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Auslander-Reiten Theory for Equivariant Sheaves on Trees

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AUSLANDER-REITEN THEORY FOR EQUIVARIANT SHEAVES ON TREES

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in

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by

Samuel A. Johnson

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Abstract

Auslander-Reiten Theory for Equivariant Sheaves on Trees

by

Samuel A. Johnson

In this thesis, we investigate the category of equivariant sheaves on a locally finite tree. We view this setting as an interesting junction between the representation theory of p-adic groups, and the geometric methods coming from quiver theory. Inspired by methods from quiver theory, we show that the category of (smooth) equivariant sheaves on a tree X is equivalent to a certain hereditary module category $\mathcal{H}_X$-mod. We construct $\mathcal{H}_X$ explicitly as the endomorphism ring of a projective generator, which emulates the construction of path algebras for quivers, and Hecke algebras for p-adic groups. In this light, we investigate the implications of Auslander-Reiten theory in this context. In particular we provide a thorough investigation of a large class of sheaves known as the pre-injective component, and give recursive formulae for their cohomology.

To my family,

Mark and Helen Johnson

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Chapter 1

Sheaves on Posets

Let $(X, \leq)$ be a partially ordered set and write $\mathrm{Vec}_{\mathbb{C}}$ for the category of complex vector spaces. The canonical reference for the following topics is Curry's thesis [Cur14], which we refer to for several important results.

Definition 1.0.1. A **sheaf** (of complex vector spaces) on X is a functor,

$$S : (X, \leq) \to \mathrm{Vec}_{\mathbb{C}}$$

This is given by the following data,

(i) For each $x \in X$, a complex vector space S_x called the **stalk** at x.

(ii) For each relation $x \leq y$, a linear map $S(x \leq y) : S_x \to S_y$ called a **restriction** map. If S is clear from the context, we will often write $\gamma_{x,y} : S_x \to S_y$ in place of the notation $S(x \leq y)$.

This data is subject to the usual associativity condition for functors. The dual notion is called a **cosheaf**, which is a functor out of the opposite category,

$$\widehat{S} : (X, \leq)^{\mathrm{op}} \to \mathrm{Vec}_{\mathbb{C}}$$

Definition 1.0.2. A morphism of sheaves $f : S \to T$ is a **natural transformation** of functors. This is given by the following data,

(i) For each $x \in X$, a linear map $f_x : S_x \to T_x$ such that whenever $x \leq y \in X$, the following square commutes,

$$\begin{tikzcd} S_x \arrow[r, "f_x"] \arrow[d, "S(x\leq y)"'] & T_x \arrow[d, "T(x\leq y)"] \\ S_y \arrow[r, "f_y"] & T_y \end{tikzcd}$$

We write $\mathrm{Sh}(X)$ for the category of sheaves on $(X, \leq)$.

Example 1.0.3. Let $U \subset X$ be a subset satisfying,

$$[x \in U \text{ and } x \leq y] \Rightarrow y \in U$$

Such a set is often called an **upper set**. We write $\mathbb{C}_U$ for the constant sheaf on U. That is the sheaf with,

$$[\mathbb{C}_U]_x = \begin{cases} \mathbb{C} \text{ if } x \in U \\ 0 \text{ otherwise} \end{cases}$$

where all relevant restriction maps are the identity. If U is connected then this sheaf is **indecomposable** in the sense that $\mathbb{C}_U \ncong S \oplus T$ for some nonzero sheaves S, T. However in general it is not **simple**. One can check that if $V \subset U$ is a subset again satisfying,

$$[x \in V \text{ and } x \leq y] \Rightarrow y \in V$$

then $\mathbb{C}_V \hookrightarrow \mathbb{C}_U$. Moreover every subsheaf of $\mathbb{C}_U$ is of this form.

Remark 1.0.4. For those who have encountered sheaves before, let us make a note justifying the above definition. Posets $(X, \leq)$ are endowed with a topology known as the **Alexandrov topology**. It is defined so that a subset $U \subset X$ is open if and only if U satisfies the condition,

$$[x \in U \text{ and } x \leq y] \Rightarrow y \in U$$

In the language of the previous example, open sets in this topology are precisely the "upper sets". One sees then that an open basis for this topology is given by the sets,

$$U_x := \{y \in X \mid x \leq y\}$$

called the **open star of** x and that any open set $U \ni x$ has $U_x \subset U$. In particular this implies that any poset $(X, \leq)$ will be locally compact in this topology. However, $(X, \leq)$ is essentially never Hausdorff. For instance if $X = \{x < y\}$, then $U_x = X$ so there is no way to pick disjoint open sets separating x and y.

Now if S is a pre-sheaf on $(X, \leq)$ in the usual sense; meaning S is a functor,

$$S : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Vec}_{\mathbb{C}}$$

then the stalk at x is given by,

$$S_x := \varinjlim_{U \ni x} S(U) = S(U_x)$$

and hence Def. 1.0.1 determines a pre-sheaf on X by setting for $U \subset X$ open,

$$S(U) = \varprojlim_{x \in U} S_x$$

In general, sheaves are just pre-sheaves satisfying some additional glueing conditions. In this setting the topology is coarse enough so that any pre-sheaf automatically satisfies these conditions,

Proposition 1.0.5. If $(X, \leq)$ is a locally finite poset and let S be a pre-sheaf on X as above. Then S is in fact a sheaf.

Proof. See [Cur14], Section 4.2.2. $\square$

Definition 1.0.6. We will say that a poset $(X, \leq)$ is **locally finite** if for all $x \in X$, both the open star $U_x = \{y \in X \mid x \leq y\}$ and the closure $\overline{x} = \{y \in X \mid y \leq x\}$ are finite sets. In particular, this condition ensures that a subset $U \subset X$ is compact if and only if it is finite.

Example 1.0.7. Consider the natural numbers $(\mathbb{N}, \leq)$ with the usual ordering. Here, $U_1 = \{n \in \mathbb{N} \mid n \geq 1\} = \mathbb{N}$ which is clearly not a finite set. However, it turns out that in this topology $\mathbb{N}$ is compact. Any open cover must cover the element 1, and any open set containing 1 has $U_1 = \mathbb{N}$ as a subset.

1.1 Push-Pull

In the following we let $f : (X, \leq) \to (Y, \leq)$ be a map of posets, i.e. a map for which $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. Note that these are precisely the maps which are continuous in the Alexandrov topology.

Definition 1.1.1. Let $f : X \to Y$ as above. Associated to f is a functor,

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$$

called the **pullback** along f. For $T \in \mathrm{Sh}(Y)$ let $f^*T \in \mathrm{Sh}(X)$ be the sheaf defined by,

(i) For each $x \in X$, $f^*T_x = T_{f(x)}$

(ii) For $x_1 \leq x_2 \in X$, $f^*T(x_1 \leq x_2) = T(f(x_1) \leq f(x_2))$

This functor has a right adjoint, called the **pushforward** along f,

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

where for $S \in \mathrm{Sh}(X)$, let $f_*S \in \mathrm{Sh}(Y)$ be the sheaf such that,

(i) For all $y \in Y$, $[f_*S]_y = \varprojlim_{x \in f^{-1}(U_y)} S_x$

(ii) When $y_1 \leq y_2 \in Y$, one has $f^{-1}(U_{y_2}) \subseteq f^{-1}(U_{y_1})$ so $[f_*S](y_1 \leq y_2) = f_*S_{y_1} \to f_*S_{y_2}$ will be the map induced by the universal property of limits.

Example 1.1.2. Let $X = \bigsqcup_{i \in I} x_i$ be a discrete set and let $\pi : X \to *$ be the projection of X to a point. Here $\mathrm{Sh}(*) \cong \mathrm{Vec}_{\mathbb{C}}$ and if V is a vector space, pulling back to X gives

$$[\pi^* V]_x = V$$

for all $x \in X$. A sheaf $S \in \mathrm{Sh}(X)$ is just the data of vector spaces S_x indexed by the points $x \in X$. So pushing forward along π gives the vector space,

$$[\pi_* S] = \prod_{x \in X} S_x$$

The universal property of products says that a map $V \to \prod_{x \in X} S_x$ is uniquely determined by a collection of maps $V \to S_x$ for all $x \in X$. In other words,

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\pi^* V, S) \cong \mathrm{Hom}_{\mathbb{C}}(V, \pi_* S)$$

Proposition 1.1.3. The functors f_* and f^* form an adjoint pair,

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^* S, T) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(S, f_* T)$$

Proof. See [Cur14], Section 5.3. ☐

Definition 1.1.4. It turns out that in this setting f^* also has a left adjoint, called the **pushforward with open support**. Let $\overline{y} := \{y' \in Y \mid y' \leq y\}$ denote the closure of a cell $y \in Y$. Given $S \in \mathrm{Sh}(X)$, let $f_\dagger S \in \mathrm{Sh}(Y)$ be the sheaf with,

1. For each $y \in Y$, $[f_\dagger S]_y = \varinjlim_{x \in f^{-1}(\overline{y})} S_x$
2. When $y_1 \leq y_2 \in Y$, one has $\overline{y_1} \subseteq \overline{y_2}$ and hence $f^{-1}(\overline{y_1}) \subseteq f^{-1}(\overline{y_2})$. So, $[f_\dagger S](y_1 \leq y_2) = f_\dagger S_{y_1} \to f_\dagger S_{y_2}$ will be the map induced by the universal property of colimits.

Example 1.1.5. Let $X = \bigsqcup_{i \in I} x_i$ and $\pi : X \to *$ as before. Given $S \in \mathrm{Sh}(X)$,

$$[\pi_\dagger S] = \bigoplus_{x \in X} S_x$$

The universal property of coproducts says that a map $\bigoplus S_x \to V$ is uniquely determined by a collection of maps $S_x \to V$ for all $x \in X$. In other words,

$$\mathrm{Hom}_{\mathbb{C}}(\pi_\dagger S, V) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(S, \pi^* V)$$

Proposition 1.1.6. The functors $f_\dagger$ and f^* form an adjoint pair,

$$\mathrm{Hom}_{\mathrm{Sh}(Y)}(f_\dagger S, T) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(S, f^* T)$$

Proof. See [Cur14], Section 5.3. ☐

Definition 1.1.7. Given $S \in \mathrm{Sh}(X)$, let $f_! S \in \mathrm{Sh}(Y)$ be the sheaf with,

1. For each $y \in Y$, $[f_! S]_y = \{s \in [f_* S]_y \mid \mathrm{supp}(s) \text{ compact}\}$

we call $f_!$ the **pushforward with compact support** along f. Note that by definition $f_! S$ is a sub-sheaf of $f_* S$. We should also note that the definition here differs from that of [Cur14], but the two are equivalent for the cases we will be interested in (i.e. for the face poset of a closed CW-complex).

Remark 1.1.8. The previous example of $X = \sqcup x_i \to *$ is somewhat misleading for the general picture. In this case points are both open and closed in X, so the pushforward with open support coincides with pushforward with compact support,

$$\pi_\dagger S = \bigoplus_{x \in X} S_x = \pi_! S$$

We will see examples later where these do not coincide. Nonetheless this example is helpful because it models the following situation.

Proposition 1.1.9. Suppose $f : X \to Y$ has discrete fibers, which is to say that for all $y \in Y$, $f^{-1}(U_y) = \bigsqcup_{f(x)=y} U_x$. Then we have,

$$\mathrm{Hom}_{\mathrm{Sh}(Y)}(f_! S, T) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(S, f^* T)$$

Proof. Since f has discrete fibers, for any $S \in \mathrm{Sh}(X)$ the pushforward has the property that,

$$[f_* S]_y = \prod_{f(x)=y} \varprojlim_{x' \in U_x} S_{x'} = \prod_{f(x)=y} S(U_x) = \prod_{f(x)=y} S_x$$

In particular if we consider the subsheaf $f_! S \subseteq f_* S$ then, $[f_! S]_y = \bigoplus_{f(x)=y} S_x$. So to give a homomorphism $F \in \mathrm{Hom}_{\mathrm{Sh}(Y)}(f_! S, T)$ is to give for each $y \in Y$ a linear map,

$$F_y : \bigoplus_{f(x)=y} S_x \to T_y$$

such that whenever $y \leq y'$ one has a commutative square,

$$\begin{tikzcd}
\bigoplus_{f(x)=y} S_x \arrow[r, "F_y"] \arrow[d] & T_y \arrow[d, "T(y\leq y')"] \\
\bigoplus_{f(x')=y'} S_{x'} \arrow[r, "F_{y'}"'] & T_{y'}
\end{tikzcd}$$

but now,

$$\mathrm{Hom}_{\mathbb{C}}(\bigoplus_{f(x)=y} S_x, T_y) \cong \prod_{f(x)=y} \mathrm{Hom}_{\mathbb{C}}(S_x, T_y) \cong \prod_{f(x)=y} \mathrm{Hom}_{\mathbb{C}}(S_x, f^*T_x)$$

which is to say that giving F is the same as giving a morphism in $\mathrm{Hom}_{\mathrm{Sh}(X)}(S, f^*T)$. So we have shown in this case that,

$$\mathrm{Hom}_{\mathrm{Sh}(Y)}(f_!S, T) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(S, f^*T)$$

$\square$

1.2 Sections

Definition 1.2.1. Let $\pi : X \to *$ denote the projection of X to a point and let $S \in \mathrm{Sh}(X)$. The **global sections** of S are defined by,

$$\Gamma(X; S) := \pi_* S = \varprojlim_{x \in X} S_x$$

$$\Gamma_c(X; S) := \pi_! S \subseteq \Gamma(X; S)$$

Here Γ_c denotes global sections with compact support. More generally for any open set $U \subset X$ we write,

$$S(U) = \Gamma(U; S|_U) = \varprojlim_{x \in U} S_x$$

for the sections of S over U.

Remark 1.2.2. We will be interested mainly in the case where $(X, \leq)$ is the face poset of a graph. By graph we mean a one dimensional simplicial complex. We will avoid the case where X has "loops", meaning a single edge attached to a single vertex, for here the notation should be modified a bit. We also make no assertions as to whether the graph is directed, labeled etc. Here we will take the convention to order the simplices such that **vertex** $\leq$ **edge**, meaning $v \leq e$ if v is a vertex attached to an edge e. So, let's make a note of what sections look like in this case. The projective limit,

$$\Gamma(X; S) := \pi_* S = \varprojlim_{x \in X} S_x$$

is characterized by a universal property. It is a vector space endowed with "projections" (not necessarily surjective) onto the stalks of each cell,

$$\begin{tikzcd}
 & \Gamma(X;S) \arrow[dl] \arrow[d] \arrow[dr] & \\
S_x \arrow[r] & S_e & S_y \arrow[l]
\end{tikzcd}$$

such that all of the relevant triangles commute. In particular, let's suppose $\overline{e} = \{x < e > y\}$ is a closed edge. Then a section on $\overline{e}$ is the data of a vector in each stalk such that the restriction maps $S_x \to S_e \leftarrow S_y$ agree on S_e. In general if X is a graph, a global section on X is the data of vectors in all vertex stalks whose restrictions agree on all edges. We should also note that by the push-pull adjunction,

$$\Gamma(X; S) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \Gamma(X; S)) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbb{C}_X, S)$$

where $\mathbb{C}_X$ denotes the constant sheaf on X. The following fact also holds quite generally for sheaves, but the proof for posets is trivial.

Proposition 1.2.3. For all $s \in S(U)$, $\mathrm{supp}(s)$ is closed in U.

Proof. For all $x \leq y \in U$ we have a commuting triangle,

$$\begin{tikzcd} S(U) \arrow[d] \arrow[dr] & \\ S_x \arrow[r] & S_y \end{tikzcd}$$

Note that if $s_x = 0 \in S_x$, then so is $s_y \in S_y$. This shows that the complement of $\mathrm{supp}(s)$ is open in U. $\square$

Remark 1.2.4. As is noted by Curry in [Cur18], the existence of $f_\dagger$ gives rise to a unusual symmetry for sheaves on posets. In particular, we will see later that it gives rise to a notion of sheaf homology; a phenomena not usually available over general topological spaces. In any case, let's take $(X, \leq)$ to be connected and let $\pi : X \to *$ as above. Then for $S \in \mathrm{Sh}(X)$ we consider homs into the constant sheaf on X,

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(S, \mathbb{C}_X) \cong \mathrm{Hom}_{\mathbb{C}}(\pi_\dagger S, \mathbb{C})$$

which we are inclined to call **global cosections** of S. By adjunction this space is dual to the vector space,

$$\widetilde{\Gamma}(X; S) := \pi_\dagger S = \varinjlim_{x \in X} S_x = \bigoplus_{x \in X} S_x \Big/ \langle s_x - \gamma_{x,y}(s_x) \rangle$$

Here we are using $\gamma_{x,y} : S_x \to S_y$ as shorthand for the restriction map $S(x \leq y)$. and we have a pairing,

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbb{C}_X, S) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sh}(X)}(S, \mathbb{C}_X) \to \mathrm{End}(\mathbb{C}_X)$$

$$\Gamma(X; S) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\widetilde{\Gamma}(X; S), \mathbb{C}) \to \mathbb{C}$$

One might ask whether this pairing is perfect, which would imply $\Gamma(X; S) \cong \widetilde{\Gamma}(X; S)$. In general it is highly unlikely for the limit and colimit of a diagram to coincide. Nonetheless, the extent to which Γ and $\widetilde{\Gamma}$ differ will be a motivating question for the discussion later on, so let us look at some easy examples.

Example 1.2.5. Let $X = \{x < e > y\}$ and let $S = \mathbb{C}_X$. Here one checks that,

$$\Gamma(X; S) = \ker(\mathbb{C}_x \oplus \mathbb{C}_y \to \mathbb{C}_e) \cong \mathbb{C}$$

$$\widetilde{\Gamma}(X; S) = [\mathbb{C}_x \oplus \mathbb{C}_e \oplus \mathbb{C}_y] \Big/ \langle [1, -1, 0], [0, -1, 1] \rangle \cong \mathbb{C}$$

It should not be surprising that these coincide in this case. After all, this corresponds to the fact that multiplication,

$$\mathrm{End}(\mathbb{C}_X) \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{C}_X) \to \mathrm{End}(\mathbb{C}_X)$$

is a perfect pairing ($\mathrm{End}(\mathbb{C}_X) \cong \mathbb{C}$). Now consider $S = \mathbb{C}_{U_x}$ which is the constant sheaf on $U_x = \{x < e\}$. In this case one sees,

$$\Gamma(X; S) = 0 \text{ and } \widetilde{\Gamma}(X; S) \cong \mathbb{C}$$

which again corresponds to the observation that,

$$\mathrm{Hom}(\mathbb{C}_X, S) = 0 \text{ but } \mathrm{Hom}(S, \mathbb{C}_X) \cong \mathbb{C}$$

Finally, consider the case $S = \mathbb{C}_x$ of the skyscraper sheaf at x. Here,

$$\Gamma(X; S) = \mathbb{C} \text{ and } \widetilde{\Gamma}(X; S) = 0$$

Definition 1.2.6. $S \in \mathrm{Sh}(X)$ is **flabby** if the restriction map $S(X) \to S(U)$ is surjective for any open set U.

Example 1.2.7. Let X be connected and let $U \subset X$ be a proper open subset. Since U is open, the constant sheaf $\mathbb{C}_U$ has no global sections, $S(X) = 0$, but over U we have $S(U) = \mathbb{C}$. So the map $S(X) \to S(U)$ is not surjective. In general one can show that flabby sheaves have closed support.

Proposition 1.2.8. Let $S \in \mathrm{Sh}(X)$ be a flabby sheaf for which every global section vanishes at at least one point $x \in X$. Then $\widetilde{\Gamma}(X; S) = 0$.

Proof. We will show that the map,

$$S_x \to \widetilde{\Gamma}(X; S) := \varinjlim S_x$$

is identically zero for all $x \in X$. The universal properties of limits/colimits induce diagrams like the following,

$$\begin{tikzcd} & \Gamma(X;S) \arrow[dl] \arrow[d] \arrow[dr] & \\ S_x \arrow[r] \arrow[dr] & S_y \arrow[d] & S_z \arrow[l] \arrow[dl] \\ & \widetilde{\Gamma}(X;S) & \end{tikzcd}$$

which depicts $x < y > z \in X$. The point being that all the triangles in this diagram commute, so if $s \in \Gamma(X; S)$ then $s_x \equiv s_y \equiv s_z \in \widetilde{\Gamma}(X; S)$. Conversely if S is flabby, then we can lift any $s_x \in S_x$ to a global section $s \in \Gamma(X; S)$. Note that if s vanishes somewhere in X, then since X is locally finite there exists a finite sequence $x < x_1 > \cdots < x_n$ with $s_x \equiv s_{x_1} \equiv \cdots \equiv s_{x_n} = 0 \in \widetilde{\Gamma}(X; S)$.

□

Proposition 1.2.9. The functors $\Gamma(X; S)$ and $\widetilde{\Gamma}(X; S) : \mathrm{Sh}(X) \to \mathrm{Vec}_{\mathbb{C}}$ are left exact.

Proof. This follows from the left exactness of $\mathrm{Hom}(\mathbb{C}_X, -)$ and $\mathrm{Hom}(-, \mathbb{C}_X)$. □

Chapter 2

Trees

In this chapter we discuss some background for group actions on trees. For a more thorough treatment of these topics one can look to [Ser02], [AB08], [BH13].

Definition 2.0.1. A **tree** X is a simply connected graph. We write $V(X)$ and $E(X)$ for the collection of vertices and edges in X respectively. We say X is **locally finite** if for each $v \in V(X)$, the set of edges $e > v$ is finite.

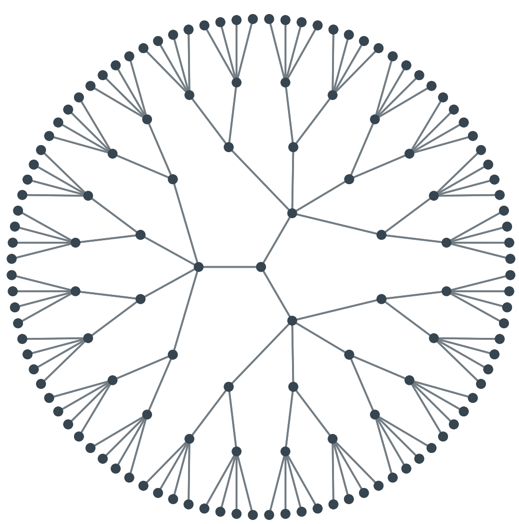


Figure 2.1: A ball of radius 4 in a tree

2.1 Trees as Euclidean buildings

Definition 2.1.1. An **apartment** $\mathcal{A} \subset X$ is a subset which is a 2-regular tree, i.e. a line in X. A **chamber** $C = \{x < e > y\}$ is a closed edge in X.

In the discussion that follows, we consider a group G acting on a tree X via graph automorphisms. Here we impose some restrictions on G via its action on X. These restrictions amount to saying G is a group with a (B, N)-pair, and that X is the Euclidean building associated to this pair. For more discussioon of the following topics see [AB08]. First and foremost we assume that,

- X is a connected, locally finite tree without leaves. In particular $X = \cup\mathcal{A}$ is a union of apartments, and any two points $x, y \in X$ are contained in a common apartment.
- G acts faithfully on X.

The most important assumption is the following,

- G acts transitively on the set of pairs $(\mathcal{A}, C)$ with $C \subset \mathcal{A}$. One says that the action of G on X is **strongly transitive**. We fix a fundamental pair $(\mathcal{A}, C)$ and consider the subgroups,

$$I := \{g \in G \mid gC = C\} = \mathrm{Stab}_G(C)$$

$$N := \{g \in G \mid g\mathcal{A} = \mathcal{A}\} = \mathrm{Stab}_G(\mathcal{A})$$

$$T := \{g \in G \mid ga = a \text{ for all } a \in \mathcal{A}\} = \mathrm{Fix}_G(\mathcal{A})$$

  We can refine the chamber structure on X such that, $I = \mathrm{Stab}_G(C) = \mathrm{Fix}_G(C)$ (meaning we can subdivide the edges if needed). In this case we get,

$$I \cap N = T$$

We assume further that I and N generate G, and that

$$W := N/T \cong D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle$$

which is generated by reflections in the vertices of a fundamental chamber C. In other words $\mathcal{A}$ is the Coxeter complex associated to the affine Coxeter group D_∞.

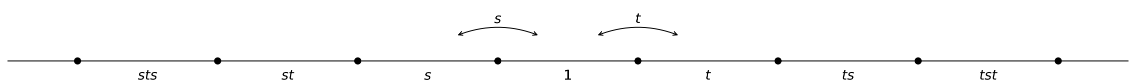


Figure 2.2: An apartment with chambers labelled by elements of W

In particular these assumptions imply that the tree X is regular or bi-regular. Another important consequence is the following theorem.

Theorem 2.1.2. Let G be as above. Then G has a Bruhat decomposition,

$$G = \bigsqcup_{w \in W} IwI$$

Proof. Strong-transitivity says that G acts transitively on pairs $(\mathcal{A}, C)$ with $C \subset \mathcal{A}$. In particular the set of chambers in X can be identified with G/I. Moreover for a fixed chamber C, I acts transitively on the set of apartments containing C, $(-, C) = \{\mathcal{A} \supset C\}$. In other words every if we fix $(\mathcal{A}, C)$ with $C \subset \mathcal{A}$, then every I orbit of chambers in X is represented by a chamber in $\mathcal{A}$. W acts faithfully-transitively on the set of chambers in $\mathcal{A}$ and hence the map,

$$W \to I \backslash G / I$$

is at least surjective. It is injective by the assumption $I \cap N = T$, which says that any element of I stabilizing $\mathcal{A}$ fixes it point-wise. It follows that the I double cosets in G

are in bijection with the set of chambers in $\mathcal{A}$,

$$I\backslash G/I \leftrightarrow (\mathcal{A}, -) := \{C \subset \mathcal{A}\} \leftrightarrow W$$

and hence that the canonical map, $w \mapsto IwI$ is a bijection.

$\square$

Corollary 2.1.3. Let $s, t \in W$ be the reflections fixing the vertices $x, y \in C$ with C a fixed fundamental chamber in $\mathcal{A} \subset X$. Then,

$$K_x := I \sqcup IsI$$

$$K_y := I \sqcup ItI$$

are the stabilizers of x, y respectively in G. Note that $K_x \cap K_y = I$.

Proof. The previous argument implies that there are two I orbits of chambers C containing x, these orbits are represented by 1 and s. The same argument applies for y. $\square$

Remark 2.1.4. If we write $W_x = \langle s \rangle$ and $W_y = \langle t \rangle$, then the above should be thought of as a "local" Bruhat decompostion for the subgroup, $K_* = \mathrm{Stab}_G(*)$,

$$K_* = \bigsqcup_{w \in W_*} IwI$$

2.2 Stabilizer filtrations

Definition 2.2.1. As above we fix a fundamental pair $(\mathcal{A}, C)$ with $C = \{x < e > y\}$ a closed edge. Let $[x, ..., x_r]$ denote a geodesic in $\mathcal{A}$ emanating from x and write,

$$K_{x,r} := \mathrm{Stab}_G([x, ..., x_r]) = K_x \cap \mathrm{Stab}_G(x_r)$$

for the stabilizer of $[x, ..., x_r]$. By strong-transitivity, one can identify the sets

$$\{\text{geodesics } \gamma \text{ emenating from x} \mid \text{length}(\gamma) = r\} \leftrightarrow K_x/K_{x,r}$$

Then since X is locally finite, the $K_{x,r}$ give a filtration of K_x by finite index subgroups,

$$K_x := K_{x,0} \supset K_{x,1} \supset K_{x,2} \supset \cdots$$

Note that the $K_{x,r}$ are not normal subgroups in K_x. We let,

$$H_{x,r} := \text{Fix}_G(B(x,r)) = \bigcap_{g \in K_x/K_{x,r}} gK_{x,r}g^{-1}$$

so the $H_{x,r}$ give a filtration of K_x by finite index normal subgroups.

Now choose a ray in the "opposite direction" emanating from y and define a similar filtration on K_y by finite index subgroups $K_{y,r}$. These filtrations are compatible in the sense that,

$$I_r := K_{x,r} \cap K_{y,r}$$

will be the stabilizer of a geodesic of length $2r+1$ with center $e \in C$ and,

$$H_{e,r} := H_{x,r} \cap H_{y,r} = \text{Fix}_G(B(e,r))$$

2.3 Action on the boundary

Definition 2.3.1. A **ray** is an infinite geodesic $[x, x_1, x_2, ...) \subset X$ emanating from a central vertex $x \in X$. Write R_x for the set of rays emanating from $x \in X$ and $R(X) = \cup_{x \in X} R_x$ for the complete set of rays in X.

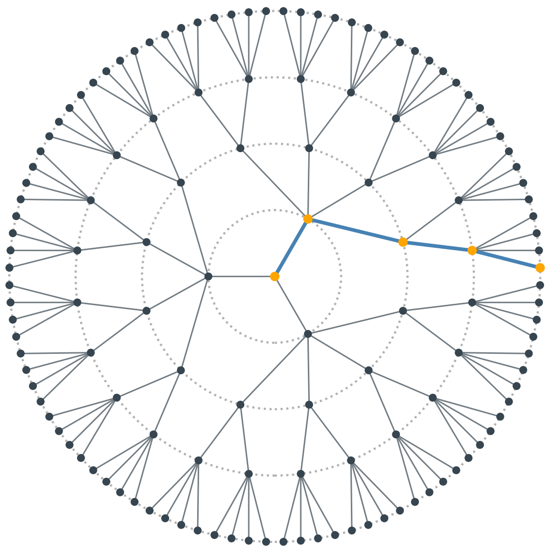


Figure 2.3: Balls of increasing radii about a central vertex, with a colored geodesic emanating from the center.

