 = (0, s_m),$$

and define a standard local trivialization $\psi_E : \pi_E^{-1}(U) \to U \times \mathbb{C}^2$ of E over U by

$$e_E^1(p) \mapsto (p, (1, 0)), \qquad e_E^2(p) \mapsto (p, (0, 1)).$$

Consider an element of $\mathcal{H}(\mathbb{CP}^1, 2r)$:

$$[E_0 \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_{2r}]{\alpha_{2r}} E_{2r}], \tag{2.2}$$

where $E_0 = \mathcal{O} \oplus \mathcal{O}$. Define rank 2 free $\mathbb{C}[z]$-modules L_i for $i = 0, \cdots, 2r$ as spaces of sections of E_i over U:

$$L_i = \Gamma(U, E_i) = \mathbb{C}[z] \cdot \{e_{E_i}^1, e_{E_i}^2\}.$$

The sequence of Hecke modifications (2.2) then yields a sequence of $\mathbb{C}[z]$-module morphisms $\bar{\alpha}_i$:

$$L_0 \xleftarrow{\bar{\alpha}_1} L_1 \xleftarrow{\bar{\alpha}_2} \cdots \xleftarrow{\bar{\alpha}_{2r}} L_{2r}.$$

We can also view $\bar{\alpha}_i$ as a holomorphic map $\bar{\alpha}_i : V \to M(2, \mathbb{C})$, defined as in Definition 2.4.1 such that

$$(\psi_{E_{i-1}} \circ \alpha_i \circ \psi_{E_i}^{-1})(q, v) = (q, \bar{\alpha}_i(\xi(q))v).$$

Define an $2r$-dimensional complex vector space V by

$$V = \operatorname{coker}(\bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \cdots \circ \bar{\alpha}_{2r}) = L_0 / (\bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \cdots \circ \bar{\alpha}_{2r})(L_{2r}).$$

One can show that an ordered basis for V is given by

$$(z^{r-1} e^1_{E_0},\, z^{r-1} e^2_{E_0},\, \cdots ,\, z e^1_{E_0},\, z e^2_{E_0},\, e^1_{E_0},\, e^2_{E_0}).$$

Note that z acts $\mathbb{C}$-linearly on V, and thus defines a $2r \times 2r$ complex matrix A relative to this basis.

Theorem 2.5.25 (Kamnitzer [Kam11]). *We have an isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{Y}(S^2, 2r)$ given by*

$$[E_0 \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_{2r}]{\alpha_{2r}} E_{2r}] \mapsto A.$$

To perform calculations, it is useful to have explicit expressions for the maps $\bar{\alpha}_i$. For each vector bundle E and point $p = [1 : \mu] \in U$, we use the standard trivialization $\psi_E : \pi^{-1}(E) \to U \times \mathbb{C}^2$ to identify $\mathbb{P}(E_p)$ with $\mathbb{CP}^1$. For each line $\ell_p \in \mathbb{P}(E_p) = \mathbb{CP}^1$, we give a holomorphic map $\bar{\alpha} : V \to M(2, \mathbb{C})$ that describes a Hecke modification $\alpha : F \to E$ corresponding to ℓ_p:

Hecke modifications of $\mathcal{O}(n) \oplus \mathcal{O}$ for $n \geq 1$:

$$\ell_p = [1:0] : \qquad \mathcal{O}(n) \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O}(n) \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z-\mu \end{pmatrix},$$

$$\ell_p = [\lambda:1] : \qquad \mathcal{O}(n) \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O}(n-1) \oplus \mathcal{O}, \qquad \bar{\alpha}(z) = \begin{pmatrix} z-\mu & \lambda \\ 0 & 1 \end{pmatrix}.$$

Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$:

$$\ell_p = [1:0] : \qquad \mathcal{O} \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O} \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z-\mu \end{pmatrix},$$

$$\ell_p = [\lambda:1] : \qquad \mathcal{O} \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O} \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} \lambda & z-\mu \\ 1 & 0 \end{pmatrix}.$$

2.5.6.2 Woodward embedding $\mathcal{Y}(S^2, 2r) \to M^s(\mathbb{CP}^1, 2r+3)$

Here we describe an embedding due to Woodward [Woo] of the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ into the space of stable rank 2 parabolic bundles $M^s(\mathbb{CP}^1, 2r+3)$. We first make the following definition:

Definition 2.5.26. Given distinct points $(p_1, \cdots, p_n) \in (\mathbb{CP}^1)^n$, define a subspace $\mathcal{P}^{ss}(\mathbb{CP}^1, n)$ of $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ consisting of semistable parabolic bundles $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$.

In particular, $\mathcal{P}^{ss}(\mathbb{CP}^1, n)$ consists of parabolic bundles $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ for which at most $n/2$ of the lines are equal to any given line under a global trivialization of $\mathcal{O} \oplus \mathcal{O}$. Given distinct points $(q_1, q_2, q_3, p_1, \cdots, p_n) \in (\mathbb{CP}^1)^{n+3}$ and distinct lines $\ell_{q_1}, \ell_{q_2}, \ell_{q_3} \in \mathbb{CP}^1$, we can define an embedding $\mathcal{P}^{ss}(\mathbb{CP}^1, n) \to M^{ss}(\mathbb{CP}^1, n+3)$,

$$(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}) \mapsto [\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}].$$

We will define an embedding $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. Composing with $\mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \to M^s(\mathbb{CP}^1, 2r+3)$ will then yield the Woodward embedding. We first define some vectors. Define vectors $x, y \in \mathbb{C}^2$ by

$$x = (1, 0) \in \mathbb{C}^2, \qquad y = (0, 1) \in \mathbb{C}^2.$$

Define vectors $x_1, \cdots, x_r, y_1, \cdots, y_r \in \mathbb{C}^{2r}$ by

$$x_1 = (x,\, 0,\, \cdots,\, 0) \in \mathbb{C}^{2r}, \quad x_2 = (0,\, x,\, 0,\, \cdots,\, 0) \in \mathbb{C}^{2r}, \quad \cdots, \quad x_r = (0,\, \cdots,\, 0,\, x) \in \mathbb{C}^{2r},$$
$$y_1 = (y,\, 0,\, \cdots,\, 0) \in \mathbb{C}^{2r}, \quad y_2 = (0,\, y,\, 0,\, \cdots,\, 0) \in \mathbb{C}^{2r}, \quad \cdots, \quad y_r = (0,\, \cdots,\, 0,\, y) \in \mathbb{C}^{2r}.$$

Define vectors $x(\mu), y(\mu) \in \mathbb{C}^{2r}$ by

$$x(\mu) = (\mu^{r-1}x,\, \mu^{r-2}x,\, \cdots,\, \mu x,\, x) = \mu^{r-1}x_1 + \mu^{r-2}x_2 + \cdots + \mu x_{r-1} + x_r \in \mathbb{C}^{2r},$$
$$y(\mu) = (\mu^{r-1}y,\, \mu^{r-2}y,\, \cdots,\, \mu y,\, y) = \mu^{r-1}y_1 + \mu^{r-2}y_2 + \cdots + \mu y_{r-1} + y_r \in \mathbb{C}^{2r}.$$

We use the vectors to define a subspace $W(s, t)$ of $\mathbb{C}^{2r}$, and we calculate its dimension:

Definition 2.5.27. Given $(s,t) \in \mathbb{C}^2$, define a subspace $W(s,t) = \mathbb{C}\cdot\{sx(\mu)+ty(\mu) \mid \mu \in \mathbb{C}\}$ of $\mathbb{C}^{2r}$.

Lemma 2.5.28. *For $(s,t) \in \mathbb{C}^2 - \{0\}$, we have that $\dim W(s,t) = r$.*

Proof. Define a vector $w(s,t,\mu) \in \mathbb{C}^{2r}$ by

$$w(s,t,\mu) := sx(\mu) + ty(\mu) = \mu^{r-1}(sx_1 + ty_1) + \cdots + \mu(sx_{r-1} + ty_{r-1}) + (sx_r + ty_r). \quad (2.3)$$

Let $S \subset \mathbb{C}^{2r}$ denote the span of the linearly independent vectors $\{sx_1+ty_1, \cdots, sx_r+ty_r\}$. Clearly $W(s,t) \subseteq S$. Form an $r\times r$ matrix V whose i-th row vector consists of the components of $w(s,t,i)$ relative to the ordered basis $(sx_1+ty_1, \cdots, sx_r+ty_r)$ of S. From equation (2.3), it follows that the (i,j) matrix element of V is given by

$$V_{ij} = (i)^{r-j}.$$

So V is a Vandermonde matrix corresponding to the distinct integers $(1,2,\cdots,r)$, and thus has nonzero determinant. It follows that the vectors $\{w(s,t,1),\cdots,w(s,t,r)\}$ are linearly independent, hence $W(s,t) = S$ and $\dim W(s,t) = \dim S = r$. $\square$

We are now ready to define the Woodward embedding. Take a matrix $A \in \mathcal{Y}(S^2, 2r)$. Let $v(\mu) \in \mathbb{C}^{2r}$ be a left-eigenvector of A with eigenvalue μ:

$$v(\mu)A = \mu v(\mu).$$

Given the form of A, it follows that

$$v(\mu) = X(\mu)x(\mu) + Y(\mu)y(\mu)$$

for some $X(\mu), Y(\mu) \in \mathbb{C}$. Since $A \in \mathcal{Y}(S^2, 2r)$, the eigenvalues of A are $\mu_1, \cdots, \mu_{2r} \in \mathbb{C}$. Define lines $\ell_{p_i} \in \mathbb{CP}^1$ for $i = 1, \cdots, 2r$ by

$$\ell_{p_i} = [X(\mu_i) : Y(\mu_i)].$$

Theorem 2.5.29 (Woodward [Woo])**.** *We have an embedding $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$, $A \mapsto (\mathcal{O}\oplus\mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_{2r}})$.*

Proof. A priori the codomain of the map is $\mathcal{P}^{tot}(\mathbb{CP}^1, 2r)$, so we need to show that the image is in fact contained in $\mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. Note that if $\ell_{p_i} = [s:t]$ then $v(\mu_i) \in W(s,t)$. Since the eigenvalues $\mu_1, \cdots, \mu_{2r}$ are distinct, the eigenvectors $\{v(\mu_1), \cdots, v(\mu_{2r})\}$ are linearly independent, so the maximum number of eigenvectors that can live in $W(s,t)$ is $\dim W(s,t) = r$ by Lemma 2.5.28. So at most r of the lines $\ell_{p_1}, \cdots, \ell_{p_{2r}}$ can be equal to any given line $[s:t]$ in $\mathbb{CP}^1$, and thus $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_{2r}})$ is semistable. $\square$

Lemma 2.5.17 states that we have an embedding $\mathcal{P}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. We can precompose this embedding with the canonical isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}(\mathbb{CP}^1, 2r)$ described in Theorem 2.5.7 to obtain an embedding $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$, and we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathbb{CP}^1, 2r) \arrow[r] \arrow[d, "\cong"'] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \arrow[d] \\
\mathcal{P}_M(\mathbb{CP}^1, 2r) \arrow[r] & M^{ss}(\mathbb{CP}^1, 2r+3),
\end{tikzcd}$$

where the bottom horizontal arrow is the embedding described in Theorem 2.5.19. It is interesting to compare the embedding $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$ to the embedding $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$ from Theorem 2.5.29. We make the following conjecture:

Conjecture 2.5.30. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}(\mathbb{CP}^1, 2r) \arrow[r] \arrow[d, "\cong"] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \arrow[d, "\cong"] \\
\mathcal{Y}(S^2, 2r) \arrow[r] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r),
\end{tikzcd}$$

where the left downward arrow is the Kamnitzer isomorphism and the right downward arrow is the map on parabolic bundles induced by $\phi \in \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = GL(2, \mathbb{C})$, where

$$\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.5.31. *Conjecture 2.5.30 holds for $r = 1$ and $r = 2$.*

Proof. This can be shown by a direct calculation. $\square$

2.6 Elliptic curves

2.6.1 Vector bundles on elliptic curves

Vector bundles on elliptic curves have been classified by Atiyah [Ati57]:

Definition 2.6.1. Define $\mathcal{E}(r, d)$ to be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d on an elliptic curve X.

The set $\mathcal{E}(r, d)$ naturally has the structure of a complex manifold, and we have the following result:

Theorem 2.6.2 (Atiyah [Ati57])**.** *There are isomorphisms* $\mathrm{Jac}(X) \to \mathcal{E}(r, d)$ *for all* r *and* d*.*

In particular, $\mathcal{E}(1, d)$ is the set of isomorphism classes of line bundles of degree d, and the isomorphism $\mathrm{Jac}(X) \to \mathcal{E}(1, d)$ is given by $[L] \mapsto [L \otimes \mathcal{O}(d \cdot e)]$ for a choice of basepoint $e \in X$. Here we summarize the facts we will need regarding line bundles and rank 2 vector bundles on elliptic curves. Results that are well-known will be stated without proof; full proofs can be found in [Ati57, Ien11, Big].

Definition 2.6.3. We say that a degree 0 line bundle L is *2-torsion* if $L^2 = \mathcal{O}$.

There are four 2-torsion line bundles on an elliptic curve. We will denote the 2-torsion line bundles by L_i for $i = 1, 2, 3, 4$, with the convention that $L_1 = \mathcal{O}$.

Definition 2.6.4. Given line bundles L and M on an elliptic curve, an *extension* of L by M is an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where E is a rank 2 vector bundle.

Lemma 2.6.5. *Given line bundles* L *and* M *on an elliptic curve, equivalence classes of extensions of* L *by* M *are classified by* $\mathrm{Ext}^1(L, M) = H^0(L \otimes M^{-1})$.

Lemma 2.6.6 (Teixidor [Big], Lemma 4.5)**.** *If $[E] \in \mathcal{E}(2,d)$, then $h^0(E) = 0$ if $d < 0$ and $h^0(E) = d$ if $d > 0$, where $h^0(E) := \dim H^0(E)$.*

We will now list the rank 2 vector bundles on an elliptic curve X. Up to tensoring with a line bundle, we have the following vector bundles:

2.6.1.1 Rank 2 decomposable vector bundles

Decomposable bundles have the form $L_1 \oplus L_2$, where L_1 and L_2 are line bundles. The instability degree of $L_1 \oplus L_2$ is $|\deg L_1 - \deg L_2|$, so $L_1 \oplus L_2$ is strictly semistable if $\deg L_1 = \deg L_2$ and unstable otherwise. The proof of the following result is straightforward:

Lemma 2.6.7. *Let $E = L_1 \oplus L_2$, where L_1 and L_2 are line bundles such that $\deg L_1 > \deg L_2$. At a point $p \in X$ the line $(L_1)_p$ is bad, and all other lines in $\mathbb{P}(E_p)$ are good.*

A semistable decomposable bundle must have even degree, so after tensoring with a suitable line bundle it has the form $L \oplus L^{-1}$, where L is a degree 0 line bundle. There are two subclasses of such bundles: the four bundles $L_i \oplus L_i$, and bundles $L \oplus L^{-1}$ such that $L^2 \neq \mathcal{O}$. These two subclasses of semistable decomposable bundles have very different properties:

Lemma 2.6.8. *The bundle $L_i \oplus L_i$ has no good lines, and $\mathrm{Aut}(L_i \oplus L_i) = GL(2, \mathbb{C})$.*

Lemma 2.6.9. *Let $E = L \oplus L^{-1}$, where L is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. The automorphism group $\mathrm{Aut}(E)$ is the subgroup of $GL(2, \mathbb{C})$ matrices of the form*

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

At a point $p \in X$ the lines L_p and $(L^{-1})_p$ are bad, and all other lines in $\mathbb{P}(E_p)$ are good. Given a pair of good lines $\ell_p, \ell'_p \in \mathbb{P}(E_p)$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \mathrm{Aut}(E)$ such that $\phi(\ell_q) = \phi(\ell'_q)$.

The proofs of Lemmas 2.6.8 and 2.6.9 are straightforward and have been omitted.

2.6.1.2 Rank 2 degree 0 indecomposable bundles

There is a unique indecomposable bundle F_2 that can be obtained via an extension of $\mathcal{O}$ by $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\alpha} F_2 \xrightarrow{\beta} \mathcal{O} \longrightarrow 0. \tag{2.4}$$

The bundle F_2 is strictly semistable, and hence has instability degree 0. The map $\mathrm{Jac}(X) \to \mathcal{E}(2,0)$, $[L] \mapsto [F_2 \otimes L]$ is an isomorphism, so in particular $F_2 \otimes L = F_2$ if and only if $L = \mathcal{O}$.

Lemma 2.6.10. *If L is a degree 0 line bundle, then*

$$\mathrm{Hom}(L, F_2) = \begin{cases} \mathbb{C}\cdot\alpha & \text{if } L = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathrm{Hom}(F_2, L) = \begin{cases} \mathbb{C}\cdot\beta & \text{if } L = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply $\mathrm{Hom}(L, -)$ to the short exact sequence (2.4) to obtain

$$0 \longrightarrow \mathrm{Hom}(L,\mathcal{O}) \xrightarrow{\alpha_*} \mathrm{Hom}(L, F_2) \xrightarrow{\beta_*} \mathrm{Hom}(L,\mathcal{O}) \xrightarrow{\delta} \mathrm{Ext}^1(L,\mathcal{O}). \tag{2.5}$$

If $L \neq \mathcal{O}$, then $\mathrm{Hom}(L,\mathcal{O}) = 0$ and the long exact sequence (2.5) implies that $\mathrm{Hom}(L, F_2) = 0$. So assume $L = \mathcal{O}$. Then $\mathrm{Hom}(L,\mathcal{O}) = \mathbb{C}\cdot 1_{\mathcal{O}}$. To prove that $\mathrm{Hom}(L, F_2) = \mathbb{C}\cdot\alpha$, it suffices to show that $\delta(1_{\mathcal{O}}) \neq 0$. Assume for contradiction that this is not the case. Then the long exact sequence (2.5) implies that there is a morphism $f \in \mathrm{Hom}(L, F_2)$ such that $\beta_*(f) = \beta \circ f = 1_{\mathcal{O}}$. It follows that the short exact sequence (2.4) splits, contradiction.

The claim regarding $\mathrm{Hom}(F_2, L)$ can be proven in a similar manner by applying $\mathrm{Hom}(-, L)$ to the short exact sequence (2.4). □

Lemma 2.6.11. *Given a point $q \in X$, there are nonzero sections t_0 and t_1 of $F_2 \otimes \mathcal{O}(q)$ such that*

1. *$H^0(F_2 \otimes \mathcal{O}(q)) = \mathbb{C}\cdot\{t_0, t_1\}$,*

2. *$\mathrm{div}\, t_0 = 0$ and $\mathrm{div}\, t_1 = q$,*

3. *$t_1(p) \in \mathcal{O}(q)_p$ for all $p \in X$, where $\mathcal{O}(q) \to F_2 \otimes \mathcal{O}(q)$ is the unique degree 1 line subbundle of $F_2 \otimes \mathcal{O}(q)$,*

4. $\{t_0(p), t_1(p)\}$ *are linearly independent for all* $p \in X$ *such that* $p \neq q$*.*

Proof. Tensoring $\alpha : \mathcal{O} \to F_2$ with $\mathcal{O}(q)$ and precomposing with the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O} \to \mathcal{O}(q)$, we obtain a section t_1 of $F_2 \otimes \mathcal{O}(q)$ such that $\operatorname{div} t_1 = q$ and $t_1(p) \in \mathcal{O}(q)_p$ for all $p \in X$. By Lemma 2.6.6 we have that $h^0(F_2 \otimes \mathcal{O}(q)) = 2$, so we can choose a section t_0 of $F_2 \otimes \mathcal{O}(q)$ linearly independent from t_1.

We claim that $\operatorname{div} t_0 = 0$. Assume for contradiction that this is not the case. We obtain a subbundle $\mathcal{O}(\operatorname{div} t_0) \to F_2 \otimes \mathcal{O}(q)$, and by semistability of $F_2 \otimes \mathcal{O}(q)$ it follows that $\operatorname{div} t_0 = p$ for some $p \in X$. We thus obtain a subbundle $\mathcal{O}(p) \to F_2 \otimes \mathcal{O}(q)$, hence a subbundle $\mathcal{O}(p-q) \to F_2$. But this contradicts Lemma 2.6.10 unless $p = q$, in which case t_0 and t_1 are linearly dependent.

We claim $t_0(p)$ and $t_1(p)$ are linearly independent at all points $p \in X$ such that $p \neq q$. Assume for contradiction that they are linearly dependent at some point p distinct from q. Then we can choose a nonzero section $s = at_0 + bt_1$ of $F_2 \otimes \mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) = 0$. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) \to F_2 \otimes \mathcal{O}(q)$. We have that $p \in \operatorname{div} s$, so semistability of $F_2 \otimes \mathcal{O}(q)$ implies that $\operatorname{div} s = p$. We thus obtain a subbundle $\mathcal{O}(p) \to F_2 \otimes \mathcal{O}(q)$, hence a subbundle $\mathcal{O}(p-q) \to F_2$. But this contradicts Lemma 2.6.10. $\square$

Lemma 2.6.12. *The automorphism group* $\operatorname{Aut}(F_2)$ *is the subgroup of* $GL(2, \mathbb{C})$ *matrices of the form*

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

At a point $p \in X$ *the line* $\mathcal{O}_p$ *is bad, where* $\mathcal{O} \to F_2$ *is the unique degree 0 subbundle of* F_2*, and all other lines in* $\mathbb{P}((F_2)_p)$ *are good. Given a pair of good lines* $\ell_p, \ell'_p \in \mathbb{P}((F_2)_p)$*, there is a unique (up to rescaling by a constant) automorphism* $\phi \in \operatorname{Aut}(E)$ *such that* $\phi(\ell_q) = \phi(\ell'_q)$*.*

Proof. Apply $\operatorname{Hom}(-, F_2)$ to the short exact sequence (2.4) to obtain

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}, F_2) \xrightarrow{\beta^*} \operatorname{Hom}(F_2, F_2) \xrightarrow{\alpha^*} \operatorname{Hom}(\mathcal{O}, F_2). \tag{2.6}$$

Note that $\alpha^*(1_{F_2}) = \alpha$, so Lemma 2.6.10 implies that α^* is surjective and thus the sequence (2.6) is in fact short exact. It follows that $\operatorname{Hom}(F_2, F_2) = \mathbb{C} \cdot \{1_{F_2}, \eta\}$, where $\eta := \beta^*(\alpha) =$

$\alpha \circ \beta$. Note that $\eta \circ \eta = 0$, so we can define an injective group homomorphism $\mathrm{Aut}(F_2) \to GL(2, \mathbb{C})$ by

$$A\,1_{F_2} + B\,\eta \mapsto \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

The fact that $\mathcal{O}_p$ is the unique bad line of $(F_2)_p$ follows from Lemma 2.6.10. Given good lines $\ell_p, \ell'_p \in \mathbb{P}((F_2)_p)$, choose nonzero vectors $v, v', w \in (F_2)_p$ such that $v \in \ell_p$, $v' \in \ell'_p$, and $w \in \mathcal{O}_p$. Since $\ell_p \neq \mathcal{O}_p$, it follows that $\{v, w\}$ is a basis for $(F_2)_p$. Define $a, b \in \mathbb{C}$ such that $v' = av + bw$; note that since $\ell'_q \neq \mathcal{O}_p$ we have that $a \neq 0$. Define $c \in \mathbb{C}$ such that $\eta_p(v) = cw$; note that since $\eta_p(w) = 0$ and $\eta_p \neq 0$, we have that $c \neq 0$. Then $v' = \phi_p(v)$, where $\phi = a1_F + (b/c)\eta \in \mathrm{Aut}(F_2)$. Hence $\phi(\ell_p) = \ell'_p$, and ϕ is clearly unique up to rescaling by a constant. $\square$

2.6.1.3 Rank 2 degree 1 indecomposable bundles

Given a point $p \in X$, there is a unique degree 1 indecomposable bundle $G_2(p)$ that can be obtained via an extension of $\mathcal{O}(p)$ by $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow G_2(p) \longrightarrow \mathcal{O}(p) \longrightarrow 0.$$

The bundle $G_2(p)$ is stable, with instability degree -1. The map $\mathcal{E}(1,1) \to \mathcal{E}(2,1)$, $[\mathcal{O}(p)] \mapsto [G_2(p)]$ is an isomorphism, with inverse isomorphism given by $\det : \mathcal{E}(2,1) \to \mathcal{E}(1,1)$, $[E] \to [\det E]$. It follows that for any degree 0 divisor D on X we have that

$$G_2(p + 2D) = G_2(p) \otimes \mathcal{O}(D),$$

and in particular $G_2(p) \otimes L \cong G_2(p)$ if and only if $L^2 = \mathcal{O}$.

