Title

EFFECTS OP A MAGNETIC FIELD ON NATURAL CONVECTION IN' A TOROIDAL CHANNEL

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Nomenclature

a	=	cross-sectional radius of torus
A	=	radius of torus
B	=	arbitrary constant
C	=	numerical constant
$\underset{\sim}{\hat{e}}_r, \underset{\sim}{\hat{e}}_\phi$	=	unit vectors in r and ϕ directions
$\underset{\sim}{E}$	=	electric field intensity
$\underset{\sim}{\mathcal{E}}$	=	dimensionless form of $\underset{\sim}{E}$
$\underset{\sim}{g}$	=	acceleration due to gravity
$\underset{\sim}{h}$	=	induced magnetic field intensity
h_z	=	z component of $\underset{\sim}{h}$
h	=	dimensionless form of h_z
h_0	=	zero-order perturbation term for h
h_c	=	value of h in core region
$\underset{\sim}{H}$	=	total magnetic field intensity
$\underset{\sim}{H}_0$	=	applied magnetic field intensity
H_0	=	magnitude of $\underset{\sim}{H}_0$
$H_n^{(1)}$	=	Hankel function of first kind
Im	=	imaginary part
$\underset{\sim}{\hat{i}}$	=	unit vector in x direction
$\underset{\sim}{J}$	=	electric current density
J_n	=	Bessel function of first kind
$\underset{\sim}{\mathcal{J}}$	=	dimensionless form of $\underset{\sim}{J}$
k	=	thermal conductivity
$\underset{\sim}{\hat{k}}$	=	unit vector in z direction
M_1	=	Hartmann number
M_r	=	reduced Hartmann number

Nu	=	Nusselt number
$(Nu)_{cv}$	=	convective part of Nu
p	=	pressure
Q	=	heat energy flow
r	=	radial coordinate
R	=	Rayleigh number
R_c	=	critical value of R
R_0	=	value of R_c when $M_r = 0$
$\mathcal{R}e$	=	real part
t	=	dimensionless form of T^*
t_r	=	real part of t
t_i	=	imaginary part of t
T	=	temperature
T_0	=	excess of boundary temperature over average
T_1	=	average temperature of system
T^*	=	complex form of T
v	=	dimensionless form of V
v_c	=	value of v in core region
v_0	=	zero-order perturbation term for v
v_1	=	first-order perturbation term for v
$\underset{\sim}{V}$	=	fluid velocity
V	=	magnitude of $\underset{\sim}{V}$ (z component)
$W(\beta)$	=	special function
W_r	=	real part of $W(\beta)$
W_i	=	imaginary part of $W(\beta)$
x, y, z	=	Cartesian coordinates
α	=	dimensionless parameter

β	=	dimensionless parameter (Graetz number)
$\log \gamma$	=	Euler's constant
ϵ	=	dielectric constant
η	=	dimensionless boundary-layer variable
κ	=	thermal diffusivity
λ	=	volume coefficient of thermal expansion
μ	=	magnetic permeability
ν	=	kinematic viscosity
ξ	=	dimensionless form of r
ρ		density of fluid
ρ_0		fluid density at temperature T_1
ρ_e		electric charge density
ρ_e'		dimensionless form of ρ_e
σ		electrical conductivity
σ_r		reduced electrical conductivity
ϕ		angular cylindrical coordinate
ψ		dimensionless form of x

Subscripts

$(\ )_1$	denotes fluid
$(\ )_2$	denotes solid
$(\ )_{av}$	denotes average value
$(\ )_\perp$	denotes perpendicular to $\hat{k}$

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ABSTRACT

The problem of the natural convection of an electrically and thermally conducting fluid within a long, narrow, vertical toroidal channel centered in a large block of an electrically and thermally conducting solid is analyzed. A uniform horizontal magnetic field is applied to the fluid, and the bottom of the solid block is maintained at a higher fixed temperature than the top. The laminar steady-state single-cell convective motion of the fluid is considered and an approximate solution is found for the heat transfer rate between the bottom and top surfaces of the block in the limiting cases of small and large Hartmann number. A numerical example is given for liquid sodium in which the application of a magnetic field of a few hundred gauss is shown to significantly reduce the rate of heat transfer.

EFFECTS OF A MAGNETIC FIELD ON NATURAL CONVECTION IN A TOROIDAL CHANNEL[1]

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Introduction

Investigation of the laminar flow of liquid metals through an external magnetic field under laboratory conditions was first begun by Hartmann [1] in 1937 when he investigated theoretically the forced flow of mercury due to an externally applied uniform pressure through a channel of rectangular cross section in the presence of a transverse magnetic field. He found that if the applied magnetic field were large enough, the velocity profile of the flow would be completely altered from that with no magnetic field, and, in addition, the speed of flow would be significantly reduced. The greater the electrical conductivity of the channel walls, the greater the reduction of flow speed for a given driving pressure. This behavior occurred because the pondermotive forces arising from the induced electric current inhibited motion of the fluid across magnetic lines of force.

During the past several years, interest in this phenomenon has become rather widespread and many more investigations have been carried out in elaborate detail for various geometrical configurations and for driving pressures arising from convective as well as mechanical and electromagnetic sources. The classical Bénard problem concerning the stability of a viscous fluid initially at rest between two infinite horizontal plane surfaces and heated from below was solved by Chandrasekhar [2] for an electrically conducting fluid in the presence of an externally applied magnetic field. He found, as did Hartmann, that fluid motion across the magnetic lines of force was inhibited, and as a result, a vertical magnetic field could stabilize the fluid so that it would withstand greater temperature gradients before giving way to convective motion.

By viewing Chandrasekhar's results in terms of the amount of heat being transferred from the lower surface to the upper, one can see that by inhibiting convection at higher temperature gradients the magnetic field has also lessened the amount of heat being transferred at these temperatures. This suggests that it would be of interest to investigate in detail the manner in which a magnetic field can control the heat-transfer rate in situations involving convection of electrically conducting fluids. It is the purpose of this paper to record such an investigation for an example in which the convective flow is restricted to being laminar.

In selecting the actual geometric configuration to analyze, one must be careful to choose one for which the complex equations of magnetohydrodynamics can be reduced to manageable proportions while still capable of giving informative results. Such a configuration, the one selected here, is that of a single vertical convective cell within block of conducting solid.

In section 1 the general set of equations of magnetohydrodynamics describing the problem is stated and reduced to a smaller set of six simplified nondimensional equations. In sections 2 and 3, these equations are solved in two steps: In section 2 the energy equation, which is the only nonlinear one, is solved approximately to give the heat transfer rate as a function of a parameter proportional to the unknown average velocity of the fluid, and the error in the solution is estimated; in section 3 the remaining equations are solved in the regions of small and large Hartmann numbers, and the unknown average fluid velocity is determined as a function of the given parameters. In section 4 the results of the previous two sections are combined to give the heat-transfer rate in terms of the given parameters, and a numerical example for liquid sodium is calculated.

1 Formulation of Problem

Consider a metallic material in the shape of a rectangular solid of dimensions $2\mathbf{A}\times4\mathbf{A}\times4\mathbf{A}$, in which a toroidal channel of circular cross section πa^2 and length $2\pi\mathbf{A}$ has been hollowed out. Let the channel be filled with a thermally and electrically conducting fluid and let the solid be so oriented that the axis of the torus is perpendicular to the direction of gravity, as shown in Fig. 1.

The bottom and top faces are kept at fixed temperatures $T_1 + T_0$ and $T_1 - T_0$, respectively, and an external magnetic field applied so that the field inside the channel, $\underset{\sim}{H}_0$, is uniform and parallel to the axis of the torus. Since the magnetic permeability of the solid may be different from that of the fluid, the field in the solid will, in general, be different from $\underset{\sim}{H}_0$.

If T_0 is made large enough, the fluid in the lower part of the channel will become gravitationally unstable with respect to the fluid in the upper part because of the thermal expansion, and as a result the fluid will flow through the channel. Let the direction of flow be counterclockwise as shown. This flow then interacts with the magnetic field, $\underset{\sim}{H}_0$, to produce an electric current throughout the solid and liquid. The thermally caused body force will thus be balanced by the resulting pondermotive force resisting the flow as well as by the viscous drag.

