Title
Subalgebras of Golod-Shafarevich algebras

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Publication Date
2006

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Subalgebras of Golod-Shafarevich Algebras

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2006

The dissertation of Thomas B. Voden is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2006

To Griselda

Verily the lust for comfort murders the passion of the soul, and then walks grinning in the funeral.

—Kahlil Gibran

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ACKNOWLEDGEMENTS

There are no words to describe the gratitude I feel towards Professor Efim Zelmanov. His teaching, guidance, words of inspiration, and endless patience have been tremendous motivating factors in my life, both inside and outside of mathematics. I am privileged for having had the chance to get to know him.

I am indebted to Professor Adrian Wadsworth for all of his help throughout my years at UCSD. From teaching algebra, to helping with my own teaching, to providing guidance on a variety of topics and carefully reading this manuscript, his help has played an important role in virtually every part of my graduate education. I also wish to thank Professor Lance Small for his numerous entertaining conversations and helpful suggestions.

My wife, mother, father, brother, and sister form the best support system imaginable. Without their love, understanding and patience, this process would have been impossible.

A large portion of this work was supported by the NSF. I am very grateful for this help.

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ABSTRACT OF THE DISSERTATION

Subalgebras of Golod-Shafarevich Algebras

by

Thomas B. Voden
Doctor of Philosophy in Mathematics

University of California San Diego, 2006

Professor Efim Zelmanov, Chair

In 1964, Golod and Shafarevich found a sufficient condition for an algebra presented by generators and relators to be infinite dimensional. This condition gives rise to an analogous condition for a pro-p group to be infinite. Groups and algebras that satisfy this condition are called GS groups and algebras.

In 1983, A. Lubotzky exhibited a class of GS groups with the property that all of their finite index subgroups are also GS. The underlying topological structure was essential in Lubotzky's examples. This dissertation is an account of our search for algebraic analogs to these examples and our exploration of conditions under which subalgebras of GS algebras are themselves GS.

In Chapter 3 it is proved that finite codimensional subalgebras of finitely presented algebras are finitely presented. However, subalgebras of finite codimension in graded GS algebras are not necessarily GS (Chapter 4).

In Chapter 5 we prove that infinitely many Veronese powers of a graded algebra presented by m generators and r relators are GS if $r < \frac{1}{4}\left(\frac{m}{2}-1\right)^2$. For quadratic algebras, the bound is improved to $r < \frac{4}{25}m^2$. We prove that for a generic quadratic algebra A presented by m generators and r relators, all Veronese powers of A are GS if $r \leq \frac{4}{25}m^2$ and all but finitely many Veronese powers of A are not GS if $r > \frac{4}{25}m^2$ (Chapter 6).

1 Introduction and Definitions

We define an *algebra* as a vector space over a fixed ground field, endowed with a bilinear multiplication which gives it the structure of an associative ring with unity. Throughout this work we use K to denote our ground field. We restrict our attention to finitely generated algebras.

An algebra A is called *graded* if

$$A = \bigoplus_{i=0}^{\infty} A_i,$$

where each A_i is a finite dimensional K-vector space, such that $A_i A_j \subseteq A_{i+j}$. We further assume that $A_0 = K$. We will have occasion to use the notation

$$A_+ = \bigoplus_{i=1}^{\infty} A_i.$$

We will also use the Hilbert Series of a graded algebra A, which is defined as

$$H_A(t) = \sum_{i=0}^{\infty} \dim_K(A_i) t^i.$$

For any graded subset S of a graded algebra A, let $S_i = S \cap A_i$.

Much of the work presented here is motivated by a search for similarities and differences between analogous theories in the category of groups, or more precisely, pro-p groups, and the category of algebras. We begin the discussion with some preliminary definitions and results.

1.1 Presentation by Generators and Relators

The free group F_X on a set X can be thought of as the set of all reduced words in X and $X^{-1} = \{x^{-1} \mid x \in X\}$, where X^{-1} is a set in one-to-one correspondence with X. Multiplication in F_X is simply by juxtaposition of words. A word is said to be reduced if every instance of xx^{-1} or $x^{-1}x$ is deleted from the word.

Free groups are characterized by the property that any function $X \to G$, where G is an arbitrary group, uniquely extends to a group homomorphism $F_X \to G$.

Definition 1. A group G is said to have presentation $\langle X|R\rangle$ if $G \cong F_X/N$ where N is the normal subgroup of F_X generated by the subset $R \subseteq F_X$.

Similarly, the free K-algebra on a set X, denoted $K\langle X\rangle$, is the algebra of all K-linear combinations of words in X. Again we have that any function $X \to A$, where A is an arbitrary algebra, uniquely extends to an algebra homomorphism $K\langle X\rangle \to A$.

Definition 2. Let X be a set and let R be a subset of $K\langle X\rangle$. $\langle X|R\rangle$ is a presentation of an algebra A if $A \cong K\langle X\rangle/I$, where I is the ideal of $K\langle X\rangle$ generated by R.

Note that the empty word in X is allowed and it plays the role of the multiplicative identity.

Suppose that X is a finite set and that each $x \in X$ is assigned a positive integer degree denoted $\deg(x)$ (if no degrees are specified, we will assume that all x have degree one). $K\langle X\rangle$ is graded by total degree. An element $f \in K\langle X\rangle_i$ is said to be homogeneous of degree i. Let R be a subset of $K\langle X\rangle$ consisting of homogeneous elements of degree at least two. By saying that $\langle X|R\rangle$ is a *graded presentation* of A (we will often omit the word *graded*) we mean that there exists an epimorphism

$$\varphi : K\langle X\rangle \longrightarrow A$$

of graded algebras such that $\ker(\varphi)$ is generated by R as an ideal of $K\langle X\rangle$. The graded vector spaces generated by X and R are respectively called the generating space and relation space for A.

It is well known that the minimal number of generators of a graded algebra A is $\dim_K(A_+/A_+^2)$ (proposition 1.5.2 of [20]). We will only consider minimal presentations (*i.e.*, presentations $\langle X|R\rangle$ with $|X| = \dim_K(A_+/A_+^2)$).

Note that we have not lost any generality by assuming that R consists of elements of degree at least two since a relator of degree one is actually an elimination of one of the generators. Also, in saying that $\langle X|R\rangle$ is a graded presentation it follows that R consists of homogeneous elements.

An algebra is said to be *finitely presented* if it admits a presentation with finitely many generators and finitely many relators (*i.e.,* the generating and relation spaces are finite dimensional). We will refer to a graded algebra A which is generated by A_1 as *one-generated*. This means that the natural map

$$\pi : K\langle\mathcal{E}\rangle \longrightarrow A$$

is surjective, where $\mathcal{E}$ denotes any basis of the vector space A_1. If in addition the kernel of π is generated, as a two-sided ideal of $K\langle\mathcal{E}\rangle$, by its subspace $\ker(\pi) \cap K\langle\mathcal{E}\rangle_2$, then A is called *quadratic*. See [19, 20] for detailed discussions of quadratic algebras.

If B is a subalgebra of A, then the codimension of B in A is defined to be $|A : B| = \dim_K(A/B)$. J. Lewin [12] proved that a subalgebra of finite codimension of a finitely generated free associative algebra is finitely presented. In Chapter 3 we show how Lewin's result implies that a subalgebra of finite codimension of a finitely presented algebra is finitely presented.

1.2 The Golod-Shafarevich Theorem

The notion of a Golod-Shafarevich (GS) algebra is central to this work. We begin the discussion in the context of profinite algebras, that is, algebras which arise as inverse limits of families of finite dimensional algebras.

Throughout this section let $X = \{x_1, \ldots, x_m\}$ be a fixed set with $m \geq 2$. Since there are situations that require it, we will allow for the possibility that the elements of X are assigned positive integer degrees other than one. If no degrees are mentioned, assume that all elements of X have degree one. Let g_i denote the number of elements of degree i in X and set $H_X(t) = \sum_i g_i t^i$. Note that $H_X(t)$ is a finite sum, and if all the elements of X have degree one, then $H_X(t) = mt$.

Let $I = K\langle X\rangle_+$, that is, $I = \bigoplus_{i\geq 1} K\langle X\rangle_i$. Then the ideals

$$I^n = \bigoplus_{i\geq n} K\langle X\rangle_i,\ n = 1, 2, \ldots$$

are a neighborhood base at 0 for a topology on $K\langle X\rangle$. The completion of $K\langle X\rangle$ with respect to this topology is $K\langle\langle X\rangle\rangle$, the algebra of infinite series over F in m noncommuting variables. We can also think of $K\langle\langle X\rangle\rangle$ as the inverse limit of the following family of finite dimensional algebras

$$\{K\langle X\rangle/I^n \,|\, n \geq 1\}.$$

We use $\widehat{I^n}$ to denote the closure of I^n in $K\langle\langle X\rangle\rangle$.

Since $\bigcap_{n\geq 1} \widehat{I^n} = (0)$, to each nonzero $f \in K\langle\langle X\rangle\rangle$ corresponds a unique integer d such that $f \in \widehat{I^d} \setminus \widehat{I^{d+1}}$. We define the *degree of* f, denoted $\deg(f)$, to be d. Observe that $\deg(f)$ is the minimal degree of all monomials involved in f.

Let R be a (linearly independent) subset of $\widehat{I^2}$ containing r_i elements of degree i, and define the Hilbert series

$$H_R(t) = \sum_{i\geq 2} r_i t^i.$$

Definition 3. $\langle X|R\rangle$ is a presentation of a profinite algebra A if $A \cong K\langle\langle X\rangle\rangle/\mathcal{I}(R)$, where $\mathcal{I}(R)$ denotes the closed ideal of $K\langle\langle X\rangle\rangle$ generated by R.