Lemma 2.6.13. *We have that* $\mathrm{Aut}(G_2(p)) = \mathbb{C}^\times$ *consists only of trivial automorphisms that scale the fibers by a constant factor.*

Proof. This follows from the fact that $G_2(p)$ is stable. $\square$

Lemma 2.6.14. *Any degree 0 line bundle L is a subbundle of $G_2(p)$ via a unique (up to rescaling by a constant) inclusion map $L \to G_2(p)$.*

Proof. Let L be a degree 0 line bundle. By Lemma 2.6.6 we have that $h^0(G_2(p) \otimes L^{-1}) = 1$, hence $G_2(p) \otimes L^{-1}$ has a nonzero section s. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) \to G_2(p) \otimes L^{-1}$. By stability of $G_2(p) \otimes L^{-1}$, we must have $\operatorname{div} s = 0$. Tensoring with L, we obtain a subbundle $L \to G_2(p)$. The claim regarding uniqueness follows from the fact that $h^0(G_2(p) \otimes L^{-1}) = 1$. □

Corollary 2.6.15. *All lines of $G_2(p)$ are bad.*

Proof. This is shown in Theorem 2.6.19. □

2.6.2 List of all possible single Hecke modifications

Here we present a list of all possible Hecke modifications at a point $p \in X$ of all possible rank 2 vector bundles on X, up to tensoring with a line bundle. We will parameterize Hecke modifications of a vector bundle E at a point p in terms of lines $\ell_p \in \mathbb{P}(E_p)$, as described in Theorem 2.4.4. Since we are always free to tensor a Hecke modification with a line bundle, it suffices to consider vector bundles of nonnegative degree.

To construct the list, we will often use the following strategy. By tensoring E with a line bundle of sufficiently high degree if necessary, we can assume without loss of generality that E is generated by global sections. Consider a Hecke modification $\alpha : F \to E$ of E at p corresponding to a line $\ell_p := \operatorname{im} \alpha_p \in \mathbb{P}(E_p)$. Since we have assumed E is generated by global sections, there is a section s of E such that $s(p) \neq 0$ and $s(p) \in \ell_p$. We then get a subbundle $\mathcal{O}(\operatorname{div} s) \to E$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(\operatorname{div} s) \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & L \otimes \mathcal{O}(-p) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathcal{O}(\operatorname{div} s) \arrow[r] & E \arrow[r] & L \arrow[r] & 0,
\end{tikzcd}$$

where f is the unique (up to rescaling by a constant) nonzero morphism $L \otimes \mathcal{O}(-p) \to L$. Thus F is an extension of $L \otimes \mathcal{O}(-p)$ by $\mathcal{O}(\operatorname{div} s)$, and we can often use this information to determine F.

2.6.2.1 Rank 2 bundles of degree greater than 1

Theorem 2.6.16. *Consider a bundle of the form $L \oplus \mathcal{O}$ for L a line bundle of degree greater than 1 (unstable, instability degree* $\deg L$*). The possible Hecke modifications are*

$$L \oplus \mathcal{O} \leftarrow \begin{cases} L \oplus \mathcal{O}(-p) & \text{if } \ell_p = L_p \text{ (a bad line)}, \\ (L \otimes \mathcal{O}(-p)) \oplus \mathcal{O} & \text{otherwise (a good line)}. \end{cases}$$

Proof. (1) The case $\ell_p = L_p$. A Hecke modification $\alpha : L \oplus \mathcal{O}(-p) \to L \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O}(-p) \to \mathcal{O}$.

(2) The case $\ell_p \neq L_p$. Since $\deg L > 1$, we can choose a nonzero section t of L such that $t(p) \neq 0$. Since t is nonvanishing at p, we can choose a section $s = (at, b)$ of $L \oplus \mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. A Hecke modification $\alpha : (L \otimes \mathcal{O}(-p)) \oplus \mathcal{O} \to L \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} f & at \\ 0 & b \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $L \otimes \mathcal{O}(-p) \to L$. $\square$

2.6.2.2 Rank 2 bundles of degree 1

Theorem 2.6.17. *Consider the bundle $\mathcal{O}(q) \oplus \mathcal{O}$ with $q \neq p$ (unstable, instability degree 1). The possible Hecke modifications are*

$$\mathcal{O}(q) \oplus \mathcal{O} \leftarrow \begin{cases} \mathcal{O}(q) \oplus \mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}(q)_p \text{ (a bad line)}, \\ \mathcal{O}(q-p) \oplus \mathcal{O} & \text{otherwise (a good line)}. \end{cases}$$

Proof. One can prove this result by using the fact that $\mathcal{O}(q)$ has a section t such that $t(p) \neq 0$ and writing down explicit Hecke modifications, as in the proof of Theorem 2.6.16. $\square$

Theorem 2.6.18. *Consider the bundle $\mathcal{O}(p)\oplus\mathcal{O}$ (unstable, instability degree 1). The possible Hecke modifications are*

$$\mathcal{O}(p)\oplus\mathcal{O} \leftarrow \begin{cases} \mathcal{O}(p)\oplus\mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}(p)_p \text{ (a bad line)}, \\ \mathcal{O}\oplus\mathcal{O} & \text{if } \ell_p = \mathcal{O}_p \text{ (a good line)}, \\ F_2 & \text{otherwise (a good line)}. \end{cases}$$

Proof. (1) The case $\ell_p = \mathcal{O}(p)_p$. A Hecke modification $\alpha : \mathcal{O}(p)\oplus\mathcal{O}(-p) \to \mathcal{O}(p)\oplus\mathcal{O}$ is given by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O}(-p) \to \mathcal{O}$.

(2) The case $\ell_p = \mathcal{O}_p$. A Hecke modification $\alpha : \mathcal{O}\oplus\mathcal{O} \to \mathcal{O}(p)\oplus\mathcal{O}$ is given by

$$\alpha = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},$$

where t is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O} \to \mathcal{O}(p)$.

(3) The case $\ell_p \neq \mathcal{O}(p)_p$ and $\ell_q \neq \mathcal{O}_p$. Pick a point $q \in X$ such that $q \neq p$. Choose a nonzero section t_0 of $\mathcal{O}(p+q)$ such that $t_0(q) \neq 0$ and $t_0(p) \neq 0$. Choose a nonzero section t_1 of $\mathcal{O}(q)$. Note that $\operatorname{div} t_1 = q$. Since $t_0(p) \neq 0$ and $t_1(p) \neq 0$, we can define a section $s = (at_0, bt_1)$ of $\mathcal{O}(p+q)\oplus\mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_p \neq \mathcal{O}(p)_p$ and $\ell_q \neq \mathcal{O}_p$, it follows that $a \neq 0$ and $b \neq 0$, thus $\operatorname{div} s = 0$ and we obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to \mathcal{O}(p+q)\oplus\mathcal{O}(q)$, $1 \mapsto s$. Thus we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(2q) \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}(p+q)\oplus\mathcal{O}(q) \arrow[r] & \mathcal{O}(2q+p) \arrow[r] & 0. \end{tikzcd}$$

The bundle F cannot split, since there are no nonzero morphisms $\mathcal{O}(2q) \to \mathcal{O}(p+q)\oplus\mathcal{O}(q)$, hence F is indecomposable. Since $\det F = \mathcal{O}(2q)$ it follows that $F = F_2 \otimes \mathcal{O}(q) \otimes L$ for a 2-torsion line bundle L. We can compose α with projection onto the second summand of $\mathcal{O}(p+q)\oplus\mathcal{O}(q)$ to obtain a nonzero morphism $F \to \mathcal{O}(q)$, so Lemma 2.6.10 implies $L = \mathcal{O}$ and $F = F_2 \otimes \mathcal{O}(q)$. $\square$

Theorem 2.6.19. *Consider the bundle $G_2(p)$ (stable, instability degree -1). There is a canonical isomorphism $\mathbb{P}(G_2(p)_p) \to M^{ss}(X) \cong \mathbb{CP}^1$ given by*

$$\ell_p \mapsto [H(G_2(p), \ell_p)].$$

All lines of $G_2(p)$ are bad.

Proof. By Lemma 2.6.14, any degree 0 line bundle L is a subbundle of $G_2(p)$ via a unique (up to rescaling by a constant) inclusion map $L \to G_2(p)$. Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & L^{-1} \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & L \arrow[r] & G_2(p) \arrow[r] & L^{-1} \otimes \mathcal{O}(p) \arrow[r] & 0.
\end{tikzcd}$$

Note that $\mathrm{Ext}^1(L^{-1}, L) = H^0(L^{-2})$. If $L^2 \neq \mathcal{O}$ then $H^0(L^{-2}) = 0$, so F splits, thus $F = L \oplus L^{-1}$.

Now suppose $L^2 = \mathcal{O}$. We claim that F is indecomposable; assume for contradiction that this is not the case. Then $F = L \oplus L$, so $\alpha : F \to G_2(p)$ gives a map $\mathcal{O} \oplus \mathcal{O} \to G_2(p) \otimes L$ that is an isomorphism away from p, so we obtain two linearly independent sections of $G_2(p) \otimes L$. But by Lemma 2.6.6 we have that $h^0(G_2(p) \otimes L) = 1$, contradiction. It follows that F is indecomposable. Since $\det F = \mathcal{O}$, it follows that $F = F_2 \otimes M$ for a 2-torsion line bundle M. Since we have a nonzero morphism $L \to F$, Lemma 2.6.10 implies that $M = L$ and $F = F_2 \otimes L$.

Our results show that we have a surjection $\mathrm{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [F]$. The vector bundle F is isomorphic to $H(G_2(p), \ell_p)$, where $\ell_p \in \mathbb{P}(G_2(p)_p)$ is the line corresponding to $[G_2(p)_p \xleftarrow[p]{\alpha} F] \in \mathcal{H}^{tot}(X, G_2(p); p)$ under the canonical isomorphism described in Theorem 2.4.4, and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Jac}(X) \arrow[r] \arrow[d] & M^{ss}(X) \\
\mathbb{P}(G_2(p)_p). \arrow[ur] &
\end{tikzcd}$$

Here $\mathrm{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ is given by $[L] \mapsto L_p$ and $\mathbb{P}(G_2(p)_p) \to M^{ss}(X)$ is given by $\ell_p \mapsto [H(G_2(p), \ell_p)]$. Since $\mathrm{Jac}(X) \to M^{ss}(X)$ is surjective, we have that $\mathrm{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ is surjective and $\mathbb{P}(G_2(p)_p) \to M^{ss}(X)$ is an isomorphism. The surjectivity of $\mathrm{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ implies that all lines of $\mathbb{P}(G_2(p)_p)$ are bad. Since $G_2(p) = G_2(q) \otimes M$ for a suitable degree 0 line bundle M, all lines of $G_2(p)$ are bad. □

2.6.2.3 Rank 2 bundles of degree 0

Theorem 2.6.20. *Consider the bundle $\mathcal{O} \oplus \mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$\mathcal{O} \oplus \mathcal{O} \leftarrow \mathcal{O} \oplus \mathcal{O}(-p) \qquad \text{for all } \ell_p \text{ (all lines are bad).}$$

Proof. We can choose a section s of $\mathcal{O} \oplus \mathcal{O}$ such that $s(p) \neq 0$ and $s = \ell_p$. We thus obtain a subbundle $\mathcal{O} \to \mathcal{O} \oplus \mathcal{O}$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(-p) \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r] & \mathcal{O} \arrow[r] & 0. \end{tikzcd}$$

Since $\mathrm{Ext}^1(\mathcal{O}(-p), \mathcal{O}) = H^0(\mathcal{O}(-p)) = 0$, we have that F splits, thus $F = \mathcal{O} \oplus \mathcal{O}(-p)$. Alternatively, one can write down explicit Hecke modifications, as in the proof of Theorem 2.6.16. □

Theorem 2.6.21. *Consider a bundle of the form $L \oplus L^{-1}$, where L is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$L \oplus L^{-1} \leftarrow \begin{cases} L \oplus (L^{-1} \otimes \mathcal{O}(-p)) & \textit{if } \ell_p = L_p \textit{ (a bad line),} \\ (L \otimes \mathcal{O}(-p)) \oplus L^{-1} & \textit{if } \ell_p = (L^{-1})_p \textit{ (a bad line),} \\ G_2(p) \otimes \mathcal{O}(-p) & \textit{otherwise (a good line).} \end{cases}$$

Proof. For $\ell_p = L_p$ or $\ell_p = (L^{-1})_p$, we can write down explicit Hecke modifications, as in the proof of Theorem 2.6.16. So assume $\ell_p \neq L_p$ and $\ell_p \neq (L^{-1})_p$. Choose a point $e \in X$

such that $(L \oplus L^{-1}) \otimes \mathcal{O}(e) = \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$ for points $q_1, q_2 \in X$ distinct from p. Since $L^2 \neq \mathcal{O}$, it follows that $q_1 \neq q_2$. Note that $q_1 + q_2 = 2e$. Let t_k be the unique (up to rescaling by a constant) nonzero section of $\mathcal{O}(q_k)$; note that $\operatorname{div} t_k = q_k$. We can define a section $s = (at_1, bt_2)$ of $\mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_p \neq L_p$ and $\ell_p \neq (L^{-1})_p$, it follows that $a \neq 0$ and $b \neq 0$, thus $\operatorname{div} s = 0$. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(q_1 + q_2 - p) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}(q_1) \oplus \mathcal{O}(q_2) \arrow[r] & \mathcal{O}(q_1 + q_2) \arrow[r] & 0.
\end{tikzcd}$$

There are no nonzero morphisms $\mathcal{O}(q_1 + q_2 - p) \to \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$, so F cannot split. Since $\det F = \mathcal{O}(q_1 + q_2 - p)$, we have that $F = G_2(q_1 + q_2 - p) = G_2(2(e-p) + p) = G_2(p) \otimes \mathcal{O}(e - p)$. $\square$

Theorem 2.6.22. *Consider the bundle F_2 (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$F_2 \leftarrow \begin{cases} \mathcal{O} \oplus \mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}_p \text{ (a bad line)}, \\ G_2(p) \otimes \mathcal{O}(-p) & \text{otherwise (a good line)}, \end{cases}$$

where $\mathcal{O} \to F_2$ is the unique degree 0 line subbundle of F_2.

Proof. (1) The case $\ell_q = \mathcal{O}_p$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(-p) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & F_2 \arrow[r] & \mathcal{O} \arrow[r] & 0.
\end{tikzcd}$$

Since $\operatorname{Ext}^1(\mathcal{O}(-p), \mathcal{O}) = H^0(\mathcal{O}(-p)) = 0$, we have that F splits, thus $F = \mathcal{O} \oplus \mathcal{O}(-p)$.

(2) The case $\ell_p \neq \mathcal{O}_p$. Pick a point $q \in X$ such that $q \neq p$. Choose sections t_0 and t_1 of $F_2 \otimes \mathcal{O}(q)$ as in Lemma 2.6.11. We can define a section $s = at_0 + bt_1$ of $F_2 \otimes \mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_q \neq \mathcal{O}_p$ it follows that $a \neq 0$, thus $\operatorname{div} s = 0$. We

thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to F_2 \otimes \mathcal{O}(q)$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(2q-p) \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & F_2 \otimes \mathcal{O}(q) \arrow[r] & \mathcal{O}(2q) \arrow[r] & 0. \end{tikzcd}$$

We claim that F cannot split. Assume for contradiction that F splits, thus $F = \mathcal{O}\oplus\mathcal{O}(2q-p)$. Then we can precompose α with the inclusion $\mathcal{O}(2q-p) \to \mathcal{O}\oplus\mathcal{O}(2q-p)$ to obtain a nonzero morphism $\mathcal{O}(2q-p) \to F_2 \otimes \mathcal{O}(q)$, contradicting Lemma 2.6.10. Since F does not split and $\det F = \mathcal{O}(2q-p)$, we have that $F = G_2(2q-p) = G_2(2(q-p)+p) = G_2(p) \otimes \mathcal{O}(q-p)$. $\square$

2.6.2.4 Observations

From this list, we make the following observations:

Lemma 2.6.23. *The following results hold for Hecke modifications of a rank 2 vector bundle E on an elliptic curve:*

1. *A Hecke modification of E changes the instability degree by ± 1.*
2. *Hecke modification of E corresponding to a line $\ell_p \in \mathbb{P}(E_p)$ changes the instability degree by -1 if ℓ_p is a good line and $+1$ if ℓ_p is a bad line.*
3. *A generic Hecke modification of E changes the instability degree by -1 unless E has the minimum possible instability degree -1, in which case all Hecke modifications of E change the instability degree by $+1$.*

2.6.3 Moduli spaces $\mathcal{P}_M^{tot}(X, m, n)$ and $\mathcal{P}_M(X, m, n)$

In Section 2.5.3 we defined a total space of marked parabolic bundles $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n) = \mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, n)$ for rational curves, and we showed that the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ could be reinterpreted as the subspace $\mathcal{P}_M(\mathbb{CP}^1, 2r) = \mathcal{P}_M(\mathbb{CP}^1, 3, 2r)$ of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 2r)$. We now want to generalize the spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ to the case of an elliptic curve X. There are obvious candidates: namely, the spaces $\mathcal{P}_M^{tot}(X, m, n)$ and $\mathcal{P}_M(X, m, n)$

for some value of m, which should be chosen to obtain the correct generalization. One possibility is to use same value $m = 3$ that we did for rational curves. But $m = 1$ is also yields a reasonable generalization, as can be understood from the following considerations.

In Section 2.2 we define a moduli space $M^{ss}(C)$ of semistable rank 2 vector bundles over a curve C with trivial determinant bundle, and in Section 2.3 we define a moduli space $M^s(C,m)$ of stable rank 2 parabolic bundles over a curve C with trivial determinant bundle and m marked points. Recall that for rational curves we chose $m = 3$ marking lines because we wanted $\mathcal{P}_M^{tot}(\mathbb{CP}^1, m, n)$ to be isomorphic to the space of parabolic bundles $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ in which the underlying vector bundle is $\mathcal{O} \oplus \mathcal{O}$, and

$$\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^s(\mathbb{CP}^1, 3) = M^{ss}(\mathbb{CP}^1) = \{[\mathcal{O} \oplus \mathcal{O}]\}.$$

For an elliptic curve X, however, the corresponding spaces $M^s(X,3)$ and $M^{ss}(X)$ are *not* isomorphic: the space $M^s(X,3)$ is a complex manifold of dimension 3, whereas $M^{ss}(X)$ is isomorphic to $\mathbb{CP}^1$. Instead we have the following results, which can be viewed as elliptic-curve analogs to Theorem 2.5.8 and Corollary 2.5.9 for rational curves:

Theorem 2.6.24. *The moduli space $M^s(X,1)$ consists of points $[E, \ell_q]$, where ℓ_q is a good line and either $E = F_2 \otimes L_i$ or $E = L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. Given any two parabolic bundles of the form (E, ℓ_q) and (E, ℓ'_q) representing points of $M^s(X,1)$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \mathrm{Aut}(E)$ such that $\phi(\ell_q) = \ell'_q$.*

Proof. If $[E, \ell_q] \in M^s(X,1)$ then E is semistable, $\det E = \mathcal{O}$, and ℓ_q is a good line. Since E is semistable and $\det E = \mathcal{O}$, it must be $L_i \oplus L_i$, $F_2 \otimes L_i$, or $L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. But Lemma 2.6.8 states that $L_i \oplus L_i$ has no good lines, so E cannot be $L_i \oplus L_i$. Lemmas 2.6.9 and 2.6.12 show that the remaining two possibilities for E do have good lines and also prove the statement regarding unique automorphisms. □

Corollary 2.6.25. *The map $M^s(X,1) \to M^{ss}(X)$, $[E, \ell_q] \mapsto [E]$ is an isomorphism.*

From these results, we see that there are *two* natural generalizations of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ to an elliptic curve. The generalization of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^{ss}(\mathbb{CP}^1)$ is

$$\mathcal{P}_M^{tot}(X, 1, 0) = M^s(X, 1) = M^{ss}(X),$$

which would lead us to choose $m = 1$ marking lines. The generalization of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^s(\mathbb{CP}^1, 3)$ is

$$\mathcal{P}_M^{tot}(X, 3, 0) = M^s(X, 3),$$

which would lead us to choose $m = 3$ marking lines. We will address the question of which of these values of m yields the correct generalization of the Seidel-Smith space in Section 2.7.

