UC Irvine

UC Irvine Previously Published Works

Title

A periodicity-induced generalized Fourier transform pair

Permalink

<https://escholarship.org/uc/item/09x43623>

Journal

IEEE Transactions on Antennas and Propagation, 50(8)

ISSN

0018-926X

Author

Capolino, F

Publication Date

2002-08-01

DOI

10.1109/TAP.2002.801395

Copyright Information

This work is made available under the terms of a Creative Commons Attribution License, available at <https://creativecommons.org/licenses/by/4.0/>

Peer reviewed

A Periodicity-Induced Generalized Fourier Transform Pair

Filippo Capolino

Abstract—The field radiated by an infinite periodic structure can be expressed in terms of Floquet waves (FWs), both in the frequency domain (FD) and time domain (TD) [1]. A new periodicity-induced generalized Fourier transform (FT) pair is derived relating FD-FWs to TD-FWs and vice versa, based on tabulated transforms and physical conditions at infinity. The new FTs are directly related to the simple canonical problem of a line array of sequentially excited dipoles that is a basic building block for more general phased periodic structures.

Index Terms—Arrays, Fourier transforms (FTs), Green function, periodic structures, transient scattering.

I. INTRODUCTION

Floquet waves (FWs) generated by one-dimensional (1-D) phased periodicity along a rectilinear coordinate z are parameterized by the dispersion relation

$$k_{zq}(\omega) = \omega\gamma_z + \alpha_q, \quad \alpha_q = 2\pi q/d, \quad q = 0, \pm 1, \pm 2, \dots \quad (1)$$

where ω is the radian frequency, k_{zq} is the z -domain wavenumber, γ_z is the interelement phase gradient, d is the interelement spacing, and q is the FW index [1]. The dispersion relation for $q \neq 0$ differs from the nondispersive case $q = 0$, i.e., $k_{z0} = \omega\gamma_z$, only through the constant term α_q . Closed form relations between frequency-domain (FD) and time-domain (TD) FWs can be established by conventional tabulated Fourier transforms (FT) when $q = 0$ [2, pp. 277]. However, no corresponding direct tabulations seem to exist for $q \neq 0$. This has motivated the study of a generalized FT pair for a class of functions that differs from those listed in the mathematical tables by involving Hankel functions with an ω dependence of the form $k_{\rho q}(\omega) = \sqrt{k^2 - k_{zq}^2}$ instead of $k_{\rho 0}(\omega) = \sqrt{k^2 - k_{z0}^2}$, with $k = \omega/c$ (c being the ambient wave speed) and $k_{zq}(\omega)$ given in (1). The periodicity-induced FT will establish direct relations between FD-FW and TD-FW with $q \neq 0$.

The important nondimensional parameter

$$\eta = \gamma_z c \quad (2)$$

permits distinction between two cases depending on $|\eta| \geq 1$, in which the phase velocity $v_z^p = \gamma_z^{-1} = c/\eta$ of the excitation of the periodic structure along z can be larger ($|\eta| < 1$) or smaller ($|\eta| > 1$) than the ambient wavespeed c .

This periodicity-induced FT is directly related to simple radiating systems such as the sequentially excited periodic line array of dipoles that has been studied in detail in [1]. There, TD-FWs have been defined and found via various analytic methods. Here, we prove this generalized periodicity-induced Fourier transform pair going from TD to FD and vice versa, in a direct manner. This constitutes the basic building block for more complicated periodic structures with sequentially excited periodicity cells.

Manuscript received September 16, 2000; revised August 27, 2001. This work was supported by The Italian Research National Council (CNR) under a Grant awarded in 1998.

The author was with the Dipartimento di Ingegneria dell'Informazione, University of Siena, 53100 Siena, Italy. He is now with the Department of Electrical and Computer Engineering, University of Houston, Houston, TX 77004 USA.

Publisher Item Identifier 10.1109/TAP.2002.801395.