Observe that by setting $A_n = (\widehat{I^n} + \mathcal{I}(R))/\mathcal{I}(R)$ for $n = 1, 2, \ldots$, we endow a profinite algebra $A = \langle X|R\rangle$ with a filtration $A = A_0 \supset A_1 \supset A_2 \supset \cdots$, such that $A_i A_j \subseteq A_{i+j}$. The graded algebra associated with A is defined to be

$$gr(A) = K \cdot 1 + \bigoplus_{i\geq 1} A_i/A_{i+1},$$

and its Hilbert series is $H_{gr(A)}(t) = 1 + \sum_{i\geq 1} \dim_K(A_i/A_{i+1})t^i$.

We formally interpret an inequality of power series with real coefficients by

$$\sum a_i t^i \geq \sum b_i t^i \iff a_i \geq b_i \text{ for all } i.$$

In [6] Golod and Shafarevich proved the following theorem.

Theorem 4.

$$\frac{H_{gr(A)}(t)}{1-t}(1 - H_X(t) + H_R(t)) \geq \frac{1}{1-t} \tag{1.1}$$

Definition 5. A presentation $\langle X|R\rangle$ is said to satisfy the *Golod-Shafarevich (GS) condition* if there exists a $t_0 \in (0,1)$ for which $H_R(t_0)$ converges and

$$1 - H_X(t_0) + H_R(t_0) < 0. \tag{1.2}$$

A profinite algebra is called *Golod-Shafarevich (GS)* if it admits a presentation which satisfies the GS condition.

A GS algebra is necessarily infinite dimensional. Indeed, suppose that there is a $t_0 \in (0,1)$ at which $H_R(t)$ converges and $1 - H_X(t_0) + H_R(t_0) < 0$. Then $H_{gr(A)}(t)$ must diverge at t_0 since otherwise the formal inequality (1.1) implies the numerical inequality

$$\frac{H_{gr(A)}(t_0)}{1-t_0}(1 - H_X(t_0) + H_R(t_0)) \geq \frac{1}{1-t_0}.$$

This is a contradiction since the left hand side is negative and the right hand side is positive.

For additional discussion of the Golod-Shafarevich theorem, see [9, 21, 23, 25, 27].

1.3 Graded Golod-Shafarevich Theorem

In the previous section we presented the Golod-Shafarevich theorem in the context of profinite algebras because it has nice applications to the analogous notions for pro-p groups. These applications will be discussed in more detail in the following section. Of course, we could have had this discussion for discrete algebras (*i.e.,* quotients of $K\langle X\rangle$ rather than $K\langle\langle X\rangle\rangle$) and all of the results mentioned above work in the same way. But rather than repeating everything already mentioned, we will turn our attention to the important special case of graded GS algebras. Virtually all of the work presented in this thesis is done in the context of graded algebras.

Again, fix a set $X = \{x_1, \ldots, x_m\}$ with $m \geq 2$. Let A be a graded algebra with presentation $\langle X|R\rangle$. Still assume that all elements of R are homogeneous of degree at least two. Let $H_X(t)$, $H_R(t)$, and $H_A(t)$ be as before. In [6] Golod and Shafarevich also proved the following stronger version of the Theorem 4.

Theorem 6.

$$H_A(t)(1 - H_X(t) + H_R(t)) \geq 1 \tag{1.3}$$

This inequality is indeed stronger than (1.1) since a formal inequality

$$\tfrac{1}{1-t}\sum_{i=0}^{\infty} a_i t^i \geq \tfrac{1}{1-t}\sum_{i=0}^{\infty} b_i t^i$$

is equivalent to the numerical inequalities

$$\sum_{i=0}^{n} a_i \geq \sum_{i=0}^{n} b_i \ \forall\ n.$$

With the definition of a GS presentation and algebra as in the last section, we see that GS algebras are infinite dimensional in this context as well.

The following well-known special case will be used heavily throughout this dissertation.

Proposition 7. *A graded presentation consisting of m generators of degree one and r relators of degree at least two satisfies the GS condition if*

$$r < m^2/4. \tag{1.4}$$

Proof. If $r < m^2/4$, then $1 - mt_0 + rt_0^2 < 0$ for some $t_0 \in (0, 1)$, and

$$1 - mt_0 + H_R(t_0) < 1 - mt_0 + rt_0^2.$$

In fact, in Lemma 33 it is shown that one of the two distinct roots of $1 - mt + rt^2$ is in the interval $(0, 1)$. □

If A is a quadratic algebra presented by m generators and r relators, then

$$1 - mt + H_R(t) = 1 - mt + rt^2,$$

which is negative for some $t \in (0,1)$ if and only if inequality (1.4) is satisfied. Also, noting that $m = \dim(A_1)$ and $r = m^2 - \dim(A_2)$, we see that inequality (1.4) is equivalent to

$$\frac{3}{4}(\dim(A_1))^2 - \dim(A_2) < 0. \tag{1.5}$$

This gives us a definition of GS algebras in the context of one-two algebras which does not make reference to a presentation.

The importance of the Golod-Shafarevich theorem became immediately evident when Golod used it to give the first counterexample to the General Burnside problem [5] and Shafarevich used it to construct the first example of an infinite tower of class fields [6].

Once a suitable definition of degree is given, these notions have meaning in the category of groups. Pro-p groups provide a natural context in which to define degree.

1.4 Pro-p Groups

For any prime number p, a *pro-p group* is the inverse limit of a family of finite p-groups. Pro-p groups are compact, Hausdorff topological groups whose open subgroups form a neighborhood base at the identity and in which every open subgroup has index a power of p.

Definition 8. If G is a group and p is a prime number, then the *pro-p completion of G*, denoted $\widehat{G}_p$, is the inverse limit of the following family of p-groups

$$\{G/N \mid N \lhd G;\ \text{the index of } N \text{ in } G \text{ is a power of } p\}.$$

The free pro-p group on m generators is the pro-p completion of the free group on m generators.

Definition 9. Let $X = \{x_1, \ldots, x_m\}$ be a set, $\widehat{F}_p = \widehat{F}_p(m)$ the free pro-p group on X, and R a subset of $\widehat{F}_p$. $\langle X|R\rangle$ is a presentation of a pro-p group G if $G \cong \widehat{F}_p/N$ where N is the normal closed subgroup of $\widehat{F}_p$ generated by R.

Let $\widehat{F}_p$ denote the free pro-p group on m generators and, for the remainder of this section only, let K be the field of order p. Consider the group algebra $K\widehat{F}_p$ and let ω be its augmentation ideal. The Zassenhaus filtration

$$\widehat{F}_p = (\widehat{F}_p)_1 > (\widehat{F}_p)_2 > \cdots$$

is defined by

$$(\widehat{F}_p)_i = \{g \in \widehat{F}_p \,|\, 1 - g \in \omega^i\}.$$

In section 4.2 of [10] it is shown that

$$\bigcap_{i\geq 1} (\omega)_i = (0),$$

and hence

$$\bigcap_{i\geq 1} (\widehat{F}_p)_i = (1).$$

So, as in the case of profinite algebras, a nontrivial element $g \in \widehat{F}_p$ is said to have *degree* i if $g \in (\widehat{F}_p)_i \setminus (\widehat{F}_p)_{i+1}$.

Fix a set $X = \{x_1, \ldots, x_m\}$ and consider $K\langle\langle X\rangle\rangle$, where K is still the field with p elements. Let the ideals $\widehat{I^n}$ be as in section 1.2. $1+\widehat{I}$ is a subgroup of the group of invertible elements of $K\langle\langle X\rangle\rangle$. $1+\widehat{I}$ is indeed a pro-p group since $1+\widehat{I}$, along with the quotient maps $1+\widehat{I} \longrightarrow (1+\widehat{I})/(1+\widehat{I^n})$, is the inverse limit of the inverse system of finite p-groups $(1+\widehat{I})/(1+\widehat{I^n})$ and surjections $(1+\widehat{I})/(1+\widehat{I^n}) \longrightarrow (1+\widehat{I})/(1+\widehat{I^m})$ for $n \geq m$.

The closed subgroup of $1+\widehat{I}$ generated by $1-x_1, \ldots, 1-x_m$ is free since any nontrivial relation among the elements $1-x_1, \ldots, 1-x_m$ in $1+\widehat{I}$ yields a nontrivial relation among the elements $x_1, \ldots, x_m$ in $K\langle\langle X\rangle\rangle$. Hence, this subgroup is isomorphic to $\widehat{F}_p = \widehat{F}_p(m)$, the free pro-$p$ group on m elements. So an element $f \in \widehat{F}_p$ can be viewed as an element of $1+\widehat{I} \subseteq K\langle\langle X\rangle\rangle$. Also observe that

$$f(1-x_1, \ldots, 1-x_m) = 1 + f'(x_1, \ldots, x_m), \tag{1.6}$$

where $f' \in \widehat{I}$. The degree of f in $\widehat{F}_p$ equals the degree of f' in $K\langle\langle X\rangle\rangle$.