Define an equivalence relation on $R(X)$ via,

$$[x, x_1, x_2, ...) \sim [y, y_1, y_2, ...)$$

iff $x_i = y_i$ for $i \gg 0$. The **boundary** of X is then defined to be the set of equivalence classes $\partial X := R(X)/\sim$. It is clear that if X is connected, then for any $x \in X$ there is a bijection

$$R_x \longleftrightarrow \partial X$$

Note that by strong-transitivity, G (or even K_x for x a vertex) acts transitively on ∂X. We write,

$$B := \mathrm{Stab}_G(\infty)$$

for a stabilizer of a point $\infty \in \partial X$ and identify ∂X with G/B. Again by strong transitivity, for any vertex x, one has $G = K_x \cdot B$. We write,

$$B_x := K_x \cap B = \bigcap_r K_{x,r}$$

Note further that there is a bijection,

$$\{\text{apartments } \mathcal{A} \subset X\} \longleftrightarrow \{\text{pairs } (-\infty, +\infty) \in \partial X\}$$

and that N is the stabilizer of such a pair $(-\infty, \infty)$.

2.4 A structure theorem for G

In the previous sections we defined subgroup filtrations for the stabilizers of cells in X. In particular, for each vertex/edge we defined a filtration of the stabilizer by normal subgroups.

$$G_x := G_{x,0} \supset H_{x,1} \supset H_{x,2} \supset \cdots$$

where $H_{x,r} := \mathrm{Fix}_G(B(x,r))$. Since X is locally finite, this gives an inverse system of finite groups,

$$\cdots \to G_x/H_{x,3} \to G_x/H_{x,2} \to G_x/H_{x,1}$$

and by the usual universal property argument one has a map,

$$f : G_x \longrightarrow \varprojlim_{r \geq 0} G_x/H_{x,r}$$

The assumption that G acts faithfully on X implies that for any $x \in X$,

$$\bigcap_{r \geq 0} H_{x,r} = \{1\}$$

which implies f is injective. It is surjective by strong-transitivity.

$$G_x \cong \varprojlim_{r \geq 0} G_x/H_{x,r}$$

So we see that for any $x \in X$, G_x is a profinite group.

Remark 2.4.1. Profinite groups are endowed with a **profinite topology**. In our setup, a neighborhood basis of $g \in G_x$ is given by the cosets $g \cdot \mathcal{B}_x = \{gH_{x,r} \mid r \geq 0\}$. This topology makes G_x into a topological group which is compact, Hausdorff, and totally disconnected.

Together these sets extend to a basis for a topology on G. Namely, a neighborhood basis of $g \in G$ is given by,

$$g \cdot \mathcal{B} = \{gH_{x,r} \mid x \in X, r \geq 0\}$$

Proposition 2.4.2. The topology generated by $\mathcal{B}$ (and it's translates) makes G into a topological group which is locally compact, Hausdorff, and totally disconnected.

Perhaps a more natural neighborhood basis of the identity is given by the subgroups,

$$\mathcal{B}' = \{G_T = \mathrm{Stab}_G(T) \mid T \subset X \text{ compact, connected}\}$$

Let us sketch the argument that these two basis generate the same topology on G. Note that by construction everything in $\mathcal{B}$ is a finite intersection of open subgroups in $\mathcal{B}'$. To see the reverse we recall the following fact about trees,

Lemma 2.4.3. Let $T \subset X$ a be a compact connected subtree of X. In particular T is bounded and one can consider its diameter,

$$\begin{aligned}\mathrm{diam}(T) &:= \min\{r \geq 0 \mid T \subset B(x,r) \text{ for some } x \in X\} \\ &= \max\{\mathrm{length}(\gamma) \mid \gamma \subset T \text{ geodesic}\}\end{aligned}$$

This minimum is attained by a unique point $c(T) \in T$ (either a vertex or edge) called the **center** of T. Moreover $c(T)$ is the midpoint of any geodesic γ attaining the maximum diameter of T.

Corollary 2.4.4. For any compact connected subtree $T \subset X$, one has $G_T \leq G_{c(T)}$.

In particular if we let $x = c(T)$ and $r = \mathrm{diam}(T)/2$ then $H_{x,r} \leq G_T \leq G_x$ and hence,

$$G_T = \bigsqcup_{g \in G_T/H_{x,r}} gH_{x,r}$$

It follows that $\mathcal{B}'$ and $\mathcal{B}$ generate the same topology on G.

Theorem 2.4.5. (Bruhat-Tits) Any compact open subgroup $K \leq G$ has a global fixed point in X, i.e. one has $K \leq G_x$ for some $x \in X$.

Proof. We saw that for all $x \in X$, G_x is a compact open subgroup of G. If $K \leq G$ is compact and open, one can consider K/K_x where $K_x = K \cap G_x$. Note that K_x is open and hence has finite index in K. This means that any K orbit $K \cdot x = \{kx \mid k \in K\} \leftrightarrow K/K_x$ is finite and bounded in X.

Let's fix an orbit $K \cdot x$ for some $x \in X$. Since X is a tree, one can take the convex hull of $K \cdot x$ and obtain a finite connected subtree $T \subset X$. Clearly then, $K \leq G_T$ which by our previous remarks implies $K \leq G_x$ for some $x \in T$. $\square$

Corollary 2.4.6. The maximal compact open subgroups are all of the form G_x for $x \in V(X)$ a vertex.

Now we say more about the algebraic structure of G. Note that by strong transitivity, there is a surjective map,

$$C \to G \backslash X$$

where $C = \{x < e > y\}$ is our fixed fundamental chamber from before. The assumption that $I = \mathrm{Stab}_G(C) = \mathrm{Fix}_G(C)$ implies that C is a fundamental domain, so we can identify $G\backslash X$ with C. "Sitting above" each point in C we have the stabilizing subgroup,

$$\begin{tikzcd}
K_x \arrow[d, dash, dashed] & I \arrow[l, hook'] \arrow[r, hook] \arrow[d, dash, dashed] & K_y \arrow[d, dash, dashed] \\
x \arrow[r] & e & y \arrow[l]
\end{tikzcd}$$

Historically people refer to this data as a **graph of groups**. In [Ser02] Bass and Serre developed this notion for general groups acting on trees. The following theorem is a special case of what is known as the fundamental theorem of Bass-Serre theory,

Theorem 2.4.7. *G can be identified with the colimit (i.e. the pushout) of $\mathcal{C} = \{K_x \hookleftarrow I \hookrightarrow K_y\}$,*

$$\begin{tikzcd}
I \arrow[r, hook, "\iota_y"] \arrow[d, hook, "\iota_x"'] & K_y \arrow[d] \arrow[ddr, bend left] & \\
K_x \arrow[r] \arrow[drr, bend right] & K_x *_I K_y \arrow[dr, "\cong"] & \\
 & & G
\end{tikzcd}$$

In particular $G \cong K_x *_I K_y$ is an amalgamated free product,

$$K_x *_I K_y := K_x * K_y \big/ \langle \iota_x(i) \cdot \iota_y(i)^{-1} \mid i \in I \rangle$$

Proof. This is Theorem 6 in [Ser02], so we will mostly follow the proof given there. The existence of a map,

$$f : \varinjlim_{x \in C} G_x \to G$$

is guaranteed by the universal property of colimits. We will show that this map is an isomorphism. First we show surjectivity, for which we use the fact that X is connected. Suppose $g \in G$ and consider the unique geodesic $[e, e_1, e_2, ..., e_{n-1}, ge]$ between e and ge. Since G acts by graph automorphisms, $e' = g^{-1} \cdot e_{n-1}$ is adjacent to e. Moreover since G acts transitively on the edges in X, there is an element $k_1 \in G$ such that $k_1 \cdot e = e'$. Note that since e' is adjacent to e, we must have $k_1 \in K_x \cup K_y$. So we have, $gk_1 \cdot e = e_{n-1}$

which is clearly unchanged if we replace k_1 by $k_1 i$ for some $i \in I = \mathrm{Stab}_G(e)$. Repeating this argument, we obtain a sequence $k_1, ..., k_n \in K_x \cup K_y$ such that,

$$(gk_1 \cdots k_n) \cdot e = e$$

which implies $gk_1 \cdots k_n = i \in I$ and hence $g = i \cdot (k_1 \cdots k_n)^{-1} \in \mathrm{Im}(f)$. So f is surjective.

Now let us show injectivity, for which we use the fact that X is a tree, so geodesics between points are unique. Choose coset representatives $R = R_x \cup R_y$ for K_x/I and K_y/I. Then any element of $K_x * K_y$ can be written as a reduced word $k_1 \cdots k_n$. The choice of representatives enables us to write,

$$k_1 \cdots k_n = \bar{k}_1 i_1 \cdots \bar{k}_n i_n$$

with $i_j \in I$. In the amalgam, these elements can be grouped however we like, and for all j, $\bar{k}_j i_j = i'_j \bar{k}'_j$ with $k'_j \in R$ and $i'_j \in I$. So, we can group all the i terms to the left and obtain,

$$k_1 \cdots k_n = i \cdot (\bar{k}_1 \cdots \bar{k}_n)$$

So every element of the amalgam has a unique expression of this form, called its **normal form**. By our previous argument, to each normal form we can associate a geodesic starting at e, and by uniqueness of geodesics two elements determine the same geodesic if and only if they have the same normal form. □

Chapter 3

Equivariant Sheaves

3.1 Group actions on posets

Definition 3.1.1. Let G be a topological group. We say G acts on $(X, \leq)$ via **poset automorphisms** if for all $x, y \in X, g \in G$, $[x \leq y] \Rightarrow [g \cdot x \leq g \cdot y]$. Such an action is **continuous** if the associated action map,

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

is continuous (where X is given the Alexandrov topology). We will always making the following additional assumptions for our group actions,

- For all $x \in X, g \in G$, $[x \leq g \cdot x] \Rightarrow [x = g \cdot x]$, i.e. $g \in G_x$. This is equivalent to the statement $[g \cdot U_x \subseteq U_x] \Leftrightarrow [g \cdot U_x = U_x]$. We say that such an action is **discrete**.

- For all $x \in X$, the stabilizer G_x acts trivially on $\overline{x} = \{y \in X \mid y \leq x\}$. That is, for all $x, y \in X$, $[x \leq y] \Rightarrow [G_y \leq G_x]$. In this case we say that G acts **without inversions**.

- For all $x, y \in X, g \in G$, $[x \leq y \text{ and } x \leq g \cdot y] \Leftrightarrow [g \in G_x]$, or equivalently, $[U_x \cap g \cdot U_x \neq \emptyset] \Leftrightarrow [g \in G_x]$. Here we say G acts **without short translations**.

Example 3.1.2. Here are some examples of actions which do not satisfy the conditions in Def 3.1.1.

- Let $(\mathbb{Z}, \leq)$ denote the integers with their usual ordering. Then $(\mathbb{Z}, +)$ acts on $(\mathbb{Z}, \leq)$ by addition, but this action is not discrete. It also has short translations. In fact by definition one can see that no short translations $\Rightarrow$ discrete.

- Let $X = \{x < y > z\}$ and let $\mathbb{Z}/2\mathbb{Z}$ act by swapping $x \leftrightarrow z$. Then $G_y = \mathbb{Z}/2\mathbb{Z}$ acts non-trivially on $X = \overline{y}$, so this is an example of an action with inversions.

- Let $(X, \leq)$ be the face poset of a regular $n-$gon ($x \leq e$ for x a vertex and e an edge). Then the cyclic group $(\mathbb{Z}/n\mathbb{Z}, +)$ acts freely on X via rotations. If we write $\bar{e} = \{x < e > y\}$ then the generator sends x to y (or y to x), so this action is has short translations.

Remark 3.1.3. If $(X, \leq)$ is locally finite (which recall means: for all $x \in X$, both U_x and $\overline{x}$ are finite sets) then discreteness is automatic. Namely if $x \leq g \cdot x$ for some $g \in G$, then $U_{g \cdot x} \subseteq U_x$. But G acts via automorphisms, so $U_{g \cdot x}$ and U_x have the same cardinality and hence $U_x = U_{g \cdot x}$. This means both $x \leq g \cdot x$ and $g \cdot x \leq x$, and hence $g \cdot x = x$. Essentially the same argument works for closures.

The short-translation condition is non-standard, and in some sense it is the least necessary for the discussion that follows. Nonetheless, let us give an explanation of why one might ask for such a condition. So far we have interchangebly used the notation $(X, \leq)$ to denote a poset both *as a set* and *as a category*. Just for this discussion let us write $(X, \leq)$ for the set and $(\mathcal{X}, \leq)$ for the category associated to $(X, \leq)$.

If G acts on $(X, \leq)$ via poset automorphisms then one can form the quotient set X/G, where $[x \sim y] \Leftrightarrow [g \cdot x = y$ for some $g \in G]$. Then X/G becomes a poset via the relation,

$$[x] \leq [y] \Leftrightarrow x \leq gy \text{ for some } g \in G$$

So one can consider the poset $(X/G, \leq)$ and the category $(\mathcal{X}/\mathcal{G}, \leq)$ as before. There is also an obvious sense in which G acts on the category $(\mathcal{X}, \leq)$. Here G acts on the objects $x \in X$ as usual, but it also acts on the morphisms $(x \leq y) \Rightarrow (gx \leq gy)$. So, one can form the quotient category $\mathcal{X}/G$. This category has the same objects as $\mathcal{X}/\mathcal{G}$, which are indexed by elements $[x] \in X/G$. The morphisms between these objects are now equivalence classes of morphisms in $(\mathcal{X}, \leq)$, namely $(x \leq y) \sim (gx \leq gy)$. Each equivalence class of such a relation gives an arrow $[x \leq y] : [x] \to [y]$ in $\mathcal{X}/G$. The issue that arises is that there are often more arrows in the categorical quotient $\mathcal{X}/G$ than in the quotient poset $\mathcal{X}/\mathcal{G}$.

Example 3.1.4. Consider our third example in Example 7. $G = \mathbb{Z}/n\mathbb{Z}$ acts transitively on both the vertices and edges in $X = \{\text{n-gon}\}$, so we write $\bar{e} = \{x < e > y\}$ for the closure of a fixed edge. In $(X/G, \leq)$ there are two elements which we denote $[x], [e]$, and $[x] \leq [e]$. So, in the category $(\mathcal{X}/\mathcal{G}, \leq)$ there is unique morphism from $[x]$ to $[e]$. However in the quotient category, $\mathcal{X}/G$ there are two equivalence classes of morphisms from $[x]$ to $[e]$ since there is no group element sending the relation $x < e$ to $y < e$. In particular one sees that $\mathcal{X}/G$ cannot be the category associated to any poset.

There is a functor,

$$F : \mathcal{X}/\mathcal{G} \to \mathcal{X}/G$$

which is the identity on objects and which sends the poset relation $[x] \leq [y]$ to its equivalence class $[x \leq y]$. This functor is always faithful. The reason for the short translations condition is the following,

Proposition 3.1.5. Suppose G acts discretely and without inversions on $(X, \leq)$. The functor F is full, i.e. surjective on morphisms if and only if G acts without short translations on $(X, \leq)$. It follows that in this case F is an isomorphism of categories.

Proof. Suppose G acts on X with short translations. Then since the action is discrete, there exists $x \neq y \in X, g \in G$ such that $x < y$ and $x < g \cdot y$ but $g \notin G_x$. Then in $\mathcal{X}/\mathcal{G}$ there is a unique arrow $[x] \leq [y] : [x] \to [y]$, but we claim that in $\mathcal{X}/G$ the relations $[x \leq y]$ and $[x \leq g \cdot y] : [x] \to [y]$ are in distinct equivalence classes. If they were equivalent, there would be an element $h \in G$ such that $(h \cdot x \leq h \cdot y) = (x \leq g \cdot y)$. This would imply $h \in G_x$ and $h \cdot y = g \cdot y$, and hence $h^{-1}g \in G_y$. But G acts without inversions, so $G_y \leq G_x$, and one sees that $g \in G_x$, which contradicts our assumption. This shows F full $\Rightarrow$ no short translations.

Now suppose F is not full. So there is some $[x] \leq [y]$ with at least two equivalence classes of morphisms $[x \leq y]$ and $[z \leq w] : [x] \to [y]$ in $\mathcal{X}/G$. In particular, there exists $h, k \in G$ such that $h \cdot z = x$ and $k \cdot w = y$. So (by replacing z with $k \cdot z$) one can choose representatives such that $x < y > z$ and $z = g \cdot x$. Now because G acts without inversions, $g \notin G_x$. So we consider $g^{-1} \cdot y > x < y$,

$$g^{-1} \cdot y \longleftarrow x \longrightarrow y \longleftarrow z$$

which is distinct from x, $x \neq g^{-1} \cdot y$ because the action is discrete. It follows that g is a short translation, and so by contrapositive no short translations $\Rightarrow F$ full

$\square$

Remark 3.1.6. For essentially everything that follows, the correct notion to use is the categorical quotient $\mathcal{X}/G$. The assumption that G acts without short translations has the following advantages. For one, we will not have to carry around equivalence classes of morphisms, which simplifies the notation and makes several arguments easier to state. It also has the advantage that $\mathcal{X}/G$ will again be the category associated to a poset, so we can continue to use some of the topological language inherent in this approach while avoiding further digression into the language of sites, topoi, etc.

Definition 3.1.7. Let G be a group acting on a poset $(X, \leq)$ as in Def 3.1.1 and let $p : X \to G\backslash X$ denote the quotient map. Assume for convenience that p admits a section $\iota : G\backslash X \hookrightarrow X$. For each $[x] \in G\backslash X$, write G_x for the stabilizer of $x = \iota([x])$ in G. If $[y] \leq [x] \in X/G$, then $\iota([y]) \leq \iota([x])$, and since G acts without inversions we have $G_x \leq G_y$. Then the subgroups G_x along with the inclusions $G_x \hookrightarrow G_y$ form a cosheaf of groups on Y which we denote,

$$\mathcal{Y} : (G\backslash X, \leq)^{\mathrm{op}} \to \mathbf{Grp}$$

We say that $\mathcal{Y}$ is the **complex of groups** associated to this action. This notion is developed more thoroughly in [BH13].

Proposition 3.1.8. Assume G acts as in Def 3.1.1 and that X is locally finite. Then the quotient map $p : X \to G\backslash X$ admits a section if and only if there is a closed **fundamental domain** D such that,

1. $G \cdot D = X$
2. $[g \cdot D \cap D \neq \emptyset] \iff [g \in G_x \text{ for some } x \in D]$

Proof. If such a fundamental domain exists, then it contains a unique representative of each G-orbit, so one can construct a section of p by sending each $[x] \in G\backslash X$ to this representative.

Now suppose there is a section $\iota : G\backslash X \hookrightarrow X$ of p and let $D = \iota(G\backslash X)$. Then D certainly satisfies $(1), (2)$, because every orbit is represented by a unique $\iota([x])$ with $[x] \in X/G$. It remains only to see that D must be closed. Suppose $y \leq x$ with $x \in D$. Then by (1) there exists $g \in G$ such that $y \in g \cdot D$. It follows that $[d] \leq [x] \in X/G$ and hence that $\iota([d]) = d \leq x = \iota([x])$. So $g \cdot d \leq x \geq d$, and hence $x \geq d \leq g^{-1} \cdot x$ which implies $g \in G_d$. It follows that $d = g \cdot d = y \in D$. So we see that D is closed. □

Corollary 3.1.9. If the G action satisfies the conditions of Def 3.1.1, and the quotient map p admits a section, then the complex of groups $\mathcal{Y}$ described above is unique up to isomorphism. That is, up to isomorphism it is independent of the chosen section ι.

Example 3.1.10. If X is the face poset of the tree described in Section 2 and G is the group described there. Then any closed chamber $C = \{x < e > y\}$ is a fundamental domain for this action. The complex of groups $\mathcal{Y}$ associated to this action is depicted as follows,

$$K_x \longleftarrow\!\!\!\rhook I \lhook\!\!\!\longrightarrow K_y$$

3.2 Equivariant sheaves

Definition 3.2.1. A G-**equivariant sheaf** on X is a sheaf $S \in \mathrm{Sh}(X)$ with the following additional data,

(i) For each $g \in G$ and $x \in X$, a linear map $g_x : S_x \to S_{gx}$ such that,

(a) $(gh)_x = g_{hx} \circ h_x$ for all $g, h \in G$

(b) $1_x = \mathrm{Id}_{S_x}$ for all $x \in X$

(c) The diagram,

$$\begin{tikzcd} S_x \arrow[r, "g_x"] \arrow[d, "\gamma_{x,y}"'] & S_{gx} \arrow[d, "\gamma_{gx,gy}"] \\ S_y \arrow[r, "g_y"'] & S_{gy} \end{tikzcd}$$

commutes for all $x \leq y$ in X.

we call this data an **equivariant structure** on S.

Definition 3.2.2. A morphism of equivariant sheaves $f : S \to T$ is a natural transformation which commutes with the action of G. That is for all $x \in X, g \in G$ the following diagram commutes,

$$\begin{tikzcd} S_x \arrow[r, "f_x"] \arrow[d, "g_x"'] & T_x \arrow[d, "g_x"] \\ S_{gx} \arrow[r, "f_{gx}"'] & T_{gx} \end{tikzcd}$$

We will write $\mathrm{Sh}_G(X)$ for the category of G-equivariant sheaves on X.

Example 3.2.3. When $X = \{*\}$ is a point carrying a trivial G-action. The data of a G-equivariant sheaf on X is the same as a representation (ρ, V) of G. In other words there is a natural identification,

$$\mathrm{Sh}_G(*) = \mathrm{Rep}(G)$$

3.3 Equivariant push-pull

Let $f : (X, \leq) \to (Y, \leq)$ be a G-equivariant map. In particular f is a map of posets, so we have already seen how one defines the various pullback/pushforward functors $\mathrm{Sh}(X) \leftrightarrow \mathrm{Sh}(Y)$. If one works with equivariant sheaves, the equivariant structures simply "come along for the ride". So, the push-pull functors associated to f can be upgraded to functors $\mathrm{Sh}_G(X) \leftrightarrow \mathrm{Sh}_G(Y)$ satisfying the usual adjunctions.

Remark 3.3.1. Let $f : (X, \leq) \to (Y, \leq)$ be a G-equivariant map and let $T \in \mathrm{Sh}_G(Y)$. The pullback f^*S inherits a G-equivariant structure via,

$$\begin{array}{ccc} (f^*T)_x & \xrightarrow{(f^*g)_x} & (f^*T)_{g\cdot x} \\ \| & & \| \\ T_{f(x)} & \xrightarrow[g_{f(x)}]{} & T_{g\cdot f(x)} \end{array}$$

which makes f^*T an equivariant sheaf on X. Morphisms of equivariant sheaves are just morphisms of sheaves which commute with this action, so by *transport du structure* one obtains a functor,

$$f^* : \mathrm{Sh}_G(Y) \to \mathrm{Sh}_G(X)$$

The pushforwards are only slightly more subtle. Recall that given $S \in \mathrm{Sh}_G(X)$, the pushforward f_*S is the sheaf with,

$$(f_*S)_y = \varprojlim_{f(x)\geq y} S_x \subseteq \prod_{f(x)\geq y} S_x$$

Namely $(f_*S)_y$ is the subspace of "restriction compatible" vectors in this product (also called sections!). By construction, the equivariant structure on S gives an isomorphism,

$$g : \prod_{f(x)\geq y} S_x \longrightarrow \prod_{g\cdot f(x)\geq g\cdot y} S_{g\cdot x}$$

but recall the G-equivariant structure necessarily commutes with all the relevant restriction maps, so it must send "compatible" vectors in this product to compatible vectors. So again we obtain a map,

$$(f_*S)_y \xrightarrow{f_*g} (f_*S)_{g\cdot y}$$

One can check that these maps satisfy the required associativity conditions, and hence that f_*S inherits a G-equivariant structure from S. In total, we get a functor at the level of equivariant sheaves as before.

$$f_* : \mathrm{Sh}_G(X) \to \mathrm{Sh}_G(Y)$$

Proposition 3.3.2. The functors (f^*, f_*) form an adjoint pair,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(f^*T, S) \cong \mathrm{Hom}_{\mathrm{Sh}_G(Y)}(T, f_*S)$$

Just for completeness let's describe the G-equivariant structure for the pushforward with open support functor. Recall that given $S \in \mathrm{Sh}_G(X)$ one defines $f_\dagger S \in \mathrm{Sh}(Y)$ to be the sheaf with,

$$(f_\dagger S)_y = \varinjlim_{x\in f^{-1}(\overline{y})} S_x = \bigoplus_{x\in f^{-1}(\overline{y})} S_x \Big/ \langle s_x - \gamma_{x,y}(s_x)\rangle$$

Here the G-equivariant structure on S gives an isomorphism,

$$g : \bigoplus_{x\in f^{-1}(\overline{y})} S_x \longrightarrow \bigoplus_{g\cdot x\in f^{-1}(\overline{g\cdot y})} S_{g\cdot x}$$

and since this map commutes with the restriction maps,

$$(s_x - \gamma_{x,y}(s_x)) \longmapsto g \cdot s_x - \gamma_{gx,gy}(g \cdot s_x)$$

So it descends to a map,

$$(f_\dagger S)_y \xrightarrow{f_\dagger g} (f_\dagger S)_{g \cdot y}$$

Again one can check that these maps satisfy the appropriate associativity conditions and that we obtain a functor,

$$f_\dagger : \mathrm{Sh}_G(X) \to \mathrm{Sh}_G(Y)$$

Moreover $f_\dagger$ satisfies the same adjunction as before.

Proposition 3.3.3. The functors $(f_\dagger, f^*)$ form an adjoint pair,

$$\mathrm{Hom}_{\mathrm{Sh}_G(Y)}(f_\dagger S, T) \cong \mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, f^* T)$$

Remark 3.3.4. Let $K \leq G$ be a topological group and suppose that the quotient G/K is discrete (so sheaves on this space fit into our framework). Let $p : G/K \to *_K$ denote the quotient map sending $gK \mapsto *_K$ where $*_K$ denotes K acting trivially on a point. Here we identify $\mathrm{Sh}_K(*_K) = \mathrm{Rep}(K)$.

By the previous discussion, given a representation (W, ρ) of K, we can pull back along p to obtain $\widetilde{W} := p^* W \in \mathrm{Sh}_K(G/K)$. Of course, the space G/K has a left G-action by translation, so in fact one can extend this K-equivariant structure to a G-equivariant structure as follows. Let us choose a system of representatives $\{h_i K\}_{i \in I}$

for the cosets in G/K and write,

$$h_i W := \widetilde{W}_{h_i K}$$

(which is really just W). In order to define a G-equivariant structure on p^*W one needs the following. Whenever $gh_iK = h_jK$, we need a linear map,

$$g_{i,j} : h_i W \to h_j W$$

Observe that $h_j^{-1} g h_i \in K$ so one can define, $g_{i,j} := \rho(h_j^{-1} g h_i) : W \to W$. These maps are clearly composable in natural way, and hence give a well defined equivariant structure on p^*W.

Now suppose we had chosen different representatives $\{h_i' K\}_{i \in I}$. Then as we have defined it above we get a map $g_{i,j}' : h_i' W \to h_j' W$ with $g_{i,j}' = \rho(h_j'^{-1} g h_i')$. These maps define another equivariant structure on p^*W, which it turns out is not so different from the first. Namely for each i, we can write $h_i = h_i' k_i$ for some $k_i \in K$. Then,

$$\rho(h_j^{-1} g h_i) = \rho(k_j^{-1} h_j'^{-1} g h_i' k_i) = \rho(k_j)^{-1} \cdot \rho(h_j'^{-1} g h_i') \cdot \rho(k_i)$$

One sees that the elements k_i provide the data of a natural transformation,

$$\begin{tikzcd} h_i W \arrow[r, "\rho(h_j^{-1} g h_i)"] \arrow[d, "\rho(k_i)"] & h_j W \arrow[d, "\rho(k_j)"] \\ h_i' W \arrow[r, "\rho(h_j'^{-1} g h_i')"] & h_j' W \end{tikzcd}$$

which gives an isomorphism between these two equivariant structures on p^*W. The result of this discussion is the following. In order to "pin-down" the points in G/K we had to choose coset representatives h_i, and once we had made these choices the G-equivariant structure on p^*T was determined. The choice of a different "pinning", i.e. of different coset representatives, gave a different equivariant structure, but these two structures differ by a canonical isomorphism. In some sense this is an artifact of the definition we give for equivariant sheaves, where the equivariant structure and the sheaf structure are seperate data. We will try to shed more light on this in the following discussion.

Definition 3.3.5. Suppose G acts on $(X, \leq)$ as in Def 3.1.1, and that this action admits a closed fundamental domain. Let $Y = G\backslash X$ and let,

$$\mathcal{Y} : (Y, \leq)^{op} \to \mathbf{Grp}$$

be the complex of groups associated to this action. A **sheaf** S on $\mathcal{Y}$ is given by the following data,

1. For each $y \in Y$, a (complex) representation (S_y, ρ_y) of the stabilizer G_y, i.e. a group homomorphism,
$$\rho_y : G_y \to GL(S_y)$$
2. For each $x \leq y$ (which recall means $G_y \leq G_x$) a G_y-equivariant restriction map $\gamma_{x,y} : S_x \to S_y$.

In other words, a sheaf on $\mathcal{Y}$ is a sheaf on $(Y, \leq)$ with compatible actions of the stabilizers G_y on each stalk $x \in Y$. Again we refer to the data of these group actions as an **equivariant structure** on S. A morphism of sheaves on $\mathcal{Y}$ is then a natural transformation $f : S \to T$ which commutes with the action of these stabilizers. We write $\mathrm{Sh}(\mathcal{Y})$ for the category of sheaves on $\mathcal{Y}$.