II. A PERIODICITY-INDUCED GENERALIZED FOURIER TRANSFORM PAIR

Defining the Fourier forward and inverse transforms as

$$I(\omega) = \int_{-\infty}^{\infty} \hat{I}(t) e^{-j\omega t} dt, \quad \hat{I}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{j\omega t} d\omega \quad (3)$$

with the caret $\hat{\cdot}$ denoting time-dependent quantities, it is demonstrated in what follows that the periodicity-induced FT pair

$$\frac{1}{j} H_0^{(2)}[k_{\rho q}(\omega)\rho] \xleftrightarrow{F} \frac{e^{j\eta\bar{\omega}_q t}}{\pi\sqrt{t^2 - \tau_0^2}} f(t) \quad (4)$$

with

$$k_{\rho q}(\omega) = \sqrt{k^2 - k_{zq}^2} = \sqrt{\left(\frac{\omega}{c}\right)^2 (1 - \eta^2) - 2\left(\frac{\omega}{c}\right)\eta\alpha_q - \alpha_q^2} \quad (5)$$

$$f(t) = \begin{cases} 2 \cos(\bar{\omega}_q \sqrt{t^2 - \tau_0^2}) U(t - \tau_0) & |\eta| < 1 \\ \exp[j \operatorname{sgn}(\eta) \bar{\omega}_q \sqrt{t^2 - \tau_0^2}] & |\eta| > 1 \end{cases} \quad (6)$$

$$\bar{\omega}_q = \frac{\alpha_q c}{1 - \eta^2}, \quad \tau_0 = \frac{\rho \sqrt{1 - \eta^2}}{c} \quad (7)$$

obeys the definitions (3).

On the FD side of (4), the top Riemann sheet of the radial wavenumber $k_{\rho q}(\omega)$ in (5) is chosen to render $\Im m k_{\rho q} \leq 0$, consistent with the radiation condition at $\rho = \infty$ (ρ is the cylindrical coordinate perpendicular to z). Furthermore, $\Re e k_{\rho q} \geq 0$ or ≤ 0 for $\omega > 0$ or < 0 , respectively, in order to satisfy the radiation condition for positive and negative real frequencies. The conditions $|k_{zq}| \geq |k|$ determine an exponentially decaying or oscillating function along ρ , respectively. On the TD side of (4), $U(\tau) = 1$ or 0 if $\tau > 0$ or $\tau < 0$, respectively. In (4)–(7) τ_0 is positive real for $|\eta| < 1$, τ_0^2 is negative for $|\eta| > 1$, and it is convenient to define the branch of the root in (7) as $\tau_0 = -j|\tau_0|$ (this is in accord with the root of $k_{\rho q}$, since $\sqrt{k^2(1 - \eta^2)} = k_{\rho q}|_{q=0}$).

A. TD $\rightarrow$ FD

In order to verify (4) with (3) directly, we derive the FD-FW by Fourier inversion of its TD counterpart. Using in the first integral in (3), a frequency shift $\omega' = \omega - \eta\bar{\omega}_q$ where the q -dependent $\bar{\omega}_q$ defined in (7) is a function of the nondimensional parameter η , the first integral in (3) is rewritten as

$$I(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{f(t)}{\sqrt{t^2 - \tau_0^2}} e^{-j\omega' t}. \quad (8)$$

Depending on $|\eta| \geq 1$, two different procedures are applied.

Case $|\eta| < 1$: Inserting the expression for $f(t)$ from (6), and noting that the unit step function $U(t - \tau_0)$ truncates the domain of integration in (8) we have

$$I(\omega) = \frac{2}{\pi} \int_{\tau_0}^{\infty} dt \frac{\cos(\bar{\omega}_q \sqrt{t^2 - \tau_0^2})}{\sqrt{t^2 - \tau_0^2}} e^{-j\omega' t}. \quad (9)$$