Let G be a pro-p group presented by m generators (of degree one) and a defining set of relators R. Define the set $R' = \{f' \,|\, f \in R\} \subset K\langle\langle X\rangle\rangle$ using (1.6). Define the algebra A presented by $\langle x_1, \ldots, x_m \,|\, R' \rangle$ in the category of profinite algebras. For any open normal subgroup H of G, define $\omega(H)$ to be the ideal of KG generated by $\{1 - h \,|\, h \in H\}$. These ideals define a topology on KG and as it turns out, the completion of KG with respect to this topology is isomorphic to A (see section 2.5 of appendix 1 in [7]). Therefore,

$$H_A(t) = 1 + \sum_{i \geq 1} \dim(\omega(G)^i / \omega(G)^{i+1}) t^i.$$

With $H_R(t)$ as in section 1.2, Theorem 4 implies that

$$\frac{H_A(t)}{1-t}(1 - mt + H_R(t)) \geq \frac{1}{1-t}.$$

See [25] and [27] for more information.

Defining a GS group as in Definition 5, this inequality implies that a GS pro-p group is infinite. We also have an analogous version of inequality 1.4 for GS groups.

Another sense in which GS groups are known to be large is given by a theorem of Zelmanov which states that any GS pro-p group contains a non-abelian free pro-p group [25].

2 Connections and Summary of Results

A. Lubotzky made the observation that if $\langle X|R\rangle$ is a presentation of a finitely generated discrete group G, then $\langle X|R\rangle$ is a presentation in the category of pro-p groups for the pro-p completion of G [13]. It is known that the fundamental group of a 3-manifold is balanced (*i.e.,* that it admits a finite presentation with equal numbers of generators and relators), and hence GS if its rank is at least five (see [13] for more details). Let X be a 3-manifold and let G be the fundamental group of X. Then any finite index subgroup H of G is the fundamental group of a finite sheeted cover of X and is therefore balanced. So if the index of H in G is large enough, then H is GS, and by the same argument, all finite index subgroups of H are also GS. Groups with this property are called *hereditary GS* groups. Lubotzky used these ideas to prove that if the fundamental group of a compact hyperbolic 3-manifold is arithmetic, then it does not have the congruence subgroup property. This was an important open problem known as Serre's conjecture [22]. Later Lubotzky and Sarnak formulated a conjecture of an even stronger result. There are some definitions required before the conjecture can be stated.

Definition 10. For $\varepsilon > 0$, a finite graph Γ is called an *ε-expander* if whenever the set of vertices is written as a disjoint union $\mathrm{vert}(\Gamma) = A\dot{\cup}B$, the number of edges between A and B is at least $\varepsilon \cdot \min\{|A|, |B|\}$.

Definition 11. Let G be a group generated by a set X with $X = X^{-1}$. The *Cayley Graph* of G (with respect to X) is the graph whose vertices are the elements of G. Two vertices g and h are connected by an edge if $g = xh$ for some $x \in X$. We use

Cay(G, X) to denote this graph.

Definition 12. A group G generated by a finite set $X = X^{-1}$ has property (τ) if there exists an $\varepsilon > 0$ such that

$$\{\mathrm{Cay}(G/N, X_N) \mid N \lhd G, |G : H| < \infty\}$$

is a family of ε-expanders, where X_N denotes the image of X in G/N.

Some examples of groups with property (τ) are $SL_n(\mathbb{Z})$ for $n \geq 3$ [8]. For more information on this and related topics, the reader is referred to [14, 17].

Lubotzky-Sarnak Conjecture [15]. *The fundamental group of a hyperbolic 3-manifold does not have property (τ).*

One hope is that a solution to this conjecture would help lead to a solution of the Virtual Haken conjecture, which predicts that every irreducible compact hyperbolic 3-manifold has a finite sheeted cover which is Haken. Some work of M. Lackenby suggests the use of techniques involving the GS condition and property (τ) for these two conjectures [11]. In this direction Zelmanov asked whether it is true that a GS group does not have property (τ) [26]. Since property (τ) is a condition inherited by finite index subgroups, an affirmative answer to this would imply the Lubotzky-Sarnak conjecture. However, in a recent paper, M. Ershov gives an explicit construction answering this question in the negative [4]. It is interesting to note that Lubotzky and Zelmanov answered the analogous question in the context of algebras in the affirmative [16]. So the interplay between these notions is proving to be quite subtle.

Lubotzky's examples of hereditary GS groups employ topological arguments with no known algebraic analogs, and not all GS groups are hereditary GS. The research in this dissertation was motivated by the natural question of what distinguishes hereditary GS groups and algebras from their non-hereditary counterparts.

It was found that not all GS algebras have the property that all of their finite codimensional subalgebras are also GS (chapter 4).

A natural analog to a subgroup of finite index in the context of graded algebras is the Veronese Power. If A is a graded algebra, define the n^{th} *Veronese power of*

A to be

$$A^{(n)} = \bigoplus_{i=0}^{\infty} A_{ni}.$$

Clearly, for any n, A is finitely generated as a module over $A^{(n)}$. Indeed, if B is any basis of $\bigoplus_{i=0}^{n-1} A_i$ (which is a finite dimensional space), then an arbitrary element of A is expressible as a linear combination of elements of B with coefficients in $A^{(n)}$. In chapters 5 and 6 we demonstrate some fairly general settings in which infinitely many (and sometimes all but finitely many) Veronese powers of a graded GS algebra are themselves GS.

2.1 Summary of Results

For convenience, we now summarize all the results obtained in this thesis.

In Chapter 3 we generalize a theorem of J. Lewin by showing that a finite codimensional subalgebra of a finitely presented algebra is finitely presented.

Let I be the ideal of the finitely generated free algebra $K\langle X\rangle$ defined in Section 1.2. Chapter 4 is devoted to showing that the GS condition is not generally inherited by subalgebras of finite codimension.

Turning to Veronese Powers of finitely presented graded algebras, in Chapter 5 we prove the following theorem.

Theorem 13. *Let A be a graded algebra with a presentation consisting of m generators and r relators. If $r < \frac{1}{4}\left(\frac{m}{2} - 1\right)^2$, then $A^{(n)}$ is GS for infinitely many n.*

In the case that A is quadratic, we have the following stronger version of this theorem.

Theorem 14. *Let A be a quadratic algebra with a presentation consisting of m generators and r relators. If $r < \frac{4}{25}m^2$, then $A^{(n)}$ is GS for infinitely many n.*

In Chapter 6 we look at generic algebras to show that the bound in the hypothesis of Theorem 14 is the best possible. Definitions and background information on generic algebras are given in Chapter 6. There we prove

Theorem 15. *Let A be a generic quadratic algebra with m generators and r relators and assume that $r < m^2/4$.*

1. *If $r \leq \frac{4}{25}m^2$, then $A^{(n)}$ is GS for all n.*
2. *If $r > \frac{4}{25}m^2$, then $A^{(n)}$ is not GS for all but finitely many n.*

This theorem provides examples of hereditary GS algebras. The results presented here are in preparation to be submitted for publication [24].

2.2 Further Questions

It would be interesting to explore whether Theorem 15 can be used to construct non-topological examples of hereditary GS pro-p groups. Such examples could shed some light on what distinguishes hereditary GS groups from non-hereditary GS groups.

It is likely that the conclusions of Theorem 15 hold for general finitely presented graded algebras (*i.e.,* that Theorems 13 and 14 can be strengthened to a result like Theorem 15). The techniques used in this thesis did not yield these stronger versions of Theorems 13 and 14, but it would certainly be interesting to find out if they hold.

3 Subalgebras of Finite Codimension

In this chapter we show that a subalgebra of finite codimension of a finitely presented algebra is finitely presented. This will be obtained as a slight extension of the following theorem of J. Lewin [12].

Theorem 16. *If A is a subalgebra of finite codimension in $K\langle X\rangle$, then A is finitely presented.*

It is interesting to note that the techniques Lewin used to prove this theorem are similar to those used by Schreier in his proof of the fact that any subgroup of a free group is itself free (which was a stronger version of a theorem of Nielsen). In this particular instance, the situation in the context of algebras is more delicate than the analogous situation for groups since of course, not all subalgebras of free algebras are free. See pages 4–13 of [18] for further discussion of the Nielsen-Schreier theorem.

Lemma 17. *Let A be a finite codimensional subalgebra of $K\langle X\rangle$ and let I be an ideal of $K\langle X\rangle$ which is contained in A. If I is finitely generated as an ideal of $K\langle X\rangle$, then I is finitely generated as an ideal of A.*

Proof. Let $R = \{r_1, \ldots, r_\ell\}$ be a set of generators for I as an ideal of $K\langle X\rangle$. Let $\mathcal{B} = \{b_1, \ldots, b_d\}$ be a basis for $K\langle X\rangle$ mod A. Define $S = R \cup \mathcal{B}R \cup R\mathcal{B} \cup \mathcal{B}R\mathcal{B}$. We will show that S generates I as an ideal of A. It suffices to show that fr_jg is in the ideal of A generated by S for all $f, g \in K\langle X\rangle$ and all $j = 1, \ldots, \ell$.

Let $f, g \in K\langle X\rangle$ and write $f = f' + \sum \alpha_i b_i$ and $g = g' + \sum \beta_k b_k$, where $f', g' \in A$, and $\alpha_i, \beta_k \in K$. Then

$$\begin{aligned} f r_j g &= (f' + \sum_i \alpha_i b_i) r_j (g' + \sum_k \beta_k b_k) \\ &= f' r_j g' + (\sum_i \alpha_i b_i r_j) g' + f'(\sum_k \beta_k r_j b_k) + \sum_{i,k} \alpha_i \beta_k b_i r_j b_k, \end{aligned}$$

which is visibly in the ideal of A generated by S. □

Recall that for algebras $B \leq A$, we use $|A : B|$ to denote the codimension of B in A (*i.e.*, $\dim_K(A/B)$).