From Theorem 2.4.20, we have that $\mathcal{P}_M^{tot}(X, 1, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(X, 1) \cong \mathbb{CP}^1$. We will show that this bundle is trivial. To prove this result, we will use the marking line of $\mathcal{P}_M^{tot}(X, 1, n)$ to canonically identify $\mathbb{P}(E_p)$ with $M^{ss}(X) \cong \mathbb{CP}^1$ for $[E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}_M^{tot}(X, 1, n)$:

Lemma 2.6.26. *Fix a parabolic bundle (E, ℓ_q) such that $[E, \ell_q] \in M^s(X, 1)$, a point $p \in X$ such that $p \neq q$, and a point $e \in X$ such that $p + q = 2e$. There is a canonical isomorphism $\mathbb{P}(E_p) \to M^{ss}(X)$ given by*

$$\ell_p \mapsto [H(E, \ell_q, \ell_p) \otimes \mathcal{O}(e)].$$

Proof. Theorem 2.6.24 implies that ℓ_q is a good line and either $E = F_2 \otimes L_i$ or $E = L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. From Theorems 2.6.21 and 2.6.22, it follows that

$$H(E, \ell_q) = G_2(q) \otimes \mathcal{O}(-q) = G_2(p + 2(q - e)) \otimes \mathcal{O}(-q) = G_2(p) \otimes \mathcal{O}(-e).$$

The result now follows from Theorem 2.6.19. $\square$

Lemma 2.6.26 can be viewed as the elliptic-curve analog to Lemma 2.5.13 for rational curves. To perform calculations, it will be useful to explicitly evaluate the map $\mathbb{P}(E_p) \to M^{ss}(X)$ for bad lines $\ell_p \in \mathbb{P}(E_p)$. In general, we prove:

Lemma 2.6.27. *Fix distinct points $p, q \in X$ and a point $e \in X$ such that $p + q = 2e$. If ℓ_q is a good line, then*

$$H(L \oplus L^{-1}, \ell_q, L_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(p - e) = L \otimes \mathcal{O}(e - q),$$
$$H(L \oplus L^{-1}, \ell_q, (L^{-1})_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(q - e) = L \otimes \mathcal{O}(e - p),$$
$$H(L \oplus L^{-1}, L_q, (L^{-1})_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(q - e) = L \otimes \mathcal{O}(e - p),$$
$$H(F_2, \ell_q, \mathcal{O}_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = \mathcal{O}(p - e) = \mathcal{O}(e - q).$$

Proof. These results are straightforward calculations using the list of Hecke modifications in Section 2.6.2. As an example, we will prove the result involving F_2. From Theorem 2.6.22 we have that

$$H(F_2, \mathcal{O}_p) = \mathcal{O} \oplus \mathcal{O}(-p).$$

Since ℓ_q is a good line, the bundle $H(F_2, \mathcal{O}_p, \ell_q) = H(F_2, \ell_q, \mathcal{O}_p)$ must be semistable, and the result now follows from Theorem 2.6.17. $\square$

Theorem 2.6.28. *There is a canonical isomorphism $h : \mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$.*

Proof. Define $h_0 : \mathcal{P}_M^{tot}(X, 1, n) \to M^{ss}(X)$ by

$$h_0([E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [E].$$

For $i = 1, \cdots, n$, choose a point $e_i \in X$ such that $q_1 + p_i = 2e_i$ and define $h_i : \mathcal{P}_M^{tot}(X, 1, n) \to M^{ss}(X)$ by

$$h_i([E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [H(E, \ell_{q_1}, \ell_{p_i}) \otimes \mathcal{O}(e_i)].$$

Then $h := (h_0, h_1, \cdots, h_n)$ is an isomorphism by Theorem 2.4.20, Corollary 2.6.25, and Lemma 2.6.26. $\square$

For $n = 1$, the isomorphism $h : \mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$ appears to be closely related to an isomorphism $M^{ss}(X, 2) \to (\mathbb{CP}^1)^2$ defined in [Var16], and our definition of h was motivated by this isomorphism.

2.6.4 Embedding $\mathcal{P}_M(X,m,n) \to M^s(X,m+n)$

We will now describe a canonical open embedding of the space $\mathcal{P}_M(X,m,n)$ into the space of stable parabolic bundles $M^s(X,m+n)$. We first need two Lemmas:

Lemma 2.6.29. *Let $(E,\ell_{p_1},\cdots,\ell_{p_n})$ be a parabolic bundle over an elliptic curve X such that E is semistable. If the lines $\ell_{p_1},\cdots,\ell_{p_n}$ are bad in the same direction then $H(E,\ell_{p_1},\cdots,\ell_{p_n})$ has instability degree n.*

Proof. Up to tensoring with a line bundle, the bundle E has one of three forms:

(1) $E = \mathcal{O} \oplus \mathcal{O}$. Since $\ell_{p_1},\cdots,\ell_{p_n}$ are bad in the same direction, we have that $\ell_{p_1} = \cdots = \ell_{p_n}$ under a global trivialization of E in which all the fibers are identified with $\mathbb{C}^2$. A sequence of Hecke modifications with $\ell_{p_1} = \cdots = \ell_{p_n}$ is given by

$$\mathcal{O}\oplus\mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O}\oplus\mathcal{O}(-p_1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O}\oplus\mathcal{O}(-p_1-\cdots-p_n).$$

Here $\mathcal{O}\oplus\mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O}\oplus\mathcal{O}(-p_1)$ is a Hecke modification corresponding to ℓ_{p_1}, and for $i = 2,\cdots,n$ we define

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & f_i \end{pmatrix},$$

where f_i is the unique (up to rescaling by a constant) morphism from $\mathcal{O}(-p_1-\cdots-p_i)$ to $\mathcal{O}(-p_1-\cdots-p_{i-1})$. Thus $H(\mathcal{O}\oplus\mathcal{O},\ell_{p_1},\cdots,\ell_{p_n}) = \mathcal{O}\oplus\mathcal{O}(-p_1-\cdots-p_n)$ has instability degree n.

(2) $E = F_2$. Then $\ell_{p_i} = \mathcal{O}_{p_i}$ for $i = 1,\cdots,n$. A sequence of Hecke modifications with $\ell_{p_i} = \mathcal{O}_{p_i}$ for $i = 1,\cdots,n$ is given by

$$F_2 \xleftarrow[p_1]{\alpha_1} \mathcal{O}\oplus\mathcal{O}(-p_1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O}\oplus\mathcal{O}(-p_1-\cdots-p_n),$$

where $F_2 \xleftarrow[p_1]{\alpha_1} \mathcal{O}\oplus\mathcal{O}(-p_1)$ is a Hecke modification corresponding to $\ell_{p_1} = \mathcal{O}_{p_1}$ and α_i is as above for $i = 1,\cdots,n$. Thus $H(F_2,\mathcal{O}_{p_1},\cdots,\mathcal{O}_{p_n}) = \mathcal{O}\oplus\mathcal{O}(-p_1-\cdots-p_n)$ has instability degree n.

(3) $E = L \oplus L^{-1}$ for a degree 0 line bundle L such that $L^2 \neq \mathcal{O}$. Then either $\ell_{p_i} = L_{p_i}$ for $i = 1, \cdots, n$ or $\ell_{p_i} = (L^{-1})_{p_i}$ for $i = 1, \cdots, n$. A sequence of Hecke modifications with $\ell_{p_i} = L_{p_i}$ for $i = 1, \cdots, n$ is given by

$$L \oplus L^{-1} \xleftarrow[p_1]{\alpha_1} L \oplus (L^{-1} \otimes \mathcal{O}(-p_1)) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} L \oplus (L^{-1} \otimes \mathcal{O}(-p_1 - \cdots - p_n)),$$

where

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \otimes f_i \end{pmatrix}$$

and f_i is as above. Thus $H(L \oplus L^{-1}, L_{p_1}, \cdots, L_{p_n}) = L \oplus (L^{-1} \otimes \mathcal{O}(-p_1 - \cdots - p_n))$ has instability degree n. We can write down a similar sequence of Hecke modifications to show that $H(L \oplus L^{-1}, (L^{-1})_{p_1}, \cdots, (L^{-1})_{p_n}) = (L \otimes \mathcal{O}(-p_1 - \cdots - p_n)) \oplus L^{-1}$ has instability degree n. $\square$

Using Lemma 2.6.29 in place of Lemma 2.5.16, the proofs of Lemma 2.5.17 and Theorem 2.5.19 for rational curves carry over to the case of elliptic curves. We thus obtain:

Lemma 2.6.30. *Let $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ be a parabolic bundle over an elliptic curve X such that E is semistable. If $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable.*

Theorem 2.6.31. *There is a canonical open embedding $\mathcal{P}_M(X, m, n) \to M^s(X, m+n)$.*

2.6.5 Examples

Here we compute the space $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$. We first make some definitions:

Definition 2.6.32. The *Abel-Jacobi isomorphism* $X \to \mathrm{Jac}(X)$ is given by $p \mapsto [\mathcal{O}(p - e)]$ for a choice of basepoint $e \in X$.

Definition 2.6.33. We define a map $\pi : \mathrm{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [L \oplus L^{-1}]$.

Note that π is surjective and $\pi(L) = \pi(L^{-1})$, so $\pi : \mathrm{Jac}(X) \cong X \to M^{ss}(X) \cong \mathbb{CP}^1$ is a 2:1 branched cover with four branch points $[L_i \oplus L_i]$ corresponding to the four 2-torsion line bundles L_i.

Definition 2.6.34. Given a degree 0 divisor D on an elliptic curve X, define the *translation map* $\tau_D : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$, $[L] \mapsto [L \otimes \mathcal{O}(D)]$.

2.6.5.1 Calculate $\mathcal{P}_M(X, 1, 0)$

We have that

$$\mathcal{P}_M(X, 1, 0) = \mathcal{P}_M^{tot}(X, 1, 0) = M^s(X, 1) = M^{ss}(X) = \mathbb{CP}^1.$$

Note that the embedding $\mathcal{P}_M(X, 1, 0) \to M^s(X, 1)$ defined in Theorem 2.6.31 is an isomorphism.

2.6.5.2 Calculate $\mathcal{P}_M(X, 1, 1)$

Theorem 2.6.35. *The map* $g : \mathrm{Jac}(X) \to (M^{ss}(X))^2$, $g = (\pi, \pi \circ \tau_{p_1 - e_1})$ *is injective and has image the complement of* $h(\mathcal{P}_M(X, 1, 1))$*, where* $h : \mathcal{P}_M^{tot}(X, 1, 1) \to (M^{ss}(X))^2$ *is the isomorphism described in Theorem 2.6.28.*

Proof. First we show that g has image the complement of $h(\mathcal{P}_M(X, 1))$ in $(M^{ss}(X))^2$ Take a point $[E, \ell_{q_1}, \ell_{p_1}] \in \mathcal{P}_M^{tot}(X, 1, 1)$. From Theorems 2.6.21, 2.6.22, and 2.6.24, it follows that $H(E, \ell_{p_1}) = G_2(p_1) \otimes \mathcal{O}(-p_1)$ is stable if ℓ_{p_1} is a good line, and $H(E, \ell_{p_1})$ is unstable if ℓ_{p_1} is a bad line. So the complement of $\mathcal{P}_M(X, 1, 1)$ in $\mathcal{P}_M^{tot}(X, 1, 1)$ consists of isomorphism classes $[E, \ell_{q_1}, \ell_{p_1}]$ such that ℓ_{p_1} is a bad line, and is thus given by the union of the sets

$$S_1 = \{[L \oplus L^{-1}, \ell_{q_1}, L_{p_1}] \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},$$
$$S_2 = \{[L \oplus L^{-1}, \ell_{q_1}, (L^{-1})_{p_1}] \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},$$
$$S_3 = \{[F_2 \otimes L_i, \ell_{q_1}, (L_i)_{p_1}] \mid i = 1, 2, 3, 4\},$$

where in each case ℓ_{q_1} is a good line. From Lemma 2.6.27, it follows that the complement of $h(\mathcal{P}_M(X, 1, 1))$ in $(M^{ss}(X))^2$ is given by the union of the sets

$$h(S_1) = \{(\pi([L]),\ (\pi \circ \tau_{p_1 - e_1})([L])) \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},$$
$$h(S_2) = \{(\pi([L]),\ (\pi \circ \tau_{e_1 - p_1})([L])) \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},$$
$$h(S_3) = \{(\pi([L_i]),\ (\pi \circ \tau_{p_1 - e_1})([L_i])) \mid i = 1, 2, 3, 4\}.$$

Note that

$$(\pi([L]),\, (\pi\circ\tau_{e_1-p_1})([L])) = (\pi([L^{-1}]),\, (\pi\circ\tau_{p_1-e_1})([L^{-1}])),$$

so $h(S_1) = h(S_2)$, and we have that

$$h(S_1)\cup h(S_2)\cup h(S_3) = \{(\pi([L]),\, (\pi\circ\tau_{p_1-e_1})([L])) \mid [L]\in \mathrm{Jac}(X)\} = \mathrm{im}\, g.$$

So the image of g is the complement of $h(\mathcal{P}_M(X,1,1))$ in $(M^{ss}(X))^2$

Next we show that g is injective. If $g(L) = g(L')$, then projection onto the first factor of $(M^{ss}(X))^2$ gives $\pi(L) = \pi(L')$, hence either $L' = L$ or $L' = L^{-1}$. Suppose $L' = L^{-1}$. Then projection onto the second factor of $(M^{ss}(X))^2$ gives $\pi(L\otimes\mathcal{O}(p_1-e_1)) = \pi(L^{-1}\otimes\mathcal{O}(p_1-e_1))$, hence either $L\otimes\mathcal{O}(p_1-e_1) = L^{-1}\otimes\mathcal{O}(p_1-e_1)$ or $L\otimes\mathcal{O}(p_1-e_1) = L\otimes\mathcal{O}(e_1-p_1)$. The first case implies $L = L^{-1}$. The second case implies $2p_1 = 2e_1$, but we chose e_1 such that $p_1+q_1 = 2e_1$, hence $p_1 = q_1$, contradiction. Thus $L' = L$, so g is injective. □

If we use the Abel-Jacobi isomorphism to identify X and $\mathrm{Jac}(X)$, the (canonical) isomorphism $h : \mathcal{P}_M^{tot}(X,1,1)\to(M^{ss}(X))^2$ to identify $\mathcal{P}_M^{tot}(X,1,1)$ and $(M^{ss}(X))^2$, and the (noncanonical) isomorphism $M^{ss}(X)\cong\mathbb{CP}^1$ to identify $M^{ss}(X)$ and $\mathbb{CP}^1$, we find that

$$\mathcal{P}_M(X,1,1) = (\mathbb{CP}^1)^2 - g(X).$$

Remark 2.6.36. Using results from the proof of Theorem 2.6.35, it is straightforward to show that

$$M^{ss}(X,2) = \mathcal{P}_M^{tot}(X,1,1) = (\mathbb{CP}^1)^2, \qquad M^{s}(X,2) = \mathcal{P}_M(X,1,1) = (\mathbb{CP}^1)^2 - g(X).$$

These calculations reproduce the results of [Var16] for $M^{ss}(X,2)$ and $M^s(X,2)$.

2.6.5.3 Calculate $\mathcal{P}_M(X,1,2)$

The same method that we used to prove Theorem 2.6.35 can be used to calculate $\mathcal{P}_M(X,1,2)$:

Theorem 2.6.37. *The map $f : \mathrm{Jac}(X)\to(M^{ss}(X))^3$, $f = (\pi,\, \pi\circ\tau_{p_1-e_1},\, \pi\circ\tau_{p_2-e_2})$ is injective and has image the complement of $h(\mathcal{P}_M(X,1,2))$, where $h : \mathcal{P}_M^{tot}(X,1,2)\to(M^{ss}(X))^3$ is the isomorphism described in Theorem 2.6.28.*

If we use the Abel-Jacobi isomorphism to identify X and $\mathrm{Jac}(X)$, the (canonical) isomorphism $h : \mathcal{P}_M^{tot}(X,1,2) \to (M^{ss}(X))^3$ to identify $\mathcal{P}_M^{tot}(X,1,2)$ and $(M^{ss}(X))^3$, and the (noncanonical) isomorphism $M^{ss}(X) \cong \mathbb{CP}^1$ to identify $M^{ss}(X)$ and $\mathbb{CP}^1$, we find that

$$\mathcal{P}_M(X,1,2) = (\mathbb{CP}^1)^3 - f(X).$$

2.7 Possible applications to topology

Here we briefly outline some possible applications of our results to topology. We have proposed complex manifolds $\mathcal{P}_M(X,1,2r)$ and $\mathcal{P}_M(X,3,2r)$ as candidates for a space $\mathcal{Y}(T^2,2r)$ that generalizes the Seidel-Smith space $\mathcal{Y}(S^2,2r)$ and that could potentially be used to construct symplectic Khovanov homology for lens spaces. The following tasks remain to be done to complete the construction:

1. We need to define a suitable symplectic form on $\mathcal{P}_M(X,m,2r)$. One possibility is to pull back the canonical symplectic form on $M^s(X,2r+m)$ using the open embedding $\mathcal{P}_M(X,m,2r) \to M^s(X,2r+m)$.

2. We need to find a suitable action of the mapping class group $\mathrm{MCG}_{2r}(T^2)$ on $\mathcal{P}_M(X,m,2r)$ that is defined up to Hamiltonian isotopy. Such an action might be obtained via symplectic monodromy by viewing $\mathcal{P}_M(X,m,2r)$ as the fiber of a larger space that fibers over the moduli space of genus 1 curves with marked points. Such an approach would be analogous the way Seidel an Smith obtain an action of the braid group on the Seidel-Smith space via monodromy around loops in the configuration space [Sei03], and similar methods are used to define mapping class group actions for constructing Reshetikhin-Turaev-Witten invariants [BK01].

3. We need to define suitable Lagrangians L_r in $\mathcal{P}(X,m,2r)$ corresponding to r unknotted arcs in a solid torus. For $\mathcal{P}_M(X,1,2r)$ we would expect L_r to be homeomorphic to $S^1 \times (S^2)^r$, and for $\mathcal{P}_M(X,3,2r)$ we would expect L_r to be homeomorphic to $S^3 \times (S^2)^r$. Perhaps such Lagrangians can be constructed in a manner analogous to Seidel-Smith by viewing $\mathcal{P}_M(X,m,2r)$ as the fiber of a larger space that fibers over the configuration

space $\mathrm{Conf}_{2r}(X)$ of $2r$ unordered points on X and looking for vanishing cycles as points successively brought together in pairs.

4. We need to prove that the Lagrangian Floer homology of a knot K in a lens space is Y invariant under different Heegaard splittings of (Y, K) into solid tori.

5. We need to verify that our construction of symplectic Khovanov homology reproduces ordinary Khovanov homology for the case of knots in S^3.

Several of our results appear to be related to a possible connection between Khovanov homology and symplectic instanton homology. Roughly speaking, symplectic instanton homology is defined as follows. Given a knot K in a 3-manifold Y, one Heegaard-splits (Y, K) along a Heegaard surface Σ to obtain handlebodies U_1 and U_2. Each handlebody U_i contains a portion of the knot $A_i := U_i \cap K$ consisting of r arcs that pairwise connect points $p_1, \cdots, p_{2r}$ in Σ. To the marked surface $(\Sigma, p_1, \cdots, p_{2r})$ one associates a character variety $R(\Sigma, 2r)$, which has the structure of a symplectic manifold, and to the handlebody pairs (U_i, A_i) one associates Lagrangians $L_i \subset R(\Sigma, 2r)$. The symplectic instanton homology of (Y, K) is then defined to be the Lagrangian Floer homology of the pair of Lagrangians (L_1, L_2).

In fact, there are several technical difficulties that must be overcome in order to get a well-defined homology theory. For example, one needs to introduce a *framing* in order to eliminate singularities in the character variety $R(\Sigma, 2r)$. One way to introduce a framing is by replacing the knot K with $K \cup \Theta$, where Θ is the theta graph shown in Figure 2.1(a); this approach is described in [Hor16]. We Heegaard-split $(Y, K \cup \Theta)$ along a Heegaard surface Σ that is chosen to transversely intersect each edge e_i of the theta graph in a single point q_i. The marked Heegaard surface is now $(\Sigma, q_1, q_2, q_3, p_1, \cdots, p_{2r})$, corresponding to the character variety $R(\Sigma, 2r+3)$, and the handlebody pairs are now $(U_i, A_i \cup \epsilon_i)$, where ϵ_i is the epsilon graph shown in Figure 2.1(b). The character variety $R(\Sigma, 2r+3)$ has the structure of a symplectic manifold that is symplectomorphic to the moduli space of stable parabolic bundles $M^s(C, 2r+3)$, where C is any complex curve homeomorphic to Σ. (The space $M^s(C, 2r+3)$ has a canonical symplectic form.)

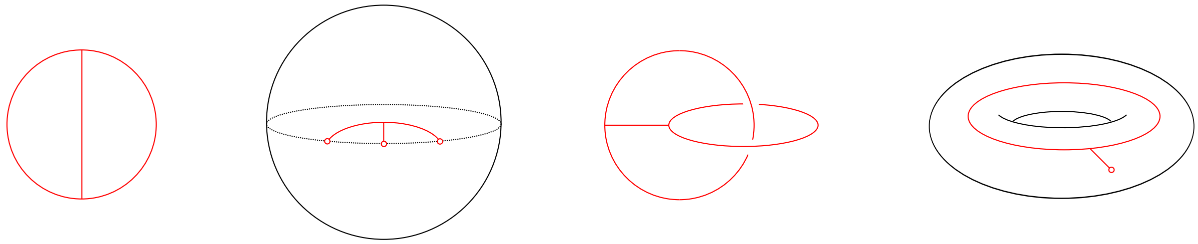

Figure 2.1: (a) The graph Θ. (b) The graph ϵ in B^3. (c) The graph $D_p \subset S^3$ for $p = 1$. (d) The graph σ in $S^1 \times D^2$.

Symplectic instanton homology can be viewed as a symplectic replacement for singular instanton homology, a knot homology theory defined using gauge theory, and the two theories are conjectured to be isomorphic. This is an example of an Atiyah-Floer conjecture, which broadly relates Floer-theoretic invariants defined using gauge theory to corresponding invariants defined using symplectic topology. Kronheimer and Mrowka constructed a spectral sequence from Khovanov homology to singular instanton homology [KM14], and the embedding $\mathcal{P}_M(\mathbb{CP}^1, 2r) \to M^s(\mathbb{CP}^1, 2r + 3)$ described in Theorem 2.5.4 suggests that it may be possible to construct an analogous spectral sequence from symplectic Khovanov homology to symplectic instanton homology. (This idea for constructing a spectral sequence was suggested to the author by Ivan Smith and Chris Woodward.) If so, perhaps the fact that we have an embedding $\mathcal{P}_M(X, 3, 2r) \to M^s(X, 2r+3)$, as described in Theorem 2.6.4, is evidence that the correct generalization of the Seidel-Smith space is $\mathcal{P}_M(X, 3, 2r)$. Indeed, a calculation of the Lagrangian intersection $L_1 \cap L_2$ in the traceless character variety $R(T^2, 3)$ for (S^3, Θ) yields a single point, a space whose cohomology is the correct Khovanov homology for the empty knot, and calculations of the Lagrangian intersections in the traceless character variety $R(T^2, 5)$ for $(S^3, \text{unknot} \cup \Theta)$ and $(S^3, \text{trefoil} \cup \Theta)$ yield S^2 and $\mathbb{RP}^3 \amalg S^2$, spaces whose cohomology gives the correct Khovanov homology for the unknot and trefoil. Based on these speculations, we make the following conjectures:

Conjecture 2.7.1. *The space $\mathcal{P}_M(C, 3, 2r)$ is the correct generalization of the Seidel-Smith space $\mathcal{Y}(S^2, 2r)$ to a curve C of arbitrary genus.*

Conjecture 2.7.2. *Given a curve C of arbitrary genus, there is a canonical open embedding $\mathcal{P}_M(C,m,n) \to M^s(C,m+n)$.*

On the other hand, perhaps the embedding $\mathcal{P}_M(X,1,2r) \to M^s(X,2r+1)$ described in Theorem 2.6.4 is related to a spectral sequence from a Khovanov-like knot homology theory to symplectic instanton homology defined with a novel framing. Rather than using a theta graph, perhaps for the lens space $L(p,q)$ one could introduce a framing specific to that lens space by using a p-linked dumbbell graph D_p, as shown in Figure 2.1(c) for the case $p=1$. There is a unique edge e_1 of the dumbbell graph that connects the two vertices, and one can choose a Heegaard surface Σ that transversely intersects e_1 in a single point q_1. The marked Heegaard surface is now $(\Sigma, q_1, p_1, \cdots, p_{2r})$, corresponding to the character variety $R(\Sigma, 2r+1)$, and the handlebody pairs (U_i, A_i) are now $(U_i, A_i \cup \sigma_i)$, where σ_i is the sigma graph shown in Figure 2.1(d). The character variety $R(\Sigma, 2r+1)$ has the structure of a symplectic manifold that is symplectomorphic to the moduli space of stable parabolic bundles $M^s(X,2r+1)$, which is the codomain of the embedding $\mathcal{P}_M(X,1,2r) \to M^s(X,2r+1)$.