It is seen that if $a/\mathbf{A} \ll 1$ and the channel is centrally placed in the block, then the curvature of the torus and the corner areas of the block will have negligible effect on the electric and thermal behavior of the flow.

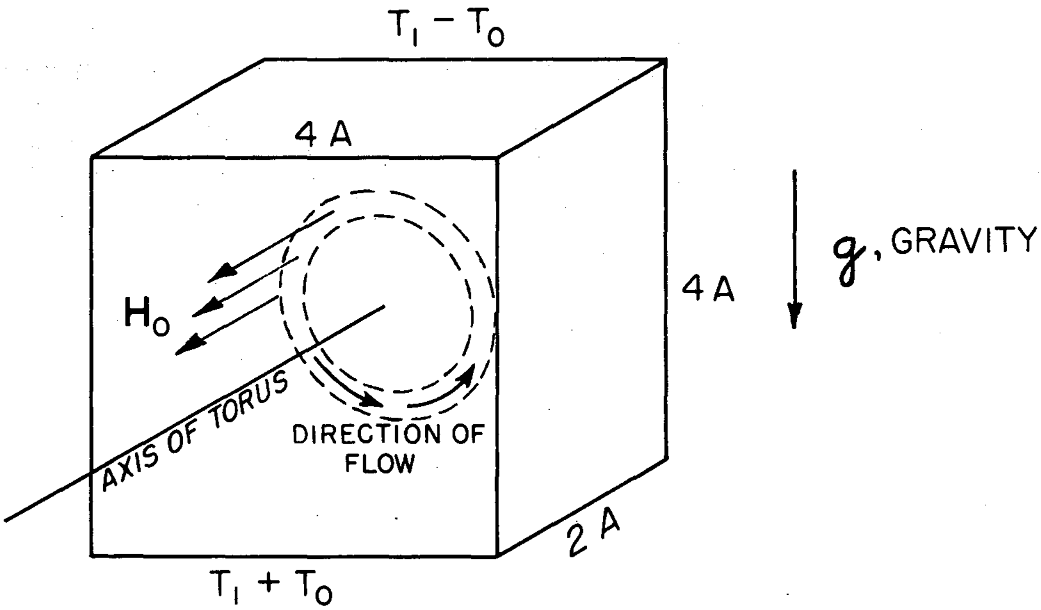

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Fig. 1. Physical configuration.

A more formal development would include a perturbation series in the parameter a/A, and the important contributions (those independent of a/A) would be the ones arising from the formulation used here. Under these assumptions, the configuration can be replaced by the one shown in Fig. 2, in which an infinitely long metallic cylinder, radius A, has fluid flowing through it inside a concentric cylindrical channel of radius a. The gravitational field will be $\underset{\sim}{g}_\perp$, normal to the z direction, plus a component which will be in the z direction and equal to $-g\sin(z/A)\,\hat{\underset{\sim}{k}}$; the temperature at the outer surface of the solid will be $T_1 + \frac{1}{2}T_0(1-\sin\phi)\cos(z/A)$; and the magnetic field in the fluid will be in the negative x direction and equal to $-H_0\hat{\underset{\sim}{i}}$. The center of the bottom of the rectangular block will correspond to $r = A$, $\phi = -\frac{1}{2}\pi$, and $z = 0$.

In writing the magnetohydrodynamic equations which govern the problem the usual approximations applicable to laboratory flow of liquid metals are made. It will be assumed that the fluid is incompressible, that the Boussinesq[3] approximation of the thermal density variations being important only in their effect on the force due to gravity is valid, that all physical properties other than the density are constant, that the electrical convection current is negligible compared to the conduction current, and that the convective velocity is sufficiently small so that Joule and viscous heating may be ignored.[2] Under these assumptions, the governing steady state equations are:

Momentum equation:

$$\rho_0(\underset{\sim}{V}\cdot\underset{\sim}{\nabla})\underset{\sim}{V} = \rho\,\underset{\sim}{g}_\perp - \rho\, g\sin(z/A)\,\hat{\underset{\sim}{k}} + \mu\,(\underset{\sim}{J\times H}) - \underset{\sim}{\nabla}p + \rho_0\nu\nabla^2\underset{\sim}{V} \qquad (1)$$

where $\underset{\sim}{V}$ is the fluid velocity, $\underset{\sim}{J}$ the electric current density, $\underset{\sim}{H}$ the magnetic field intensity, p the pressure, ρ the fluid density, ρ_0 its density at temperature T_1, ν the kinematic viscosity, and μ the magnetic permeability;

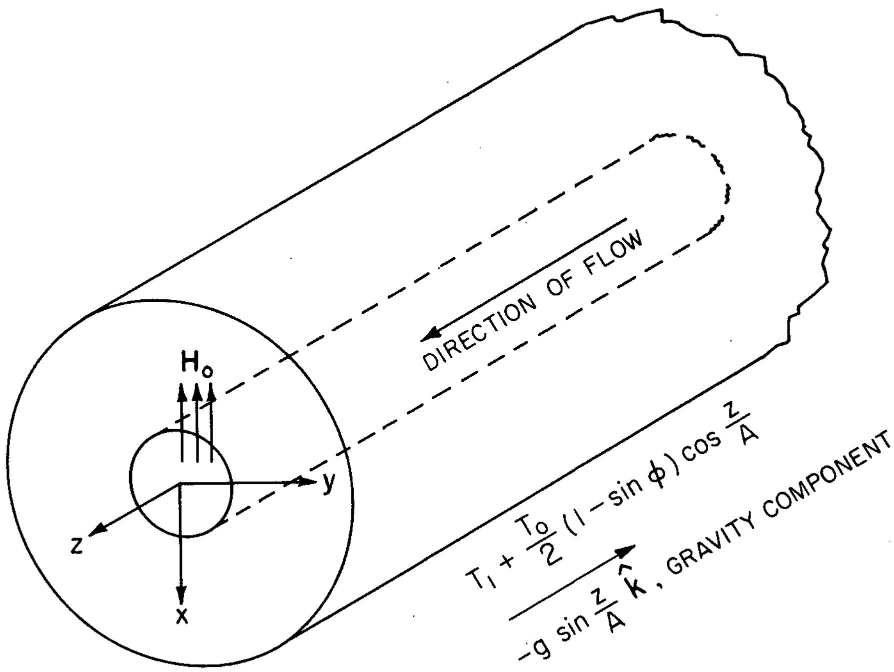

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Fig. 2. Cylindrical approximation.

Continuity equation:

$$\nabla \cdot \mathbf{V} = 0 \tag{2}$$

Energy equation:

$$(\mathbf{V} \cdot \nabla) T = \kappa \nabla^2 T \tag{3}$$

where Joule heating, viscous heating, and kinetic energy transport are neglected, where κ is the thermal diffusivity, and T the temperature;

Equation of state:

$$\rho = \rho_0 [1 - \lambda(T - T_1)] , \tag{4}$$

where λ is the volume coefficient of thermal expansion;

Maxwell's equations:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (5) \qquad \nabla \times \mathbf{E} = 0 \tag{6}$$

$$\nabla \cdot \mathbf{H} = 0 \quad (7) \qquad \nabla \cdot \mathbf{E} = \rho_e / \epsilon \tag{8}$$

where $\mathbf{E}$ is the electric field intensity, ϵ the dielectric constant, and ρ_e the electric charge density, convection current being neglected;

Ohm's law:

$$\mathbf{J} = \sigma (\mathbf{E} + \mu \mathbf{V} \times \mathbf{H}) , \tag{9}$$

where σ is the electrical conductivity; and

Electrical continuity equation:

$$\nabla \cdot \mathbf{J} = 0 , \tag{10}$$

which results from combining the first and last of Maxwell's equations. Rational mks units are used throughout.

This system of equations must be solved subject to the boundary conditions:

(a) at $r = 0$ all quantities are finite;

(b) at the fluid-metal interface $(r = a)$, $\mathbf{V} = 0$; $\mathbf{J} \cdot \hat{\mathbf{e}}_r$, $\mathbf{E} \times \hat{\mathbf{e}}_r$, $\mathbf{H} \times \hat{\mathbf{e}}_r$, T, and $k \partial T / \partial r$ are all continuous, where k is the thermal conductivity and $\hat{\mathbf{e}}_r$ the unit vector along r.