Corollary 18. *If A is a finitely presented K-algebra and B is a subalgebra of finite codimension in A, then B is finitely presented.*

Proof. Let $\langle X|R\rangle$ be a finite presentation of A and let I be the ideal of $K\langle X\rangle$ generated by R. Let B' be the pre-image of B in $K\langle X\rangle$. So B' is a subalgebra of $K\langle X\rangle$ containing I. Now $|K\langle X\rangle : B'| = |A : B| < \infty$. So by Theorem 16, B' is finitely presented. By Lemma 17, I is finitely generated as an ideal of B'. Therefore, B is finitely presented. A finite set of generators is given by Lewin's theorem. A finite set of relators is given in combination by Lewin's theorem and Lemma 17. □

The hope here was that if A and B are as in the previous corollary and A is GS, then the proof will give us bounds on the number generators and relators ensuring that B is also GS. This turned out to be too hopeful. In the next chapter we give counterexamples to a general result of this type.

4 Subalgebras of $K\langle X\rangle$

In this section we exhibit a finite codimensional subalgebra of $K\langle X\rangle$ which is not GS. Let $X = \{x_1, \ldots, x_m\}$ and let $I = K\langle X\rangle_+$. For any positive integer n we have the ideal

$$I^n = \bigoplus_{i=n}^{\infty} K\langle X\rangle_i$$

We show that the graded subalgebra $K \cdot 1 + I^2$ is not GS with respect to the grading inherited from $K\langle X\rangle$. Compelling computer evidence suggests that this is true of all $K \cdot 1 + I^n$ for all $n \geq 2$.

4.1 A Presentation of $K \cdot 1 + I^n$

Here we take the convention that each x_i has degree one. For every sequence of integers $\pi = i_1 i_2 \cdots i_k$ with the $i_j \in \{1, \ldots, m\}$, define $a_\pi = x_{i_1} \cdots x_{i_k}$ and use the following notation for the length of π: $|\pi| = k$. Given two such sequences π and ρ, we form a third sequence $\pi\rho$ by juxtaposition. Use the usual degree function from $K\langle X\rangle$ as the degree function for I^n.

The set

$$Y = \{a_\pi \,|\, \pi \text{ is any sequence in } \{1, \ldots, m\} \text{ with } n \leq |\pi| \leq 2n-1\}$$

is a generating set for $K \cdot 1 + I^n$. Notice that this generating set is not contained in $K\langle X\rangle_n$. In fact, $K \cdot 1 + I^n$ cannot be presented as a graded subalgebra of $K\langle X\rangle$ in such a way that the generating set is contained in $K\langle X\rangle_n$. This is why we allowed degrees of generators to be different from one in the formulation of the GS theorem.

We say that a word w in X of length at least n is in *canonical form* if $w = w'a_\pi$, where w' is a product of elements in Y_n and $a_\pi \in Y$ (of course w' may be the empty word). An arbitrary element $f = \sum \alpha_v v$ of I^n is in canonical form if all v are canonical words. Recall that in the introduction we defined the notation $S_i = \{s \in S | \deg s = i\}$ for any graded subset of a graded algebra.

We have two basic types of relations among elements of Y:

Type 1: $a_\pi a_\rho = a_{\pi'} a_{\rho'}$ where $\pi\rho = \pi'\rho'$ and $|\pi'|$ is minimal while ensuring that $n \leq |\pi'|, |\rho'| \leq 2n-1$. These occur in each degree $2n+1, 2n+2, \ldots, 4n-3$.

Type 2: $a_{\pi'} a_{\rho'} = a_\pi a_\rho a_\sigma$ where $\pi\rho\sigma = \pi'\rho'$, $a_\pi a_\rho a_\sigma$ is in canonical form (*i.e.*, $|\pi| = |\rho| = n$), and $|\pi'|$ is minimal while ensuring that $n \leq |\pi'|, |\rho'| \leq 2n-1$. These occur in each degree $3n, 3n+1, \ldots, 4n-2$, for $n \geq 3$. No relations of this type occur for $n = 2$.

Let R denote the set of all relators described by relations of types 1 and 2. Note that elements in R are homogeneous as elements of $K\langle X \rangle$. If one simply defined all the elements of Y to have degree one, relators of type 2 would not be homogeneous.

Example 19. Consider the case $n = 3$. The following are examples of relations of type 1.

$$a_{(1,2,3,4)} a_{(5,6,7)} = a_{(1,2,3)} a_{(4,5,6,7)}$$

$$a_{(1,2,3,4,5)} a_{(6,7,8)} = a_{(1,2,3)} a_{(4,5,6,7,8)} \tag{4.1}$$

$$a_{(1,2,3,4,5)} a_{(6,7,8,9)} = a_{(1,2,3,4)} a_{(5,6,7,8,9)} \tag{4.2}$$

Note that (4.2) does not include a canonical word, but the word on the right-hand-side can be put in canonical form by a relation of type 2:

$$a_{(1,2,3,4)} a_{(5,6,7,8,9)} = a_{(1,2,3)} a_{(4,5,6)} a_{(7,8,9)}$$

Also note the following relation, which is neither of type 1 nor 2.

$$a_{(1,2,3,4,5)} a_{(6,7,8)} = a_{(1,2,3,4)} a_{(5,6,7,8)} \tag{4.3}$$

This does not qualify under our definition of a relation of type 1 since we may place a sequence of length three in place of $(1,2,3,4)$ while keeping the length of all involved sequences between three and five as in (4.1). However, relation (4.3) follows from relations of type 1:

$$a_{(1,2,3,4,5)}a_{(6,7,8)} = a_{(1,2,3)}a_{(4,5,6,7,8)} = a_{(1,2,3,4)}a_{(5,6,7,8)}$$

Claim 20. *Any word in Y can be put into canonical form using relations of types 1 and 2.*

Proof. Let $w = a_{\pi_1} \cdots a_{\pi_k}$ be any word in Y. First observe that the claim holds for $k = 2$:

- If $d(w) < 3n$, then applying one relation of type 1 (once) yields

  $$w = a_{\pi_1}a_{\pi_2} = a_{\pi'_1}a_{\pi'_2}$$

  where $|\pi'_1| = n$. This puts w in canonical form.

- If $d(w) \geq 3n$, then applying one relation of type 1 followed by one relation of type 2 yields

  $$w = a_{\pi_1}a_{\pi_2} = a_{\pi'_1}a_{\pi'_2} = a_\pi a_\rho a_\sigma$$

  where $|\pi|, |\rho| = n$. The result is in canonical form.

Suppose $k > 2$, and $w = a_{\pi_1} \cdots a_{\pi_{k-1}}a_{\pi_k}$. By induction on k, we can put $a_{\pi_1} \cdots a_{\pi_{k-1}}$ into canonical form using relations of types 1 and 2. This gives

$$w = a_{\pi_1} \cdots a_{\pi_{k-1}}a_{\pi_k} = w'a_\pi a_{\pi_k}$$

and as already seen, $a_\pi a_{\pi_k}$ can be put into canonical form using relations of types 1 and 2. □

Since two distinct canonical words in Y are distinct as elements of $K\langle X\rangle$, we see that the set of all canonical words in Y is linearly independent. Therefore, $\langle Y|R\rangle$ is indeed a presentation for $K \cdot 1 + I^n$.

4.2 Minimality of $\langle Y|R\rangle$

Now we check that Y is indeed a minimal set of generators.

Claim 21. *If Z is any homogeneous set of generators for $K \cdot 1 + I^n$, then*

$$Y_k \subseteq \operatorname{span}_K(Z_k)$$

for all $k = n, \dots, 2n-1$.

Proof. Suppose that Z is any set of homogeneous generators for $K \cdot 1 + I^n$ and let $a_\pi \in Y_k$. We can write

$$a_\pi = \sum_v \alpha_v v$$

where the $\alpha_v \in K$ and the v are distinct words in Z. By comparing degrees in X, we can see that all the words v can be assumed to have degree k. The claim now follows. □

We also have the following minimality condition on R.

Claim 22. *Let S be any set of relators in Y which define $K \cdot 1 + I^n$, and let $r \in R_k$ with $2n+1 \leq k \leq 3n$. Then $r \in \operatorname{span}_K(S_k)$.*

Proof. As S is a set of defining relators for $K \cdot 1 + I^n$, we have

$$r = \sum_i \alpha_i v_i s_i v_i' \tag{4.4}$$

where $\alpha_i \in K$ and the v_i, v_i' are words in Y. Note that S_i is empty for $i \leq 2n$. If any of the v_i or v_i' are nonempty words, then the right hand side of (4.4) has degree $> 3n$, which is a contradiction. Therefore,

$$r = \sum_i \alpha_i s_i \in \operatorname{span}_K(S_k).$$

□

4.3 $\langle Y|R\rangle$ is not a GS presentation

We need to count the number of generators and relators in $\langle Y|R\rangle$. For each $i = n, \ldots, 2n-1$, there are m^i elements of degree i in Y, since any sequence of i elements of X gives us such an element, and the order of X is m.

Elements of R arise in every degree $2n+1, 2n+2, \ldots, 4n-2$. Let $r \in R$ be of degree k. Distinguish between two cases:

- $2n+1 \leq k \leq 3n-1$: In this case r must be of type 1, and it arises from a partition of a sequence π, of length k, into two subsequences: $\pi = \tau_1\tau_2$. Since $n \leq |\tau_i| \leq k-n$, there are $k-2n+1$ such partitions, yielding $k-2n$ distinct relations. $|X| = m$ so there are m^k sequences π, therefore there are $(k-2n)m^k$ distinct possibilities for r.