CHAPTER 3

Singular instanton homology for knots in lens spaces

3.1 Introduction

We describe here a scheme for constructing generating sets for Kronheimer and Mrowka's singular instanton knot homology for the case of knots in lens spaces. In outline, our approach is as follows. We Heegaard-split a lens space Y containing a knot K into two solid tori U_1 and U_2. The solid torus U_1 contains a portion of the knot consisting of an unknotted arc, together with a specific holonomy perturbation. The solid torus U_2 contains the remainder of the knot. From the Heegaard splitting of Y we obtain a pair of Lagrangians L_1 and L_2 in the traceless $SU(2)$-character variety of the twice-punctured torus $R(T^2, 2)$, and in many cases the points of intersection of L_1 and L_2 constitute a generating set for the (reduced) singular instanton homology $I^\natural(Y, K)$.

To explain the details of our scheme, we must first define several character varieties and explain their relationship to the Chern-Simons functional. Critical points of the unperturbed Chern-Simons functional are flat connections. Gauge-equivalence classes of flat connections correspond to conjugacy classes of homomorphisms $\rho : \pi_1(Y - K \cup H \cup W) \to SU(2)$, where H is a small loop around K and W is an arc connecting K to H, as shown in Figure 3.1, and the homomorphisms are required to take loops around K and H to traceless matrices and loops around W to -1. The space of such conjugacy classes form a character variety that we will denote by $R^\natural(Y, K)$. We will refer to $H \cup W$ as an *earring* that has been added to the knot K. The conditions on ρ involving the earring are imposed in order to avoid reducible connections; such connections prevent us from obtaining a chain complex for singular instanton homology, with a differential that squares to zero. It will also be useful to

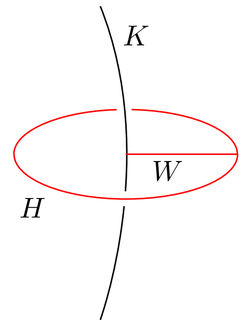

Figure 3.1: The knot K, loop H, and arc W.

define a character variety $R(Y,K)$ in which we do not impose these conditions, and which consists of conjugacy classes of homomorphisms $\rho : \pi_1(Y-K) \to SU(2)$ that take loops around K to traceless matrices.

The character variety $R^\natural(Y,K)$ is typically degenerate, in which case the unperturbed Chern-Simons functional is not Morse. We can render the Chern-Simons functional Morse by introducing a suitable holonomy perturbation term that vanishes outside of a small solid torus obtained by thickening a loop $P \subset Y$. The net effect of the perturbation is to modify the corresponding character variety: the critical points of the perturbed Chern-Simons functional correspond to conjugacy classes of homomorphisms $\rho : \pi_1(Y - K \cup H \cup W \cup P) \to SU(2)$, where ρ obeys the same conditions as for $R^\natural(Y,K)$ as well as an additional condition involving the loop P that we will describe in Section 3.3.3. We will denote the character variety corresponding to the perturbed Chern-Simons functional by $R^\natural_\pi(Y,K)$.

Example 3.1.1. For the trefoil K in S^3, one can show that

$$R(S^3,K) = \{2 \text{ points}\}, \qquad R^\natural(S^3,K) = \{1 \text{ point}\} \amalg S^1, \qquad R^\natural_\pi(S^3,K) = \{3 \text{ points}\},$$

where the perturbation used to define $R^\natural_\pi(S^3,K)$ is as described in Section 3.6.1.

Our goal, then, is to devise an effective means of calculating $R^\natural_\pi(Y,K)$. We will view (Y,K) as the result of gluing together two solid tori $U_1 = S^1 \times D^2$ and $U_2 = S^1 \times D^2$. The solid torus U_1 contains an unknotted arc A_1, the earring $H \cup W$, and the holonomy perturbation loop P, as shown in Figure 3.6. The solid torus U_2 contains a (possibly knotted) arc A_2. We glue the two tori together via a homeomorphism $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ to obtain (Y, K).

We define character varieties $R^\natural_\pi(U_1, A_1)$ and $R(U_2, A_2)$ in analogy with $R^\natural_\pi(Y, K)$ and $R(Y, K)$. The character variety $R^\natural_\pi(U_1, A_1)$ consists of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that take loops around A_1 and H to traceless matrices and loops around W to -1, and satisfy an additional requirement involving P as described in Section 3.3.3. The character variety $R(U_2, A_2)$ consists of conjugacy classes of homomorphisms $\rho : \pi_1(U_2 - A_2) \to SU(2)$ that take loops around A_2 to traceless matrices. We define a torus $T^2 := \partial U_1$ containing points $\{p_1, p_2\} = \partial A_1$, and we define a corresponding character variety $R(T^2, 2)$ that consists of conjugacy classes of homomorphisms $\rho : \pi_1(T^2 - \{p_1, p_2\}) \to SU(2)$ that take loops around p_1 and p_2 to traceless matrices.

We define a map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ by pulling back along the inclusion $(\partial U_1, \partial A_1) \hookrightarrow (U_1, A_1)$. We define a map $R(U_2, A_2) \to R(T^2, 2)$ by pulling back along the composition of $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ with the inclusion $(\partial U_2, \partial A_2) \hookrightarrow (U_2, A_2)$. We similarly define maps $R^\natural_\pi(Y, K) \to R^\natural_\pi(U_1, A_1)$ and $R^\natural_\pi(Y, K) \to R(U_2, A_2)$ by pulling back along inclusions. We have a commutative diagram:

$$\begin{tikzcd}
R^\natural_\pi(Y,K) \arrow[dr, "p"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R^\natural_\pi(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2) \arrow[r] \arrow[d] & R(U_2, A_2) \arrow[d] \\
& R^\natural_\pi(U_1, A_1) \arrow[r] & R(T^2, 2).
\end{tikzcd} \tag{3.1}$$

Here p is an induced map from $R^\natural_\pi(Y, K)$ to the fiber product $R^\natural_\pi(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2)$. The character variety $R(T^2, 2)$ is a symplectic manifold that generalizes the pillowcase, and the images of the maps $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ and $R(U_2, A_2) \to R(T^2, 2)$ define Lagrangians L_1 and L_2 in $R(T^2, 2)$. We want to use diagram (3.1) to describe $R^\natural_\pi(Y, K)$ in terms of the intersection points of these Lagrangians. Our first task is to obtain an explicit description of the character variety $R(T^2, 2)$. We prove:

Theorem 3.1.2. *The character variety $R(T^2, 2)$ is the union of two pieces P_4 and P_3, where P_4 is homeomorphic to $S^2 \times S^2 - \bar{\Delta}$ and $\bar{\Delta} = \{(\hat{r}, -\hat{r})\}$ is the antidiagonal, and P_3 deformation retracts onto the pillowcase. (The spaces P_4 and P_3 are described in Theorems 3.3.10 and 3.3.13.)*

Our next task is to explicitly describe the Lagrangian L_1. We prove:

Theorem 3.1.3. *The character variety $R^\natural_\pi(U_1, A_1)$ is homeomorphic to S^2. The map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ is an injective immersion away from the points of $R^\natural_\pi(U_1, A_1)$ corresponding to the north and south pole of S^2, which are mapped to the same point. All representations in the image L_1 of the map are nonabelian. (An explicit parameterization of L_1 is given in Theorem 3.3.25.)*

Corollary 3.1.4. *The map $p : R^\natural_\pi(Y, K) \to R^\natural_\pi(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2)$ in diagram (3.1) is injective.*

Proof. Consider a point $([\rho_1], [\rho_2])$ in $R^\natural_\pi(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2)$, so ρ_1 and ρ_2 pull back to the same homomorphism $\rho_{12} : \pi_1(T^2 - \{p_1, p_2\}) \to SU(2)$. One can show (see [HHK14] Lemma 4.2) that the fiber $p^{-1}([\rho_1], [\rho_2])$ is homeomorphic to the double coset space $\operatorname{Stab}(\rho_1) \backslash \operatorname{Stab}(\rho_{12}) / \operatorname{Stab}(\rho_2)$, where

$$\operatorname{Stab}(\rho) = \{g \in SU(2) \mid g\rho(x)g^{-1} = \rho(x) \text{ for all } x \text{ in the domain of } \rho\}.$$

The center of $SU(2)$ is $Z(SU(2)) = \{\pm 1\}$. By Theorem 3.1.3 we have that $\operatorname{Stab}(\rho_{12}) = Z(SU(2))$, and $Z(SU(2)) \subseteq \operatorname{Stab}(\rho_i) \subseteq \operatorname{Stab}(\rho_{12})$, so $\operatorname{Stab}(\rho_i) = \operatorname{Stab}(\rho_{12}) = Z(SU(2))$ and thus the fibers of p are points. $\square$

By introducing a suitable holonomy perturbation, we obtain a finite character variety $R^\natural_\pi(Y, K)$, each point of which corresponds to a gauge-orbit of connections that are critical points of the perturbed Chern-Simons functional. In order for $R^\natural_\pi(Y, K)$ to serve as a generating set for singular instanton homology, each point in $R^\natural_\pi(Y, K)$ must be nondegenerate; that is, at each connection representing a point in $R^\natural_\pi(Y, K)$ we want the Hessian of the perturbed Chern-Simons functional to be nondegenerate when restricted to a complement of the tangent space to the gauge-orbit of that connection. We show that there is a simple criterion for determining when a point $R^\natural_\pi(Y,K)$ is nondegenerate. Recall that we defined the Lagrangian L_2 to be the image of $R(U_2,A_2) \to R(T^2,2)$. If $R(U_2,A_2) \to R(T^2,2)$ is injective and $[\rho] \in L_1 \cap L_2 \subset R(T^2,2)$ is not the double-point of L_1, then by Corollary 3.1.4 the point $[\rho]$ is the image of a unique point $[\tilde{\rho}] \in R^\natural_\pi(Y,K)$ under the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$. We prove:

Theorem 3.1.5. *Suppose $R(U_2,A_2) \to R(T^2,2)$ is an injective immersion and $[\rho] \in L_1 \cap L_2$ is the image of a regular point of $R(U_2,A_2)$ and is not the double-point of L_1. Then the unique preimage $[\tilde{\rho}]$ of $[\rho]$ under the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$ is nondegenerate if and only if the intersection of L_1 with L_2 at $[\rho]$ is transverse.*

Collecting these results, we find if the hypotheses of Theorem 3.1.5 are satisfied for every point in $L_1 \cap L_2$, then every point in $R^\natural_\pi(Y,K)$ is nondegenerate and the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$ is injective with image $L_1 \cap L_2$. Thus we obtain:

Corollary 3.1.6. *If the hypotheses of Theorem 3.1.5 are satisfied for every point in $L_1 \cap L_2$, then $R^\natural_\pi(Y,K)$ is a generating set for $I^\natural(Y,K)$ consisting of $|L_1 \cap L_2|$ generators.*

Our scheme is particularly well-suited for the case of $(1,1)$-knots. By definition, a $(1,1)$-knot is a knot K in a lens space Y that has a Heegaard splitting into a pair of solid tori $U_1, U_2 \subset Y$ such that the components $U_1 \cap K$ and $U_2 \cap K$ of the knot in each solid torus are unknotted arcs. It is known that $(1,1)$-knots include all torus knots and 2-bridge knots.

We can construct $(1,1)$-knots by taking (U_2,A_2) to be a copy of (U_1,A_1) without the earring $H \cup W$ or the perturbation loop P, and we can explicitly describe the corresponding Lagrangian L_2 as follows. We first define a character variety $R(U_1,A_1)$ that consists of conjugacy classes of homomorphisms $\pi_1(U_1 - A_1) \to SU(2)$ that take loops around A_1 to traceless matrices. We define a map $R(U_1,A_1) \to R(T^2,2)$ by pulling back along the inclusion $(\partial U_1, \partial A_1) \hookrightarrow (U_1,A_1)$. The image of this map defines a Lagrangian L_d in $R(T^2,2)$. We can view $R(U_1,A_1)$ and L_d as "unperturbed" versions of $R^\natural_\pi(U_1,A_1)$ and L_1. Since $(T^2,\{p_1,p_2\}) := (\partial U_1, \partial A_1)$ and there is a natural identification $(\partial U_2, \partial A_2) \xrightarrow{\sim} (T^2,\{p_1,p_2\})$, the gluing map $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ defines an element $[\phi]$ of the mapping class group $\mathrm{MCG}_2(T^2)$ of the twice-punctured torus. The group $\mathrm{MCG}_2(T^2)$ acts on $R(T^2, 2)$ from the right in a way that we explicitly describe in Section 3.5, and the Lagrangian L_2, which we defined to be the image of the map $R(U_2, A_2) \to R(T^2, 2)$, is given by $L_2 = L_d \cdot [\phi]$. We prove results that explicitly describe the character variety $R(U_1, A_1)$ and the Lagrangian L_d:

Theorem 3.1.7. *The character variety $R(U_1, A_1)$ is homeomorphic to the closed disk D^2. The map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. (An explicit parameterization of the image L_d of the map is given in Theorem 3.3.20.)*

Theorem 3.1.8. *The character variety $R(U_1, A_1)$ is regular on its interior.*

From Theorems 3.1.7 and 3.1.8, we obtain a Corollary to Theorem 3.1.5 for the special case of $(1, 1)$-knots:

Corollary 3.1.9. *For a $(1, 1)$-knot K, if L_1 intersects $L_2 = L_d \cdot [\phi]$ transversely away from the double-point of L_1, then $R^\natural_\pi(Y, K)$ is a generating set for $I^\natural(Y, K)$ consisting of $|L_1 \cap L_2|$ generators.*

Since we have explicit descriptions of the character variety $R(T^2, 2)$, the Lagrangians L_1 and L_d, and the action of the mapping class group $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$, Corollary 3.1.9 provides us with a practical scheme for calculating generating sets for $I^\natural(Y, K)$ for any $(1, 1)$-knot K in any lens space Y.

3.2 The group $SU(2)$

Here we briefly review some basic facts about the group $SU(2)$. We define $SU(2)$-matrices $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ by

$$\mathbf{i} = -i\sigma_x, \qquad \mathbf{j} = -i\sigma_y, \qquad \mathbf{k} = -i\sigma_z,$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy the quaternion multiplication laws $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Any $SU(2)$-matrix A can be uniquely expressed as

$$A = t + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

where $(t, x, y, z) \in S^3 = \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 + x^2 + y^2 + z^2 = 1\}$, and thus we may identify $SU(2)$ with the space of unit quaternions. We will refer to t and $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ as the *scalar* and *vector* parts of the matrix A, respectively. Note that $\mathrm{tr}(A) = 2t$, so traceless $SU(2)$-matrices are precisely those for which the scalar part is zero. It follows that traceless $SU(2)$-matrices are parameterized by unit vectors in $\mathbb{R}^3$, and we will frequently pass back and forth between traceless matrices $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \in SU(2)$ and their corresponding unit vectors $\hat{a} = (a_x, a_y, a_z) \in S^2$.

We can define a surjective group homomorphism $SU(2) \to SO(3)$ by $g \mapsto (\hat{v} \mapsto \hat{v}')$, where the unit vectors $\hat{v} = (v_x, v_y, v_z)$ and $\hat{v}' = (v'_x, v'_y, v'_z)$ are related by

$$g(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k})g^{-1} = v'_x\,\mathbf{i} + v'_y\,\mathbf{j} + v'_z\,\mathbf{k}.$$

In general, conjugating an arbitrary $SU(2)$-matrix preserves the scalar part of the matrix and rotates the vector part of the matrix:

$$g(t + r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})g^{-1} = t + r'_x\,\mathbf{i} + r'_y\,\mathbf{j} + r'_z\,\mathbf{k},$$

where (r'_x, r'_y, r'_z) is given by multiplying (r_x, r_y, r_z) by the $SO(3)$-matrix corresponding to $g \in SU(2)$. We will thus sometimes describe conjugation in terms of the corresponding rotation performed on the vector part of an $SU(2)$-matrix.

3.3 Character varieties

3.3.1 The character variety $R(T^2, 2)$

Our first task is to understand the structure of $R(T^2, 2)$, the traceless $SU(2)$-character variety of the twice-punctured torus. In general, we make the following definition:

Definition 3.3.1. Given a surface S with n distinct marked points $p_1, \cdots, p_n \in S$, we define the character variety $R(S, n)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(S - \{p_1, \cdots, p_n\}) \to SU(2)$ that take loops around the marked points to traceless $SU(2)$-matrices.

Before examining the space $R(T^2, 2)$, we first consider the simpler space $R(T^2) := R(T^2, 0)$, which is known as the *pillowcase*. We have the following well-known result:

Theorem 3.3.2. *The pillowcase $R(T^2)$ is homeomorphic to S^2.*

Proof. The fundamental group of T^2 is $\pi_1(T^2) = \langle A, B \mid ABA^{-1}B^{-1} = 1 \rangle$, where A and B are represented by the two fundamental cycles. A homomorphism $\rho : \pi_1(T^2) \to SU(2)$ is uniquely determined by the pair of matrices $(\rho(A), \rho(B))$, which for simplicity we will also denote by (A, B). Since A and B commute, any conjugacy class $[\rho] \in R(T^2)$ has a representative (A, B) of the form

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad\qquad B = \cos\beta + \sin\beta\,\mathbf{k}$$

for some angles α and β. These equations are invariant under the replacements $\alpha \to \alpha + 2\pi$ and $\beta \to \beta + 2\pi$, and we can simultaneously flip the signs of α and β by conjugating by $\mathbf{i}$, so we obtain the following identifications:

$$(\alpha, \beta) \sim (\alpha + 2\pi, \beta), \qquad (\alpha, \beta) \sim (\alpha, \beta + 2\pi), \qquad (\alpha, \beta) \sim (-\alpha, -\beta).$$

We can thus restrict to a fundamental domain in which $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$, with edges identified as shown in Figure 3.2. From Figure 3.2 it is clear that this space is homeomorphic to S^2. $\square$

Definition 3.3.3. We will refer to the four points $[A, B] = [\pm 1, \pm 1] \in R(T^2)$ as *pillowcase points.*

Remark 3.3.4. One can show that the character variety $R(S^2, 4)$ that is used in the work of Hedden, Herald, and Kirk is also described by a rectangle with edges identified as shown in Figure 3.2 (see, for example, [HHK14] Section 3.1), so both $R(T^2)$ and $R(S^2, 4)$ are referred to as the pillowcase.

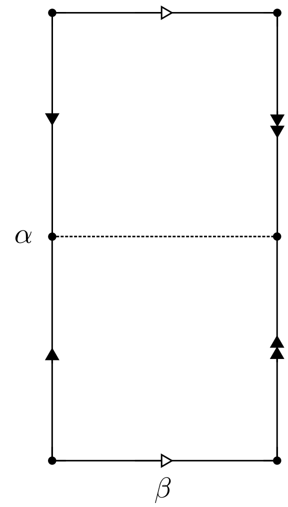

Figure 3.2: The pillowcase $R(T^2)$. The black dots indicate the four pillowcase points.

We now consider the space $R(T^2, 2)$. The fundamental group of the twice-punctured torus is

$$\pi_1(T^2 - \{p_1, p_2\}) = \langle A, B, a, b \mid ABA^{-1}B^{-1}ab = 1 \rangle,$$

where p_1 and p_2 denote the puncture points, A and B denote the fundamental cycles of the torus, and a and b denote loops around the punctures p_1 and p_2, as shown in Figure 3.3. As above, we will use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. A homomorphism $\rho : \pi_1(T^2 - \{p_1, p_2\}) \to SU(2)$ is thus specified by $SU(2)$-matrices (A, B, a, b) such that a and b are traceless and $ABA^{-1}B^{-1}ab = 1$, and we will sometimes denote a homomorphism ρ by the corresponding list of matrices (A, B, a, b).

The structure of $R(T^2, 2)$ can be understood by considering the fibers of the following map:

Definition 3.3.5. We define a map $\mu : R(T^2, 2) \to [-1, 1]$ by

$$\mu([A, B, a, b]) = (1/2)\,\mathrm{tr}(ABA^{-1}B^{-1}) = (1/2)\,\mathrm{tr}((ab)^{-1}).$$

In particular, it is convenient to decompose $R(T^2, 2)$ into the disjoint union of an open piece $P_4 = \mu^{-1}([-1, 1))$ and a closed piece $P_3 = \mu^{-1}(1)$. The notation for these pieces is

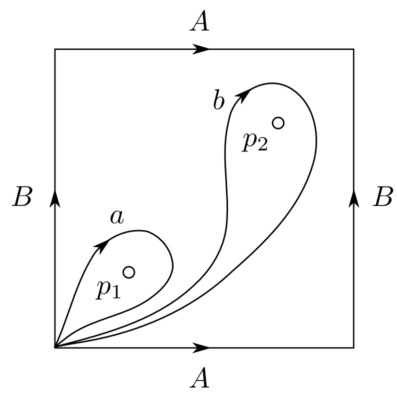

Figure 3.3: Cycles corresponding to the generators A, B, a, b of the fundamental group $\pi_1(T^2 - \{p_1, p_2\})$.

motivated by the fact that, as we will see, the piece P_4 is four-dimensional and the piece P_3 is three-dimensional. We will describe the topology of the pieces P_3 and P_4 and define coordinate systems on each piece that are useful for performing calculations.

3.3.1.1 The piece $P_4 \subset R(T^2, 2)$

We define the piece $P_4 \subset R(T^2, 2)$ to be the set of conjugacy classes $[\rho] \in R(T^2, 2)$ such that $\mu([\rho]) \in [-1, 1)$. For any representative (A, B, a, b) of a given conjugacy class $[\rho] \in P_4$, the matrices A and B do not commute. This fact can be used to choose a canonical representative of each conjugacy class in P_4:

Lemma 3.3.6. *Any conjugacy class $[\rho] \in P_4$ has a unique representative (A, B, a, b) for which*

$$A = r\cos\alpha + \sqrt{1-r^2}\,\mathbf{i} + r\sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \tag{3.2}$$

where $\alpha \in [0, 2\pi]$, $\beta \in (0, \pi)$, and $r \in [0, 1)$.