(c) at the outer boundary of the metal ($r = \mathbb{A}$), $\underset{\sim}{J} \cdot \underset{\sim}{e_r} = 0$, $\underset{\sim}{H}$ equals the applied magnetic field, and $T = T_1 + \frac{1}{2} T_0 (1 - \sin\phi) \cos(z/\mathbb{A})$.

Solutions to these equations will be sought for the condition of laminar flow in which the only nonvanishing fluid velocity component is in the z direction, i.e., along the length of the channel. Equation (2) then implies that the velocity is independent of z, and hence the resulting induced magnetic field will also be independent of z. This allows $\underset{\sim}{V}$ to be written as $\underset{\sim}{V} = V(r, \phi)\, \underset{\sim}{\hat{k}}$, and $\underset{\sim}{H}$ to be written as the sum of the applied constant field $\underset{\sim}{H}_0$ and an induced variable field, $\underset{\sim}{h}(r, \phi)$. Placing these forms into the curl of (9) and substituting into it (6), (7), and the curl of (5) will give an equation for $\underset{\sim}{h}$ in terms of $\underset{\sim}{H}_0$ and $\underset{\sim}{V}$. The x and y components of this equation yield the trivial solution for the x and y components of $\underset{\sim}{h}$, and the z component is

$$\nabla^2 h_z = \sigma \mu H_0 \, \partial V/\partial x \, . \tag{11}$$

Similarly, upon substitution of (4) and (5) into (1), its z component becomes

$$0 = - g \rho_0 \, [1 - \lambda(T - T_1)] \sin(z/\mathbb{A}) - \mu H_0 \partial h_z/\partial x - \partial p/\partial z + \rho_0 \nu \nabla^2 V . \tag{12}$$

In (11) and Eq. (12) the Laplacian is, of course, only two-dimensional.

The z dependence of the Laplacian of (3) can also be removed by assigning to T, in accordance with its periodic boundary conditions at $r = \mathbb{A}$, a periodic dependence on z in the form $T(r, \phi, z) = T_1 + \mathcal{R}e(T^*(r, \phi) e^{iz/\mathbb{A}})$, where $T^*(r, \phi)$ is a complex variable and the real part of $T^*(r, \phi) e^{iz/\mathbb{A}}$ is the function of interest. Then (3) becomes

$$i V T^*/\mathbb{A} = \kappa(\nabla^2 - 1/\mathbb{A}^2) T^* \tag{13}$$

where the Laplacian is now two-dimensional.

The complex form of T may be substituted into (12), and upon integration over one period in z from 0 to $2\pi A$, the equation becomes

$$0 = -\pi A g \rho_0 \lambda \operatorname{Im}(T^*) - 2\pi A \mu H_0 \partial h_z/\partial x + 2\pi A \rho_0 \nu \nabla^2 V . \tag{14}$$

The third term of (12) yields zero upon integration, since the pressure must be a continuous function of z; the second and last terms are independent of z.

Equations (11) (13) and (14) give the necessary relationships between the three unknown functions, V, T^*, and h_z for the interior of the channel. By proceeding in a similar manner, the relationships for the region exterior to the channel will be found to be $V = 0$, $(\nabla^2 - 1/A^2)T^* = 0$, and $\nabla^2 h_z = 0$. Once T^*, V, and h_z are found, the electrical quantities $\underset{\sim}{J}$, $\underset{\sim}{E}$, and ρ_e can be calculated directly from (5) (9) and (8) respectively, for each of the two regions.

At this point it will be convenient to place the problem in dimensionless form by means of the following substitutions:

$$t = 2T^*/T_0 = t_r + i\, t_i, \quad v = \pi a^2 V \Big/ \int_0^1 \int_0^{2\pi} V \xi \, d\phi \, d\xi = V/V_{av} ,$$

$$h = h_z/\mu_1 \sigma_1 a H_0 V_{av} , \quad \underset{\sim}{\mathcal{J}} = \underset{\sim}{J}/\mu_1 \sigma_1 H_0 V_{av}, \quad \underset{\sim}{\mathcal{E}} = \underset{\sim}{E}/\mu_1 H_0 V_{av},$$

$$\rho_e = \rho_e\, a/\mu_1 \epsilon_1 H_0 V_{av}, \quad \xi = r/a, \quad \text{and} \quad \psi = x/a .$$

Here, as well as in the remainder of the paper, subscript-1 refers to the fluid quantities, and subscript-2 to the solid. It will also be helpful to define the following nondimensional parameters: $M_1 = \mu_1 H_0 a \sigma_1^{1/2}/(\rho_0 \nu)^{1/2}$ (Hartmann number), $R = \lambda g T_0 a^4 / 4 A \nu \kappa_1$ (one-half the conventional Rayleigh number), and $\beta = a^2 V_{av}/A \kappa_1$ (one-half the conventional Graetz number).

Upon substitution of the above quantities, the governing equations become:

Inside the channel (continued)

$$\nabla^2 v = R\, t_i/\beta + M_1^2 \partial k/\partial\psi = R\, t_i/\beta + M_1^2(\cos\phi\, \partial h/\partial\xi - \xi^{-1} \sin\phi \partial h/\partial\phi) \tag{15}$$

Inside the channel

$$\begin{cases} \nabla^2 t - i\beta v t = 0 & (16) \\ \nabla^2 h = \partial v/\partial\psi = \cos\phi\, \partial v/\partial\xi - \xi^{-1} \sin\phi\, \partial v/\partial\phi & (17) \\ \mathfrak{J} = \nabla \times (h\,\hat{k}) & (18) \\ \mathcal{E} = \mathfrak{J} + v\,\hat{j} = \mathfrak{J} + v\,(\sin\phi\, \hat{e}_r + \cos\phi\, \hat{e}_\phi) & (19) \\ \rho_e' = \nabla \cdot \mathcal{E} & (20) \end{cases}$$

Outside the channel

$$\begin{cases} \nabla^2 t - (a/\mathbf{A})^2\, t = 0 & (21) \\ \nabla^2 h = 0 & (22) \\ \mathfrak{J} = \nabla \times (h\,\hat{k}) & (23) \\ \mathcal{E} = (\sigma_1/\sigma_2)\, \mathfrak{J} & (24) \\ \rho_e' = (\epsilon_1/\epsilon_2)\nabla \cdot \mathcal{E} = 0 & (25) \end{cases}$$

In (16) the term $(a/\mathbf{A})^2 t$ has been omitted since it is negligible. It must be retained in (21) since the independent variable is allowed to become as large as $\mathbf{A}/a$. ρ_e' is set equal to zero in (25) by using (24) and (10). The Laplacian refers to the dimensionless independent variables and is two-dimensional.

The boundary conditions for the dimensionless variables will be:

(a) at $\xi = 0$: $v, t,$ and h are finite; (26)

(b) at $\xi = 1$: $v = 0$ and $t, h, k\ \partial t/\partial\xi$, and $\sigma^{-1}\partial h/\partial\xi$ are continuous;

and (27)

(c) at $\xi = \mathbf{A}/a$: $h = 0$ and $t = 1 - \sin\phi$.

2 Solution of Energy Equation

Equations (16) and (21) will now be solved to give an expression for t in terms of the parameter β. This will determine how the temperature (and hence the heat transfer) depends upon the average velocity of the fluid.

Because of the nonlinearity of (16) the usual methods for solving systems of linear differential equations cannot be directly applied. If the nonlinear equation is first approximated by one which is linear, however, the techniques can be used. Let (16) be approximated by an equation in which the velocity, v, has been replaced by its average value, unity. The equation is then

$$\nabla^2 t - i\beta t = 0 \tag{28}$$

one that is linear and easily solved. The error introduced by this approximation will be shown later to be small.

Upon separation of variables, one finds that (28) and (21) have as solutions the product of Bessel functions in ξ and circular harmonics in ϕ. However, because of the boundary condition (27) on t, only terms independent of ϕ and those proportional to $\sin\phi$ will have nonzero coefficients.