Theorem 3.3.6. Suppose G acts on $(X, \leq)$ as in Def 3.1.1, and that there is a closed fundamental domain D for this action. Let $\mathcal{Y}$ be the associated complex of groups. Then there is an equivalence of categories,

$$\mathrm{Sh}_G(X) \cong \mathrm{Sh}(\mathcal{Y})$$

Proof. The functor $R : \mathrm{Sh}_G(X) \longrightarrow \mathrm{Sh}(\mathcal{Y})$ simply restricts an equivariant sheaf to it's values on the fundamental domain D. If you like, this is the pullback functor along the inclusion $\iota : D \hookrightarrow X$

Now we describe a functor in the opposite direction. Since D is a closed fundamental domain the action map,

$$\begin{aligned} \alpha : G \times D &\longrightarrow X \\ (g, \diamond) &\longmapsto g \cdot \diamond \end{aligned}$$

is surjective, and has $\alpha(g, \diamond) = \alpha(h, \bullet)$ if and only if $\diamond = \bullet \in D$ and $g^{-1}h \in G_\diamond$. In other words, we can identify $X = G \times D/\sim$ where $(gk, \diamond) \sim (g, \diamond)$ for $k \in G_\diamond$. Now, given $T \in \mathrm{Sh}(\mathcal{Y})$ we get a G-equivariant sheaf on $G \times D$, which for the moment we will call $G \times T$. Here the stalk at $(g, \diamond)$ is given by, $(G \times T)_{(g,\diamond)} := T_\diamond$ and the restriction maps are defined only for pairs for which $\diamond \leq \bullet \in D$. The equivariant structure is induced by the permutation action on G. The data of the local representations $\rho_\diamond$ for $\diamond \in D$ should be thought of as glueing data which ensures that $G \times T$ descends to a G-equivariant sheaf on $G \times D/\sim = X$. We will refer to this sheaf as $\widetilde{T}$. Here each $x \in X$ is represented by a unique equivalence class $[g, \diamond]$, and the stalk of $\widetilde{T}$ at x is,

$$\widetilde{T}_x := gT_\diamond$$

which as a vector space is just $T_\diamond$. Note that changing coset representatives $[g, \diamond] = [gk, \diamond]$ "twists" $T_\diamond$ by the action $\rho_\diamond(k) \cdot T_\diamond$. Now suppose $x \leq y \in X$. Then we have $x = [g, \diamond] \leq [h, \bullet] = y$ for some equivalence classes representing x, y. Then $h^{-1}x \leq \bullet$

and since D is closed, this implies $h^{-1}x \in D$. But $h^{-1}x$ is in the same G-orbit as $\diamond$, so the fact that D is a fundamental domain implies $h^{-1}x = \diamond$ and hence that $h^{-1}g \in G_\diamond$. So, we define the restriction map $\widetilde{\gamma}_{x,y} : \widetilde{T}_x \to \widetilde{T}_y$ via $\widetilde{\gamma}_{x,y} := \gamma_{\diamond,\bullet} \circ h^{-1}g$,

$$\begin{tikzcd}
\widetilde{T}_x = T_\diamond \arrow[r, "\widetilde{\gamma}_{x,y}"] \arrow[d, "h^{-1}g"'] & \widetilde{T}_y = T_\bullet \\
T_\diamond \arrow[ur, "\gamma_{\diamond,\bullet}"'] &
\end{tikzcd}$$

One should check that this is independent of the chosen representatives so that $\widetilde{T}$ is well defined. The equivariant structure on $\widetilde{T}$ is inherited from the permutation action on G, and can be described explicitly by choosing coset representatives as in Remark 3.3.4. In total, this construction gives a covariant functor, $I : \mathrm{Sh}(\mathcal{Y}) \to \mathrm{Sh}_G(X)$

Note that the composition $R \circ I$ is the identity functor on $\mathrm{Sh}(\mathcal{Y})$. It remains only to show that $I \circ R \cong \mathrm{Id}_{\mathrm{Sh}_G(X)}$. For this, suppose $S \in \mathrm{Sh}_G(X)$ and let $\widetilde{S} = [I \circ R](S)$. We define a natural transformation, $\alpha : \widetilde{S} \to S$ as follows. The component at $x = [g, \diamond]$ is given by,

$$\alpha_x : \widetilde{S}_x = gS_\diamond \longrightarrow S_x$$
$$s_\diamond \longmapsto g \cdot s_\diamond$$

where $g \cdot s_\diamond$ denotes the map $g : S_\diamond \to S_x$ coming from the equivariant structure on S. One can check that this gives a canonical isomorphism $\widetilde{S} \cong S$, and hence this completes the proof of the equivalence $\mathrm{Sh}(\mathcal{Y}) \cong \mathrm{Sh}_G(X)$ as desired.

□

Remark 3.3.7. It is perhaps more natural to state the previous theorem in the following way. One can define $\mathrm{Sh}_G(X)$ to be the functor category $\mathrm{Fun}([X//G], \mathbf{Vec}_\mathbb{C})$ where $[X//G]$ is the category with objects $x \in X$ and morphisms $\mathrm{Mor}(x, y) := \{g \in G \mid gx \leq y\}$. Here we can view the complex of groups as the full subcategory $\iota : [\mathcal{Y}] \hookrightarrow [X//G]$ generated by the objects $\diamond \in D$. Since D is a fundamental domain, $[\mathcal{Y}]$ turns out to be a "skeleton" of the category $[X//G]$. Then $\mathrm{Sh}(\mathcal{Y}) := \mathrm{Fun}([\mathcal{Y}], \mathbf{Vec}_{\mathbb{C}})$ and the equivalence we describe in the previous theorem is given by pulling back functors along $\iota : [\mathcal{Y}] \hookrightarrow [X//G]$ and $p : [X//G] \twoheadrightarrow [\mathcal{Y}]$. We will somewhat abuse notation and write,

$$\begin{tikzcd} D \arrow[r, bend right, "\iota"'] & X \arrow[l, bend right, "p"'] \end{tikzcd}$$

Then we will write $p^* : \mathrm{Sh}(\mathcal{Y}) \to \mathrm{Sh}_G(X)$ for the pullback construction I we describe in various terms in the previous theorem, and in Remark 3.3.4. These "stacky" pullbacks satisfy the same adjunctions as the push-pull functors associated to poset maps, although they are somehow slightly different in their behavior. In any case, it is just a matter of being careful when needed.

3.4 Global Sections and Local Systems

As before we let $\pi : X \to *$ be the projection to a point. This map is clearly G-equivariant, and we saw that there is a natural identification $\mathrm{Sh}_G(*) \cong \mathrm{Rep}(G)$. So, taking global sections gives a functor,

$$\begin{aligned} \Gamma(X; -) : \mathrm{Sh}_G(X) &\longrightarrow \mathrm{Rep}(G) \\ S &\longmapsto \pi_* S \end{aligned}$$

Here we take note of a class of equivariant sheaves which arise from the push-pull adjunction.

Definition 3.4.1. Let $\mathrm{Loc}(X) \subseteq \mathrm{Sh}_G(X)$ denote the full subcategory of consisting of sheaves which are locally constant on X, i.e. where all restriction maps are isomorphisms. We call such sheaves **local systems** on X.

Example 3.4.2. Let $V \in \mathrm{Sh}_G(*) \cong \mathrm{Rep}(G)$ be a representation of G and let,

$$\widetilde{V} := \pi^* V \in \mathrm{Sh}_G(X)$$

Then $\widetilde{V}$ is a local system (in fact just a constant sheaf) on X. Moreover by the push-pull adjunction,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(\widetilde{V}, S) \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(V, \Gamma(X; S))$$

In classical algebraic topology, it is well known that for a sufficiently nice space X (something like a connected manifold or CW-complex), there is an equivalence of categories,

$$\mathrm{Rep}(\pi_1(X)) \cong \mathrm{Loc}(X)$$

where $\mathrm{Loc}(X)$ is the category of locally constant sheaves on X (with its usual topology). In particular if X is simply connected, then any locally constant sheaf on X is isomorphic to a constant sheaf.

For us the setup will be the following. Let X be a locally finite CW-complex such that G acts by cellular automorphisms. Then the cells in X naturally form a poset with $x \leq y$ if x is a co-dimension one face of y, and one can consider the category $\mathrm{Sh}_G(X)$ as before. Here we prove something like the usual correspondence between local systems and representations of an appropriate fundamental group. For simplicity, we will assume that the quotient $G\backslash X$ is simply connected. This is not needed, but it avoids further digression into HNN-extensions, etc.

Theorem 3.4.3. Suppose G acts without inversions on a locally finite CW-complex X, such that the quotient $Y = G\backslash X$ is simply connected (in the usual sense). As before write $\mathcal{Y}$ for the complex of groups associated to this action. Moreover let,

$$\pi_1(\mathcal{Y}) := \varinjlim \mathcal{Y}$$

be the **fundamental group** of $\mathcal{Y}$. Then there is an equivalence of categories,

$$\mathrm{Rep}(\pi_1(\mathcal{Y})) \cong \mathrm{Loc}(X)$$

Proof. By Theorem 3.3.6, $\mathrm{Sh}_G(X) \cong \mathrm{Sh}(\mathcal{Y})$. One can check that under this equivalence $\mathrm{Loc}(X)$ corresponds to $\mathrm{Loc}(\mathcal{Y})$, where $\mathrm{Loc}(\mathcal{Y}) \subset \mathrm{Sh}(\mathcal{Y})$ is the subcategory of locally constant sheaves on $\mathcal{C}$. It suffices then to show that $\mathrm{Loc}(\mathcal{Y}) \cong \mathrm{Rep}(\pi_1(\mathcal{Y}))$.

Let $S \in \mathrm{Loc}(\mathcal{Y})$ and $V := \Gamma(Y; S)$ (which at the moment is just a vector space). First we show that for all $y \in Y$ the map,

$$\pi_y : V \longrightarrow S_y$$

is an isomorphism. Since all restriction maps are isomorphisms, any global section is necessarily supported on all of $(Y, \leq)$, so π_y is injective for all $y \in Y$. To see that π_y is surjective, note that any path $\tilde{p} : [0,1] \to X$ is compact and hence intersects only finitely many cells in $(X, \leq)$. So $\tilde{p}$ descends to a path in $(X, \leq)$ $p : y_0 < y_{t_1} > y_{t_2} < \cdots < y_1$. The restriction maps then induce an isomorphism $f_{0,1} : S_{y_0} \to S_{y_1}$ which (by associativity of S) depends only on the homotopy class of $\tilde{p}$. In particular, for any $s_0 \in S_{y_0}$ there exists a unique section $s \in \Gamma(U_p; S|_{U_p})$ such that $\pi_{y_1}(s) = \gamma_{0,1}(s_0)$ and $\pi_{y_0}(s) = s_0$. Moreover since Y is simply connected, any section $s \in S(U)$ with $p \subset U$ has this property. Hence if for each $y' \in Y$, if we choose a path p from y to y' and writes $s_{y'} = f_{y,y'}(s_y)$, then $s = (s_{y'})$ is a global section on Y lifting s_y.

By adjunction this can be upgraded to an isomorphism in $\mathrm{Sh}(Y)$, $\pi : \widetilde{V} \xrightarrow{\sim} S$ where $\widetilde{V}$ is the constant sheaf taking value V on all of Y. We endow $\widetilde{V}$ with an equivariant structure as follows. For each $y \in Y$ define a homomorphism $\rho_y : G_y \to GL(V)$ via,

$$\rho_y(g) \cdot v := \pi_y^{-1}(g \cdot \pi_y(v))$$

where $g \cdot \pi_y(v)$ is the action coming from the equivariant structure on S. One checks that this is a homomorphism as follows,

$$\begin{aligned}\rho_y(gh)\cdot v &= \pi_y^{-1}((gh)\cdot\pi_y(v))\\ &= \pi_y^{-1}(g\cdot(h\cdot\pi_y(v)))\\ &= \pi_y^{-1}(g\cdot\pi_y(\rho_y(h)\cdot v))\\ &= \rho_y(g)\cdot(\rho_y(h)\cdot v)\end{aligned}$$

Now suppose $x \leq y \in Y$ and let $\gamma_{x,y} : S_x \to S_y$ denote the restriction map (which recall is G_y-equivariant). By definition for all $v \in V$, $\gamma_{x,y}(\pi_x(v)) = \pi_y(v)$. Also since $\pi : \widetilde{V} \to S$ is a sheaf morphism, the following square of isomorphisms commutes,

$$\begin{tikzcd} V \arrow[r, "\mathrm{Id}_{x,y}"] \arrow[d, "\pi_x"] & V \arrow[d, "\pi_y"] \\ S_x \arrow[r, "\gamma_{x,y}"] & S_y \end{tikzcd}$$

and hence for all $g \in G_y \leq G_x, v \in V$,

$$\begin{aligned}\mathrm{Id}_{x,y}(\rho_x(g)\cdot v) &= [\pi_y^{-1}\circ\gamma_{x,y}\circ\pi_x](\rho_x(g)\cdot v)\\ &= [\pi_y^{-1}\circ\gamma_{x,y}\circ\pi_x](\pi_x^{-1}(g\cdot\pi_x(v)))\\ &= \pi_y^{-1}(\gamma_{x,y}(g\cdot\pi_x(v)))\\ &= \pi_y^{-1}(g\cdot\gamma_{x,y}(\pi_x(v)))\\ &= \pi_y^{-1}(g\cdot\pi_y(v))\\ &= \rho_y(g)\cdot v\end{aligned}$$

which shows that $\mathrm{Id}_{x,y}$ is G_y-equivariant and that ρ_y is the pullback of ρ_x along the inclusion $G_y \hookrightarrow G_x$. So by the universal property of colimits there exists a unique homomorphism $\rho : \pi_1(\mathcal{Y}) \to GL(V)$,

$$\begin{tikzcd}
G_x \arrow[dr] \arrow[ddr] & G_y \arrow[l, hook'] \arrow[r, hook] \arrow[d] & G_z \arrow[dl] \arrow[ddl] \\
& \pi_1(\mathcal{C}) \arrow[d, dotted, "\exists!"] & \\
& GL(V) &
\end{tikzcd}$$

such that the pull back of ρ to G_x is ρ_x for all $x \in Y$. This construction gives a functor, $\mathrm{Loc}(\mathcal{Y}) \to \mathrm{Rep}(\pi_1(\mathcal{Y}))$. Conversely if one has a representation (V, ρ) of $\pi_1(\mathcal{Y})$, then by pullback one obtains the local system $\widetilde{V} \in \mathrm{Sh}(\mathcal{Y})$. By our previous argument, these functors are quasi inverses and hence we obtain the equivalence as desired. $\square$

3.5 Frobenius Reciprocity

Let G be a locally compact, Hausdorff, totally disconnected group. Then for any open subgroup $K \leq G$ the quotient G/K is discrete. So by Theorem 3.3.6 there is an equivalence of categories,

$$\mathrm{Sh}_G(G/K) \cong \mathrm{Sh}_K(*_K) \cong \mathrm{Rep}(K)$$

Let us write $p_K : G/K \to *_K$ for the quotient map and $\iota_K : *_K \hookrightarrow G/K$ for the section sending $* \mapsto K$. Similarly let $p_G : G/K \to *_G$ denote the G-equivariant projection. If W is a smooth representation of K, we write $\widetilde{W} = p_K^* W$ for the associated sheaf in $\mathrm{Sh}_G(G/K)$.

$$\begin{tikzcd}
 & G/K \arrow[dl, bend right, "p_K"'] \arrow[dr, "p_G"] & \\
*_K \arrow[ur, bend right, "\iota_K"'] & & *_G
\end{tikzcd}$$

Pushing forward along the projection $p_G : G/K \to *$ gives a representation of G.

Definition 3.5.1. Let K be an open subgroup of G, and let (ρ, W) be a representation of K. The **compact induction** of W from K to G is the representation,

$$c\,\mathrm{Ind}_K^G(W) := \Gamma_c(G/K; \widetilde{W}) = [(p_G)_! \circ p_K^*]W$$

The **induction** of W to G is given by,

$$\mathrm{Ind}_K^G(W) = \Gamma(X; \widetilde{W}) = [(p_G)_* \circ p_K^*]W$$

Conversely given a representation $(\pi, V) \in \mathrm{Sh}_G(*)$ one can consider the pullback $[\iota_K^* \circ$

$p_G^*]V \in \mathrm{Sh}_K(*)$ which is called the **restriction** of V to K,

$$\mathrm{Res}_K^G(V) := [\iota_K^* \circ p_G^*]V$$

Note that $p_G \circ \iota_K = \mathrm{Id}_K : *_K \to *_G$, so $\mathrm{Res}_K^G(V) = \mathrm{Id}_K^* V$ is just the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Rep}(K)$.

Remark 3.5.2. $\mathrm{Ind}_K^G(W)$ can be identified with the following space of functions,

$$\mathrm{Ind}_K^G(W) \cong \{f : G \to W \mid f(gk) = \rho(k^{-1}) \cdot f(g)\}$$

with the left G-action given by, $[g \cdot f](x) = f(g^{-1} \cdot x)$. The subspace $c\,\mathrm{Ind}_K^G(W)$ is comprised of those functions whose support is finite modulo K.

Proof. We just sketch the proof. By definition, $\mathrm{Ind}_K^G(W) := \Gamma(X; \widetilde{W}) = \varprojlim_{G/K} W_{gK}$. So, using the notation of Remark 3.3.4 we have,

$$\mathrm{Ind}_K^G(W) = \prod_{h_i \in G/K} h_i W$$

where the $\{h_i\}_{i \in I}$ are a complete set of coset representatives for G/K, and $h_i W = W$. Choosing these representatives allows us to identify,

$$\{f : G \to W \mid f(gk) = \rho(k^{-1}) \cdot f(g)\} \cong \{\overline{f} : \{h_i\}_{i \in I} \to W\}$$

(as vector spaces). That is, any f is uniquely determined by its values on the h_i via the relation,

$$f(x) = f(h_i k) = \rho(k^{-1}) \cdot \overline{f}(h_i)$$

where $x = h_i k \in G$. In particular we get an isomorphism of vector spaces,

$$\begin{aligned}\Psi : \{f : G \to W \mid f(gk) = \rho(k^{-1}) \cdot f(g)\} \to \prod_{h_i \in G/K} h_i W \\ f \mapsto (\bar{f}(h_i))_{i \in I}\end{aligned}$$

One can check using the equivariant structure discussed in Remark 3.3.4 that Ψ is an isomorphism of G-representations. Essentially the same argument implies,

$$c\operatorname{Ind}_K^G(W) \cong \bigoplus_{h_i} h_i W$$

is the subspace of functions in $\operatorname{Ind}_K^G(W) \cong \{f : G \to W \mid f(gk) = \rho(k^{-1}) \cdot f(g)\}$ whose support modulo K is finite.

□

Theorem 3.5.3. (Frobenius Reciprocity) Let $K \leq G$ be an open subgroup. The functor $c\operatorname{Ind}_K^G$ is left adjoint to Res_K^G and Ind_K^G is its right adjoint i.e. there are natural isomorphisms,

$$\operatorname{Hom}_G(c\operatorname{Ind}_K^G(W), V) \cong \operatorname{Hom}_K(W, \operatorname{Res}_K^G(V))$$
$$\operatorname{Hom}_G(V, \operatorname{Ind}_K^G(W)) \cong \operatorname{Hom}_K(\operatorname{Res}_K^G(V), W)$$

Proof. Since $K \leq G$ is open, G/K is discrete, so by Prop. 1.4, $(p_G)_!$ is left adjoint to p_G^*. Therefore,

$$\begin{aligned}\operatorname{Hom}_{\operatorname{Rep}(G)}(c\operatorname{Ind}_K^G(W), V) &\cong \operatorname{Hom}_{\operatorname{Sh}_G(G/K)}(p_K^* W, p_G^* V) \\ &\cong \operatorname{Hom}_{\operatorname{Rep}(K)}(W, \iota_K^* p_G^* V) \\ &= \operatorname{Hom}(W, \operatorname{Res}_K^G(V))\end{aligned}$$

The second isomorphism uses the adjunction $(p_G^*, (p_G)_*)$. □

3.6 Sheaf Induction

Here we describe the associated induction functors for equivariant sheaves. As above let $K \leq G_x$ be a compact open subgroup and let X_K, X_G denote $(X, \leq)$ with the respective actions of these groups. Then we have the following diagram,

$$\begin{tikzcd}
& G \times_K X_K \arrow[dl, bend right, "p_K"'] \arrow[dr, "p_G"] & \\
X_K \arrow[ur, bend right, "\iota_K"'] \arrow[rr, "\mathrm{Id}_X"] & & X_G
\end{tikzcd}$$

where now $G \times_K X_K$ is the fiber product,

$$G \times_K X_K := G \times X / \sim = \{(gk, x) \sim (g, k \cdot x)\}$$

which is the quotient of $G \times X$ by the action $k \cdot (g, x) = (gk^{-1}, k \cdot x)$. $G \times X$ is a poset via $(g_1, x) \leq (g_2, y) \iff g_1 = g_2$ and $x \leq y$. Here the open star of a pair (g, x) is as you would expect $U_{(g,x)} = G \times U_x$. The quotient inherits the usual poset structure given by,

$$\begin{aligned}
[g_1, x] \leq [g_2, y] &\iff \exists k \in K : (g_1, x) \leq (g_2 k^{-1}, k \cdot y) \\
&\iff \exists k \in K : g_1 = g_2 k^{-1} \text{ and } x \leq k \cdot y
\end{aligned}$$

which makes the open star of an equivalence class $[g, x]$ the quotient of the orbit $K \cdot (G \times U_x) = G \times (K \cdot U_x)$ by this K-action,

$$U_{[g,x]} = (G \times K \cdot U_x)/K = G \times_K U_x$$

G acts on the fiber product by left multiplication on the component G. Moreover the quotient of this action $p_K : G \times_K X_K \to X_K$ has a natural section sending $\iota_K : x \mapsto [1, x]$. So as before there is an equivalence of categories, $\mathrm{Sh}_G(G \times_K X_K) \cong \mathrm{Sh}_K(X_K)$. Note also that the fibers of p_G are discrete, $p_G^{-1}(x) \cong G/(K \cap K_x) \times \{x\}$. So all of the previous adjunctions apply.

Definition 3.6.1. Let $T \in \mathrm{Sh}_K(X)$, the **compact induction** of T to G is the sheaf,

$$c\,\mathrm{Ind}_K^G(T) := [(p_G)_! \circ p_K^*]T$$

and the **induction** of T to G is given by,

$$c\,\mathrm{Ind}_K^G(T) := [(p_G)_* \circ p_K^*]T$$

Similarly given $S \in \mathrm{Sh}_G(X)$ one defines the restriction of S to K via,

$$\mathrm{Res}_K^G(S) := [\iota_K^* \circ p_G^*]S = \mathrm{Id}_X^* S$$

which is again the forgetful functor $\mathrm{Sh}_G(X) \to \mathrm{Sh}_K(X)$. These functors satisfy the same adjunction as described in Theorem 4.16.

Proposition 3.6.2. Let $T \in \mathrm{Sh}_K(X)$ with K a compact open subgroup of G. Then,

$$\Gamma(X; \mathrm{Ind}_K^G(T)) \cong \mathrm{Ind}_K^G(\Gamma(X;T))$$
$$\Gamma_c(X; c\,\mathrm{Ind}_K^G(T)) \cong c\,\mathrm{Ind}_K^G(\Gamma_c(X;T))$$

In other words, (up to isomorphism) induction commutes with taking global sections.

Proof. Consider the map,

$$\begin{aligned} G \times_K *_K &\longrightarrow G/K \\ [g,*] &\longmapsto gK \end{aligned}$$

This map is an isomorphism of G-sets, so we identify $G \times_K *_K = G/K$. The claim follows then from the commutativity of the following diagram,

$$\begin{tikzcd}
X_K \arrow[d, "\pi_K"'] & G \times_K X_K \arrow[l, "p_K"'] \arrow[r, "p_G"] \arrow[d, "{(1,\pi_K)}"] & X_G \arrow[d, "\pi_G"] \\
*_K & G \times_K *_K \arrow[l, "\overline{p}_K"'] \arrow[r, "\overline{p}_G"] & *_G
\end{tikzcd}$$

By definition, the global sections of the induced sheaf $\mathrm{Ind}_K^G(T)$ are given by the following formula,

$$\Gamma(X; \mathrm{Ind}_K^G(T)) = (\pi_G)_* \circ \big((p_G)_* \circ p_K^*\big)T$$

But then by the commutativity of the right square we have a natural isomorphism,

$$(\pi_G)_* \circ (p_G)_* \cong (\overline{p}_G)_* \circ (1, \pi_K)_*$$

and by the commutativity of the left square we have,

$$(1, \pi_K)_* \circ p_K^* \cong \overline{p}_K^* \circ (\pi_K)_*$$

Therefore,

$$\begin{aligned}
\Gamma(X; \mathrm{Ind}_K^G(T)) &= (\pi_G)_* \circ (p_G)_* \circ p_K^* T \\
&\cong (\overline{p}_G)_* \circ (1, \pi_K)_* \circ p_K^* T \\
&\cong (\overline{p}_G)_* \circ \overline{p}_K^* \circ (\pi_K)_* T \\
&= \mathrm{Ind}_K^G(\Gamma(X; T))
\end{aligned}$$

A similar argument proves the claim for compactly supported sections. $\square$

Chapter 4

Hecke Sheaves

4.1 Hecke Algebras

For more discussion on Hecke algebras in the context of totally disconnected groups, we recommend the lecture notes [Châ16] by Ngô Bàu Châu.

Let G be a finite group and let $K \leq G$. Then the space of functions,

$$\mathbb{C}[G/K] := \{f : G/K \to \mathbb{C}\} = \{f : G \to \mathbb{C} \mid f(gk) = f(g)\}$$

comes with a left G-action via, $[g \cdot f](x) = f(g^{-1} \cdot x)$. By Remark 3.5.2, this representation can be identified with $c\,\mathrm{Ind}_K^G(\mathbb{C})$, where $\mathbb{C}$ denotes the trivial representation of K. So for any $V \in \mathrm{Rep}(G)$, Frobenius Reciprocity says that,

$$\mathrm{Hom}_G(\mathbb{C}[G/K], V) \cong \mathrm{Hom}_K(\mathbb{C}, \mathrm{Res}_K^G(V)) = V^K$$

where $V^K = \{v \in V \mid k \cdot v = v \text{ for all } k \in K\}$ is the subspace of K-fixed vectors in V.

To be explicit, this isomorphism is given by,

$$\begin{aligned} \operatorname{Hom}_G(\mathbb{C}[G/K], V) &\longrightarrow V^K \\ \varphi &\longmapsto \varphi(\mathbf{1}_K) \end{aligned}$$

where $\mathbf{1}_K$ is the characteristic function on K. Here $\operatorname{Hom}_G(\mathbb{C}[G/K], V)$ is a left module over the finite dimensional algebra,

$$\mathcal{H}(G, K) := \operatorname{End}_G(\mathbb{C}[G/K])^{\text{op}}$$

with the action given by pre-composition. In particular, given $\psi \in \operatorname{End}_G(\mathbb{C}[G/K])^{op}$ the image $\psi(\mathbf{1}_K)$ is again fixed by K, and hence the same is true of, $[\varphi \circ \psi](\mathbf{1}_K) \in V^K$. So by naturality, V^K becomes a left module over $\mathcal{H}(G, K)$. By the same reasoning, $\mathcal{H}(G, K)$ can be identified with the space of K fixed vectors in $\mathbb{C}[G/K]$. These are the K bi-invariant functions,

$$\mathbb{C}[K\backslash G/K] = \mathbb{C}[G/K]^K = \{f : G \to \mathbb{C} \mid f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \in K\}$$

This space of functions inherits a product from the endomorphism ring as follows. Let's pick $\psi_1, \psi_2 \in \operatorname{End}_G(\mathbb{C}[G/K])^{\text{op}}$ and define $f_1 := \psi_1(\mathbf{1}_K), f_2 := \psi_2(\mathbf{1}_K) \in \mathbb{C}[K\backslash G/K]$. We'd like the evaluation at $\mathbf{1}_K$ map to be a ring homomorphism. So, if we take the product $\psi_1 \circ \psi_2 \in \operatorname{End}_G(\mathbb{C}[G/K])^{\text{op}}$ then the product $f_1 * f_2$ should be given by,

$$f_1 * f_2 := [\psi_2 \circ \psi_1](\mathbf{1}_K)$$

but what is this function? Just for convenience, let us view $f \in \mathbb{C}[K\backslash G/K]$ as a function $G/K \to \mathbb{C}$. Then we can express f_1 as a linear combination,

$$f_1 = \sum_{gK \in G/K} f_1(gK) \cdot \mathbf{1}_{gK}$$

Here we have $\mathbf{1}_{gK} = g \cdot \mathbf{1}_K$, so we can use the fact the ψ are G-equivariant to see that,

$$\begin{aligned} f_1 * f_2 &= \psi_2\Big(\sum_{gK \in G/K} f_1(gK) \cdot (g \cdot \mathbf{1}_K) \Big) \\ &= \sum_{gK \in G/K} f_1(gK) \cdot (g \cdot \psi_2(\mathbf{1}_K)) \\ &= \sum_{gK \in G/K} f_1(gK) \cdot [g \cdot f_2] \end{aligned}$$

Evaluating at $xK \in G/K$ gives us a formula for the **convolution product**,

$$[f_1 * f_2](xK) = \sum_{gK \in G/K} f_1(gK) \cdot f_2(g^{-1}xK)$$

It is more common to view f_1, f_2 as functions on G, in which case,

$$[f_1 * f_2](x) = \frac{1}{|K|} \sum_{g \in G} f_1(g) f_2(g^{-1}x)$$

The result of this discussion is that we have an algebra isomorphism, $\mathcal{H}(G, K) \cong (\mathbb{C}[K\backslash G/K], *)$. Just for completeness, let's see how a function $f \in \mathbb{C}[K\backslash G/K]$ acts on V^K. Each f determines an element $\mathrm{mult}_f \in \mathrm{End}_G(\mathbb{C}[G/K])^{\mathrm{op}}$ by left multiplication. If we identify $v \in V^K$ with $\varphi(\mathbf{1}_K)$ for $\varphi : \mathbb{C}[G/K] \to V$, then the action of f on v is given by,

$$f \cdot v := [\varphi \circ \mathrm{mult}_f](\mathbf{1}_K) = \varphi(f)$$

Again we write f as a linear combination $f = \sum_{gK \in G/K} f(gK) \cdot (g \cdot \mathbf{1}_K)$. Then applying φ gives,

$$f \cdot v = \varphi(f) = \sum_{gK} f(gK) \cdot (g \cdot \varphi(\mathbf{1}_K)) = \sum_{gK} f(gK) \cdot (g \cdot v)$$

where now $g \cdot v$ is the action coming from V as a representation of G. In total, this construction gives a functor,

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{C}[G/K], -) : \mathrm{Rep}(G) &\longrightarrow \mathcal{H}(G,K)\text{-mod} \\ V &\longmapsto V^K \end{aligned}$$

In the case where $K = \{1\}$ we get,

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{C}[G], -) : \mathrm{Rep}(G) &\longrightarrow \mathbb{C}[G]\text{-mod} \\ V &\longmapsto V \end{aligned}$$

which is well known to be an equivalence of categories (here $\mathbb{C}[G] = \mathcal{H}(G,1)$ is the usual group algebra). In the language of Morita theory, one says that that $\mathbb{C}[G]$ is a **compact projective generator** for the category $\mathrm{Rep}(G)$.