Proof. Since $[\rho] \in P_4$, for any representative of $[\rho]$ the matrices A and B do not commute. Given an arbitrary representative, first conjugate so that the coefficients of $\mathbf{i}$ and $\mathbf{j}$ in B are zero and the coefficient of $\mathbf{k}$ is positive, and then rotate about the z-axis so the coefficient of $\mathbf{j}$ in A is zero and the coefficient of $\mathbf{i}$ in A is positive. The restrictions on the ranges of β and

r follow from the fact that the matrices A and B do not commute. The uniqueness of the representative follows from the fact that the coefficients of $\mathbf{i}$ in A and $\mathbf{k}$ in B are nonzero. $\square$

We can use the canonical representatives of conjugacy classes in P_4 to define the following maps:

Definition 3.3.7. Define maps $q_1 : P_4 \to SU(2) \times SU(2)$ and $q_2 : P_4 \to S^2 \times S^2$ by

$$q_1([\rho]) = (A, B), \qquad\qquad q_2([\rho]) = (\hat{a}, \hat{b}),$$

where (A, B, a, b) is the canonical representative of $[\rho]$, and $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$ are the unit vectors corresponding to the traceless matrices a and b:

$$a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}, \qquad\qquad b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}.$$

Note that we cannot extend the maps q_1 and q_2 to all of $R(T^2, 2)$, since our choice of canonical representative relies on the fact that the matrices A and B do not commute.

To describe the structure of the piece P_4, we will show that the map $q_2 : P_4 \to S^2 \times S^2$ is injective and identify its image. This requires two Lemmas that describe the image of $q_1 : P_4 \to SU(2) \times SU(2)$ on the fibers of $\mu : P_4 \to [-1, 1)$:

Lemma 3.3.8. *The space $q_1(\mu^{-1}(-1))$ consists of the single point $(\mathbf{i}, \mathbf{k})$.*

Proof. Consider a point $[\rho] \in \mu^{-1}(-1)$. From equation (3.2) for the canonical representative (A, B, a, b) of $[\rho]$, we find that

$$\mu([\rho]) = -1 = (1/2)\,\mathrm{tr}(ABA^{-1}B^{-1}) = \cos 2\beta + r^2(1 - \cos 2\beta).$$

Thus $r = 0$ and $\beta = \pi/2$. Substituting these values into equation (3.2), we obtain the desired result. $\square$

Lemma 3.3.9. *For $t \in (-1, 1)$ we can define a map $q_1(\mu^{-1}(t)) \to S^2$, $(A, B) \mapsto \hat{v}$, where the unit vector $\hat{v} = (v_x, v_y, v_z)$ is the direction of the vector part of $ABA^{-1}B^{-1}$:*

$$ABA^{-1}B^{-1} = t + \sqrt{1 - t^2}\,(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}).$$

This map is a homeomorphism.

Proof. Consider a point $[\rho] \in \mu^{-1}(t)$ for $t \in (-1,1)$. From equation (3.2) for the canonical representative (A, B, a, b) of $[\rho]$, we find that

$$\mu([\rho]) = t = (1/2)\operatorname{tr}(ABA^{-1}B^{-1}) = \cos 2\beta + r^2(1-\cos 2\beta). \tag{3.3}$$

We solve equation (3.3) for r to obtain

$$r = \left(\frac{t-\cos 2\beta}{1-\cos 2\beta}\right)^{1/2}. \tag{3.4}$$

From equation (3.4), we see that for a fixed value of $t \in (-1,1)$ the parameter β must lie in the range $[\beta_0, \pi-\beta_0]$, where we have defined

$$\beta_0 := (1/2)\cos^{-1} t \in (0, \pi/2). \tag{3.5}$$

Using equations (3.2), (3.4), and (3.5), we find that the matrices A and B can be expressed as

$$A = \left(\frac{\cos 2\beta_0 - \cos 2\beta}{1-\cos 2\beta}\right)^{1/2}(\cos\alpha + \sin\alpha\,\mathbf{k}) + \left(\frac{1-\cos 2\beta_0}{1-\cos 2\beta}\right)^{1/2}\mathbf{i}, \quad B = \cos\beta + \sin\beta\,\mathbf{k}, \tag{3.6}$$

where $(\alpha,\beta) \in [0,2\pi]\times[\beta_0,\pi-\beta_0]$. Define a space

$$X = \{(\alpha,\beta) \in [0,2\pi]\times[\beta_0,\pi-\beta_0]\}/\!\sim,$$

where the equivalence relation $\sim$ is defined such that the bottom edge of the rectangle $[0,2\pi]\times[\beta_0,\pi-\beta_0]$ is collapsed to a point, the top edge is collapsed to a point, and the left and right edges are identified:

$$(\alpha,\beta_0)\sim(0,\beta_0), \qquad (\alpha,\pi-\beta_0)\sim(0,\pi-\beta_0), \qquad (0,\beta)\sim(2\pi,\beta).$$

Define a map $X \to q_1(\mu^{-1}(t))$, $[\alpha,\beta]\mapsto(A,B)$, where A and B are given by equation (3.6). From equation (3.6), it is clear that this map is well-defined and is a homeomorphism.

Using equations (3.2) and (3.4), a calculation shows that

$$ABA^{-1}B^{-1} = t + \sqrt{1-t^2}\,(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}),$$

where the unit vector $\hat{v} = (v_x, v_y, v_z) \in S^2$ is given by

$$\hat{v} = (\sqrt{1 - z(\beta)^2}\,\sin(\alpha+\beta),\ \sqrt{1 - z(\beta)^2}\,\cos(\alpha+\beta),\ z(\beta)) \tag{3.7}$$

and we have defined a diffeomorphism $z : [\beta_0, \pi - \beta_0] \to [-1, 1]$ by

$$z(\beta) = -\sqrt{\frac{1-t}{1+t}}\cot\beta.$$

Define a map $X \to S^2$, $[\alpha, \beta] \mapsto \hat{v}$, where $\hat{v}$ is given by equation (3.7). From equation (3.7), it is clear that this map is well-defined and is a homeomorphism. Composing the inverse of the map $X \to q_1(\mu^{-1}(t))$ with the map $X \to S^2$, we obtain the desired result. □

We can now describe the topology of the piece P_4:

Theorem 3.3.10. *The space P_4 is homeomorphic to $S^2 \times S^2 - \bar{\Delta}$, where $\bar{\Delta} = \{(\hat{r}, -\hat{r})\}$ is the antidiagonal. All representations in P_4 are nonabelian.*

Proof. Consider the map $q_2 : P_4 \to S^2 \times S^2$. Clearly the image of q_2 lies in $S^2 \times S^2 - \bar{\Delta}$, since if $q_2([\rho]) \in \bar{\Delta}$ then $b = a^{-1}$, which implies that $\mu([\rho]) = (1/2)\,\mathrm{tr}((ab)^{-1}) = 1$ and hence $[\rho] \notin P_4$. We can define an inverse map $S^2 \times S^2 - \bar{\Delta} \to P_4$ as follows. Given a point $(\hat{a}, \hat{b}) \in S^2 \times S^2 - \bar{\Delta}$, define traceless matrices $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$ corresponding to $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$. Then

$$ab = t - v_x\,\mathbf{i} - v_y\,\mathbf{j} - v_z\,\mathbf{k},$$

where $t := -\hat{a} \cdot \hat{b}$ and $\vec{v} = (v_x, v_y, v_z) := -\hat{a} \times \hat{b}$. If $t = -1$ then map $(\hat{a}, \hat{b})$ to $[\mathbf{i}, \mathbf{k}, a, b]$, otherwise map $(\hat{a}, \hat{b})$ to $[A, B, a, b]$, where A and B are determined from t and $\hat{v} := \vec{v}/|\vec{v}| \in S^2$ via the homeomorphism $q_1(\mu^{-1}(t)) \to S^2$ defined in Lemma 3.3.9. By Lemmas 3.3.8 and 3.3.9, this inverse map is well-defined. The fact that all representations in P_4 are nonabelian is clear from the definition of the space P_4. □

Our main application of Theorem 3.3.10 will be to use $(\hat{a}, \hat{b})$ as coordinates on the piece P_4.

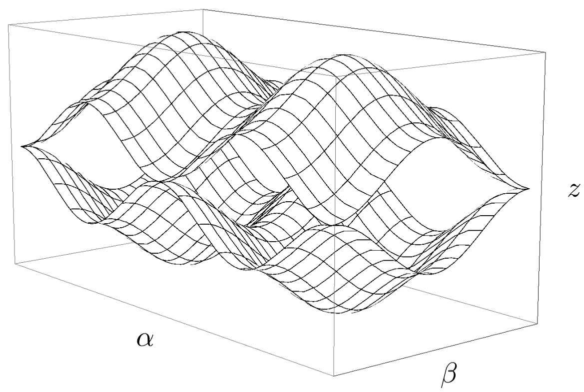

Figure 3.4: The space Y, which is homeomorphic to the piece P_3, is the region between the pair of surfaces. The vertical faces are identified as described in Definition 3.3.12.

3.3.1.2 The piece $P_3 \subset R(T^2, 2)$

We define the piece $P_3 \subset R(T^2, 2)$ to be the set of conjugacy classes $[\rho] \in R(T^2, 2)$ such that $\mu([\rho]) = 1$. For any representative (A, B, a, b) of a given conjugacy class $[\rho] \in P_3$, the matrices A and B commute. We can therefore make the following definition:

Definition 3.3.11. We define a map $q : P_3 \to R(T^2)$, $[A, B, a, b] \mapsto [A, B]$.

We will describe the topology of the piece P_3 by considering the fibers of the map q. In particular, we will show that P_3 is homeomorphic to the following space:

Definition 3.3.12. We define a space Y by

$$Y = \{(\alpha, \beta, z) \mid \alpha \in [0, 2\pi],\ \beta \in [0, \pi],\ |z| \leq \sin^2 \alpha + \sin^2 \beta\}/\sim,$$

where the equivalence relation $\sim$ is defined such that

$$(\alpha, 0, z) \sim (2\pi - \alpha, 0, -z), \qquad (\alpha, \pi, z) \sim (2\pi - \alpha, \pi, -z), \qquad (0, \beta, z) \sim (2\pi, \beta, z).$$

The space Y is depicted in Figure 3.4.

Theorem 3.3.13. *The space P_3 is homeomorphic to Y. Representations on the boundary of P_3 are abelian, and representations on the interior of P_3 are nonabelian.*

Proof. We first determine the fibers of the map $q : P_3 \to R(T^2)$. Given a conjugacy class $[\rho] \in P_3$, we can always choose a representative of the form

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \qquad a = \cos\gamma\,\mathbf{i} + \sin\gamma\,\mathbf{k}, \qquad b = a^{-1}, \qquad (3.8)$$

where $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$ and $\gamma \in [-\pi/2, \pi/2]$. For $(A, B) = (\pm 1, \pm 1)$, we can conjugate so as to force $\gamma = 0$. From these considerations it follows that the fibers of q are points ($\gamma = 0$) over the four pillowcase points $[A, B] = [\pm 1, \pm 1]$, and intervals ($\gamma \in [-\pi/2, \pi/2]$) over all other points. We can thus define a homeomorphism $P_3 \to Y$ by

$$(\alpha,\ \beta,\ \gamma) \mapsto (\alpha,\ \beta,\ (2\gamma/\pi)(\sin^2\alpha + \sin^2\beta)),$$

where $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$, and $\gamma \in [-\pi/2, \pi/2]$ are chosen such that equations (3.8) are satisfied. The statement regarding abelian and nonabelian representations is clear from equation (3.8). $\square$

Our main application of Theorem 3.3.13 will be to use (α, β, γ) as coordinates on P_3, subject to the identifications

$$(\alpha, \beta, \gamma) \sim (\alpha + 2\pi, \beta, \gamma), \qquad (\alpha, \beta, \gamma) \sim (\alpha, \beta + 2\pi, \gamma), \qquad (\alpha, \beta, \gamma) \sim (-\alpha, -\beta, -\gamma),$$

and if $(\alpha, \beta) \in \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$, corresponding to the four pillowcase points of $R(T^2)$, then $(\alpha, \beta, \gamma) \sim (\alpha, \beta, 0)$. Note that P_3 deformation retracts onto $P_3 \cap \{\gamma = 0\}$, which may be identified with the pillowcase $R(T^2)$.

Remark 3.3.14. Theorems 3.3.10 and 3.3.13 imply Theorem 3.1.2 from the introduction.

Remark 3.3.15. The character variety $R(T^2, 2)$ is smooth away from the reducible locus ∂P_3. We note that ∂P_3 is homeomorphic to T^2; a specific homeomorphism $T^2 \to \partial P_3$ is given by $(\alpha, \beta) \mapsto [A, B, a, b]$, where

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \qquad a = b^{-1} = \mathbf{k}.$$

Remark 3.3.16. The character variety $R(T^2, 2)$ is homeomorphic to the moduli space of semistable parabolic bundles $M^{ss}(T^2, 2)$, which is known to have the structure of an algebraic variety isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ (see [Boo18, Var16]). It follows that $R(T^2, 2)$ is homeomorphic to $S^2 \times S^2$, although this does not seem to be easy to show from our description of this space. We use the character variety $R(T^2, 2)$, rather than the moduli space $M^{ss}(T^2, 2)$, since it is only for $R(T^2, 2)$ that we can explicitly describe the Lagrangians L_1 and L_d, and the action of the mapping class group $\mathrm{MCG}_2(T^2)$.

3.3.2 The character variety $R(U_1, A_1)$ and Lagrangian $L_d \subset R(T^2, 2)$

Our next task is to determine the Lagrangian L_d in $R(T^2, 2)$ that corresponds to a solid torus $U_1 = S^1 \times D^2$ containing an unknotted arc A_1 connecting distinct points $p_1, p_2 \in \partial U_1$. We first define and describe a character variety $R(U_1, A_1)$ for (U_1, A_1). The Lagrangian L_d is then given by the image of a pullback map $R(U_1, A_1) \to R(T^2, 2)$.

Definition 3.3.17. We define the character variety $R(U_1, A_1)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1) \to SU(2)$ that map loops around the arc A_1 to traceless matrices.

Theorem 3.3.18. *The space $R(U_1, A_1)$ is homeomorphic to the closed unit disk D^2. Representations on the boundary of $R(U_1, A_1)$ are abelian, and and representations on the interior of $R(U_1, A_1)$ are nonabelian.*

Proof. The fundamental group of $U_1 - A_1$ is given by

$$\pi_1(U_1 - A_1) = \langle A, B, a, b \mid B = 1,\ b = a^{-1} \rangle,$$

where A and B are the longitude and meridian of the boundary of the solid torus and a and b are loops in the boundary encircling the points p_1 and p_2, respectively.

We now consider homomorphisms $\rho : \pi_1(U_1 - A_1) \to SU(2)$ that satisfy the requirements described in Definition 3.3.17 for $R(U_1, A_1)$. As usual, we use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. Given an arbitrary representative of a conjugacy class $[\rho] \in R(U_1, A_1)$, we will argue that we can always conjugate so as to obtain a representative of the form

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = 1, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}, \tag{3.9}$$

where $(\chi, \psi) \in [0, \pi] \times [-\pi/2, \pi/2]$. We first conjugate so the coefficients of $\mathbf{i}$ and $\mathbf{j}$ in A are zero and the coefficient of $\mathbf{k}$ is nonnegative, and then rotate about the z-axis so the coefficient of $\mathbf{j}$ in a is zero and the coefficient of $\mathbf{i}$ is nonnegative. We have thus obtained a representative of the form given in equation (3.9). If $\chi \in (0, \pi)$, then it is clear from these equations that the representative is unique. If $\chi \in \{0, \pi\}$ then $A = \pm 1$ and we can conjugate so that $a = b^{-1} = \mathbf{i}$, so we obtain the identifications $(0, \psi) \sim (0, 0)$ and $(\pi, \psi) \sim (\pi, 0)$. It follows that $R(U_1, A_1)$ is homeomorphic to the square $[0, \pi] \times [-\pi/2, \pi/2]$ with the left and right edges each collapsed to a point, and this space is homeomorphic to the closed disk D^2. The statement regarding abelian and nonabelian representations is clear from equation (3.9). $\square$

Given a representation of $\pi_1(U_1 - A_1)$, we can pull back along the inclusion $T^2 - \{p_1, p_2\} \hookrightarrow U_1 - A_1$ to obtain a representation of $\pi_1(T^2 - \{p_1, p_2\})$. This induces a map $R(U_1, A_1) \to R(T^2, 2)$.

Definition 3.3.19. We will refer to the image of $R(U_1, A_1) \to R(T^2, 2)$ as the *disk Lagrangian* L_d, and we will denote the image in $R(T^2, 2)$ of the point in $R(U_1, A_1)$ with coordinates (χ, ψ) by $L_d(\chi, \psi)$.

The following Theorem gives an explicit description of the disk Lagrangian L_d in terms of the coordinates $(\chi, \psi) \in [0, \pi] \times [-\pi/2, \pi/2]$:

Theorem 3.3.20. *The map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. The image $L_d(\chi, \psi) = [A, B, a, b] \in R(T^2, 2)$ of the point in $R(U_1, A_1)$ with coordinates (χ, ψ) is given by*

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = 1, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}.$$

The image L_d of the map lies entirely in the piece P_3, and the (α, β, γ) coordinates of $L_d(\chi, \psi)$ are

$$\alpha(L_d(\chi, \psi)) = \chi, \qquad \beta(L_d(\chi, \psi)) = 0, \qquad \gamma(L_d(\chi, \psi)) = \psi.$$

Representations on the boundary of L_d are abelian, and representations on the interior of L_d are nonabelian.

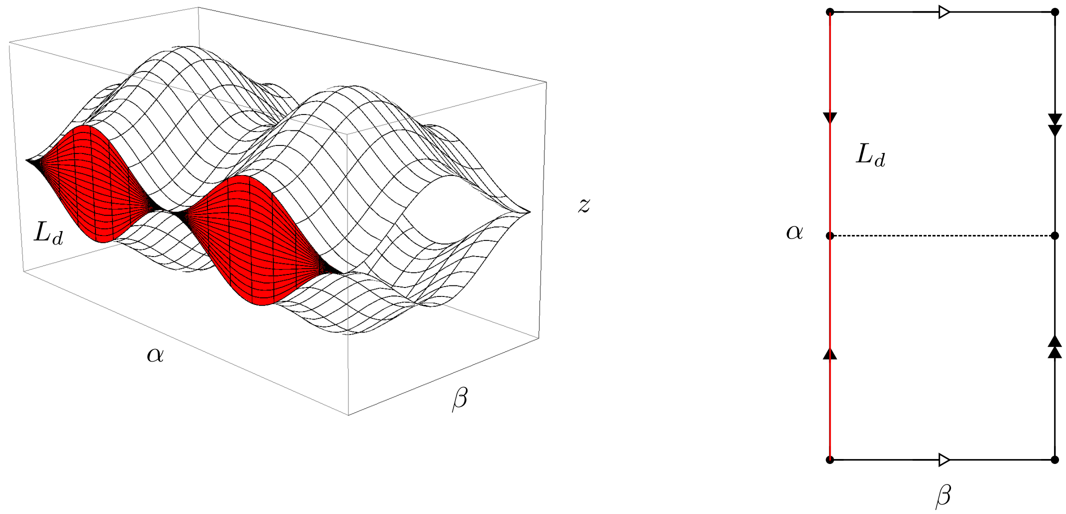

Figure 3.5: (Left) The Lagrangian L_d in the piece P_3. (Right) The intersection of the Lagrangian L_d with the pillowcase $P_3 \cap \{\gamma = 0\}$.

Proof. The representative (A, B, a, b) of $L_d(\chi, \psi)$ follows directly from equation (3.9), and the statement regarding abelian and nonabelian representations is clear from the form of this representative. The (α, β, γ) coordinates of $L_d(\chi, \psi)$ can be read off from equation (3.8). It is clear from these expressions that the map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. $\square$

We plot the Lagrangian L_d in Figure 3.5.

Remark 3.3.21. Theorems 3.3.18 and 3.3.20 imply Theorem 3.1.7 from the introduction.

3.3.3 The character variety $R^\natural_\pi(U_1, A_1)$ and Lagrangian $L_1 \subset R(T^2, 2)$

We now want to modify the character variety $R(U_1, A_1)$ in order to address the technical issues described in the Introduction. Specifically, we want to (1) eliminate reducible connections, and (2) introduce a suitable holonomy perturbation so as to render the Chern-Simons functional Morse. These modifications yield a perturbed character variety $R^\natural_\pi(U_1, A_1)$. We define a corresponding perturbed Lagrangian L_1 given by the image of a pullback map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$.

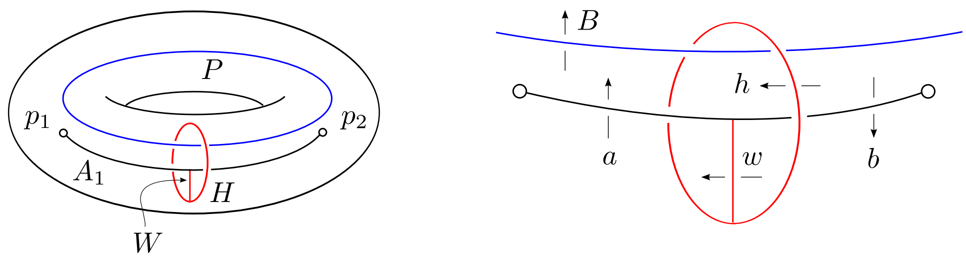

Figure 3.6: (Left) Solid torus U_1 used to define $R^\natural_\pi(S^1 \times D^2, A_1)$. Shown are the arc A_1, the loop H and arc W, and the perturbation loop P. (Right) Loops B, a, b, h, and w.

We eliminate reducible connections by adding an *earring* consisting of a small loop H around A_1 and an arc W connecting A_1 to H, as shown in Figure 3.6. We require that representations take loops around A_1 and H to traceless matrices and loops around W to -1. One can show that representations satisfying these requirements must be nonabelian, corresponding to irreducible connections.

We render the Chern-Simons functional Morse by adding a holonomy perturbation term [KM11b, KM11a]. We choose a perturbation that vanishes outside of a small solid torus obtained by thickening the loop P shown in Figure 3.6. The net effect of the perturbation is to impose an additional requirement on the representations. Specifically, letting $\lambda_P = h^{-1}A$ and $\mu_P = B$ denote the homotopy classes of the longitude and meridian of the solid torus obtained by thickening P, we require that if $\rho(\lambda_P)$ has the form

$$\rho(\lambda_P) = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}) \tag{3.10}$$

for some angle ϕ and some unit vector $\hat{r} = (r_x, r_y, r_z) \in S^2$, then $\rho(\mu_P)$ must have the form

$$\rho(\mu_P) = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}), \tag{3.11}$$

where $\nu = \epsilon f(\phi)$. Here $\epsilon > 0$ is a small parameter that controls the magnitude of the perturbation and $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(-x) = -f(x)$, f is 2π-periodic, and $f(x)$ is zero if and only if x is a multiple of π. We will usually take $f(\phi) = \sin\phi$.

We define a character variety $R^\natural_\pi(U_1, A_1)$ that includes both of these modifications to $R(U_1, A_1)$:

Definition 3.3.22. We define the character variety $R^{\natural}_{\pi}(U_1, A_1)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that take loops around A_1 and H to traceless matrices and loops around W to -1, and are such that if $\rho(\lambda_P)$ has the form given in equation (3.10) then $\rho(\lambda_P)$ must have the form given in equation (3.11).