The solution for the temperature in the exterior region as determined by (21) subject to the boundary condition (27) is thus

$$t = [1 - BH_0^{(1)}(i)]\, J_0(i\xi a/A)/J_0(i) + BH_0^{(1)}(i\xi a/A) - \sin\phi\, J_1(i\xi a/A)/J_1(i). \tag{29}$$

Terms proportional to $H_1^{(1)}(i\xi a/A)\sin\phi$ are not included because their coefficients must be negligibly small owing to the smallness of a/A and the finite boundary conditions on the interior temperature distribution at $\xi = 0$. In order to evaluate the arbitrary constant B, the exterior solution must be matched with the interior solution at $\xi = 1$, and this may be done by using the approximate behavior of Eq. (29) near $\xi = 1$. Replacing the

Bessel and Hankel functions by their approximations for small argument gives

$$t\,\Big|_{\xi=1} = [\,1 - BH_0^{(1)}(i)]\,/\,J_0(i) - (2i/\pi)B\log(2\mathbb{A}/\gamma a)\ ,$$

$\log\gamma = 0.577\cdots$ and $dt/d\xi\,\Big|_{\xi=1^+} = 2iB/\pi$.

It is necessary to denote that the derivative is evaluated on the exterior side of $\xi = 1$, since it need not be continuous across this boundary.

Using this form for the exterior temperature and applying the boundary conditions (26) and (27) to the solution of Eq. (28), one obtains for the interior temperature

$$t = \left\{C\ W(\beta)/(W(\beta) + i\,\alpha\,\beta/2)\right\}\cdot\left\{J_0\,[\,(-i\beta)^{1/2}\,\xi]\,/J_0[(-i\beta)^{1/2}]\right\} \tag{30}$$

where

$$C = [\,J_0(i)]^{-1}\ ,$$

$$\alpha = \frac{1}{2}\,\pi k_1/k_2\,[\,(2/\pi)\log(2\mathbb{A}/\gamma a) - iH_0^{(1)}(i)/J_0(i)]$$

and

$$W(\beta) = \frac{1}{2}\,(-i\beta)^{1/2}J_0[\,(-i\beta)^{1/2}]/J_1[\,(-i\beta)^{1/2}]$$

is the complex function tabulated in Table XXI of Jahnke and Emde [4]. The value of B is also obtained, and by substitution into (29) one obtains for the exterior temperature

$$t = C\,J_0(i\xi a/\mathbb{A}) - \sin\phi\,J_1(i\xi a/\mathbb{A})/J_1(i) + \frac{1}{2}\,\pi(k_1/k_2)[\,\frac{1}{2}\,i\,C\beta/W(\beta) + \frac{1}{2}i\alpha\beta]\,\cdot$$
$$[\,iCH_0^{(1)}(i)\,J_0(i\xi a/\mathbb{A}) - iH_0^{(1)}(i\xi a/\mathbb{A})]\ . \tag{31}$$

The general dependence of these temperature distributions on the parameter β may be described as follows: When $\beta=0$ the fluid is not moving and heat is transferred only by conduction. The temperature distribution is purely real so that it is everywhere in phase with the boundary temperature in the z direction, and the temperature distribution inside the channel is merely $t = C$, a constant.

As β increases from zero, t becomes complex and the maximum temperature for a given ξ will no longer occur at $z = 0$ where the maximum boundary temperature occurs, but will be shifted along the direction of flow of the fluid. It is the convection of heat in the fluid that causes this shift in extreme temperature position. The amount by which the position of the maximum temperature is shifted increases rapidly with β and then approaches the value $\frac{1}{2}\pi\mathbf{A}$ asymptotically as $\beta \to \infty$.

This shift in the position of the maximum temperature of the fluid in the direction of flow causes the fluid in the cooler half of the metal to be warmer than it would be were it not flowing. Part of this additional heat which the fluid has transported from the warmer half of the metal will therefore conduct to the cold top surface, causing an increase in the total rate of heat transfer.

A convenient method for calculating this total rate of heat transfer from the lower face of the block to the upper is to consider separately the rate of heat flow due to convection and the rate of heat flow due to conduction across the horizontal plane midway between the two faces. If the thermal conductivity of the fluid is not excessively larger than that of the solid, the presence of the channel will have negligible effect on the part of the heat-flow rate across the plane due to conduction, and it will be simply $4\mathbf{A}k_2T_0$. The additional heat-flow rate due to convection must equal

$$\int_{-\frac{1}{2}\pi\mathbf{A}}^{\frac{1}{2}\pi\mathbf{A}} \int_0^{2\pi} k_1 [\partial T/\partial \xi]_{\xi=1^-} \, d\phi dz$$, which is the net rate of heat flow into the channel from the lower half of the block. The total rate of heat flow Q is then the sum of these two expressions, and by writing T in terms of $t\, e^{iz/\mathbf{A}}$ and performing the integrations, one obtains

$$Q = 4\mathbf{A}k_2 T_0 + 2\pi\mathbf{A}\, k_1\, T_0\, [dt_r/d\xi]_{\xi=1^-} .$$

After first substituting the derivative of t_r as calculated from (30), the equation may be rewritten in dimensionless form as

$$Nu = 1.0 + 1.24\ (\beta k_1/2k_2) \cdot (W_i + \frac{1}{2}\alpha\beta)/[W_r^2 + (W_i + \frac{1}{2}\alpha\beta)^2], \qquad (32)$$

where $Nu = Q/4Ak_2T_0$ (Nusselt number) and W_r and W_i are respectively the real and imaginary parts of $W(\beta)$.

The first term of (32) gives the rate, unity, at which heat is transferred purely by conduction and the second term, $(Nu)_{c.v.}$, the additional rate due to convection. This latter term increases with β from its value of zero at $\beta = 0$ to its maximum value of $1.24\ k_1/\alpha k_2$ at $\beta = \infty$. Its variation with β is plotted for three different values of α in Fig. 3.

It must now be determined under what ranges of α and β the approximation of substituting the average velocity for the actual velocity into Eq. (16) is a valid one. The closer the actual velocity distribution is to a uniform flow, the smaller will be the error introduced. An upper bound on the error can thus be estimated by considering the case in which the velocity distribution differs most widely from a uniform flow. This case, as will be seen later from solutions for the velocity distribution, is that of a paraboloid of revolution. That is, by taking $v = 2\ (1 - \xi^2)$, an upper bound on the error can be estimated.

It will be most meaningful to consider the error introduced in $[dt_r/d\xi]_{\xi=1^-}$, the quantity on which the total heat flow depends, rather than in t itself. Using the above form for v, integration of (16) shows that

$$[dt_r/d\xi]_{\xi=1^-} = -2\beta \int_0^1 (\xi - \xi^3) t_i\, d\xi\ .$$

Thus a measure of the error is the amount by which the left side of the equation differs from the right when the approximate temperature distribution (30) is used. The left side is

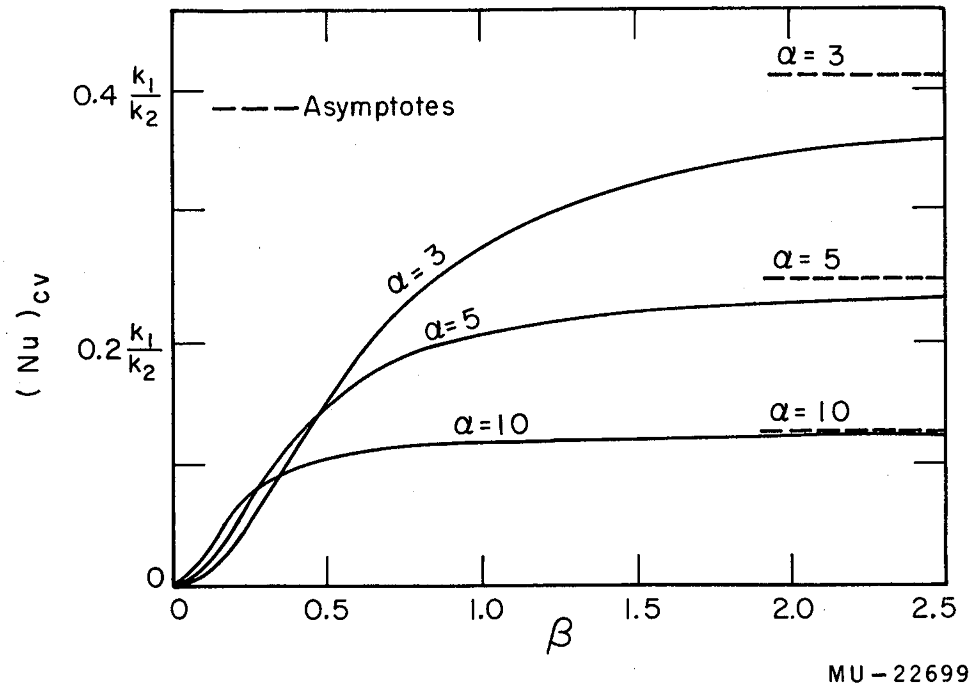


Fig. 3. $(Nu)_{cv}$, convective Nusselt number, as a function of β, the Graetz number, for certain values of α.