Now suppose that K is a profinite group. Here K is endowed with a filtration by finite index normal subgroups,

$$K \supset H_1 \supset H_2 \supset H_3 \supset \cdots$$

such that $\cap_i H_i = \{1\}$ and $K \cong \varprojlim K/H_i$. We saw in Chapter 2 that K inherits a topology in which it is compact, Hausdorff and totally disconnected. In the presence of this additional structure, one usually considers **smooth representations** (ρ, V). These are representations for which the associated action map,

$$K \times V \to V$$

is continuous. Here V is given the discrete topology and K its usual profinite topology. So, this map is continuous iff for each $v \in V$, $\mathrm{Stab}_K(v)$ is an open subgroup of K. In particular, since the H_i form a neighborhood basis of the identity, V is smooth if and only if,

$$V = \bigcup_{i \geq 0} V^{H_i}$$

If V is finite dimensional, then $V = V^{H_i}$ for some $i \geq 0$, in which case ρ factors through the finite group K/H_i. If d is the minimal such i, we say that V has **depth** d. We write $\mathrm{Rep}^{\infty}(K)$ for the full subcategory of $\mathrm{Rep}(K)$ consisting of smooth representations. Here each H_i is an open subgroup of K, so Frobenius reciprocity says that,

$$\mathrm{Hom}_K(\mathbb{C}[K/H_i], V) \cong \mathrm{Hom}_{H_i}(\mathbb{C}, V) \cong V^{H_i}$$

where,

$$\mathbb{C}[K/H_i] := \{f : K/H_i \to \mathbb{C}\} = \{f : K \to \mathbb{C} \mid f(kh) = f(k) \text{ for all } h \in H_i\}$$

and hence V^{H_i} is again a left module over,

$$\mathcal{H}(K, H_i) = \mathrm{End}_K(\mathbb{C}[K/H_i])^{\mathrm{op}}$$

Where $\mathcal{H}(K, H_i)$ can be identified with the algebra of bi-invariant functions on K,

$$\mathcal{H}(K, H_i) \cong \Big(\mathbb{C}[H_i \backslash K/H_i], *\Big)$$

and the convolution product is given by,

$$[f_1 * f_2](xH_i) = \sum_{kH_i \in K/H_i} f_1(kH_i) f_2(k^{-1}xH_i)$$

In fact, since we chose the H_i to be normal subgroups, we can identify $H_i \backslash K/H_i = K/H_i$ and hence $\mathcal{H}(K, H_i) \cong (\mathbb{C}[K/H_i], *)$. Functions $K \to \mathbb{C}$ which are invariant under an open subgroup are called **smooth functions**. As the H_i form a neighborhood basis of the identity, the $\mathbb{C}[K/H_i]$ assemble to form the space of all smooth functions on K.

This is called the K-**Hecke algebra**,

$$\mathcal{H}(K) := \{f : K \to \mathbb{C} \mid f \in \mathcal{H}(K, H_i) \text{ for some } i \geq 0\} = \bigcup_{i \geq 0} \mathbb{C}[K/H_i]$$

In order to lift the convolution product to functions on the full group K, one needs to be able to assign a "volume" to each open subgroup H_i. In other words, we must choose a K-invariant measure μ, called a **Haar measure**. Here μ can be thought of as an element of the dual space $\mathbb{C}[K/K]^\vee$, which is one dimensional, so such a measure exists and is unique up to scaling. Then any smooth function f is necessarily constant on cosets in K/H_i for some i, and hence we can integrate f over K via,

$$\int_K f(k)\ d\mu(k) = \mu(H_i) \cdot \sum_{kH_i} f(kH_i)$$

which is a finite sum since H_i has finite index in K. Then the convolution product on $\mathcal{H}(K)$ is given by,

$$[f_1 * f_2](x) := \int_K f_1(k) f_2(k^{-1}x)\ d\mu(k)$$

Note that if a function is invariant under H_i, then it is also invariant under $H_{i+1} \leq H_i$. So $\mathcal{H}(K)$ is canonically a direct limit taken over the sequence of (K-equivariant) inclusions,

$$\mathbb{C}[K/K] \hookrightarrow \mathbb{C}[K/H_1] \hookrightarrow \mathbb{C}[K/H_2] \hookrightarrow \mathbb{C}[K/H_3] \hookrightarrow \cdots$$

In any case, just as we had for finite groups, for each $i \geq 0$ we get a functor,

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{C}[K/H_i], -) : \mathrm{Rep}^\infty(K) &\longrightarrow \mathcal{H}(K, H_i) - \mathrm{mod} \\ V &\longmapsto V^{H_i} \end{aligned}$$

Unfortunately $\{1\}$ is no longer an open subgroup, so the characteristic function at $1 \in K$ is not smooth. In particular, there is no way to "evaluate at 1" as we did for finite groups. One way to see this is to note that,

$$\begin{aligned}\operatorname{Hom}_K(\mathcal{H}(K), V) &= \operatorname{Hom}_K(\varinjlim \mathbb{C}[K/H_i], V)\\ &\cong \varprojlim \operatorname{Hom}_K(\mathbb{C}[K/H_i], V)\\ &\cong \varprojlim V^{H_i}\end{aligned}$$

which is general not a smooth representation. This issue can be dealt with as follows. For each $i \geq 0$ we have an idempotent,

$$e_i := \frac{1}{\mu(H_i)} \cdot \mathbf{1}_i$$

where $\mathbf{1}_i$ is the characteristic function on $H_i \subset K$. Convolution with e_i averages $f \in \mathcal{H}(K)$ over the subgroup H_i,

$$\begin{aligned}[f * e_i](x) &= \frac{1}{\mu(H_i)} \int_K f(xk^{-1}) e_i(k) \; d\mu(k)\\ &= \frac{1}{\mu(H_i)} \int_{H_i} f(xh^{-1}) \; d\mu(h)\end{aligned}$$

which makes f invariant under H_i. In fact since $H_i \trianglelefteq K$, these turn out to be *central* idempotents. In any case, we have a (left K-equivariant) quotient map,

$$\begin{aligned}\operatorname{avg}_i : \mathcal{H}(K) &\longrightarrow \mathbb{C}[K/H_i]\\ f &\longmapsto f * e_i\end{aligned}$$

The fix for the Hom space issue is to consider the subspace of **finite rank** Homs, consisting of the morphisms which factor through one of these averaging maps,

$$\operatorname{Hom}_K^\infty(\mathcal{H}(K), V) := \{\varphi : \mathcal{H}(K) \to V \mid \varphi = \varphi_i \circ \operatorname{avg}_i \text{ for some } i \geq 0\}$$

Then essentially by definition,

$$\begin{aligned}
\mathrm{Hom}_K^\infty(\mathcal{H}(K), V) &\cong \varinjlim_{i \geq 0} \mathrm{Hom}_K(\mathbb{C}[K/H_i], V) \\
&\cong \varinjlim_{i \geq 0} V^{H_i} = V
\end{aligned}$$

Which by naturality is a left module over,

$$\mathrm{End}_K^\infty(\mathcal{H}(K))^{\mathrm{op}} \cong \varinjlim_{i \geq 0} \mathbb{C}[K/H_i] = \mathcal{H}(K)$$

Just to sell the point, $\mathcal{H}(K)$ is a non-unital algebra (since $\{1\}$ is not an open subgroup), but the identity endomorphism is clearly K-equivariant. This is reflected in the fact that $\mathrm{Id}_{\mathcal{H}(K)}$ does not factor through a finite level, so this additional technicality with finite Hom spaces is genuinely necessary. The result of this discussion is that we have a fully-faithful functor,

$$\mathrm{Hom}_K^\infty(\mathcal{H}(K), -) : \mathrm{Rep}^\infty(K) \to \mathcal{H}(K)\text{-mod}$$

but what modules appear in its essential image? The smoothness condition on V says that, $V = \cup V^{H_i}$. With some thought, one sees that this is equivalent to saying that every $v \in V$ is a *finite* sum of fixed vectors. Here we say that a module M is **non-degenerate** if $M = \sum_i e_i \cdot M$. This is generally the appropriate condition for non-unital rings with "enough idempotents". In particular, M is non-degenerate if and only if the map,

$$\begin{aligned}
\mathcal{H}(K) \otimes_{\mathcal{H}(K)} M &\longrightarrow M \\
f \otimes m &\longmapsto f \cdot m
\end{aligned}$$

is an isomorphism (the inverse sends m to the appropriate finite sum $\sum e_i \otimes m_i$). We have just about outlined the proof of the equivalence between smooth representations and non-degenerate modules over the Hecke algebra. Let us state this as a theorem so we can refer back to it later,

Theorem 4.1.1. The functors,

$$\operatorname{Hom}_K^\infty(\mathcal{H}(K), -) : \operatorname{Rep}^\infty(K) \longrightarrow \mathcal{H}(K)\text{-mod}$$
$$\operatorname{Rep}^\infty(K) \longleftarrow \mathcal{H}(K)\text{-mod} : \mathcal{H}(K) \otimes_{\mathcal{H}(K)} (-)$$

form an adjoint equivalence between the categories of smooth representations and non-degenerate $\mathcal{H}(K)$-modules.

Proof. Let $V \in \operatorname{Rep}^\infty(K)$ and let M be a non-degenerate left $\mathcal{H}(K)$-module. We will sketch the proof that the standard unit and co-unit maps are isomorphisms. Let us start with the counit,

$$\varepsilon_V : \mathcal{H}(K) \otimes_{\mathcal{H}(K)} \operatorname{Hom}_K^\infty(\mathcal{H}(K), V) \to V$$
$$f \otimes \varphi \mapsto \varphi(f)$$

Since V is smooth, each vector $v \in V$ is fixed by an open subgroup H_i. So for each $v \in V$ we can choose the unique $\varphi_v \in \operatorname{Hom}_K(\mathbb{C}[K/H_i], V)$ such that $\varphi_v(\mathbf{1}_i) = v$. Then one can check that, $v \longmapsto \mathbf{1}_i \otimes \varphi_v$ provides an explicit inverse to ε_V. Now we consider the unit map,

$$\eta_M : M \longrightarrow \operatorname{Hom}_K^\infty(\mathcal{H}(K), \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M)$$
$$m \longmapsto [\varphi_m : f \mapsto f \otimes m]$$

First we show that this map actually lands in the space of finite homs. Since M is non-degenerate, we can write any $m \in M$ as a finite linear combination $m = \sum_i m_i$ with $m_i \in e_i \cdot M$. Let j be the smallest i for which $m_i \neq 0$. Then $e_j \cdot m = m$ and hence,

$f \otimes m = f \otimes e_j \cdot m = f * e_j \otimes m$. So φ_m factors through a finite level as desired,

$$\begin{tikzcd} \mathcal{H}(K) \arrow[rr, "\varphi_m"] \arrow[dr, "(-)*e_j"'] & & \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M \\ & \mathbb{C}[K/H_j] \arrow[ur] & \end{tikzcd}$$

Now let's describe an inverse to η_M. Given $\varphi \in \mathrm{Hom}_K^\infty(\mathcal{H}(K), \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M)$ we can find $\varphi_i : \mathbb{C}[K/H_i] \to \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M$ such that $\varphi = \varphi_i \circ \mathrm{avg}_i$. Evaluating φ_i at $\mathbf{1}_i$ gives an H_i fixed point in $\mathcal{H}(K) \otimes_{\mathcal{H}(K)} M$. Here G acts only on the left component, so we have

$$\varphi_i(\mathbf{1}_i) \in \mathbb{C}[K/H_i] \otimes_{\mathcal{H}(K)} M$$

The image of $\varphi_i(\mathbf{1}_i)$ under the map $\mathcal{H}(K) \otimes_{\mathcal{H}(K)} M \to M$ gives us an element of M. This process is precisely the sequence of isomorphisms we discussed previously,

$$\begin{aligned} \mathrm{Hom}_K^\infty(\mathcal{H}(K), \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M) &\cong \bigcup_{i \geq 0} \mathbb{C}[K/H_i] \otimes_{\mathcal{H}(K)} M \\ &\cong \mathcal{H}(K) \otimes_{\mathcal{H}(K)} M \cong M \end{aligned}$$

Where the final isomorphism uses the fact that M is nondegenerate. One can check that this describes an explicit inverse to the unit map η_M. $\square$

Remark 4.1.2. When working over a field of characteristic zero as we are here, compactness implies that any smooth representation of K is isomorphic to a direct sum of irreducible representations. Moreover one can show that every smooth irreducible representation is finite dimensional over $\mathbb{C}$. In other words the Hecke algebra $\mathcal{H}(K)$ is a **semi-simple** algebra. In particular, every smooth representation is both projective and injective in $\mathrm{Rep}^\infty(K)$.

In the following sections we will also use the notion of the **smooth dual** $V^\vee$ of a smooth representation V. If V is finite dimensional this notion is the usual dual representation. Namely G acts on $\phi \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ via, $[g \cdot \phi](v) := \phi(g^{-1} \cdot v)$. When V has infinite dimension we take the subspace of *smooth vectors* in $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, i.e. those which have open stabilizer.

$$V^{\vee} := \mathrm{Hom}^{\infty}_{\mathbb{C}}(V, \mathbb{C}) = \{\phi : V \to \mathbb{C} \mid \exists i \geq 0\,, \phi(h^{-1} \cdot v) = \phi(v)\ \forall h \in H_i\} = \bigcup_{i \geq 0} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^{H_i}$$

In other words $\phi \in V^{\vee}$ if and only if for some $i \geq 0$, ϕ factors through the space of co-invariants $V_{H_i} := V \big/ \langle v - h \cdot v \mid h \in H_i \rangle$,

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, two heads] & \mathbb{C} \\ V_{H_i} \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

In characteristic zero, convolution with $e_i \in \mathcal{H}(K)$ identifies co-invariants with invariants,

$$\begin{aligned} V_{H_i} &\longrightarrow V^{H_i} \\ v &\longmapsto e_i \cdot v := \frac{1}{\mu(H_i)} \int_{H_i} h \cdot v \ d\mu(h) \end{aligned}$$

So this map splits V as a direct sum $V = V^{H_i} \oplus V[H_i]$ where $V[H_i] = \langle v - h \cdot v \mid h \in H_i \rangle$. In particular this shows that $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^{H_i} \cong \mathrm{Hom}_{\mathbb{C}}(V^{H_i}, \mathbb{C})$. So we see,

$$V^{\vee} := \bigcup_{i \geq 1} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^{H_i} \cong \bigcup_{i \geq 1} \mathrm{Hom}_{\mathbb{C}}(V^{H_i}, \mathbb{C}) = \bigcup_{i \geq 1} (V^{H_i})^{\vee}$$

This construction is particularly well behaved for so called **admissible** representations, i.e. those for which for all $i \geq 0$, V^{H_i} has finite dimension as a $\mathbb{C}$-vector space. For instance, one can check that V is smooth and admissible if and only if $(V^{\vee})^{\vee} \cong V$ via the usual evaluation map. Admissible representations form an abelian subcategory of $\mathrm{Rep}^{\infty}(K)$, and restricted to this subcategory the smooth dual functor is naturally isomorphic to $V \mapsto \mathrm{Hom}^{\infty}_{K}(V, \mathcal{H}(K))$. Here again $\mathrm{Hom}^{\infty}_{K}(V, \mathcal{H}(K))$ refers to the subspace of K-equivariant maps which factor through a finite quotient,

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, two heads] & \mathcal{H}(K) \\ V^{H_i} \arrow[r] & \mathbb{C}[K/H_i] \arrow[u, hook] \end{tikzcd}$$

Again we make this assumption so that,

$$\mathrm{Hom}_K^\infty(V, \mathcal{H}(K)) \cong \varinjlim \mathrm{Hom}_{K/H_i}(V^{H_i}, \mathbb{C}[K/H_i])$$

If V is admissible, then V^{H_i} is finite dimensional, so we have an isomorphism (of right K-representations),

$$\begin{aligned} (V^{H_i})^\vee \otimes_{\mathbb{C}[K/H_i]} \mathbb{C}[K/H_i] &\longrightarrow \mathrm{Hom}_{K/H_i}(V^{H_i}, \mathbb{C}[K/H_i]) \\ \phi \otimes f &\longmapsto [v \mapsto \phi(v) \cdot f] \end{aligned}$$

$(V^{H_i})^\vee$ is a right module over $\mathbb{C}[K/H_i]$ via, $[\phi \cdot f](v) = \phi(f \cdot v)$. Now $\mathbb{C}[K/H_i]$ is unital, so pre-composing with the isomorphism $(V^{H_i})^\vee \to (V^{H_i})^\vee \otimes_{\mathbb{C}[K/H_i]} \mathbb{C}[K/H_i]$ given by $\phi \mapsto \phi \otimes e_i$ we see that,

$$\mathrm{Hom}_K^\infty(V, \mathcal{H}(K)) \cong \varinjlim (V^{H_i})^\vee = V^\vee$$

From this perspective, despite being infinite dimensional, $\mathcal{H}(K)$ is self dual, meaning $\mathcal{H}(K)^\vee \cong \mathcal{H}(K)$ with the left-right actions reversed. It is also clear here that $V \mapsto V^\vee$ is an exact functor. If one prefers to think in terms of $\mathcal{H}(K)$-modules, then the smooth dual is given by,

$$V^\vee := \mathrm{Hom}_{\mathcal{H}(K)}^\infty(V, \mathcal{H}(K)) = \{\varphi : V \to \mathcal{H}(K) \mid \varphi(e_i \cdot V) = 0 \text{ for almost all } i\}$$

we will say a bit more about this perspective later.

4.2 Hecke Sheaves

Here we return to the case where $(X, \leq)$ is the tree described in Ch. 2, and G is the group described there. Throughout this section we fix a fundamental chamber $C = \{x < e > y\}$ which we identify with the quotient $Y = G\backslash X$.

Let $K = K_\diamond$ for $\diamond \in C$, and write $\iota(\diamond) : G/K \hookrightarrow X$ for the inclusion sending $gK \mapsto g{\cdot}\diamond$. As before we write, $\iota_K : *_K \hookrightarrow G/K$ for the canonical section of the quotient map $p_K : G/K \to *_K$.

$$\begin{tikzcd} & G/K \arrow[dl, bend right, "p_K"'] \arrow[dr, "\iota(\diamond)"] & \\ *_K \arrow[ur, bend right, "\iota_K"'] & & X \end{tikzcd}$$

In light of the discussion in the previous section we make the following definitions,

Definition 4.2.1. A sheaf $S \in \mathrm{Sh}_G(X)$ is **smooth** if for all $x \in X$, S_x is a smooth representation of K_x. We write $\mathrm{Sh}^\infty_G(X)$ for the full subcategory of $\mathrm{Sh}_G(X)$ consisting of smooth sheaves.
Given $S, T \in \mathrm{Sh}^\infty_G(X)$ let,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(S, T) \subseteq \mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, T)$$

denote the subspace of **finite rank** natural transformations $f : S \to T$, i.e. those for which for all $x \in X$, the component, $f_x : S_x \to T_x$ has finite rank. As we saw in the previous section, these are precisely the morphisms for which each $S_x \to T_x$ factors through a finite quotient.

With the necessary notation in place, we are now ready to define sheaves which will fill the role of the Hecke algebras discussed in the previous section.

Definition 4.2.2. Let $\mathcal{H}(K) \in \mathrm{Rep}^{\infty}(K) = \mathrm{Sh}_K^{\infty}(*_K)$ be the Hecke algebra of K. We define the **projective Hecke sheaf** at $\diamond$,

$$P(\diamond) := [\iota(\diamond)_{\dagger} \circ p_K^*]\mathcal{H}(K_{\diamond})$$

and the **injective Hecke sheaf** at $\diamond$,

$$I(\diamond) := [\iota(\diamond)_{*} \circ p_K^*]\mathcal{H}(K_{\diamond})$$

Proposition 4.2.3. $P(\diamond)$ is projective and $I(\diamond)$ is injective in $\mathrm{Sh}_G^{\infty}(X)$.

Proof. By the push-pull adjunctions,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}_G(X)}(P(\diamond), S) &\cong \mathrm{Hom}_{\mathrm{Sh}_G(G/K)}(p_K^*\mathcal{H}(K), \iota(\diamond)^*S) \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(K)}([\iota_K^* \circ p_K^*]\mathcal{H}(K), [\iota_K^* \circ \iota(\diamond)^*]S) \\
&= \mathrm{Hom}_{\mathrm{Rep}(K)}(\mathcal{H}(K), S_{\diamond})
\end{aligned}$$

In total, $\mathrm{Hom}_{\mathrm{Sh}_G(X)}(P(\diamond), -) \cong \mathrm{Hom}_{\mathrm{Rep}(K_{\diamond})}(\mathcal{H}(K_{\diamond}), (-)_{\diamond})$ which is exact since $\mathrm{Rep}^{\infty}(K)$ is semi-simple. So, $P(\diamond)$ is projective in $\mathrm{Sh}_G^{\infty}(X)$. Note that this adunction simply says that a morphism $P(\diamond) \to S$ is determined by its component at the stalk $\diamond \in C$. If we restrict to the subspace of finite rank morphisms $\mathrm{Hom}_{\mathrm{Sh}_G}^{\infty}(P(\diamond), -)$, then we obtain,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}^{\infty}(P(\diamond), S) \cong \mathrm{Hom}_{\mathrm{Rep}(K)}^{\infty}(\mathcal{H}(K), S_{\diamond}) \cong S_{\diamond}$$

The dual argument for the injective Hecke sheaf shows that,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, I(\diamond)) &\cong \mathrm{Hom}_{\mathrm{Sh}_G(G/K)}(\iota(\diamond)^*S, p_K^*\mathcal{H}(K)) \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(K)}([\iota_K^* \circ \iota(\diamond)^*]S, [\iota_K^* \circ p_K^*]\mathcal{H}(K)) \\
&= \mathrm{Hom}_{\mathrm{Rep}(K)}(S_{\diamond}, \mathcal{H}(K))
\end{aligned}$$

So we have a natural isomorphism $\mathrm{Hom}_{\mathrm{Sh}_G(X)}((-), I(\diamond)) \cong \mathrm{Hom}_{\mathrm{Rep}(K_\diamond)}((-)_\diamond, \mathcal{H}(K_\diamond))$ which is exact again because $\mathrm{Rep}^\infty(K)$ is semi-simple. So $I(\diamond)$ is injective in $\mathrm{Sh}_G^\infty(X)$. Here if we restrict to the subspace of finite rank morphisms, and assume that the stalk of S at $\diamond$ is admissible then,

$$\begin{aligned}\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(S, I(\diamond)) &\cong \mathrm{Hom}^\infty_{\mathrm{Rep}(K)}(S_\diamond, \mathcal{H}(K))\\ &\cong S_\diamond^\vee\end{aligned}$$

which recall denotes the smooth dual of $S_\diamond$. □

Definition 4.2.4. We define the (projective) **Hecke sheaf** on X to be,

$$P_X := P(x) \oplus P(e) \oplus P(y)$$

and the **sheaf Hecke algebra** to be,

$$\mathcal{H}_X := \mathrm{End}^\infty_{\mathrm{Sh}_G(X)}(P_X)^{\mathrm{op}}$$

Remark 4.2.5. Let's examine the structure of the algebra $\mathcal{H}_X = \mathrm{End}^\infty_{\mathrm{Sh}_G(X)}(P_X)^{\mathrm{op}}$. Let $\diamond \in \{x, y\}$ and $K = K_\diamond$ as before. Then we consider the space of morphisms,

$$\begin{aligned}\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P(e), P(\diamond)) &\cong \mathrm{Hom}^\infty_{\mathrm{Sh}_G(G/I)}(p_I^*\mathcal{H}(I), \iota(e)^*P(\diamond))\\ &\cong \mathrm{Hom}^\infty_{\mathrm{Rep}(I)}(\mathcal{H}(I), \mathrm{Res}_I^K \mathcal{H}(K))\end{aligned}$$

Note that $\iota(x)^*P(y) = 0$ and vice versa, so the only non-trivial components are the diagonal endomorphisms and the ones described above. In other words, there is a nontrivial map $P(\bullet) \to P(\diamond)$ if and only if $\diamond \leq \bullet$ in the fundamental chamber $C = \{x <$

$e > y\}$. If you like, now we can arrange the endomorphisms of P_X into a matrix,

$$\mathrm{End}^{\infty}_{\mathrm{Sh}_G(X)}(P_X) \cong \begin{bmatrix} \mathrm{End}^{\infty}_{K_x}(\mathcal{H}(K_x)) & \mathrm{Hom}^{\infty}_{I}(\mathcal{H}(I), \mathcal{H}(K)) & 0 \\ 0 & \mathrm{End}^{\infty}_{I}(\mathcal{H}(I)) & 0 \\ 0 & \mathrm{Hom}^{\infty}_{I}(\mathcal{H}(I), \mathcal{H}(K)) & \mathrm{End}^{\infty}_{K_y}(\mathcal{H}(K_y)) \end{bmatrix}$$

Here we view $\mathcal{H}(K_\diamond)$ as a representation via its left $K_\diamond$ action. Taking the opposite algebra structure, we see that $\mathcal{H}_X$ is isomorphic to the following matrix algebra,

$$\mathcal{H}_X \cong \begin{bmatrix} \mathcal{H}(K_x) & 0 & 0 \\ \mathrm{Hom}^{\infty}_{I}(\mathcal{H}(K_x), \mathcal{H}(I)) & \mathcal{H}(I) & \mathrm{Hom}^{\infty}_{I}(\mathcal{H}(K_y), \mathcal{H}(I)) \\ 0 & 0 & \mathcal{H}(K_y) \end{bmatrix}$$

Now we describe the structure of a (non-degenerate) left $\mathcal{H}_X$-module M. Unfortunately we cannot escape the fact that the local Hecke algebras are non-unital. So for $\diamond \in \{x < e > y\}$, $i \geq 0$ let $e_{\diamond,i} \in \mathcal{H}_X$ denote the idempotent projection corresponding to $e_i \in \mathcal{H}(K_\diamond)$. Then let $M_\diamond = \varinjlim e_{\diamond,i} \cdot M = \sum_i e_{\diamond,i} \cdot M$. The idempotents $e_\diamond$ are pairwise orthogonal on distinct cells, so we get a unique decomposition of M of the form,

$$M = M_x \oplus M_e \oplus M_y$$

Where each $M_\diamond$ is a left module over $\mathcal{H}(K_\diamond) \cong \varinjlim e_{\diamond,i} \cdot \mathcal{H}_X \cdot e_{\diamond,i}$. For the off-diagonal Hom spaces, note that there is a natural I-equivariant projection map,

$$\mathrm{res}_{\diamond,e} : \mathcal{H}(K_\diamond) \longrightarrow \mathcal{H}(I)$$

which just restricts a smooth function on $K_\diamond$ to the subgroup I. Recall that we defined $H_{\diamond,i} = \mathrm{Fix}(B(\diamond, r))$, so for $\diamond \leq e$ we have $\mathrm{res}_{\diamond,e}(e_{\diamond,i+1}) = e_{e,i}$. It follows that $\mathrm{res}_{\diamond,e}$

induces an $\mathcal{H}(I)$-equivariant linear map,

$$\gamma_{\diamond,e} : M_\diamond \longrightarrow M_e$$

Moreover, Frobenius reciprocity implies that every I-equivariant map from $\mathcal{H}(K_\diamond)$ to $\mathcal{H}(I)$ is in the span of the $K_\diamond$ orbit of $\mathrm{res}_{\diamond,e}$. In other words, $\mathrm{res}_{\diamond,e}$ generates $\mathrm{Hom}_I(\mathcal{H}(K_\diamond), \mathcal{H}(I))$ as a left $\mathcal{H}(K_\diamond)$ module. It follows that a left $\mathcal{H}_X$-module M is uniquely determined by the following data,

$$M_x \xrightarrow{\gamma_{x,e}} M_e \xleftarrow{\gamma_{y,e}} M_y$$

where each $M_\diamond$ is a left module over $\mathcal{H}(K_\diamond)$, and $\gamma_{\diamond,e} : M_\diamond \to M_e$ is a map of $\mathcal{H}(I)$-modules (where $\mathcal{H}(I) \subset \mathcal{H}(K_\diamond)$ is viewed as a subalgebra via the extension by zero map). Notice that the data of a left $\mathcal{H}_X$-module is essentially that of a sheaf on the complex of groups $\mathcal{Y} = \{K_x \hookleftarrow I \hookrightarrow K_y\}$. By a similar argument, the data of a right $\mathcal{H}_X$-module corresponds to a cosheaf on $\mathcal{Y}$.

Now, given a left $\mathcal{H}_X$-module M we can construct a sheaf,

$$P_X \otimes_{\mathcal{H}_X} M \in \mathrm{Sh}_G(X)$$

Here we must simultaneously think of P_X as a right $\mathcal{H}_X$-module, and M as a left $\mathcal{H}_X$-module. I find it easier to think of this construction in two steps. First, recall that for each $\diamond \in C = \{x < e > y\}$ we have an idempotent $\mathbf{1}_\diamond \in \mathcal{H}_X$ which projects onto the component $P(\diamond)$. So by definition, $P_X \cdot \mathbf{1}_\diamond = P(\diamond)$ and by the adjunction isomorphisms $\mathbf{1}_\diamond \cdot \mathcal{H}_X \cdot \mathbf{1}_\diamond \cong \mathcal{H}(K_\diamond)$. Here we can form the sheaves,

$$P(\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond$$

Recall that $P(\diamond)$ is the constant sheaf on $G \cdot U_\diamond$ with value $\mathcal{H}(K_\diamond)$. So, tensoring with $M_\diamond$ makes $P(\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond$ the constant sheaf on this orbit with value $\mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond$.