Theorem 3.3.23. *For $\epsilon > 0$ sufficiently small, the space $R^{\natural}_{\pi}(U_1, A_1)$ is homeomorphic to S^2. All representations in $R^{\natural}_{\pi}(U_1, A_1)$ are nonabelian.*

Proof. We define homotopy classes of loops A, B, a, b, and h as shown in Figure 3.6, and read off relations from Figure 3.6 to obtain a presentation of $\pi_1(U_1 - A_1 \cup H \cup W \cup P)$:

$$\pi_1(U_1 - A_1 \cup H \cup W \cup P) = \langle A, B, a, b, h, w \mid hwaB = aBh,\, b = ha^{-1}w^{-1}h^{-1} \rangle$$

We now consider homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that satisfy the requirements described in Definition 3.3.22 for $R^{\natural}_{\pi}(U_1, A_1)$. As usual, we use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. Given an arbitrary representative of a conjugacy class $[\rho] \in R^{\natural}_{\pi}(U_1, A_1)$, we will argue that we can always conjugate so as to obtain a unique representative of the form given by

$$\begin{aligned} A &= h(\cos\phi + \sin\phi\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})), & B &= \cos\nu + \sin\nu\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}), \\ a &= \mathbf{k}, & b &= -ha^{-1}h^{-1}, \\ h &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k}), & w &= -1, \end{aligned}$$

where $\nu = \epsilon\sin\phi$ and $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$ are spherical-polar coordinates on S^2. We first conjugate so that $a = \mathbf{k}$. Next, we rotate about the z-axis so that the coefficient of $\mathbf{j}$ in h is zero. After these operations have been performed, we can express λ_P as

$$\lambda_P = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$$

for some angle ϕ and some unit vector $\hat{r} = (r_x, r_y, r_z) \in S^2$. The relationship between λ_P and μ_P described in equations (3.10) and (3.11) then implies that

$$B = \mu_P = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}),$$

where $\nu = \epsilon \sin \phi$. We also find that

$$A = h\lambda_P = h(\cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})).$$

Since $w = -1$, the relation $b = ha^{-1}w^{-1}h^{-1}$ implies that $b = -ha^{-1}h^{-1}$, and the relation $hwaB = aBh$ implies that aB and h anticommute. Since $a = \mathbf{k}$, the fact that aB and h anticommute implies that $r_z = 0$, so $\hat{r} = (\cos\theta, \sin\theta, 0)$ for some angle θ. The fact that aB and h anticommute, in conjunction with the fact that the coefficient of $\mathbf{j}$ in h is zero, further implies that h must have the form

$$h = \pm(\cos^2\nu + \sin^2\nu \sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k}). \tag{3.12}$$

In fact, we can assume that the plus sign obtains in equation (3.12), since if not then we can conjugate by $\mathbf{k}$ and redefine $\theta \mapsto \theta + \pi$; this operation flips the signs of h and A and leaves B, a, b, and w invariant. We have thus obtained a representative of the desired form. Since $a = \mathbf{k}$ and the coefficient of $\mathbf{i}$ in h is nonzero for ϵ sufficiently small, this representative is unique and nonabelian.

We note that the unique representative is invariant under the transformations

$$(\phi,\theta) \mapsto (\phi + 2\pi, \theta), \qquad (\phi,\theta) \mapsto (\phi, \theta + 2\pi), \qquad (\phi,\theta) \mapsto (-\phi, \theta + \pi).$$

By invariance under the first transformation we can assume that $\phi \in [-\pi, \pi]$, by invariance under the third transformation we can further assume that $\phi \in [0, \pi]$, and by invariance under the second transformation we can assume that $\theta \in [0, 2\pi]$. From the equations defining the unique representative, it is clear that the map $S^2 \to R_\pi^\natural(U_1, A_1)$, $(\phi,\theta) \mapsto [\rho]$ is a homeomorphism, where (ϕ, θ) are spherical-polar coordinates on S^2. □

Given a representation of $\pi_1(U_1 - A_1 \cup H \cup W \cup P)$, we can pull back along the inclusion $U_1 - A_1 \cup H \cup W \cup P \hookrightarrow T^2 - \{p_1, p_2\}$ to obtain a representation of $\pi_1(T^2 - \{p_1, p_2\})$. This induces a map $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$.

Definition 3.3.24. We define the Lagrangian L_1 to be the image of $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$, and we will denote the image in $R(T^2, 2)$ of the point in $R_\pi^\natural(U_1, A_1)$ with coordinates (ϕ, θ) by $L_1(\phi, \theta)$.

We can view the Lagrangian L_1 as a perturbation of L_d, which we defined to be the image of $R(U_1, A_1) \to R(T^2, 2)$. The following Theorem gives an explicit description of the Lagrangian L_1 in terms of the spherical-polar coordinates $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$:

Theorem 3.3.25. *The map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ is an injective immersion except at the north pole ($\phi = 0$) and south pole ($\phi = \pi$), both of which get mapped to the same point $(\alpha, \beta, \gamma) = (\pi/2, 0, 0)$ in the piece P_3. The image $L_1(\phi, \theta) = [A, B, a, b] \in R(T^2, 2)$ of the point in $R^\natural_\pi(U_1, A_1)$ with coordinates (ϕ, θ) is given by*

$$\begin{aligned}
A &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k})(\cos\phi + \sin\phi\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})),\\
B &= \cos\nu + \sin\nu\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),\\
a &= \mathbf{k},\\
b &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\sin 2\nu\sin\theta\,\mathbf{i} - (\cos^2\nu - \sin^2\nu\sin^2\theta)\mathbf{k}),
\end{aligned}$$

where $\nu = \epsilon\sin\phi$ and $\epsilon > 0$ is a small control parameter that determines the strength of the perturbation. Points $L_1(\phi, \theta)$ with $\phi \in (0, \pi)$, $\theta \notin \{0, \pi\}$ lie in the piece P_4, and the $(\hat{a}, \hat{b})$ coordinates of such points are

$$\begin{aligned}
\hat{a}(L_1(\phi,\theta)) &= (\sin(\phi+\nu),\, -\cos(\phi+\nu),\, 0),\\
b_x(L_1(\phi,\theta)) &= -(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\sin(\phi+\nu) + \sin^2\theta\sin(\phi-\nu)),\\
b_y(L_1(\phi,\theta)) &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\cos(\phi+\nu) + \sin^2\theta\cos(\phi-\nu)),\\
b_z(L_1(\phi,\theta)) &= (1/2)(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}\sin(2\nu)\sin(2\theta)
\end{aligned}$$

for $\theta \in (0, \pi)$, and

$$\begin{aligned}
\hat{a}(L_1(\phi,\theta)) &= (-\sin(\phi+\nu),\, \cos(\phi+\nu),\, 0),\\
b_x(L_1(\phi,\theta)) &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\sin(\phi+\nu) + \sin^2\theta\sin(\phi-\nu)),\\
b_y(L_1(\phi,\theta)) &= -(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos\nu^2\cos^2\theta\cos(\phi+\nu) + \sin^2\theta\cos(\phi-\nu)),\\
b_z(L_1(\phi,\theta)) &= (1/2)(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}\sin(2\nu)\sin(2\theta)
\end{aligned}$$

for $\theta \in (\pi, 2\pi)$. Points $L_1(\phi, \theta)$ with $\theta \in \{0, \pi\}$ lie in the piece P_3, and the (α, β, γ) coordi-

nates of such points are

$$\alpha(L_1(\phi,0)) = \phi + \pi/2, \qquad \beta(L_1(\phi,0)) = \nu = \epsilon\sin\phi, \qquad \gamma(L_1(\phi,0)) = 0,$$
$$\alpha(L_1(\phi,\pi)) = \phi - \pi/2, \qquad \beta(L_1(\phi,\pi)) = \nu = \epsilon\sin\phi, \qquad \gamma(L_1(\phi,\pi)) = 0.$$

All representations in L_1 are nonabelian.

Proof. The representative (A, B, a, b) of $L_1(\phi, \theta)$ follows directly from the proof of Theorem 3.3.23. The fact that all representations in L_1 are nonabelian follows from the fact that $a = \mathbf{k}$ and the coefficient of either $\mathbf{i}$ or $\mathbf{j}$ in B is nonzero. We find the $(\hat{a}, \hat{b})$ coordinates for points $L_1(\phi,\theta) \in P_4$ by conjugating the representative of $L_1(\phi,\theta)$ so that A and B have the form given in equation (3.2), then reading off $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$ from $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$. We find the (α, β, γ) coordinates for points $L_1(\phi,\theta) \in P_3$ by substituting $\theta = 0$ and $\theta = \pi$ into the representative of $L_1(\phi,\theta)$ and then conjugating the resulting equations so they have the form given in equation (3.8).

We will prove that $R^{\natural}_{\pi}(U_1, A_1) \to R(T^2, 2)$ is an injective immersion on $\phi \in (0,\pi)$, $\theta \neq \{0,\pi\}$ by showing that the coordinates (ϕ,θ) can be recovered from certain functions defined on $R(T^2, 2)$. Define functions $h_1 : R(T^2,2) \to \mathbb{R}$ and $h_2 : R(T^2,2) \cap \{\operatorname{tr} Aa \neq 0\} \to \mathbb{R}$ by

$$h_1([A,B,a,b]) = -\operatorname{tr} AB - (i/2)(\operatorname{tr} B)(\operatorname{tr} Aa), \qquad h_2([A,B,a,b]) = -\frac{\operatorname{tr} Ab}{\operatorname{tr} Aa}.$$

A calculation shows that

$$h_1(L_1(\phi,\theta)) = \frac{2\cos\nu\sin(\phi+\nu)e^{i\theta}}{\sqrt{\cos^2\nu + \sin^2\nu\sin^2\theta}}.$$

We note that if $\phi \in (0,\pi)$ then $h_1(L_1(\phi,\theta)) \neq 0$ and $\operatorname{Arg}(h_1(L_1(\phi,\theta))) = \theta$. A calculation shows that $(\operatorname{tr} Aa)(L_1(\phi,\theta)) \neq 0$ for $\phi \in (0,\pi)$, $\theta \neq \{0,\pi\}$, and for such values of (ϕ,θ) we have

$$h_2(L_1(\phi,\theta)) = \frac{\sin(\phi-\nu)}{\sin(\phi+\nu)} = \frac{\sin(\phi - \epsilon\sin\phi)}{\sin(\phi+\epsilon\sin\phi)}.$$

Define $\tilde{h}_2(\phi)$ to be the right-hand-side of this equation. It is straightforward to show that if ϵ is sufficiently small then $\tilde{h}_2'(\phi) > 0$ for all $\phi \in (0,\pi)$, hence $\tilde{h}_2 : (0,\pi) \to \mathbb{R}$ is a

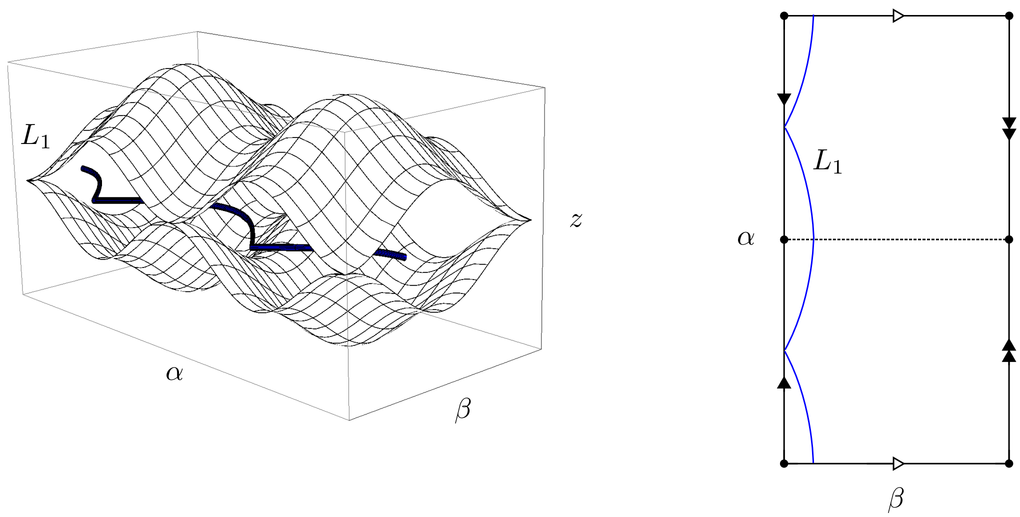

Figure 3.7: (Left) The intersection of the Lagrangian L_1 with the piece P_3. (Right) The intersection of the Lagrangian L_1 with the pillowcase $P_3 \cap \{\gamma = 0\}$.

diffeomorphism onto its image. We conclude that $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ is an injective immersion on $\phi \in (0, \pi)$, $\theta \neq \{0, \pi\}$.

We similarly prove that $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ is an immersion on $\phi \in (0, \pi)$, $\theta \in \{0, \pi\}$ by using the functions $h_1 : R(T^2, 2) \to \mathbb{R}$ and $\alpha : P_3 \to \mathbb{R}$. The statements regarding the injectivity of the map for $\theta \in \{0, \pi\}$ are clear from the expressions for the (α, β, γ) coordinates. $\square$

We plot the intersection of the Lagrangian L_1 with the piece P_3 in Figure 3.7.

Remark 3.3.26. Theorems 3.3.23 and 3.3.25 imply Theorem 3.1.3 from the introduction.

Remark 3.3.27. One can also define a character variety $R^\natural(U_1, A_1)$ that includes the earring but not the holonomy perturbation. It is straightforward to show that the space $R^\natural(U_1, A_1)$ is homeomorphic to S^3 and all representations in $R^\natural(U_1, A_1)$ are nonabelian. We will not use the character variety $R^\natural(U_1, A_1)$ here.

3.4 Nondegeneracy

In this section, we adapt an argument from [AM17] to obtain a simple criterion for determining when a point $[\rho] \in R^\natural_\pi(Y, K)$ is nondegenerate; namely, it is nondegenerate if and only if the Lagrangians L_1 and L_2 in $R(T^2, 2)$ corresponding to the Heegaard splitting of (Y, K) intersect transversely at the image of $[\rho]$ under the pullback map $R^\natural_\pi(Y, K) \to R(T^2, 2)$. The argument relies on several results involving group cohomology and the regularity of character varieties, which we discuss first.

3.4.1 Constrained group cohomology

Consider a finitely presented group $\Gamma = \langle S \mid R\rangle$ with generators $S = \{s_1, \cdots, s_n\}$ and relations $R = \{r_1, \cdots, r_m\}$. In defining character varieties, we often want to consider a space $X(\Gamma) \subseteq \mathrm{Hom}(\Gamma, SU(2))$ consisting of homomorphisms that satisfy certain constraints; for example, we may require the homomorphisms to map certain generators to traceless matrices. Provided the constraints are algebraic, the space $X(\Gamma)$ has the structure of a real algebraic variety, and we can define a corresponding scheme $\mathcal{X}(\Gamma)$ whose set of closed points is $X(\Gamma)$. The group $SU(2)$ acts on the variety $X(\Gamma)$ by conjugation, and we define the character variety $R(\Gamma)$ and character scheme $\mathcal{R}(\Gamma)$ to be the GIT quotients $X(\Gamma)//SU(2)$ and $\mathcal{X}(\Gamma)//SU(2)$. Generalizing a result due to Weil for the unconstrained case [Wei64], we have that the Zariski tangent space $T_{[\rho]}\mathcal{R}(\Gamma)$ of the character scheme $\mathcal{R}(\Gamma)$ at a closed point $[\rho]$ can be identified with the constrained group cohomology $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$, which we define here.

Roughly speaking, the constrained group cohomology $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$ describes deformations of homomorphisms $\rho : \Gamma \to SU(2)$ that satisfy the relevant constraints, modulo deformations that can be obtained by the conjugation action of $SU(2)$. The precise definition of $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$ that we will use is as follows. Define a function $F_r : \mathrm{Hom}(\langle S\rangle, SU(2)) \to SU(2)^m$, where $\langle S\rangle$ is the free group on S, by

$$F_r(\rho) = (\rho(r_1), \cdots, \rho(r_m)).$$

Thus $F_r(\rho) = (1, \cdots, 1)$ if and only if $\rho : \langle S \rangle \to SU(2)$ preserves all the relations in R and thus descends to a homomorphism $\rho : \Gamma \to SU(2)$. Given a homomorphism $\rho : \Gamma \to SU(2)$ and a function $\eta : S \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $SU(2)$, define a homomorphism $\rho_t : \langle S \rangle \to SU(2)$ such that

$$\rho_t(s_k) = e^{t\eta(s_k)}\rho(s_k).$$

Note that we can view η as a vector in $\mathfrak{g}^{\oplus n}$. We define a linear map $c_r : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m}$ by

$$c_r(\eta) = \frac{d}{dt}F_r(\rho_t)|_{t=0}.$$

Thus $c_r(\eta) = 0$ if and only if η describes a deformation of ρ that is a homomorphism $\Gamma \to SU(2)$.

Homomorphisms $\Gamma \to SU(2)$ that represent points in a character variety may be required to satisfy certain constraints; for example, that they take particular generators to traceless matrices. Define a function $F_c : \mathrm{Hom}(\langle S \rangle, SU(2)) \to \mathbb{R}^q$ such that $F_c(\rho) = 0$ if and only if ρ satisfies these constraints; for example, if we require that ρ take the generator s_1 to a traceless matrix, we would define $F_c : \mathrm{Hom}(\langle S \rangle, SU(2)) \to \mathbb{R}$ by

$$F_c(\rho) = \mathrm{tr}(\rho(s_1)).$$

We define a linear map $c_c : \mathfrak{g}^{\oplus n} \to \mathbb{R}^q$ by

$$c_c(\eta) = \frac{d}{dt}F_c(\rho_t)|_{t=0}.$$

Thus $c_c(\eta) = 0$ if and only if η describes a deformation of ρ that satisfies the constraints.

We now combine the linear maps for the relations and constraints to obtain a linear map $c : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$, $c(\eta) = (c_r(\eta), c_c(\eta))$. Given a homomorphism $\rho : \Gamma \to SU(2)$ that satisfies the constraints, we define the space of 1-cocycles to be

$$Z^1_c(\Gamma; \mathrm{Ad}\,\rho) = \ker c,$$

so a vector $\eta \in \mathfrak{g}^n$ is a 1-cocycle if and only if it describes a deformation of ρ that is a homomorphism that preserves the constraints. We define the space of 1-coboundaries to be deformations of ρ that are obtained via the conjugation action of $SU(2)$:

$$B_c^1(\Gamma; \operatorname{Ad}\rho) =$$
$$\{\eta : S \to \mathfrak{g} \mid \text{there exists } u \in \mathfrak{g} \text{ such that } \eta(s_k) = u - \operatorname{Ad}_{\rho(s_k)} u \text{ for } k = 1, \cdots, n\}.$$

Here $\operatorname{Ad}_g u := gug^{-1}$ for $g \in SU(2)$ and $u \in \mathfrak{g}$. We define the constrained group cohomology $H_c^1(\Gamma; \operatorname{Ad}\rho)$ to be

$$H_c^1(\Gamma; \operatorname{Ad}\rho) = Z_c^1(\Gamma; \operatorname{Ad}\rho)/B_c^1(\Gamma; \operatorname{Ad}\rho).$$

3.4.2 Regularity

Definition 3.4.1. We say that a point $[\rho]$ of a character variety $R(\Gamma)$ is *regular* if $\dim_{[\rho]} R(\Gamma) = \dim H_c^1(\Gamma; \operatorname{Ad}\rho)$.

We define $R'(\Gamma)$ to be the subspace of regular points of $R(\Gamma)$. The space $R'(\Gamma)$ has the structure of a smooth manifold, and the tangent space of this manifold at a point $[\rho] \in R'(\Gamma)$ is given by $T_{[\rho]}R'(\Gamma) = H_c^1(\Gamma; \operatorname{Ad}\rho)$. We will prove theorems that describe the regular points of the character varieties $R(U_1, A_1)$, $R_\pi^\natural(U_1, A_1)$, and $R(T^2, 2)$:

Theorem 3.4.2. *The character variety $R(U_1, A_1)$ is regular at all points represented by nonabelian homomorphisms.*

Proof. Using results from the proof of Theorem 3.3.18, we find that we can take the set of generators of the fundamental group Γ to be $S = \{A, a\}$, with no relations, and we can take the constraint function $F_c : \operatorname{Hom}(\langle S \rangle, SU(2)) \to \mathbb{R}$ to be

$$F_c(\rho) = \operatorname{tr}(\rho(a)).$$

Using the expressions for the homomorphisms $\rho : \Gamma \to SU(2)$ given in the proof of Theorem 3.3.18, we obtain a linear map $c : \mathbb{R}^6 \to \mathbb{R}$. A straightforward calculation shows that $\dim H_c^1(\Gamma; \operatorname{Ad}\rho) = \dim R(U_1, A_1) = 2$ for all $[\rho] \in R(U_1, A_1)$ such that ρ is nonabelian. $\square$

We would next like to determine the regular points of the perturbed character variety $R_\pi^\natural(U_1, A_1)$, but there are two difficulties that must be overcome. The first difficulty involves the function $f(\phi)$ that defines the perturbation. Recall that points $[\rho] \in R^\natural_\pi(U_1, A_1)$ are constrained by the requirement that if $\rho(\lambda_P)$ has the form $\rho(\lambda_P) = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$, then $\rho(\mu_P)$ must have the form $\rho(\mu_P) = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$, where $\nu = \epsilon f(\phi)$. In order to give $R^\natural_\pi(U_1, A_1)$ the structure of a real algebraic variety, and to define the corresponding character scheme, this constraint must be algebraic. We will therefore choose $f(\phi)$ to be

$$f(\phi) = \frac{1}{\epsilon}\sin^{-1}(\epsilon\sin\phi). \tag{3.13}$$

Then the constraint on ρ becomes

$$\epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{i}) = \mathrm{tr}(\rho(\mu_P)\mathbf{i}), \quad \epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{j}) = \mathrm{tr}(\rho(\mu_P)\mathbf{j}), \quad \epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{k}) = \mathrm{tr}(\rho(\mu_P)\mathbf{k}). \tag{3.14}$$

Remark 3.4.3. In fact, the constraint given in equation (3.14) yields a variety with two connected components, one with $\rho(\mu_P)$ near 1 and one with $\rho(\mu_P)$ near -1, and only the first component corresponds to $R^\natural_\pi(U_1, A_1)$. To calculate the constrained group cohomology, however, we consider only infinitesimal deformations of homomorphisms, hence the extraneous second component is irrelevant.