$$[dt_r/d\xi]_{\xi=1^-} = \frac{1}{2} C\beta(W_i + \frac{1}{2}\alpha\beta)/[W_r^2 + (W_i + \frac{1}{2}\alpha\beta)^2] .$$

Using integration by parts for the Bessel functions, one obtains for the right side

$$-2\beta\int_0^1 (\xi-\xi^3)t_i d\xi = 4C[W_i(W_i + \frac{1}{2}\alpha\beta) - W_r(1-W_r)]/[W_r^2+(W_i + \frac{1}{2}\alpha\beta)^2].$$

If $\beta < 1$, the difference between these two expressions depends only on the size of α; as α increases from 2 to 4, the error decreases from 4 percent to 2 percent and is less than 2 percent for $\alpha > 4$. As β becomes larger an additional error is introduced depending only on the size of β; as β increases from 2 to 4, the additional error increases from 1 percent to 4.5 percent. For $\beta > 5$ and $\alpha < 1$, the errors are no longer independent and become larger quite rapidly.

The lower limit on α and the upper limit on β which accuracy place are quite adequate to cover most of the regions of interest. For example, for $a/A < 0.1$ the Reynolds number for liquid sodium will exceed the critical value for transition to turbulent flow in the absence of a magnetic field for β greater than unity. Also, for $a/A < 0.1$, α cannot be less than 2 or 3 or else unreasonably large velocities would be required to achieve a sizable convective heat transfer, velocities so large as to again make turbulent flow likely.

However, in the case where a large magnetic field is applied, the fluid is stabilized [5, 6] so that laminar flow can be expected to extend to higher velocities and hence higher values of β. This means that solutions for larger values of β and smaller values of α would be desirable under this condition. Fortunately, the velocity profile in a large magnetic field will be found to be nearly uniform, so that in this case it will be allowable to extend the solution to somewhat larger values of β

and smaller values of α because the erorrs so introduced will be smaller than those estimated for the paraboloid of revolution profile.

3 Solution of Remaining Equations

In this Section, equations (15), (17), and (22) will be solved for v and h and for the relationship between β and the given parameters. Once v and h are found, the remaining variables $\underset{\sim}{\mathcal{J}}$, $\underset{\sim}{\mathcal{E}}$, and ρ_e' can be found by direct substitution into (18) through (20) and (23) through (24). The solution will be carried out for two limiting regions of interest, $M_1 \ll 1$ (small applied magnetic field), and $M_1 \gg 1$ (large applied magnetic field). For intermediate values of M_1, all of the inherent complexities of the equations are present and so involved as to make impractical any attempt at an analytic solution.

Small Magnetic Field

As a special case of the small-magnetic-field region, first consider the case where the field is zero. If $\underset{\sim}{H_0} = 0$, then the resulting simplifications are that $M_1 = 0$, $h_z = 0$, and that (15) reduces to $\nabla^2 v = Rt_i/\beta$, where t_i is the imaginary part of t. Substituting (30) for t_i, the solution for v subject to the boundary conditions (26) and (27) is obtained as

$$v = (RC/\beta^2)\,\mathcal{R}e\left\{(1-J_0[(-i\beta)^{1/2}\xi]/J_0[(-i\beta)^{1/2}])\cdot W(\beta)/[W(\beta)-\tfrac{1}{2}i\alpha\beta]\right\}. \tag{33}$$

Because only those values of β for which laminar flow can be anticipated are to be considered, the distribution given by (33) can be approximated by a simpler form. Assuming that the Reynolds number $V_{av}a/\nu$ must be less than 1600, one sees that the maximum permissible value of β for liquid sodium will be less than unity for $a/\mathbf{A} < 0.1$. Under this condition, the Bessel Function $J_0[(-i\beta)^{1/2}\xi]$ can be closely approximated by $1 + \frac{1}{4}\, i\beta\xi^2$ so that the velocity distribution reduces to a paraboloid of revolution.

By taking the average of the velocity distribution and setting it equal to unity, as required by the definition of v, the relationship between β and the other parameters can be derived. Integrating Eq. (33) over the cross-sectional area gives

$$1 = (RC/\beta^2) \cdot \mathcal{R}e\left\{[W(\beta)-1]/[W(\beta) + \tfrac{1}{2} i\alpha\beta]\right\} . \tag{34}$$

For $\beta < 1$ this relationship becomes

$$\beta^2 = [1/16\, RC(\alpha + 1/4) - 1] \cdot 4/(\alpha+1/4)^2 \tag{35}$$

when the approximation that $W(\beta) \approx 1 + 1/8\, i\beta$ for β less than unity is used.

Equations (35) gives the desired relationship between β, α, and **R**. It shows that for values of **R** above the critical value of $16/[C(\alpha+\frac{1}{4})]$ convection will take place, and the average convective velocity will be proportional to the square root of the increase in the applied temperature gradient.

If now a small magnetic field is applied so that M_1 remains less than unity, the effect of the magnetic field on the flow can be found in terms of a perturbation series in powers of M_1. Values of β larger than unity will have to be considered, however, since the transition from laminar to turbulent flow takes place at a greater Reynolds number due to the stabilization provided by the magnetic field. The only available experimental investigations of this stabilization are for nonconducting walls [5, 6] but they can provide a guide in estimating the maximum value of β for which laminar flow can be expected. For M_1 less than 1, laminar flow can be expected only if $\beta < 3$.

Before solving the equations for v and h in the interior region, an expression for h exterior to the channel must be found. It is given by the solution to (22) subject to the boundary condition (28). Using separation of variables one finds this solution to be

$$h = \sum_{n=1}^{\infty} \xi^{-n} (A_{1n} \sin(n\phi) + A_{2n} \cos(n\phi)) \tag{36}$$

where terms proportional to $(a/A)^{2n}$ are neglected because they are small.

The parameter relationship can now be found by solving the perturbation equations for v and h in the interior region. Let v and h be expanded as

$$v = v_0 + M_1{}^2 v_1 + \cdots \qquad \text{and} \quad h = h_0 + M_1{}^2 h_1 + \cdots$$

then the lowest-order equations in $M_1{}^2$ resulting from substitution into (15) and (17) are

$$\left.\begin{aligned} &\nabla^2 v_0 = R\, t_i/\beta \\ &\nabla^2 v_1 = \cos\phi\, \partial h_0/\partial\xi - \xi^{-1} \sin\phi\, \partial h_0/\partial\phi \\ \text{and}\quad &\nabla^2 h_0 = \cos\phi\, \partial v_0/\partial\xi - \xi^{-1} \sin\phi\, \partial v_0/\partial\phi \end{aligned}\right\} \quad . \tag{37}$$

The solution to the first of the above equations is given by (33). Substitution of this expression for v_0 into the last equation and solving for h_0 subject to the boundary conditions (26) and (27) along with (36) yields

$$h_0 = \frac{RC}{\beta^2}\ \mathcal{R}e \left\{ [\, W(\beta) + \tfrac{1}{2} i\alpha\beta]^{-1} \cdot [\, ([\, \tfrac{1}{2} - W(\beta)] \ \frac{\sigma_2}{\sigma_1} - \tfrac{1}{2}) \ \frac{\xi}{1+\sigma_2/\sigma_1} + \frac{J_1[(-i\beta)^{1/2}\xi}{2J_1[\,(-i\beta)^{1/2}} \right\} \cos\phi \tag{38}$$