Rewriting $\exp(-j\omega't) = \cos(\omega't) - j\sin(\omega't)$ yields the two tabulated integrals when $\omega' > 0$, [2, pp. 28,86]

$$\int_{\tau_0}^{\infty} dt \cos(\omega't) \frac{\cos(b_q \sqrt{t^2 - \tau_0^2})}{\sqrt{t^2 - \tau_0^2}} = \begin{cases} K_0(\tau_0 \sqrt{\bar{\omega}_q^2 - \omega'^2}) \\ -\frac{\pi}{2} Y_0(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2}) \end{cases} \quad (10)$$

$$\int_{\tau_0}^{\infty} dt \sin(\omega't) \frac{\cos(\bar{\omega}_q \sqrt{t^2 - \tau_0^2})}{\sqrt{t^2 - \tau_0^2}} = \begin{cases} 0, \\ \frac{\pi}{2} J_0(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2}) \end{cases} \quad (11)$$

where, in both (10) and (11), the upper equality apply when $0 < \omega' < |\bar{\omega}_q|$, while the lower applies when $|\bar{\omega}_q| < \omega'$. Also, J_0 and Y_0 are the zero-order Bessel functions of the first and second kind, respectively, and K_0 is the modified Bessel function [3, p. 358]. Using the relations $K_0(\zeta) = -j(\pi/2)H_0^{(2)}(-j\zeta)$ and $Y_0(\zeta) + jJ_0(\zeta) = jH_0^{(2)}(\zeta)$, (10) and (11) can be combined and (9) becomes

$$I(\omega) = \frac{1}{j} \begin{cases} H_0^{(2)}(-j\tau_0 \sqrt{\bar{\omega}_q^2 - \omega'^2}), & 0 < \omega' < |\bar{\omega}_q| \\ H_0^{(2)}(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2}), & |\bar{\omega}_q| < \omega'. \end{cases} \quad (12)$$

Substituting the value of $\omega' = \omega - \eta\bar{\omega}_q$ and $\bar{\omega}_q$ from (7), we have

$$\omega'^2 - \bar{\omega}_q^2 = \omega^2 - 2\omega \frac{\alpha_q \eta c}{1 - \eta^2} - \frac{\alpha_q^2 c^2}{1 - \eta^2} = \frac{c^2}{1 - \eta^2} (k^2 - k_{zq}^2). \quad (13)$$

Recalling that $\tau_0 = \rho\sqrt{1 - \eta^2}/c$, the arguments of the Hankel functions in (12) are re-expressed as follows:

$$k_{\rho q} \rho = \begin{cases} -j\rho \sqrt{k_{zq}^2 - k^2} = -j\tau_0 \sqrt{\bar{\omega}_q^2 - \omega'^2}, & 0 < \omega' < |\bar{\omega}_q| \\ \rho \sqrt{k^2 - k_{zq}^2} = \tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2}, & |\bar{\omega}_q| < \omega' \end{cases} \quad (14)$$

where we have chosen the branch of the square root as that for $k_{\rho q}(\omega)$ in (5) [defined after (7)] thus establishing the result in (4). For $\omega' < 0$, the same considerations apply, after noting that (10) is independent of the sign of ω' while in (11), the substitution $\omega' = -|\omega'|$ relates the result to the one for $\omega' > 0$.

Case $|\eta| > 1$: We recall that in this case we have $\tau_0^2 = -|\tau_0|^2$, with $|\tau_0| = \rho\sqrt{\eta^2 - 1}/c$, in (6) and (8), resulting in

$$I(\omega) = \int_{-\infty}^{\infty} dt e^{-j\omega't} \frac{e^{j \operatorname{sgn}(\eta) \bar{\omega}_q \sqrt{t^2 + |\tau_0|^2}}}{\pi \sqrt{t^2 + |\tau_0|^2}}. \quad (15)$$

This transform is given in [4, pp. 493] as

$$I(\omega) = \frac{1}{j} H_0^{(2)}(|\tau_0| \sqrt{\bar{\omega}_q^2 - \omega'^2}) \quad (16)$$

Using the equality in (13), substituting for $|\tau_0|$ and choosing the branch of the square root in (16) as that for $k_{\rho q}$, we again obtain the FD part in (4).