Moreover this tensor product inherits a G-equivariant structure from that on $P(\diamond)$.

So far we have taken into account the relations coming from the diagonal terms in $\mathcal{H}_X$, so we must quotient out by the appropriate relations coming from the off-diagonal terms. Here if we view P_X as a *right* $\mathcal{H}_X$-module (i.e. a left $\mathrm{End}_{\mathrm{Sh}_G}(P_X)$-module) there are canonical inclusions,

$$\mathrm{ext}_{e,\diamond} : P(e) \hookrightarrow P(\diamond)$$

induced by the extension by zero map $\mathcal{H}(I) \hookrightarrow \mathcal{H}(K_\diamond)$ (the corresponding "opposite" of the restriction map). As a left $\mathcal{H}_X$-module this is precisely the map which induces the $\mathcal{H}(I)$-equivariant map $\gamma_{\diamond,e} : M_\diamond \to M_e$. In order to introduce these relations to our sheaf $P_X \otimes_{\mathcal{H}_X} M$ let us define,

$$P(e) \otimes_{\mathcal{H}(I)} M_\diamond$$

which is the constant sheaf on $G \cdot e$ with value $\mathcal{H}(I) \otimes_{\mathcal{H}(I)} M_\diamond$, where $M_\diamond$ is viewed as a left $\mathcal{H}(I)$ module by restriction. This sheaf comes with morphisms,

$$\begin{tikzcd} & P(e) \otimes M_\diamond \arrow[dl, "\mathrm{ext}_{e,\diamond}\otimes 1"'] \arrow[dr, "1\otimes\gamma_{\diamond,e}"] & \\ P(\diamond) \otimes M_\diamond & & P(e) \otimes M_e \end{tikzcd}$$

When forming the full tensor product $P_X \otimes_{\mathcal{H}_X} M$, the image of these two morphisms should be identified. The most concise way to state this gluing condition is to say that $P_X \otimes_{\mathcal{H}_X} M$ is the colimit of the following diagram,

$$\begin{tikzcd} P(e) \otimes M_x \arrow[r, "1\otimes\gamma_{x,e}"] \arrow[d, "\mathrm{ext}_{e,x}\otimes 1"'] & P(e) \otimes M_e \arrow[d] & P(e) \otimes M_y \arrow[l, "1\otimes\gamma_{y,e}"'] \arrow[d, "\mathrm{ext}_{e,y}\otimes 1"] \\ P(x) \otimes M_x \arrow[r] & P_X \otimes_{\mathcal{H}_X} M & P(y) \otimes M_y \arrow[l] \end{tikzcd}$$

In total we see that $P_X \otimes_{\mathcal{H}_X} M \in \mathrm{Sh}_G^\infty(X)$ is the sheaf with,

- For each $x = g \cdot \diamond \in X$, the stalk of $P_X \otimes_{\mathcal{H}_X} M$ is given by,

$$\mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond$$

- For each $x \leq y$ we have the restriction map,

$$\mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond \xrightarrow{\mathrm{res}_{\diamond,e} \otimes \gamma_{\diamond,e}} \mathcal{H}(I) \otimes_{\mathcal{H}(I)} M_e$$

As usual the G-equivariant structure should be understood by G acting on the left component P_X. We hope that this reminds the reader of the pullback construction from Chapter 3. We conclude this discussion by showing that the functors $\mathrm{Hom}(P_X, -)$ and $P_X \otimes -$ induce an equivalence of categories between $\mathrm{Sh}_G^\infty(X)$ and $\mathcal{H}_X$-mod.

Theorem 4.2.6. The functor,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, -) : \mathrm{Sh}_G^\infty(X) \longrightarrow \mathcal{H}_X - \mathrm{mod}$$

is an equivalence between the category of smooth equivariant sheaves $\mathrm{Sh}_G^\infty(X)$ and the category of non-degenerate $\mathcal{H}_X$-modules. The quasi-inverse is given by,

$$P_X \otimes_{\mathcal{H}_X} (-) : \mathcal{H}_X - \mathrm{mod} \longrightarrow \mathrm{Sh}_G^\infty(X)$$

Proof. The proof here is essentially a combination of Theorem 3.3.6 and Theorem 4.1.1, so we will lean on those results a bit. Let $S \in \mathrm{Sh}_G^\infty(X)$ be a smooth sheaf and $M \in \mathcal{H}_X$-mod a non-degenerate left $\mathcal{H}_X$-module. Here we will show that the unit and co-unit maps are isomorphisms. Let us start with the unit, which in this case is a map of left $\mathcal{H}_X$-modules,

$$M \xrightarrow{\quad \eta_M \quad} \mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, P_X \otimes_{\mathcal{H}_X} M)$$

Here we recall that $P_X \otimes_{\mathcal{H}_X} M$ is the sheaf whose stalk at $\diamond \in C$ is, $\mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond$. Moreover by the argument given in Prop. 4.2.1,

$$\begin{aligned}
\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, P_X \otimes_{\mathcal{H}_X} M) &= \bigoplus_{\diamond \in C} \mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P(\diamond), P_X \otimes_{\mathcal{H}_X} M) \\
&\cong \bigoplus_{\diamond \in C} \mathrm{Hom}^\infty_{Rep(K_\diamond)}(\mathcal{H}(K_\diamond), \mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond)
\end{aligned}$$

So η descends to the unit isomorphism described in Theorem 4.1.1,

$$\begin{aligned}
\eta_\diamond : M_\diamond &\longrightarrow \mathrm{Hom}_{\mathcal{H}(K_\diamond)}(\mathcal{H}(K_\diamond), \mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond) \\
m &\longmapsto [\varphi_m : f \mapsto f \otimes m]
\end{aligned}$$

It remains to see that for $\diamond \leq e$ the following diagram commutes,

$$\begin{tikzcd}
M_\diamond \arrow[r, "\eta_\diamond"] \arrow[d, "\gamma_{\diamond,e}"'] & \mathrm{Hom}^\infty_{Rep(K_\diamond)}(\mathcal{H}(K_\diamond), \mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} M_\diamond) \arrow[d] \\
M_e \arrow[r, "\eta_e"] & \mathrm{Hom}^\infty_{Rep(I)}(\mathcal{H}(I), \mathcal{H}(I) \otimes_{\mathcal{H}(I)} M_e)
\end{tikzcd}$$

with the vertical arrows viewed as maps of $\mathcal{H}(I)$-modules, but this follows from the naturality of the unit.

Now consider the co-unit, which in this case is a natural transformation between sheaves,

$$P_X \otimes_{\mathcal{H}_X} \mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, S) \xrightarrow{\quad \varepsilon \quad} S$$

Here given a natural transformation $F \in \mathrm{Hom}_{\mathrm{Sh}_G(X)}(P_X, S)$ we should "evaluate" F on P_X. It will be easier to see this is an isomorphism on the fundamental domain $C = \{x < e > y\}$, and by Theorem 3.3.4, this is enough to see that ε is an isomorphism on all of X. Here for each $\diamond \in C$ we saw by the push-pull adjunction that,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P(\diamond), S) \cong \mathrm{Hom}^\infty_{\mathrm{Rep}(K_\diamond)}(\mathcal{H}(K_\diamond), S_\diamond)$$

So, by the discussion above, ε is the natural transformation which on the fundamental domain simply pieces together the "local" co-unit maps,

$$\begin{tikzcd}
\mathcal{H}(K_\diamond) \otimes_{\mathcal{H}(K_\diamond)} \operatorname{Hom}^\infty_{K_\diamond}(\mathcal{H}(K_\diamond), S_\diamond) \arrow[r] \arrow[d, "\varepsilon_\diamond"] & \mathcal{H}(I) \otimes_{\mathcal{H}(I)} \operatorname{Hom}^\infty_I(\mathcal{H}(I), S_e) \arrow[d, "\varepsilon_e"'] \\
S_\diamond \arrow[r, "\gamma_{x,e}"] & S_e
\end{tikzcd}$$

So ε is an isomorphism by Theorem 4.1.1 and by the naturality of the co-unit. □

Remark 4.2.7. One should think of the previous theorem as a "module-theoretic" version of Theorem 3.3.6. In fact, the above construction works for any compact open subgroup $K \leq G$. By the Bruhat-Tits fixed point theorem, any such K has $K \leq K_x$ where K_x is the stabilizer of a point in X. In this case the K action admits a closed fundamental domain, the only difference being that this domain is no longer compact. Here we give a sketch of the analogous construction, mimicking that which is given above.

Fix a compact open subgroup $K = K_x \leq G$, which for convenience we assume is the full stabilizer of a point (i.e. a vertex or an edge) $x \in X$. Again we fix a fundamental domain $D \subset X$ for this action and write $\diamond \in D$ for an element of this domain. We also write $H_\diamond := K \cap K_\diamond$ for the subgroup of K stabilizing $\diamond$. Then as above we have,

$$\begin{tikzcd}
& K/H_\diamond \arrow[dl, "p_\diamond"'] \arrow[dr, "\iota(\diamond)"] & \\
*_{H_\diamond} & & X_K
\end{tikzcd}$$

where $p_\diamond$ is the projection to a point, and $\iota(\diamond)$ is the inclusion of the K-orbit of $\diamond$ into X_K which denotes X with its K action. (Here we are trying be somewhat consistent with notation, although completely mimicking the notation above would make things a bit cluttered) So we can define,

$$P^K(\diamond) := [\iota(\diamond)_\dagger \circ p_\diamond^*]\mathcal{H}(H_\diamond)$$

and hence the K-equivariant projective generator is given by,

$$P_X^K := \bigoplus_{\diamond \in D} P^K(\diamond)$$

Note here if $\diamond, \bullet \in D$ then by the usual adjunctions we get,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_K(X)}(P^K(\diamond), P^K(\bullet)) \cong \mathrm{Hom}^\infty_{\mathrm{Sh}_K(K/H_\diamond)}(p_\diamond^*\mathcal{H}(H_\diamond), \iota(\diamond)^* P^K(\bullet))$$

which is zero unless $\bullet \leq \diamond$. In the case where $\bullet \leq \diamond$ we get,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_K(X)}(P^K(\diamond), P^K(\bullet)) \cong \mathrm{Hom}^\infty_{\mathrm{Rep}(H_\bullet)}(\mathcal{H}(K_\bullet), \mathcal{H}(K_\diamond))$$

Here because D is generally non-compact, we need even more finite conditions. Namely we restrict to the subspace of endomorphisms for which only finitely many components $P^K(\bullet) \to P^K(\diamond)$ is non-zero,

$$\begin{aligned}\mathcal{H}_X^K &:= \mathrm{End}^{\infty,f}_{\mathrm{Sh}_K(X)}(P_X^K)^{\mathrm{op}} \\ &\cong \bigoplus_{\bullet \leq \diamond \in X/K} \mathrm{Hom}^\infty_{H_\bullet}(\mathcal{H}(H_\diamond), \mathcal{H}(H_\bullet))\end{aligned}$$

Again one should restrict to the subspace of “finite” Homs,

$$\mathrm{Hom}^{\infty,f}_{\mathrm{Sh}_K(X)}(P_X^K, S) := \{f \in \mathrm{Hom}_{\mathrm{Sh}_K(X)}(P_X^K, S) \mid f = 0 \text{ for almost all } \diamond \in D\}$$

Then,

$$\mathrm{Hom}^{\infty,f}_{\mathrm{Sh}_K(X)}(P_X^K, S) \cong \bigoplus_{\diamond \in D} \mathrm{Hom}^\infty_{H_\diamond}(\mathcal{H}(H_\diamond), S_\diamond) \cong \bigoplus_{\diamond \in D} S_\diamond$$

is a non-degenerate left module over,

$$\mathcal{H}_X^K := \mathrm{End}^{\mathrm{fin}}_{\mathrm{Sh}_K(X)}(P_X^K)$$

provided each $S_\diamond$ is a smooth representation of $H_\diamond$. If one has the patience for this notation, then one can check that we the same equivalence,

$$\mathrm{Hom}^{\infty,f}_{\mathrm{Sh}_K(X)}(P_X^K, -) : \mathrm{Sh}_K(X) \to \mathcal{H}_X^K - \mathrm{mod}$$

between smooth equivariant sheaves and non-degenerate modules over the K-Hecke algebra $\mathcal{H}_X^K$. We hope this gives an idea for the "recipe" used to construct a projective generator P_X^K in $\mathrm{Sh}_K^\infty(X)$. In any case, let us show that this construction exhibits the desired behavior with the following example.

Example 4.2.8. Here we describe the algebra $\mathcal{H}_X^K$ in the case where $K = K_x$ is the stabilizer of a vertex $x \in X$. By strong transitivity, the quotient X/K can be identified with a ray R extending from x to the boundary ∂X. We defined the filtration subgroups,

$$K \supset K_1 \supset K_2 \supset \cdots$$

Here K_r is the stabilizer of the geodesic R_r of length r emanating from x. Then using the notation above,

$$\mathcal{H}_X^K \cong \bigoplus_{\bullet \leq \diamond \in X/K} \mathrm{Hom}^\infty_{H_\bullet}(\mathcal{H}(H_\diamond), \mathcal{H}(H_\bullet))$$

Note that if $d(x, \diamond) = r$ then $H_\diamond := K_x \cap K_\diamond = K_r$. Here the only non-trivial morphisms are between the subgroups $r - 1 < r < r + 1$. So, if you prefer the matrix algebra presentation of $\mathcal{H}_X$ above, then in this case we see something of the form,

$$\mathcal{H}_X^K \cong \begin{bmatrix} \mathcal{H}(K) & 0 & 0 & 0 & \cdots \\ \mathrm{Hom}_{K_1}(\mathcal{H}(K), \mathcal{H}(K_1)) & \mathcal{H}(K_1) & \mathcal{H}(K_1) & 0 & \cdots \\ 0 & 0 & \mathcal{H}(K_1) & 0 & \cdots \\ 0 & 0 & \mathrm{Hom}_{K_2}(\mathcal{H}(K_1), \mathcal{H}(K_2)) & \mathcal{H}(K_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which should look something like the path algebra of (the poset of) an infinite ray R about x.

$$\begin{tikzcd} \mathcal{H}(K) \arrow[r] & \mathcal{H}(K_1) & \mathcal{H}(K_1) \arrow[l] \arrow[r] & \mathcal{H}(K_2) & \mathcal{H}(K_2) \arrow[l] \arrow[r] & \mathcal{H}(K_3) & \cdots \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r, dash, "e_1"'] & \underset{x_1}{\bullet} \arrow[r, dash, "e_2"'] & \underset{x_2}{\bullet} \arrow[r, dash, "e_3"'] & \cdots \end{tikzcd}$$

4.3 Elementary Sheaves

In this section we give a complete description of the indecomposable injective/projective objects in $\mathrm{Sh}_G^\infty(X)$. The arguments here are inspired by the description of injective/projective sheaves given in in [Cur14].

Definition 4.3.1. Let $W \in \mathrm{Rep}^\infty(K) = \mathrm{Sh}_K^\infty(*_K)$ be a smooth representation of $K = K_\diamond$ for some $\diamond \in C = \{x < e > y\}$. As above, we have the following diagram,

$$\begin{tikzcd} & G/K \arrow[dl, "p_K"'] \arrow[dr, "\iota(\diamond)"] & \\ *_K & & X \end{tikzcd}$$

The left hand side being the quotient projection to a point, and the right hand side identifying G/K with the orbit of $\diamond$ in X. We define the **elementary projective** associated to W,

$$P(W_\diamond) := [\iota(\diamond)_\dagger \circ p_K^*]W$$

$$I(W_\diamond) = [\iota(\diamond)_* \circ p_K^*]W$$

and the **elementary injective** associated to W. By definition, $P(W_\diamond)$ is the constant sheaf with value W on $G \cdot U_\diamond$ and $I(W_\diamond)$ the constant sheaf on $G \cdot \overline{\diamond}$.

Proposition 4.3.2. Let $\diamond \in C = \{x < e > y\}$ and let W be a smooth irreducible representation of $K = K_\diamond$. Then,

1. If $\diamond = \{x, y\}$ is a vertex (i.e. a closed point in C) then $I(W_\diamond)$ is simple and injective, and if $\diamond = e$, then $P(W_\diamond)$ is simple and projective. Moreover every simple sheaf $S \in \mathrm{Sh}_G^\infty(X)$ is of one of these forms.

2. Every indecomposable injective is of the form $I(W_\diamond)$ with W irreducible.

3. Every indecomposable projective is of the form $P(W_\diamond)$ with W irreducible.

Proof. One can repeat the push-pull adjunction argument given in Prop 4.2.3 to see that $I(W_\diamond)$ is injective and $P(W_\diamond)$ is projective, since $\mathrm{Rep}^\infty(K_\diamond)$ is semi-simple.

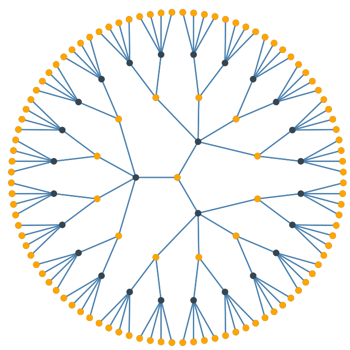
Now we prove the first claim regarding simple sheaves. Let $\diamond$ be a vertex. If $S \subset I(W_\diamond)$ is a subsheaf, then its support must be contained in $G \cdot \diamond$ and must have $S_\diamond \subset W$. Since W is irreducible, it follows that either $S = I(W_\diamond)$ or $S = 0$. So $I(W_\diamond)$ is simple. Now consider $\diamond = e$. Here if $S \subset P(W_\diamond)$ then its support must be contained in $G \cdot e$, and must have $S_\diamond \subseteq W$, so either $S = 0$ or $S = P(W_\diamond)$. Hence $P(W_\diamond)$ is simple.

Now if $S \in \mathrm{Sh}_G^\infty(X)$ is simple then one can take a maximal element $\diamond \in \{x < e > y\}$ such that $S_\diamond$ is nonzero. Taking the G-orbit of this stalk generates a subsheaf either of the form $I(S_\diamond)$ (if $\diamond = \{x, y\}$) or $P(S_\diamond)$ (if $\diamond = e$). Since S is simple, it must be that S is equal to this nonzero subsheaf. Note a sheaf which is supported only on the orbit of $\diamond$ is simple iff $S_\diamond$ is simple as a representation of K. This finishes the proof of (1).

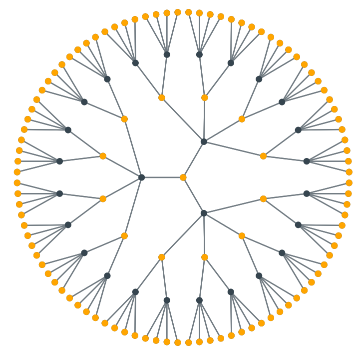
Now we prove (2). Suppose I is indecomposable and injective. Then as above, we can take $\diamond$ to be a maximal element in $\{x < e > y\}$ such that $W := I_\diamond \neq 0$. Taking the orbit of this stalk determines a subsheaf $S(W_\diamond) \subset I$ whose support is contained only in the orbit $G \cdot \diamond$. Now $S(W_\diamond)$ is a subsheaf of both $I(W_\diamond)$ and I so by the injectivity of these sheaves there exist maps α, β making the following diagram commute.

$$\begin{tikzcd}
 & & I(W_\diamond) \arrow[dd, dotted, shift right, "\alpha"'] \\
0 \arrow[r] & S(W_\diamond) \arrow[ur, hook] \arrow[dr, hook] & \\
 & & I \arrow[uu, dotted, shift right, "\beta"']
\end{tikzcd}$$

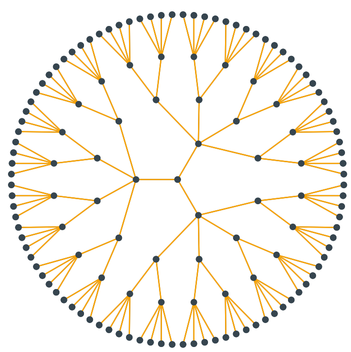
which implies $I \cong I(W_\diamond) \oplus \ker(\beta)$. Now by indecomposability we must have $I = I(W_\diamond)$ which is indecomposable iff $W_\diamond$ is irreducible as a representation of K (because $\mathrm{Rep}^\infty(K)$ is semisimple). The dual argument proves the third claim for projectives. $\square$



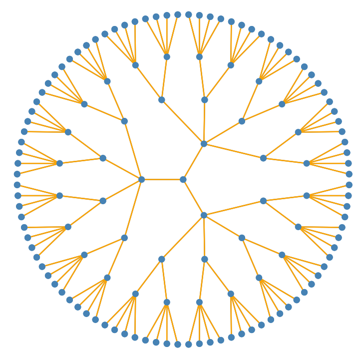
(a) The support of a vertex projective.



(b) The support of a vertex injective



(c) The support of an edge projective.



(d) The support of an edge injective.

Figure 4.1: In each figure the support is colored, with the yellow indicating the orbit along which we pushforward. These images were made with the help of Google Gemini.

Proposition 4.3.3. Let W be a smooth irreducible representation of $K_\diamond$ with $\diamond \in \{x, y\}$. Then $P(W_\diamond)$ is the projective cover of $I(W_\diamond)$. Dually if $\diamond = e$, then $I(W_\diamond)$ is the injective hull of $P(W_\diamond)$.

Proof. Suppose $\diamond = \{x, y\}$. Then by the standard adjunctions we have,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(P(W_\diamond), I(W_\diamond)) \cong \mathrm{End}_{\mathrm{Rep}(K)}(W)$$

which is one dimensional by Schur's lemma. Recall that $P(W_\diamond)$ is the constant sheaf on

$G \cdot U_x$, and that $I(W_\diamond)$ is the constant sheaf on $G \cdot x$. It follows that if f is any nonzero map here, then $P = \ker(f)$ has $\operatorname{supp}(K) \subset (G \cdot U_x) \setminus (G \cdot x) = G \cdot e$. In particular, one can check that $P \cong P(\operatorname{Res}_I^K(W)_e)$ which contains as direct summands all the simple subsheaves of $P(W_\diamond)$. It follows that $P(W_\diamond)$ is the projective cover of $I(W_\diamond)$. The dual argument proves the second statement.

□

Corollary 4.3.4. Suppose $P \in \mathrm{Sh}_G^\infty(X)$ is projective. Then any subsheaf $S \subset P$ is projective. Dually if $I \in \mathrm{Sh}_G^\infty(X)$ is injective, then every quotient $I \twoheadrightarrow S$ is again injective. In other words, one says that the category $\mathrm{Sh}_G^\infty(X)$ is **hereditary**.

Proof. This fact is entirely dependent on the fact that X is a tree. Here it suffices to check the case where $P = P(W_\diamond)$ is indecomposable as above. If $\diamond = e$ then by Prop 4.3.2 $P(W_\diamond)$ is simple, so we are done. If $\diamond = \{x, y\}$ then $P(W_\diamond)$ is the constant sheaf on the open orbit $G \cdot U_\diamond$. If $S \subset P(W_\diamond)$ is a subsheaf such that $S_\diamond \neq 0$ then by irreducibility $S_\diamond = W$ and since $P(W_\diamond)$ is constant on the open neighborhood of $\diamond$, $S = P(W_\diamond)$. So the only proper non-trivial subsheaves $S \subset P(W_\diamond)$ are those for which $S_\diamond = 0$ and $S_e \neq 0$. This implies that S is supported only on the open orbit $G \cdot e$, which is to say that $S \cong P(S_e)$. The dual argument for injectives is similar. □

Proposition 4.3.5. Let $I(W_\diamond)$ be an elementary injective in $\mathrm{Sh}_G^\infty(X)$ as above. Then,

$$I(W_\diamond) \cong c\,\mathrm{Ind}_K^G(I_\diamond(W_\diamond))$$

where $I_\diamond(W_\diamond)$ is an elementary injective in $\mathrm{Sh}_{K_\diamond}^\infty(X)$. The analogous statement holds for projectives.

Proof. As in Prop 3.6.2, let us identify $G/K = G \times_K *_K$ via the map $[g, *] \mapsto gK$. Moreover, let us write $\iota(\diamond) : G \times_K *_K \to X_G$ for the map $[g, *] \mapsto g \cdot \diamond$. The claim follows then from the commutativity of the following diagram,

$$\begin{tikzcd}
*_K \arrow[d, hook, "\beta"'] & G \times_K *_K \arrow[l, "p_K"'] \arrow[d, "\alpha"] \arrow[dr, "\iota(\diamond)"] & \\
X_K & G \times_K X_K \arrow[l, "\overline{p}_K"] \arrow[r, "\overline{\iota(\diamond)}"'] & X_G
\end{tikzcd}$$

The top row we have just discussed. Here we defined,

$$I(W_\diamond) := [\iota(\diamond)_* \circ p_K^*]W$$

$\beta : *_K \hookrightarrow X_K$ is the map sending $* \to \diamond \in X_K$ (which recall denotes X with its K-action). α is the map $[g, *] \mapsto [g, \diamond]$. So we define,

$$I_\diamond(W_\diamond) := \beta_* W \in \mathrm{Sh}_K^\infty(X)$$

Then by the usual push-pull adjunction $I_\diamond(W_\diamond)$ is injective in $\mathrm{Sh}_K^\infty(X)$. We then defined the compactly induced sheaf,

$$c\,\mathrm{Ind}_K^G(I_\diamond(W_\diamond)) := [(\overline{\iota(\diamond)})_! \circ \overline{p}_K^*]I_\diamond(W_\diamond)$$

Here the support of $I_\diamond(W_\diamond)$ in X is closed and compact (either a vertex or the closure of an edge). So, since X is locally finite the pushforward with compact support $(\pi_G)_!$ can be replaced with the usual pushforward $(\pi_G)_*$ (there are only finitely many nonzero stalks in the fiber of each point). In total what we will show is that,

$$I(W_\diamond) := [\iota(\diamond)_* \circ p_K^*]W \cong [(\overline{\iota(\diamond)})_* \circ \overline{p}_K^* \circ \beta_*]W =: c\,\mathrm{Ind}_K^G(I_\diamond(W_\diamond))$$

By the commutativity of the left square we have a natural isomorphism,

$$\overline{p}_K^* \circ \beta_* \cong \alpha_* \circ p_K^*$$

and by the the commutativity of the right triangle we have a natural isomorphism,

$$\iota(\diamond)_* \cong (\overline{\iota(\diamond)})_* \circ \alpha_*$$

This gives,

$$\begin{aligned} c\,\mathrm{Ind}_K^G(I_\diamond(W_\diamond)) &= [(\overline{\iota(\diamond)})_* \circ \overline{p}_K^* \circ \beta_*]W \\ &\cong [(\overline{\iota(\diamond)})_* \circ \alpha_* \circ p_K^*]W \\ &\cong [\iota(\diamond)_* \circ p_K^*]W \\ &= I(W_\diamond) \end{aligned}$$

A similar argument will prove the statement for projectives. □

Corollary 4.3.6. Let $I(W_\diamond) \in \mathrm{Sh}_G^\infty(X)$ be an elementary injective as above. Then,

$$\Gamma_c(X; I(W_\diamond)) \cong c\,\mathrm{Ind}_{K_\diamond}^G(W)$$

Proof. By the previous proposition and Prop. 3.6.2,

$$\Gamma_c(X; I(W_\diamond)) \cong c\,\mathrm{Ind}_K^G(\Gamma_c(X; I_\diamond(W_\diamond)))$$

where $I_\diamond(W_\diamond) := \beta_* W \in \mathrm{Sh}_K(X)$. To compute the global sections of $I_\diamond(W_\diamond)$ we can use the following,

$$\begin{tikzcd} *_K \arrow[r, bend right, "\beta"'] & X_K \arrow[l, bend right, "\pi"'] \end{tikzcd}$$

Here $\pi \circ \beta = \mathrm{Id}$, where π is the projection to a point. Moreover, the support of $I_\diamond(W_\diamond)$ is closed and compact, so we can replace the pushforward with compact support $\pi_!$ with the usual pushforward π_*. It follows that there is a natural isomorphism,

$$\Gamma_c(X; I_\diamond(W_\diamond)) = \pi_* \circ \beta_* W \cong \mathrm{Id}_* W = W$$

$\square$

Remark 4.3.7. In Ch. 1 we showed that global sections always have closed support. Here we saw that the projectives $P(W_\diamond)$ are constant sheaves on open orbits $G \cdot U_\diamond$, so in particular they cannot have any global sections. One thing we can do is consider the pushforward with open support functor $\widetilde{\Gamma}(X; -) = \pi_\dagger : \mathrm{Sh}_G(X) \to \mathrm{Rep}(G)$, which takes a colimit over X rather than a limit. Later we will discuss cohomology, but let us just note here that another way to say the following statement is that $P(W_\diamond)$ has cohomology concentrated in degree 1. In any case, it is not difficult to adjust the argument in the previous corollary to see the following.

Corollary 4.3.8. Let $P(W_\diamond)$ be an elementary projective and let $K = K_\diamond$. If $\diamond = e$ is an edge, then $\widetilde{\Gamma}(X; P(W_\diamond)) = \pi_\dagger P(W_\diamond) \cong c\,\mathrm{Ind}_K^G(W)$. If $\diamond = x, y$ is a vertex then,

$$\widetilde{\Gamma}(X; P(W_\diamond)) \cong c\,\mathrm{Ind}_K^G(\mathrm{St}_\diamond \otimes W)$$

where $\mathrm{St}_\diamond := \mathrm{coker}(\Delta : \mathbb{C}_\diamond \hookrightarrow \bigoplus_{e>\diamond} \mathbb{C}_e)$ and Δ is the diagonal embedding. The notation $\mathrm{St}_\diamond$ refers to the **Steinberg representation** at $\diamond$.