One need not solve for the exact form of v_1. The average value is the only information required in order to find the parameter relationship. Substitution of the expression for h_0 from (38) into (37) and multiplying by $\frac{1}{4}(1-\xi^2)$ and integrating over the cross section from $\xi = 0$ to 1 and from $\phi = 0$ to 2π gives by virtue of Green's theorem and the vanishing of v_1 at $\xi = 0$,

$$(v_1)_{av} = \frac{RC}{\beta^2}\ \mathcal{R}e \left\{ [\, W(\beta) + \tfrac{1}{2}\ i\alpha\beta]^{-1} \cdot [\, -\tfrac{1}{16}\, (\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}) + i\,\frac{\beta}{2} - W(\beta)\ (\frac{\sigma_2}{8(\sigma_1+\sigma_2)}) + i\ \frac{\beta}{2}\,)] \right\} .$$

The parameter relationship to first order in $M_1{}^2$ can be found by setting $(v_0)_{av} + M_1{}^2 (v_1)_{av} = 1$. Doing this, one obtains

$$1 = \frac{RC}{\beta^2} \mathcal{R}e \left\{ [W(\beta) + \frac{1}{2} i a\beta]^{-1} \cdot [W(\beta) - 1 - M_1{}^2 \left\{ \frac{1}{16} \left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \right) - i \frac{\beta}{2} + W(\beta) \left(\frac{\sigma_2}{8(\sigma_1 + \sigma_2)} + i \frac{\beta}{2} \right) \right] \right\} . \qquad (39)$$

For values of β less than unity this expression can be approximated by

$$\beta^2 = [4/(a + \tfrac{1}{4})^2] \; [\tfrac{1}{16} RC(a + \tfrac{1}{4}) \; (1 - \tfrac{1}{8} M_1{}^2 \sigma_2/[\sigma_1 + \sigma_2]) - 1] \; .$$

The last relationship shows that the magnetic field decreases the average speed of flow for a given temperature gradient. In doing so it also raises the critical point at which convection just begins. The amount by which this critical point is raised depends not only on the Hartmann number of the fluid but on the conductivity of the surrounding metal, as well. If a reduced conductivity σ_r is defined as $\sigma_r = \sigma_1 \sigma_2/(\sigma_1 + \sigma_2)$, and this conductivity is used to compute a reduced Hartmann number, $M_r = \mu_1 H_0 a(\sigma_r/\rho_0 \nu)^{1/2}$ then it is seen that this is the quantity on which the velocity change depends. Making this substitution yields

$$\beta^2 = [4/(a + \tfrac{1}{4})^2] \; [\tfrac{1}{16} RC(a + \tfrac{1}{4}) \; (1 - \tfrac{1}{8} M_r{}^2) - 1] \; . \qquad (40)$$

This equation is valid, of course, only for $\beta < 1$. For larger values of β (39) assumes a more complex form.

This paper is concerned mainly with heat-transfer phenomena, and detailed expressions for the remaining electromagnetic quantities, which can be found by direct substitution of v and h into (18) through (20) and (23) and (24), will not be given.[3]

Large Magnetic Field

In the case in which the applied magnetic field is large and $M_1 >> 1$, the entire character of the flow changes. For the small magnetic field, the velocity distribution v was determined mainly by the viscous drag, and the electric current merely assumed the form determined by this distribution. In the case of a large magnetic field, however, the opposite is true. The velocity distribution is determined mainly by the pondermotive force which the electric current produces, and the viscous drag acts merely as the means by which the distribution adjusts to the boundary conditions.

The equations governing the flow are (15) and (17) along with the boundary conditions (26) and (27). Of course, the exterior magnetic field solution is still given by (36). Equation (15) is of a type which can be solved by using a boundary layer approximation[7]. Because M_1^2 is large, it is assumed that a core region exists covering the major part of the fluid in which the magnitude of the $\nabla^2 v$ term is, by comparison, negligibly small. A thin boundary-layer region is, in addition, assumed to exist along the boundary in which the $\nabla^2 v$ term becomes of comparable magnitude to the other terms as the velocity changes rapidly from its value in the core to its value of zero at the boundary.

In boundary-layer analysis the boundary-layer thickness is assumed inversely proportional to a power of the large parameter, and the exponent is adjusted so that the magnitude of the viscous term is the same as that of the other dominant terms. Let

$$\eta = (1-\xi)\, M_1^n .$$

Then (15) and (17) become respectively,

$$M_1^{2n}\, \xi^{-1} \frac{\partial}{\partial \eta}\left(\xi \frac{\partial v}{\partial \eta}\right) + \xi^{-2} \frac{\partial^2 v}{\partial \phi^2} = \frac{R}{\beta} t_i + M_1^2\left(-M_1^n \cos\phi \frac{\partial h}{\partial \eta} - \xi^{-1} \sin\phi \frac{\partial h}{\partial \phi}\right) \tag{41}$$

and

$$M_1^{2n}\,\xi^{-1}\frac{\partial}{\partial\eta}\left(\xi\frac{\partial h}{\partial\eta}\right) = -\,M_1^{n}\cos\phi\,\frac{\partial v}{\partial\eta} - \xi^{-1}\sin\phi\,\frac{\partial v}{\partial\phi}\,, \tag{42}$$

where the terms multiplied by lower powers of M_1 have been neglected. If ϕ is not near $\pm\frac{1}{2}\pi$ then $M_1^n \cos\phi >> \sin\phi$ so that (41) and (42) can be combined to give

$$M_1^{2n}\ \partial^3 v/\partial\eta^3 = M_1^2 \cos^2\phi\ \partial v/\partial\eta$$

where ξ is treated as constant over the boundary layer, and terms of lesser magnitude in M_1 are neglected. Hence the correct choice for n is $n = 1$. This makes the equation become

$$\partial^3 v/\partial\eta^3 = \cos^2\phi\ \partial v/\partial\eta \tag{43}$$

where now $\eta = M_1(1 - \xi)$.

The solution of this equation, subject to boundary condition (27) and the condition that the velocity must approach its core value v_c as $\eta\rightarrow\infty$, is

$$v = v_c[\,1-\exp(-\eta\,|\cos\phi|\,)]\ . \tag{44}$$

The corresponding solution for h is

$$h = h_c - (v_c/M_1)\,(\cos\phi/|\cos\phi|\,)\,\exp(-\eta\,|\cos\phi|\,) \tag{45}$$

where h_c denotes the core value of h.

These solutions are valid only so long as $M_1 \cos\phi >> \sin\phi$. For large M_1 this will cover most of the region except for a small area near $\phi = \pm\frac{1}{2}\pi$. The behavior very near $\phi = \frac{1}{2}\pi$ can be found by assuming $M_1^n \cos\phi << \sin\phi$ in (41) and (42), which will give as the final equation for the velocity distribution

$$\partial^4 v/\partial\eta^4 = \sin^2\phi\ \partial^2 v/\partial\phi^2\ .$$

In this case it is necessary to choose $n = \frac{1}{2}$ to give the required correspondence of orders of magnitude, thus making $\eta = M_1^{1/2}(1-\xi)$. Hence v still has a boundary-layer behavior near $\phi = \frac{1}{2}\pi$, and hence also near $\phi = -\frac{1}{2}\pi$, with the boundary layer thickness proportional to $M_1^{1/2}$ rather than M_1 as it is in the remainder of the channel.