- $3n \leq k \leq 4n-2$: Here we have exactly one relation of type 2 for each sequence of length k, giving us m^k distinct possibilities of type 2 for r. Again, type 1 possibilities for r arise from partitions of a sequence of length k into two parts whose lengths are between n and $2n-1$ (inclusive). There are $4n-k-1$ such partitions, yielding $4n-k-2$ distinct relations of types 1. In all, there are $(4n-k-1)m^k$ distinct possibilities for r.

By the minimality claims in Section 4.2, we see that

$$1 - H_Y(t) + H_R(t) > 1 - \sum_{i=n}^{2n-1}(mt)^i + \sum_{i=2n+1}^{3n-1}(i-2n)(mt)^i.$$

So, in order to show that $\langle Y|R\rangle$ is not a GS presentation, it suffices to show that for all t in the interval $(0,1)$,

$$1 - \sum_{i=n}^{2n-1}(mt)^i + \sum_{i=2n+1}^{3n-1}(i-2n)(mt)^i \geq 0. \tag{4.5}$$

For any real number x and positive integer n, define

$$S(x,n) = 1 - \sum_{i=n}^{2n-1} x^i + \sum_{i=2n+1}^{3n-1}(i-2n)x^i. \tag{4.6}$$

Lemma 23. *$S(x,2) \geq 0$ for all $x > 0$.*

Proof. $S(x,2) = 1 - x^2 - x^3 + x^5 = (1-x^2)(1-x^3) \geq 0$ for all $x \geq -1$. □

Computer experiments strongly support the following conjecture.

Conjecture 24. *$S(x,n) \geq 0$ for all $x > 0$ and all $n \geq 2$.*

In conclusion, we see that any set of generators of $K \cdot 1 + I^n$ contains Y, and any set of relators in Y contains $\bigcup_{i=2n+1}^{3n} R_i$, which in the $n = 2$ case is already too large for $\langle Y | R \rangle$ to be GS. Moreover, there is no redundancy of elements in $\bigcup_{i=2n+1}^{3n} R_i$. Therefore, there is no GS presentation for $K \cdot 1 + I^2$ (and probably for $K \cdot 1 + I^n$) consisting of generators which are homogeneous in $K\langle X \rangle$.

This result makes it clear that in the search for a setting in which the GS condition is inherited by subalgebras, finite codimensional algebras are not good analogs of finite index subgroups. Veronese powers are better analogs of finite index subgroups and in the next two chapters we see that they behave better with respect to the GS condition.

5 Veronese Powers of Graded Algebras

In this chapter and the next we turn our attention to Veronese powers of graded associative algebras. Let us recall that if $A = \bigoplus_{i=0}^{\infty} A_i$ is a graded algebra, then the n^{th} *Veronese power of* A is defined to be

$$A^{(n)} = \bigoplus_{i=0}^{\infty} A_{ni}.$$

In section 5.2 we prove Theorem 13. That is, we show that if a graded algebra A is finitely presented by m generators and r relators and

$$r < \frac{1}{4}\left(\frac{m}{2} - 1\right)^2 \tag{5.1}$$

then infinitely many Veronese powers of A are GS.

In the case that this is specialized to quadratic algebras, we prove that the same conclusion holds, but with slightly weaker assumptions (section 5.3). We show that if a quadratic algebra A is finitely presented by m generators and r relators and

$$r < \frac{4}{25}m^2 \tag{5.2}$$

then infinitely many Veronese powers of A are GS.

For further discussions of Veronese powers, see [2, 3, 20].

5.1 A Presentation

Let A be a one-generated graded algebra and set $a_i = \dim(A_i)$. Suppose that A has a finite presentation $\langle X = \{x_1, \ldots, x_m\} | R \rangle$ with r_i relators of degree $i \geq 2$.

Let k be maximal such that $r_k \neq 0$. Set $r = \sum_{i=2}^{k} r_i$.

It is easy to see that any basis for A_n will generate $A^{(n)}$ as an algebra.

By proposition 3.2.2 of [20], $A^{(n)}$ is quadratic for all $n \geq k-1$. We can see that if $f \in R$ is of degree i, then ufv is a relator for $A^{(n)}$, where u and v are any two elements of $K\langle X\rangle$ whose degrees add up to $2n-i$. This proposition is proved by obtaining a presentation for $A^{(n)}$ via the following construction (although they use slightly different language). The set X is identified with a basis of A_1, call it $\mathcal{E}_1$. Then we can write bases of $A_2, \ldots, A_n$ in terms of $\mathcal{E}_1$ (as linear combinations of words in $\mathcal{E}_1$). Denote these bases by $\mathcal{E}_2, \ldots, \mathcal{E}_n$, respectively. As already remarked, we may choose $\mathcal{E}_n$ as a set of generators. So there are a_n generators of $A^{(n)}$. For any given relator $f \in R$ of degree i, consider the set of relators in $A^{(n)}$

$$\mathcal{S}(f) = \{ufv \mid u \in \mathcal{E}_j, v \in \mathcal{E}_\ell, j + \ell = 2n - i\}.$$

The proof of proposition 3.2.2 of [20] is completed by showing that

$$\bigcup\{\mathcal{S}(f) \mid f \in R\}$$

is a defining set of relators for $A^{(n)}$. Note that all of these relations are quadratic in the generators $\mathcal{E}_n$.

For $n \geq k-1$, this construction has yielded a presentation of $A^{(n)}$ with a_n generators and

$$R_n = \sum_{i=2}^{k} \left(\sum_{j=1}^{i-1} a_{n-j} a_{n-i+j} \right) r_i \tag{5.3}$$

quadratic relators.

Remark 25. Throughout the rest of the dissertation, let us assume that $m > 2$ and $r \geq 1$. The case with $r = 0$ is far simpler since this means that A is free. Since this implies that $R_n = 0$ for all n, we have that $A^{(n)}$ is also free for all n.

5.2 The GS Condition

In this section we prove the following theorem.

Theorem 13. *Let A be a graded algebra with a presentation consisting of m generators and r relators. If $r < \frac{1}{4}\left(\frac{m}{2}-1\right)^2$, then $A^{(n)}$ is GS for infinitely many n.*

If A is GS, then inequality (1.3) implies

$$H_A(t) \geq (1 - mt + H_R(t))^{-1}. \tag{5.4}$$

Lemma 26. *If $r < m^2/4$, then $(1-mt+rt^2)^{-1} \geq \sum_{i=0}^{\infty} \alpha^{-i}t^i$, where α is the smaller root of $1-mt+rt^2$. Further, this inequality is strict for all the non-constant terms (the constant terms on both sides are 1).*

Proof. Let α and β be the roots of $1 - mt + rt^2$ and $\alpha < \beta$. Note that these are distinct and $0 < \alpha < 1$ (see proof of Lemma 33), since $r < m^2/4$. So we can factor $1 - mt + rt^2 = r(\alpha - t)(\beta - t)$. Hence,

$$\begin{aligned}
(1 - mt + rt^2)^{-1} &= (1 - \alpha^{-1}t)^{-1}(1 - \beta^{-1}t)^{-1} \\
&= \left(\sum_{i=0}^{\infty} \alpha^{-i}t^i\right)\left(\sum_{j=0}^{\infty} \beta^{-j}t^j\right) \\
&= \sum_{\ell=0}^{\infty}\left(\sum_{i+j=\ell} \alpha^{-i}\beta^{-j}\right)t^{\ell}.
\end{aligned}$$

Now compute:

$$\begin{aligned}
\sum_{i+j=\ell} \alpha^{-i}\beta^{-j} &= \alpha^{-\ell} + \alpha^{-(\ell-1)}\beta^{-1} + \cdots + \alpha^{-1}\beta^{-(\ell-1)} + \beta^{-\ell} \\
&= \alpha^{-\ell}\sum_{i=0}^{\ell} \alpha^i\beta^{-i} \\
&> \alpha^{-\ell}, \text{ for all } \ell \geq 1.
\end{aligned}$$

□

The following formula, which was obtained in the proof of the previous lemma, is highlighted for later use.

$$(1 - mt + rt^2)^{-1} = \sum_{\ell=0}^{\infty}\left(\sum_{i+j=\ell} \alpha^{-i}\beta^{-j}\right)t^{\ell} \tag{5.5}$$

Lemma 27. *Let $i_1, \ldots, i_r \geq 2$ be integers, suppose $r \leq m^2/4$, $m \geq 2$, and write*

$$(1 - mt + t^{i_1} + \cdots + t^{i_r})^{-1} = \sum_{j=0}^{\infty} b_j t^j. \tag{5.6}$$

Then $b_{j+1} \geq \frac{m}{2} b_j$ for all $j \geq 0$.