4.4 Nakayama Duality

In the previous sections we saw that there is an equivalence $\mathrm{Sh}_G^\infty(X) \cong \mathcal{H}_X$-mod between smooth equivariant sheaves and non-degenerate $\mathcal{H}_X$ modules. Here we describe how this observation leads to some natural dualities between sheaves and cosheaves.

Definition 4.4.1. Here we describe the **smooth dual** functor D,

$$\mathrm{Sh}_G^\infty(X) \xleftrightarrow{D(-)} \mathrm{coSh}_G^\infty(X)$$

For $S \in \mathrm{Sh}_G^\infty(X)$ we let $D(S)$ denote the cosheaf with,

- $D(S)_x = \mathrm{Hom}_{\mathbb{C}}^\infty(S_x, \mathbb{C}) = S_x^\vee$, the smooth dual of S_x.

- Whenever $x \leq y \in X$ (so $y \leq x \in X^{\mathrm{op}}$) let $\gamma_{x,y} : S_x \to S_y$ denote the restriction map from S. Then $D(\gamma_{x,y}) : D(S)_y \to D(S_x)$ is the adjoint map,

$$\mathrm{Hom}_{\mathbb{C}}^\infty(S_y, \mathbb{C}) \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(\gamma_{x,y}, \mathbb{C})} \mathrm{Hom}_{\mathbb{C}}^\infty(S_x, \mathbb{C})$$

If $S \in \mathrm{Sh}_G(X)$ is G-equivariant, then $D(S)$ inherits an equivariant structure from S. That is for each $g \in G, x, y = gx \in X$ we have a map $g_{x,y} : S_x \to S_y$. We let $D(g)_{x,y}$ denote the induced map,

$$\begin{aligned} D(g)_{x,y} : \mathrm{Hom}_{\mathbb{C}}^\infty(S_x, \mathbb{C}) &\longrightarrow \mathrm{Hom}^\infty(S_y, \mathbb{C}) \\ f &\mapsto \Big[[g \cdot f](s_y) = f(g_{y,x}^{-1} \cdot s_y) \Big] \end{aligned}$$

One can check that these maps commute with the relevent (co)restriction maps, and hence that this action gives a well defined equivariant structure on the cosheaf $D(S)$. Moreover, given $F : S \to T$, one can check that D gives a natural transformation $D(F) : D(T) \to D(S)$. In total, we get a contravariant functor,

$$D(-) : \mathrm{Sh}_G^\infty(X) \to \mathrm{coSh}_G^\infty(X)$$

which is exact because the smooth dual functor is, and exactness for sheaves can be checked at the stalks. The reverse direction from cosheaves to sheaves is described similarly.

Remark 4.4.2. For $S \in \mathrm{Sh}_G^\infty(X)$ we have a canonical evaluation morphism,

$$\mathrm{ev} : S \to D^2 S$$

which at the level of stalks is given by the usual one,

$$\begin{aligned} \mathrm{ev}_x : S_x &\to (S_x^\vee)^\vee \\ v &\mapsto [f \mapsto f(v)] \end{aligned}$$

It is not too difficult to check that this map is an isomorphism if and only if for all $x \in X$, S_x is an admissible representation of K_x. In particular if we restrict to the (abelian) subcategory of such sheaves D is an antiequivalence.

Definition 4.4.3. We say that $S \in \mathrm{Sh}_G^\infty(X)$ is **admissible** if for all $x \in X$, S_x is admissible as a representation of K_x.

Proposition 4.4.4. Let W be a smooth admissible representation of $K_\diamond$ for some $\diamond \in X$ and let $P(W_\diamond), I(W_\diamond)$ be the associated elementary projective/injective sheaves described above. We write $\widehat{P}(W_\diamond), \widehat{I}(W_\diamond)$ for the respective projective/injective cosheaves. Then,

$$P(W_\diamond) \xleftrightarrow{D(-)} \widehat{I}(W_\diamond^\vee)$$

$$I(W_\diamond) \longleftrightarrow \widehat{P}(W_\diamond^\vee)$$

Proof. By definition $P(W_\diamond)$ is the constant sheaf with value W on the open orbit $G \cdot U_\diamond \subset (X, \leq)$. So, $D(P(W_\diamond))$ is the constant cosheaf on this orbit in $(X, \leq^{op})$ with value $W^\vee$. Note then that $G \cdot U_\diamond$ is closed in $(X, \leq^{\mathrm{op}})$, which is to say that $D(P(W_\diamond))$ is the elementary injective cosheaf $\widehat{I}(W_\diamond^\vee)$. A similar argument shows that $D(I(W_\diamond)) = \widehat{P}(W_\diamond)$. $\square$

Remark 4.4.5. Let $\diamond \in C = \{x < e > y\}$ as usual. Although we have avoided it thus far, in a moment we will need **Hecke cosheaves** analogous to the sheaves in the previous section. Here we define, $\widehat{I}(\diamond) := D(P(\diamond))$ and $\widehat{P}(\diamond) := D(I(\diamond))$. Moreover let,

$$\widehat{I}_X := D(P_X)\ ,\ \widehat{P}_X := D(I_X)$$

Then D is an anti-equivalence and $I_X \in \mathrm{Sh}^\infty_G(X)$ is admissible, so we can identify,

$$\mathrm{End}^\infty_{\mathrm{co\,Sh}_G(X)}(\widehat{P}_X)^{\mathrm{op}} \cong \mathrm{End}^\infty_{\mathrm{Sh}_G(X)}(I_X) \cong \mathcal{H}_X^{\mathrm{op}}$$

Moreover by essentially the same argument as was given for sheaves, the following pair of functors define an adjoint equivalence,

$$\mathrm{Hom}^\infty_{\mathrm{coSh}_G(X)}(\widehat{P}_X, -) : \mathrm{coSh}^\infty_G(X) \to \text{mod-}\mathcal{H}_X$$

$$(-) \otimes_{\mathcal{H}_X} \widehat{P}_X : \text{mod-}\mathcal{H}_X \to \mathrm{coSh}^\infty_G(X)$$

Recall that at the level of admissible representations, the smooth dual functor was naturally isomorphic to $V \mapsto \mathrm{Hom}^\infty_K(V, \mathcal{H}(K))$. Moreover in Proposition 4.2.1 we showed that,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(S, I(\diamond)) \cong \mathrm{Hom}^\infty_{K_\diamond}(S_\diamond, \mathcal{H}(K_\diamond)) \cong S_\diamond^\vee$$

So taking Hom into $I_X = \oplus_C I(\diamond)$ gives,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(S, I_X) \cong \bigoplus_{\diamond \in C} S_\diamond^\vee$$

Which is now a right $\mathcal{H}_X$-module. Tensoring with the dual projective generator results in a cosheaf which is naturally isomorphic to DS,

$$DS \cong \mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(S, I_X) \otimes_{\mathcal{H}_X} \widehat{P}_X$$

This might look strange, but perhaps it is familiar if we imagine X is a point with trivial action by a profinite group K. In this case we just recover the smooth dual of an admissible repsentation,

$$V^\vee \cong \mathrm{Hom}_K^\infty(V, \mathcal{H}(K)) \otimes_{\mathcal{H}(K)} \mathcal{H}(K)$$

which is what we saw at the end of Section 4.1. Here $\mathcal{H}(K)$ was both a projective generator and an injective cogenerator for the category of smooth representations.

Remark 4.4.6. The next construction needs some introduction, so let us suppose we have an (associative) ring A. Then we have the so called **algebraic duality** functor,

$$\mathrm{Hom}_A(-, A) : A\text{-mod} \to \text{mod-}A$$

It is well known that P is a left indecomposable projective module if and only if $P = Ae$ for some primitive idempotent $e \in A$. Here the algebraic dual of Ae comes with an isomorphism,

$$\begin{aligned} \mathrm{Hom}_A(Ae, A) &\to eA \\ f &\mapsto f(e) \end{aligned}$$

Which is now indecomposable and projective as a right A-module. This breaks down if P is not finitely generated. For instance if $P = \bigoplus_{i=1}^\infty Ae_i$ then,

$$\mathrm{Hom}_A(P, A) \cong \prod_i e_i A$$

which is now degenerate as a right A-module. As we have done before, we avoid this issue by restricting to the subspace of finite homs,

$$\mathrm{Hom}_A^\infty(P, A) := \{f : P \to A \mid f(e_i) = 0 \text{ for almost all } i\}$$

which is then a non-degenerate A-module. We are inclined to call this the **smooth algebraic dual**, which we denote by $(-)^* = \mathrm{Hom}_A^\infty(-, A)$. In particular if $A = \varinjlim Ae$, then $A^* = A$, with the left/right actions swapped. The discussion at the end of Section 4.1 was precisely to show that the smooth dual of a left $\mathcal{H}(K)$-module is this smooth algebraic dual. At the level of sheaves these this algebraic dual differs from the linear dual, and gives rise to what is known as Nakayama duality.

Now let's try to transport this to our setting. Unlike $D(-)$, it is unclear to me how to give a clean definition of the algebraic dual entirely at the level of sheaves. Instead we consider the following diagram,

$$\begin{tikzcd}
\mathrm{Sh}_G(X) \arrow[r, "(-)^*"] \arrow[d, "\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X,-)"'] & \mathrm{coSh}_G(X) \\
\mathcal{H}_X - \mathrm{mod} \arrow[r, "\mathrm{Hom}^\infty_{\mathcal{H}_X}(-,\mathcal{H}_X)"] & \mathrm{mod} - \mathcal{H}_X \arrow[u, "(-)\otimes_{\mathcal{H}_X}\widehat{P_X}"']
\end{tikzcd}$$

Definition 4.4.7. We define the **smooth algebraic dual** $(-)^* : \mathrm{Sh}_G^\infty(X) \to \mathrm{co}\,\mathrm{Sh}_G^\infty(X)$ to be the composition along the bottom path of the diagram above,

$$(-)^* := \mathrm{Hom}^\infty_{\mathcal{H}_X}(\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, -), \mathcal{H}_X) \otimes_{\mathcal{H}_X} \widehat{P_X}$$

Whatever this functor is, since both the vertical arrows are equivalences it should be the "natural" lift of the smooth algebraic dual $\mathrm{Hom}^\infty_{\mathcal{H}_X}(-, \mathcal{H}_X)$ present at the level of $\mathcal{H}_X$-modules. Luckily, it is possible to clean things up using the equivalence proved in Theorem 4.2.5. Namely we can tensor both components on the left with P_X, then use the counit isomorphism to see that,

$$\begin{tikzcd}
\mathrm{Hom}^{\infty}_{\mathcal{H}_X}(\mathrm{Hom}^{\infty}_{\mathrm{Sh}_G}(P_X, -), \mathcal{H}_X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G}(-, P_X) \\
\mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(P_X \otimes \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(P_X, -), P_X \otimes_{\mathcal{H}_X} \mathcal{H}_X) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

So the algebraic dual of $S \in \mathrm{Sh}^{\infty}_G(X)$ is given by,

$$S^* := \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(S, P_X) \otimes_{\mathcal{H}_X} \widehat{P}_X \in \mathrm{coSh}_G(X)$$

Recall that any subobject of a projective object in $\mathrm{Sh}_G(X)$ is again projective, so $S^* = 0$ whenever S has no projective summands. Moreover, applying $(-)^*$ to the generator P_X one sees that,

$$(P_X)^* = \mathcal{H}_X \otimes_{\mathcal{H}_X} \widehat{P}_X \cong \widehat{P}_X$$

Similarly we can go backwards and define,

$$(-)^{*,-1} := P_X \otimes_{\mathcal{H}_X} \mathrm{Hom}_{\mathrm{co\,Sh}_G(X)}(-, \widehat{P_X})$$

and see that applying this functor to the dual generator gives,

$$(\widehat{P_X})^{*,-1} := P_X \otimes_{\mathcal{H}_X} \mathcal{H}_X \cong P_X$$

Proposition 4.4.8. The functors $(-)^*, (-)^{*,-1}$ induce a duality,

$$\mathrm{Proj}(\mathrm{Sh}^{\infty}_G(X)) \longleftrightarrow \mathrm{Proj}(\mathrm{co\,Sh}^{\infty}_G(X))$$

$$P(W_\diamond) \leftrightarrow \widehat{P}(W_\diamond^\vee)$$

Here Proj refers to the full subcategory of admissible projective objects, and $\widehat{P}(W_\diamond^\vee)$

denotes the projective cosheaf associated to the smooth dual of W.

Proof. Let $\diamond \in C$ and let $W \in \mathrm{Rep}^\infty(K_\diamond)$ be admissible. In light of Proposition 4.4.1, the proof boils down to showing that $(P(W_\diamond))^* \cong D(I(W_\diamond)) = \widehat{P}(W_\diamond^\vee)$. So we work backwards from $D(I(W_\diamond))$. Here we use the discussion in Remark 4.4.4 to see that,

$$D(I(W_\diamond)) \cong \mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(I(W_\diamond)), I_X) \otimes_{\mathcal{H}_X} \widehat{P}_X$$

Now using the push-pull adjunction arguments given in Proposition 4.2.1,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(I(W_\diamond), I_X) \cong \bigoplus_{\bullet \in C} \mathrm{Hom}^\infty_{\mathrm{Rep}(K_\diamond)}((I(W_\diamond))_\bullet, \mathcal{H}(K_\bullet))$$

By definition $I(W_\diamond)$ is the constant sheaf with value W on $G \cdot \overline{\diamond}$. So,

$$I(W_\diamond)_\bullet = \begin{cases} W \text{ if } \bullet \leq \diamond \in C \\ 0 \text{ otherwise} \end{cases}$$

So we get, $\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(I(W_\diamond), I_X) \cong \bigoplus_{\bullet \leq \diamond \in C} \mathrm{Hom}^\infty_{K_\bullet}(W, \mathcal{H}(K_\bullet))$. Moreover by the dual push-pull argument,

$$\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P(W_\diamond), P_X) \cong \bigoplus_{\bullet \in C} \mathrm{Hom}^\infty_{K_\diamond}(W, P(\bullet)_\diamond)$$

Here $P(\bullet)$ is the constant sheaf on $G \cdot U_\bullet$ with value $\mathcal{H}(K_\bullet)$,

$$P(\bullet)_\diamond = \begin{cases} \mathcal{H}(K_\bullet) \text{ if } \bullet \leq \diamond \in C \\ 0 \text{ otherwise} \end{cases}$$

So the push-pull adjunctions give rise to an isomorphism between,

$$\begin{aligned} \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(P(W_\diamond), P_X) &\cong \bigoplus_{\bullet \leq \diamond \in C} \mathrm{Hom}^{\infty}_{K_\diamond}(W, \mathcal{H}(K_\bullet)) \\ &\cong \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(I(W_\diamond), I_X) \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{P}(W_\diamond^\vee) = D(I(W_\diamond)) &\cong \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(I(W_\diamond)), I_X) \otimes_{\mathcal{H}_X} \widehat{P}_X \\ &\cong \mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(P(W_\diamond)), P_X) \otimes_{\mathcal{H}_X} \widehat{P}_X \\ &=: (P(W_\diamond))^* \end{aligned}$$

The dual argument from cosheaves to sheaves can be done similarly. □

Definition 4.4.9. We define the **Nakayama functor**,

$$\nu(-) := D((-)^*)$$

and the **Nakayama inverse**,

$$\nu(-)^{-1} := (D(-))^{*,-1}$$

Remark 4.4.10. We saw in Corollary 4.8.1 that the category $\mathrm{Sh}^{\infty}_G(X)$ is hereditary, meaning that every subobject of a projective object is again projective. It follows that if $S \in \mathrm{Sh}^{\infty}_G(X)$ has no projective summands then,

$$\mathrm{Hom}^{\infty}_{\mathrm{Sh}_G(X)}(S, P_X) = 0$$

and hence $\nu(S)$ is the zero object in $\mathrm{Sh}^{\infty}_G(X)$. Dually if S has no injective summands, then DS has no projective summands and hence $\nu^{-1}(S)$ is the zero object in $\mathrm{Sh}_G^\infty(X)$.

Theorem 4.4.11. The Nakayama functors induce an equivalence between the full subcategories consisting of admissible injective and projective sheaves,

$$\mathrm{Proj}(\mathrm{Sh}_G^\infty(X)) \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathrm{Inj}(\mathrm{Sh}_G^\infty(X))$$

$$P(W_\diamond) \longleftrightarrow I(W_\diamond)$$

Proof. This follows from Propositions 4.4.1 and 4.4.2. □

Proposition 4.4.12. ν is right exact and ν^{-1} is left exact.

Proof. We saw that $D(-)$ is exact since the smooth dual functor is, and exactness for sheaves can be checked at the stalks. $\mathrm{Hom}(-, P_X)$ is always right exact, so the composition, $D((-)^*)$ is right exact. For ν^{-1}, we apply D first, which is contravariant, so the composition $(D(-))^{*,-1}$ is left exact. □

Remark 4.4.13. We will see later that the Nakayama functor is what enables us to explore the implications of Auslander-Reiten theory in this context. The approach given here is fairly involved, so let us just remark that there is a cleaner way to describe this functor as follows. Namely we could restrict ourselves to the full subcategories $\mathrm{Proj}(\mathrm{Sh}_G^\infty(X))$ and $\mathrm{Inj}(\mathrm{Sh}_G^\infty(X))$ as above and simply define ν, ν^{-1} such that, $\nu(P(W_\diamond)) := I(W_\diamond)$ and $\nu^{-1}(I(W_\diamond))$. Here if W is finite dimensional (which by Prop 4.3.2 is certainly the case if $P(W_\diamond)$ is indecomposable), then by the adjunction arguments given in Prop 4.2.1,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}_G(X)}(-, \nu P(W_\diamond)) &= \mathrm{Hom}_{\mathrm{Sh}_G(X)}(-, I(W_\diamond)) \\
&\cong \mathrm{Hom}_{\mathrm{Rep}(K_\diamond)}((-)_\diamond, W) \\
&\cong D\,\mathrm{Hom}_{\mathrm{Rep}(K_\diamond)}(W, (-)_\diamond) \\
&\cong D\,\mathrm{Hom}_{\mathrm{Sh}_G(X)}(P(W_\diamond), -)
\end{aligned}$$

where now D is the usual linear dual. In particular this isomorphism makes the assignment $\nu(P(W_\diamond)) = I(W_\diamond)$ functorial when restricted to finitely generated projective sheaves $\mathrm{Proj}^{\mathrm{fg}}(\mathrm{Sh}_G^\infty(W))$. In an abelian category (perhaps with a few more adjectives), any functor ν satisfying $\mathrm{Hom}(-, \nu P) \cong D\,\mathrm{Hom}(P, -)$ for P projective is called a Nakayama functor. The perspective given above has the minor advantage that the functors we define make sense on the full subcategory of admissible objects in $\mathrm{Sh}_G^\infty(X)$. In any case, later we will use the Nakayama functor both at the level of $\mathrm{Sh}_G^\infty(X)$ and in $\mathrm{Sh}_K^\infty(X)$ where $K = K_\diamond$ is the stabilizer of a point. It is not hard to adjust the arguments above to see that in this case every indecomposable injective/projective is of the form $I_\diamond(W_x), P_\diamond(W_x)$ where W is an irreducible smooth representation of $K_x \cap K_\diamond$. These are just constant sheaves on a closed/open $K_\diamond$-orbit as opposed to a G-orbit. Then $\nu : \mathrm{Sh}_K^\infty(X) \to \mathrm{Sh}_K^\infty(X)$ just swaps $I_\diamond(W_x) \leftrightarrow P_\diamond(W_x)$. Rather than repeat these arguments in both categories, we can safely lean on this more conceptual definition. In any case, this perspective on the Nakayama functor is explored in the paper *Auslander-Reiten Theory via Nakayama Duality in Abelian Categories* [LL25], and later we will cite their results for a few theorems.

Chapter 5

Cohomology

5.1 Resolutions

Proposition 5.1.1. Suppose $S \in \mathrm{Sh}^{\infty}_{G}(X)$ has no injective or projective summands. Then there is an exact sequence of the form,

$$0 \to S \to I(S_e) \to I(\widetilde{\Gamma}(U_x; S|_{U_x})) \oplus I(\widetilde{\Gamma}(U_x; S|_{U_x})) \to 0$$

called a **injective co-presentation** of S. Moreover one also has an exact sequence of the form,

$$0 \to P(\Gamma(\bar{e}; S|_{\bar{e}})) \to P(S_x) \oplus P(S_y) \to S \to 0$$

called a **projective presentation** of S.

Proof. Since S has no projective summands, every vector $v_e \in S_e$ has $v_e = \gamma_{\diamond,e}(v_\diamond)$ for some $\diamond \in \{x, y\}$ (if not, then the subspace of such vectors would generate a semi-simple projective summand of S). So there is a surjection,

$$P(S_x) \oplus P(S_y) \twoheadrightarrow S$$

The kernel of this map is supported only on the open orbit $G \cdot e$, so it is semisimple and projective. Namely if we fix an edge e then we have,

$$0 \to S(U_x \cup U_y) \hookrightarrow S_x \oplus S_y \xrightarrow{\gamma_{x,e}-\gamma_{y,e}} S_e$$

where by definition $\Gamma(\bar{e}; S|_{\bar{e}}) := S(U_x \cup U_y)$ is the kernel of the difference map, $\gamma_{x,e}-\gamma_{y,e} : S_x \oplus S_y \to S_e$. It follows that,

$$0 \to P(\Gamma(\bar{e}; S)) \to P(S_x) \oplus P(S_y) \to S \to 0$$

is an exact sequence as above. Now since S has no injective summands, every vector $v_x \in S_x$ must have $\gamma_{x,e}(v_x) \neq 0$ for some $e > x$ (otherwise the subspace of these vectors would split off as a semi-simple injective summand). In other words, there is an injection,

$$S_x \hookrightarrow \bigoplus_{e>x} S_e$$

and similarly for S_y. It follows that there is an injection, $S \hookrightarrow I(S_e)$ which at the level of stalks is precisely the map $S_x \hookrightarrow \bigoplus_{e>x} S_e$ described above. The cokernel of this map is supported only at the vertices, so it must be semisimple and injective. In particular if we fix a vertex x then we have,

$$S_x \hookrightarrow \bigoplus_{e>x} S_e \twoheadrightarrow \bigoplus_{e>x} S_e/\mathrm{im}(S_x)$$

where by definition,

$$\widetilde{\Gamma}(U_x; S|_{U_x}) := \varinjlim_{U_x} S_x \cong \bigoplus_{e>x} S_e/\mathrm{img}(S_x)$$

(see Remark 1.2.4). It follows that,

$$0 \to S \to I(S_e) \to I(\widetilde{\Gamma}(U_x; S)) \oplus I(\widetilde{\Gamma}(U_x; S)) \to 0$$

is an exact sequence as above. $\square$

Remark 5.1.2. If S has injective/projective summands, then one can simply place these in the first step of the above presentations to obtain a resolution for arbitrary S.

Remark 5.1.3. Let $S \in \mathrm{Sh}_G^\infty(X)$ and let,

$$0 \to S \to I_0 \to I_1 \to 0$$

be an injective copresentation as above. We saw that the functor $\Gamma(X; -) = \pi_*(-)$ is left exact, so taking global sections gives an exact sequence,

$$0 \to \Gamma(X; S) \to \Gamma(X; I_0) \xrightarrow{d} \Gamma(X; I_1)$$

In this setting the fix is simple, we simply repair this exactness by defining $H^1(X; S) := \mathrm{coker}(d)$. Let us also write $H^0(X; S) := \Gamma(X; S)$. Then we obtain an exact sequence,

$$0 \to H^0(X; S) \to \Gamma(X; I_0) \xrightarrow{d} \Gamma(X; I_1) \to H^1(X; S) \to 0$$

Taking global sections with compact support gives a similar sequence,

$$0 \to H^0_c(X; S) \to \Gamma_c(X; I_0) \xrightarrow{d} \Gamma_c(X; I_1) \to H^1_c(X; S) \to 0$$

where now we write $H^0_c(X; S) := \Gamma_c(X; S) = \ker(d)$ and $H^1_c(X; S) := \mathrm{coker}(d)$. Dually, if we take a projective presentation of S,

$$0 \to P_1 \to P_0 \to S \to 0$$

Applying $\pi_\dagger(-)$ gives a right exact sequence,

$$\pi_\dagger P_1 \xrightarrow{\partial} \pi_\dagger P_0 \to \pi_\dagger S \to 0$$

Here we let $H_0(X;S) := \pi_\dagger S$ and $H_1(X;S) := \ker(\partial)$. These are computed via the complex,

$$0 \to H_1(X;S) \to \pi_\dagger P_1 \xrightarrow{\partial} \pi_\dagger P_0 \to H_0(X;S) \to 0$$

Theorem 5.1.4. Up to isomorphism, the representations $H^i(X;S)$, $H_i(X;S)$ are independent of the chosen injective resolution of S.

Proof. For cohomology, this is Theorem 1.4.2 in Shephard's thesis [She85]. See Curry's thesis [Cur14] Section 7.4 for the analogous result regarding sheaf homology. □

Definition 5.1.5. The representations,

$$H^i(X;S)\ ,\ i = 0, 1$$

are the **cohomology** spaces associated to S. The subspaces,

$$H^i_c(X;S) \subseteq H^i(X;S)$$

are the **compactly supported cohomology** of S. Finally,

$$H_i(X;S)\ ,\ i = 0, 1$$

are the **homology** spaces associated to S.

Remark 5.1.6. Let $S \in \mathrm{Sh}^\infty_G(X)$ without injective or projective summands. Referring back to Section 4, and using the resolutions provided in Prop. 5.11, one sees that the complex which computes the cohomology of S is given by,

$$0 \to H^0(X;S) \to \mathrm{Ind}_e^G(S_e) \xrightarrow{d} \mathrm{Ind}_x^G(\widetilde{\Gamma}(U_x;S)) \oplus \mathrm{Ind}_y^G(I(\widetilde{\Gamma}(U_y;S)) \to H^1(X;S) \to 0$$

which if you like is the complex,

$$0 \to H^0(X;S) \to \prod_{e\in E(X)} S_e \xrightarrow{d} \prod_{x\in V(X)} \widetilde{S}(x) \to H^1(X;S) \to 0$$

Similarly the compactly supported cohomology of S is computed via the complex,

$$0 \to H^0_c(X;S) \to c\operatorname{Ind}_e^G(S_e) \xrightarrow{d} c\operatorname{Ind}_x^G(\widetilde{S}(x)) \oplus c\operatorname{Ind}_y^G(\widetilde{S}(y)) \to H^1_c(X;S) \to 0$$

which can be written,

$$0 \to H^0_c(X;S) \to \bigoplus_{e\in E(X)} S_e \xrightarrow{d} \bigoplus_{x\in V(X)} \widetilde{S}(x) \to H^1_c(X;S) \to 0$$

Finally, the homology of S is computed via the complex,

$$0 \to H_1(X;S) \to \bigoplus_{e\in E(X)} \Gamma(\overline{e};S|_{\overline{e}}) \xrightarrow{d} \bigoplus_{x\in V(X)} \mathrm{St}_x \otimes S_x \to H_0(X;S) \to 0$$

Where recall St_x refers to the **Steinberg representation** at x. By definition it is given by,

$$\widetilde{\Gamma}(U_x;\mathbb{C}) := \varinjlim_{U_x} \mathbb{C}_x = \operatorname{coker}(\Delta : \mathbb{C} \hookrightarrow \bigoplus_{e>x} \mathbb{C}_e)$$

where Δ is the diagonal embedding. Choosing a fundamental domain gives,

$$0 \to H_1(X;S) \to c\operatorname{Ind}_e^G(\Gamma(\overline{e};S|_{\overline{e}})) \xrightarrow{d} c\operatorname{Ind}_x^G(St_x \otimes S_x) \oplus c\operatorname{Ind}_y^G(\mathrm{St}_y \otimes S_y) \to H_0(X;S) \to 0$$

Theorem 5.1.7. Let $S \in \mathrm{Sh}_G(X)$ with no injective/projective summands. Then,

$$H^0_c(X;S) \cong H_1(X;S)$$

$$H^1_c(X;S) \cong H_0(X;S)$$

Proof. The first isomorphism is a matter of unraveling some definitions. As we discuss above, the homology of S is computed via the complex,

$$0 \to H_1(X;S) \to \bigoplus_{e\in E(X)} \Gamma(\bar{e};S|_{\bar{e}}) \xrightarrow{\partial} \bigoplus_{x\in V(X)} \mathrm{St}_x \otimes S_x \to H_0(X;S) \to 0$$

where,

$$\mathrm{St}_x \otimes S_x \cong \mathrm{coker}(\Delta : S_x \hookrightarrow \bigoplus_{e>x} S_x)$$

and $\Delta : S_x \hookrightarrow \bigoplus_{e>x} S_x$ is the diagonal embedding. Here ∂ sends a section $f_e \in \Gamma(\bar{e};S|_{\bar{e}})$ to,

$$\partial f_e = (..., 0, f_{e,x}, f_{e,y}, 0, ...) \in \Big(\bigoplus_{e>x} S_x \Big/ \Delta(S_x)\Big) \oplus \Big(\bigoplus_{e>y} S_y \Big/ \Delta(S_y)\Big)$$

In other words, ∂ sends f_e to its values on the vertices $x < e > y$, in the e component of $\bigoplus_{e>x} S_x/\Delta(S_x)$. So for $f = (f_e) \in \bigoplus_e \Gamma(\bar{e};S)$ to be in $H_1(X;S) := \ker(\partial)$ it means that for all $x \in V(X)$, $e > x$,

$$(f_{e,x})_{e>x} = 0 \in \bigoplus_{e>x} S_x \Big/ \Delta(S_x)$$

which is to say that $f_{e,x} = f_{e',x}$ for all $e, e' > x$. In other words, f is comprised of sections on a finite number of edges whose values at all vertices agree. It follows that f constitutes a global section with compact support on X.