The solution for v near $\phi = \frac{1}{2}\pi$ will be similar to that for the rest of the region but will contain a more involved function than an exponential. This function can be found by substitution of the functional form which Shercliff [8] used for the boundary layer in the end regions of a rectangular channel. By letting

$$v = (\tfrac{1}{2}\pi - \phi)\left[1 - f\left\{\eta(\tfrac{1}{2}\pi - \phi)^{-1/2}\right\}\right]$$

the partial differential equation becomes an ordinary one for $f\left\{\eta(\frac{1}{2}\pi - \phi)^{-1/2}\right\}$ and the desired exponential-type solution can be derived. However, since the region near $\phi = \frac{1}{2}\pi$ where this solution holds is very small, the effect of this region on the average velocity will be small and it will not be necessary to actually solve for the velocity distribution. In fact to derive the core velocity, the boundary-layer velocity will be taken to be that given by (44) as if it held over the entire range of ϕ. The same is true for the induced magnetic field and (45).

In the core region (15) simplifies to

$$0 = R t_i/\beta + M_1^2\, \partial h_c/\partial\psi\,. \tag{46}$$

This equation will determine h_c, and v_c can then be found by applying the boundary conditions on the total solution for h at $\xi = 1$ and by solving (17) in the core region, where it remains unchanged. It would be rather complex to solve for h_c using the general expression for t_i as given by (30), so the equation will be solved only in the region for which $\beta < 1$ which, after all, is the region of greatest interest. Using the simplified form which (30) assumes for $\beta < 1$, (46) becomes

$$0 = -\frac{RC}{2}\, \frac{\alpha + \frac{1}{2} - \frac{1}{2}\xi^2}{1 + \frac{1}{4}\beta^2(\alpha + \frac{1}{4})^2} + M_1^2\, \frac{\partial h_c}{\partial\psi}\,.$$

Solving for h_c , subject to the symmetry condition resulting from the evenness of v and t_i in ξ that h be odd in ψ , gives

$$h_c = \frac{RC}{2M_1^2[1 + \frac{1}{4}\beta^2(\alpha+\frac{1}{4})^2]} \left\{ [(\alpha + \frac{1}{2})\xi - \frac{1}{4}\xi^3] \cos\phi + \frac{1}{12}\xi^3 \cos 3\phi \right\}. \quad (47)$$

The small third harmonic must be kept to give the correct value for the electric current, which depends on the derivatives of h.

Substituting this solution for h_c into (45), and matching the resulting expression for h at $\xi = 1$ with the exterior magnetic field as given by (36) according to the boundary conditions (27) gives, to the boundary-layer degree of approximation,

$$v_c \Big|_{\xi=1} = \frac{RC}{2M_1^2[1+\frac{1}{4}\beta^2(\alpha+\frac{1}{4})^2]} [\alpha(1+\frac{\sigma_1}{\sigma_2}) + \frac{\sigma_1}{2\sigma_2} \cos 2\phi]$$

where it has been assumed that both $\sigma_2 M_1/\sigma_1$ and M_1 are large compared with unity. This assumption requires that the surrounding metal be at least as good an electrical conductor as the fluid. Equation (47) may now be substituted into (17) for the core region and the above boundary condition imposed on the resulting solution for v_c to yield

$$v_c = \frac{RC}{2M_r^2[1+\frac{1}{4}\beta^2(\alpha+\frac{1}{4})^2]} [\alpha + \frac{1}{2} - \frac{1}{2}\xi^2 + (\frac{1}{2} - \xi^2)\frac{\sigma_r}{\sigma_1} + \frac{1}{2}\xi^2 \cos 2\phi] .$$

Because of the magnitude of α, the major contribution to the core velocity distribution is uniform; additional contributions of the other terms include a paraboloid of revolution profile and terms of average zero.

The parameter relationship can be most easily derived by assuming the core velocity distribution to extend throughout the cross section, thereby neglecting the presence of the boundary layer which affects the result only to order M_1^{-1}. Setting the average of the core velocity equal to unity gives

$$\beta^2 = 4(\alpha + \frac{1}{4})^{-2}[\frac{1}{2}RCM_r^{-2}(\alpha + \frac{1}{4}) - 1]. \quad (48)$$

This relationship shows that for values of R above the critical value of $2M_r^2/[C(\alpha + \frac{1}{4})]$ convection will take place, and that the average convective velocity will be proportional to the square root of the increase in the applied temperature gradient.

The parameter relationship (48) is valid only for $\beta < 1$. Since the magnetic field stabilizes the fluid against transition to turbulent flow, laminar flow can be expected to hold for values of β larger than unity. As mentioned in section 2, it is permissible to extend our solution to these larger values of β in this case because the velocity distribution is nearly uniform. To determine the correct parameter relationship for the larger values of β, replace t_i in (46) by its average value over the cross section as calculated from (30). Then the corresponding expressions for h_c and v_c may be easily derived by the methods of this section, and setting $v_c = 1$ will give the desired expression, which is

$$1 = RC\,\beta^{-1}\,M_r^{-2}[W_i + \tfrac{1}{2}\alpha\beta]/[W_r^2 + (W_i + \tfrac{1}{2}\alpha\beta)^2] . \qquad (49)$$

Although explicit expressions will not be given for the remaining electromagnetic quantities,[3] it will be of interest to describe briefly their general behavior, since they affect the fluid velocity distribution so strongly. The induced magnetic field does not exhibit rapid changes in the boundary layer, whereas the ϕ component of electric current, radial component of electric field, and electric charge density do. In the core region the electric current is nearly uniform and in the negative y direction, whereas the electric field is nearly uniform and in the positive y direction. In fact the magnitude of the electric current is approximately that which the electric field $\underset{\sim}{E}$ would produce, but it is oppositely directed. This arises from the fact that the induced electric field $\mu\, \underset{\sim}{V} \times \underset{\sim}{H}$ is in the opposite direction to the electric field $\underset{\sim}{E}$ and about twice as large as the electric field itself.

Shercliff [9] has investigated forced flow through a channel of circular cross section in a transverse magnetic field for thin conducting walls, and for large Hartmann numbers he found a uniform electric current in the core region. The electric current in the present problem was found to depart somewhat from uniformity, however, largely because the channel walls were not thin and to a lesser extent because the flow was driven by natural convection.

4 Final Solution

The mathematical details of solving the equations presented in section 1 have now been completed. In section 2 the solution was found for the temperature distribution and total heat transfer in terms of the parameter β. In section 3 the relationship between β and the given parameters was determined and solutions were indicated for the remaining variables. It is now necessary only to express the total heat transfer in terms of the given parameters using the relationship found in section 3 to have in explicit form the solution to the problem. A numerical example can then be worked out to determine the typical magnitudes of the various quantities.

The parametric relationships for $\beta < 1$ are given by (40) for $M_r << 1$ and (48) for $M_r >> 1$. Comparison between these relationships shows that the applied magnetic field increases the critical temperature gradient at which convection first takes place. In addition, it causes the velocity of the fluid to increase more slowly as the temperature gradient is increased. These effects will lessen the amount of heat transferred by convection as the magnetic field is increased.

The expression for the total heat transfer in each case may be found by placing the appropriate relationship for β, depending on the magnitudes of M_r and β, into (32). The effect of the magnetic field on the part of the heat transferred by convection is shown in Fig. 4, where the convective heat transfer is plotted as a function of $R^{1/2}$. This figure, of course, represents the same quantity as Fig. 3, but it is shown here as a function of the given parameter $R^{1/2}$ rather than the parameter β. Only the cases of zero applied magnetic field are shown because the curves for small applied magnetic field lie too close to those for zero magnetic field. The non-magnetic-field curves are drawn only to the point where turbulence may begin for liquid sodium. The large-magnetic-field curves are drawn for $M_r = 20$. Asymptotes representing the maximum possible heat transfer are also shown for each value of α plotted.

Figure 4 shows the pronounced effect the magnetic field has on heat transferred by convection. Generally speaking, as the magnetic field is increased, the curve for a given value of α is stretched along the $R^{1/2}$ coordinate and shifted to the right. Thus by application of a large enough magnetic field, the convective heat transfer in a given situation can be completely stopped.

In Fig. 5 the ratio R_c/R_0 is plotted as a function of the reduced Hartmann number, where R_c represents the critical value of the Rayleigh number at which convection just begins and R_0 is the value of R_c when $M_r = 0$. Equation (35) shows R_0 to be equal to $16/[C(\alpha+\frac{1}{4})]$. Solid portions of the curve for $M_r < 1$ and $M_r > 10$ represent values calculated from (40) and (48) repsectively, whereas the dotted portion merely suggests a probable way in which the solid portions connect.