A second difficulty in determining the regular points of $R^\natural_\pi(U_1, A_1)$ is that a direct calculation of the constrained group cohomology for $R^\natural_\pi(U_1, A_1)$ does not appear to be practical, because the perturbed representations, as described in Theorem 3.3.23, are rather complicated. Instead, we will apply the following theorem, which simplifies the necessary calculations by allowing us to extrapolate from unperturbed representations:

Theorem 3.4.4. *Consider a character variety $R_\epsilon(\Gamma)$ in which the homomorphisms are required to satisfy an algebraic constraint that depends on a control parameter $\epsilon \in \mathbb{R}$. Given a homomorphism $\rho_\epsilon : \Gamma \to SU(2)$ representing a point $[\rho_\epsilon] \in R_\epsilon(\Gamma)$, let $c_\epsilon : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$ denote the corresponding linear map used to define the constrained group cohomology. Define $c_0, c_1 : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$ such that $c_\epsilon = c_0 + \epsilon c_1 + \cdots$. The following string of inequalities holds for $\epsilon > 0$ sufficiently small:*

$$\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon) \le \dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0) \le \dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_0). \tag{3.15}$$

Proof. Since the dimension of the Zariski tangent space is upper semi-continuous, for $\epsilon > 0$ sufficiently small we have that

$$\dim(\ker c_\epsilon) = \dim Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon) \leq \dim Z^1_c(\Gamma; \operatorname{Ad}\rho_0) = \dim(\ker c_0).$$

Thus any vector $w_\epsilon \in \ker c_\epsilon$ must have the form $w_\epsilon = w_0 + \epsilon w_1 + \cdots$, where

$$c_\epsilon(w_\epsilon) = c_0(w_0) + \epsilon(c_0(w_1) + c_1(w_0)) + \cdots = 0. \tag{3.16}$$

The space of vectors $w_0 \in \mathfrak{g}^{\oplus n}$ that satisfy equation (3.16) up to first order in ϵ is

$$V = \{w_0 \in \ker c_0 \mid c_1(w_0) \in \operatorname{im} c_0\}.$$

Since $\ker c_\epsilon = Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon)$ is the space of vectors that satisfies equation (3.16) to all orders in ϵ, it follows that $Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon) \subseteq V \subseteq \ker c_0 = Z^1_c(\Gamma; \operatorname{Ad}\rho_0)$, and we have the string of inequalities

$$\dim Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon) \leq \dim V \leq \dim Z^1_c(\Gamma; \operatorname{Ad}\rho_0).$$

Equation (3.15) now follows from the fact that

$$\dim V = \dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \operatorname{im} c_0).$$

□

Example 3.4.5. Take $\Gamma = \mathbb{Z}$, and consider the character varieties $R^i_\epsilon(\Gamma)$ for $i = 1, 2, 3$ with constraint functions $F^i_c : \operatorname{Hom}(\Gamma, SU(2)) \to \mathbb{R}$ given by

$$F^1_c(\rho) = \epsilon \operatorname{tr}\rho(1), \qquad F^2_c(\rho) = \epsilon(\operatorname{tr}\rho(1))^2, \qquad F^3_c(\rho) = \epsilon(\operatorname{tr}\rho(1))^2 + \epsilon^2 \operatorname{tr}\rho(1).$$

The character varieties are given by

$$R^1_\epsilon(\Gamma) = R^2_\epsilon(\Gamma) = R^3_\epsilon(\Gamma) = \begin{cases} S^2 & \text{if } \epsilon \neq 0, \\ S^3 & \text{if } \epsilon = 0. \end{cases}$$

Consider the homomorphism $\rho_\epsilon : \mathbb{Z} \to SU(2)$, $\rho_\epsilon(1) = \mathbf{k}$. Then $\dim Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon)$ and $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \operatorname{im} c_0)$ are given by

	F^1_c	F^2_c	F^3_c
$\dim Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon)$	2	3	2
$\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \operatorname{im} c_0)$	2	3	3

From the expressions for $\dim Z^1_c(\Gamma; \operatorname{Ad}\rho_\epsilon)$, we find that for $\epsilon \neq 0$ the character schemes $\mathcal{R}^1_\epsilon(\Gamma)$ and $\mathcal{R}^3_\epsilon(\Gamma)$ are reduced, and the character scheme $\mathcal{R}^2_\epsilon(\Gamma)$ is not reduced. We can use Theorem 3.4.4 to show that $\mathcal{R}^1_\epsilon(\Gamma)$ is reduced, but not that $\mathcal{R}^3_\epsilon(\Gamma)$ is reduced.

Theorem 3.4.6. *The character variety $R^\natural_\pi(U_1, A_1)$ is regular everywhere.*

Proof. Using results from the proof of Theorem 3.3.23, we find that we can take the set of generators for the fundamental group Γ to be $S = \{a, A, B, h\}$, the relations function $F_r : \operatorname{Hom}(\langle S\rangle, SU(2)) \to SU(2)$ to be

$$F_r(\rho) = -\rho([h, aB]),$$

and the constraint function $F_c : \operatorname{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}^6$ to be

$$F_c(\rho) = (\operatorname{tr}(\rho(a)),\ \operatorname{tr}(\rho(ha^{-1}h^{-1})),\ \operatorname{tr}(\rho(h)),\ f(\rho, \mathbf{i}),\ f(\rho, \mathbf{j}),\ f(\rho, \mathbf{k})),$$

where

$$f(\rho, q) = \epsilon \operatorname{tr}(\rho(h^{-1}A)q) - \operatorname{tr}(\rho(B)q).$$

Using the expressions for the homomorphisms $\rho_\epsilon : \Gamma \to SU(2)$ given in the proof of Theorem 3.3.23, we obtain a linear map $c_\epsilon : \mathbb{R}^{12} \to \mathbb{R}^9$. We now apply Theorem 3.4.4. A straightforward, but rather lengthy, calculation shows that $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \operatorname{im} c_0) = 5$ for all homomorphisms representing points in $R^\natural_\pi(U_1, A_1)$. Since these homomorphisms are all nonabelian, we conclude that $\dim H^1_c(\Gamma; \operatorname{Ad}\rho) = \dim R^\natural_\pi(U_1, A_1) = 2$ for all $[\rho] \in R^\natural_\pi(U_1, A_1)$, and thus $R^\natural_\pi(U_1, A_1)$ is regular everywhere. □

Theorem 3.4.7. *The character variety $R(T^2, 2)$ is regular on L_1.*

Proof. Using results from Section 3.3.1, we find that we can take the set of generators for the fundamental group Γ to be $S = \{a, A, B\}$, with no relations, and we can take the constraint function $F_c : \operatorname{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}^2$ to be

$$F_c(\rho) = (\operatorname{tr}(\rho(a)),\ \operatorname{tr}(\rho(ABA^{-1}B^{-1}a))).$$

Using results from the proof of Theorem 3.3.23, we obtain a linear map $c_\epsilon : \mathbb{R}^9 \to \mathbb{R}^2$ for homomorphisms representing points in L_1. A straightforward, but rather lengthy, calculation shows that $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \operatorname{im} c_0) = 7$ for all homomorphisms representing points in L_1. Since these homomorphisms are all nonabelian, we conclude that $\dim H^1_c(\Gamma; \operatorname{Ad}\rho) = \dim R(T^2, 2) = 4$ for all $[\rho] \in R(T^2, 2)$, and thus $R(T^2, 2)$ is regular on L_1. $\square$

Remark 3.4.8. We conjecture that $R(T^2, 2)$ is in fact regular at all points represented by nonabelian homomorphisms, but Theorem 3.4.7 will suffice for our purposes.

3.4.3 Transversality

We are now ready to prove our key result that relates nondegeneracy to transversality. Recall that we defined the Lagrangian L_2 to be the image of $R(U_2, A_2) \to R(T^2, 2)$. If $R(U_2, A_2) \to R(T^2, 2)$ is injective, and $[\rho] \in L_1 \cap L_2 \subset R(T^2, 2)$ is not the double-point of L_1, then by Corollary 3.1.4 the point $[\rho]$ is the image of a unique point in $R^\natural_\pi(Y, K)$ under the pullback map $R^\natural_\pi(Y, K) \to R(T^2, 2)$, which for simplicity we will also denote by $[\rho]$. The following is a restatement of Theorem 3.1.5 from the introduction:

Theorem 3.4.9. *Suppose $R(U_2, A_2) \to R(T^2, 2)$ is an injective immersion and $[\rho] \in L_1 \cap L_2$ is the image of a regular point of $R(U_2, A_2)$ and is not the double-point of L_1. Then the unique preimage $[\rho] \in R^\natural_\pi(Y, K)$ of $[\rho]$ under the pullback map $R^\natural_\pi(Y, K) \to R(T^2, 2)$ is nondegenerate if and only if the intersection of L_1 with L_2 at $[\rho] \in L_1 \cap L_2$ is transverse.*

Proof. We introduce the notation $K' = K \cup W \cup H \cup P$, $Y' = Y - K'$, $U_i' = U_i - K'$, and $\Sigma' = T^2 - \{p_1, p_2\}$. We have the following Mayer-Vietoris sequence:

$$H^0_c(\Sigma'; \operatorname{Ad}\rho) \longrightarrow H^1_c(Y'; \operatorname{Ad}\rho) \longrightarrow H^1_c(U_1'; \operatorname{Ad}\rho) \oplus H^1_c(U_2'; \operatorname{Ad}\rho) \longrightarrow H^1_c(\Sigma'; \operatorname{Ad}\rho)$$

Here $H^0_c(\Sigma'; \operatorname{Ad}\rho)$ is

$$H^0_c(\Sigma'; \operatorname{Ad}\rho) = \{x \in \mathfrak{g} \mid [\rho(\lambda), x] = 0 \text{ for all } \lambda \in \pi_1(\Sigma')\},$$

and $H^1_c(Y';\mathrm{Ad}\,\rho)$, $H^1_c(U_1';\mathrm{Ad}\,\rho)$, $H^1_c(U_2';\mathrm{Ad}\,\rho)$, $H^1_c(\Sigma';\mathrm{Ad}\,\rho)$ are the constrained group cohomology for the character varieties $R^\natural_\pi(Y,K)$, $R^\natural_\pi(U_1,A_1)$, $R(U_2,A_2)$, and $R(T^2,2)$, respectively. For notational simplicity, we are using ρ to denote a homomorphism representing a point in $R^\natural_\pi(Y,K)$, as well as its pullbacks to homomorphisms representing points in $R^\natural_\pi(U_1,A_1)$, $R(U_2,A_2)$, and $R(T^2,2)$. From Theorem 3.3.25 we have that all points in L_1 are represented by nonabelian homomorphisms, thus $H^0_c(\Sigma';\mathrm{Ad}\,\rho)=0$. From Theorems 3.4.6 and 3.4.7, we have the identifications

$$H^1_c(U_1';\mathrm{Ad}\,\rho)=T_{[\rho]}R^\natural_\pi(U_1,A_1), \qquad H^1_c(\Sigma';\mathrm{Ad}\,\rho)=T_{[\rho]}R(T^2,2).$$

Since we have assumed that $[\rho]\in R(U_2,A_2)$ is regular, we have the identification

$$H^1_c(U_2';\mathrm{Ad}\,\rho)=T_{[\rho]}R(U_2,A_2).$$

By Theorem 3.3.25, the map $R^\natural_\pi(U_1,A_1)\to R(T^2,2)$ is an immersion (with image L_1), and we have assumed that $R(U_2,A_2)\to R(T^2,2)$ is an immersion (with image L_2), so we can identify

$$T_{[\rho]}R^\natural_\pi(U_1,A_1)=T_{[\rho]}L_1, \qquad T_{[\rho]}R(U_2,A_2)=T_{[\rho]}L_2.$$

We conclude that the constrained group cohomology $H^1_c(Y';\mathrm{Ad}\,\rho)$ is given by

$$H^1_c(Y';\mathrm{Ad}\,\rho)=T_{[\rho]}\mathcal{R}^\natural_\pi(Y,K)=T_{[\rho]}L_1\cap T_{[\rho]}L_2.$$

The constrained group cohomology $H^1_c(Y';\mathrm{Ad}\,\rho)$ is zero if and only if $[\rho]$ is nondegenerate (see [Don04] Section 2.5.4). Thus $[\rho]$ is nondegenerate if and only if L_1 intersects L_2 transversely at $[\rho]$. $\square$

Example 3.4.10. Consider the algebraic functions $f,g:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$, $g(x)=x^3$. The schemes corresponding to the critical loci of f and g are $\operatorname{Spec}F=\{(0)\}$ and $\operatorname{Spec}G=\{(x)\}$, where

$$F=\mathbb{R}[x]/(f'(x))=\mathbb{R}, \qquad G=\mathbb{R}[x]/(g'(x))=\mathbb{R}[x]/(x^2).$$

The fact that 0 is a nondegenerate critical point of f, but a degenerate critical point of g, is reflected in the fact that F is reduced, but G is nonreduced, which in turn is reflected in the fact that $T_{(0)}\operatorname{Spec}F=0$, but $T_{(x)}\operatorname{Spec}G=\mathbb{R}$.

Since nondegeneracy is a stable property, for sufficiently small $\epsilon > 0$ we can use the function $f(\phi) = \sin\phi$ to define the perturbation, rather than the function $f(\phi)$ given in equation (3.13).

3.5 The group $\mathrm{MCG}_2(T^2)$ and its action on $R(T^2, 2)$

An important property of the character variety $R(T^2, 2)$ is that it admits an action of the mapping class group $\mathrm{MCG}_2(T^2)$. Here we describe the group $\mathrm{MCG}_2(T^2)$ and its action on $R(T^2, 2)$.

3.5.1 The mapping class group $\mathrm{MCG}_2(T^2)$

Definition 3.5.1. Given a surface S and n distinct marked points $p_1, \cdots, p_n \in S$, we define the *mapping class group* $\mathrm{MCG}_n(S)$ to be the group of isotopy classes of orientation-preserving homeomorphisms of S that fix $\{p_1, \cdots, p_n\}$ as a set.

Presentations for mapping class groups are described in [CM04, Ger01, LP01]. The mapping class group $\mathrm{MCG}_2(T^2)$ for the twice-punctured torus is generated by Dehn twists T_a, T_A, T_b, and T_B around the simple closed curves a, A, b, and B shown in Figure 3.8, together with a π-rotation ω of the square shown in Figure 3.8. The mapping class group $\mathrm{MCG}(T^2) := \mathrm{MCG}_0(T^2)$ for the unpunctured torus is generated by the Dehn twists T_a and T_b.

It is useful to relate the mapping class groups $\mathrm{MCG}_2(T^2)$ and $\mathrm{MCG}(T^2)$ to the braid group $B_2(T^2)$, which we define as follows:

Definition 3.5.2. Given a surface S, we define the *configuration space for ordered points* $\mathrm{Conf}'_n(S)$ to be the space $\{(p_1, \cdots, p_n) \in S^n \mid p_i \neq p_j \text{ if } i \neq j\}$. We define the *configuration space for unordered points* $\mathrm{Conf}_n(S)$ to be the space $\mathrm{Conf}'_n(S)/\Sigma_n$, where the fundamental group on n letters Σ_n acts on $\mathrm{Conf}'_n(S)$ by permutation.

Definition 3.5.3. Given a surface S and n distinct marked points $p_1, \cdots, p_n \in S$, we define the *braid group* $B_n(S)$ to be the fundamental group of $\mathrm{Conf}_n(S)$ with base point

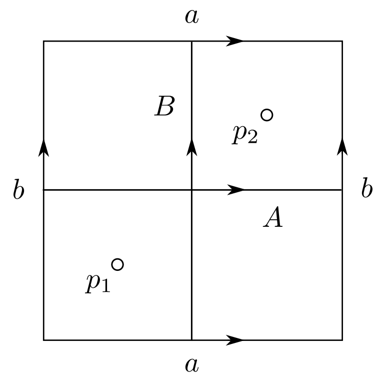

Figure 3.8: Cycles a, A, b, and B corresponding to generators T_a, T_A, T_b, and T_B of $\mathrm{MCG}_2(T^2)$.

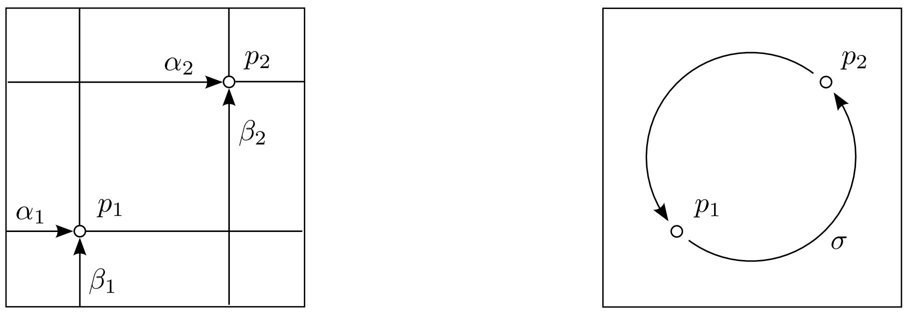

Figure 3.9: (Left) Generators α_1, α_2, β_1, and β_2 of $B_2(T^2)$. (Right) Generator σ of $B_2(T^2)$.

$[(p_1, \cdots, p_n)]$.

Presentations for braid groups are described in [Bel04]. The braid group $B_2(T^2)$ for the twice-punctured torus is generated by braids α_i and β_i for $i = 1, 2$ that drag marked the point p_i rightward and upward around a cycle, together with a braid σ that interchanges the marked points p_1 and p_2 via a counterclockwise π-rotation. These generators are depicted in Figure 3.9.

The braid group $B_2(T^2)$ and the mapping class groups $\mathrm{MCG}_2(T^2)$ and $\mathrm{MCG}(T^2)$ are related by the Birman exact sequence [Bir69]:

$$1 \longrightarrow \pi_1(\mathrm{Homeo}_0(T^2)) \longrightarrow B_2(T^2) \xrightarrow{p} \mathrm{MCG}_2(T^2) \xrightarrow{g} \mathrm{MCG}(T^2) \longrightarrow 1.$$

Here $\mathrm{Homeo}_0(T^2)$ is the group of orientation-preserving homeomorphisms of T^2 that are isotopic to the identity. The group $\mathrm{Homeo}_0(T^2)$ deformation retracts onto T^2 [Ham65], so

$\pi_1(\mathrm{Homeo}_0(T^2)) = \pi_1(T^2) = \mathbb{Z}^2$. The two free abelian generators of $\pi_1(\mathrm{Homeo}_0(T^2))$ can be identified with the elements $\alpha_1\alpha_2$ and $\beta_1\beta_2$ of $B_2(T^2)$ under the injection $\pi_1(\mathrm{Homeo}_0(T^2)) \to B_2(T^2)$. The push homomorphism $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$ is given by

$$p(\alpha_1) = p(\alpha_2)^{-1} = T_a T_A^{-1}, \qquad p(\beta_1) = p(\beta_2)^{-1} = T_b T_B^{-1}, \qquad p(\sigma) = (T_a T_b^{-1} T_a)^2 \omega.$$

The forgetful homomorphism $g : \mathrm{MCG}_2(T^2) \to \mathrm{MCG}(T^2)$ is given by

$$g(T_a) = g(T_A) = T_a, \qquad g(T_b) = g(T_B) = T_b, \qquad g(\omega) = (T_a T_b^{-1} T_a)^2.$$

In what follows we will use the generators of $B_2(T^2)$ to also denote their images in $\mathrm{MCG}_2(T^2)$ under $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$.

We will use elements of the group $\mathrm{MCG}_2(T^2)$ to describe gluing data for constructing $(1,1)$-knots. By definition, a $(1,1)$-knot K in a lens space Y can be obtained by gluing together two copies of a solid torus containing an unknotted arc via a homeomorphism that represents an element $f \in \mathrm{MCG}_2(T^2)$. The Birman sequence is useful for understanding the relationship between elements $f \in \mathrm{MCG}_2(T^2)$ and the corresponding pairs (Y, K). The lens space Y can be recovered from the image of f under $g : \mathrm{MCG}_2(T^2) \to \mathrm{MCG}(T^2)$, so this map can be viewed as forgetting the part of the gluing data used to construct the knot and preserving the part of the data used to construct the lens space. If we multiply f by an element in the image of the map $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$, the resulting element $f' \in \mathrm{MCG}_2(T^2)$ yields a pair (Y, K') consisting of a potentially different knot K' in the same lens space Y. The braid group $B_2(T^2)$ is thus useful for constructing different knots in a fixed lens space.

3.5.2 The action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$

We will define an action of the group $\mathrm{MCG}_2(T^2)$ on the character variety $R(T^2, 2)$ via a homomorphism from $\mathrm{MCG}_2(T^2)$ to $\mathrm{Out}(\pi_1(T^2 - \{p_1, p_2\}))$, the group of outer automorphisms of $\pi_1(T^2 - \{p_1, p_2\})$. In general, we define a group homomorphism from $\mathrm{MCG}_n(T^2)$ to $\mathrm{Out}(\pi_1(T^2 - \{p_1, \cdots, p_n\}))$, the group of outer automorphisms of $\pi_1(T^2 - \{p_1, \cdots, p_n\})$, as follows. Define $X = T^2 - \{p_1, \cdots, p_n\}$. Choose a base point $x_0 \in X$ and consider the fundamental group $\pi_1(X, x_0)$. Given an element $[\phi] \in \mathrm{MCG}_n(X)$ represented by a homeomorphism $\phi : X \to X$, there is an induced isomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(X, \phi(x_0))$, $[\alpha] \mapsto [\phi \circ \alpha]$. Choose a path $\gamma : I \to X$ from x_0 to $\phi(x_0)$; this induces an isomorphism $\gamma_* : \pi_1(X, \phi(x_0)) \to \pi_1(X, x_0)$, $[\alpha] \mapsto [\gamma\alpha\bar{\gamma}]$. We now define a map $\mathrm{MCG}_n(T^2) \to \mathrm{Out}(\pi_1(X, x_0))$ by $[\phi] \mapsto [\gamma_*\phi_*]$. One can show that this map is well-defined and is a homomorphism (see [FM12] Chapter 8.1).

Remark 3.5.4. A version of the Dehn-Nielsen-Baer theorem states that the homomorphism $\mathrm{MCG}_n(T^2) \to \mathrm{Out}(\pi_1(T^2 - \{p_1, \cdots, p_n\}))$ is injective (see [FM12] Theorem 8.8), and one can use this result to obtain the expressions for the homomorphisms p and g in the Birman sequence.

We define a right action of $\mathrm{MCG}_2(T^2)$ on the character variety $R(T^2, 2)$ by

$$[\rho] \cdot f = [\rho \circ \tilde{f}],$$

where $[\rho] \in R(T^2, 2)$, $f \in \mathrm{MCG}_2(T^2)$, and $\tilde{f} \in \mathrm{Aut}(\pi_1(T^2 - \{p_1, p_2\}))$ is a representative of the image of f under the homomorphism $\mathrm{MCG}_2(T^2) \to \mathrm{Out}(\pi_1(T^2 - \{p_1, p_2\}))$. We find that the action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ is given by

$$\begin{aligned}
&[A,\, B,\, a,\, b] \cdot T_a = [A,\, BA,\, a,\, b],\\
&[A,\, B,\, a,\, b] \cdot T_b = [AB,\, B,\, a,\, b],\\
&[A,\, B,\, a,\, b] \cdot T_A = [A,\, aAB,\, a,\, Aaba^{-1}A^{-1}],\\
&[A,\, B,\, a,\, b] \cdot T_B = [a^{-1}BA,\, B,\, a,\, a^{-1}BbB^{-1}a],\\
&[A,\, B,\, a,\, b] \cdot \omega = [A^{-1},\, B^{-1},\, B^{-1}A^{-1}bAB,\, A^{-1}B^{-1}aBA].
\end{aligned}$$

The action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ fixes the reducible locus ∂P_3 of $R(T^2, 2)$ as a set. The homomorphism $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$ in the Birman sequence induces a right action of $B_2(T^2)$ on $R(T^2, 2)$.