B. FD $\rightarrow$ TD

Using in the second integral in (3) with (4), a frequency shift $\omega' = \omega - \eta\bar{\omega}_q$, with the q -dependent $\bar{\omega}_q$ defined in (7), leads to

$$\hat{I}(t) = \frac{e^{j\bar{\omega}_q t}}{2\pi j} \int_{-\infty}^{\infty} d\omega' e^{j\omega' t} H_0^{(2)} \left(\rho \sqrt{\frac{(\omega'^2 - \bar{\omega}_q^2)(1 - \eta^2)}{c^2}} \right) \quad (17)$$

Fig. 1. Topology of the ω' complex plane for the case (a) $|\eta| < 1$ and (b) $|\eta| > 1$. Branch points are located at $\omega' = \pm\bar{\omega}_q$. The vertical dotted line at $\omega' = -\eta\bar{\omega}_q$ ($\omega = 0$) separates positive and negative ω frequencies (here, $\eta\bar{\omega}_q > 0$ for simplicity). The dashed region denotes the side of the cuts where $\Re(\omega'^2 - \bar{\omega}_q^2)^{1/2} > 0$, according to the choice of the root for $k_{\rho q}$ in the text after (6). In order to FT the outgoing/decaying FD function, the integration path is moved to the real axis and indented accordingly with respect to the singularities.

in which we have used (13) for $k_{\rho q}$ in the argument of the Hankel function. The integrand has branch points at $\omega' = \pm\bar{\omega}_q$, shown in Fig. 1. Due to the frequency shift $\omega' = \omega - \eta\bar{\omega}_q$, the vertical dashed line at $\omega' = -\eta\bar{\omega}_q$ ($\omega = 0$) separates positive and negative ω frequencies (in the figure, $\eta\bar{\omega}_q > 0$ for simplicity). Depending on $|\eta| \gtrless 1$, two different procedures are applied.

Case $|\eta| < 1$: Using the definition $\tau_0 = \rho\sqrt{1 - \eta^2}/c$, (17) is rewritten as

$$\hat{I}(t) = \frac{e^{j\bar{\omega}_q t}}{2\pi j} \int_{-\infty}^{\infty} d\omega' e^{j\omega' t} H_0^{(2)} \left(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2} \right) \quad (18)$$

in which the square root is defined as $\Im m \sqrt{\omega'^2 - \bar{\omega}_q^2} < 0$ and $\Re e \sqrt{\omega'^2 - \bar{\omega}_q^2} \geq 0$ or ≤ 0 for $\omega > 0$ or $\omega < 0$, respectively, in accord with that for $k_{\rho q}$ in the text after (7). Separation of positive and negative ω occurs at $\omega' = -\eta\bar{\omega}_q$ between the two branch points. In order to Fourier-invert the FD function that satisfies the radiation condition at ∞ [see text after (7)] for any ω , the integration path from $-\infty$ to $+\infty$ is shifted below the branch cuts [see Fig. 1(a)] where the sign of the square root is in accord with the radiation condition at ∞ . This choice is also in agreement with [4, pp. 35] where to ensure the existence of the Fourier pair in (3) the ω' variable in (3) and, therefore, the ω' contour of integration in (18), is shifted slightly from the real ω' axis into $\Im m \omega' < 0$.

Using the large argument asymptotic approximation $H_0^{(2)}(\zeta) \sim (2/(\pi\zeta))^{1/2} \exp(-j\zeta - j\pi/4)$, it is easy to see that for $t < \tau_0$ the integrand decays exponentially for $\Im m \omega' < 0$. Therefore, for $t < \tau_0$, the integration contour can be deformed onto $P_{-\infty}$ where the integral vanishes by Jordan's lemma, and since no singularities are included in the deformation, the integral in (18) vanishes by Cauchy's theorem. For $t > \tau_0$, the integration contour is deformed onto $P_1 + P_2 + P_{\infty}$, with the integral on P_{∞} vanishing. The integration of the even part of the integrand on the symmetric integration path $P_1 + P_2$ vanishes.