Proof. First observe that from (5.6) we have the formal equality

$$\begin{aligned} 1 &= (1 - mt + t^{i_1} + \cdots + t^{i_r}) \sum_{j=0}^{\infty} b_j t^j \\ &= b_0 + (b_1 - mb_0)t + \sum_{j=2}^{\infty} (b_j - mb_{j-1} + b_{j-i_1} + \cdots + b_{j-i_r}) t^j, \end{aligned}$$

where we set $b_j = 0$ for all $j < 0$. Thus, $b_0 = 1$, $b_1 = m$, and

$$b_j = mb_{j-1} - (b_{j-i_1} + \cdots + b_{j-i_r}), \tag{5.7}$$

for all $j > 0$. Now proceed with the proof by induction on j. The case $j = 0$ is done. Suppose $b_{j+1} \geq \frac{m}{2} b_j$ for all $j = 0, 1, \ldots, n-2$. From (5.7), our induction hypothesis, and our assumption that $r \leq m^2/4$, we have

$$\begin{aligned} b_n &= mb_{n-1} - (b_{n-i_1} + \cdots + b_{n-i_r}) \\ &\geq mb_{n-1} - rb_{n-2} \\ &\geq mb_{n-1} - \frac{m^2}{4} b_{n-2} \\ &\geq mb_{n-1} - \frac{m}{2} b_{n-1} \\ &= \frac{m}{2} b_{n-1}. \end{aligned}$$

□

Remark 28. Consider the setting of the previous lemma in the case that $i_1 = i_2 = \cdots = i_r = 2$. From the proof of the lemma we can notice the following recursion relation for the coefficients b_j.

$$\begin{aligned} b_0 &= 1 \\ b_1 &= m \\ b_j &= mb_{j-1} - rb_{j-2}, \text{ for all } j \geq 2. \end{aligned}$$

This fact will be useful in our study of generic quadratic algebras in Chapter 6.

Lemma 29. *If $r < m^2/4$, then $(1 - mt + H_R(t))^{-1} \geq (1 - mt + rt^2)^{-1}$.*

Proof. Write

$$\begin{aligned}(1 - mt + H_R(t))^{-1} &= (1 - mt + t^{i_1} + \cdots + t^{i_r})^{-1} \\ &= \sum_{j=0}^{\infty} b_j t^j\end{aligned}$$

and

$$(1 - mt + rt^2)^{-1} = \sum_{k=0}^{\infty} c_k t^k.$$

For each $n \geq 0$ define $d_n = b_n c_0 + b_{n-1} c_1 + \cdots + b_0 c_n$. From the previous lemma we have $b_{j+1} \geq \frac{m}{2} b_j$ and $c_{k+1} \geq \frac{m}{2} c_k$. This gives

$$\begin{aligned}d_{n+1} &= b_{n+1} c_0 + b_n c_1 + \cdots + b_1 c_n + b_0 c_{n+1} \\ &\geq \frac{m}{2}(b_n c_0 + b_{n-1} c_1 + \cdots + b_0 c_n) + b_0 c_{n+1} \\ &\geq \frac{m}{2} d_n.\end{aligned}$$

Now compute

$$\begin{aligned}\frac{1}{1 - mt + t^{i_1} + \cdots + t^{i_r}} - \frac{1}{1 - mt + rt^2} &= \frac{rt^2 - t^{i_1} - \cdots - t^{i_r}}{(1 - mt + t^{i_1} + \cdots + t^{i_r})(1 - mt + rt^2)} \\ &= \sum_{\ell=1}^{r} (t^2 - t^{i_\ell}) \sum_{j=0}^{\infty} b_j t^j \sum_{k=0}^{\infty} c_k t^k \\ &= \sum_{\ell=1}^{r} (t^2 - t^{i_\ell}) \sum_{n=0}^{\infty} d_n t^n \\ &= \sum_{n=0}^{\infty} \sum_{\ell=1}^{r} (d_{n-2} - d_{n-i_\ell}) t^n,\end{aligned}$$

where again we set $d_j = 0$ for all $j < 0$. From this we can see that all the coefficients are non-negative since $i_j \geq 2$ and the d_n are increasing. □

By applying (5.4) along with the previous two lemmas, and still assuming $r < m^2/4$, we have

$$H_A(t) \geq (1 - mt + H_R(t))^{-1} \geq (1 - mt + rt^2)^{-1} \geq \sum_{i=0}^{\infty} \alpha^{-i} t^i.$$

Hence,

$$a_i > \alpha^{-i}, \text{ for all } i \geq 1. \tag{5.8}$$

Lemma 30. *For any integer $k \geq 2$ and any positive real number c,*

$$f_k(x) = x^k - c(x^{k-2} + 2x^{k-3} + 3x^{k-4} + \cdots + (k-2)x + k - 1)$$

has a root in the interval $[\sqrt{c}, 1 + \sqrt{c})$.

Proof. We will begin by showing that $f_k(\sqrt{c}) < 0$ for all $k \geq 3$ (note that $f_2(\sqrt{c}) = 0$). Observe that we have the recursion relation

$$f_k(x) = x f_{k-1}(x) - c(k-1)$$

for all $k \geq 3$. For $k = 3$ we have

$$f_3(\sqrt{c}) = c^{3/2} - c(c^{1/2} + 3 - 1) = -2c < 0.$$

Inductively, if $k \geq 4$ and $f_{k-1}(\sqrt{c}) < 0$, then

$$f_k(\sqrt{c}) = \sqrt{c} f_{k-1}(\sqrt{c}) - c(k-1) < 0 + 0 = 0.$$

Now we will show that $f_k(1 + \sqrt{c}) > 0$ for all $k \geq 2$. Again we'll use the same recursion relation to show that

$$f_k(1 + \sqrt{c}) \geq 1 + k\sqrt{c}$$

for all $k \geq 2$. To start induction observe that

$$\begin{aligned} f_2(1 + \sqrt{c}) &= (1 + \sqrt{c})^2 - c \\ &= 1 + 2\sqrt{c}. \end{aligned}$$

Suppose that $f_{k-1}(1 + \sqrt{c}) \geq 1 + (k-1)\sqrt{c}$. Then

$$\begin{aligned} f_k(1 + \sqrt{c}) &= (1 + \sqrt{c}) f_{k-1}(1 + \sqrt{c}) - c(k-1) \\ &\geq (1 + \sqrt{c})(1 + (k-1)\sqrt{c}) - c(k-1) \\ &= 1 + k\sqrt{c} + c(k-1) - c(k-1) \\ &= 1 + k\sqrt{c}. \end{aligned}$$

The lemma now follows from the Intermediate Value Theorem. □

Remark 31. It can be shown that for any $\varepsilon > 0$, there exists a K such that $f_k(1-\varepsilon) < 0$ for all $k \geq K$. This is mentioned only to point out that $1+\sqrt{c}$ is the best possible upper bound (not depending on k) on the roots of all the $f_k(x)$, but since it is not required for the argument, a proof is omitted.

To show that $A^{(n)}$ is GS, by Proposition 7 it suffices to show that $R_n < a_n^2/4$. That is,

$$a_n^2 > 4\sum_{i=2}^{k}\left(\sum_{j=1}^{i-1} a_{n-j}a_{n-i+j}\right) r_i. \tag{5.9}$$

We will do this by proving the following stronger inequality.

$$a_n^2 > 4r\sum_{i=2}^{k}\left(\sum_{j=1}^{i-1} a_{n-j}a_{n-i+j}\right), \tag{5.10}$$

where as before, $r = \sum_i r_i$.

Lemma 32. *Let $b, c > 1$ be constants with $b > 1+\sqrt{c}$ and let (a_n) be a sequence such that $a_n > b^n$ for all $n \geq 1$. Then $a_n^2 > c\sum_{i=2}^{k}\left(\sum_{j=1}^{i-1} a_{n-j}a_{n-i+j}\right)$ for infinitely many n.*

Proof. Suppose that the conclusion of the lemma is false. So there exists an N such that $a_n^2 \leq c\sum_{i=2}^{k}\left(\sum_{j=1}^{i-1} a_{n-j}a_{n-i+j}\right)$ for all $n \geq N$. Define a sequence $s_N, s_{N+1}, s_{N+2}, \ldots$ by choosing

$$s_N \geq a_N, s_{N+1} \geq a_{N+1}, \ldots, s_{N+k-1} \geq a_{N+k-1}, \tag{5.11}$$

then defining

$$s_n^2 = c\sum_{i=2}^{k}\left(\sum_{j=1}^{i-1} s_{n-j}s_{n-i+j}\right) \tag{5.12}$$

for all $n \geq N+k$. It follows that $s_n \geq a_n$ for all $n \geq N$.

For this choice of the sequence (s_n), we want to find constants d and s such that $s_n = ds^n$ for all $n \geq N$. From (5.12) it follows that s must satisfy

$$s^{2n} = c(s^{2n-2} + 2s^{2n-3} + 3s^{2n-4} + \cdots + (k-1)s^{2n-k}),$$

which reduces to

$$s^k = c(s^{k-2} + 2s^{k-3} + \cdots + (k-2)s + k - 1). \tag{5.13}$$

Using Lemma 30, let us choose a number s in the interval $[\sqrt{c}, 1 + \sqrt{c})$ which satisfies equation (5.13). Now choose a constant d large enough to satisfy (5.11).