For the second isomorphism let us recall that,

$$H_0(X;S) := \varinjlim_{X} S_x = \bigoplus_{x\in X} S_x \Big/ \langle s_x - \gamma_{x,e}(s_x)\rangle$$

Thn since S has no injective summands (in particular no sections supported only on vertices) we can identify,

$$\bigoplus_{x\in X} S_x \Big/ \langle s_x - \gamma_{x,e}(s_x)\rangle \cong \bigoplus_{e} S_e \Big/ d(\bigoplus_{x} S_x) = H^1_c(X;S)$$

$\square$

Theorem 5.1.8. (Long exact sequences) Let,

$$0 \to S \to T \to R \to 0$$

be a short exact sequence in $\mathrm{Sh}_G(X)$. Then one has a long exact sequence in cohomology,

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X;S) \arrow[r] & \Gamma(X;T) \arrow[r] & \Gamma(X;R) \arrow[dll] & \\
 & H^1(X;S) \arrow[r] & H^1(X;T) \arrow[r] & H^1(X;R) \arrow[r] & 0
\end{tikzcd}$$

and a long exact sequence in homology,

$$\begin{tikzcd}
0 \arrow[r] & H_1(X;S) \arrow[r] & H_1(X;T) \arrow[r] & H_1(X;R) \arrow[dll] & \\
 & H_0(X;S) \arrow[r] & H_0(X;T) \arrow[r] & H_0(X;R) \arrow[r] & 0
\end{tikzcd}$$

Proof. This is a consequence of the Snake Lemma. $\square$

Chapter 6

Auslander-Reiten Theory

6.1 Minimal Presentations

Definition 6.1.1. Let $S, T \in \mathrm{Sh}_G^\infty(X)$.

- A morphism $f : S \to T$ is **left minimal** if for all $\varphi \in \mathrm{End}_{\mathrm{Sh}_G(X)}(T)$, $\varphi \circ f = f$ implies φ is an isomorphism. A **minimal injective copresentation** is an injective copresentation,

$$0 \to S \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \to 0$$

in which d_0, d_1 are left minimal.

- $f : S \to T$ is **right minimal** if for all $\psi \in \mathrm{End}_{\mathrm{Sh}_G(X)}(S)$, $f \circ \psi = f$ implies ψ is an isomorphism. A **minimal projective presentation** is a projective presentation,

$$0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} S \to 0$$

in which ∂_1, ∂_0 are right minimal.

Remark 6.1.2. In our setting $\mathrm{Sh}_G^\infty(X)$ is hereditary, so these presentations can always be chosen to be short exact sequences.

Example 6.1.3. Here are some canonical examples of maps which are not left/right minimal. Let R be a ring and let $M = M_1 \oplus M_2$ be a decomposable left R module. Then let $f : M_1 \hookrightarrow M_1 \oplus 0 \subset M$ and $g : M \twoheadrightarrow M_1 \oplus 0 \subset M$ be the obvious inclusion/projection maps. Then f is not left minimal because $g \circ f = f$ and g is not an isomorphism $M \to M$. Similarly g is not right minimal because $g \circ g = g$ and g is not an isomorphism. In particular, idempotent projections can never be left/right minimal.

Proposition 6.1.4. If $S \in \mathrm{Sh}_G^\infty(X)$ has no injective summand, then the injective copresentation described in Prop 5.1.1 is minimal. Similarly if S has no projective summand, then the projective presentation described in Prop 5.1.1 is minimal.

Proof. We only prove that the projective presentation is minimal, as the injective copresentation follows a similar argument. So, assume S has no projective summand and take the projective presentation we describe in Prop 5.1.1,

$$0 \to P(\Gamma(\bar{e}; S|_{\bar{e}})) \xrightarrow{\partial_1} P(S_x) \oplus P(S_y) \xrightarrow{\partial_0} S \to 0$$

As a sheaf (forgetting the equivariant structure) we have $P(S_x) \oplus P(S_y) \cong \bigoplus_{v \in V(X)} P(S_v)$ where $P(S_v)$ is the constant sheaf on U_v with value S_v. The map ∂_0 is meant to indicate the transformation which is the identity at vertices, and the sum map $\gamma_{x,e} + \gamma_{y,e} : S_x \oplus S_y \to S_e$ whenever $x < e > y$. In particular, if $\psi \in \mathrm{End}_{\mathrm{Sh}_G(X)}(S)$ satisfies $\partial_0 \circ \psi = \partial_0$ then ψ is the identity map on vertices. If $x \leq e$ this forces ψ to be the identity map on $\gamma_{x,e}(S_x)$ via the diagram,

$$\begin{tikzcd} S_x \arrow[r, "\mathrm{Id}"] \arrow[d, "\gamma_{x,e}"'] & S_x \arrow[d, "\gamma_{x,e}"] \\ S_e \arrow[r] & S_e \end{tikzcd}$$

Moreover since S has no projective summands, for all $e \in E(X)$, $S_e = \gamma_{x,e}(S_x) + \gamma_{y,e}(S_y)$. This implies $\psi = \mathrm{Id}_S$, and hence that ∂_0 is right minimal. Now for each edge $e \in E(X)$, $\partial_{1,e}$ is the inclusion $\Gamma(\bar{e}; S) := \ker(S_x \oplus S_y \to S_e) \hookrightarrow S_x \oplus S_y$. So if

$\psi \in \mathrm{End}_{\mathrm{Sh}_G(X)}(P(S_x) \oplus P(S_y))$ satisfies $\psi \circ \partial_1 = \partial_1$, then it preserves the values of sections at every edge. Again since S has no projective summands, for every $v \in V(X)$, the map $\oplus_{e>v}\Gamma(\bar{e}; S) \to S_v$ is surjective. The cokernel of this map is precisely the subspace of vectors $s_v \in S_v$ which fail to produce a section on any edge $e > v$, and this subspace would generate a projective subsheaf. So since every vector $s_v \in S_v$ comes from $\Gamma(\bar{e}; S)$ for some $e > v$, and ψ must act as the identity on such vectors, $\psi = \mathrm{Id}_{P(S_x)\oplus P(S_y)}$. It follows that ∂_1 is right minimal, and hence that the projective presentation is minimal. $\square$

Suppose we have two minimal projective presentations of $S \in \mathrm{Sh}_G^\infty(X)$,

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\partial_0"] & S \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P_1' \arrow[r, "\partial_1'"] & P_0' \arrow[r, "\partial_0'"] & S \arrow[r] & 0 \end{tikzcd}$$

Then using projectivity, one can find maps making the following diagram commute,

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, dashed, shift left, "g_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, dashed, shift left, "g_0"] & S \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P_1' \arrow[r, "\partial_1'"] \arrow[u, dashed, shift left, "f_1"] & P_0' \arrow[r, "\partial_0'"] \arrow[u, dashed, shift left, "f_0"] & S \arrow[r] & 0 \end{tikzcd}$$

In particular $f_0 \circ g_0 \in \mathrm{End}_{\mathrm{Sh}_G(X)}(P_0)$ and $\partial_0 \circ f_0 \circ g_0 = \partial_0' \circ g_0 = \partial_0$. By definition, minimality implies $f_0 \circ g_0$ is an isomorphism. If you like, we can see that $P_1 \cong P_1'$ because f_0, g_0 send $P_1 \cong \ker(\partial_0) \cong \ker(\partial_0') \cong P_1'$. It follows that minimal projective presentations are unique up to isomorphism, and hence the ones we describe represent this unique isomorphism class. A similar argument shows that minimal injective copresentations are unique up to isomorphism. Let us end this section by stating the following fact, which is essentially obvious by looking at the presentations we describe in Prop. 5.1.1,

Proposition 6.1.5. If $S \in \mathrm{Sh}_G^\infty(X)$ is admissible, then both its minimal injective copresentation, and minimal projective presentation are comprised of admissible objects.

6.2 The Auslander-Reiten translates

Let $S \in \mathrm{Sh}_G^\infty(X)$ be indecomposable and admissible. Then we can choose an admissible minimal projective presentation of S,

$$0 \to P_1 \to P_0 \to S \to 0$$

Applying the Nakayama functor ν (which by Prop. 4.4.12 is right exact) gives,

$$\nu(P_1) \to \nu(P_0) \to \nu(S) \to 0$$

In particular if S is not projective, then $\nu(S) = 0$, so we have,

$$\nu(P_1) \to \nu(P_0) \to 0$$

which by construction is an injective resolution of $\ker(\nu(P_1) \to \nu(P_0))$.

Definition 6.2.1.

$$\tau S := \ker(\nu(P_0) \to \nu(P_1)) \in \mathrm{Sh}_G(X)$$

is the **Auslander-Reiten translate** of S.

Dually one could take an injective resolution of S,

$$0 \to S \to I_0 \to I_1 \to 0$$

and apply the Nakayama inverse $\nu^{-1}(-)$ (which is left exact) to get,

$$0 \to \nu^{-1}(S) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1)$$

In particular if S is not injective, then $\nu^{-1}(S) = 0$, so we have,

$$0 \to \nu^{-1} I_0 \to \nu^{-1} I_1$$

which is a projective resolution of $\operatorname{coker}(\nu^{-1} I_0 \to \nu^{-1} I_1)$.

Definition 6.2.2.

$$\tau^{-1} S := \operatorname{coker}(\nu^{-1} I_0 \to \nu^{-1} I_1)$$

is the **inverse Auslander-Reiten translate** of S.

Proposition 6.2.3. Let $S \in \mathrm{Sh}_G^{\infty}(X)$ be indecomposable and admissible. Then,

- Up to isomorphism, $\tau^{\pm} S$ is independent of the chosen presentation/copresentation.
- $\tau S = 0$ if and only if S is projective.
- $\tau^{-1} S = 0$ if and only S is injective.
- τ, τ^{-1} are inverse bijections between the sets of admissible, indecomposable, non-projective and non-injective objects in $\mathrm{Sh}_G^{\infty}(X)$.

Proof. The first statement follows from the fact that minimal projective/injective presentations are unique up to isomorphism. The second and third follow immediately from the definition. Namely if S is injective/projective, then $S \xrightarrow{\mathrm{Id}} S$ is its own minimal presentation, and hence the second term in its minimal presentation is 0.

To see that τ, τ^{-1} are inverses, it suffices to show that they send minimal presentations to minimal presentations. In our setting we can just see this directly. Namely take S indecomposable and not projective, then take the projective presentation as in Prop 5.1.1. Then by Prop 4.4.11 we get an injective copresentation of τS of the form,

$$0 \to \tau S \to I(\Gamma(\bar{e}; S)) \to I(S_x) \oplus I(S_y) \to 0$$

So $(\tau S)_e = \Gamma(\bar{e}; S)$ and $(\tau S)_x = \ker(\oplus_{e>x}\Gamma(\bar{e}; S) \twoheadrightarrow S_x)$. But then,

$$\operatorname{coker}(\tau S_x \hookrightarrow \bigoplus_{e>x} \Gamma(\bar{e}; S)) \cong S_x$$

So this copresentation is precisely of the form in Prop 5.1.1. A similar argument works in the opposite direction. Since ν, ν^{-1} are inverses, this implies τ, τ^{-1} are inverses.

To see that they preserve indecomposablility, suppose that $\tau S = T \oplus R$. Then we could choose an injective resolution $\tau S \to I_T^\bullet \oplus I_R^\bullet$ which respects this splitting. Applying ν^{-1} would give a direct sum presentation of S implying S is decomposable. □

6.3 Almost split sequences

Definition 6.3.1. A morphism, $g : S \to T$ is **right almost split** if,

(i) g is not a split epimorphism.

(ii) If $h : R \to T$ is not a split epimorphism then h factors through g,

$$\begin{tikzcd} S \arrow[r, "g"] & T \\ R \arrow[u, dotted, "\exists"] \arrow[ur, "h"'] & \end{tikzcd}$$

We say that g is **minimal right almost split** if g is right almost split and right minimal. Dually, a morphism $f : T \to S$ is **left almost split** if,

(i) f is not a split monomorphism.

(ii) If $g : T \to R$ is not a split monomorphism then g factors through f,

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[d, dotted, "\exists"] \\ & R \end{tikzcd}$$

again we say f is **minimal left almost split** if f is left almost split, and left minimal.

Proposition 6.3.2. (Lemma 1.7 [ARS95])

Let $g : T \to R$ and $f : S \to T$ be morphisms in $\mathrm{Sh}_G^{\infty}(X)$.

(i) if g is right almost split, then R is indecomposable. If R is not projective, then g is an epimorphism.

(ii) if f is left almost split, then S is indecomposable. If S is not injective, then f is a monomorphism.

Proof. Assume g is right almost split. Then suppose for contradiction $R = R_1 \oplus R_2$ is decomposable. Then the inclusion maps, $i_1 : R_1 \hookrightarrow R$, $i_2 : R_2 \hookrightarrow R$ are not epimorphisms. So they factor through g. In other words there exist $h_1 : R_1 \to T$ and $h_2 : R_2 \to T$ such that $g \circ h_j = i_j$. It follows that, $h = h_1 \oplus h_2$ gives a factorization of $\mathrm{Id}_R = i_1 \oplus i_2$ through g

$$\begin{tikzcd} T \arrow[r, "g"] & R \\ R_1 \oplus R_2 \arrow[u, "h"] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

That is $\mathrm{Id}_R = g \circ h$. It would follow then that g is a split epimorphism, which contradicts the assumption that g is right almost split. So R must be indecompoosable. Now suppose R is not projective. Then we saw that $\mathrm{Sh}_G^{\infty}(X)$ has enough projectives, so there is a projective cover, $p : P(R) \to R$ which is a non-split epimorphism. So p factors through g and hence g is an epimorphism. The dual argument proves (ii). $\square$

Definition 6.3.3. An **almost split sequence** or **AR-sequence** is a short exact sequence,

$$0 \longrightarrow S \xrightarrow{g} T \xrightarrow{f} R \longrightarrow 0$$

in which g is left almost split and f is right almost split.

Remark 6.3.4. Until now we have been able to work at the level of smooth, admissible sheaves in $\mathrm{Sh}_G^\infty(X)$. For what follows it is necessary to make even more finiteness assumptions. Here it is in our best interest to restrict to the case of locally finite dimensional sheaves, i.e. those for which $\dim_{\mathbb{C}}(S_x) < \infty$ for all $x \in X$. Let us write $\mathrm{Sh}_G^{\mathrm{fd}}(X)$ for the (abelian) subcategory of $\mathrm{Sh}_G^\infty(X)$ consisting of locally finite dimensional sheaves. In light of the discussion in Chapter 4, it is not difficult to see that $S \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$ if and only if $\mathrm{Hom}^\infty_{\mathrm{Sh}_G(X)}(P_X, S)$ is finitely generated as a left $\mathcal{H}_X$-module, and in particular finite dimensional as $\mathbb{C}$-vector space. For locally finite sheaves, it is known that every object decomposes uniquely into indecomposables [BCB20]. Moreover by a standard argument using Fitting's lemma, $S \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$ is indecomposable if and only if $\mathrm{End}_{\mathrm{Sh}_G(X)}(S)$ is a local ring.

Lemma 6.3.5. (Ch. I, Section 2 [ARS95]) Suppose,

$$0 \longrightarrow S \xrightarrow{g} T \xrightarrow{f} R \longrightarrow 0$$

is an almost split sequence in $\mathrm{Sh}_G^{\mathrm{fd}}(X)$. Then f is right minimal and g is left minimal.

Proof. The proof here follows essentially several lemmas in Ch. 1 Section 2 of [ARS95]. We only prove the first claim, since the second follows by a dual argument. Suppose for contradiction that f is not right minimal and let $K = g(S) = \ker(f)$. Since f is not right minimal, there exists a non-invertible endomorphism $\varphi \in \mathrm{End}(T)$ such that $f \circ \varphi = f$. Then by Fitting's lemma, there exists some $n > 0$ such that,

$$T = \mathrm{Im}(\varphi^n) \oplus \ker(\varphi^n)$$

(this is where we use that T has finite length). Let us write $A = \mathrm{Im}(\varphi^n)$ and $B = \ker(\varphi^n)$. Then $f \circ \varphi^n = f$ so, $B \subset K$. Moreover,

$$K = \ker(f|_A) \oplus B$$

If $\ker(f|_A) = 0$, then $f|_A : A \to R$ is an isomorphism (f is an epimorphism and hence $f|_A$ would be both an epimorphism and a monomorphism). This would make f a split epimorphism which is a contradiction. Thus $K = \ker(f|_A) \oplus B$ would give a nonzero decomposition of K, which contradicts the assumption that K is indecomposable. It follows that f must be right minimal. $\square$

Proposition 6.3.6. (Uniqueness of almost-split sequences) Almost split sequences in $\mathrm{Sh}_G^{\mathrm{fd}}(X)$ starting or ending with the same object are unique up to isomorphism.

Proof. Suppose we have two almost split sequences ending with $R \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$.

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "g"] & T \arrow[r, "f"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S' \arrow[r, "g'"] & T' \arrow[r, "f'"] & R \arrow[r] & 0
\end{tikzcd}$$

Then $f : T \to R$ and $f' : T' \to R$ are right almost split, so there exist maps α, β making the following diagram commute,

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "g"] & T \arrow[r, "f"] \arrow[d, shift left, "\alpha"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S' \arrow[r, "g'"] & T' \arrow[r, "f'"] \arrow[u, shift left, "\beta"] & R \arrow[r] & 0
\end{tikzcd}$$

That is where, $f = f' \circ \alpha$ and $f' = f \circ \beta$. Therefore,

$$f' = f \circ \beta = f' \circ \alpha \circ \beta$$

$$f = f' \circ \alpha = f \circ \beta \circ \alpha$$

Now by Lemma 6.3.5, f is right minimal, so $\alpha \circ \beta$ and $\beta \circ \alpha$ are automorphisms. It follows that α, β are isomorphisms and hence that $T \cong T'$. In particular α, β send $S \cong \ker(f) \cong S' \cong \ker(f')$. So, the sequences are isomorphic. A dual argument proves the statement if two sequences have $S = S'$. $\square$

Theorem 6.3.7. (Existence of almost-split sequences) Let $S \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$ be indecomposable and not projective. Then there is an almost split sequence of the form,

$$0 \longrightarrow \tau S \longrightarrow T \longrightarrow S \longrightarrow 0$$

Dually if S is indecomposable and not injective, then there is an almost split sequence of the form,

$$0 \longrightarrow S \longrightarrow R \longrightarrow \tau^{-1} S \longrightarrow 0$$

Proof. This follows from Theorem 2.5.1 in [LL25]. $\square$

Theorem 6.3.8. (Auslander-Reiten Duality) Let $S, T \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$, where S has no projective summands. Then we have the following isomorphisms, natural in both S, T,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(T, \tau S) \cong D\,\mathrm{Ext}^1_{\mathrm{Sh}_G(X)}(S, T)$$
$$D\,\mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, T) \cong \mathrm{Ext}^1_{\mathrm{Sh}_G(X)}(T, \tau S)$$

Here D refers to the linear dual of the Hom/Ext spaces, which are necessarily finite dimensional since $S \in \mathrm{Sh}_G^{\mathrm{fd}}(X)$.

Proof. This follows from Theorem 2.3.3 in [LL25]. $\square$

6.4 The AR quiver

Proposition 6.4.1. Let S, T be terms in an almost split sequence as above. Then given any commutative diagram,

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & S \\ & R \arrow[ur, "\beta"'] & \end{tikzcd}$$

or dually given any diagram of the form,

$$\begin{tikzcd}
\tau S \arrow[rr, "g"] \arrow[dr, "\alpha"'] & & T \\
& Q \arrow[ur, "\beta"'] &
\end{tikzcd}$$

either α is a split monomorphism or β is a split epimorphism.

Proof. Let us focus on the first diagram, since the argument for the second is similar. Suppose β is not a split epimorphism. Then f is right almost split, so β factors through f. That is $\beta = f \circ \gamma$ for some $\gamma : R \to T$. But then,

$$f = \beta \circ \alpha = (f \circ \gamma) \circ \alpha = f \circ (\gamma \circ \alpha)$$

Now f is also right minimal, so this implies $\alpha \circ \gamma$ is an automorphism of T. So there is a left inverse $h : T \to T$ such that $h \circ (\gamma \circ \alpha) = (h \circ \gamma) \circ \alpha = \mathrm{Id}_T$. It follows that α is a split monomorphism. $\square$

Corollary 6.4.2. Such a factorization through R exists if and only if the Krull-Schmidt decomposition of R contains some of the same indecomposable objects as T. The same holds for Q.

Definition 6.4.3. A morphism $f : T \to S$ is said to be **irreducible** if it is not an isomorphism, and for any factorization,

$$\begin{tikzcd}
T \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & S \\
& R \arrow[ur, "\beta"'] &
\end{tikzcd}$$

either α is a split monomorphism or β is a split epimorphism.

Proposition 6.4.4. An irreducible morphism $f : S \to T$ exists if and only if either S or T occurs as a direct summand in the middle term of an almost split sequence with S, T as the first or last term respectively.

Proof. See Chapter V, Theorem 5.3 in [ARS95]. □

Definition 6.4.5. Let us now describe how the irreducible maps give a sort of "atlas" on the category $\mathrm{Sh}_G(X)$. We construct a directed graph (i.e. a quiver) $Q_G(X)$ called the **Auslander-Reiten quiver** as follows,

- The vertices of $Q_G(X)$ are indexed by isomorphism classes of indecomposable sheaves $[S]$ in $\mathrm{Sh}_G(X)$.
- There is an arrow $[T] \to [S]$ in $Q_G(X)$ if and only if there is an irreducible morphism $f : T \to S$ in $\mathrm{Sh}_G(X)$.

There is one extra decoration (which we will mostly ignore for our purposes). Suppose there is a right minimal almost split map,

$$T^{\oplus a} \oplus T' \to S$$

where T is not a summand of T'. Moreover, suppose there is a left minimal almost split map,

$$T \to S^{\oplus b} \oplus S'$$

where S is not a direct summand of S'. Then we decorate the arrow $[T] \to [S]$ with the pair (a, b),

$$[T] \xrightarrow{(a,b)} [S]$$

to indicate the multiplicity of S, T in the almost split sequences with S, T as direct summands of their middle terms respectively.

Proposition 6.4.6. Suppose there is an irreducible morphism $T \to S$ in $\mathrm{Sh}^{\mathrm{fd}}_G(X)$. If S, T are indecomposable and not projective, then there is an irreducible morphism,

$$\tau T \longrightarrow \tau S$$

Proof. This follows from Prop 6.2.3. $\square$

Corollary 6.4.7. Suppose $0 \longrightarrow \tau S \longrightarrow T \longrightarrow S \longrightarrow 0$ is an almost split sequence in $\mathrm{Sh}^{\mathrm{fd}}_G(X)$. Then there is an almost split sequence of the form,

$$0 \longrightarrow \tau^2 S \longrightarrow \tau T \longrightarrow \tau S \longrightarrow 0$$

Proof. This follows from the previous proposition, in conjunction with Prop 6.4.4. $\square$

Remark 6.4.8. The previous remarks here imply that the Auslander-Reiten quiver splits into connected components, and that the translates act "by translations" on each component. In applications of this theory, it is common to study the "shape" of a particular component, or to describe how objects within a component are woven together. In the following section we investigate this question for the so called pre-injective component of $Q_G(X)$.

Chapter 7

Preinjective sheaves

Definition 7.0.1. Let $S \in \mathrm{Sh}_G^\infty(X)$ be indecomposable. We say that,

- S is **pre-injective** if $S \cong \tau^k I(W_\diamond)$ for some $k \geq 0$, $W \in \mathrm{Rep}(K_\diamond)$.
- S is **pre-projective** if $S \cong \tau^{-k} P(W_\diamond)$ for some $k \geq 0$, $W \in \mathrm{Rep}(K_\diamond)$.
- S is **regular** otherwise.

By definition, any indecomposable object in $\mathrm{Sh}_G^\infty(X)$ falls into one of these three categories.

Remark 7.0.2. For essentially all of what follows we focus on the case of pre-injective sheaves. This might seem somewhat arbitrary, so let us give some informal reasons why we decided to look at this particular class of sheaves.

In [Wei19a] Weissman showed that, for X a tree, if $S \in \mathrm{Sh}_G(X)$ has $V = \Gamma_c(X; S)$ an irreducible representation of G, then $V \cong c\,\mathrm{Ind}_K^G(W)$ is compactly induced from an (irreducible) representation W of the stabilizer of a point $\diamond \in X$. At least naïvely, this means that the cohomology of S looks-like that of an injective sheaf. Later it was realized that the argument there proves something slightly stronger. If one supposes that $S \in \mathrm{Sh}_G(X)$ has (compactly supported) cohomology concentrated in degree zero, then the argument in [Wei19a] shows that there exists a surjection, $S \twoheadrightarrow I$ onto an injective sheaf, and that this surjection induces an isomorphism in cohomology. This observation is what motivated us to investigate the cohomology of pre-injective sheaves. In particular, for what follows we give recursive formulas which enable one to compute the cohomology of any pre-injective sheaf, as well as some recursive formulas for their "relative" dimension (i.e. the dimension of their inducing data.)

Definition 7.0.3. Let $S \in \mathrm{Sh}_G^\infty(X)$. We say that S is **generated by its compactly supported global sections** if for all $x \in X$, the canonical map $\Gamma_c(X; S) \to S_x$ is surjective.

Example 7.0.4. Let $I(W_\diamond) \in \mathrm{Sh}_G^\infty(X)$ be an elementary injective sheaf. Then one can easily check via Prop 4.3.6 that $I(W_\diamond)$ is generated by its compactly supported global sections.

Proposition 7.0.5. Suppose $S \in \mathrm{Sh}_G^\infty(X)$ is generated by its compactly supported global sections. Then S has no projective summands. Moreover τS is again generated by its compactly supported global sections, and $H_c^1(X; \tau S) = 0$.

Proof. $P(W_\diamond)$ is a constant sheaf on the open orbit $G \cdot U_\diamond$ and hence by Prop 1.2.3 has no global sections. So if S is generated by its global sections, then it has no projective summands. We now prove the statements regarding τS. By Prop. 5.1.1, S has a projective resolution of the form,

$$0 \longrightarrow P(\Gamma(\bar{e}; S)) \longrightarrow P(S_x) \oplus P(S_y) \longrightarrow S \longrightarrow 0$$

Applying the Nakayama functor gives an injective resolution of τS,

$$0 \longrightarrow \tau S \longrightarrow I(\Gamma(\bar{e}; S)) \longrightarrow I(S_x) \oplus I(S_y) \longrightarrow 0$$

Finally, taking global sections with compact support give a left exact sequence,

$$0 \longrightarrow \Gamma_c(X;\tau S) \longrightarrow \bigoplus_{e\in E(X)} \Gamma(\bar{e};S) \xrightarrow{d} \bigoplus_{v\in V(X)} S_v$$

where d sends $s \in \Gamma(\bar{e};S|_{\bar{e}}))$ to its values on the vertices $x < e > y$. We would like to show that d is surjective. By assumption $\Gamma_c(X;S) \twoheadrightarrow S_v$ for all $v \in V(X)$, so let us fix $v \in V(X)$, $s_v \in S_v$ and choose a lift $s \in \Gamma_c(X;S)$ whose value at v is s_v. Then $\mathrm{supp}(s) \subset X$ is compact and closed, so we can choose a finite geodesic $\gamma = [v,w] \subset \mathrm{supp}(s)$ with $w \in \partial\mathrm{supp}(s)$. Let $e_0, ..., e_n$ denote the edges with $R = \cup_{i=0}^{n-1} e_i$ and $e_n \cap \gamma = \{w\}$. So the final edge e_n intersects $\mathrm{supp}(s)$ only at w. Let $s_i \in \Gamma(\overline{e_i};S|_{\bar{e}_i})$ be the restriction of s to $\overline{e_i}$. Then we send s to,

$$s \longmapsto \sum_{i=0}^{n} (-1)^i s_i \in \bigoplus_{e\in E(X)} \Gamma(\bar{e};S|_{\bar{e}}))$$

Now for $x \in V(R)$ a vertex, let $s_{i,x}$ be the value of s_i at x. Note that since s is a global section $s_{i,x} = s_{i+1,x}$. Then by construction,

$$d(\sum_{i=0}^{n} (-1)^i s_i) = s_v$$

This shows that d is surjective and hence that $H^1_c(X;\tau S) = 0$. Now we show that for all $x \in X$, the map $\pi_x : \Gamma_c(X;\tau S) \to \tau S_x$ is surjective. Since τS has no projective summands, it suffices to show this for $x \in V(X)$ a vertex. By definition of τ, for each $x \in V(X)$ we have a short exact sequence of the form,

$$0 \longrightarrow \tau S_x \longrightarrow \bigoplus_{e\geq x} \Gamma(\bar{e};S) \longrightarrow S_x \longrightarrow 0$$

So, an element of τS_x is given by $(s_e) \in \bigoplus_{e>x} \Gamma(\bar{e};S|_{\bar{e}})$ such that $\sum_{e>x} s_{e,x} = 0$. If for all $e > x$, s_e vanishes at the opposing vertex $s_{e,y} = 0$ then we are done, for then (s_e) constitutes a global section in $\Gamma_c(X;\tau S)$. If not, then for each $y < e > x$ with

$s_{e,y} \neq 0$ let us choose a global section $s_y \in \Gamma_c(X; S)$ whose value at y is $s_{e,y}$. Then by the same logic as above, we can construct an element $s \in \bigoplus_{e \in E(X)} \Gamma(\bar{e}; S|_{\bar{e}})$ such that $d(s) = (s_{e,y}) \in \bigoplus_{x \in V(X)} S_x$. It follows then that $d((s_e) - s) = 0$ and hence that $v = (s_e) - s \in \Gamma_c(X; \tau S)$. So we have shown that π_x is surjective. $\square$

Corollary 7.0.6. Let,

$$0 \longrightarrow \tau S \longrightarrow T \longrightarrow S \longrightarrow 0$$

be an almost split sequence in which S is generated by its compactly supported global sections. Then by the previous proposition, and the long exact sequence in cohomology, we obtain a short exact sequence,

$$0 \longrightarrow \Gamma_c(X; \tau S) \longrightarrow \Gamma_c(X; T) \longrightarrow \Gamma_c(X; S) \longrightarrow 0$$

Corollary 7.0.7. If $S \cong \tau^k I(W_\diamond)$, then the compactly supported cohomology of S is concentrated in degree zero, and S is generated by $\Gamma_c(X; S)$.