The integration of the odd part on P_1 is equal to the contribution from P_2 , and $\hat{I}(t)$ in (18) can be evaluated as twice the integral on P_2

$$\hat{I}(t) = \frac{e^{j\bar{\omega}_q t}}{\pi} \int_{P_2} d\omega' \sin(\omega' t) H_0^{(2)} \left(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2} \right). \quad (19)$$

Since on the upper and lower side of the branch cut the square root assumes opposite negative/positive values, then using the relation $H_0^{(2)}(\xi e^{-j\pi}) = -H_0^{(1)}(\xi)$, reversing the integration path above the real ω' axis, and combining $H_0^{(2)}(\xi) + H_0^{(1)}(\xi) = 2J_0(\xi)$, leads to

$$\hat{I}(t) = e^{j\bar{\omega}_q t} \frac{2}{\pi} \int_{|\bar{\omega}_q|}^{\infty} d\omega' e^{j\omega' t} J_0 \left(\tau_0 \sqrt{\omega'^2 - \bar{\omega}_q^2} \right) \quad (20)$$

in which the square root assumes positive values. This sine transform is given in [2, pp. 113], yielding directly the right-hand side of (4).

Case $|\eta| > 1$: Using the definition $|\tau_0| = \rho \sqrt{\eta^2 - 1}/c$, (17) is rewritten as

$$\hat{I}(t) = \frac{e^{j\bar{\omega}_q t}}{2\pi j} \int_{-\infty}^{\infty} d\omega' e^{j\omega' t} H_0^{(2)} \left(|\tau_0| \sqrt{\bar{\omega}_q^2 - \omega'^2} \right). \quad (21)$$

Since $|\tau_0| > 0$, the square root in (21) is still defined as $\Im m \sqrt{\bar{\omega}_q^2 - \omega'^2} < 0$, and $\Re e \sqrt{\bar{\omega}_q^2 - \omega'^2} \geq 0$ or ≤ 0 for $\omega > 0$ or $\omega < 0$, respectively. Branch points are still located at $\omega' = \pm \bar{\omega}_q$, with the only difference that the branch cuts are now located as in Fig. 1(b). Note that now the separation of positive and negative ω frequencies at $\omega' = -\eta \bar{\omega}_q$ occurs outside the branch point

region, as shown in Fig. 1(b), for the case $\eta \bar{\omega}_q > 0$; similarly, at $\omega' = |\eta \bar{\omega}_q|$ for the case $\eta \bar{\omega}_q < 0$. Moreover, from Fig. 1(b), the whole region $-|\bar{\omega}_q| < \omega' < |\bar{\omega}_q|$ corresponds to $\omega > 0$. Therefore, in order to Fourier-invert the FD function that satisfies the radiation condition at ∞ [see text after (7)] for any ω , the integration path from $-\infty$ to $+\infty$ is indented between the cuts as in Fig. 1(b), where $\Re e \sqrt{\bar{\omega}_q^2 - \omega'^2} \geq 0$ (for $\eta \bar{\omega}_q > 0$). In the case of $\eta \bar{\omega}_q < 0$, the whole region $-|\bar{\omega}_q| < \omega' < |\bar{\omega}_q|$ corresponds to $\omega < 0$, and the integration path is still defined in between the cuts, where now $\Re e \sqrt{\bar{\omega}_q^2 - \omega'^2} \leq 0$. Since for any deformation a branch point singularity is included, the integral is nonvanishing for any value of $t \gtrless |\tau_0|$, and its evaluation is given in [4, pp. 481] which leads directly to the right-hand side of (4).

ACKNOWLEDGMENT

The author would like to thank Prof. L. B. Felsen for his suggestions and encouragement.

REFERENCES

- [1] L. B. Felsen and F. Capolino, "Time domain Green's function for an infinite sequentially excited periodic line array of dipoles," *IEEE Trans. Antennas Propagat.*, vol. 48, pp. 921–931, June 2000.
- [2] A. Erdélyi *et al.*, *Tables of Integral Transforms*. New York: McGraw-Hill, 1954, vol. I.
- [3] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1970.
- [4] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*. Englewood Cliffs, NJ: Prentice-Hall, 1973.