Overall we have

$$b^n < a_n \leq ds^n < d(1 + \sqrt{c})^n$$

for all $n \geq N$. This implies that

$$b \leq 1 + \sqrt{c},$$

which contradicts our assumption. □

As before, set

$$\alpha = \frac{m - \sqrt{m^2 - 4r}}{2r}.$$

Lemma 33. *Fix $m \geq 2$ and let $r \leq m^2/4$. Then $\alpha \leq 2/m$.*

Proof. Let $f(t) = 1 - mt + rt^2$. Notice that we have

$$f(1/m) = \frac{r}{m^2} > 0$$

and

$$f(2/m) = -1 + \frac{4r}{m^2} \leq -1 + \frac{4}{m^2} \cdot \frac{m^2}{4} = 0.$$

Hence, by the Intermediate Value Theorem, f has a root in the interval $(1/m, 2/m]$. □

Proof of Theorem 13. Assuming that $r < m^2/4$, the inequalities in (5.8) hold. So using $b = \alpha^{-1}$ and $c = 4r$ in Lemma 32, we see that (5.10) holds for infinitely many n (and hence $A^{(n)}$ is GS for infinitely many n) if

$$\alpha^{-1} > 1 + 2\sqrt{r}. \tag{5.14}$$

It is possible to solve for r in this inequality, but the solution is a cumbersome (but quadratic) expression in m. Instead, notice that Lemma 33 implies that $\alpha^{-1} > m/2$. Thus

$$\begin{aligned} r < \frac{1}{4}\left(\frac{m}{2} - 1\right)^2 \quad &\Rightarrow \quad \frac{m}{2} > 1 + 2\sqrt{r} \\ &\Rightarrow \quad \alpha^{-1} > 1 + 2\sqrt{r} \\ &\Rightarrow \quad A^{(n)} \text{ is GS for infinitely many } n. \end{aligned}$$

□

5.3 Quadratic Case

It is interesting to point out the following special case of the above situation. Suppose that A is a quadratic algebra (*i.e.,* $r = r_2$). In this case, the technique used to prove the above proposition requires that $r < \frac{1}{16}m^2$ in order to conclude that infinitely many Veronese powers of A are GS. This bound comes from the fact that if $k = 2$, Lemma 30 is greatly simplified because the root is actually $\sqrt{c}$, not merely in the interval $[\sqrt{c}, 1 + \sqrt{c})$. Thus, Lemma 32 can be proved with the weaker assumption that $b > \sqrt{c}$ (rather than using the assumption $b > 1 + \sqrt{c}$). So the argument can be finished in the same way it was done above:

$$\begin{aligned} r < \frac{1}{16}m^2 \quad &\Rightarrow \quad \frac{m}{2} > 2\sqrt{r} \\ &\Rightarrow \quad \alpha^{-1} > 2\sqrt{r} \\ &\Rightarrow \quad A^{(n)} \text{ is GS for infinitely many } n. \end{aligned}$$

However, we can get a slightly better result.

Theorem 14. *Suppose A is a quadratic algebra with m generators and r relators. If $r < \frac{4}{25}m^2$, then $A^{(n)}$ is GS for infinitely many n.*

With A being a quadratic algebra, the count of relators for $A^{(n)}$ given in (5.3) becomes ra_{n-1}^2. So $A^{(n)}$ will be GS if $R_n < a_n^2/4$ by Proposition 7. That is,

$$ra_{n-1}^2 < \frac{1}{4}a_n^2. \tag{5.15}$$

Let α and a_n still be defined as above and note that (5.8) still holds assuming that $r < m^2/4$. That is, $a_i > \alpha^{-i}$ for all i.

Lemma 34. *Let $\varepsilon > 0$. Then $a_n > (\alpha^{-1} - \varepsilon)a_{n-1}$ for infinitely many n.*

Proof. Assume that the lemma is false. Then there exists an ε in the interval $(0, \alpha^{-1})$ and a natural number N such that

$$a_n \leq (\alpha^{-1} - \varepsilon)a_{n-1}$$

for all $n \geq N$. Thus,

$$\alpha^{-(N+k)} < a_{N+k} \leq a_N(\alpha^{-1} - \varepsilon)^k,$$

and hence,

$$\alpha^{-k} < (a_N\alpha^N)(\alpha^{-1} - \varepsilon)^k,$$

for all $k \geq 0$. This is a contradiction since for any constant c,

$$\alpha^{-k} > c(\alpha^{-1} - \varepsilon)^k.$$

for k sufficiently large, as $\alpha^{-1}/(\alpha^{-1} - \varepsilon) > 1$. □

Proof of Theorem 14. Recall that

$$\alpha = \frac{m - \sqrt{m^2 - 4r}}{2r}.$$

Observe that we have the following equivalence of inequalities.

$$\begin{aligned}
\alpha^{-2} > 4r \quad &\iff \quad 4\alpha^2 < 1/r = \alpha\beta \\
&\iff \quad 4\alpha < \beta = \frac{m}{r} - \alpha \\
&\iff \quad \frac{m}{r} > 5\alpha \\
&\iff \quad 2m > 5m - 5\sqrt{m^2 - 4r} \\
&\iff \quad 3m < 5\sqrt{m^2 - 4r} \\
&\iff \quad 9m^2 < 25m^2 - 100r \\
&\iff \quad r < \frac{4}{25}m^2
\end{aligned}$$

So for $r < \frac{4}{25}m^2$ and positive ε sufficiently small, Lemma 34 gives

$$\left(\frac{a_n}{a_{n-1}}\right)^2 > (\alpha^{-1} - \varepsilon)^2 > 4r$$

for infinitely many n. With this, Theorem 14 is proved using (5.15). □

This is a curious situation because intuitively, the case where A is quadratic should be the worst case. Loosely speaking, quadratic relators should have more consequences in $A^{(n)}$ than relators of higher degree. Also, it is likely that the conclusions of Theorems 13 and 14 can be strengthened to, "*$A^{(n)}$ is GS for all n.*" We state this as the following conjecture, the proof of which has been elusive to the techniques employed here.

Conjecture 35. *Let A be a graded algebra with a presentation consisting of m generators and r relators. If $r \leq \frac{4}{25}m^2$, then $A^{(n)}$ is GS for all n.*

In the next chapter it is shown that this conjecture is true if A is a generic quadratic algebra.

6 Generic Quadratic Algebras

In [1], the author considers a general view of the notion of a generic, finitely presented graded algebra. Such an algebra is defined to be an algebra with a coefficient-wise minimal Hilbert series. The following question immediately presents itself. Given non-negative integers $d_1, \ldots, d_k$ and $r_1, \ldots, r_\ell$, what is the minimal Hilbert series among algebras presented by d_i generators of degree i and r_j homogeneous relators of degree j? This and other related questions are explored in considerable detail in [1].

Since we will only deal with the case of generic quadratic algebras, we will not discuss generic algebras in full generality. Instead, we shall think of a generic quadratic algebra as any algebra arising in the following way. Fix positive integers m and r. Consider an extension of our ground field by finitely many algebraically independent variables

$$\{t_{ij}^{(k)} \,|\, i, j = 1, \ldots, m; k = 1, \ldots, r\}.$$

Denote it by $\widetilde{K} = K(t_{ij}^{(k)})$. Then the K-algebra presented by (degree-one) generators $X = \{x_1, \ldots, x_m\}$ and quadratic relators

$$R = \{f_k = \sum_{i,j=1}^{m} t_{ij}^{(k)} x_i x_j \,|\, k = 1, \ldots, r\}$$

is the generic quadratic K-algebra.

In [1] D. Anick proves that there exists a quadratic algebra with m generators and r relators whose Hilbert series is precisely $(1 - mt + rt^2)^{-1}$ if and only if $r < m^2/4$. By the Golod-Shafarevich theorem, if A is a quadratic algebra presented by m generators and less than $m^2/4$ relators, then $H_A(t) \geq (1 - mt + rt^2)^{-1}$. So Anick's result shows us that in the case that $r < m^2/4$, generic quadratic algebras presented by m generators and r relators are precisely those algebras with Hilbert series equal to $(1 - mt + rt^2)^{-1}$.

For a graded algebra A, we will continue to use the notation $a_i = \dim(A_i)$. As mentioned in the introduction in (1.5), we have the following nice characterization of quadratic GS algebras.

Remark 36. The number of relators in any presentation of a quadratic algebra is $a_1^2 - a_2$. Therefore, a quadratic algebra is GS if and only if

$$\frac{3}{4}a_1^2 - a_2 < 0. \tag{6.1}$$

Another useful application of this count of the number relators of a quadratic algebra arises when considering Veronese powers of quadratic algebras. If A is a quadratic GS algebra presented by $m = a_1$ generators and r relators, then all of its Veronese powers are also quadratic. So $A^{(n)}$ is presented by a_n generators and $a_n^2 - a_{2n}$ relators. On the other hand we already saw that $A^{(n)}$ can be presented by the same a_n generators and ra_{n-1}^2 relators. Hence, $a_n^2 - a_{2n} = ra_{n-1}^2$. So we have the following handy recursion relation.

$$a_{2n} = a_n^2 - ra_{n-1}^2$$

Remark 37. If A is a quadratic algebra, then $A^{(n)}$ is also quadratic. However, if A is a generic algebra, then $A^{(n)}$ is not necessarily generic. Consider the following example. Let A be a generic quadratic algebra presented by m generators and r relators, and assume that $r < m^2/4$. Then, by following Remark 28 and the previous remark, we see that

$$a_2 = m^2 - r,\ a_3 = m^3 - 2rm,\ a_4 = m^4 - 3rm^2 + r^2,\ a_6 = m^6 - 5rm^4 + 6r^2m^2 - r^3$$

But notice that

$$H_{A^{(2)}}(t) = 1 + a_2t + a_4t^2 + a_6t^3 + \cdots \neq (1 - (m^2 - r)t + rm^2t^2)^{-1}$$

since the coefficient of t^3 in $H_{A^{(2)}}(t)(1 - (m^2 - r)t + rm^2t^2)$ is

$$a_6 - a_4(m^2 - r) + a_2rm^2 = r^2m^2 \neq 0.$$

Similar computations can be done for higher values of n to show that $A^{(n)}$ is not generic.

6.1 Veronese Powers

In this section we prove Theorem 15. Let $\beta > \alpha$ be the roots of $1 - mt + rt^2$. In the case that $r \leq m^2/4$, it is easy to see that α is positive.