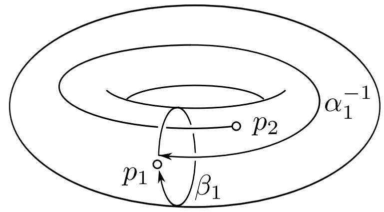

Figure 3.10: The trefoil in S^3 is constructed by gluing together (U_1, A_1) and (U_2, A_2) using the mapping class group element $f = s\beta_1\alpha_1^{-1}$.

3.6 Examples

We will now compute generating sets for $I^\natural(Y, K)$ for several example $(1,1)$-knots K in lens spaces Y. As described in the Introduction, we Heegaard-split (Y, K) into a pair of handlebodies (U_1, A_1) and (U_2, A_2). The handlebodies are glued together via a homeomorphism $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$, which defines an element $f = [\phi]$ of the mapping class group $\mathrm{MCG}_2(T^2)$. We define a character variety $R(T^2, 2)$ corresponding to the Heegaard surface $(T^2, \{p_1, p_2\}) := (\partial U_1, \partial A_1)$, and we define Lagrangians L_1 and $L_2 = L_d \cdot f$ in $R(T^2, 2)$ corresponding to the handlebodies (U_1, A_1) and (U_2, A_2). To obtain a generating set for $I^\natural(Y, K)$, we count the intersection points $L_1 \cap L_2$ and show that the intersection is transverse at each point. The calculations needed to accomplish this task rely on the parameterizations $L_1(\phi, \theta)$ and $L_d(\chi, \psi)$ of the Lagrangians L_1 and L_d given in Theorems 3.3.25 and 3.3.20, together with the description of the action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ given in Section 3.5.2. To describe the intersection, we will use the coordinates $(\hat{a}, \hat{b})$ that we defined on the piece $P_4 \subset R(T^2, 2)$ in Section 3.3.1.1, and the coordinates (α, β, γ) that we defined on the piece $P_3 \subset R(T^2, 2)$ in Section 3.3.1.2.

3.6.1 Trefoil in S^3

As shown in Figure 3.10, we can construct the trefoil in S^3 by gluing the two handlebodies together using the mapping class group element $f = s\beta_1\alpha_1^{-1} \in \mathrm{MCG}_2(T^2)$, where $s := T_a T_b^{-1} T_a$ exchanges the longitude and meridian of T^2. We first prove a Lemma that constrains the possible intersection points of L_1 and $L_2 = L_d \cdot f$:

Lemma 3.6.1. *If $L_1(\phi, \theta) = L_2(\chi, \psi)$, then $\chi = \pi/2$ and either $\theta \in \{\pi/2, 3\pi/2\}$ or $\phi \in \{0, \pi\}$*

Proof. Define functions $h_1, h_2 : R(T^2, 2) \to \mathbb{R}$ by

$$h_1([A, B, a, b]) = \operatorname{tr} A, \qquad h_2([A, B, a, b]) = \operatorname{tr} Ba.$$

We evaluate the functions h_1 and h_2 at the points $L_1(\phi, \theta)$ and $L_2(\chi, \psi)$. If we require that each function give the same value at both points, we obtain the desired result. $\square$

Theorem 3.6.2. *The rank of $I^\natural(S^3, K)$ for the trefoil K in S^3 is at most 3.*

Proof. From Lemma 3.6.1, we know that if $L_1(\phi, \theta) = L_2(\chi, \psi)$ then $\chi = \pi/2$. A calculation shows that $L_2(\pi/2, \psi) = L_d(\pi/2, \psi) \cdot f = [A, B, a, b]$, where

$$A = \mathbf{i}, \qquad B = \sin 3\psi + \cos 3\psi\, \mathbf{k}, \tag{3.17}$$

$$a = -\cos 2\psi\, \mathbf{i} + \sin 2\psi\, \mathbf{j}, \qquad b = -\cos 4\psi\, \mathbf{i} - \sin 4\psi\, \mathbf{j}. \tag{3.18}$$

We will first show that the intersection $L_1 \cap L_2$ takes place entirely in the piece P_4. Suppose $L_2(\pi/2, \psi)$ lies in the piece P_3. Then the matrices A and B in equation (3.17) must commute, so $\cos 3\psi = 0$, corresponding to $\psi \in \{\pm\pi/6, \pm\pi/2\}$. From equations (3.17) and (3.18), we find that

$$\gamma(L_2(\pi/2, \pm\pi/6)) = -\pi/6, \qquad \gamma(L_2(\pi/2, \pm\pi/2)) = \pi/2.$$

But Theorem 3.3.25 states that all of the points in $L_1 \cap P_3$ have $\gamma = 0$. It follows that L_1 does not intersect L_2 in the piece P_3.

We now consider the intersection $L_1 \cap L_2$ in the piece P_4. Using equations (3.17) and (3.18), we find that the $(\hat{a}, \hat{b})$ coordinates of $L_2(\pi/2, \psi)$ are

$$\hat{a}(L_2(\pi/2, \psi)) = (-\cos 2\psi,\, \sin 2\psi,\, 0), \quad \hat{b}(L_2(\pi/2, \psi)) = (-\cos 4\psi,\, -\sin 4\psi,\, 0) \tag{3.19}$$

for $\psi \in (-\pi/6, \pi/6)$, and

$$\hat{a}(L_2(\pi/2, \psi)) = (-\cos 2\psi,\, -\sin 2\psi,\, 0), \quad \hat{b}(L_2(\pi/2, \psi)) = (-\cos 4\psi,\, \sin 4\psi,\, 0) \tag{3.20}$$

for $\psi \in (-\pi/2, -\pi/6) \cup (\pi/6, \pi/2)$. From Lemma 3.6.1, we know that either $\theta \in \{\pi/2, 3\pi/2\}$ or $\phi \in \{0, \pi\}$. But $\phi = 0$ and $\phi = \pi$ correspond to the double-point of L_2, which lies in P_3, and we have already shown that L_1 does not intersect L_2 in P_3. Thus $\theta \in \{\pi/2, 3\pi/2\}$. Substituting $\theta = \pi/2$ and $\theta = 3\pi/2$ into the expressions for the $(\hat{a}, \hat{b})$ coordinates of $L_1(\phi, \theta)$ given in Theorem 3.3.25, we find that

$$\hat{a}(L_1(\phi, \pi/2)) = (-\sin(\phi + \nu),\, -\cos(\phi + \nu),\, 0), \tag{3.21}$$

$$\hat{b}(L_1(\phi, \pi/2)) = (\sin(\phi - \nu),\, \cos(\phi - \nu),\, 0), \tag{3.22}$$

$$\hat{a}(L_1(\phi, 3\pi/2)) = (\sin(\phi + \nu),\, \cos(\phi + \nu),\, 0), \tag{3.23}$$

$$\hat{b}(L_1(\phi, 3\pi/2)) = (-\sin(\phi - \nu),\, -\cos(\phi - \nu),\, 0). \tag{3.24}$$

From equations (3.19)–(3.24), it follows that the intersection $L_1 \cap L_2$ fact takes place in a torus $T^2 - \bar{\Delta} \subset S^2 \times S^2 - \bar{\Delta}$, where $\bar{\Delta} \subset T^2$ is the antidiagonal. In Figure 3.11 we use equations (3.19)–(3.24) to plot the intersection of L_1 and L_2 in $T^2 - \bar{\Delta}$. We see that L_1 and L_2 intersect in three points.

We will now show that the intersection is transverse at each of these three points. A calculation shows that at each point we have

$$\begin{aligned}
&\partial_\phi h_1(L_1(\phi, \theta)) = 0, \quad \partial_\theta h_1(L_1(\phi, \theta)) \neq 0, \quad \partial_\chi h_1(L_2(\chi, \psi)) = 0, \quad \partial_\psi h_1(L_2(\chi, \psi)) = 0, \\
&\partial_\phi h_2(L_1(\phi, \theta)) = 0, \quad \partial_\theta h_2(L_1(\phi, \theta)) = 0, \quad \partial_\chi h_2(L_2(\chi, \psi)) \neq 0, \quad \partial_\psi h_2(L_2(\chi, \psi)) = 0.
\end{aligned}$$

These equations, together with Figure 3.11, show that the intersection is transverse at each intersection point. $\square$

For knots K in S^3, one can show (see [HHK14], Section 12.1) that

$$\operatorname{rank} I^\natural(S^3, K) \geq \sum_i |a_i|,$$

where a_i is the coefficient of t^i in the Alexander polynomial $\Delta_K(t)$ of K:

$$\Delta_K(t) = \sum_i a_i t^i.$$

This inequality, together with Theorem 3.6.2, gives the singular instanton homology for the trefoil. This result was already known, since, as shown by Kronheimer and Mrowka,

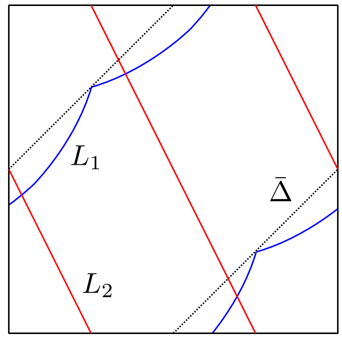

Figure 3.11: The trefoil in S^3. The space depicted is $T^2 - \bar{\Delta} \subset S^2 \times S^2 - \bar{\Delta}$. Shown are the Lagrangian L_1, the Lagrangian L_2, and the antidiagonal $\bar{\Delta}$.

the singular instanton homology of an alternating knot in S^3 is isomorphic to the reduced Khovanov homology of its mirror [KM14].

3.6.2 Unknot in $L(p,1)$ for p not a multiple of 4

We can construct the unknot U in the lens space $L(p,1)$ by gluing the two handlebodies together using the mapping class group element $f = T_a^p \in \mathrm{MCG}_2(T^2)$. The following is a restatement of Theorem 1.2.1 from the Introduction:

Theorem 3.6.3. *If p is not a multiple of 4, then the rank of $I^\natural(L(p,1),U)$ for the unknot U in the lens space $L(p,1)$ is at most p.*

Proof. A calculation shows that $L_2(\chi,\psi) = L_d(\chi,\psi) \cdot f = [A,B,a,b]$, where

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = \cos p\chi + \sin p\chi\,\mathbf{k}, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}. \tag{3.25}$$

Since A and B commute, the Lagrangian L_2 lies in the piece P_3. From equation (3.25), it follows that the (α,β,γ) coordinates of the point $L_2(\chi,\psi)$ are

$$\alpha(L_2(\chi,\psi)) = \chi, \qquad \beta(L_2(\chi,\psi)) = p\chi, \qquad \gamma(L_2(\chi,\psi)) = \psi.$$

Comparing with the parameterization of L_1 in P_3 given in Theorem 3.3.25, we find that the intersection $L_1 \cap L_2$ in fact takes place in the pillowcase $P_3 \cap \{\gamma = 0\}$. In Figure 3.12 we plot the intersection of L_1 with L_2 in the pillowcase $P_3 \cap \{\gamma = 0\}$ for $p = 1,2,3$. We find that if p is not a multiple of 4 then we obtain a generating set with p generators. If p is a multiple of 4 then $L_1 \cap L_2$ contains the double-point $(\alpha, \beta, \gamma) = (\pi/2, 0, 0)$ of L_1, and thus our scheme for counting generators fails.

We will now show that the intersection is transverse at each intersection point. Define functions

$$h_1([A, B, a, b]) = \operatorname{tr} Aa, \qquad\qquad h_2([A, B, a, b]) = \operatorname{tr} Ba.$$

A straightforward calculation shows that at each point of $L_1 \cap L_2$ we have that

$$\begin{aligned}
&\partial_\phi h_1(L_1(\phi,\theta)) = 0, \quad \partial_\theta h_1(L_1(\phi,\theta)) \neq 0, \quad \partial_\chi h_1(L_2(\chi,\psi)) = 0, \quad \partial_\psi h_1(L_2(\chi,\psi)) \neq 0,\\
&\partial_\phi h_2(L_1(\phi,\theta)) = 0, \quad \partial_\theta h_2(L_1(\phi,\theta)) = 0, \quad \partial_\chi h_2(L_2(\chi,\psi)) = 0, \quad \partial_\psi h_2(L_2(\chi,\psi)) \neq 0.
\end{aligned}$$

These equations, together with Figure 3.12, show that the intersection is transverse at each point of $L_1 \cap L_2$. $\square$

For the case $p = 1$ we have that $L(p, 1) = S^3$, and our results imply that the unknot in S^3 has a generating set with a single generator. Since there is a single generator, this amounts to a calculation of the singular instanton homology.

Remark 3.6.4. It is interesting to note that for the unknot U in the lens space $Y = L(p, q)$, the knot Floer homology $\widehat{HFK}(Y, U)$ has rank p (see [Hed11]).

3.6.3 Simple knot in $L(p, 1)$ in homology class $1 \in \mathbb{Z}_p = H_1(L(p, 1); \mathbb{Z})$

Definition 3.6.5. A knot K in a lens space $L(p, q)$ is said to be *simple* if the lens space has a Heegaard splitting into solid tori U_1 and U_2 with meridian disks D_1 and D_2 such that D_1 intersects D_2 in p points and $K \cap U_i$ is an unknotted arc in disk D_i for $i = 1, 2$ (see [Hed11]).

One can show that there is exactly one simple knot in each nonzero homology class of $H_1(L(p, q); \mathbb{Z}) = \mathbb{Z}_p$ [Hed11]. For the case $q = 1$, we can view the lens space $L(p, 1)$ is a circle bundle over S^2, and a loop that winds n times around a circle fiber is a simple knot in homology class $n \in \mathbb{Z}_p = H_1(L(p, 1); \mathbb{Z})$. For $p \geq 2$, we can construct the simple knot

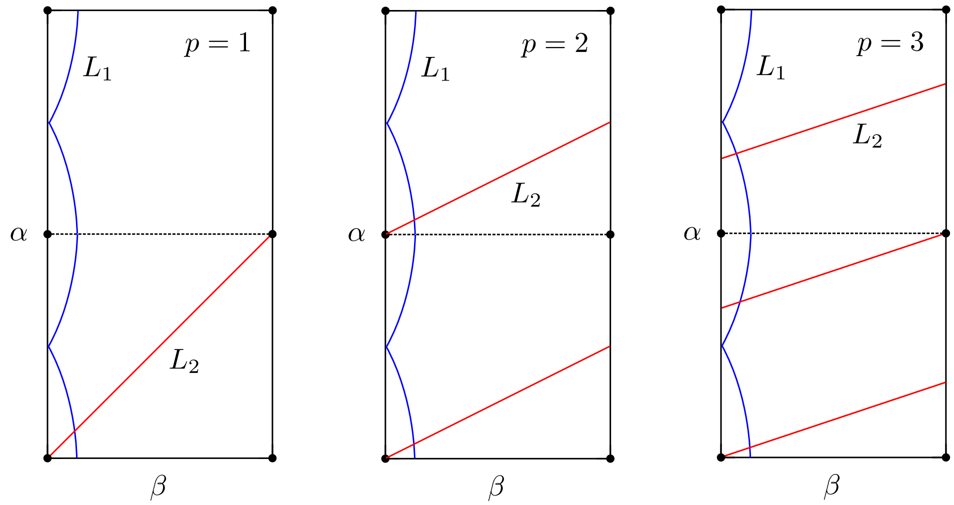

Figure 3.12: The unknot in $L(p,1)$ for $p = 1,2,3$. The space depicted is the pillowcase $P_3 \cap \{\gamma = 0\}$. Shown are the Lagrangians L_1 and L_2.

K in the lens space $L(p,1)$ corresponding to the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$ by gluing the two handlebodies together using the mapping class group element $f = \alpha_1^{-1}T_a^p \in \mathrm{MCG}_2(T^2)$. We first prove a result that constrains the possible intersection points of L_1 and $L_2 = L_d \cdot f$:

Lemma 3.6.6. *If $L_1(\phi,\theta) = L_2(\chi,\psi)$ then $(\chi,\psi) \in \{(\chi_0,\psi_0),\cdots,(\chi_{p-1},\psi_{p-1})\}$ and $\phi = \pi/2$, where $\chi_n := (n+1/2)(\pi/p)$ and $\psi_n := (-1)^{n+1}(\pi/2-\epsilon)$.*

Proof. Define a function $h_1 : R(T^2,2)\cap\{\mathrm{tr}\,Ab \neq 0\} \to \mathbb{R}$ and functions $h_2, h_3 : R(T^2,2) \to \mathbb{R}$ by

$$h_1([A,B,a,b]) = -\frac{\mathrm{tr}\,Aa}{\mathrm{tr}\,Ab}, \qquad h_2([A,B,a,b]) = \mathrm{tr}\,Ba, \qquad h_3([A,B,a,b]) = \mathrm{tr}\,B.$$

Using straightforward calculations, one can show that if $h_3(L_1(\phi,\theta)) = h_3(L_2(\chi,\psi))$ then $(\mathrm{tr}\,Ab)(L_2(\chi,\psi)) \neq 0$, and thus the function h_1 is defined everywhere on $L_1 \cap L_2$. We evaluate the functions h_1, h_2, and h_3 at the points $L_1(\phi,\theta)$ and $L_2(\chi,\phi)$. If we require that each function give the same value at both points, we obtain the desired result. $\square$

The following is a restatement of Theorem 1.2.2 from the introduction:

Theorem 3.6.7. *If K is the unique simple knot in the lens space $L(p,1)$ representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$, then the rank of $I^\natural(L(p,1),K)$ is at most p.*

Proof. We will argue that each of the p potential intersection points described by Lemma 3.6.6 is an actual intersection point. A calculation shows that $L_2(\chi_n,\psi_n) = L_d(\chi_n,\psi_n)\cdot f = [A,B,a,b]$, where

$$A = \cos\chi_n + \sin\chi_n\,\mathbf{i}, \qquad B = \cos\epsilon + \sin\epsilon\,\mathbf{k}, \tag{3.26}$$

$$a = (-1)^{n+1}(\cos\epsilon\,\mathbf{i} + \sin\epsilon\,\mathbf{j}), \qquad b = (-1)^n\cos\epsilon\,\mathbf{i} + \sin\epsilon\cos\eta_n\,\mathbf{j} + \sin\epsilon\sin\eta_n\,\mathbf{k}, \tag{3.27}$$

and $\eta_n := (1+n(p+2))(\pi/p)$. We note that A and B do not commute, since the coefficient of $\mathbf{i}$ in A and the coefficient of $\mathbf{k}$ in B are both nonzero, so the intersection $L_1 \cap L_2$ takes place entirely in the piece P_4. From equations (3.26) and (3.27), we find that the $(\hat{a},\hat{b})$ coordinates of $L_2(\chi_n,\psi_n)$ are given by

$$\hat{a}(L_2(\chi_n,\psi_n)) = (-1)^{n+1}(\cos\epsilon,\, \sin\epsilon,\, 0), \tag{3.28}$$

$$\hat{b}(L_2(\chi_n,\psi_n)) = ((-1)^n\cos\epsilon,\, \sin\epsilon\cos\eta_n,\, \sin\epsilon\sin\eta_n). \tag{3.29}$$

From Lemma 3.6.6, we know that if $L_1(\phi,\theta) = L_2(\chi,\psi)$ then $\phi = \pi/2$. Substituting $\phi = \pi/2$ into the expressions for the $(\hat{a},\hat{b})$ coordinates of $L_1(\phi,\theta)$ given in Theorem 3.3.25, we find that

$$\hat{a}(L_1(\pi/2,\theta)) = (\cos\epsilon,\, \sin\epsilon,\, 0), \quad \hat{b}(L_1(\pi/2,\theta)) = (-\cos\epsilon,\, -\sin\epsilon\cos\bar{\theta},\, \sin\epsilon\sin\bar{\theta}) \tag{3.30}$$

for $\theta \in (0,\pi)$, and

$$\hat{a}(L_1(\pi/2,\theta)) = (-\cos\epsilon,\, -\sin\epsilon,\, 0), \quad \hat{b}(L_1(\pi/2,\theta)) = (\cos\epsilon,\, \sin\epsilon\cos\bar{\theta},\, \sin\epsilon\sin\bar{\theta}) \tag{3.31}$$

for $\theta \in (\pi, 2\pi)$, where $\bar{\theta}$ is defined such that

$$\cos\bar{\theta} = \frac{\cos^2\epsilon\cos^2\theta - \sin^2\theta}{\cos^2\epsilon\cos^2\theta + \sin^2\theta}, \qquad \sin\bar{\theta} = \frac{\cos\epsilon\sin 2\theta}{\cos^2\epsilon\cos^2\theta + \sin^2\theta}.$$

It is straightforward to verify that for small enough values of ϵ, the maps $(0,\pi)\to(0,2\pi)$, $\theta\mapsto\bar{\theta}$ and $(\pi,2\pi)\to(0,2\pi)$, $\theta\mapsto\bar{\theta}$ are diffeomorphisms. Thus we can always solve equations (3.28)–(3.31) to obtain a unique value of θ such that $L_1(\pi/2,\theta) = L_2(\chi_n,\psi_n)$. Specifically, if n is even, then $\theta \in (0, \pi)$ is given by $\bar{\theta}(\theta) = \eta_n$, and if n is odd then $\theta \in (\pi, 2\pi)$ is given by $\bar{\theta}(\theta) = \pi - \eta_n$. We conclude that L_1 and L_2 intersect in p points.

We will now show that L_1 intersects L_2 transversely at each of these p points. A straightforward calculation shows that at each point of $L_1 \cap L_2$ we have

$$\begin{aligned}
&\partial_\phi h_1(L_1(\phi,\theta)) \neq 0, \quad \partial_\theta h_1(L_1(\phi,\theta)) = 0, \quad \partial_\chi h_1(L_2(\chi,\psi)) = 0, \quad \partial_\psi h_1(L_2(\chi,\psi)) = 0,\\
&\partial_\phi h_2(L_1(\phi,\theta)) = 0, \quad \partial_\theta h_2(L_1(\phi,\theta)) = 0, \quad \partial_\chi h_2(L_2(\chi,\psi)) \neq 0, \quad \partial_\psi h_2(L_2(\chi,\psi)) = 0,\\
&\partial_\phi h_3(L_1(\phi,\theta)) = 0, \quad \partial_\theta h_3(L_1(\phi,\theta)) = 0, \quad \partial_\chi h_3(L_2(\chi,\psi)) = 0, \quad \partial_\psi h_3(L_2(\chi,\psi)) \neq 0.
\end{aligned}$$

These equations, together with Theorem 3.3.25, show that the intersection is transverse at each point. □

For the case $p = 0$, the knot we have constructed is $K = S^1 \times \{pt\}$ in $S^1 \times S^2$, and our above result implies that this knot has a generating set with zero generators. This result holds even in the absence of the perturbation, since there are no homomorphisms $\rho : \pi_1(S^1 \times S^2 - K) \to SU(2)$ that take loops around K to traceless matrices.

For the case $p = 1$, the knot we have constructed is the unknot in S^3, and we have have reproduced the result of Section 3.6.2 for this knot.

Remark 3.6.8. It is interesting to note that for a simple knot K in the lens space $Y = L(p, q)$, the knot Floer homology $\widehat{HFK}(Y, K)$ has rank p (see [Hed11]).

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