Proposition 7.0.8. Let $K = K_\diamond$ be with $\diamond \in X$ and suppose $T \in \mathrm{Sh}_K^\infty(X)$ is admissible. Then,

$$\tau(c\,\mathrm{Ind}_K^G(T)) \cong c\,\mathrm{Ind}_K^G(\tau T)$$

Proof. It suffices to consider the case where $T \in \mathrm{Sh}_K(X)$ is indecomposable and not projective. Take a minimal projective presentation of $T \in \mathrm{Sh}_K^\infty(X)$,

$$0 \to P_1 \to P_0 \to T \to 0$$

then since $c\,\mathrm{Ind}_K^G(-)$ is exact and proserves projectivity,

$$0 \to c\,\mathrm{Ind}_K^G(P_1) \to c\,\mathrm{Ind}_K^G(P_0) \to S \to 0$$

is a minimal projective presentation of $S := c\,\mathrm{Ind}_K^G(T) \in \mathrm{Sh}_G^\infty(X)$. Applying the

Nakayama functor gives an injective copresentation of τS,

$$0 \to \tau S \to \nu(c\,\mathrm{Ind}_K^G(P_1)) \to \nu(c\,\mathrm{Ind}_K^G(P_0)) \to 0$$

Let us now show that ν commutes with induction. By Prop. 4.3.5, if $P(W_\diamond)$ is an elementary projective (similarly for $I(W_\diamond)$ an elementary injective) then,

$$P(W_\diamond) = c\,\mathrm{Ind}_K^G(P_\diamond(W_\diamond))$$

where $P_\diamond(W_\diamond)$ is an elementary projective in $\mathrm{Sh}_K(X)$. Here $\nu(P_\diamond(W_\diamond)) = I_\diamond(W_\diamond)$, so we have the following commutative diagram,

$$\begin{tikzcd}
P(W_\diamond) \arrow[r, "\nu"] & I(W_\diamond) \\
P_\diamond(W_\diamond) \arrow[u, "\mathrm{cInd}_K^G"] \arrow[r, "\nu"'] & I_\diamond(W_\diamond) \arrow[u, "\mathrm{cInd}_K^G"']
\end{tikzcd}$$

It follows that ν commutes with induction, so we have,

$$0 \to \tau S \to c\,\mathrm{Ind}_K^G(\nu(P_1)) \to c\,\mathrm{Ind}_K^G(\nu(P_0)) \to 0$$

but by exactness of $c\,\mathrm{Ind}_K^G(-)$ this is also a minimal resolution of $c\,\mathrm{Ind}_K^G(\tau T)$. It follows that $c\,\mathrm{Ind}_K^G(\tau T) \cong \tau(c\,\mathrm{Ind}_K^G(T))$. $\square$

7.1 Knitting

Here we describe the almost split sequences associated to injective (and hence pre-injective sheaves). Using the results of the previous section, this will enable us to describe recursive formulas for their cohomology.

Proposition 7.1.1. Let $I(W_x)$ be an elementary vertex injective in $\mathrm{Sh}_G^\infty(X)$, with W an irreducible representation of K_x. Then there is an almost-split sequence of the form,

$$0 \longrightarrow \tau I(W_x) \longrightarrow I(\mathrm{Res}_e^x W) \longrightarrow I(W_x) \longrightarrow 0$$

Proof. By Prop. 5.1.1, $I(W_x)$ has a minimal projective presentation of the form,

$$0 \to P(\mathrm{Res}_e^x(W_x)) \to P(W_x) \to I(W_x) \to 0$$

Applying the Nakayama functor gives a minimal injective copresentation of $\tau I(W_x)$,

$$0 \to \tau I(W_x) \to I(\mathrm{Res}_e^x(W_x)) \to I(W_x) \to 0$$

The claim is that this sequence is almost-split. Note that it is not split, and that both $I(W_x)$ and $\tau I(W_x)$ are indecomposable. So, suppose S is a sheaf with a nonsplit epimorphism,

$$\begin{tikzcd} I(\mathrm{Res}_e^x(W)) \arrow[rr] & & I(W_x) \\ & S \arrow[ur, "f"] & \end{tikzcd}$$

We would like to show that f factors through $I(\mathrm{Res}_e^x(W))$, which is an edge injective. By adjunction,

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, I(W_x)) \cong \mathrm{Hom}_{K_x}(S_x, W)$$

Then since $\mathrm{Rep}(K_x)$ is semi-simple, and W is irreducible, S_x contains W as a direct summand. Let us write $S(W_x) \subset S_x$ for this summand. Then since f is non-split, and we must have $\gamma_{x,e}(S(W_x)) \neq 0 \subset S_e$ for some $e > x$. Hence we have a commutative diagram of the form,

$$\begin{tikzcd} S_x \arrow[r, "f_x"] \arrow[d] & W \\ S_e \arrow[ur] & \end{tikzcd}$$

Again by adjunction, this induces a map,

$$\mathrm{Hom}_{K_e}(S_e, \mathrm{Res}^x_e(W)) \cong \mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, I(\mathrm{Res}^x_e(W)))$$

It follows that f factors through $I(\mathrm{Res}^x_e(W))$,

$$\begin{tikzcd} I(\mathrm{Res}^x_e(W)) \arrow[r] & I(W_x) \\ S \arrow[u, dotted, "\exists"] \arrow[ur, "f"'] & \end{tikzcd}$$

and hence that the map $I(\mathrm{Res}^x_e(W)) \to I(W_x)$ is minimal right almost split. It follows that the minimal injective presentation associated to $\tau I(W_x)$ is an almost split sequence.

$\square$

Corollary 7.1.2. If $I(W_x)$ is an elementary vertex injective as above, then the almost split sequence ending in $I(W_x)$ arises via compact induction from an almost split sequence in $\mathrm{Sh}^\infty_{K_x}(X)$.

Proof. Note that the first two terms in the sequence are compactly induced, and by Prop 7.0.8 compact induction commutes with the AR-translate. So we have,

$$\begin{tikzcd} 0 \arrow[r] & c\,\mathrm{Ind}^G_x(\tau I_x(W_x)) \arrow[r] & I(\mathrm{Res}^x_e(W_x)) \arrow[r] & I(W_x) \arrow[r] & 0 \\ 0 \arrow[r] & \tau I_x(W_x) \arrow[r] & I_x(\mathrm{Res}^x_e(W_x)) \arrow[r] \arrow[u, "c\,\mathrm{Ind}^G_x"] & \tau I_e(V_e) \arrow[r] & 0 \end{tikzcd}$$

where the bottom sequence is in $\mathrm{Sh}_{K_x}(X)$. Here the middle term is an edge injective supported only on the K_x orbit of edges $e > x$. If you like, one can check that the bottom sequence is almost split via the same proof given in Prop. 7.1.1. □

Corollary 7.1.3. Let $V_\diamond^k(W) = \Gamma_c(X;\tau^k I_\diamond(W_\diamond))$ where $I_\diamond(W_\diamond)$ is the corresponding elementary vertex injective in $\mathrm{Sh}^\infty_{K_\diamond}(X)$. Then for all $k \geq 0$,

$$\Gamma_c(X;\tau^k I(W_x)) \cong c\,\mathrm{Ind}^G_{K_\diamond}(V_\diamond^k(W))$$

Proof. This follows from Prop. 3.6.2, where we show that the global sections functor commutes with induction. Here we are just giving a name to the inducing data, $V_\diamond^k(W) = \Gamma_c(X;\tau^k I_\diamond(W_\diamond))$. □

Remark 7.1.4. We remarked previously that the middle term in an almost split sequence is generally decomposable. In this case, since $\mathrm{Rep}(K_e)$ is semi-simple we can decompose,

$$I(\mathrm{Res}^x_e(W)) \cong \bigoplus_{V \subset \mathrm{Res}^x_e(W)} I(V_e)$$

where this sum runs over all of the irreducible constituents V of $\mathrm{Res}^x_e(W)$. By Frobenius reciprocity, these are precisely the irreps of K_e for which, $W \subset \mathrm{Ind}^x_e(V)$. If we now fix $I(V_e)$ an indecomposable edge injective. Then the stalk of $I(V_e)$ at $\diamond = \{x, y\}$ for $x < e > y$ is given by $\mathrm{Ind}^\diamond_e(V)$. So there are two non-split epimorphisms,

$$\begin{tikzcd}
 & I(\mathrm{Ind}^x_e(V)) \\
I(V_e) \arrow[ur] \arrow[dr] & \\
 & I(\mathrm{Ind}^y_e(V))
\end{tikzcd}$$

Taking the AR-translates of these vertex injectives gives the middle terms in the AR-sequence ending in $I(V_e)$,

$$\begin{tikzcd}
 & \tau I(\mathrm{Ind}_e^y(V)) \arrow[dr] & & I(\mathrm{Ind}_e^x(V)) \\
\tau I(V_e) \arrow[ur] \arrow[dr] & & I(V_e) \arrow[ur] \arrow[dr] & \\
 & \tau I(\mathrm{Ind}_e^y(V)) \arrow[ur] & & I(\mathrm{Ind}_e^y(V))
\end{tikzcd}$$

which we prove in the following proposition.

Proposition 7.1.5. Let $I(V_e)$ be an elementary edge injective in $\mathrm{Sh}_G^\infty(X)$, with V an irreducible representation of K_e. Then there is an AR-sequence of the form,

$$0 \longrightarrow \tau I(V_e) \longrightarrow \tau I(\mathrm{Ind}_e^x(V)) \oplus \tau I(\mathrm{Ind}_e^y(V))) \longrightarrow I(V_e) \longrightarrow 0$$

Proof. The discussion in the previous remark impliess that the $I(\mathrm{Ind}_e^x(V)), I(\mathrm{Ind}_e^y(V))$ contain as direct summands all the indecomposable injectives with $I(V_e)$ as a direct summand in the middle term in their AR-sequence. Now suppose S is indecomposable with $I(V_e)$ as a direct summand in the middle term of its AR-sequence,

$$0 \longrightarrow \tau S \longrightarrow T \oplus I(V_e) \longrightarrow S \longrightarrow 0$$

Applying $\tau \circ \tau^{-1}$ to the above sequence "deletes" $I(V_e)$, so it does not preserve the AR-sequence above. It follows that $\tau^{-1} S = 0$ and hence that $S \subset I(\mathrm{Ind}_e^x(V)$. In other words every irreducible map out of $I(V_e)$ factors through one of the projections, $I(V_e) \twoheadrightarrow I(\mathrm{Ind}_e^\diamond(V))$. It follows that every irreducible map into $I(V_e)$ factors through the map,

$$\tau I(\mathrm{Ind}_e^x(V)) \oplus \tau I(\mathrm{Ind}_e^x(V))) \longrightarrow I(V_e) \longrightarrow 0$$

which is a non-split epimorphism because the projections $I(V_e) \twoheadrightarrow I(\mathrm{Ind}_e^\diamond(V))$ are. So this map is minimal right almost split, and $\tau I(V_e)$ is its kernel. It follows that the almost split sequence ending in $I(V_e)$ is of the form in the proposition. $\square$

Corollary 7.1.6. Let $I(V_e)$ be an indecomposable elementary edge injective as above. Then the almost split sequence ending in $I(V_e)$ arises via compact induction from an almost split sequence in $\mathrm{Sh}_I(X)$ where $I = K_e$ is the stabilizer e.

$$\begin{tikzcd}
0 \arrow[r] & c\operatorname{Ind}_e^G(\tau I_e(V_e)) \arrow[r] & c\operatorname{Ind}_e^G(\tau I_e(V_x) \oplus \tau I_e(V_y)) \arrow[r] & I(V_e) \arrow[r] & 0 \\
0 \arrow[r] & \tau I_e(V_e) \arrow[r] & \tau I_e(V_x) \oplus \tau I_e(V_y) \arrow[r] \arrow[u, dashed, "c\operatorname{Ind}_I^G"] & \tau I_e(V_e) \arrow[r] & 0
\end{tikzcd}$$

Proof. This again becuause τ commutes with induction, and the right-most term is compactly induced.

$\square$

Corollary 7.1.7. Let $E^k(V) := H_c^0(X; \tau^k I_e(V_e))$ where $I_e(V_e)$ is the corresponding elementary vertex injective in $\mathrm{Sh}_{K_e}(X)$. Then for all $k \geq 0$,

$$H_c^0(X; \tau^k I(V_e)) \cong c\operatorname{Ind}_{K_e}^G(E^k(V))$$

Proof. Again this follows from Prop. 3.6.2, and we are just giving a name to the inducing data, $E^k(V) := H_c^0(X; \tau^k I_e(V_e))$. $\square$

Remark 7.1.8. In Section 6.3 we described the Auslander-Reiten quiver, which provides a sort of "atlas" for the category $\mathrm{Sh}_G^\infty(X)$. The discussion above enables us to at least qualitatively describe the so called **pre-injective component** of this quiver, that is the (connected) component consisting of all preinjective sheaves. The inductive procedure we describe here is called the **knitting algorithm**.

Suppose we start with $I(W_x)$ an indecomposable vertex injective. Then by Prop. 7.1.1, there is an almost split sequence of the form,

$$\tau I(W_x) \longrightarrow I(\operatorname{Res}_e^x(W)) \longrightarrow I(W_x)$$

We will use colors to indicate which objects are generally decomposable. In order to compute the $\tau I(W_x)$, one needs to compute the translates of the middle term. This involves "knitting" together the almost split sequence for the vertex and edge injectives. Recall that for the edge injective there are two non-split epimorphisms, $I(\mathrm{Res}_e^x(W)) \twoheadrightarrow I(\mathrm{Ind}_e^{\diamond}(\mathrm{Res}_e^x(W)))$ with $\diamond \in \{x, y\}$. So we have the following situation,

$$\begin{tikzcd}
 & & I(\mathrm{Ind}_e^y(\mathrm{Res}_e^x(W))) \\
\tau I(W_x) \arrow[r] & I(\mathrm{Res}_e^x(W)) \arrow[ur] \arrow[r] \arrow[dr] & I(W_x) \\
 & & I(\mathrm{Ind}_e^y(\mathrm{Res}_e^x(W)))
\end{tikzcd}$$

Note that $I(W_x)$ is really a subsheaf in the bottom blue injective. We then argued that taking the translate of the blue terms gives the middle term in the sequence ending with $I(\mathrm{Res}_e^x(W))$. So we can extend the diagram to see that,

$$\begin{tikzcd}
 & \tau I(\mathrm{Ind}_e^y(\mathrm{Res}_e^x(W))) \arrow[dr] & & I(\mathrm{Ind}_e^y(\mathrm{Res}_e^x(W))) \\
\tau I(\mathrm{Res}_e^x(W)) \arrow[ur] \arrow[r] \arrow[dr] & \tau I(W_x) \arrow[r] & I(\mathrm{Res}_e^x(W)) \arrow[ur] \arrow[r] \arrow[dr] & I(W_x) \\
 & \tau I(\mathrm{Ind}_e^x(\mathrm{Res}_e^x(W))) \arrow[ur] & & I(\mathrm{Ind}_e^x(\mathrm{Res}_e^x(W)))
\end{tikzcd}$$

Now one needs to know how to compute the terms $\tau I(\mathrm{Ind}_e^y(\mathrm{Res}_e^x(W)))$. Here we the middle term of the associated sequence would be given by, $I(\mathrm{Res}_e^y(\mathrm{Ind}_e^y(\mathrm{Res}_e^y(W))))$. In total, the terms which appear on the right will be successive induction/restrictions of the representation W we start with. Moreover the number of terms needed to compute the successsive translates of $I(W_\diamond)$ increases exponentially. Unfortunately a complete description of the quiver would require complete knowledge of the irreducible representations of the stabilizers, which is of course the thing we would like to investigate.

7.2 Recursive formulas in cohomology

In the previous section we,

- Explicitly described the almost split sequences associated to preinjective sheaves, i.e. those of the form $\tau^k I(W_\diamond)$.
- Showed that the compactly supported cohomology of $\tau^k I(W_\diamond)$ is concentrated in degree zero, and these global sections are always compactly induced from a representation of $K_\diamond$. Here we gave names to the inducing data. Namely, we wrote $V_\diamond^k(W)$ for the inducing data if $\diamond \in \{x, y\}$ is a vertex and $E^k(W)$ if $\diamond = e$.

Here we apply the results of the previous section to give recursive formulae for the inducing data. At the level of stabilizers, all smooth representations are semi-simple, so it will be convenient to write these formulae in the Grothendieck group $K^0(\mathrm{Rep}^\infty(K_\diamond))$. The central case will be that where $W = \mathbb{C}$ is the trivial representation of $K_\diamond$. For this we will use the notation,

$$E^k := E^k(\mathbb{C}) \quad V_\diamond^k := V_\diamond^k(\mathbb{C})$$

Proposition 7.2.1. For all $k \geq 0$,

$$V_\diamond^{k+1} \cong \mathrm{Ind}_e^\diamond(E^k) - V_\diamond^k$$

where the right hand side should be thought of as living in the Grothendieck group $K^0(\mathrm{Rep}^\infty(K_\diamond))$

Proof. By Prop. 7.1.1, for all $k \geq 0$ we have an almost split sequence of the form,

$$0 \longrightarrow \tau^{k+1} I(\mathbb{C}_\diamond) \longrightarrow \tau^k I(\mathbb{C}_e) \longrightarrow \tau^k I(\mathbb{C}_\diamond) \longrightarrow 0$$

By Prop. 7.0.6, taking global sections gives a short exact sequence,

$$0 \longrightarrow c\,\mathrm{Ind}_{\diamond}^{G}(V_{\diamond}^{k+1}) \longrightarrow c\,\mathrm{Ind}_{e}^{G}(E^{k}) \longrightarrow c\,\mathrm{Ind}_{\diamond}^{G}(V_{\diamond}^{k}) \longrightarrow 0$$

Which by the exactness and transitivity of induction implies,

$$V_{\diamond}^{k+1} \cong \mathrm{Ind}_{e}^{\diamond}(E^{k}) - V_{\diamond}^{k}$$

□

Proposition 7.2.2. Let $\diamond \in \{x, y\}$, $W \in \mathrm{Rep}^{\infty}(K_{\diamond})$ and $U \in \mathrm{Rep}^{\infty}(I)$. Then for all $k \geq 0$,

$$V_{\diamond}^{k}(W) \cong V_{\diamond}^{k} \otimes W \tag{7.1}$$

$$E^{k}(U) \cong E^{k} \otimes U \tag{7.2}$$

$$E^{k+1} \cong \mathrm{Res}_{e}^{x}(V_{x}^{k+1}) \oplus \mathrm{Res}_{e}^{y}(V_{y}^{k+1}) - E^{k} \tag{7.3}$$

Proof. The proof is by induction on $k \geq 0$. In the base case $k = 0$ we have, $V_{\diamond}^{0} = \mathbb{C}$, $V_{\diamond}^{0}(W) = W$ and $E^{0} = \mathbb{C}$, $E^{0}(U) = U$. So the base case holds. Now assume (1) and (2) are true for $k \geq 0$. By Prop. 7.1.1 we have an almost split sequence of the form,

$$0 \longrightarrow \tau^{k+1} I(W_{\diamond}) \longrightarrow \tau^{k} I(\mathrm{Res}_{e}^{\diamond}(W)) \longrightarrow \tau^{k} I(W_{\diamond}) \longrightarrow 0$$

By the inductive hypothesis and Prop. 7.1, taking global sections gives a short,

$$0 \longrightarrow c\,\mathrm{Ind}_{\diamond}^{G}(V_{\diamond}^{k+1}(W)) \longrightarrow c\,\mathrm{Ind}_{e}^{G}(E^{k} \otimes \mathrm{Res}_{e}^{\diamond}(W)) \longrightarrow c\,\mathrm{Ind}_{\diamond}^{G}(V_{\diamond}^{k}(W)) \longrightarrow 0$$

Here we observe that in the middle term,

$$\begin{aligned} c\,\mathrm{Ind}_e^G(E^k \otimes \mathrm{Res}_e^\diamond(W)) &\cong c\,\mathrm{Ind}_\diamond^G(\mathrm{Ind}_e^\diamond(E^k \otimes \mathrm{Res}_e^\diamond(W))) \\ &\cong c\,\mathrm{Ind}_\diamond^G(\mathrm{Ind}_e^\diamond(E^k) \otimes W) \end{aligned}$$

Here exactness of induction along with the previous proposition implies,

$$\begin{aligned} V_\diamond^{k+1}(W) &\cong \mathrm{Ind}_e^\diamond(E^k) \otimes W - V_\diamond^k \otimes W \\ &\cong (\mathrm{Ind}_e^\diamond(E^k) - V_\diamond^k) \otimes W \\ &\cong V_\diamond^{k+1} \otimes W \end{aligned}$$

which proves (1) in the case $k+1$. We are now able to prove (3) as stated in the proposition. Here by Prop. 7.1.5, we have an almost split sequence in $\mathrm{Sh}_G(X)$ of the form,

$$0 \longrightarrow \tau^{k+1} I(\mathbb{C}_e) \longrightarrow \tau^{k+1} I(\mathrm{Ind}_e^x(\mathbb{C}_e)) \oplus \tau^{k+1} I(\mathrm{Ind}_e^y(\mathbb{C}_e)) \longrightarrow \tau^k I(\mathbb{C}_e) \longrightarrow 0$$

Note that the middle terms are translates of vertex injectives, so by the proof of our first claim and Prop. 7.2.1, taking global sections gives a short exact sequence of the form,

$$0 \longrightarrow c\,\mathrm{Ind}_e^G(E^{k+1}) \longrightarrow c\,\mathrm{Ind}_x^G(V_x^{k+1} \otimes \mathrm{Ind}_e^x(\mathbb{C})) \oplus c\,\mathrm{Ind}_x^G(V_y^{k+1} \otimes \mathrm{Ind}_e^y(\mathbb{C})) \longrightarrow c\,\mathrm{Ind}_e^G(E^k) \longrightarrow 0$$

Here we again use the fact that,

$$V_\diamond^k \otimes \mathrm{Ind}_e^\diamond(\mathbb{C}) \cong \mathrm{Ind}_e^\diamond(\mathrm{Res}_e^\diamond(V_\diamond^k))$$

and so exactness of induction implies,

$$E^{k+1} \cong \mathrm{Res}_e^x(V_x^{k+1}) \oplus \mathrm{Res}_e^y(V_y^{k+1}) - E^k$$

We can finally prove the second claim in the case $k+1$, which follows just by swapping the $\mathbb{C}$'s in the above sequence with U's. Namely, to compute $E^{k+1}(U)$ we have the following short exact sequence,

$$\begin{aligned} 0 \longrightarrow c\operatorname{Ind}_e^G(E^{k+1}(U)) \longrightarrow & c\operatorname{Ind}_x^G(V_x^{k+1} \otimes \operatorname{Ind}_e^x(U)) \\ & \oplus c\operatorname{Ind}_x^G(V_y^{k+1} \otimes \operatorname{Ind}_e^y(U)) \longrightarrow c\operatorname{Ind}_e^G(E^k(U)) \longrightarrow 0 \end{aligned}$$

Here again,

$$V_\diamond^k \otimes \operatorname{Ind}_e^\diamond(U) \cong \operatorname{Ind}_e^\diamond(\operatorname{Res}_e^\diamond(V_\diamond^k) \otimes U)$$

which by exactness and transitivity of induction gives,

$$E^{k+1}(U) \cong \left[\operatorname{Res}_e^x(V_x^{k+1}) \oplus \operatorname{Res}_e^y(V_y^{k+1}) - E^k\right] \otimes U \cong E^{k+1} \otimes U$$

so we have proved (2) in the case $k+1$, and hence completed the inductive step. This concludes the proof. $\square$

Remark 7.2.3. The proof of the previous proposition is somewhat confusing, as the three claims are somehow recursively knitted together. We hope that the following diagram will aid in following the progression of the inductive argument.

$$\begin{tikzcd}
\text{Claim 1: [k]} \arrow[dr] & & & \text{Claim 1: [k+1]} \\
& \text{Claim 1: [k+1]} \arrow[dr] \arrow[urr, leftrightarrow] & & \\
\text{Claim 2: [k]} \arrow[ur] \arrow[rr] & & \text{Claim 3: [k]} \arrow[r] & \text{Claim 2: [k+1]}
\end{tikzcd}$$

Here each claim follows from its predecessors, as indicated by the arrows pointing into the respective node. We write Claim n : $[k]$ to refer to the claim as stated in Prop. 7.2.2.

Remark 7.2.4. With the recursive formulas in hand, let us try to give some of the informal geometric intuition which allowed us to compute them. The idea is that we are recursively "knitting" together the global sections of injectives centered around the fundamental domain $\{x < e > y\}$, in a way that reflects the knitting of their respective almost split sequences. Recall that for $k = 0$ we had,

$$V_x^0 = V_y^0 = E^0 = \mathbb{C}$$

so that in the first step, we start with essentially one global section centered at each point in the fundamental domain. The vertex recursion in Prop 7.2.1 then says that, $V_\diamond^1 = \mathrm{Ind}_e^x(E^0) - V_\diamond^0$,

$$V_x^1 = \mathrm{Ind}_e^x(\mathbb{C}) - \mathbb{C}$$
$$V_y^1 = \mathrm{Ind}_e^y(\mathbb{C}) - \mathbb{C}$$

which we informally think of as spinning the global sections on $\bar{e}$ about the vertices x, y and then taking a sort of "trace-zero" sum at each vertex. This feels as though we are pulling loops while knitting a piece of fabric. In the next step, the edge recursion formula in Prop 7.2.2 says that,

$$E^1 = \mathrm{Res}_e^x(V_x^1) \oplus \mathrm{Res}_e^y(V_y^1) - E^0$$

which one might think of as taking the loops in our fabric V_x^1, V_y^1 and interweaving them again by taking a "trace-zero" sum at the edge. This process then continues on recursively, at each step we "spin" the edge sections about each vertex, take a trace-zero sum, forming the loops in our fabric. We then interweave the loops by taking a trace zero sum at the edge. At each step the sections grow, both in their dimension, and in their support as they spread further from the central chamber. Unfortunately it seems intractable to give explicit decompositions of these representations as k grows, as doing so would require explicit descriptions of certain double coset spaces associated to the filtration subgroups. We will be content in ending this discussion with a simple recursive formula for the dimensions of these representations, as well as a description of their depth.

Corollary 7.2.5. Suppose X is a d-regular tree and let $q = d-1$. Let $p_k(q) := \dim_{\mathbb{C}}(V_\diamond^k)$ and $r_k(q) := \dim_{\mathbb{C}}(E^k)$. Then both $p_k(q), r_k(q)$ are polynomials in q, and for all $k \geq 1$ they satisfy the following recurrence relation,

$$p_{k+1}(q) = 2q \cdot p_k(q) - p_{k-1}(q)$$
$$r_{k+1}(q) = 2q \cdot r_k(q) - r_{k-1}(q)$$

Proof. When $k = 0$, $p_0(q) = r_0(q) = 1$, and by the recursive formulae given in Prop 7.2.1, 7.2.2 one has,

$$p_{k+1}(q) = (q+1) \cdot r_k(q) - p_k(q)$$
$$r_k(q) = 2 \cdot p_k(q) - r_{k-1}(q)$$

So by substitution,

$$\begin{aligned} p_{k+1}(q) &= 2(q+1) \cdot p_k(q) - (q+1) \cdot r_{k-1}(q) - p_k(q) \\ &= (2q+1) \cdot p_k(q) - (q+1) \cdot r_{k-1}(q) \\ &= (2q+1) \cdot p_k(q) - (p_k(q) - p_{k-1}(q)) \\ &= 2q \cdot p_k(q) - p_{k-1}(q) \end{aligned}$$

and by a similar argument one has,

$$\begin{aligned} r_{k+1}(q) &= 2(q+1)\cdot r_k(q) - 2\cdot p_k(q) - r_k(q) \\ &= (2q+1)\cdot r_k(q) - 2\cdot p_k(q) \\ &= (2q+1)\cdot r_k(q) - (r_k(q) + r_{k-1}(q)) \\ &= 2q\cdot r_k(q) - r_{k-1}(q) \end{aligned}$$

□

Proposition 7.2.6. For all $k \geq 0$, $V_\diamond^k$ and E^k have depth k as representations of the respective stabilizer $K_\diamond$.

Proof. Recall that if V is a smooth finite dimensional representation of $K_\diamond$, the depth of V is the minimal $i \geq 0$ such that $H_{\diamond,i}$ acts trivially on V. In the base case both $V_\diamond^0$ and E^0 are the trivial representation of $K_\diamond$ and hence have depth 0. Now suppose for induction that $V_\diamond^k$ and E^k have depth k as representations of their respective stabilizing subgroups. By definition of the filtration subgroups,

$$H_{\diamond,k+1} = \bigcap_{g\in K_\diamond/I} gH_{e,k}g^{-1}$$

So if we examine the formula,

$$V_\diamond^{k+1} = \mathrm{Ind}_e^\diamond(E^k) - V_\diamond^k$$

$\mathrm{Ind}_e^\diamond(E^k) = \bigoplus_{g\in K_\diamond/I} gE^k$ has depth $k+1$, since $H_{\diamond,k+1}$ acts trivially but $H_{\diamond,k}$ does not. Moreover every irreducible constituent of $V_\diamond^k$ has depth less than or equal to k. So, $V_\diamond^{k+1} = \mathrm{Ind}_e^\diamond(E^k) - V_\diamond^k$ has depth $k+1$, since the depth of a finite dimensional representation is always equal to the maximal depth among its irreducible constituents.

The edge filtrations were defined so that,

$$H_{e,k+1} = H_{x,k+1} \cap H_{y,k+1}$$

and hence by a similar reasoning,

$$E^{k+1} = \mathrm{Res}_e^x(V_x^{k+1}) \oplus \mathrm{Res}_e^y(V_y^{k+1}) - E^k$$

has depth $k+1$ as a representation of I. $\square$

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