Lemma 38. *If $r \leq m^2/4$, then*

$$\left(\frac{4}{r} - \alpha^2\right)\alpha^{2n} < \frac{4}{r^{n+1}}, \text{ for all } n \geq 0$$

Proof. The following inequalities are equivalent.

$$\begin{aligned} \left(\frac{4}{r} - \alpha^2\right)\alpha^{2n} &< \frac{4}{r^{n+1}} \\ (4 - r\alpha^2)\alpha^{2n}r^n &< 4. \end{aligned}$$

We will verify the second one. Begin by noting that since $r\alpha\beta = 1$ and $\alpha \leq \beta$, we have

$$r\alpha^2 \leq r\alpha\beta = 1. \tag{6.2}$$

Hence,

$$(r\alpha^2)^n \leq 1 \Rightarrow (4 - r\alpha^2)\alpha^{2n}r^n \leq 4 - r\alpha^2 < 4.$$

□

Theorem 15. *Let A be a generic quadratic algebra with m generators and r relators and assume that $r < m^2/4$.*

1. *If $r \leq \frac{4}{25}m^2$, then $A^{(n)}$ is GS for all n.*

2. *If $r > \frac{4}{25}m^2$, then $A^{(n)}$ is not GS for all but finitely many n.*

Proof. Following Remark 36, and referring to the presentation described in Chapter 5, we see that $A^{(n)}$ is GS if and only if

$$\frac{3}{4}a_n^2 - a_{2n} < 0. \tag{6.3}$$

From formula (5.5) we have

$$\sum_{n=0}^{\infty} a_n t^n = (1 - mt + rt^2)^{-1} = \sum_{n=0}^{\infty}\left(\sum_{i+j=n}\alpha^{-i}\beta^{-j}\right)t^n.$$

Using this along with the familiar formula

$$\sum_{i+j=n} x^i y^j = \tfrac{x^{n+1}-y^{n+1}}{x-y},$$

we see that

$$a_n = \sum_{i+j=n} \alpha^{-i}\beta^{-j} = \tfrac{\beta^{-(n+1)}-\alpha^{-(n+1)}}{\beta^{-1}-\alpha^{-1}} = \tfrac{\beta^{n+1}-\alpha^{n+1}}{\alpha^n\beta^n(\beta-\alpha)}.$$

With this description of the coefficients a_n we check to see when (6.3) is satisfied. Each of the following inequalities are equivalent.

$$\begin{aligned}
\frac{3}{4}\left(\frac{\beta^{n+1}-\alpha^{n+1}}{\alpha^n\beta^n(\beta-\alpha)}\right)^2 - \frac{\beta^{2n+1}-\alpha^{2n+1}}{\alpha^{2n}\beta^{2n}(\beta-\alpha)} &< 0 \\
3(\beta^{n+1}-\alpha^{n+1})^2 - 4(\beta-\alpha)(\beta^{2n+1}-\alpha^{2n+1}) &< 0 \\
3(\beta^{2n+2}-2(\beta\alpha)^{n+1}+\alpha^{2n+2}) - 4(\beta^{2n+2}-\beta\alpha^{2n+1}-\alpha\beta^{2n+1}+\beta^{2n+2}) &< 0 \\
-\beta^{2n+2}-\alpha^{2n+2}+4\beta\alpha^{2n+1}-6(\beta\alpha)^{n+1}+4\alpha\beta^{2n+1} &< 0 \\
\left(\frac{4}{r}-\beta^2\right)\beta^{2n} + \left(\frac{4}{r}-\alpha^2\right)\alpha^{2n} - \frac{6}{r^{n+1}} &< 0 \quad (6.4)
\end{aligned}$$

So $A^{(n)}$ is GS if and only if inequality (6.4) is satisfied.

Recall that

$$\alpha = \frac{m-\sqrt{m^2-4r}}{2r},$$

and notice

$$\frac{4}{r}-\alpha^2 = \frac{4-r\alpha^2}{r} > 0,$$

since $r\alpha^2 \leq 1$ by (6.2).

Using

$$\beta = \frac{m+\sqrt{m^2-4r}}{2r}$$

along with our assumption that $r < m^2/4$ we see that we have the following equivalence of inequalities.

$$\begin{aligned}
\frac{4}{r}-\beta^2 \leq 0 \quad &\Longleftrightarrow \quad \beta^2 \geq 4/r = 4\alpha\beta \\
&\Longleftrightarrow \quad \beta \geq 4\alpha \\
&\Longleftrightarrow \quad r \leq \frac{4}{25}m^2
\end{aligned}$$

Note that the last equivalence was seen in the proof of Theorem 14.

Case 1: Assume that $r \leq \frac{4}{25}m^2$. By the above computation,

$$\frac{4}{r} - \beta^2 \leq 0.$$

By Lemma 38,

$$\left(\frac{4}{r} - \alpha^2\right)\alpha^{2n} - \frac{6}{r^{n+1}} \leq 0, \text{ for all } n \geq 0.$$

Therefore, inequality (6.4) holds for all $n \geq 0$.

Case 2: Assume that $r > \frac{4}{25}m^2$. In this case

$$\beta,\ \alpha,\ \frac{4}{r} - \alpha^2,\ \frac{4}{r} - \beta^2 > 0.$$

Note that $\beta^2 > \alpha\beta = 1/r$ and hence

$$\left(\frac{4}{r} - \beta^2\right)\beta^{2n} - \frac{6}{r^{n+1}} = \left(\frac{4}{r} - \beta^2\right)(\beta^2)^n - \frac{6}{r} \cdot \frac{1}{r^n} > 0$$

for all $n \gg 0$. Therefore, inequality (6.4) fails to hold for all $n \gg 0$.

□

Bibliography

[1] D. J. Anick. Generic algebras and CW complexes. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 247–321. Princeton Univ. Press, Princeton, NJ, 1987.

[2] J. Backelin. On the rates of growth of the homologies of Veronese subrings. In *Algebra, algebraic topology and their interactions (Stockholm, 1983)*, volume 1183 of *Lecture Notes in Math.*, pages 79–100. Springer, Berlin, 1986.

[3] J. Backelin and R. Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.*, 30(2):85–97, 1985.

[4] M. Ershov. Golod-Shafarevich groups with property (T) and Kac-Moody groups, preprint.

[5] E. S. Golod. On nil-algebras and finitely approximable p-groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:273–276, 1964.

[6] E. S. Golod and I. R. Šafarevič. On the class field tower. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:261–272, 1964.

[7] K. Haberland. *Galois cohomology of algebraic number fields*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978. With two appendices by Helmut Koch and Thomas Zink.

[8] D. A. Každan. On the connection of the dual space of a group with the structure of its closed subgroups. *Funkcional. Anal. i Priložen.*, 1:71–74, 1967.

[9] H. Koch. Zum Satz von Golod-Schafarewitsch. *Math. Nachr.*, 42:321–333, 1969.

[10] H. Koch. *Galoissche Theorie der p-Erweiterungen*. Springer-Verlag, Berlin, 1970.

[11] M. Lackenby. Large groups, Property (tau) and the homology growth of subgroups, preprint. *arXiv:math.GR/0509036*.

[12] J. Lewin. Free modules over free algebras and free group algebras: The Schreier technique. *Trans. Amer. Math. Soc.*, 145:455–465, 1969.

[13] A. Lubotzky. Group presentation, p-adic analytic groups and lattices in $SL_2(\mathbf{C})$. *Ann. of Math. (2)*, 118(1):115–130, 1983.

[14] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.

[15] A. Lubotzky. Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. *Ann. of Math. (2)*, 144(2):441–452, 1996.

[16] A. Lubotzky and E. Zelmanov. Dimension expanders. *To appear in the J. of Algebra.*

[17] A. Lubotzky and A. Zuk. *On property (τ).* Book in preparation—available at http://www.ma.huji.ac.il/∼alexlub.

[18] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[19] Y. I. Manin. *Quantum groups and noncommutative geometry.* Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.

[20] A. Polishchuk and L. Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005.

[21] P. Roquette. On class field towers. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965, editors: J. Cassels and A. Fröhlich)*, pages 231–249. Thompson, Washington, D.C., 1967.

[22] J.-P. Serre. Le problème des groupes de congruence pour SL2. *Ann. of Math. (2)*, 92:489–527, 1970.

[23] È. B. Vinberg. On the theorem concerning the infinite-dimensionality of an associative algebra. *Izv. Akad. Nauk SSSR Ser. Mat.*, 29:209–214, 1965.

[24] T. Voden. Subalgebras of Golod-Shafarevich algebras, preprint.

[25] E. Zelmanov. On groups satisfying the Golod-Shafarevich condition. In *New horizons in pro-p groups (editors: M. du Sautoy, et. al.)*, volume 184 of *Progr. Math.*, pages 223–232. Birkhäuser Boston, Boston, MA, 2000.

[26] E. Zelmanov. Infinite algebras and pro-p groups. In *Infinite groups: geometric, combinatorial and dynamical aspects (editors: L. Bartholdi, et. al.)*, volume 248 of *Progr. Math.*, pages 403–413. Birkhäuser, Basel, 2005.

[27] E. I. Zel′manov. Lie ring methods in the theory of nilpotent groups. In C. M. Campbell, editor, *Groups '93 Galway/St. Andrews, Vol. 2 (editors: C. M. Campbell, et. al.)*, volume 212 of *London Math. Soc. Lecture Note Ser.*, pages 567–585. Cambridge Univ. Press, Cambridge, 1995.