It may now be seen what size magnetic fields are required in a practical example to produce these effects. Consider the case in which the channel is filled with liquid sodium at an average temperature of 300°C. The properties of the sodium at this temperature are $\rho_0 = 8.8\times10^2$ kg/m^3,

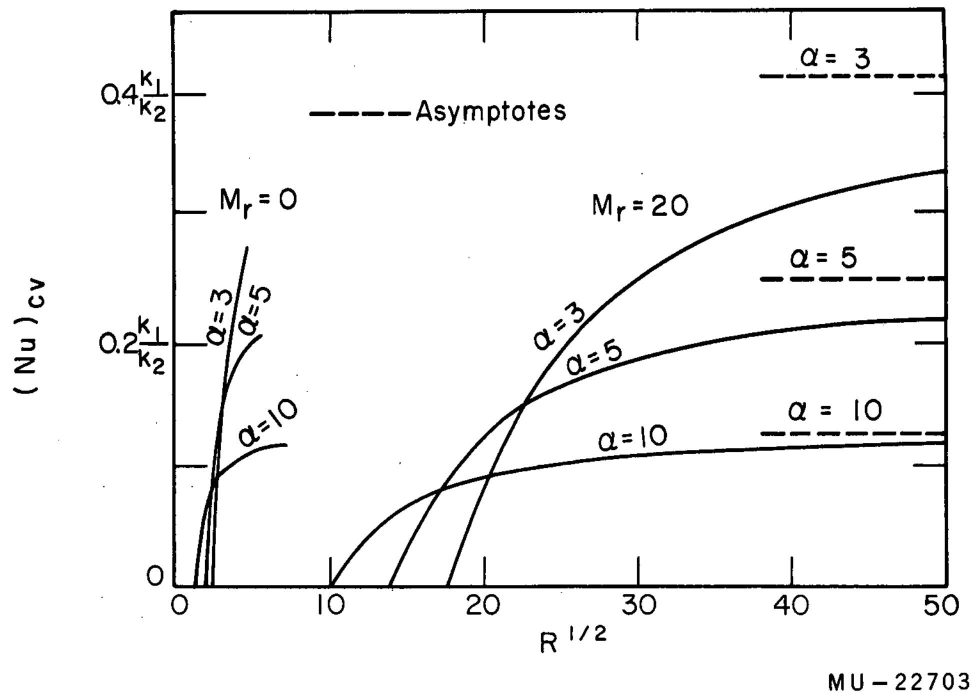
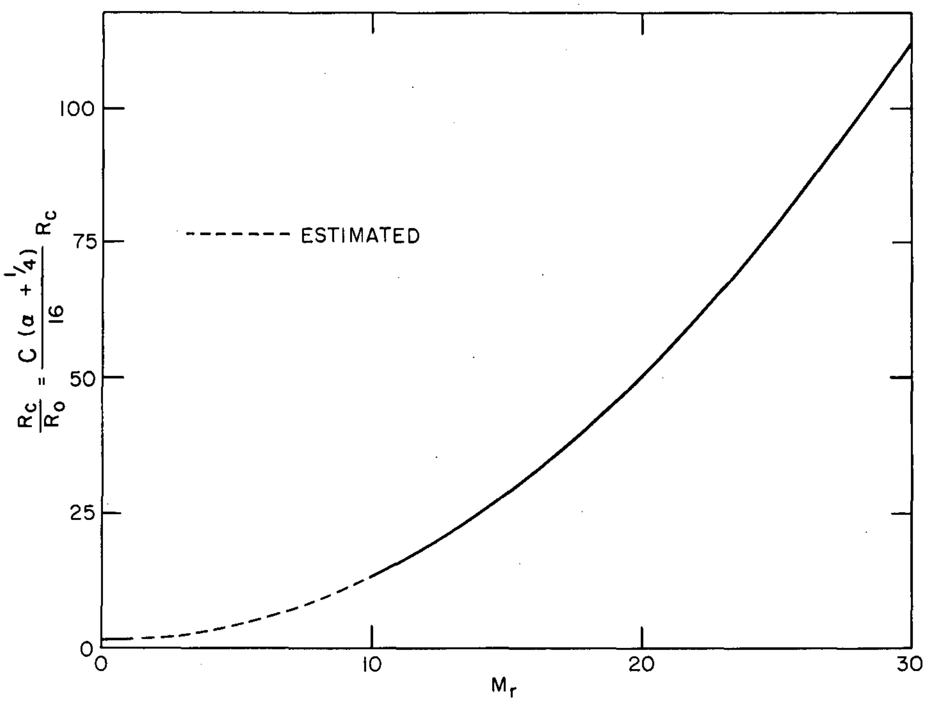


Fig. 4. $(Nu)_{cv}$, the convective Nusselt number, as a function of $R^{1/2}$, the square root of the Rayleigh Number, for certain values of α and of M_r, the reduced Hartmann Number.



$\frac{R_c}{R_o}$, THE INCREASE IN CRITICAL RAYLEIGH NUMBER, vs. M_r, THE REDUCED HARTMANN NUMBER

MU-22702

Fig. 5. R_c/R_0, the increase in critical Rayleigh Number, as a function of M_r, the reduced Hartmann Number.

$\nu = 3.9\times10^{-7}$ m^2/sec, $\lambda = 2.7\times10^{-4}$ $(^0C)^{-1}$, $\kappa_1 = 6.6\times10^{-5}$ m^2/sec, $\sigma_1 = 6.0\times10^6$ $(ohm-m)^{-1}$, and $\mu_1=\mu_0$, the magnetic permeability of free space. Let the solid block be such that its thermal and electrical conductivity are the same as those of the liquid sodium so that $k_1 = k_2$ and $\sigma_1 = \sigma_2$. It should be noted that the block should not be made of a ferromagnetic substance, since this would make the achievement of a large magnetic field in the sodium very difficult.

Reasonable dimensions for the channel and the block are $a = 0.01m$ and $A = 0.25m$. This makes $\alpha = 3.0$. Applying a magnetic field of $\mu_0 H_0 = 0.021$ weber/m^2 (210 gauss) results in a reduced Hartmann number of $M_r = 20$. Thus this example is described by the curve for $\alpha = 3$ and $M_r = 20$, in Fig. 4. The effect of the magnetic field can be seen by comparing this curve with the one for $\alpha = 3$ and $M_r = 0$. Figure 4 shows that the critical value of $R^{1/2}$ at which convection just begins is raised by the magnetic field from 2.5 to 17.7. The corresponding increase in critical temperature gradient is from $13^0C/m$ to $630^0C/m$. This illustrates how extremely effective the magnetic field is in inhibiting convection under these conditions.

This study has shown that for the given example of a narrow toroidal channel of conducting liquid inside a metallic block, even a moderate applied magnetic field has a large effect in stabilizing the liquid against convection. Reference to (32) shows that the heat transferred by convection can affect the total heat transfer by at most 30 or 40 percent. One might increase the number or size of convective channels in the block to provide greater control by the magnetic field. Of course, the more fluid there is inside the block, the more difficult it is to predict its exact mode of instability, and having a large body of fluid would not necessarily increase the amount of convective heat transfer, because of the cellular flow patterns which would develop.

In the present example no allowance was made for other types of instability than that of a single convective cell. That is, it was assumed that in the region of interest, the fluid would flow only in the prescribed pattern when it was gravitationally unstable. This appears to be quite a resonable assumption, at least at the lower temperatures, and can most easily be verified by experiment.

Acknowledgment

The author wishes to express his gratitude to Professor George F. Carrier for originally suggesting this problem and for providing invaluable guidance during its solution.

Footnotes

[1]This paper is condensed from the author's doctoral thesis, "Effects of a Magnetic Field on Natural Convection in a Toroidal Channel," March 1959, submitted to the Division of Engineering and Applied Physics , Harvard University, Cambridge, Mass. The research was supported in part by the Office of Naval Research.

[2]Verification of the last assumption may be found in the author's thesis.

[3]Such expressions for both small and large Hartmann numbers can be found in the author's thesis.

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